First occurrence indices of tempered representations of metaplectic groups

Ivan Matić

Abstract

We explicitly determine the first occurrence indices of tempered representations of metaplectic groups over a non-archimedean local field of characteristic zero with odd residual characteristic.

Abstract

We explicitly determine the first occurrence indices of tempered representations of metaplectic groups over a non-archimedean local field of characteristic zero.

1 Introduction

In this paper we study the theta correspondence for the metaplectic odd-orthogonal reductive dual pair and determine the first occurrence indices of tempered representations of the metaplectic group over a non-archimedean local field of characteristic zero. It is well known that a tempered representation can be obtained as a subrepresentation of the representation parabolically induced from discrete series of general linear groups (or their two-fold covers) and a discrete series of the group of same type and smaller rank. Therefore, we provide a description of the first occurrence indices of tempered representations in terms of the first occurrence indices of discrete series of metaplectic groups, which have been obtained in our previous work ([13]).

In the last several years, a substantial progress has been made in studying the local theta correspondence. In particular, the conservation relation,

MSC2000: primary 22E35; secondary 11F27, 22E50

Keywords: theta correspondence, metaplectic groups, tempered representations

which states that the first occurrence indices of an irreducible admissible representation of the rank n metaplectic group sum up to $2n$, has recently been proved by Sun and Zhu (and independently by Gan and Ichino in [3, Theorem 9.3]).

This deep result reduces the investigation of the first occurrence indices to determination of one of them. For tempered representations, we determine the larger first occurrence index, i.e., the first occurrence index corresponding to the orthogonal group of the space of larger dimension. To determine such first occurrence index, we use methods for pushing down the theta lifts in the same way as in [12] and [13], together with description of tempered representations of metaplectic groups arising from the work of Gan and Savin, and some elementary properties of Jacquet modules of tempered representations. Also, since our description is given in terms of the first occurrence indices of discrete series representations, an important role is played by precise knowledge of the structure of the first non-zero lifts of discrete series of metaplectic groups.

An analogous problem for the symplectic even-orthogonal dual pair has been addressed in [19]. However, in an exceptional case we also provide a simple criterion for differentiating between tempered representations with different first occurrence indices, in terms of their Jacquet modules.

We now describe the contents of the paper in more detail. In the next section we set up notation and terminology, while in the third section we recall standard facts on the theta correspondence and review basic techniques for determining the first occurrence indices. In Section 4 we provide a precise description of the first non-zero theta lifts of discrete series in the orthogonal tower corresponding to the larger first occurrence index. Section 5 is devoted to determination of the first occurrence indices of tempered representations of metaplectic groups.

The author would like to thank Šime Ungar for help with English language. The author would also like to thank the referee for a number of corrections and very useful suggestions.

This work has been supported by Croatian Science Foundation under the project 9364.

2 Notation

Let F stand for a non-archimedean local field of characteristic zero with odd residual characteristic.

Let $\widetilde{Sp(n)}$ denote the metaplectic group of rank n, the unique non-trivial two-fold central extension of the symplectic group $Sp(n, F)$. In other words, we have

$$1 \to \mu_2 \to \widetilde{Sp(n)} \to Sp(n, F) \to 1,$$

where $\mu_2 = \{1, -1\}$.

In this paper we are interested only in genuine representations of $\widetilde{Sp(n)}$ (i.e., those which do not factor through μ_2). Thus, let $\mathrm{Irr}(\widetilde{Sp(n)})$ stand for the set of isomorphism classes of irreducible admissible genuine representations of the group $\widetilde{Sp(n)}$. Furthermore, let $\mathcal{S}(\widetilde{Sp(n)})$ denote the Grothendieck group of the category of all admissible genuine finite length representations of $\widetilde{Sp(n)}$ and we define $\mathcal{S} = \bigoplus_{n\geq 0} \mathcal{S}(\widetilde{Sp(n)})$.

Let V_0 be an anisotropic quadratic space over F of odd dimension (recall that its dimension can only be 1 or 3). To obtain the odd orthogonal tower, for each non-negative integer r let V_r be the orthogonal direct sum of V_0 with r hyperbolic planes. We assume that V_r comes equipped with a fixed Witt decomposition $V_r = V_r' \oplus V_0 \oplus V_r''$ and with bases $\{v_1', \ldots, v_r'\}$ for V_r' and $\{v_1'', \ldots, v_r''\}$ for V_r'' satisfying $(v_i', v_j') = (v_i'', v_j'') = 0$ and $(v_i', v_j'') = \delta_{ij}$. The corresponding orthogonal group will be denoted by $O(V_r)$. Set $m_r = \frac{1}{2}\mathrm{dim}V_r$.

To a fixed quadratic character χ of $F^\times$ one can attach two odd orthogonal towers, one with $\dim V_0 = 1$ ($+$-tower) and the other with $\dim V_0 = 3$ ($-$-tower), as in Chapter V of [10]. The corresponding orthogonal groups will be denoted by $O(V_r^+)$ and $O(V_r^-)$.

Similarly as before, let $\mathrm{Irr}(O(V_r))$ denote the set of isomorphism classes of irreducible admissible representations of the orthogonal group $O(V_r)$.

Let $\widetilde{GL(n, F)}$ denote the double cover of the general linear group $GL(n, F)$, where the multiplication is given by $(g_1, \epsilon_1)(g_2, \epsilon_2) = (g_1 g_2, \epsilon_1\epsilon_2(\mathrm{det}g_1, \mathrm{det}g_2)_F)$. Here $\epsilon_i \in \mu_2$, $i = 1, 2$, and $(\cdot, \cdot)_F$ denotes the Hilbert symbol of the field F.

In what follows, we fix a non-trivial additive character ψ of F. We denote by χ_ψ the genuine character of $\widetilde{GL(n, F)}$ given by $\chi_\psi(g, \epsilon) = \epsilon\gamma(\mathrm{det}g, \psi_{\frac{1}{2}})^{-1}$, where γ denotes the Weil index, and for $a \in F^\times$ we have $\psi_a(x) = \psi(ax)$.

For $V = V_r$ we denote by χ_V the quadratic character of $F^\times$ given by $\chi_V(x) = (x, (-1)^k \det(V))_F$, where $\det(V)$ is the determinant of the matrix of the bilinear form on V and $k = \dim V \cdot (\dim V - 1)/2$. We note that this character is, in fact, independent of r, by [9, page 240]. To simplify the notation, the character of $GL(n, F)$ given by $g \mapsto \chi_V(\det g)$ will also be denoted by χ_V. Furthermore, we define the character $\chi_{V,\psi}$ of $\widetilde{GL(n,F)}$ by $\chi_{V,\psi}(g, \epsilon) = \chi_V(g)\chi_\psi(g, \epsilon)$. We write $\alpha = \chi_\psi^2$ and note that [10, Lemma 4.1] implies that α is a quadratic character of $GL(n, F)$.

We define $\mathcal{R}^{gen} = \oplus_n \mathcal{R}(\widetilde{GL(n,F)})_{gen}$, where $\mathcal{R}(\widetilde{GL(n,F)})_{gen}$ denotes the Grothendieck group of the category of all admissible genuine finite length representations of $\widetilde{GL(n,F)}$. Also, we denote by $\mathrm{Irr}(GL(n, F))$ the set of isomorphism classes of irreducible admissible representations of $GL(n, F)$ and by $\mathrm{Irr}(\widetilde{GL(n,F)})$ the set of isomorphism classes of irreducible admissible representations of $\widetilde{GL(n,F)}$. Using the genuine character χ_ψ, in the same way as in [4, Section 2] or in [8, Section 4], one obtains a bijection between $\mathrm{Irr}(GL(n, F))$ and $\mathrm{Irr}(\widetilde{GL(n,F)})$ via $\rho \mapsto \chi_\psi \rho = \chi_\psi \otimes \rho$.

Throughout the paper, ν stands for the character of $GL(n, F)$ defined by $|\det|_F$. If ρ is an irreducible cuspidal representation of $GL(n_\rho, F)$ (this defines n_ρ), or such genuine representation of $\widetilde{GL(n_\rho,F)}$, we call the set $\Delta = \{\nu^a\rho, \nu^{a+1}\rho, \ldots, \nu^{a+k}\rho\}$ a segment, where $a \in \mathbb{R}$ and $k \in \mathbb{Z}_{\geq 0}$. In the sequel, we abbreviate $\{\nu^a\rho, \nu^{a+1}\rho, \ldots, \nu^{a+k}\rho\}$ to $[\nu^a\rho, \nu^{a+k}\rho]$. We denote by $\delta(\Delta)$ the unique irreducible quotient of $\nu^a\rho \times \nu^{a+1}\rho \times \cdots \times \nu^{a+k}\rho$. The representation $\delta(\Delta)$ is an essentially square-integrable representation attached to the segment Δ. If ρ is a genuine representation, then so is $\delta(\Delta)$ (by [8, Proposition 4.2]).

To shorten the notation, for a non-negative half integer x and an irreducible cuspidal representation ρ of $GL(n_\rho, F)$ we denote by $\delta(x, \rho)^k$ the induced representation

$$\delta([\nu^{-x}\rho, \nu^x\rho]) \times \delta([\nu^{-x}\rho, \nu^x\rho]) \times \cdots \times \delta([\nu^{-x}\rho, \nu^x\rho]),$$

where $\delta([\nu^{-x}\rho, \nu^x\rho])$ appears k times. Also, the induced representation $\nu^x 1_{F^\times} \times \nu^x 1_{F^\times} \times \cdots \times \nu^x 1_{F^\times}$, where $\nu^x 1_{F^\times}$ appears k times, will be denoted by $(\nu^x 1_{F^\times})^k$.

We define $\delta(x, \chi_\psi\rho)^k$ and $(\nu^x\chi_\psi 1_{F^\times})^k$ in the completely analogous way. It follows from [25, Theorem 9.7] that all these induced representations are irreducible. Also, all these representations are non-degenerate in the sense of [2, page 455].

