The unitary dual of p-adic $\widetilde{Sp(2)}$

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Abstract

In this paper, we investigate the composition series of the induced admissible representations of the metaplectic group $\widetilde{Sp(2)}$ over a p–adic field F. In this way, we determine the non–unitary and unitary duals of $\widetilde{Sp(2)}$ modulo cuspidal representations.

1 Introduction

The admissible representations of reductive groups over p–adic fields were studied intensively by many authors, but the knowledge about the unitary dual of such groups is still incomplete. Besides some results concerning specific parts of the unitary dual of some classical and exceptional groups (i.e., spherical, generic ([13]), etc.), there are also some situations where, for some low - rank groups, the complete unitary dual is described ([21], [17], [6] or [15]).

In this paper, we completely describe the non–cuspidal unitary dual of the double cover of the symplectic group of split rank two. Although this is not an algebraic group, some recent results enabled the authors to study this group in the same spirit as the classical split groups. More concretely, recent paper [9] of the first named author and Muić relates reducibilities of the induced representations of metaplectic groups with those of the odd orthogonal groups (using theta correspondence), while their second paper [8] describes the extension of the Jacquet module techniques of Tadić for classical groups to metaplectic groups. More specifically, Tadić's structure formula for symplectic and odd-orthogonal groups ([24]) (which is a version of a geometric lemma of [3]) is extended to metaplectic groups. These ingredients made

the determination of the irreducible subquotients of the principal series for $\widetilde{Sp(2)}$ very similar to the one obtained for $SO(5)$ in [14], but this happens to be insufficient tool in some cases. In these cases, the authors use the theta correspondence to again obtain the formal similarity to the $SO(5)$–case. We note that this similarity was expected (e.g., [28]). After determining complete non-unitary dual, modulo cuspidal representations, the unitary dual follows in the almost the same way as in [15], but after discussion of some exceptional cases (e.g., the discussion on the unitary principal series): in the case of odd orthogonal group $SO(5)$ the irreducibility of the unitary principal series follows from the considerations about R–groups, and in the case of $\widetilde{Sp(2)}$, since the R–group theory for metaplectic groups is not available in it's full generality, irreducibility is obtained using theta correspondence. In the forthcoming paper, we extended the methods used here to prove the irreducibility of unitary principal series for $\widetilde{Sp(n)}$, for general n ([7]). We hope that these results will have applications in the theory of automorphic forms.

We now describe the content of the paper in more detail. In the second section we recall the definition of the metaplectic double cover $\widetilde{Sp(n)}$. We also recall the notions of parabolic subgroups, Jacquet functor, and parabolic induction in the context of metaplectic groups. We then recall the notion of the dual pair, and the lifts of an irreducible representations of one member of the dual pair to the Weil representation of the ambient metaplectic group. We recall the criteria for the square–integrability and temperedness of the irreducible representations of metplectic groups, due to Ban and Jantzen ([1]) and recall the classification of the irreducible genuine representations of $\widetilde{Sp(n)}$ obtained in ([8]). In the third section we analyze the principal series for $\widetilde{Sp(2)}$, using both theta correspondence and Tadić's methods applied to metaplectic groups. In the fourth section we determine the unitary dual of $\widetilde{Sp(n)}$ supported in the minimal parabolic subgroup. In the fifth section we describe irreducible representations of $\widetilde{Sp(n)}$ supported on maximal parabolic subgroups, and unitary dual of $\widetilde{Sp(n)}$ supported on maximal parabolic subgroups.

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2 Preliminaries

Let $\widetilde{Sp(2)}$ be the unique non-trivial two-fold central extension of symplectic group $Sp(2, F)$, where F is a non-Archimedean local field of characteristic different from two. In other words, the following holds:

$$1 \rightarrow \mu_2 \rightarrow \widetilde{Sp(2)} \rightarrow Sp(2, F) \rightarrow 1.$$

The multiplication in $\widetilde{Sp(2)}$ (which is as a set given by $Sp(2, F) \times \mu_2$) is given by the Rao's cocycle ([20]). Observe that in [8] the metaplectic group $\widetilde{Sp(2)}$ was denoted by $\widetilde{Sp(W_2)}$. In this paper we are interested only in genuine representations of $\widetilde{Sp(n)}$ (i.e., those which do not factor through μ_2). So, let $R(n)$ be the Grothendieck group of the category of all admissible genuine representations of finite length of $\widetilde{Sp(n)}$ (i.e., a free abelian group over the set of all irreducible genuine representations of $\widetilde{Sp(n)}$) and define $R = \bigoplus_{n \geq 0} R(n)$. By ν we denote a character of $GL(k, F)$ defined by $|\det|_F$. Further, for an ordered partition $s = (n_1, n_2, \ldots, n_j)$ of some $m \leq n$, we denote by P_s a standard parabolic subgroup of $Sp(n, F)$ (consisting of block upper-triangular matrices), whose Levi factor equals $GL(n_1) \times GL(n_2) \times \cdots \times GL(n_j) \times Sp(n-|s|, F)$, where $|s| = \sum_{i=1}^{j} n_i$. Then the standard parabolic subgroup $\widetilde{P}_s$ of $\widetilde{Sp(n)}$ is the preimage of P_s in $\widetilde{Sp(n)}$. We have the analogous notation for the Levi subgroups of the metaplectic groups, which are described in more detail in Section 2.2 of [8]. Following the notation introduced there, for a representation σ of $\widetilde{Sp(2)}$ the normalized Jacquet module with respect to $\widetilde{M}_{(1,1)}$ is denoted by $R_{\widetilde{P}_{(1,1)}}(\sigma)$, the normalized Jacquet module with respect to $\widetilde{M}_{(1)}$ is denoted by $R_{\widetilde{P}_1}(\sigma)$, while the normalized Jacquet module with respect to $\widetilde{M}_{(2)}$ is denoted by $R_{\widetilde{P}_2}(\sigma)$.

We fix a non-trivial additive character ψ of F and let $\omega_{n,r}$ be the pullback of the Weil representation $\omega_{n(2r+1),\psi}$ of the group $Sp(n(2r+1))$, restricted to the dual pair $\widetilde{Sp(n)} \times O(2r + 1)$ ([12], chapter II). Here $O(2r + 1)$ denotes the split odd-orthogonal group of the split rank r, with the one–dimensional

anisotropic space sitting at the bottom of the orthogonal tower ([12], chapter III.1.). The standard parabolic subgroups (containing the upper triangular Borel subgroup) of $O(2r+1)$ have the analogous description as the standard parabolic subgroups of $Sp(n, F)$; we use the similar notation for the normalized Jacquet functors.

Let σ be an irreducible smooth genuine representation of $\widetilde{Sp(n)}$. We write $\Theta(\sigma, r)$ for the smooth isotypic component of σ in $\omega_{n,r}$ (we view it as a representation of $O(2r+1)$). Denote with r_0 the smallest r such that $\Theta(\sigma, r) \neq 0$. When σ is cuspidal, we know that $\Theta(\sigma, r_0)$ is an irreducible cuspidal representation of $O(2r+1)$.

Let $\widetilde{GL(n, F)}$ be a double cover of $\widetilde{GL(n, F)}$, where the multiplication is given by $(g_1, \epsilon_1)(g_2, \epsilon_2) = (g_1 g_2, \epsilon_1 \epsilon_2 (\det g_1, \det g_2)_F)$. Here $\epsilon_i \in \mu_2$, $i = 1, 2$ and $(\cdot, \cdot)_F$ denotes the Hilbert symbol of the field F, and this cocycle on $GL(n, F)$ is actually a restriction of the Rao's cocycle on $Sp(n, F)$ to $GL(n, F)$, if we view this group as the Siegel Levi subgroup of $Sp(n, F)$ ([11], p. 235). From now on, we fix a character $\chi_{V,\psi}$ of $\widetilde{GL(n, F)}$ which is given by $\chi_{V,\psi}(g, \epsilon) = \chi_V(\det g)\epsilon\gamma(\det g, \psi_{\frac{1}{2}})^{-1}$. Here γ denotes the Weil invariant, while χ_V is a character related to the quadratic form on $O(2r+1)$ ([12], p. 17 and 37) and for $a \in F^*$, $\psi_a(x) = \psi(ax)$. We may suppose $\chi_V \equiv 1$ (but the arguments which follow are valid without this assumption). We denote by $\alpha = \chi^2_{V,\psi}$. Observe that α is a quadratic character on $GL(n)$.

The following fact, which follows directly from [8], we use frequently while determining composition series of induced representations: for an irreducible genuine representation π of $\widetilde{GL(k, F)}$ and an irreducible genuine representation σ of $\widetilde{Sp(n)}$ we have (in R)

$$\pi \rtimes \sigma = \widetilde{\pi}\alpha \rtimes \sigma,$$

where $\pi \rtimes \sigma$ denotes the representation of the group $\widetilde{Sp(n+k)}$ parabolically induced from the representation $\pi \otimes \sigma$ of the maximal Levi subgroup $\widetilde{M_{(k)}}$. We follow here the usual notation for parabolic induction for classical groups, adopted to the metaplectic case ([23],[8]). We also freely use Zelevinsky's notation for the parabolic induction for general linear groups ([27]). We denote the Steinberg representation of the reductive algebraic group G by St_G and the trivial representation of that group by 1_G. Also, following [12], let $\omega^+_{\psi_a,n}$ denote the even part of the Weil representation of $\widetilde{Sp(n)}$ determined

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by the additive character $\underbrace{\psi_a}$. The non–trivial character of μ_2, when we view it as a representation of $\widetilde{Sp(0)}$, is denoted by ω_0.

If ζ is a quadratic character of $F^\times$, then we can write $\zeta(x) = (x, a)_F$, for some $a \in F^\times$. Let $sp_{\zeta,1}$ be an irreducible (square–integrable, according to the criterion for the square–integrability which we recall below) subrepresentation of $\chi_{V,\psi}\zeta\nu^{\frac{1}{2}} \rtimes \omega_0$. Then, as in [12], page 89, we have the following exact sequence:

$$0 \longrightarrow sp_{\zeta,1} \longrightarrow \chi_{V,\psi}\zeta\nu^{\frac{1}{2}} \rtimes \omega_0 \longrightarrow \omega^+_{\psi_a,1} \longrightarrow 0.$$

The results of Ban and Jantzen ([1]) imply that Casselmans criteria for square-integrability and temperedness hold for metaplectic groups in a similar form as for the classical groups (for example symplectic). We take a moment to recall these criteria.

Let π be an admissible irreducible representation of $\widetilde{Sp(n)}$ and let $\widetilde{P}_s$ be any standard parabolic subgroup minimal with respect to the property that $R_{\widetilde{P}_s}(\pi) \neq 0$. Write $s = (n_1, \ldots, n_k)$ and let σ be any irreducible subquotient of $R_{\widetilde{P}_s}(\pi)$. We can write $\sigma = \rho_1 \otimes \rho_2 \otimes \cdots \otimes \rho_k \otimes \rho$. Define $e(\rho_i)$ by $\rho_i = \nu^{e(\rho_i)}\rho_i^u$, where ρ_i^u is unitary for $1 \leq i \leq n$.

Assume that all of the following inequalities hold for every s and σ as above:

$$
\begin{aligned}
n_1 e(\rho_1) &> 0, \\
n_1 e(\rho_1) + n_2 e(\rho_2) &> 0, \\
&\vdots \\
n_1 e(\rho_1) + n_2 e(\rho_2) + \cdots + n_k e(\rho_k) &> 0.
\end{aligned}
$$

Then, π is a square integrable representation. Also, if π is a square integrable representation, then all of given inequalities hold for any s and σ as above.

The criterion for tempered representations is given by replacing every inequality above with $\geq$.

We also recall the definition of a negative representation ([8], Definition 4.1).

Let σ be an admissible irreducible representation of $\widetilde{Sp(n)}$. Then σ is a strongly negative (resp., negative) representation if and only if for every embedding $\sigma \hookrightarrow \rho_1 \times \rho_2 \times \cdots \times \rho_k \rtimes \rho$, where $\rho_i, 1 \leq i \leq k$ and ρ are irreducible

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supercuspidal representations, we have the following:

$$
\begin{aligned}
n_1 e(\rho_1) &< 0 \text{ (respectively, } \leq 0), \\
n_1 e(\rho_1) + n_2 e(\rho_2) &< 0 \text{ (respectively, } \leq 0), \\
&\vdots \\
n_1 e(\rho_1) + n_2 e(\rho_2) + \cdots + n_k e(\rho_k) &< 0 \text{ (respectively, } \leq 0).
\end{aligned}
$$

For the purpose of the notation we recall two useful results (Theorems 4.5 and 4.6 from [8]). We remind the reader that, for a cuspidal representation ρ of some $GL(m_\rho, F)$, a segment Δ is a set of cuspidal representations $\Delta = \{\rho, \nu\rho, \ldots, \nu^{k-1}\rho\}$ and $\langle \Delta \rangle$ is a unique irreducible subrepresentation of $\rho \times \nu\rho \times \cdots \times \nu^{k-1}\rho$.

- Suppose that $\Delta_1, \ldots, \Delta_k$ is a sequence of segments such that $e(\Delta_1) \geq \cdots \geq e(\Delta_k) > 0$ (we also allow $k = 0$). Let σ_{neg} be a negative representation. Then the induced representation $\langle \Delta_1 \rangle \times \langle \Delta_2 \rangle \times \cdots \times \langle \Delta_k \rangle \rtimes \sigma_{neg}$ has a unique irreducible subrepresentation, we denote it by $\langle \Delta_1, \ldots, \Delta_k; \sigma_{neg} \rangle$.

- If σ is an irreducible admissible genuine representation of $\widetilde{Sp(n)}$, then there exists a sequence of segments $\Delta_1, \ldots, \Delta_k$ satisfying $e(\Delta_1) \geq \cdots \geq e(\Delta_k) > 0$ and a negative representation σ_{neg} such that $\sigma \simeq \langle \Delta_1, \ldots, \Delta_k; \sigma_{neg} \rangle$.

In the same way as in [21], Chapter 2, with additional help of the subsection 4.2 (especially Proposition 4.5) of [8] we get very useful technical results:

- Fix an admissible representation π of $\widetilde{GL(2)}$, suppose that π is of finite length. Let $m^*(\pi) = 1 \otimes \pi + \sum_i \pi_i^1 \otimes \pi_i^2 + \pi \otimes 1$, where $\sum_i \pi_i^1 \otimes \pi_i^2$ is a decomposition into a sum of irreducible representations. Now we have:

$$
\begin{aligned}
\mu^*(\pi \rtimes \omega_0) &= 1 \otimes \pi \rtimes \omega_0 + \sum_i \pi_i^1 \otimes \pi_i^2 \rtimes \omega_0 + \sum_i \alpha\widetilde{\pi}_i^2 \otimes \pi_i^1 \rtimes \omega_0 + \\
&\quad + \pi \otimes \omega_0 + \alpha\widetilde{\pi} \otimes \omega_0 + \sum_i \pi_i^1 \times \alpha\widetilde{\pi}_i^2 \otimes \omega_0
\end{aligned}
$$

- Fix an admissible representation π of $\widetilde{GL(1)}$ and an admissible representation σ of $\widetilde{Sp(1)}$. If we have

$$\mu^*(\sigma) = 1 \otimes \sigma + \sum_i \sigma_i^1 \otimes \sigma_i^2,$$

where σ_i^1 and σ_i^2 are irreducible representations, then

$$\mu^*(\pi \rtimes \sigma) = 1 \otimes \pi \rtimes \sigma + \pi \otimes \sigma + \alpha\widetilde{\pi} \otimes \sigma + \sum_i \sigma_i^1 \otimes \pi \rtimes \sigma_i^2 +$$

$$+ \sum_i \pi \times \sigma_i^1 \otimes \sigma_i^2 + \sum_i \sigma_i^1 \times \alpha\widetilde{\pi} \otimes \sigma_i^2$$

From now on, $\widehat{F^\times}$ denotes the set of the unitary characters, while $\widetilde{F^\times}$ denotes the set of the not-necessarily unitary characters of F^*.

3 Principal series

We first state important reducibility result that follows directly from [9], Theorems 3.5. and 4.2.

Proposition 3.1. *Let $\chi \in \widehat{F^\times}$ and $s \in \mathbb{R}$, $s \geq 0$. The representation $\chi_{V,\psi}\nu^s\chi \rtimes \omega_0$ of $\widetilde{Sp(1)}$ reduces if and only if $\chi^2 = 1_{F^\times}$ and $s = \frac{1}{2}$.*
Let $\zeta \in \widehat{F^\times}$ such that $\zeta^2 = 1_{F^\times}$. In R we have (cf. [12], p. 89)

$$\chi_{V,\psi}\nu^{\frac{1}{2}}\zeta \rtimes \omega_0 = sp_{\zeta,1} + \omega^+_{\psi_a,1}.$$

The following proposition is well–known and follows easily from the analogous results for the split $SO(3)$ and $SO(5)$.

Proposition 3.2.   *1. Let $\chi \in \widehat{F^\times}$ and $s \in \mathbb{R}$, $s \geq 0$. The representation $\nu^s\chi \rtimes 1$ of $O(3)$ reduces if and only if $\chi^2 = 1_{F^\times}$ and $s = \frac{1}{2}$. In that situation, the length of $\nu^{\frac{1}{2}}\chi \rtimes 1$ is two, and this representation has the unique subrepresentation which is square integrable.*

2. Let $\zeta_1, \zeta_2 \in \widehat{F^\times}$. Then, the unitary principal series $\zeta_1 \times \zeta_2 \rtimes 1$ of $O(5)$ is irreducible.

We use the previous two propositions in the sequel without explicitly mentioning them.

3.1 Unitary principal series

In this subsection we prove irreducibility of the unitary principal series $\chi_{V,\psi}\chi_1 \times \chi_{V,\psi}\chi_2 \rtimes \omega_0$, $(\chi_i \in \widehat{F^\times}, \ i = 1, 2)$.

Let Π denote the representation $\chi_{V,\psi}\chi_1 \times \chi_{V,\psi}\chi_2 \rtimes \omega_0$. Using the structure formula for $\mu^*(\Pi)$ from the end of the previous section, we get

$$
\begin{aligned}
R_{\widetilde{P_1}}(\Pi) \quad = \quad & \chi_{V,\psi}\chi_1^{-1} \otimes \chi_{V,\psi}\chi_2 \rtimes \omega_0 + \chi_{V,\psi}\chi_1 \otimes \chi_{V,\psi}\chi_2 \rtimes \omega_0 + \\
& \chi_{V,\psi}\chi_2^{-1} \otimes \chi_{V,\psi}\chi_1 \rtimes \omega_0 + \chi_{V,\psi}\chi_2 \otimes \chi_{V,\psi}\chi_1 \rtimes \omega_0.
\end{aligned}
$$

Remark 3.3. *Let π be an irreducible subrepresentation of Π. Because of irreducibility of the representations $\chi_{V,\psi}\chi_1 \times \chi_{V,\psi}\chi_2$ and $\chi_{V,\psi}\chi_i \rtimes \omega_0$, $i = 1, 2$, we get:*

$\pi \hookrightarrow \Pi \simeq \chi_{V,\psi}\chi_1^{-1} \times \chi_{V,\psi}\chi_2 \rtimes \omega_0 \simeq \chi_{V,\psi}\chi_2^{-1} \times \chi_{V,\psi}\chi_1 \rtimes \omega_0.$

If $\chi_i \neq \chi_i^{-1}$ holds for both $i = 1, 2$ and $\chi_1 \neq \chi_2^{\pm 1}$, then Frobenius reciprocity implies that $R_{\widetilde{P_1}}(\pi) = R_{\widetilde{P_1}}(\Pi)$, so $\pi = \Pi$ and the representation Π is irreducible.

Now we prove the irreducibility of the unitary principal series for general unitary characters. Let ζ_1, ζ_2 be the unitary characters of $F^\times$. We prove irreducibility of the representation $\chi_{V,\psi}\zeta_1 \times \chi_{V,\psi}\zeta_2 \rtimes \omega_0$ using theta correspondence. We start with the following important lemma:

Lemma 3.4. *Let π_1 be an irreducible subrepresentation of $\chi_{V,\psi}\zeta_1 \times \chi_{V,\psi}\zeta_2 \rtimes \omega_0$. Then $\Theta(\pi_1, 2) = \zeta_1 \times \zeta_2 \rtimes 1$.*

Proof. According to the stable range condition (cf. [12], p. 48), $\Theta(\pi_1, 4) \neq 0$ (observe that $\Theta(\pi_1, 4)$ is a smooth representation of $O(9)$). We have epimorphisms $\omega_{2,4} \to \pi_1 \otimes \Theta(\pi_1, 4)$ and $R_{P_1}(\omega_{2,4}) \to \pi_1 \otimes R_{P_1}(\Theta(\pi_1, 4))$. If τ is an irreducible quotient of $\Theta(\pi_1, 4)$, then [11] (Corollary 2.6.) implies $[\tau] = [\nu^{-\frac{3}{2}}, \nu^{-\frac{1}{2}}, \zeta_1, \zeta_2; 1]$, where $[\tau]$ denotes the cuspidal support of τ. Clearly, $R_{P_{(1,1,1,1)}}(\tau) \geq \nu^{\frac{l_1}{2}} \otimes \nu^{\frac{l_2}{2}} \otimes \zeta_1^{\pm 1} \otimes \zeta_2^{\pm 1}$ or $R_{P_{(1,1,1,1)}}(\tau) \geq \zeta_1^{\pm 1} \otimes \nu^{\frac{l_1}{2}} \otimes \zeta_2^{\pm 1} \otimes \nu^{\frac{l_2}{2}}$ (or we have some order of factors), for some $l_1, l_2 \in \{\pm 1, \pm 3\}$. If we assume that, in the Jacquet module $R_{P_{(1,1,1,1)}}(\tau)$ there is an irreducible subquotient as above which has the first factor consisting of a unitary character, then, using [4] (Lemma 26), together with Frobenius reciprocity, easily follows that $\mathrm{Hom}(\tau, \zeta_1^{\pm 1} \times \nu^{\frac{l_1}{2}} \times \zeta_2^{\pm 1} \times \nu^{\frac{l_2}{2}} \rtimes 1) \neq 0$. But since $\zeta_i^{\pm 1} \times \nu^{\frac{l_i}{2}} \cong \nu^{\frac{l_i}{2}} \times \zeta_i^{\pm 1}$, we have $\mathrm{Hom}(\tau, \nu^{\frac{l_1}{2}} \times \zeta_1^{\pm 1} \times \zeta_2^{\pm 1} \times \nu^{\frac{l_2}{2}} \rtimes 1) \neq 0$. So, there is some irreducible subquotient τ' of $\zeta_1 \times \zeta_2 \times \nu^{\frac{l_2}{2}} \rtimes 1$ such that τ is subrepresentation of $\nu^{\frac{l_1}{2}} \rtimes \tau'$.

This implies that $R_{P_1}(\tau)(\nu^{\frac{l_1}{2}})$ (the isotypic component of $R_{P_1}(\tau)$ along the generalized character $\nu^{\frac{l_1}{2}}$) is non-zero and $R_{P_1}(\Theta(\pi_1,4))(\nu^{\frac{l_1}{2}})$ is also non-zero.

Observations above imply that there is an irreducible representation τ_1 of $O(3)$ such that the mappings $R_{P_1}(\omega_{2,4}) \to \pi_1 \otimes R_{P_1}(\Theta(\pi_1,4)) \to \pi_1 \otimes \nu^{\frac{l_1}{2}} \otimes \tau_1$ are epimorphisms. We denote the epimorphism $R_{P_1}(\omega_{2,4})) \to \pi_1 \otimes \nu^{\frac{l_1}{2}} \otimes \tau_1$ by T. $R_{P_1}(\omega_{2,4})$ has the following filtration ([12], p. 57, see also Proposition 3.3 of [9], where all the notation here is detaily explained)

$I_{10} = \nu^{-\frac{3}{2}} \otimes \omega_{2,3}$ (the quotient),

$I_{11} = \mathrm{Ind}_{GL(1) \times \widetilde{P}_1 \times O(3)}^{M_1 \times \widetilde{Sp(2)}}(\chi_{V,\psi}\Sigma_1' \otimes \omega_{1,3})$ (the subrepresentation).

Suppose $T|_{I_{11}} \neq 0$. Since the isotypic component of $\nu^{\frac{l_1}{2}}$ in the $\widetilde{GL(1,F)} \times GL(1,F)$–module $\chi_{V,\psi}\Sigma_1'$ is $\chi_{V,\psi}\nu^{-\frac{l_1}{2}}$, by applying the second Frobenius we get a non-zero $\widetilde{GL(1,F)} \times GL(1,F) \times \widetilde{Sp(1)} \times O(3)$–homomorphism

$$\nu^{\frac{l_1}{2}} \otimes \chi_{V,\psi}\nu^{-\frac{l_1}{2}} \otimes \omega_{1,3} \to \nu^{\frac{l_1}{2}} \otimes \tau_1 \otimes \widetilde{R_{\widetilde{P_1}}(\widetilde{\pi}_1)},$$

which implies that $\widetilde{R_{\widetilde{P_1}}(\widetilde{\pi}_1)}(\chi_{V,\psi}\nu^{-\frac{l_1}{2}}) \neq 0$. Because $l_1 \neq 0$, this contradicts our assumption $\pi_1 \hookrightarrow \chi_{V,\psi}\zeta_1 \times \chi_{V,\psi}\zeta_2 \rtimes \omega_0$, hence $T|_{I_{11}} = 0$. Therefore, we can consider T as an epimorphism $I_{10} \to \pi_1 \otimes \nu^{\frac{l_1}{2}} \otimes \tau_1$. Consequently, $l_1 = -3$ and there is an epimorphism $\omega_{2,3} \to \pi \otimes \tau_1$. Obviously, $\Theta(\pi_1,3) \neq 0$.

Repeating the same procedure once again, we obtain $\Theta(\pi_1,2) \neq 0$. Since the cuspidal support of each irreducible quotient of $\Theta(\pi_1,2)$ equals $[\zeta_1,\zeta_2;1]$, it follows that all of the irreducible quotients of $\Theta(\pi_1,2)$ are equal to $\zeta_1 \times \zeta_2 \rtimes 1$. $\qquad\square$

Proposition 3.5. *Let $\zeta_1, \zeta_2 \in \widehat{F^\times}$. Then the unitary principal series representation $\chi_{V,\psi}\zeta_1 \times \chi_{V,\psi}\zeta_2 \rtimes \omega_0$ is irreducible.*

We present two proofs of this proposition, both based on the previous lemma. The first proof is much simpler that the second one, it also uses some known results about Whittaker models for the principal series for metaplectic groups, but we have to assume that the residue characteristic of F is odd. The second proof is more technical, but it doesn't depend on the residue characteristic of F. We feel that presenting both proofs may be useful.