For an ordered partition $s = (n_1, n_2, \ldots, n_i)$ of some $m \leq n$, we denote by P_s the standard parabolic subgroup of $Sp(n, F)$ (consisting of block upper-triangular matrices) whose Levi subgroup M_s equals $GL(n_1, F) \times GL(n_2, F) \times \cdots \times GL(n_i, F) \times Sp(n-m, F)$. Then the standard parabolic subgroup $\widetilde{P_s}$ of $\widetilde{Sp(n)}$ is the preimage of P_s in $\widetilde{Sp(n)}$ and its Levi subgroup $\widetilde{M_s}$ differs from the product $\widetilde{GL(n_1, F)} \times \widetilde{GL(n_2, F)} \times \cdots \times \widetilde{GL(n_i, F)} \times \widetilde{Sp(n-m, F)}$ by a finite subgroup. This enables us to write every irreducible genuine representation π of $\widetilde{M_s}$ in the form $\pi_1 \otimes \pi_2 \otimes \cdots \otimes \pi_i \otimes \sigma$ where all representations $\pi_1, \pi_2, \ldots, \pi_i, \sigma$ are genuine at the same time. We will denote the representation of $\widetilde{Sp(n)}$ parabolically induced from $\pi_1 \otimes \pi_2 \otimes \cdots \otimes \pi_i \otimes \sigma$ by $\pi_1 \times \pi_2 \times \cdots \times \pi_i \rtimes \sigma$. If $s = (k)$, for some $0 \leq k \leq n$, we denote P_s briefly by P_k and $\widetilde{P_s}$ briefly by $\widetilde{P_k}$. The standard parabolic subgroups of $O(V_n)$ have an analogous description as those of $Sp(n, F)$. The normalized Jacquet module of a smooth representation σ of $\widetilde{Sp(n)}$ with respect to the standard parabolic subgroup $\widetilde{P_s}$ will be denoted by $R_{\widetilde{P_s}}(\sigma)$. It can be easily checked that for an irreducible representation σ of $\widetilde{Sp(n)}$, $R_{\widetilde{P_s}}(\sigma)$ is a genuine representation of $\widetilde{M_s}$ and, as such, can be interpreted as an element of $\mathcal{R}^{gen} \otimes \mathcal{S}$, the Grothendieck group of the category of all admissible genuine finite length representations of Levi subgroups of the maximal parabolic subgroups of $\widetilde{Sp(n)}$, for all n. The normalized Jacquet module of a smooth representation σ of $O(V_r)$ with respect to the standard parabolic subgroup Q_s will be denoted by $R_{Q_s}(\sigma)$. For a cuspidal representation ρ of $\mathrm{Irr}(GL(n_\rho, F))$ and a genuine representation σ of $\widetilde{Sp(n)}$, we write $R_{\widetilde{P}_{n_\rho}}(\sigma)(\chi_\psi \rho)$ for the maximal $\chi_\psi \rho$-isotypic quotient of $R_{\widetilde{P}_{n_\rho}}(\sigma)$. It is a maximal direct summand of $R_{\widetilde{P}_{n_\rho}}(\sigma)$ on which $\widetilde{GL(n_\rho, F)}$ acts by $\chi_\psi \rho$. Also, let $R_{Q_{n_\rho}}(\tau)(\rho)$ denote the maximal ρ-isotypic quotient of $R_{Q_{n_\rho}}(\tau)$ for representation τ of $O(V_r)$.

Let $\sigma \in \mathrm{Irr}(\widetilde{Sp(n)})$. We define $\mu^*(\sigma) \in \mathcal{R}^{gen} \otimes \mathcal{S}$ by

$$\mu^*(\sigma) = \sum_{k=0}^{n} \text{s.s.}(R_{\widetilde{P}_k}(\sigma))$$

(s.s. denotes the semisimplification), and extend μ^* linearly to the whole of $\mathcal{S}$. In the same way μ^* can be defined for irreducible representations of classical groups. If $\mu^*(\sigma)$ contains a constituent of the form $\pi \otimes \sigma'$, we write $\mu^*(\sigma) \geq \pi \otimes \sigma'$.

In the following theorem we recall the metaplectic version of the structure formula (obtained in [8]), due to Tadić in the classical group case ([22]).

Theorem 2.1. *Let $\rho \in \mathcal{R}^{gen}$ be an irreducible cuspidal representation, and $a, b \in \mathbb{R}$ be such that $a + b \in \mathbb{Z}_{\geq 0}$. Let σ be an admissible genuine representation of finite length of $\widetilde{Sp(n)}$. Let $\mu^*(\sigma) = \sum_{\pi,\sigma'} \pi \otimes \sigma'$. Then we have:*

$$\mu^*(\delta([\nu^{-a}\rho, \nu^b\rho]) \rtimes \sigma) = \sum_{i=-a-1}^{b} \sum_{j=i}^{b} \sum_{\pi,\sigma'} \delta([\nu^{-i}\alpha\widetilde{\rho}, \nu^a\alpha\widetilde{\rho}]) \times \delta([\nu^{j+1}\rho, \nu^b\rho]) \times \pi \otimes \delta([\nu^{i+1}\rho, \nu^j\rho]) \rtimes \sigma'$$

where $\widetilde{\rho}$ denotes the contragredient of ρ. We omit $\delta([\nu^x\rho, \nu^y\rho])$ if $x > y$.

Also, we recall that by the Mœglin-Tadić classification, which is given in [14, 16], several invariants are attached to a discrete series representation σ of an orthogonal group. One of them is the partial cuspidal support of σ, which we denote by σ_{sc}. The other invariant attached to σ is its Jordan block, denotes by $\mathrm{Jord}(\sigma)$, and defined as the set of all pairs (a, ρ) where a is a positive integer and $\rho \in \mathrm{Irr}(GL(n_\rho, F))$ is a cuspidal self-contragredient representation such that the following two conditions are satisfied:

1. $\frac{a-1}{2} - s_{\rho,\sigma_{sc}}$ is an integer, for the unique non-negative $s_{\rho,\sigma_{sc}}$ such that $\nu^{s_{\rho,\sigma_{sc}}}\rho \rtimes \sigma_{sc}$ reduces (such $s_{\rho,\sigma_{sc}}$ exists by [20]),
2. the induced representation $\delta([\nu^{-\frac{a-1}{2}}\rho, \nu^{\frac{a-1}{2}}\rho]) \rtimes \sigma$ is irreducible.

For a cuspidal self-contragredient representation $\rho \in \mathrm{Irr}(GL(n_\rho, F))$, we set $\mathrm{Jord}_\rho(\sigma) = \{a \in \mathbb{Z} : (a, \rho) \in \mathrm{Jord}(\sigma)\}$.

We emphasize that the Mœglin-Tadić classification now holds unconditionally, since the natural hypothesis on which this classification is based now follows from the results of [1] and is proved in [15, Théorème 3.1.1].

3 Preliminary results on theta correspondence

In this section we review some results about the theta correspondence which will be used later.

The pair $(Sp(n, F), O(V_r))$ is a reductive dual pair in $Sp(n \cdot \mathrm{dim}V_r, F)$ and the theta correspondence relates the representations of the metaplectic group $\widetilde{Sp(n)}$ and those of the orthogonal group $O(V_r)$. Define $n_1 = n \cdot \mathrm{dim} V_r$ and let $\omega_{n,r}$ denote the pull-back of the Weil representation $\omega_{n_1,\psi}$ of the group $\widetilde{Sp(n_1)}$, restricted to the dual pair $\widetilde{Sp(n)} \times O(V_r)$ (as in [10]).

For $\sigma \in \mathrm{Irr}(\widetilde{Sp(n)})$, we let $\Theta(\sigma, r)$ denote the smooth representation of $O(V_r)$ given as the big theta lift of σ to the r-th level of the orthogonal tower. The big theta lift $\Theta(\sigma, r)$ is the maximal σ-isotypic quotient of $\omega_{n,r}$. Specially, we write $\Theta^+(\sigma, r)$ for the big theta lift of σ to the r-th level of the $+$-orthogonal tower and $\Theta^-(\sigma, r)$ for the big theta lift of σ to the r-th level of the $-$-orthogonal tower,

Similarly, for $\tau \in \mathrm{Irr}(O(V_r))$ we denote by $\Theta(\tau, n)$ the big theta lift of the representation τ, which is a smooth genuine representation of $\widetilde{Sp(n)}$.

In the following theorem we summarize important results on the theta correspondence, proved in [10], [17], [21] and [24]. We note that the second part of the theorem, which is on the Howe duality, is now proved for any residual characteristic by Gan and Takeda ([6]).

Theorem 3.1. *For $\sigma \in Irr(\widetilde{Sp(n)})$ there is a non-negative integer r such that $\Theta(\sigma, r) \neq 0$. The smallest such r is called the first occurrence index of σ in the orthogonal tower and will be denoted by $r(\sigma)$. Also, $\Theta(\sigma, r') \neq 0$ for $r' \geq r$. We write $r^+(\sigma)$ for the first occurrence index of σ in the $+$-orthogonal tower and $r^-(\sigma)$ for the first occurrence index of σ in the $-$-orthogonal tower.*

The first occurrence indices satisfy the following equality, called the conservation relation:

$$r^+(\sigma) + r^-(\sigma) = 2n.$$

The representation $\Theta(\sigma, r)$ is either zero or it has a unique irreducible quotient. We denote this unique irreducible quotient by $\theta(\sigma, r)$. Also, we write $\theta^+(\sigma, r)$ for this irreducible quotient in the $+$-orthogonal tower and $\theta^-(\sigma, r)$ for this irreducible quotient in the $-$-orthogonal tower.

If σ_1 and σ_2 are irreducible genuine representations of $\widetilde{Sp(n)}$ such that $\theta(\sigma_1, r) \neq 0$ and $\theta(\sigma_1, r) \cong \theta(\sigma_2, r)$, then $\sigma_1 \cong \sigma_2$.

For $\tau \in Irr(O(V_r))$, the representation $\Theta(\tau, n)$ is either zero or it has a unique irreducible quotient, which we denote by $\theta(\tau, n)$. If τ_1 and τ_2 are irreducible representations of $O(V_r)$ such that $\theta(\tau_1, n) \neq 0$ and $\theta(\tau_1, n) \cong \theta(\tau_2, n)$, then $\tau_1 \cong \tau_2$.

In the rest of this section we fix an odd orthogonal tower and denote by $\chi_{V,\psi}$ the character of $\widetilde{GL(n,F)}$ related to this orthogonal tower and to character ψ.