Proof. (The first proof) We denote the representation $\chi_{V,\psi}\zeta_1 \times \chi_{V,\psi}\zeta_2 \rtimes \omega_0$ by Π. Suppose that the residue characteristic of F is not 2. Howe's duality conjecture and Lemma then implies that the representation $\Theta(\zeta_1 \times \zeta_2 \rtimes 1, 2)$ has a unique irreducible quotient, so, by Lemma 3.4, all the irreducible subrepresentations of Π are isomorphic, i.e.,

$$\Pi = \pi \oplus \cdots \oplus \pi. \tag{1}$$

Now, observe that the representation Π has the unique Whittaker model ([2], [22]). In more words, for a nondegenerate character θ of the unipotent radical U of Borel subgroup of $Sp(n)$ (observe that $\widetilde{Sp(n)}$ splits over U, and the mapping $n \mapsto (n,1)$ is the splitting) and a genuine character $\chi_{V,\psi}\zeta_1 \otimes \cdots \otimes \chi_{V,\psi}\zeta_n$ of $\widetilde{T}$ (where $\widetilde{T}$ denotes the preimage of maximal diagonal torus in $Sp(n)$), we have:

$$\dim_{\mathbb{C}}\mathrm{Hom}_{\widetilde{Sp_n}}(\chi_{V,\psi}\zeta_1 \times \cdots \times \chi_{V,\psi}\zeta_n \rtimes \omega_0, \mathrm{Ind}_U^{\widetilde{Sp(n)}}(\theta)) = 1$$

This forces that the number of copies of π in (1) is one, and this finishes the first proof.

The second proof: We have already seen that there is an epimorphism $R_{\widetilde{P}_1}(\omega_{2,2}) \to \chi_{V,\psi}\zeta_1 \otimes \chi_{V,\psi} \rtimes \omega_0 \otimes \zeta_1 \times \zeta_2 \rtimes 1$, so $\Theta(\chi_{V,\psi}\zeta_1 \otimes \chi_{V,\psi}\zeta_2 \rtimes \omega_0 \otimes \zeta_1 \times \zeta_2 \rtimes 1, R_{\widetilde{P}_1}(\omega_{2,2})) \neq 0$. $R_{\widetilde{P}_1}(\omega_{2,2})$ has the following filtration:
$J_{10} = \chi_{V,\psi}\nu^{\frac{1}{2}} \otimes \omega_{1,2}$ (the quotient),
$J_{11} = \mathrm{Ind}_{\widetilde{GL(1)} \times P_1 \times \widetilde{Sp(1)}}^{\widetilde{M}_1 \times O(2)}(\chi_{V,\psi}\Sigma_1' \otimes \omega_{1,1})$ (the subrepresentation).
In what follows, we use the next

Lemma 3.6. *There is an isomorphism of vector spaces*

$\mathrm{Hom}_{\widetilde{M}_1}(R_{\widetilde{P}_1}(\omega_{2,2}), \chi_{V,\psi}\zeta_1 \otimes \chi_{V,\psi}\zeta_2 \rtimes \omega_0) \cong \mathrm{Hom}_{\widetilde{M}_1}(J_{11}, \chi_{V,\psi}\zeta_1 \otimes \chi_{V,\psi}\zeta_2 \rtimes \omega_0)$,

which is given by restriction $(T \mapsto T|_{J_{11}})$.

Proof. (of Lemma 3.6) The map obtained by the restriction is obviously an homomorphism, while the injectivity follows directly. Surjectivity is proved in the following way:
We consider the filtration $0 \subseteq W_2 \subseteq W_1 \subseteq R_{\widetilde{P}_1}(\omega_{2,2})$, where W_1 is the representation J_{11}, while $W_1/W_2 \cong \chi_{V,\psi}\zeta_1 \otimes \chi_{V,\psi}\zeta_2 \rtimes \omega_0 \otimes \Theta(\chi_{V,\psi}\zeta_1 \otimes \chi_{V,\psi}\zeta_2 \rtimes \omega_0, J_{11})$. Observe that

$$(R_{\widetilde{P}_1}(\omega_{2,2})/W_2)/(W_1/W_2) \cong R_{\widetilde{P}_1}(\omega_{2,2})/W_1 \cong J_{10}.$$

Using standard argument, it can be proved that the representation $R_{\widetilde{P}_1}(\omega_{2,2})/W_2$ is $\widetilde{GL(1)}$-finite. Because of that, using the decomposition along the generalized central characters, which in this case coincide with the central characters (because W_1/W_2 i J_{10} have different central characters), we obtain:

$$R_{\widetilde{P}_1}(\omega_{2,2})/W_2 \cong W_1/W_2 \oplus J_{10}.$$

Now an element of $\mathrm{Hom}_{\widetilde{M}_1}(J_{11}, \chi_{V,\psi}\zeta_1 \otimes \chi_{V,\psi}\zeta_2 \rtimes \omega_0)$ is trivial on W_2, so it can be extended to $R_{\widetilde{P}_1}(\omega_{2,2})$ in an obvious way and surjectivity is proved. $\quad\square$

Using a standard relation between taking a smooth part of the isotypic component of a representation and the homomorphism functor ([9], p. 10), from the previous lemma it follows that

$$\Theta(\chi_{V,\psi}\zeta_1 \otimes \chi_{V,\psi}\zeta_2 \rtimes \omega_0, R_{\widetilde{P}_1}(\omega_{2,2})) \cong \Theta(\chi_{V,\psi}\zeta_1 \otimes \chi_{V,\psi}\zeta_2 \rtimes \omega_0, J_{11}),$$

if we prove that $\Theta(\chi_{V,\psi}\zeta_1 \otimes \chi_{V,\psi}\zeta_2 \rtimes \omega_0, J_{11})$ is admissible.

Lemma 3.7. *We have* $\Theta(\chi_{V,\psi}\zeta_1 \otimes \chi_{V,\psi}\zeta_2 \rtimes \omega_0, J_{11}) = \zeta_1 \times \zeta_2 \rtimes 1$.

Proof. (of Lemma 3.7): Since $\chi_{V,\psi}\zeta_1 \otimes \chi_{V,\psi}\zeta_2 \rtimes \omega_0 \otimes \zeta_1 \times \zeta_2 \rtimes 1$ is a quotient of J_{11}, there is an epimorphism $\Theta(\chi_{V,\psi}\zeta_1 \otimes \chi_{V,\psi}\zeta_2 \rtimes \omega_0, J_{11}) \to \zeta_1 \times \zeta_2 \rtimes 1$.

Applying Lemma 3.2. from [9], we have:

$$\mathrm{Hom}_{\widetilde{M}_1 \times O(2)}(J_{11}, \chi_{V,\psi}\zeta_1 \otimes \chi_{V,\psi}\zeta_2 \rtimes \omega_0 \otimes \Theta(\chi_{V,\psi}\zeta_1 \otimes \chi_{V,\psi}\zeta_2 \rtimes \omega_0, J_{11})) \cong$$

$$\mathrm{Hom}_{\widetilde{M}_1 \times M(1)}(\chi_{V,\psi}\Sigma_1' \otimes \omega_{1,1}, \chi_{V,\psi}\zeta_1 \otimes \chi_{V,\psi}\zeta_2 \rtimes \omega_0 \otimes R_{\overline{P}_1}(\Theta(\chi_{V,\psi}\zeta_1 \otimes \chi_{V,\psi}\zeta_2 \rtimes \omega_0, J_{11}))).$$

For every intertwining map T from the first space, let T_0 be the corresponding intertwining map from the second space. Let φ be a natural homomorphism belonging to the first space.

Since $\chi_{V,\psi}\zeta_1 \otimes \zeta_1^{-1}$ (respectively, $\chi_{V,\psi}\zeta_2 \rtimes \omega_0 \otimes \zeta_2 \rtimes 1$) are the corresponding isotypic components in the $\widetilde{GL(1,F)} \times GL(1,F)$–module $\chi_{V,\psi}\Sigma_1'$ (respectively, in the $\widetilde{Sp(1)} \times O(3)$–module $\omega_{1,1}$), irreducibility of these isotypic components implies that the image of φ_0 is isomorphic to $\chi_{V,\psi}\zeta_1 \otimes \chi_{V,\psi}\zeta_2 \rtimes \omega_0 \otimes \zeta_1^{-1} \otimes \zeta_2 \rtimes 1$. Now, we write $\varphi_0 = \varphi'' \circ \varphi'$, where φ' is a canonical epimorphism $\chi_{V,\psi}\Sigma_1' \otimes \omega_{1,1} \to \chi_{V,\psi}\zeta_1 \otimes \chi_{V,\psi}\zeta_2 \rtimes \omega_0 \otimes \zeta_1^{-1} \otimes \zeta_2 \rtimes 1$, and φ'' is an inclusion of the representation $\chi_{V,\psi}\zeta_1 \otimes \chi_{V,\psi}\zeta_2 \rtimes \omega_0 \otimes \zeta_1^{-1} \otimes \zeta_2 \rtimes 1$ in $\chi_{V,\psi}\zeta_1 \otimes \chi_{V,\psi}\zeta_2 \rtimes \omega_0 \otimes$

$R_{\overline{P}_1}(\Theta(\chi_{V,\psi}\zeta_1 \otimes \chi_{V,\psi}\zeta_2 \rtimes \omega_0, J_{11}))$. Observe that $\mathrm{Ind}(\varphi')$ is a homomorphism $\mathrm{Ind}_{\widetilde{GL(1) \times P_1 \times Sp(1)}}^{\widetilde{M_1 \times O(2)}}(\chi_{V,\psi}\Sigma_1' \otimes \omega_{1,1}) \to \chi_{V,\psi}\zeta_1 \otimes \chi_{V,\psi}\zeta_2 \rtimes \omega_0 \otimes \zeta_1^{-1} \times \zeta_2 \rtimes 1$.

Let φ_1 be an operator belonging to $\mathrm{Hom}(\chi_{V,\psi}\zeta_1 \otimes \chi_{V,\psi}\zeta_2 \rtimes \omega_0 \otimes \zeta_1 \times \zeta_2 \rtimes 1, \chi_{V,\psi}\zeta_1 \otimes \chi_{V,\psi}\zeta_2 \rtimes \omega_0 \otimes \Theta(\chi_{V,\psi}\zeta_1 \otimes \chi_{V,\psi}\zeta_2 \rtimes \omega_0, J_{11}))$, such that $(\varphi_1)_0 = \varphi''$.

Lemma 3.8. *Under the above assumptions $(\varphi_1 \circ \mathrm{Ind}(\varphi'))_0 = \varphi_0$.*

Proof. (of Lemma 3.8): We prove this lemma much more generally. Let (π, V) be a smooth representation of some Levi subgroup M' in the parabolic P' and the opposite parabolic $\overline{P'}$ of the group G' (which is one of the groups we are considering, i.e., metaplectic or odd orthogonal) and (Π, W) a smooth representation of G'. Then the second Frobenius isomorphism asserts

$$\mathrm{Hom}_{G'}(\mathrm{Ind}_{M'}^{G'}(\pi), \Pi) \cong \mathrm{Hom}_{M'}(\pi, R_{\overline{P'}}(\Pi)).$$

Let $\psi \hookrightarrow R_{\overline{P'}}(\mathrm{Ind}_{M'}^{G'}(\pi))$ be an embedding corresponding to the open cell $P'\overline{P'}$ in G' given in the following way:

For an open compact subgroup K of G', which has Iwahori decomposition with respect to both P' and $\overline{P'}$, and $v \in V^{K \cap M'}$, we define:

$$f_{v,K}(g) = \frac{1}{\mathrm{meas}_{\overline{N'}}(K \cap \overline{N'})} \begin{cases} 0, & g \notin P'K \\ \delta_P^{\frac{1}{2}}(m)\pi(m)v, & g = mnk, m \in M', n \in N', k \in K. \end{cases}$$

Then $\psi : v \mapsto f_{v,K} + \mathrm{Ind}_{M'}^{G'}(\pi)(\overline{N'})$ is independent on the choice of K.

For $\varphi \in \mathrm{Hom}_{G'}(\mathrm{Ind}_{M'}^{G'}(\pi), \Pi)$, let φ_0 be the corresponding element of $\mathrm{Hom}_{M'}(\pi, R_{\overline{P}}(\Pi))$. It follows that $\varphi_0(v) = \varphi(f_{v,K}) + \Pi(\overline{N'})$. Write $\varphi_0 = \varphi'' \circ \varphi'$, where φ' denotes the canonical epimorphism $\pi \to \pi/\mathrm{Ker}\varphi_0$, while φ'' denotes the embedding $\pi/\mathrm{Ker}\varphi_0 \hookrightarrow R_{\overline{P'}}$. So, we are able to construct the mapping $\mathrm{Ind}(\varphi') : \mathrm{Ind}_{M'}^{G'}(\pi) \to \mathrm{Ind}_{M'}^{G'}(\pi/\mathrm{Ker}\varphi_0)$. Since

$$\mathrm{Hom}_{M'}(\pi/\mathrm{Ker}\varphi_0, R_{\overline{P'}}(\Pi)) \cong \mathrm{Hom}_{G'}(\mathrm{Ind}_{M'}^{G'}(\pi/\mathrm{Ker}\varphi_0), \Pi),$$

analogously as above, we conclude that there is an element $\varphi_1 \in \mathrm{Hom}_{G'}(\mathrm{Ind}_{M'}^{G'}(\pi/\mathrm{Ker}\varphi_0), \Pi)$ such that $(\varphi_1)_0 = \varphi''$.

To prove $(\varphi_1 \circ \mathrm{Ind}(\varphi'))_0 = \varphi_0$, it is enough to prove $(\varphi_1 \circ \mathrm{Ind}(\varphi'))_0 = (\varphi_1)_0 \circ \varphi'$.

Let $v \in V$. Clearly, $\varphi'(v) = v + \mathrm{Ker}\varphi_0$. Further, $(\varphi_1)_0(\varphi'(v)) = \varphi_1(f_{v+\mathrm{Ker}\varphi_0, K}) + \Pi(\overline{N'})$ and $(\varphi_1 \circ \mathrm{Ind}(\varphi'))_0(v) = \varphi_1(\mathrm{Ind}(\varphi')f_{v,K}) + \Pi(\overline{N'})$. It follows easily that $f_{v+\mathrm{Ker}\varphi_0, K} = f_{v,K} + \mathrm{Ker}\varphi_0$ and $\mathrm{Ind}(\varphi')f_{v,K} = f_{v,K} + \mathrm{Ker}\varphi_0$, and the lemma follows. $\square$

We are now able to complete the proof of the Lemma 3.7. Lemma 3.8 implies $\varphi_1 \circ \mathrm{Ind}(\varphi') = \varphi$, so the image of φ is a quotient of $\chi_{V,\psi}\zeta_1 \otimes \chi_{V,\psi}\zeta_2 \rtimes \omega_0 \otimes \zeta_1^{-1} \times \zeta_2 \rtimes 1$. This implies that $\Theta(\chi_{V,\psi}\zeta_1 \otimes \chi_{V,\psi}\zeta_2 \rtimes \omega_0, J_{11})$ is a quotient of $\zeta_1^{-1} \times \zeta_2 \rtimes 1$. Since $\zeta_1^{-1} \times \zeta_2 \rtimes 1 \simeq \zeta_1 \times \zeta_2 \rtimes 1$ is an irreducible representation, $\Theta(\chi_{V,\psi}\zeta_1 \otimes \chi_{V,\psi}\zeta_2 \rtimes \omega_0, J_{11}) = \zeta_1 \times \zeta_2 \rtimes 1$.

$\square$

Lemma 3.9. *There is an epimorphism* $\Theta(\zeta_1 \times \zeta_2 \rtimes 1, 2) \to \chi_{V,\psi}\zeta_1 \times \chi_{V,\psi}\zeta_2 \rtimes \omega_0$.

Proof. (of Lemma 3.9): We have an isomorphism of vector spaces

$$\mathrm{Hom}_{O(2)}(\omega_{2,2}, \zeta_1^{-1} \times \zeta_2 \rtimes 1) \cong \mathrm{Hom}(R_{P_1}(\omega_{2,2}), \zeta_1^{-1} \otimes \zeta_2 \rtimes 1),$$

which also an isomorphism of $\widetilde{Sp(2)}$-modules. By taking the smooth parts we obtain

$$\mathrm{Hom}_{\widetilde{Sp(2)} \times O(2)}(\omega_{2,2}, \zeta_1^{-1} \times \zeta_2 \rtimes 1)_\infty \cong \mathrm{Hom}(R_{P_1}(\omega_{2,2}), \zeta_1^{-1} \otimes \zeta_2 \rtimes 1)_\infty,$$

so that $\Theta(\zeta_1^{-1} \times \zeta_2 \rtimes 1, 2)\widetilde{} \cong \Theta(\zeta_1^{-1} \otimes \zeta_2 \rtimes 1, R_{P_1}(\omega_{2,2}))\widetilde{}$.

In the same way as before, we get $\Theta(\zeta_1^{-1} \otimes \zeta_2 \rtimes 1, R_{P_1}(\omega_{2,2}))\widetilde{} \cong \Theta(\zeta_1^{-1} \otimes \zeta_2 \rtimes 1, I_{11})\widetilde{}$. Now, epimorphism $I_{11} \to \zeta_1^{-1} \otimes \zeta_2 \rtimes 1 \otimes \chi_{V,\psi}\zeta_1 \times \chi_{V,\psi}\zeta_2 \rtimes \omega_0$ gives an epimorphism $\Theta(\zeta_1^{-1} \otimes \zeta_2 \rtimes 1, I_{11})) \to \chi_{V,\psi}\zeta_1 \times \chi_{V,\psi}\zeta_2 \rtimes \omega_0$. Since the representations $\zeta_1^{-1} \times \zeta_2 \rtimes 1$ and $\zeta_1 \times \zeta_2 \rtimes 1$ are isomorphic, we obtain the epimorphism $\Theta(\zeta_1 \times \zeta_2 \rtimes 1, 2) \to \chi_{V,\psi}\zeta_1 \times \chi_{V,\psi}\zeta_2 \rtimes \omega_0$, which proves the lemma. $\square$

Now we finish the second proof of the Proposition 3.5. Suppose that the representation $\chi_{V,\psi}\zeta_1 \times \chi_{V,\psi}\zeta_2 \rtimes \omega_0$ reduces. Suppose also that it is the representation of length 2 and write $\chi_{V,\psi}\zeta_1 \times \chi_{V,\psi}\zeta_2 \rtimes \omega_0 = \pi_1 \oplus \pi_2$. Obviously, $R_{\widetilde{P_1}}(\chi_{V,\psi}\zeta_1 \times \chi_{V,\psi}\zeta_2 \rtimes \omega_0) = R_{\widetilde{P_1}}(\pi_1) \oplus R_{\widetilde{P_1}}(\pi_2)$.

We have, by Lemma 3.9, an epimorphism $\omega_{2,2} \to \zeta_1 \times \zeta_2 \rtimes 1 \otimes \chi_{V,\psi}\zeta_1 \times \chi_{V,\psi}\zeta_2 \rtimes \omega_0$ which leads to the epimorphisms $R_{\widetilde{P_1}}(\omega_{2,2}) \to \zeta_1 \times \zeta_2 \rtimes 1 \otimes (R_{\widetilde{P_1}}(\pi_1) \oplus R_{\widetilde{P_1}}(\pi_2))$ and $R_{\widetilde{P_1}}(\omega_{2,2}) \to \chi_{V,\psi}\zeta_1 \otimes \chi_{V,\psi}\zeta_2 \rtimes \omega_0 \otimes (\zeta_1 \times \zeta_2 \rtimes 1 \oplus \zeta_1 \times \zeta_2 \rtimes 1)$.

Finally, we obtain an epimorphism

$$\Theta(\chi_{V,\psi}\zeta_1 \otimes \chi_{V,\psi}\zeta_2 \rtimes \omega_0, R_{\widetilde{P_1}}(\omega_{2,2})) \to \zeta_1 \times \zeta_2 \rtimes 1 \oplus \zeta_1 \times \zeta_2 \rtimes 1,$$

which contradicts Lemma 3.6 and Lemma 3.7.

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The same proof remains valid if we suppose that $\chi_{V,\psi}\zeta_1 \times \chi_{V,\psi}\zeta_2 \rtimes \omega_0$ is the representation of the length 4. This completes the second proof of the Proposition 3.5. $\qquad\square$

3.2 Non-unitary principal series

First we determine the reducibility points of the representations with cuspidal support in the minimal parabolic subgroup $\widetilde{P}_{(1,1)}$.

Let $\chi_1, \chi_2 \in \widehat{F^\times}$ and $s_i \geq 0$, $i = 1, 2$ such that $s_i > 0$ for at least one i. Define $\Pi = \chi_{V,\psi}\nu^{s_1}\chi_1 \times \chi_{V,\psi}\nu^{s_2}\chi_2 \rtimes \omega_0$. We have the following:

$$
\begin{aligned}
\mu^*(\Pi) \;=\; & \chi_{V,\psi}\nu^{s_1}\chi_1 \otimes \chi_{V,\psi}\nu^{s_2}\chi_2 \rtimes \omega_0 + \chi_{V,\psi}\nu^{-s_1}\chi_1^{-1} \otimes \chi_{V,\psi}\nu^{s_2}\chi_2 \rtimes \omega_0 + \\
& \chi_{V,\psi}\nu^{s_2}\chi_2 \otimes \chi_{V,\psi}\nu^{s_1}\chi_1 \rtimes \omega_0 + \chi_{V,\psi}\nu^{-s_2}\chi_2^{-1} \otimes \chi_{V,\psi}\nu^{s_1}\chi_1 \rtimes \omega_0 + \\
& \chi_{V,\psi}\nu^{-s_1}\chi_1^{-1} \times \chi_{V,\psi}\nu^{s_2}\chi_2 \otimes \omega_0 + \chi_{V,\psi}\nu^{s_1}\chi_1 \times \chi_{V,\psi}\nu^{-s_2}\chi_2^{-1} \otimes \omega_0 + \\
& \chi_{V,\psi}\nu^{s_1}\chi_1 \times \chi_{V,\psi}\nu^{s_2}\chi_2 \otimes \omega_0 + \chi_{V,\psi}\nu^{-s_1}\chi_1^{-1} \times \chi_{V,\psi}\nu^{-s_2}\chi_2^{-1} \otimes \omega_0 + \\
& 1 \otimes \chi_{V,\psi}\nu^{s_1}\chi_1 \times \chi_{V,\psi}\nu^{s_2}\chi_2 \rtimes \omega_0
\end{aligned}
$$

We prove that irreducibility of all the above representations implies irreducibility of the representation Π. We keep this assumption throughout this subsection.

First, suppose that $\nu^{s_1}\chi_1 \neq \nu^{-s_1}\chi_1^{-1}$, $\nu^{s_2}\chi_2 \neq \nu^{-s_2}\chi_2^{-1}$ and $\nu^{s_1}\chi_1 \neq \nu^{\pm s_2}\chi_2^{\pm 1}$ (i.e., Jacquet modules of Π are multiplicity one).

Let τ be an irreducible subquotient of Π such that $\chi_{V,\psi}\nu^{-s_1}\chi_1^{-1} \times \chi_{V,\psi}\nu^{s_2}\chi_2 \otimes \omega_0 \leq R_{\widetilde{P}_2}(\tau)$. From transitivity of Jacquet modules we get $\chi_{V,\psi}\nu^{-s_1}\chi_1^{-1} \otimes \chi_{V,\psi}\nu^{s_2}\chi_2 \otimes \omega_0 + \chi_{V,\psi}\nu^{s_2}\chi_2 \otimes \chi_{V,\psi}\nu^{-s_1}\chi_1^{-1} \otimes \omega_0 \leq R_{\widetilde{P}_{(1,1)}}(\tau)$. This implies $\chi_{V,\psi}\nu^{-s_1}\chi_1^{-1}\otimes\chi_{V,\psi}\nu^{s_2}\chi_2\rtimes\omega_0 + \chi_{V,\psi}\nu^{s_2}\chi_2\otimes\chi_{V,\psi}\nu^{s_1}\chi_1\rtimes\omega_0 \leq R_{\widetilde{P}_1}(\tau)$. We get directly that $R_{\widetilde{P}_2}(\tau) = \chi_{V,\psi}\nu^{-s_1}\chi_1^{-1} \times \chi_{V,\psi}\nu^{s_2}\chi_2\otimes\omega_0 + \chi_{V,\psi}\nu^{s_1}\chi_1 \times \chi_{V,\psi}\nu^{-s_2}\chi_2^{-1}\otimes\omega_0 + \chi_{V,\psi}\nu^{s_1}\chi_1 \times \chi_{V,\psi}\nu^{s_2}\chi_2 \otimes \omega_0 + \chi_{V,\psi}\nu^{-s_1}\chi_1^{-1} \times \chi_{V,\psi}\nu^{-s_2}\chi_2^{-1} \otimes \omega_0$, so $\tau = \Pi$ and Π is irreducible.