Now we state a criterion ([12, Proposition 5.1] and [13, Corollary 6.4]) for pushing down the lifts of irreducible representations.

Lemma 3.2. *Suppose that σ is an irreducible genuine representation of $\widetilde{Sp(n)}$. Then $\Theta(\sigma, r) \neq 0$ implies $R_{P_1}(\Theta(\sigma, r+1))(\nu^{-(m_{r+1}-n-1)}1_{F^\times}) \neq 0$.*

Furthermore, if $R_{\widetilde{P_1}}(\sigma)(\nu^{-(m_{r+1}-n-1)}\chi_{V,\psi}1_{F^\times}) = 0$, then $\Theta(\sigma, r) \neq 0$ if and only if $R_{P_1}(\Theta(\sigma, r+1))(\nu^{-(m_{r+1}-n-1)}1_{F^\times}) \neq 0$.

Also, if σ is a discrete series representation of $\widetilde{Sp(n)}$ and $\Theta(\sigma, r) \neq 0$, then $\theta(\sigma, r+1)$ is a subrepresentation of the induced representation $\nu^{-(m_{r+1}-n-1)}1_{F^\times} \rtimes \theta(\sigma, r)$.

We take a moment to state several results which will be frequently used in the paper. The first one is ([13, Proposition 3.7]).

Proposition 3.3. *Suppose that an irreducible representation $\sigma \in Irr(\widetilde{Sp(n)})$ can be written as an irreducible subrepresentation of the induced representation $\delta([\nu^a \chi_{V,\psi}\rho, \nu^b \chi_{V,\psi}\rho]) \rtimes \sigma'$, where ρ is an irreducible cuspidal representation of $GL(n_\rho, F)$, $\sigma' \in Irr(\widetilde{Sp(n')})$ and $b-a \geq 0$. If $\Theta(\sigma, r) \neq 0$ and $(a, \rho) \neq (m_r - n, 1_{F^\times})$, then there exists an irreducible representation τ of some $O(V_{r'})$ such that $\theta(\sigma, r)$ is a subrepresentation of $\delta([\nu^a\rho, \nu^b\rho]) \rtimes \tau$. Furthermore, suppose that if $\mu^*(\sigma)$ contains the representation $\delta([\nu^a \chi_{V,\psi}\rho, \nu^b \chi_{V,\psi}\rho]) \otimes \sigma''$, for some irreducible genuine representation σ'' of $\widetilde{Sp(n')}$, then $\sigma'' \cong \sigma'$. Then $\theta(\sigma, r)$ is a subrepresentation of*

$$\delta([\nu^a\rho, \nu^b\rho]) \rtimes \theta(\sigma', r - n + n').$$

We note that there is an analogous result for irreducible admissible representations of odd orthogonal groups ([13, Proposition 3.8]).

The following two propositions play an important role in determination of theta lifts of tempered representations and can be proved in an analogous way as [19, Theorem 3.8] and [19, Theorem 3.9].

Proposition 3.4. *Let σ denote an irreducible genuine representation of $\widetilde{Sp(n)}$. If $\mu^*(\sigma) \geq (\nu^x \chi_{V,\psi}1_{F^\times})^k \otimes \sigma'$ for some irreducible representation σ' and $\Theta(\sigma, r) \neq 0$ for $x \neq m_r - n$, then $\mu^*(\theta(\sigma, r)) \geq (\nu^x 1_{F^\times})^k \otimes \pi$ for some irreducible representation π. Also, if σ is a subrepresentation of the induced* *representation* $\delta(x,\chi_{V,\psi}\rho)^k \rtimes \sigma'$, *for some* $\sigma' \in Irr(\widetilde{Sp(n')})$ *such that* $\mu^*(\sigma) \geq \delta(x,\chi_{V,\psi}\rho)^k \otimes \sigma''$ *leads to* $\sigma'' \cong \sigma'$, *then* $\Theta(\sigma, r) \neq 0$ *and* $(x,\rho) \neq (m_r - n, 1_{F^\times})$ *imply that* $\theta(\sigma, r)$ *is a subrepresentation of* $\delta(x,\rho)^k \rtimes \theta(\sigma', r-n+n')$.

Proposition 3.5. *Let* τ *denote an irreducible representation of* $O(V_r)$. *If* $\mu^*(\tau) \geq (\nu^x 1_{F^\times})^k \otimes \tau'$ *for some irreducible representation* τ' *and* $\Theta(\tau, n) \neq 0$ *for* $x \neq n - m_r + 1$, *then* $\mu^*(\theta(\tau, r)) \geq (\nu^x \chi_{V,\psi} 1_{F^\times})^k \otimes \pi$ *for some irreducible representation* π. *Also, if* τ *is a subrepresentation of the induced representation* $\delta(x,\rho)^k \rtimes \tau'$, *for some* $\tau' \in Irr(O(V_{r'}))$ *such that* $\mu^*(\tau) \geq \delta(x,\rho)^k \otimes \tau''$ *implies* $\tau'' \cong \tau'$, *then* $\Theta(\tau, n) \neq 0$ *and* $(x,\rho) \neq (n - m_r + 1, 1_{F^\times})$ *imply that* $\theta(\tau, r)$ *is a subrepresentation of* $\delta(x,\chi_{V,\psi}\rho)^k \rtimes \theta(\tau', n-r+r')$.

4 Theta lifts of discrete series

In this section we discuss the theta lifts of discrete series representations of metaplectic groups. Set $t_+ = 0$ and $t_- = 1$ and let $\sigma \in \mathrm{Irr}(\widetilde{Sp(n)})$ denote a discrete series representation. By [4, Theorem 1.1], there is a unique $\epsilon \in \{+,-\}$ such that $\Theta^\epsilon(\sigma, n - t_\epsilon) \neq 0$. We will determine the structure of the first non-zero theta lift of the representation σ in the $-\epsilon$-tower. Since for strongly positive σ the structure of $\theta^{-\epsilon}(\sigma, r^{-\epsilon}(\sigma))$ is given in Section 4 of [13], we assume that σ is a non-strongly positive discrete series. In what follows, we denote by χ_V the quadratic character of $GL(n,F)$ related to $-\epsilon$-tower and define $\chi_{V,\psi}$ in the same way as before.

It has been proved in Section 6 of [13] that for each σ there is an ordered s-tuple $S = (\sigma_0, \sigma_1, \ldots, \sigma_{s-1})$ of discrete series representations, $\sigma_i \in \mathrm{Irr}(\widetilde{Sp(n_i)})$, where $\sigma_{s-1} \cong \sigma$ and σ_0 is strongly positive, such that the following properties hold:

(i) for every $i \in \{1,2,\ldots,s-1\}$ there exist a self-contragredient cuspidal representations $\rho_i \in \mathrm{Irr}(GL(m_i,F))$ and non-negative half-integers a_i, b_i with $b_i - a_i \in \mathbb{Z}_{>0}$ and $a_i - c \in \mathbb{Z}$ for all $2c-1 \in \mathrm{Jord}_{\rho_i}(\theta^\epsilon(\sigma_{i-1}, n_{i-1} - t_\epsilon))$, such that σ_i is a subrepresentation of $\delta([\nu^{-a_i}\chi_{V,\psi}\rho_i, \nu^{b_i}\chi_{V,\psi}\rho_i]) \rtimes \sigma_{i-1}$ and $\mathrm{Jord}_{\rho_i}(\theta^\epsilon(\sigma_{i-1}, n_{i-1} - t_\epsilon)) \cap [2a_i+1, 2b_i+1] = \emptyset$ (we note that this also gives $R_{\widetilde{P}_{m_i}}(\sigma_{i-1})(\nu^x \chi_{V,\psi}\rho_i) = 0$ for $a_i \leq x \leq b_i$);

(ii) if $\rho_i \cong \rho_j$ for $i < j$, then $\rho_i \cong \rho_l$ for $l \in \{i+1, i+2, \ldots, j\}$;

(iii) if $\rho_i \cong \rho_{i+1}$ then $a_i < a_{i+1}$;

(iv) if there is some $i \in \{1, 2, \ldots, s-1\}$ such that $\rho_i \cong 1_{F^\times}$, then $\rho_1 \cong 1_{F^\times}$.

We note that $\Theta^\epsilon(\sigma_i, n_i - t_\epsilon) \neq 0$ for all $i \in \{0, 1, \ldots, s-1\}$. Also, if $\mu^*(\sigma_i)$ contains some irreducible constituent of the form $\delta([\nu^{-a_i}\chi_{V,\psi}\rho_i, \nu^{b_i}\chi_{V,\psi}\rho_i]) \otimes \sigma'$, then $\sigma' \cong \sigma_{i-1}$.

Let us denote by $U(\sigma)$ the set of all such ordered s-tuples of discrete series representations. To each $S \in U(\sigma)$ we attach a non-negative half-integer $\min(S)$ which is the minimal x such that σ_0 can be written as the unique irreducible subrepresentation of the induced representation of the form $\delta([\nu^x\chi_{V,\psi}1_{F^\times}, \nu^y\chi_{V,\psi}1_{F^\times}]) \rtimes \sigma_{sp}$, $y \geq x$, for a strongly positive discrete series σ_{sp}, or zero if such x does not exist.

We call an ordered s-tuple $S \in U(\sigma)$ minimal if $\min(S) \leq \min(S')$ for every $S' \in U(\sigma)$.

In what follows, we fix a minimal ordered s-tuple $S = (\sigma_0, \sigma_1, \ldots, \sigma_{s-1})$ and write $\sigma_i \hookrightarrow \delta([\nu^{-a_i}\chi_{V,\psi}\rho_i,\ \nu^{b_i}\chi_{V,\psi}\rho_i]) \rtimes \sigma_{i-1}$ in the same way as in (i) above. Set $m = n_1 - t_\epsilon - r^\epsilon(\sigma_1)$. We denote by k the largest integer j, $1 \leq j \leq s-1$, such that $(a_i, \rho_i) = (m + i - \frac{1}{2}, 1_{F^\times})$ for $i = 1, 2, \ldots, j$. If there is no such j, we set $k = 0$. If $k > 0$, we denote by l the largest integer j, $1 \leq j \leq k$, such that $R_{\widetilde{P_1}}(\sigma)(\nu^{a_i}\chi_{V,\psi}1_{F^\times}) = 0$ for $i = 1, 2, \ldots, j$. If there is no such j, or $k = 0$, we set $l = 0$.