Now we assume that there is some i such that $\nu^{s_i}\chi_i \neq \nu^{-s_i}\chi_i^{-1}$. Without loss of generality, let $i = 1$. So, $s_1 = 0$ and $\chi_1 = \chi_1^{-1}$, i.e., $\chi_1^2 = 1_{F^\times}$. We prove that in this case Π is also irreducible. Again, we start by writing corresponding Jacquet modules:

$$
\begin{aligned}
R_{\widetilde{P}_1}(\Pi) \;=\; & 2\chi_{V,\psi}\chi_1 \otimes \chi_{V,\psi}\nu^{s_2}\chi_2 \rtimes \omega_0 + \chi_{V,\psi}\nu^{s_2}\chi_2 \otimes \chi_{V,\psi}\chi_1 \rtimes \omega_0 + \\
& \chi_{V,\psi}\nu^{-s_2}\chi_2^{-1} \otimes \chi_{V,\psi}\chi_1 \rtimes \omega_0,
\end{aligned}
$$

$$R_{\widetilde{P}_2}(\Pi) \;\; = \;\; 2\chi_{V,\psi}\chi_1 \times \chi_{V,\psi}\nu^{s_2}\chi_2 \otimes \omega_0 + \chi_{V,\psi}\chi_1 \times \chi_{V,\psi}\nu^{-s_2}\chi_2^{-1} \otimes \omega_0.$$

Let τ be an irreducible subquotient of Π such that $R_{\widetilde{P}_1}(\tau) \geq \chi_{V,\psi}\nu^{s_2}\chi_2 \otimes \chi_{V,\psi}\chi_1 \rtimes \omega_0$. Of course, $R_{\widetilde{P}_{(1,1)}}(\tau) \geq 2\chi_{V,\psi}\nu^{s_2}\chi_2 \otimes \chi_{V,\psi}\chi_1 \otimes \omega_0$, so $R_{\widetilde{P}_2}(\tau) \geq 2\chi_{V,\psi}\chi_1 \times \chi_{V,\psi}\nu^{s_2}\chi_2 \otimes \omega_0$. Continuing in the same way, we get $R_{\widetilde{P}_{(1,1)}}(\tau) \geq 2\chi_{V,\psi}\chi_1 \otimes \chi_{V,\psi}\nu^{s_2}\chi_2 \otimes \omega_0 + 2\chi_{V,\psi}\nu^{s_2}\chi_2 \otimes \chi_{V,\psi}\chi_1 \otimes \omega_0$ and $R_{\widetilde{P}_1}(\tau) \geq 2\chi_{V,\psi}\chi_1 \otimes \chi_{V,\psi}\nu^{s_2}\chi_2 \rtimes \omega_0 + \chi_{V,\psi}\nu^{s_2}\chi_2 \otimes \chi_{V,\psi}\chi_1 \rtimes \omega_0$. Finally, $R_{\widetilde{P}_{(1,1)}}(\tau) \geq 2\chi_{V,\psi}\chi_1 \otimes \chi_{V,\psi}\nu^{s_2}\chi_2 \otimes \omega_0 + 2\chi_{V,\psi}\chi_1 \otimes \chi_{V,\psi}\nu^{-s_2}\chi_2^{-1} \otimes \omega_0 + 2\chi_{V,\psi}\nu^{s_2}\chi_2 \otimes \chi_{V,\psi}\chi_1 \otimes \omega_0$ and $R_{\widetilde{P}_2}(\tau) \geq 2\chi_{V,\psi}\chi_1 \times \chi_{V,\psi}\nu^{s_2}\chi_2 \otimes \omega_0 + \chi_{V,\psi}\chi_1 \times \chi_{V,\psi}\nu^{-s_2}\chi_2^{-1} \otimes \omega_0 = R_{\widetilde{P}_2}(\Pi)$. So, $\Pi = \tau$ and Π is irreducible.

If $\nu^{s_1}\chi_1 = \nu^{s_2}\chi_2$ or $\nu^{s_1}\chi_1 = \nu^{-s_2}\chi_2^{-1}$, then the irreducibility of Π follows in the same way as above. Observe that equalities $\nu^{s_1}\chi_1 = \nu^{-s_1}\chi_1^{-1}$ and $\nu^{s_2}\chi_2 = \nu^{-s_2}\chi_2^{-1}$ lead to unitary principal series.

It is worth pointing out that in this way we have proved irreducibility of the principal series in all but the following cases:

- some of the representations $\chi_{V,\psi}\nu^{s_1}\chi_1 \rtimes \omega_0$ or $\chi_{V,\psi}\nu^{s_2}\chi_2 \rtimes \omega_0$ reduce (so-called $\widetilde{Sp(1)}$ reducibility),

- some of the representations $\chi_{V,\psi}\nu^{-s_1}\chi_1^{-1} \times \chi_{V,\psi}\nu^{s_2}\chi_2$, $\chi_{V,\psi}\nu^{s_1}\chi_1 \times \chi_{V,\psi}\nu^{-s_2}\chi_2^{-1}$, $\chi_{V,\psi}\nu^{s_1}\chi_1 \times \chi_{V,\psi}\nu^{s_2}\chi_2$ or $\chi_{V,\psi}\nu^{-s_1}\chi_1^{-1} \times \chi_{V,\psi}\nu^{-s_2}\chi_2^{-1}$ reduce (so-called $\widetilde{GL(2)}$ reducibility).

3.2.1 $\widetilde{Sp(1)}$ reducibility

Let $\chi, \zeta \in \widehat{F^\times}$, $\zeta^2 = 1_{F^\times}$, and $s \geq 0$. It is well - known that in R holds:

$$\chi_{V,\psi}\nu^s\chi \times \chi_{V,\psi}\nu^{\frac{1}{2}}\zeta \rtimes \omega_0 = \chi_{V,\psi}\nu^s \rtimes sp_{\zeta,1} + \chi_{V,\psi}\nu^s \rtimes \omega_{\psi_a,1}^+.$$

Let Π denote $\chi_{V,\psi}\nu^s \rtimes sp_{\zeta,1}$.
Calculating Jacquet modules we find:

$$\begin{aligned} R_{\widetilde{P}_1}(\Pi) \;\; = \;\; & \chi_{V,\psi}\nu^{-s}\chi^{-1} \otimes sp_{\zeta,1} + \chi_{V,\psi}\nu^s\chi \otimes sp_{\zeta,1} + \\ & \chi_{V,\psi}\nu^{\frac{1}{2}}\zeta \otimes \chi_{V,\psi}\nu^s\chi \rtimes \omega_0 \end{aligned}$$

and
$$R_{\widetilde{P}_2}(\Pi) = \chi_{V,\psi}\nu^{-s}\chi^{-1} \times \chi_{V,\psi}\nu^{\frac{1}{2}}\zeta \otimes \omega_0 + \chi_{V,\psi}\nu^s\chi \times \chi_{V,\psi}\nu^{\frac{1}{2}}\zeta \otimes \omega_0.$$

If the representation $\chi_{V,\psi}\nu^s\chi \rtimes \omega_0$ is irreducible (that is, when $\nu^s\chi \neq \nu^{\pm\frac{1}{2}}\zeta_2$, where $\zeta_2^2 = 1_{F^\times}$), we proceed in the following way:

Let ρ be an irreducible subquotient of Π such that $\chi_{V,\psi}\nu^{\frac{1}{2}}\zeta \otimes \chi_{V,\psi}\nu^s\chi \rtimes \omega_0 \leq s_1(\rho)$. We directly get that $\chi_{V,\psi}\nu^{\frac{1}{2}}\zeta \otimes \chi_{V,\psi}\nu^s\chi \otimes \omega_0 + \chi_{V,\psi}\nu^{\frac{1}{2}}\zeta \otimes \chi_{V,\psi}\nu^{-s}\chi^{-1} \otimes \omega_0 \leq R_{\widetilde{P}_{(1,1)}}(\rho)$. If both $\chi_{V,\psi}\nu^{-s}\chi^{-1} \times \chi_{V,\psi}\nu^{\frac{1}{2}}\zeta$ and $\chi_{V,\psi}\nu^s\chi \times \chi_{V,\psi}\nu^{\frac{1}{2}}\zeta$ are irreducible, Π is also irreducible.

For the reducibility of the $\widetilde{Sp(1)}$-part we still have to determine the composition factors of the following representations:

(i) $\chi_{V,\psi}\nu^{\frac{1}{2}}\zeta_1 \times \chi_{V,\psi}\nu^{\frac{1}{2}}\zeta_2 \rtimes \omega_0$,

(ii) $\chi_{V,\psi}\nu^{\frac{1}{2}}\zeta \times \chi_{V,\psi}\nu^{\frac{1}{2}}\zeta \rtimes \omega_0$,

(iii) $\chi_{V,\psi}\nu^{\frac{3}{2}}\zeta \times \chi_{V,\psi}\nu^{\frac{1}{2}}\zeta \rtimes \omega_0$,

where $\zeta^2 = \zeta_1^2 = \zeta_2^2 = 1_{F^*}$.

Thus, we have proved the following result:

Proposition 3.10. *Let* $\chi \in \widehat{F^\times}$, $s \in \mathbb{R}$, $s \geq 0$, $\zeta \in \widehat{F^\times}$ *such that* $\zeta^2 = 1_{F^\times}$. *The representations* $\chi_{V,\psi}\nu^s\chi \rtimes sp_{\zeta,1}$ *and* $\chi_{V,\psi}\nu^s\chi \rtimes \omega_{\psi_a,1}^+$ *are irreducible unless* $(s,\chi) = (\frac{3}{2},\zeta)$ *or* $(\frac{1}{2},\zeta_1)$, *where* $\zeta_1^2 = 1_{F^\times}$. *In R we have* $\chi_{V,\psi}\nu^s\chi \times \chi_{V,\psi}\nu^{\frac{1}{2}}\zeta \rtimes \omega_0 = \chi_{V,\psi}\nu^s\chi \rtimes sp_{\zeta,1} + \chi_{V,\psi}\nu^s\chi \rtimes \omega_{\psi_a,1}^+$. *Also, if* $(s,\chi) \neq (\frac{3}{2},\zeta)$ *and* $(s,\chi) \neq (\frac{1}{2},\zeta_1)$, *then*

$$\chi_{V,\psi}\nu^s\chi \rtimes sp_{\zeta,1} = \begin{cases} \langle \chi_{V,\psi}\nu^{\frac{1}{2}}\zeta; \chi_{V,\psi}\chi \rtimes \omega_0 \rangle & \textit{if } s = 0, \\ \langle \chi_{V,\psi}\nu^{\frac{1}{2}}\zeta, \chi_{V,\psi}\nu^s\chi; \omega_0 \rangle & \textit{if } 0 < s \leq \frac{1}{2}, \\ \langle \chi_{V,\psi}\nu^s\chi, \chi_{V,\psi}\nu^{\frac{1}{2}}\zeta; \omega_0 \rangle & \textit{if } s > \frac{1}{2}, \end{cases}$$

and

$$\chi_{V,\psi}\nu^s\chi \rtimes \omega_{\psi_a,1}^+ = \begin{cases} \chi_{V,\psi}\chi \rtimes \omega_{\psi_a,1}^+ & \textit{if } s = 0, \\ \langle \chi_{V,\psi}\nu^s\chi; \omega_{\psi_a,1}^+ \rangle & \textit{if } s > 0. \end{cases}$$

3.2.2 $\widetilde{GL(2)}$ reducibility

Let $\chi \in \widehat{F^\times}$ and $s \in \mathbb{R}$, $s \geq 0$. In R we have

$$\chi_{V,\psi}\nu^{s+\frac{1}{2}}\chi \times \chi_{V,\psi}\nu^{s-\frac{1}{2}}\chi \rtimes \omega_0 = \chi_{V,\psi}\nu^s\chi St_{GL(2)} \rtimes \omega_0 + \chi_{V,\psi}\nu^s\chi 1_{GL(2)} \rtimes \omega_0.$$

Let Π denote the representation $\chi_{V,\psi}\nu^s\chi St_{GL(2)} \rtimes \omega_0$. Calculation of $\mu^*(\Pi)$ gives:

$$\begin{aligned} R_{\widetilde{P}_1}(\Pi) &= \chi_{V,\psi}\nu^{s+\frac{1}{2}}\chi \otimes \chi_{V,\psi}\nu^{s-\frac{1}{2}}\chi \rtimes \omega_0 + \\ & \quad \chi_{V,\psi}\nu^{\frac{1}{2}-s}\chi^{-1} \otimes \chi_{V,\psi}\nu^{s+\frac{1}{2}}\chi \rtimes \omega_0 \end{aligned}$$

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$$R_{\widetilde{P_2}}(\Pi) \;=\; \chi_{V,\psi}\nu^s\chi St_{GL(2)} \otimes \omega_0 + \chi_{V,\psi}\nu^{-s}\chi^{-1}St_{GL(2)} \otimes \omega_0 +$$
$$\chi_{V,\psi}\nu^{s+\frac{1}{2}}\chi \times \chi_{V,\psi}\nu^{\frac{1}{2}-s}\chi^{-1} \otimes \omega_0$$

Looking at Jacquet modules with respect to different parabolic subgroups we can conclude, in the same way as in the $\widetilde{Sp(1)}$-reducibility case, that if all the representations $\chi_{V,\psi}\nu^{s-\frac{1}{2}}\chi \rtimes \omega_0$, $\chi_{V,\psi}\nu^{s+\frac{1}{2}}\chi \rtimes \omega_0$ and $\chi_{V,\psi}\nu^{s+\frac{1}{2}}\chi \times \chi_{V,\psi}\nu^{\frac{1}{2}-s}\chi^{-1}$ are irreducible, then also the representation Π is irreducible.

The representation $\chi_{V,\psi}\nu^{s+\frac{1}{2}}\chi \rtimes \omega_0$ reduces for $(\chi,s) = (\zeta,0)$, while $\chi_{V,\psi}\nu^{s-\frac{1}{2}}\chi \rtimes \omega_0$ reduces for $(\chi,s) = (\zeta,1)$, where $\zeta^2 = 1_{F^\times}$.

The representation $\chi_{V,\psi}\nu^{s+\frac{1}{2}}\chi \times \chi_{V,\psi}\nu^{\frac{1}{2}-s}\chi^{-1}$ reduces for $(\chi,s) = (\zeta,\frac{1}{2})$, where $\zeta^2 = 1_{F^\times}$. These observations imply

Proposition 3.11. *Let $\chi \in \widehat{F^\times}$, $s \in \mathbb{R}$, $s \geq 0$. The representations $\chi_{V,\psi}\nu^s\chi St_{GL(2)} \rtimes 1$ and $\chi_{V,\psi}\nu^s\chi 1_{GL(2)} \rtimes 1$ are irreducible unless $(s,\chi) = (\frac{1}{2},\zeta)$, $(s,\chi) = (1,\zeta)$ or $(s,\chi) = (0,\zeta)$, where $\zeta^2 = 1_{F^\times}$. In R we have $\chi_{V,\psi}\nu^{s+\frac{1}{2}}\chi \times \chi_{V,\psi}\nu^{s-\frac{1}{2}}\chi \rtimes \omega_0 = \chi_{V,\psi}\nu^s\chi St_{GL(2)} \rtimes \omega_0 + \chi_{V,\psi}\nu^s\chi 1_{GL(2)} \rtimes \omega_0$. Also, if $\chi_{V,\psi}\nu^{s+\frac{1}{2}}\chi \times \chi_{V,\psi}\nu^{s-\frac{1}{2}}\chi \rtimes \omega_0$ is a representation of length 2, then $\chi_{V,\psi}\nu^s\chi 1_{GL(2)} \rtimes \omega_0 = \langle \chi_{V,\psi}\nu^s\chi 1_{GL(2)}; \omega_0 \rangle$ and*

$$\chi_{V,\psi}\nu^s\chi St_{GL(2)} \rtimes \omega_0 = \begin{cases} \langle \chi_{V,\psi}\nu^{s+\frac{1}{2}}\chi, \chi_{V,\psi}\nu^{\frac{1}{2}-s}\chi; \omega_0 \rangle & \text{if } s < \frac{1}{2}, \\ \langle \chi_{V,\psi}\nu^{s+\frac{1}{2}}\chi; \chi_{V,\psi}\nu^{s-\frac{1}{2}}\chi \rtimes \omega_0 \rangle & \text{if } s = \frac{1}{2}, \\ \langle \chi_{V,\psi}\nu^{s+\frac{1}{2}}\chi, \chi_{V,\psi}\nu^{s-\frac{1}{2}}\chi; \omega_0 \rangle & \text{if } s > \frac{1}{2}. \end{cases}$$

For the reducibility of the $\widetilde{GL(2)}$-part we still have to determine the composition factors of the following representations:

(i) $\chi_{V,\psi}\nu^{\frac{1}{2}}\zeta \times \chi_{V,\psi}\nu^{\frac{1}{2}}\zeta \rtimes \omega_0$,

(ii) $\chi_{V,\psi}\nu^{\frac{3}{2}}\zeta \times \chi_{V,\psi}\nu^{\frac{1}{2}}\zeta \rtimes \omega_0$,

(iii) $\chi_{V,\psi}\nu\zeta \times \chi_{V,\psi}\zeta \rtimes \omega_0$,

where $\zeta^2 = 1_{F^\times}$.

All together, this leaves us next four exceptional cases of the representations whose composition series we have to determine:

(a) $\chi_{V,\psi}\nu^{\frac{1}{2}}\zeta_1 \times \chi_{V,\psi}\nu^{\frac{1}{2}}\zeta_2 \rtimes \omega_0$,

(b) $\chi_{V,\psi}\nu^{\frac{1}{2}}\zeta \times \chi_{V,\psi}\nu^{\frac{1}{2}}\zeta \rtimes \omega_0$,

(c) $\chi_{V,\psi}\nu^{\frac{3}{2}}\zeta \times \chi_{V,\psi}\nu^{\frac{1}{2}}\zeta \rtimes \omega_0$,

(d) $\chi_{V,\psi}\nu\zeta \times \chi_{V,\psi}\zeta \rtimes \omega_0$,

where $\zeta^2 = \zeta_1^2 = \zeta_2^2 = 1_{F^\times}$, $\zeta_1 \neq \zeta_2$.

These cases are treated in the following subsection.

3.2.3 Exceptional cases

All the following equalities are given in semisimplifications. We obtain desired composition series using case-by-case examination:

(a) Write $\chi_{V,\psi}\nu^{\frac{1}{2}}\zeta_i \rtimes \omega_0 = \chi_{V,\psi}sp_{\zeta_i,1} + \omega^+_{\psi_{a_i},1}$, $i = 1,2$. In R we have:
$\chi_{V,\psi}\nu^{\frac{1}{2}}\zeta_1 \times \chi_{V,\psi}\nu^{\frac{1}{2}}\zeta_2 \rtimes \omega_0 = \chi_{V,\psi}\nu^{\frac{1}{2}}\zeta_2 \times \chi_{V,\psi}\nu^{\frac{1}{2}}\zeta_1 \rtimes \omega_0 = \chi_{V,\psi}\nu^{\frac{1}{2}}\zeta_1 \rtimes sp_{\zeta_2,1} +$
$\chi_{V,\psi}\nu^{\frac{1}{2}}\zeta_1 \rtimes \omega^+_{\psi_{a_2},1} = \chi_{V,\psi}\nu^{\frac{1}{2}}\zeta_2 \rtimes sp_{\zeta_1,1} + \chi_{V,\psi}\nu^{\frac{1}{2}}\zeta_2 \rtimes \omega^+_{\psi_{a_1},1}$

Using standard calculations, we obtain:

$$
\begin{aligned}
R_{\widetilde{P}_1}(\chi_{V,\psi}\nu^{\frac{1}{2}}\zeta_1 \rtimes sp_{\zeta_2,1}) &= \chi_{V,\psi}\nu^{-\frac{1}{2}}\zeta_1 \otimes sp_{\zeta_1,1} + \chi_{V,\psi}\nu^{\frac{1}{2}}\zeta_1 \otimes sp_{\zeta_2,1} \\
&\quad + \chi_{V,\psi}\nu^{\frac{1}{2}}\zeta_2 \otimes sp_{\zeta_1,1} + \chi_{V,\psi}\nu^{\frac{1}{2}}\zeta_2 \otimes \omega^+_{\psi_{a_1},1}
\end{aligned}
$$

and

$$
\begin{aligned}
R_{\widetilde{P}_2}(\chi_{V,\psi}\nu^{\frac{1}{2}}\zeta_1 \rtimes sp_{\zeta_2,1}) &= \chi_{V,\psi}\nu^{-\frac{1}{2}}\zeta_1 \times \chi_{V,\psi}\nu^{\frac{1}{2}}\zeta_2 \otimes \omega_0 + \\
&\quad \chi_{V,\psi}\nu^{\frac{1}{2}}\zeta_1 \times \chi_{V,\psi}\nu^{\frac{1}{2}}\zeta_2 \otimes \omega_0
\end{aligned}
$$

Last equality implies that the length of $\chi_{V,\psi}\nu^{\frac{1}{2}}\zeta_1 \rtimes sp_{\zeta_2,1}$ is less than or equal 2. If $\chi_{V,\psi}\nu^{\frac{1}{2}}\zeta_1 \rtimes sp_{\zeta_2,1}$ would be an irreducible representation, then it would have to be equal either to $\chi_{V,\psi}\nu^{\frac{1}{2}}\zeta_2 \rtimes sp_{\zeta_1,1}$ or to $\chi_{V,\psi}\nu^{\frac{1}{2}}\zeta_2 \rtimes \omega^+_{\psi_{a_1},1}$, but Jacquet modules of those two representations show that this is not the case. So, we write $\chi_{V,\psi}\nu^{\frac{1}{2}}\zeta_1 \rtimes sp_{\zeta_2,1} = \rho_1 + \rho_2$, where ρ_1 and ρ_2 are irreducible representations such that $R_{\widetilde{P}_2}(\rho_1) = \chi_{V,\psi}\nu^{-\frac{1}{2}}\zeta_1 \times \chi_{V,\psi}\nu^{\frac{1}{2}}\zeta_2 \otimes \omega_0$ and $R_{\widetilde{P}_2}(\rho_2) = \chi_{V,\psi}\nu^{\frac{1}{2}}\zeta_1 \times \chi_{V,\psi}\nu^{\frac{1}{2}}\zeta_2 \otimes \omega_0$. Clearly, ρ_2 is square-integrable ($\rho_2 = \langle \chi_{V,\psi}\nu^{\frac{1}{2}}\zeta_1, \chi_{V,\psi}\nu^{\frac{1}{2}}\zeta_2; \omega_0\rangle$) and $\rho_1 = \langle \chi_{V,\psi}\nu^{\frac{1}{2}}\zeta_2; \omega^+_{\psi_{a_1},1}\rangle$.

Reasoning in the same way, we obtain that $\chi_{V,\psi}\nu^{\frac{1}{2}}\zeta_1 \rtimes \omega^+_{\psi_{a_2},1} = \rho_3 + \rho_4$, where ρ_3 and ρ_4 are irreducible representations such that $R_{\widetilde{P}_2}(\rho_3) = \chi_{V,\psi}\nu^{-\frac{1}{2}}\zeta_1 \times \chi_{V,\psi}\nu^{-\frac{1}{2}}\zeta_2 \otimes \omega_0$ and $R_{\widetilde{P}_2}(\rho_4) = \chi_{V,\psi}\nu^{\frac{1}{2}}\zeta_1 \times \chi_{V,\psi}\nu^{-\frac{1}{2}}\zeta_2 \otimes \omega_0$. So, ρ_3 is a strongly negative representation, while $\rho_4 = \langle \chi_{V,\psi}\nu^{\frac{1}{2}}\zeta_1; \omega^+_{\psi_{a_2},1}\rangle$. Using Jacquet modules again, we easily obtain the composition factors of the above representations. Thus, we conclude:

Proposition 3.12. *Let* $\zeta_1, \zeta_2 \in \widehat{F^\times}$ *such that* $\zeta_i^2 = 1_{F^\times}$, $i = 1,2$ ($\zeta_1 \neq \zeta_2$). *Than the representations* $\chi_{V,\psi}\nu^{\frac{1}{2}}\zeta_2 \rtimes \omega^+_{\psi_{a_1},1}$, $\chi_{V,\psi}\nu^{\frac{1}{2}}\zeta_2 \rtimes sp_{\zeta_1,1}$, $\chi_{V,\psi}\nu^{\frac{1}{2}}\zeta_1 \rtimes$ $\omega^+_{\psi_{a_2},1}$ *and* $\chi_{V,\psi}\nu^{\frac{1}{2}}\zeta_1 \rtimes sp_{\zeta_2,1}$ *are reducible and* $\chi_{V,\psi}\nu^{\frac{1}{2}}\zeta_1 \times \chi_{V,\psi}\nu^{\frac{1}{2}}\zeta_2 \rtimes \omega_0$ *is a representation of length 4.* $\chi_{V,\psi}\nu^{\frac{1}{2}}\zeta_1 \rtimes sp_{\zeta_2,1}$ *and* $\chi_{V,\psi}\nu^{\frac{1}{2}}\zeta_2 \rtimes sp_{\zeta_1,1}$ *have exactly*

one irreducible subquotient in common, that subquotient is square-integrable, we denote it with σ (i.e., $\sigma = \langle \chi_{V,\psi}\nu^{\frac{1}{2}}\zeta_1, \chi_{V,\psi}\nu^{\frac{1}{2}}\zeta_2; \omega_0 \rangle$). Also, the unique irreducible common subquotient of $\chi_{V,\psi}\nu^{\frac{1}{2}}\zeta_1 \rtimes \omega^+_{\psi_{a_2,1}}$ and $\chi_{V,\psi}\nu^{\frac{1}{2}}\zeta_2 \rtimes \omega^+_{\psi_{a_1,1}}$ is a strongly negative representation, we denote it by ρ_{sneg}. In R we have:

$\chi_{V,\psi}\nu^{\frac{1}{2}}\zeta_1 \rtimes sp_{\zeta_2,1} = \sigma + \langle \chi_{V,\psi}\nu^{\frac{1}{2}}\zeta_2; \omega^+_{\psi_{a_1,1}} \rangle$,

$\chi_{V,\psi}\nu^{\frac{1}{2}}\zeta_1 \rtimes \omega^+_{\psi_{a_2,1}} = \langle \chi_{V,\psi}\nu^{\frac{1}{2}}\zeta_1; \omega^+_{\psi_{a_2,1}} \rangle + \rho_{sneg}$,

$\chi_{V,\psi}\nu^{\frac{1}{2}}\zeta_2 \rtimes sp_{\zeta_1,1} = \sigma + \langle \chi_{V,\psi}\nu^{\frac{1}{2}}\zeta_1; \omega^+_{\psi_{a_2,1}} \rangle$,

$\chi_{V,\psi}\nu^{\frac{1}{2}}\zeta_2 \rtimes \omega^+_{\psi_{a_1,1}} = \langle \chi_{V,\psi}\nu^{\frac{1}{2}}\zeta_2; \omega^+_{\psi_{a_1,1}} \rangle + \rho_{sneg}$.