By [13, Proposition 6.2], we have

$$r^{-\epsilon}(\sigma) = n - t_{-\epsilon} + m + l + 1. \tag{1}$$

To describe the representation $\theta^{-\epsilon}(\sigma, r^{-\epsilon}(\sigma))$ we need the following two lemmas. The first one presents an important part of Mœglin-Tadić classification of discrete series and its proof can be found in Sections 9 and 10 of [16] and in [18, Theorem 2.1].

Lemma 4.1. *Suppose that π is a discrete series of an orthogonal group and $\rho \in Irr(GL_{n_\rho}, F)$ is a self-contragredient cuspidal representation. Let a and b denote non-negative half integers such that $b - a \in \mathbb{Z}_{>0}$ and $a - c \in \mathbb{Z}$ for all $2c+1 \in Jord_\rho(\pi)$. Also, suppose $Jord_\rho(\pi) \cap [2a+1, 2b+1] = \emptyset$. Then the induced representation*

$$\delta([\nu^{-a}\rho, \nu^b\rho]) \rtimes \pi$$

contains two irreducible subrepresentations which are non-isomorphic and square-integrable.

Lemma 4.2. *Let π denote a discrete series of an orthogonal group and let $\rho \in Irr(GL_{n_\rho}, F)$, $\rho' \in Irr(GL_{n_{\rho'}}, F)$ denote self-contragredient cuspidal representations. Let a and b stand for non-negative half integers such that $b - a \in \mathbb{Z}_{>0}$ and $a - c \in \mathbb{Z}$ for all $2c + 1 \in Jord_\rho(\pi)$. Also, suppose $Jord_\rho(\pi) \cap [2a+1, 2b+1] = \emptyset$. Let x denote a non-negative half integer such that $x - c \in \mathbb{Z}$ for all $2c+1 \in Jord_{\rho'}(\pi)$ and let τ be an irreducible (tempered) subrepresentation of*

$$\delta([\nu^{-x}\rho', \nu^{x}\rho']) \rtimes \pi.$$

If $\rho \cong \rho'$, we additionally assume $x \notin [a, b]$. Then every irreducible subrepresentation of $\delta([\nu^{-a}\rho, \nu^{b}\rho]) \rtimes \tau$ is an irreducible tempered subrepresentation of $\delta([\nu^{-x}\rho', \nu^{x}\rho']) \rtimes \pi'$ for some irreducible square-integrable subrepresentation π' of $\delta([\nu^{-a}\rho, \nu^{b}\rho]) \rtimes \pi$.

Proof. Let π_1 denote an irreducible subrepresentation of $\delta([\nu^{-a}\rho, \nu^{b}\rho]) \rtimes \tau$. The assumption of the lemma gives

$$\pi_1 \hookrightarrow \delta([\nu^{-a}\rho, \nu^{b}\rho]) \times \delta([\nu^{-x}\rho', \nu^{x}\rho']) \rtimes \pi \cong \delta([\nu^{-x}\rho', \nu^{x}\rho']) \times \delta([\nu^{-a}\rho, \nu^{b}\rho]) \rtimes \pi.$$

It can be easily seen, using the structural formula for μ^*, that the irreducible representation $\delta([\nu^{-x}\rho', \nu^{x}\rho']) \otimes \delta([\nu^{-a}\rho, \nu^{b}\rho]) \otimes \pi$ appears with multiplicity four in the Jacquet module of $\delta([\nu^{-x}\rho', \nu^{x}\rho']) \times \delta([\nu^{-a}\rho, \nu^{b}\rho]) \rtimes \pi$ with respect to the appropriate parabolic subgroup.

We will denote non-isomorphic irreducible subrepresentations of the induced representation $\delta([\nu^{-a}\rho, \nu^{b}\rho]) \rtimes \pi$ by π_2 and π_3. Obviously, $\delta([\nu^{-x}\rho', \nu^{x}\rho']) \rtimes \pi_i$ is a subrepresentation of $\delta([\nu^{-x}\rho', \nu^{x}\rho']) \times \delta([\nu^{-a}\rho, \nu^{b}\rho]) \rtimes \pi$ for $i = 2, 3$. Also, $\delta([\nu^{-x}\rho', \nu^{x}\rho']) \rtimes \pi_2$ reduces if and only if $\delta([\nu^{-x}\rho', \nu^{x}\rho']) \rtimes \pi_3$ reduces.

If x is an integer, set $s_1 = (2x + 1) \cdot n_{\rho'}$, otherwise set $s_1 = 2x \cdot n_{\rho'}$. Also, set $s_2 = (a + b + 1) \cdot n_\rho$ and let $s = (s_1, s_2)$.

If the induced representation $\delta([\nu^{-x}\rho', \nu^{x}\rho']) \rtimes \pi_2$ is irreducible, then the irreducible representation $\delta([\nu^{-x}\rho', \nu^{x}\rho']) \otimes \delta([\nu^{-a}\rho, \nu^{b}\rho]) \otimes \pi$ appears with multiplicity two in the Jacquet modules of both $\delta([\nu^{-x}\rho', \nu^{x}\rho']) \rtimes \pi_2$ and $\delta([\nu^{-x}\rho', \nu^{x}\rho']) \rtimes \pi_3$ with respect to the standard parabolic subgroup Q_s.

If the induced representation $\delta([\nu^{-x}\rho', \nu^{x}\rho']) \rtimes \pi_2$ reduces, both representations $\delta([\nu^{-x}\rho', \nu^{x}\rho']) \rtimes \pi_2$ and $\delta([\nu^{-x}\rho', \nu^{x}\rho']) \rtimes \pi_3$ are direct sums of two tempered irreducible subrepresentations and Jacquet modules with respect to to the standard parabolic subgroup Q_s of each of these subrepresentations contains $\delta([\nu^{-x}\rho', \nu^{x}\rho']) \otimes \delta([\nu^{-a}\rho, \nu^{b}\rho]) \otimes \pi$ with multiplicity one.

In any case, there is an $i \in \{2,3\}$ such that π_1 is an irreducible subrepresentation of $\delta([\nu^{-x}\rho', \nu^x\rho']) \rtimes \pi_i$ and π_1 is obviously tempered. This proves the lemma. □

Now we are ready to provide our description of the first non-zero theta lift of discrete series σ in the $-\epsilon$-tower. We note that the first statement of the following theorem is subsumed under [5, Proposition 3.1].

Theorem 4.3. *Let $\sigma \in Irr(\widetilde{Sp(n)})$ be a discrete series representation and let ϵ denote $+$ or $-$ such that $\Theta^\epsilon(\tau, n - t_\epsilon) \neq 0$, where $t_+ = 0$ and $t_- = 1$. The first non-zero theta lift $\theta^{-\epsilon}(\sigma, r^{-\epsilon}(\sigma))$ in the $-\epsilon$-tower is a tempered representation. Let χ_V denote the quadratic character related to $-\epsilon$-tower and let $\chi_{V,\psi} = \chi_V\chi_\psi$. For a minimal ordered s-tuple $S = (\sigma_0, \sigma_1, \dots, \sigma_{s-1})$, $\sigma_i \in Irr(\widetilde{Sp(n_i)})$ and $\sigma_i \hookrightarrow \delta([\nu^{-a_i}\chi_{V,\psi}\rho_i, \nu^{b_i}\chi_{V,\psi}\rho_i]) \rtimes \sigma_{i-1}$, we denote by k the largest integer j, $1 \leq j \leq s-1$, such that $(a_i, \rho_i) = (n_1 - t_\epsilon - r^\epsilon(\sigma_1) + i - \frac{1}{2}, 1_{F^\times})$ for $i = 1, 2, \dots, j$. If there is no such j, we set $k = 0$. If $k > 0$, we denote by l the largest integer j, $1 \leq j \leq k$, such that $R_{\widetilde{P_1}}(\sigma)(\nu^{a_i}\chi_{V,\psi}1_{F^\times}) = 0$ for $i = 1, 2, \dots, j$. If there is no such j, or $k = 0$, we set $l = 0$. Then $\theta^{-\epsilon}(\sigma, r^{-\epsilon}(\sigma))$ is a discrete series representation if and only if $k = l$ and $b_k > a_k + 1$, if $k > 0$.*

Proof. Several possibilities will be considered separately. Let us first discuss the case $k = 0$. In this case, it follows directly from the result obtained in Section 4 of [13] that $\mathrm{Jord}_{\rho_i}(\theta^{-\epsilon}(\sigma_0, r^{-\epsilon}(\sigma_0))) \cap [2a_i + 1, 2b_i + 1] = \emptyset$ for $i = 1, 2, \dots, s-1$.

Using the description of the first occurrence indices given by (1) and Proposition 3.3, we deduce that $\theta^{-\epsilon}(\sigma_i, r^{-\epsilon}(\sigma_i))$ is an irreducible subrepresentation of

$$\delta([\nu^{-a_i}\rho_i, \nu^{b_i}\rho_i]) \rtimes \theta^{-\epsilon}(\sigma_{i-1}, r^{-\epsilon}(\sigma_{i-1}))$$

for all $i = 1, 2, \dots, s-1$. Now the description of Jordan blocks of an induced representation, given in [16, Proposition 2.1], enables us to use Lemma 4.1 to conclude that the representation $\theta^{-\epsilon}(\sigma_i, r^{-\epsilon}(\sigma_i))$ is a discrete series representation for all $i = 1, 2, \dots, s-1$ and $\mathrm{Jord}_{\rho_i}(\theta^{-\epsilon}(\sigma_{i-1}, r^{-\epsilon}(\sigma_{i-1}))) \cap [2a_i + 1, 2b_i + 1] = \emptyset$ for $i = 2, 3, \dots, s-1$.