(b) $\chi_{V,\psi}\nu^{\frac{1}{2}}\zeta \times \chi_{V,\psi}\nu^{-\frac{1}{2}}\zeta \rtimes \omega_0 = \chi_{V,\psi}\nu^{\frac{1}{2}}\zeta \rtimes sp_{\zeta,1} + \chi_{V,\psi}\nu^{\frac{1}{2}}\zeta \rtimes \omega^+_{\psi_a,1} = \chi_{V,\psi}\zeta St_{GL(2)} \rtimes \omega_0 + \chi_{V,\psi}\zeta 1_{GL(2)} \rtimes \omega_0$

From Jacquet modules we get:

$R_{\widetilde{P_1}}(\chi_{V,\psi}\nu^{\frac{1}{2}}\zeta \rtimes sp_{\zeta,1}) = 2\, \chi_{V,\psi}\nu^{\frac{1}{2}}\zeta \otimes sp_{\zeta,1} + \chi_{V,\psi}\nu^{-\frac{1}{2}}\zeta \otimes sp_{\zeta,1} + \chi_{V,\psi}\nu^{\frac{1}{2}}\zeta \otimes \omega^+_{\psi_a,1}$

$R_{\widetilde{P_1}}(\chi_{V,\psi}\nu^{\frac{1}{2}}\zeta \rtimes \omega^+_{\psi_1}) = 2\, \chi_{V,\psi}\nu^{-\frac{1}{2}}\zeta \otimes \omega^+_{\psi_a,1} + \chi_{V,\psi}\nu^{\frac{1}{2}}\zeta \otimes \omega^+_{\psi_a,1} + \chi_{V,\psi}\nu^{-\frac{1}{2}}\zeta \otimes sp_{\zeta,1}$

$R_{\widetilde{P_1}}(\chi_{V,\psi}\zeta St_{GL(2)} \rtimes \omega_0) = 2\, \chi_{V,\psi}\nu^{\frac{1}{2}}\zeta \otimes sp_{\zeta,1} + 2\, \chi_{V,\psi}\nu^{\frac{1}{2}}\zeta \otimes \omega^+_{\psi_a,1}$

$R_{\widetilde{P_1}}(\chi_{V,\psi}\zeta 1_{GL(2)} \rtimes \omega_0) = 2\, \chi_{V,\psi}\nu^{-\frac{1}{2}}\zeta \otimes sp_{\zeta,1} + 2\, \chi_{V,\psi}\nu^{-\frac{1}{2}}\zeta \otimes \omega^+_{\psi_a,1}$

From preceding Jacquet modules we conclude (as in [23], Chapter 3) that $\chi_{V,\psi}\nu^{\frac{1}{2}}\zeta \rtimes \omega^+_{\psi_a,1}$ and $\chi_{V,\psi}\zeta St_{GL(2)} \rtimes \omega_0$ have an irreducible subquotient in common , which is different from both $\chi_{V,\psi}\nu^{\frac{1}{2}}\zeta \rtimes \omega^+_{\psi_a,1}$ and $\chi_{V,\psi}\zeta St_{GL(2)} \rtimes \omega_0$. For simplicity of the notation, we let ρ_1 stand for this subqoutient. $R_{\widetilde{P_1}}(\rho_1) = \chi_{V,\psi}\nu^{\frac{1}{2}}\zeta \otimes \omega^+_{\psi_a,1}$.

In the same way, let ρ_2 be an irreducible common subquotient that $\chi_{V,\psi}\nu^{\frac{1}{2}}\zeta \rtimes sp_{\zeta,1}$ and $\chi_{V,\psi}\zeta 1_{GL(2)} \rtimes \omega_0$ have in common. $R_{\widetilde{P_1}}(\rho_2) = \chi_{V,\psi}\nu^{-\frac{1}{2}}\zeta \otimes sp_{\zeta,1}$

The representations $\chi_{V,\psi}\zeta 1_{GL(2)} \otimes \omega_0$ and $\chi_{V,\psi}\zeta St_{GL(2)} \otimes \omega_0$ are irreducible and unitary, multiplicity of $\chi_{V,\psi}\zeta 1_{GL(2)} \otimes \omega_0$ in $R_{\widetilde{P_2}}(\chi_{V,\psi}\zeta 1_{GL(2)} \rtimes \omega_0)$ equals 2, which implies that length of $\chi_{V,\psi}\zeta 1_{GL(2)} \rtimes \omega_0$ is 2. Analogously, length of $\chi_{V,\psi}\zeta St_{GL(2)} \rtimes \omega_0$ also equals 2.

Now we write $\chi_{V,\psi}\zeta St_{GL(2)} \rtimes \omega_0 = \rho_1 + \rho_3$ and $\chi_{V,\psi}\zeta 1_{GL(2)} \rtimes \omega_0 = \rho_2 + \rho_4$. Observe that $R_{\widetilde{P_1}}(\rho_3) = 2\, \chi_{V,\psi}\nu^{\frac{1}{2}}\zeta \otimes sp_{\zeta,1} + \chi_{V,\psi}\nu^{\frac{1}{2}}\zeta \otimes \omega^+_{\psi_a,1}$ and $R_{\widetilde{P_1}}(\rho_4) = \chi_{V,\psi}\nu^{-\frac{1}{2}}\zeta \otimes sp_{\zeta,1} + 2\, \chi_{V,\psi}\nu^{-\frac{1}{2}}\zeta \otimes \omega^+_{\psi_a,1}$.

We immediately get

Proposition 3.13. *Let $\zeta \in \widehat{F^\times}$ such that $\zeta^2 = 1_{F^\times}$. Then the representations $\zeta 1_{\widetilde{GL(2)}} \rtimes \omega_0$, $\chi_{V,\psi}\zeta St_{GL(2)} \rtimes \omega_0$, $\chi_{V,\psi}\nu^{\frac{1}{2}}\zeta \rtimes \omega^+_{\psi_a,1}$ and $\chi_{V,\psi}\nu^{\frac{1}{2}}\zeta \rtimes sp_{\zeta,1}$ are*

reducible and $\chi_{V,\psi}\nu^{\frac{1}{2}}\zeta \times \chi_{V,\psi}\nu^{\frac{1}{2}}\zeta \rtimes \omega_0$ is a representation of length 4. The representations $\chi_{V,\psi}\zeta St_{GL(2)} \rtimes \omega_0$ and $\nu^{\frac{1}{2}}\chi_{V,\psi}\zeta \rtimes \omega^+_{\psi_a,1}$ (respectively $\chi_{V,\psi}\nu^{\frac{1}{2}}\zeta \rtimes sp_{\zeta,1}$) have exactly one irreducible subquotient in common, which is tempered and denoted by τ_1 (respectively τ_2). Observe that $\tau_1 = \langle \chi_{V,\psi}\nu^{\frac{1}{2}}\zeta; \omega^+_{\psi_a,1}\rangle$ and $\tau_2 = \langle \chi_{V,\psi}\nu^{\frac{1}{2}}\zeta, \chi_{V,\psi}\nu^{\frac{1}{2}}\zeta; \omega_0\rangle$. Also, the unique irreducible common subquotient of $\chi_{V,\psi}\zeta 1_{GL(2)}\rtimes\omega_0$ and $\chi_{V,\psi}\nu^{\frac{1}{2}}\zeta\rtimes\omega^+_{\psi_a,1}$ is a negative representation, we denote it by ρ_{neg}. In R we have:

$\chi_{V,\psi}\zeta St_{GL(2)} \rtimes \omega_0 = \tau_1 + \tau_2$,

$\chi_{V,\psi}\zeta 1_{GL(2)} \rtimes \omega_0 = \rho_{neg} + \langle \chi_{V,\psi}\zeta 1_{GL(2)}; \omega_0\rangle$,

$\chi_{V,\psi}\nu^{\frac{1}{2}}\zeta \rtimes sp_{\zeta,1} = \tau_2 + \langle \chi_{V,\psi}\zeta 1_{GL(2)}; \omega_0\rangle$,

$\chi_{V,\psi}\nu^{\frac{1}{2}}\zeta \rtimes \omega^+_{\psi_a,1} = \tau_1 + \rho_{neg}$.

(c) $\chi_{V,\psi}\nu^{\frac{3}{2}}\zeta \times \chi_{V,\psi}\nu^{\frac{1}{2}}\zeta \rtimes \omega_0 = \chi_{V,\psi}\nu^{\frac{3}{2}}\zeta \rtimes sp_{\zeta,1} + \chi_{V,\psi}\nu^{\frac{3}{2}}\zeta \rtimes \omega^+_{\psi_a,1} = \chi_{V,\psi}\nu\zeta St_{GL(2)} \rtimes \omega_0 + \chi_{V,\psi}\nu\zeta 1_{GL(2)} \rtimes \omega_0$

Observe that $\chi_{V,\psi}\nu^{\frac{3}{2}}\zeta \times \chi_{V,\psi}\nu^{\frac{1}{2}}\zeta \rtimes \omega_0$ is a regular representation. So, [26] implies that it is a representation of the length $2^2 = 4$ (we recall that there are used only the technics of Jacquet modules which can also be applied in our case). Since the irreducible subquotients of $\chi_{V,\psi}\nu^{\frac{3}{2}}\zeta \times \chi_{V,\psi}\nu^{\frac{1}{2}}\zeta \rtimes \omega_0$ are $\langle \chi_{V,\psi}\nu^{\frac{3}{2}}\zeta, \chi_{V,\psi}\nu^{\frac{1}{2}}\zeta; \omega_0\rangle$, $\langle \chi_{V,\psi}\nu\zeta 1_{GL(2)}; \omega_0\rangle$, $\omega^+_{\psi_a,2}$ and $\langle \chi_{V,\psi}\nu^{\frac{3}{2}}\zeta; \omega^+_{\psi_a,1}\rangle$, using Jacquet modules we easily obtain the following proposition:

Proposition 3.14. *Let* $\zeta \in \widehat{F^\times}$ *such that* $\zeta^2 = 1_{F^\times}$. *Then the representations* $\chi_{V,\psi}\nu^{\frac{3}{2}}\zeta \rtimes sp_{\zeta,1}$, $\chi_{V,\psi}\nu^{\frac{3}{2}}\zeta \rtimes \omega^+_{\psi_a,1}$, $\chi_{V,\psi}\nu\zeta St_{GL(2)} \rtimes \omega_0$ *and* $\chi_{V,\psi}\nu\zeta 1_{GL(2)} \rtimes \omega_0$ *are reducible and* $\chi_{V,\psi}\nu^{\frac{3}{2}}\zeta \times \chi_{V,\psi}\nu^{\frac{1}{2}}\zeta \rtimes \omega_0$ *is a representation of length 4. Unique irreducible common subquotient of the representations* $\chi_{V,\psi}\nu^{\frac{3}{2}}\zeta \rtimes sp_{\zeta,1}$ *and* $\chi_{V,\psi}\nu\zeta St_{GL(2)} \rtimes \omega_0$ *is square-integrable. In R we have:*

$\chi_{V,\psi}\nu^{\frac{3}{2}}\zeta \rtimes sp_{\zeta,1} = \langle \chi_{V,\psi}\nu^{\frac{3}{2}}\zeta, \chi_{V,\psi}\nu^{\frac{1}{2}}\zeta; \omega_0\rangle + \langle \chi_{V,\psi}\nu\zeta 1_{GL(2)}; \omega_0\rangle$,

$\chi_{V,\psi}\nu^{\frac{3}{2}}\zeta \rtimes \omega^+_{\psi_a,1} = \omega^+_{\psi_a,2} + \langle \chi_{V,\psi}\nu^{\frac{3}{2}}\zeta; \omega^+_{\psi_a,1}\rangle$,

$\chi_{V,\psi}\nu\zeta St_{GL(2)} \rtimes \omega_0 = \langle \chi_{V,\psi}\nu^{\frac{3}{2}}\zeta, \chi_{V,\psi}\nu^{\frac{1}{2}}\zeta; \omega_0\rangle + \langle \chi_{V,\psi}\nu^{\frac{3}{2}}\zeta; \omega^+_{\psi_a,1}\rangle$,

$\chi_{V,\psi}\nu\zeta 1_{GL(2)} \rtimes \omega_0 = \langle \chi_{V,\psi}\nu\zeta 1_{GL(2)}; \omega_0\rangle + \omega^+_{\psi_a,2}$.

(d) $\chi_{V,\psi}\nu\zeta \times \chi_{V,\psi}\zeta \rtimes \omega_0 = \chi_{V,\psi}\nu^{\frac{1}{2}}\zeta St_{GL(2)} \rtimes \omega_0 + \chi_{V,\psi}\nu^{\frac{1}{2}}\zeta 1_{GL(2)} \rtimes \omega_0$

Since it isn't known yet if the results related to the $R-$groups [5] also hold for metaplectic groups, this case will not be solved using only Jacquet modules method. Namely, the combination of Jacquet modules techniques and the knowledge about $R-$groups for symplectic groups was used in ([25]) to determine the composition series of the representations similar to this one

(for symplectic groups). We resolve this case using theta correspondence. We start with the following lemma:

Lemma 3.15. *The following equalities hold:*
(1) $\Theta(\zeta\nu \otimes \zeta \rtimes 1, R_{P_1}(\omega_{2,2})) = \chi_{V,\psi}\nu^{-1}\zeta \times \chi_{V,\psi}\zeta \rtimes \omega_0,$
(2) $\Theta(\zeta\nu^{-1} \otimes \zeta \rtimes 1, R_{P_1}(\omega_{2,2})) = \chi_{V,\psi}\nu\zeta \times \chi_{V,\psi}\zeta \rtimes \omega_0,$
(3) $\Theta(\zeta \otimes \zeta\nu \rtimes 1, R_{P_1}(\omega_{2,2})) = \chi_{V,\psi}\zeta \times \chi_{V,\psi}\nu\zeta \rtimes \omega_0,$
(4) $\Theta(\chi_{V,\psi}\nu\zeta \otimes \chi_{V,\psi}\zeta \rtimes \omega_0, R_{\widetilde{P}_1}(\omega_{2,2})) = \zeta\nu^{-1} \times \zeta \rtimes 1,$
(5) $\Theta(\chi_{V,\psi}\nu^{-1}\zeta \otimes \chi_{V,\psi}\zeta \rtimes \omega_0, R_{\widetilde{P}_1}(\omega_{2,2})) = \zeta\nu \times \zeta \rtimes 1,$
(6) $\Theta(\chi_{V,\psi}\zeta \otimes \chi_{V,\psi}\nu\zeta \rtimes \omega_0, R_{\widetilde{P}_1}(\omega_{2,2})) = \zeta \times \zeta\nu \rtimes 1.$

Proof. Recall that $R_{P_1}(\omega_{2,2})$ has a following filtration:
$I_{10} = \nu^{\frac{1}{2}} \otimes \omega_{2,1}$ (the quotient),
$I_{11} = \mathrm{Ind}_{GL(1)\times \widetilde{P}_1 \times O(3)}^{M_1 \times \widetilde{Sp(2)}}(\chi_{V,\psi}\Sigma_1' \otimes \omega_{1,1})$ (the subrepresentation).