Now we consider the case $k > 0$. If $l = 0$, then $\theta^{-\epsilon}(\sigma_1, r^{-\epsilon}(\sigma_1))$ is an irreducible subrepresentation of

$$\delta([\nu^{-a_1}1_{F^\times}, \nu^{b_1}1_{F^\times}]) \rtimes \theta^{-\epsilon}(\sigma_0, r^{-\epsilon}(\sigma_0)) \tag{2}$$

and $2a_1+1 \in \mathrm{Jord}_{1_{F^\times}}(\theta^{-\epsilon}(\sigma_0, r^{-\epsilon}(\sigma_0))$. It can be easily seen that $\theta^{-\epsilon}(\sigma_1, r^{-\epsilon}(\sigma_1))$ is not a discrete series representation. Let us assume that it is a non-tempered representation. Then there is an irreducible constituent of $\mu^*(\theta^{-\epsilon}(\sigma_1, r^{-\epsilon}(\sigma_1)))$ of the form $\delta([\nu^{-c}\rho, \nu^d\rho]) \otimes \pi$, such that $-c+d<0$. Since $\theta^{-\epsilon}(\sigma_0, r^{-\epsilon}(\sigma_0))$ is a discrete series representation, applying the formula for μ^* to (2), we obtain $\rho \cong 1_{F^\times}$ and $c > a_1$. Using Proposition 3.8 of [13], we get a contradiction with the square-integrability of σ_1. Thus, $\theta^{-\epsilon}(\sigma_1, r^{-\epsilon}(\sigma_1))$ is a tempered representation and it is a subrepresentation of

$$\delta([\nu^{-a_1}1_{F^\times}, \nu^{a_1}1_{F^\times}]) \rtimes \sigma',$$

where σ' is a discrete series subquotient of the induced representation

$$\delta([\nu^{a_1+1}1_{F^\times}, \nu^{b_1}1_{F^\times}]) \rtimes \theta^{-\epsilon}(\sigma_0, r^{-\epsilon}(\sigma_0)).$$

We note that $\mathrm{Jord}(\sigma') = \mathrm{Jord}(\theta^{-\epsilon}(\sigma_0, r^{-\epsilon}(\sigma_0))) \setminus \{(2a_1+1, 1_{F^\times})\} \cup \{(2b_1+1, 1_{F^\times})\}$. Also, [23, Theorem 8.2] can be used to prove that σ' is a subrepresentation of $\delta([\nu^{a_1+1}1_{F^\times}, \nu^{b_1}1_{F^\times}]) \rtimes \theta^{-\epsilon}(\sigma_0, r^{-\epsilon}(\sigma_0))$, and that it is the unique discrete series subquotient of this induced representation.

Proposition 3.3 and (1) show that $\theta^{-\epsilon}(\sigma_i, r^{-\epsilon}(\sigma_i))$ is an irreducible subrepresentation of

$$\delta([\nu^{-a_i}\rho_i, \nu^{b_i}\rho_i]) \rtimes \theta^{-\epsilon}(\sigma_{i-1}, r^{-\epsilon}(\sigma_{i-1}))$$

for all $i = 2, \ldots, s-1$. Furthermore, Lemma 4.2 can be used to deduce that there is an ordered $(s-1)$-tuple of discrete series $(\sigma_1', \sigma_2', \ldots, \sigma_{s-1}')$ such that $\sigma_1' \cong \sigma'$ and σ_j' is an irreducible subrepresentation of $\delta([\nu^{-a_j}\rho_j, \nu^{b_j}\rho_j]) \rtimes \sigma_{j-1}'$ with a_j such that $a_j - c \in \mathbb{Z}$ for all $c \in \mathrm{Jord}_{\rho_j}(\sigma_{j-1}')$ and $\mathrm{Jord}_{\rho_j}(\sigma_{j-1}') \cap [2a_j+1, 2b_j+1] = \emptyset$ for all $j = 2, \ldots, s-1$ and, for every $i = 2, \ldots, s-1$, $\theta^{-\epsilon}(\sigma_i, r^{-\epsilon}(\sigma_i))$ is an irreducible tempered subrepresentation of

$$\delta([\nu^{-a_1}1_{F^\times}, \nu^{a_1}1_{F^\times}]) \rtimes \sigma_{i-1}'.$$

In the rest of the proof we may assume $l > 0$. Using the results from Section 4 of [13], we deduce $\mathrm{Jord}_{1_{F^\times}}(\theta^{-\epsilon}(\sigma_0, r^{-\epsilon}(\sigma_0))) \cap [2a_1+1, 2b_1+1] = 2a_1+1$ and $\mathrm{Jord}_{\rho_i}(\theta^{-\epsilon}(\sigma_0, r^{-\epsilon}(\sigma_0))) \cap [2a_i+1, 2b_i+1] = \emptyset$ for $i = 2, \ldots, s-1$. We have

$$\theta^{-\epsilon}(\sigma_1, r^{-\epsilon}(\sigma_1)) \hookrightarrow \delta([\nu^{-a_1}1_{F^\times}, \nu^{b_1}1_{F^\times}]) \times \nu^{-a_1-1}1_{F^\times} \rtimes \theta^{-\epsilon}(\sigma_0, r^{-\epsilon}(\sigma_0))$$

and $\theta^{-\epsilon}(\sigma_1, r^{-\epsilon}(\sigma_1))$ is obviously contained in the kernel of the intertwining operator

$$\delta([\nu^{-a_1}1_{F^\times}, \nu^{b_1}1_{F^\times}]) \times \nu^{-a_1-1}1_{F^\times} \rtimes \theta^{-\epsilon}(\sigma_0, r^{-\epsilon}(\sigma_0)) \to \nu^{-a_1-1}1_{F^\times} \times \delta([\nu^{-a_1}1_{F^\times}, \nu^{b_1}1_{F^\times}]) \rtimes \theta^{-\epsilon}(\sigma_0, r^{-\epsilon}(\sigma_0)),$$

i.e., $\theta^{-\epsilon}(\sigma_1, r^{-\epsilon}(\sigma_1))$ is a subrepresentation of the induced representation

$$\delta([\nu^{-a_1-1}1_{F^\times}, \nu^{b_1}1_{F^\times}]) \rtimes \theta^{-\epsilon}(\sigma_0, r^{-\epsilon}(\sigma_0)).$$

If $b_1 > a_1 + 1$, Lemma 4.1 implies that $\theta^{-\epsilon}(\sigma_1, r^{-\epsilon}(\sigma_1))$ is a discrete series representation, and otherwise it is tempered. For $i < k$, obviously $b_i > a_i + 1$.

Repeating the same procedure, we obtain that, for $i \leq l$, $\theta^{-\epsilon}(\sigma_i, r^{-\epsilon}(\sigma_i))$ is a discrete series subrepresentation of $\delta([\nu^{-a_i-1}1_{F^\times}, \nu^{b_i}1_{F^\times}]) \rtimes \theta^{-\epsilon}(\sigma_{i-1}, r^{-\epsilon}(\sigma_{i-1}))$. Observe that this gives $(2(a_i + 1) + 1, 1_{F^\times}) \in \mathrm{Jord}(\theta^{-\epsilon}(\sigma_i, r^{-\epsilon}(\sigma_i)))$.

If $k = l$ and $b_k > a_k + 1$, in the same way as before we deduce that $\theta^{-\epsilon}(\sigma_j, r^{-\epsilon}(\sigma_j))$ is a discrete series subrepresentation of $\delta([\nu^{-a_j}\rho_j, \nu^{b_j}\rho_j]) \rtimes \theta^{-\epsilon}(\sigma_{j-1}, r^{-\epsilon}(\sigma_{j-1}))$ for all $j = k+1, k+2, \ldots, s-1$. If $k = l$ and $b_k = a_k + 1$, $\theta^{-\epsilon}(\sigma_k, r^{-\epsilon}(\sigma_k))$ is a tempered subrepresentation of $\delta([\nu^{-b_k}1_{F^\times}, \nu^{b_k}1_{F^\times}]) \rtimes \theta^{-\epsilon}(\sigma_{k-1}, r^{-\epsilon}(\sigma_{k-1}))$. An inductive procedure, based on Proposition 3.3 and Lemma 4.2, shows that, for $j = k+1, k+2, \ldots, s-1$, $\theta^{-\epsilon}(\sigma_j, r^{-\epsilon}(\sigma_j))$ is a tempered subrepresentation of $\delta([\nu^{-b_k}1_{F^\times}, \nu^{b_k}1_{F^\times}]) \rtimes \sigma'_j$, where each σ'_j is a discrete series subrepresentation of $\delta([\nu^{-a_j}\rho_j, \nu^{b_j}\rho_j]) \rtimes \sigma'_{j-1}$ with $a_j - c \in \mathbb{Z}$ for all $2c + 1 \in \mathrm{Jord}_{\rho_j}(\sigma'_{j-1})$ and $\mathrm{Jord}_{\rho_j}(\sigma'_{j-1}) \cap [2a_j + 1, 2b_j + 1] = \emptyset$.

It remains to consider the case $k > l$. Now $\theta^{-\epsilon}(\sigma_{l+1}, r^{-\epsilon}(\sigma_{l+1}))$ is a subrepresentation of $\delta([\nu^{-a_{l+1}}1_{F^\times}, \nu^{b_{l+1}}1_{F^\times}]) \rtimes \theta^{-\epsilon}(\sigma_l, r^{-\epsilon}(\sigma_l))$. Since $a_{l+1} = a_l + 1$ and $(2(a_l+1)+1, 1_{F^\times}) \in \mathrm{Jord}(\theta^{-\epsilon}(\sigma_l, r^{-\epsilon}(\sigma_l)))$, it follows that $\theta^{-\epsilon}(\sigma_{l+1}, r^{-\epsilon}(\sigma_{l+1}))$ is not a discrete series. In the same way as in the case $k > l = 0$, we deduce that $\theta^{-\epsilon}(\sigma_{l+1}, r^{-\epsilon}(\sigma_{l+1}))$ is a tempered subrepresentation of the induced representation $\delta([\nu^{-a_{l+1}}1_{F^\times}, \nu^{a_{l+1}}1_{F^\times}]) \rtimes \sigma'$ for a unique discrete series subquotient σ' of $\delta([\nu^{a_{l+1}+1}1_{F^\times}, \nu^{b_{l+1}}1_{F^\times}]) \rtimes \theta^{-\epsilon}(\sigma_l, r^{-\epsilon}(\sigma_l))$. Using Proposition 3.3 and Lemma 4.2 we obtain that $\theta^{-\epsilon}(\sigma_j, r^{-\epsilon}(\sigma_j))$ is tempered for all $j = l+1, l+2, \ldots, s-1$ and the theorem is proven. $\square$

In the rest of this section, we denote by $\chi_{V'}$ the quadratic character of $GL(n, F)$ attached to the ϵ–tower in the same way as in Section 2.

Directly from the proof of the previous theorem, we obtain

Corollary 4.4. *Let* $y = r^{-\epsilon}(\sigma) - n + \frac{1}{2} + t_{-\epsilon}$. *If* $\theta^{-\epsilon}(\sigma, r^{-\epsilon}(\sigma))$ *is not a discrete series representation, then* $(2 \cdot y - 1, \chi_V \chi_{V'} 1_{F^\times}) \in$ *Jord*$(\theta^{\epsilon}(\sigma, n - t_\epsilon))$. *If* $\theta^{-\epsilon}(\sigma, r^{-\epsilon}(\sigma))$ *is a discrete series representation, then* $(2 \cdot y - 1, 1_{F^\times}) \in$ *Jord*$(\theta^{-\epsilon}(\sigma, r^{-\epsilon}(\sigma))$.

We will also need the following result regarding Jacquet modules of discrete series and their theta lifts:

Lemma 4.5. *Suppose* $(2z + 1, \chi_V \chi_{V'} \rho) \notin$ *Jord*$(\theta^{\epsilon}(\sigma, n - t_\epsilon))$ *for a cuspidal self-contragredient representation* $\rho \in$ *Irr*$(GL(n_\rho, F))$. *Then* $R_{\widetilde{P_1}}(\sigma)(\nu^z \chi_{V,\psi} \rho) = 0$.

Proof. By [16, Lemma 3.6] we have $R_{P_1}(\theta^{\epsilon}(\sigma, n - t_\epsilon))(\nu^z \chi_V \chi_{V'} \rho) = 0$. If $(z, \rho) \neq (\frac{1}{2}, \chi_V \chi_{V'} 1_{F^\times})$, Proposition 3.5 directly implies $R_{\widetilde{P_1}}(\sigma)(\nu^z \chi_{V,\psi} \rho) = 0$.

It remains to consider the case $(z, \rho) = (\frac{1}{2}, \chi_V \chi_{V'} 1_{F^\times})$. Suppose, contrary to our assumption, that $R_{\widetilde{P_1}}(\sigma)(\nu^{\frac{1}{2}} \chi_{V,\psi} 1_{F^\times}) \neq 0$. Now [13, Theorem 6.1] and [12, Theorem 6.1] can be used to deduce $(2z + 1, \rho) \in \mathrm{Jord}(\theta^{\epsilon}(\sigma, n - t_\epsilon))$, which is impossible. This proves the lemma. $\square$

5 First occurrence indices of tempered representations

Let τ denote an irreducible tempered representation of the metaplectic group $\widetilde{Sp(n)}$. As before, we set $t_+ = 0$ and $t_- = 1$ and let ϵ denote an element of $\{+, -\}$ such that $\Theta^{\epsilon}(\tau, n - t_\epsilon) \neq 0$. As in Section 2, we denote by χ_V the quadratic character of $GL(n, F)$ related to the $-\epsilon$-tower and by $\chi_{V'}$ the quadratic character of $GL(n, F)$ related to the ϵ-tower.

Using well-known results for classical groups (i.e., Section 2 of [19]), together with [4, Theorem 1.3] and Proposition 3.5, we obtain that there exists an ordered t-tuple $(\tau_0, \tau_1, \ldots, \tau_{t-1})$ of tempered representations, $\tau_i \in \mathrm{Irr}(\widetilde{Sp(n_i)})$, such that $\tau \cong \tau_{t-1}$, τ_0 is a discrete series representation and for $i \in \{1, 2, \ldots, t-1\}$ there is an irreducible cuspidal representation ρ_i of $GL(n_{\rho_i}, F)$, a non-negative half-integer x_i and a positive integer m_i such that

$$\tau_i \hookrightarrow \delta(x_i, \chi_{V,\psi} \rho_i)^{m_i} \rtimes \tau_{i-1}$$

and $\mu^*(\tau_{i-1})$ does not contain an irreducible constituent of the form $\delta(x_i, \chi_{V,\psi} \rho_i) \otimes \pi$.

We note that the last fact implies that if $\mu^*(\tau_i)$ contains an irreducible constituent of the form $\delta(x_i, \chi_{V,\psi}\rho_i)^{m_i} \otimes \tau'$, then $\tau' \cong \tau_{i-1}$.

Proposition 3.3 implies $\Theta^\epsilon(\tau_i, n_i - t_\epsilon) \neq 0$ for $i = 0, 1, \ldots, t-1$.

Define $y = r^{-\epsilon}(\tau_0) - n_0 + \frac{1}{2} + t_{-\epsilon}$. Observe that $\theta^{-\epsilon}(\tau_0, r^{-\epsilon}(\tau_0)+1)$ is a subrepresentation of $\nu^{-y}1_{F^\times} \rtimes \theta^{-\epsilon}(\tau_0, r^{-\epsilon}(\tau_0))$.

The following lemma follows directly from [7] and Section 8 of [4] (Mackey theory can be used to extend Goldberg's results to the non-connected case, as in [11, 16]):

Lemma 5.1. *The induced representation $\delta(x, \chi_{V,\psi}\rho)^m \rtimes \tau_0$ is a direct sum of tempered representations. It contains at most two non-isomorphic irreducible subquotients and it contains exactly two non-isomorphic irreducible subquotients if and only if the representation $\delta(x, \chi_V \chi_{V'}\rho)^m \rtimes \theta^\epsilon(\tau_0, n_0 - t_\epsilon)$ reduces.*

The following theorem is the main result of this paper:

Theorem 5.2. *Let $\tau \in Irr(\widetilde{Sp(n)})$ be a tempered representation and let $\epsilon \in \{+,-\}$ denote $+$ or $-$ such that $\Theta^\epsilon(\tau, n - t_\epsilon) \neq 0$, where $t_+ = 0$ and $t_- = 1$. Denote by χ_V the quadratic character related to $-\epsilon$-tower and let $\chi_{V,\psi} = \chi_V \chi_\psi$. Let $(\tau_0, \tau_1, \ldots, \tau_{t-1})$ denote an ordered t-tuple of tempered representations, $\tau_i \in Irr(\widetilde{Sp(n_i)})$, such that $\tau \cong \tau_{t-1}$, τ_0 is a discrete series and for $i \in \{1, 2, \ldots, t-1\}$ we have $\tau_i \hookrightarrow \delta(x_i, \chi_{V,\psi}\rho_i)^{m_i} \rtimes \tau_{i-1}$ where $\rho_i \in Irr(GL(n_{\rho_i}, F))$ is a cuspidal representation, x_i is a non-negative half-integer, and m_i is a positive integer such that $\mu^*(\tau_{i-1})$ does not contain an irreducible constituent of the form $\delta(x_i, \chi_{V,\psi}\rho_i) \otimes \pi$. Set $y = r^{-\epsilon}(\tau_0) - n_0 + \frac{1}{2} + t_{-\epsilon}$.*

If $\mu^*(\tau)$ does not contain an irreducible constituent of the form $\delta(y - 1, \chi_{V,\psi}1_{F^\times}) \otimes \pi$, then $r^\epsilon(\tau) = n - n_0 + r^\epsilon(\tau_0)$ and $r^{-\epsilon}(\tau) = n - n_0 + r^{-\epsilon}(\tau_0)$.

If some irreducible constituent of the form $\delta(y-1, \chi_{V,\psi}1_{F^\times}) \otimes \pi$ appears in $\mu^*(\tau)$, let us denote by m the largest integer such that $\mu^*(\tau)$ contains an irreducible constituent of the form $\delta(y-1, \chi_{V,\psi}1_{F^\times})^m \otimes \pi$. Then there is an $i \in \{1, 2, \ldots, t-1\}$ such that $(x_i, \rho_i, m_i) = (y-1, 1_{F^\times}, m)$. We can assume $i = 1$. There are two possibilities:

(i) Suppose that the induced representation $\delta(y-1, \chi_{V,\psi}1_{F^\times})^m \rtimes \tau_0$ is a direct sum of mutually isomorphic tempered representations. Then $r^\epsilon(\tau) = n - n_0 + r^\epsilon(\tau_0)$ and $r^{-\epsilon}(\tau) = n - n_0 + r^{-\epsilon}(\tau_0)$.

(ii) Suppose that the induced representation $\delta(y-1, \chi_{V,\psi}1_{F^\times})^m \rtimes \tau_0$ is a direct sum of copies of two non-isomorphic tempered representations. If $\mu^*(\tau_1)$

contains an irreducible constituent of the form $(\nu^{y-1}\chi_{V,\psi}1_{F^\times})^{2m} \otimes \pi$, then $r^\epsilon(\tau) = n - n_0 + r^\epsilon(\tau_0)$ and $r^{-\epsilon}(\tau) = n - n_0 + r^{-\epsilon}(\tau_0)$. Otherwise, $r^\epsilon(\tau) = n - n_0 + r^\epsilon(\tau_0) - 1$ and $r^{-\epsilon}(\tau) = n - n_0 + r^{-\epsilon}(\tau_0) + 1$.

We divide the proof of Theorem 5.2 in a sequence of propositions. The conservation relation shows that it is enough to determine one of the first occurrence indices. In each case, we determine the larger one.

Proposition 5.3. *Suppose that $\mu^*(\tau)$ does not contain an irreducible constituent of the form $\delta(y-1, \chi_{V,\psi}1_{F^\times}) \otimes \pi$. Then $r^{-\epsilon}(\tau_i) = n_i - n_0 + r^{-\epsilon}(\tau_0)$ for $i = 1, 2, \ldots, t-1$.*

Proof. The proof is similar to that of [13, Theorem 4.1]. Let us first consider the case $i = 1$ and let z be such that $\Theta^{-\epsilon}(\tau_1, z) \neq 0$. Proposition 3.3 gives

$$\theta^{-\epsilon}(\tau_1, z) \hookrightarrow \delta(x_i, \rho_i)^{m_i} \rtimes \theta^{-\epsilon}(\tau_0, z - n_1 + n_0).$$

If $z \neq n_1 - n_0 + r^{-\epsilon}(\tau_0)$, Lemma 3.2 implies $\theta^{-\epsilon}(\tau_0, z - n_1 + n_0) \hookrightarrow \nu^{n_1 - z + \frac{1}{2} - t_{-\epsilon}}1_{F^\times} \rtimes \theta^{-\epsilon}(\tau_0, z - n_1 + n_0 - 1)$ and, if $z > n_1 - n_0 + r^{-\epsilon}(\tau_0) + 1$, $\theta^{-\epsilon}(\tau_0, z - n_1 + n_0) \hookrightarrow \nu^{n_1 - z + \frac{1}{2} - t_{-\epsilon}}1_{F^\times} \times \nu^{n_1 - z + \frac{3}{2} - t_{-\epsilon}}1_{F^\times} \rtimes \theta^{-\epsilon}(\tau_0, z - n_1 + n_0 - 2)$.

If $(x_1, \rho_1) \neq (-(n_1 - z + \frac{3}{2} - t_{-\epsilon}), 1_{F^\times})$ or $z = n_1 - n_0 + r^{-\epsilon}(\tau_0) + 1$ we have

$$\delta(x_1, \rho_1)^{m_1} \times \nu^{n_1 - z + \frac{1}{2} - t_{-\epsilon}}1_{F^\times} \cong \nu^{n_1 - z + \frac{1}{2} - t_{-\epsilon}}1_{F^\times} \times \delta(x_1, \rho_1)^{m_1}$$

(note that $(x_1, \rho_1) \neq (y - 1, 1_{F^\times})$ by the assumption of the proposition). Otherwise, we have

$$\theta^{-\epsilon}(\tau_1, z) \hookrightarrow \delta(x_1, 1_{F^\times})^{m_1} \times \nu^{-x_1 - 1}1_{F^\times} \times \nu^{-x_1}1_{F^\times} \rtimes \theta^{-\epsilon}(\tau_0, z - n_1 + n_0 - 2).$$

Assume that $\theta^{-\epsilon}(\tau_1, z)$ is contained in the kernel of the intertwining operator

$$\delta(x_1, 1_{F^\times})^{m_1} \times \nu^{-x_1 - 1}1_{F^\times} \times \nu^{-x_1}1_{F^\times} \rtimes \theta^{-\epsilon}(\tau_0, z - n_1 + n_0 - 2) \to \nu^{-x_1 - 1}1_{F^\times} \times \delta(x_1, 1_{F^\times})^{m_1} \times \nu^{-x_1}1_{F^\times} \rtimes \theta^{-\epsilon}(\tau_0, z - n_1 + n_0 - 2).$$

Then $\theta^{-\epsilon}(\tau_1, z)$ is a subrepresentation of

$$\delta([\nu^{-x_1 - 1}1_{F^\times}, \nu^{x_1}1_{F^\times}]) \times \delta(x_1, 1_{F^\times})^{m_1 - 1} \times \nu^{-x_1}1_{F^\times} \rtimes \theta^{-\epsilon}(\tau_0, z - n_1 + n_0 - 2),$$

which is isomorphic to

$$\nu^{-x_1}1_{F^\times} \times \delta([\nu^{-x_1 - 1}1_{F^\times}, \nu^{x_1}1_{F^\times}]) \times \delta(x_1, 1_{F^\times})^{m_1 - 1} \rtimes \theta^{-\epsilon}(\tau_0, z - n_1 + n_0 - 2),$$

and Proposition 3.5 can be used to obtain a contradiction with the temperedness of τ_1.

Consequently, $R_{P_1}(\Theta^{-\epsilon}(\tau_1, z))(\nu^{n_1-z+\frac{1}{2}-t_{-\epsilon}}1_{F^\times}) \neq 0$ and, since $n_1 - z + \frac{1}{2} - t_{-\epsilon} < 0$, we obtain $\Theta^{-\epsilon}(\tau_1, z-1) \neq 0$.

Repeating the same procedure, we get $\Theta^{-\epsilon}(\tau_1, n_1 - n_0 + r^{-\epsilon}(\tau_0)) \neq 0$. Also, $\theta^{-\epsilon}(\tau_1, n_i - n_0 + r^{-\epsilon}(\tau_0))$ is a subrepresentation of $\delta(x_i, \rho_i)^{m_i} \rtimes \theta^{-\epsilon}(\tau_0, r^{-\epsilon}(\tau_0))$. By Theorem 4.3, this representation is tempered, and the structural formula for μ^* gives $R_{P_1}(\Theta^{-\epsilon}(\tau_1, z))(\nu^{-y+1}1_{F^\times}) = 0$. Consequently, $r^{-\epsilon}(\tau_1) = n_1 - n_0 + r^{-\epsilon}(\tau_0)$.

Let us now assume that $r^{-\epsilon}(\tau_i) = n_i - n_0 + r^{-\epsilon}(\tau_0)$ for all $i < j$. Since τ_i is tempered, this also gives $\theta^{-\epsilon}(\tau_i, r+1) \hookrightarrow \nu^{-(m_{r+1}-n_i-1)}1_{F^\times} \rtimes \theta^{-\epsilon}(\tau_i, r)$ for $r \geq r^{-\epsilon}(\tau_i)$. Now it can be proved that $r^{-\epsilon}(\tau_j)$ equals $n_j - n_0 + r^{-\epsilon}(\tau_0)$ following the same lines as in the case $j = 1$.

This ends the proof. □

In the rest of this section we assume that an irreducible constituent of the form $\delta(y-1, \chi_{V,\psi}1_{F^\times}) \otimes \pi$ appears in $\mu^*(\tau)$, and we denote by m the largest integer such that $\mu^*(\tau)$ contains an irreducible constituent of the form $\delta(y-1, \chi_{V,\psi}1_{F^\times})^m \otimes \pi$. We may, and we will, assume $(x_1, \rho_1, m_1) = (y-1, 1_{F^\times}, m)$.

Proposition 5.4. *Suppose that the induced representation $\delta(y-1, \chi_{V,\psi}1_{F^\times})^m \rtimes \tau_0$ is a direct sum of mutually isomorphic tempered representations. Then $r^{-\epsilon}(\tau_1) = n_1 - n_0 + r^{-\epsilon}(\tau_0)$.*

Proof. In this case, $\delta(y-1, \chi_V \chi_{V'} 1_{F^\times})^m \rtimes \theta^\epsilon(\tau_0, n_0 - t_\epsilon)$ is irreducible. Let m' denote the largest integer such that $\mu^*(\tau_0)$ contains an irreducible constituent of the form $(\nu^{y-1}\chi_{V,\psi}1_{F^\times})^{m'} \otimes \pi$. Propositions 3.4 and 3.5 show that $\mu^*(\theta^{-\epsilon}(\tau_0, r^{-\epsilon}(\tau_0)))$ contains an irreducible constituent of the form $(\nu^{y-1}1_{F^\times})^{m'} \otimes \pi_1$ and does not contain an irreducible constituent of the form $(\nu^{y-1}1_{F^\times})^{m'+1} \otimes \pi_2$. Furthermore, the structural formula directly implies that $\mu^*(\tau_1)$ contains an irreducible constituent of the form $(\nu^{y-1}\chi_{V,\psi}1_{F^\times})^{2m+m'} \otimes \pi_3$.

Following the same steps as in the proof of the previous proposition, we deduce $r^{-\epsilon}(\tau_1) \in \{n_1 - n_0 + r^{-\epsilon}(\tau_0), n_1 - n_0 + r^{-\epsilon}(\tau_0) + 1\}$. Proposition 3.4 shows that

$$\mu^*(\theta^{-\epsilon}(\tau_1, n_1 - n_0 + r^{-\epsilon}(\tau_0) + 1)) \geq (\nu^{y-1}1_{F^\times})^{2m+m'} \otimes \pi_4$$

for some irreducible representation π_4. If $r^{-\epsilon}(\tau_1)$ equals $n_1 - n_0 + r^{-\epsilon}(\tau_0) + 1$, then $\theta^{-\epsilon}(\tau_1, n_1 - n_0 + r^{-\epsilon}(\tau_0) + 1)$ is a subrepresentation of

$$\delta(y-1, 1_{F^\times})^{m-1} \times \delta([\nu^{-y} 1_{F^\times}, \nu^{y-1} 1_{F^\times}]) \rtimes \theta^{-\epsilon}(\tau_0, r^{-\epsilon}(\tau_0))$$

and it follows at once that no representation of the form $(\nu^{y-1}\chi_{V,\psi} 1_{F^\times})^{2m+m'} \otimes \pi_5$ appears in $\mu^*(\theta^{-\epsilon}(\tau_1, n_1 - n_0 + r^{-\epsilon}(\tau_0) + 1))$. Consequently, $r^{-\epsilon}(\tau_1)$ equals $n_1 - n_0 + r^{-\epsilon}(\tau_0)$ and the proposition is proved. □

The remaining possibility is treated in the following proposition.

Proposition 5.5. *Suppose that the induced representation $\delta(y-1, \chi_{V,\psi} 1_{F^\times})^m \rtimes \tau_0$ is a direct sum of copies of two non-isomorphic tempered representations. If $\mu^*(\tau_1)$ contains an irreducible constituent of the form $(\nu^{y-1}\chi_{V,\psi} 1_{F^\times})^{2m} \otimes \pi$, then $r^{-\epsilon}(\tau_1) = n_1 - n_0 + r^{-\epsilon}(\tau_0)$. Otherwise $r^{-\epsilon}(\tau_1) = n_1 - n_0 + r^{-\epsilon}(\tau_0) + 1$.*

Proof. In this case, $\delta(y-1, \chi_V \chi_{V'} 1_{F^\times})^m \rtimes \theta^{\epsilon}(\tau_0, n_0 - t_\epsilon)$ reduces and it follows that $(2y-1, \chi_V \chi_{V'} 1_{F^\times}) \notin \mathrm{Jord}(\theta^{\epsilon}(\tau_0, n_0 - t_\epsilon))$. Corollary 4.4 implies that the representation $\theta^{-\epsilon}(\tau_0, r^{-\epsilon}(\tau_0))$ is a discrete series representation. Also, Lemma 4.5 implies that $R_{\widetilde{P_1}}(\tau_0)(\nu^{y-1}\chi_{V,\psi} 1_{F^\times}) = 0$ and it follows from Proposition 3.4 that $R_{P_1}(\theta^{-\epsilon}(\tau_0, r^{-\epsilon}(\tau_0))(\nu^{y-1} 1_{F^\times}) = 0$.

Thus, $\mu^*(\delta(y-1, \chi_{V,\psi} 1_{F^\times})^m \rtimes \tau_0)$ contains an irreducible constituent of the form $(\nu^{y-1}\chi_{V,\psi} 1_{F^\times})^{2m} \otimes \pi_1$, and it does not contain an irreducible constituent of the form $(\nu^{y-1}\chi_{V,\psi} 1_{F^\times})^{2m+1} \otimes \pi_2$.

Let us denote by τ_a and τ_b the two not isomorphic irreducible subrepresentations of $\delta(y-1, \chi_{V,\psi} 1_{F^\times})^m \rtimes \tau_0$. There is some $i \in \{a, b\}$ such that $\mu^*(\tau_i)$ contains an irreducible constituent of the form $(\nu^{y-1}\chi_{V,\psi} 1_{F^\times})^{2m} \otimes \pi_1$. There is no loss of generality in assuming $i = a$.

In the same way as in the proof of Proposition 5.3 we obtain $r^{-\epsilon}(\tau_a)$, $r^{-\epsilon}(\tau_b) \in \{n_1 - n_0 + r^{-\epsilon}(\tau_0), n_1 - n_0 + r^{-\epsilon}(\tau_0) + 1\}$. The assumption $r^{-\epsilon}(\tau_a) = n_1 - n_0 + r^{-\epsilon}(\tau_0) + 1$ implies

$$\theta^{-\epsilon}(\tau_a, r^{-\epsilon}(\tau_a)) \hookrightarrow \delta(y-1, 1_{F^\times})^{m-1} \times \delta([\nu^{-y} 1_{F^\times}, \nu^{y-1} 1_{F^\times}]) \rtimes \theta^{-\epsilon}(\tau_0, r^{-\epsilon}(\tau_0)),$$

and, since $R_{P_1}(\theta^{-\epsilon}(\tau_0, r^{-\epsilon}(\tau_0))(\nu^{y-1} 1_{F^\times}) = 0$, it can be seen that this is impossible in the same way as in the proof of Proposition 5.4. Consequently, $r^{-\epsilon}(\tau_a) = n_1 - n_0 + r^{-\epsilon}(\tau_0)$. Also, $\theta^{-\epsilon}(\tau_a, r^{-\epsilon}(\tau_a))$ is a subrepresentation of

$$\delta(y-1, 1_{F^\times})^m \rtimes \theta^{-\epsilon}(\tau_0, r^{-\epsilon}(\tau_0)),$$

which is irreducible by Corollary 4.4 and [19, Lemma 2.3].

We will now prove that $r^{-\epsilon}(\tau_b)$ equals $n_1 - n_0 + r^{-\epsilon}(\tau_0) + 1$. Suppose, on the contrary, that $r^{-\epsilon}(\tau_b) = n_1 - n_0 + r^{-\epsilon}(\tau_0)$. Then $\theta^{-\epsilon}(\tau_b, r^{-\epsilon}(\tau_b))$ is also a subrepresentation of $\delta(y-1, 1_{F^\times})^m \rtimes \theta^{-\epsilon}(\tau_0, r^{-\epsilon}(\tau_0))$ and

$$\theta^{-\epsilon}(\tau_a, n_1 - n_0 + r^{-\epsilon}(\tau_0)) \cong \theta^{-\epsilon}(\tau_b, n_1 - n_0 + r^{-\epsilon}(\tau_0)),$$

which is impossible since τ_a and τ_b are not isomorphic. Consequently, $r^{-\epsilon}(\tau_b) = n_1 - n_0 + r^{-\epsilon}(\tau_0) + 1$ and $\theta^{-\epsilon}(\tau_b, r^{-\epsilon}(\tau_b))$ is a subrepresentation of

$$\delta(y-1, 1_{F^\times})^{m-1} \times \delta([\nu^{-y} 1_{F^\times}, \nu^{y-1} 1_{F^\times}]) \rtimes \theta^{-\epsilon}(\tau_0, r^{-\epsilon}(\tau_0)).$$

Now the fact that $R_{P_1}(\theta^{-\epsilon}(\tau_0, r^{-\epsilon}(\tau_0))(\nu^{y-1} 1_{F^\times}) = 0$, the structural formula for μ^* and Proposition 3.4 can be used to show that $\mu^*(\tau_b)$ does not contain an irreducible constituent of the form $(\nu^{y-1} \chi_{V,\psi} 1_{F^\times})^{2m} \otimes \pi_1$. This completes the proof. $\square$

The following proposition completes the proof of Theorem 5.2. It can be proved in an analogous way as Proposition 5.3, details being left to the reader.

Proposition 5.6. *Suppose that $\mu^*(\tau)$ contains an irreducible constituent of the form $\delta(y-1, \chi_{V,\psi} 1_{F^\times}) \otimes \pi$ and denote by m the largest integer such that $\mu^*(\tau)$ contains $\delta(y-1, \chi_{V,\psi} 1_{F^\times})^m \otimes \pi$ for an irreducible π. We assume $(x_1, \rho_1, m_1) = (y-1, 1_{F^\times}, m)$. Then $r^{-\epsilon}(\tau_i) = n_i - n_1 + r^{-\epsilon}(\tau_1)$ for $i = 2, 3, \ldots, t-1$.*

References

[1] J. Arthur, *The endoscopic classification of representations*, vol. 61 of American Mathematical Society Colloquium Publications, American Mathematical Society, Providence, RI, 2013. Orthogonal and symplectic groups.

[2] I. N. Bernstein and A. V. Zelevinsky, *Induced representations of reductive p-adic groups. I*, Ann. Sci. École Norm. Sup. (4), 10 (1977), pp. 441–472.

[3] W. T. Gan and A. Ichino, *Formal degrees and local theta correspondence*, Invent. Math., 195 (2014), pp. 509–672.

[4] W. T. Gan and G. Savin, *Representations of metaplectic groups I: epsilon dichotomy and local Langlands correspondence*, Compos. Math., 148 (2012), pp. 1655–1694.

[5] W. T. Gan and S. Takeda, *On the Howe duality conjecture in classical theta correspondence*, Contemp. Math., to appear, 2015.

[6] ——, *A proof of the Howe duality conjecture*, J. Amer. Math. Soc., to appear, (2015).

[7] D. Goldberg, *Reducibility of induced representations for Sp(2n) and SO(n)*, Amer. J. Math., 116 (1994), pp. 1101–1151.

[8] M. Hanzer and G. Muić, *Parabolic induction and Jacquet functors for metaplectic groups*, J. Algebra, 323 (2010), pp. 241–260.

[9] S. S. Kudla, *On the local theta-correspondence*, Invent. Math., 83 (1986), pp. 229–255.

[10] ——, *Notes on the local theta correspondence (lectures at the European School in Group Theory)*, preprint, available at http://www.math.toronto.edu/~skudla/castle.pdf, (1996).

[11] E. Lapid, G. Muić, and M. Tadić, *On the generic unitary dual of quasisplit classical groups*, Int. Math. Res. Not., (2004), pp. 1335–1354.

[12] I. Matić, *Theta lifts of strongly positive discrete series: the case of* $(\widetilde{Sp(n)}, O(V))$, Pacific J. Math., 259 (2012), pp. 445–471.

[13] ——, *Conservation relation for discrete series of metaplectic groups*, Int. Math. Res. Not. IMRN, (2013), pp. 5227–5269.

[14] C. Mœglin, *Sur la classification des séries discrètes des groupes classiques p-adiques: paramètres de Langlands et exhaustivité*, J. Eur. Math. Soc. (JEMS), 4 (2002), pp. 143–200.

[15] ——, *Paquets stables des séries discrètes accessibles par endoscopie tordue; leur paramètre de Langlands*, in Automorphic forms and related geometry: assessing the legacy of I. I. Piatetski-Shapiro, vol. 614 of Contemp. Math., Amer. Math. Soc., Providence, RI, 2014, pp. 295–336.

[16] C. Mœglin and M. Tadić, *Construction of discrete series for classical p-adic groups*, J. Amer. Math. Soc., 15 (2002), pp. 715–786.

[17] C. Mœglin, M.-F. Vignéras, and J.-L. Waldspurger, *Correspondances de Howe sur un corps p-adique*, vol. 1291 of Lecture Notes in Mathematics, Springer-Verlag, Berlin, 1987.

[18] G. Muić, *Composition series of generalized principal series; the case of strongly positive discrete series*, Israel J. Math., 140 (2004), pp. 157–202.

[19] ——, *Theta lifts of tempered representations for dual pairs* $(Sp_{2n}, O(V))$, Canad. J. Math., 60 (2008), pp. 1306–1335.

[20] A. J. Silberger, *Special representations of reductive p-adic groups are not integrable*, Ann. of Math. (2), 111 (1980), pp. 571–587.

[21] B. Sun and C.-B. Zhu, *Conservation relations for local theta correspondence*, J. Amer. Math. Soc., 28 (2015), pp. 939–983.

[22] M. Tadić, *Structure arising from induction and Jacquet modules of representations of classical p-adic groups*, J. Algebra, 177 (1995), pp. 1–33.

[23] ——, *On tempered and square integrable representations of classical p-adic groups*, Sci. China Math., 56 (2013), pp. 2273–2313.

[24] J.-L. Waldspurger, *Démonstration d'une conjecture de dualité de Howe dans le cas p-adique, $p \neq 2$*, in Festschrift in honor of I. I. Piatetski-Shapiro on the occasion of his sixtieth birthday, Part I (Ramat Aviv, 1989), vol. 2 of Israel Math. Conf. Proc., Weizmann, Jerusalem, 1990, pp. 267–324.

[25] A. V. Zelevinsky, *Induced representations of reductive p-adic groups. II. On irreducible representations of $GL(n)$*, Ann. Sci. École Norm. Sup. (4), 13 (1980), pp. 165–210.