We will prove (1), the proofs of $(2)-(6)$ are analogous. In the same way as in the second proof of the Proposition 3.5, we get $\Theta(\zeta\nu\otimes\zeta\rtimes 1, R_{P_1}(\omega_{2,2})) = \Theta(\zeta\nu \otimes \zeta \rtimes 1, I_{11})$, so it is sufficient to show $\Theta(\zeta\nu \otimes \zeta \rtimes 1, I_{11}) = \chi_{V,\psi}\nu^{-1}\zeta \times \chi_{V,\psi}\zeta \rtimes \omega_0$. It can be seen easily that there is an $GL(1)\times \widetilde{M}_1 \times O(3)$-invariant epimorphism

$$\chi_{V,\psi}\Sigma_1' \otimes \omega_{1,1} \to \chi_{V,\psi}\zeta\nu^{-1} \otimes \chi_{V,\psi}\zeta \rtimes \omega_0 \otimes \zeta\nu \otimes \zeta \rtimes 1.$$

Consequently, we get an $M_1 \times \widetilde{Sp(2)}$-invariant epimorphism
$I_{11} = \mathrm{Ind}_{GL(1)\times\widetilde{P}_1\times O(3)}^{M_1\times\widetilde{Sp(2)}}(\chi_{V,\psi}\Sigma_1'\otimes\omega_{1,1}) \to \zeta\nu\otimes\zeta\rtimes 1 \otimes \chi_{V,\psi}\zeta\nu^{-1} \times \chi_{V,\psi}\zeta \rtimes \omega_0,$
so we conclude that $\chi_{V,\psi}\zeta\nu^{-1} \times \chi_{V,\psi}\zeta \rtimes \omega_0$ is a quotient of $\Theta(\zeta\nu\otimes\zeta\rtimes 1, I_{11})$.

We prove that $\Theta(\zeta\nu\otimes\zeta\rtimes 1, I_{11})$ is also a quotient of $\chi_{V,\psi}\zeta\nu^{-1}\times\chi_{V,\psi}\zeta\rtimes\omega_0$. Let $\varphi \in \mathrm{Hom}(I_{11}, \zeta\nu\otimes\zeta\rtimes 1\otimes\Theta(\zeta\nu\otimes\zeta\rtimes 1, I_{11}))$. Using the second Frobenius reciprocity, as before, we get $\mathrm{Hom}(I_{11}, \zeta\nu \otimes \zeta \rtimes 1 \otimes \Theta(\zeta\nu \otimes \zeta \rtimes 1, I_{11})) \cong \mathrm{Hom}(\chi_{V,\psi}\Sigma_1'\otimes\omega_{1,1}, \zeta\nu\otimes\zeta\rtimes 1\otimes R_{\overline{\widetilde{P}}_1}(\Theta(\zeta\nu\otimes\zeta\rtimes 1, I_{11})))$; let φ_0 be an element corresponding to φ. Since the representations $\zeta \rtimes 1$ and $\chi_{V,\psi}\zeta \rtimes \omega_0$ are irreducible, we obtain that the image of φ_0 equals $\zeta\nu\otimes\zeta\rtimes 1 \otimes \chi_{V,\psi}\zeta\nu^{-1}\otimes\chi_{V,\psi}\zeta\rtimes\omega_0$. Reasoning as before, we get that the image of φ is a quotient of $\zeta\nu\otimes\zeta\rtimes 1\otimes\chi_{V,\psi}\zeta\nu^{-1}\times\chi_{V,\psi}\zeta\rtimes\omega_0$. Finally, $\Theta(\zeta\nu\otimes\zeta\rtimes 1, I_{11})$ is a quotient of $\chi_{V,\psi}\zeta\nu^{-1}\times\chi_{V,\psi}\zeta\rtimes\omega_0$. Therefore $\Theta(\zeta\nu\otimes\zeta\rtimes 1, I_{11}) = \chi_{V,\psi}\zeta\nu^{-1}\times\chi_{V,\psi}\zeta\rtimes\omega_0$. $\square$

Proposition 3.16. *Let $\zeta \in \widehat{F^\times}$ such that $\zeta^2 = 1_{F^\times}$. Then the representations* $\chi_{V,\psi}\nu^{\frac{1}{2}}\zeta St_{GL(2)} \rtimes \omega_0$ *and* $\chi_{V,\psi}\nu^{\frac{1}{2}}\zeta 1_{GL(2)} \rtimes \omega_0$ *are irreducible and* $\chi_{V,\psi}\nu\zeta \times$

$\chi_{V,\psi}\zeta \rtimes \omega_0$ is a representation of length 2. Observe that $\chi_{V,\psi}\nu^{\frac{1}{2}}\zeta St_{GL(2)} \rtimes \omega_0 = \langle \chi_{V,\psi}\nu\zeta; \chi_{V,\psi}\zeta \rtimes \omega_0 \rangle$ and $\chi_{V,\psi}\nu^{\frac{1}{2}}\zeta 1_{GL(2)} \rtimes \omega_0 = \langle \chi_{V,\psi}\nu^{\frac{1}{2}}\zeta 1_{GL(2)}; \omega_0 \rangle$.

Proof. Suppose on the contrary that the representation $\chi_{V,\psi}\nu^{\frac{1}{2}}\zeta St_{GL(2)} \rtimes \omega_0$ reduces. Jacquet modules imply that length of this representation is at most 2. Choose π_1 and π_2 such that the following equality holds in R: $\chi_{V,\psi}\nu^{\frac{1}{2}}\zeta St_{GL(2)} \rtimes \omega_0 = \pi_1 + \pi_2$; further suppose $R_{\widetilde{P}_1}(\pi_1) = \chi_{V,\psi}\zeta \otimes \chi_{V,\psi}\zeta\nu \rtimes \omega_0$, $R_{\widetilde{P}_1}(\pi_2) = \chi_{V,\psi}\zeta\nu \otimes \chi_{V,\psi}\zeta \rtimes \omega_0$.
Frobenius reciprocity implies
$$\text{Hom}(\omega_{2,2}, \pi_1 \otimes \zeta \times \zeta\nu^{-1} \rtimes 1) \cong \text{Hom}\left(R_{P_1}(\omega_{2,2}), \pi_1 \otimes \zeta \otimes \zeta\nu^{-1} \rtimes 1\right).$$
Using Lemma 3.15 we obtain $\text{Hom}(R_{P_1}(\omega_{2,2}), \pi_1 \otimes \zeta \otimes \zeta\nu^{-1} \rtimes 1) \cong \text{Hom}(\chi_{V,\psi}\zeta \times \chi_{V,\psi}\nu^{-1}\zeta \rtimes \omega_0, \pi_1) \neq 0$, because π_1 is a quotient of $\chi_{V,\psi}\zeta \times \chi_{V,\psi}\nu\zeta \rtimes \omega_0$. So, $\Theta(\pi_1, 2) \neq 0$.

The representation $\chi_{V,\psi}\zeta \otimes \chi_{V,\psi}\zeta\nu \rtimes \omega_0 \otimes \Theta(\pi_1, 2)$ is a quotient of $R_{\widetilde{P}_1}(\omega_{2,2})$. Lemma 3.15 implies that $\Theta(\pi_1, 2)$ is a quotient of $\zeta \times \zeta\nu \rtimes 1$. Listing quotients of $\zeta \times \zeta\nu \rtimes 1$ we get the following possibilities:

(a) $\Theta(\pi_1, 2) = \zeta \times \zeta\nu \rtimes 1$,

(b) $\Theta(\pi_1, 2) = \nu^{\frac{1}{2}}\zeta St_{GL(2)} \rtimes 1$,

(c) $\Theta(\pi_1, 2) = \nu^{-\frac{1}{2}}\zeta 1_{GL(2)} \rtimes 1$.

Suppose that (a) holds. Obviously, $\pi_1 \otimes \zeta\nu^{-1} \otimes \zeta \rtimes 1$ is then a quotient of $R_{P_1}(\omega_{2,2})$, since it is a quotient of $\pi_1 \otimes R_{P_1}(\zeta \times \zeta\nu \rtimes 1)$. This implies that π_1 is a quotient of $\chi_{V,\psi}\zeta\nu \times \chi_{V,\psi}\zeta \rtimes \omega_0$ and $R_{\widetilde{P}_1}(\pi_1)$ contains $\chi_{V,\psi}\zeta\nu^{-1} \otimes \chi_{V,\psi}\zeta \rtimes \omega_0$. This contradicts our assumption on π_1.

Similarly, using Jacquet modules, we obtain contradiction with (b) and (c). So, $\chi_{V,\psi}\nu^{\frac{1}{2}}\zeta St_{GL(2)} \rtimes \omega_0$ is irreducible.

Irreducibility of $\chi_{V,\psi}\nu^{\frac{1}{2}}\zeta 1_{GL(2)} \rtimes \omega_0$ can be proved in the same way. $\square$

4 Unitary dual supported in minimal parabolic subgroup

Let π be an irreducible genuine admissible representation of $\widetilde{Sp(n)}$. We recall that the contragredient representation is denoted by $\widetilde{\pi}$. We write $\overline{\pi}$ for

the complex conjugate representation of the representation π. The representation π is called Hermitian if $\pi \simeq \widetilde{\overline{\pi}}$. It is well-known that every unitary representation is Hermitian. For a deeper discussion we refer the reader to [19].

Suppose that $\Delta_1, \ldots, \Delta_k$ is a sequence of segments such that $\widetilde{e(\Delta_1)} \geq \cdots \geq e(\Delta_k) > 0$, let σ_{neg} be a negative representation of some $\widetilde{Sp(n')}$. From [8], Theorem 4.5 (v), we get directly: $\overline{\langle \Delta_1, \ldots, \Delta_k; \sigma_{neg} \rangle} = \langle \overline{\Delta}_1, \ldots, \overline{\Delta}_k; \overline{\sigma}_{neg} \rangle$.

Also, we have an epimorphism

$$\langle \widetilde{\Delta_1} \rangle \times \cdots \times \langle \widetilde{\Delta_k} \rangle \rtimes \widetilde{\sigma}_{neg} \to \langle \Delta_1, \ldots, \Delta_k; \sigma_{neg} \rangle^{\widetilde{}}.$$

We know that the group $GSp(n)$ acts on $\widetilde{Sp(n)}$ (Second chapter, II.1 (3) of [16]). Moreover, by (p. 92 of [16]), this action extends to the action on irreducible representations, which is equivalent to taking contragredients. We choose an element $\eta' = (1, \eta) \in GSp(n)$, where $\eta \in GSp(n')$ is an element with similitude equal to -1 and 1 denotes the identity acting on the GL–part. Thus, we obtain an epimorphism

$$\langle \widetilde{\Delta_1} \rangle \times \cdots \times \langle \widetilde{\Delta_k} \rangle \rtimes \widetilde{\sigma}_{neg}^{\eta} \to \langle \Delta_1, \ldots, \Delta_k; \sigma_{neg} \rangle^{\widetilde{\eta}'}.$$

Since $\widetilde{\sigma}_{neg}^{\eta} \simeq \sigma_{neg}$, we have $\langle \Delta_1, \ldots, \Delta_k; \sigma_{neg} \rangle^{\widetilde{}} = \langle \Delta_1, \ldots, \Delta_k; \widetilde{\sigma}_{neg} \rangle$.

Now we have the following proposition, which has a straightforward proof:

Proposition 4.1. *Let $\chi, \zeta, \zeta_1, \zeta_2 \in \widehat{F^\times}$ such that $\zeta^2 = \zeta_i^2 = 1_{F^\times}$, $i = 1, 2$ (ζ_1 and ζ_2 not necessarily different). Let $s, s_1, s_2 > 0$. The following families of the representations are Hermitian and they exhaust all irreducible Hermitian genuine representations of $\widetilde{Sp(2)}$ which are supported in the minimal parabolic subgroup:*

(1) irreducible tempered representations supported in the minimal parabolic subgroup,

(2) $\langle \chi_{V,\psi} \nu^s \chi, \chi_{V,\psi} \nu^s \chi^{-1}; \omega_0 \rangle$,

(3) $\langle \chi_{V,\psi} \nu^{s_1} \zeta_1, \chi_{V,\psi} \nu^{s_2} \zeta_2; \omega_0 \rangle$,

(4) $\langle \chi_{V,\psi} \nu^s \zeta 1_{GL(2)}; \omega_0 \rangle$,

(5) $\langle \chi_{V,\psi} \nu^s \zeta; \chi_{V,\psi} \chi \rtimes \omega_0 \rangle$,

(6) $\langle \chi_{V,\psi} \nu^s \zeta; \omega_{\psi_a, 1}^+ \rangle$,

(7) $\omega_{\psi_a, 2}^+$.

Theorem 4.2. *Let $\chi, \zeta, \zeta_1, \zeta_2 \in (\widehat{F^\times})$ such that $\zeta^2 = \zeta_i^2 = 1_{F^\times}$, $i = 1, 2$ (ζ_1 and ζ_2 not necessarily different). The following families of representations*

are unitary and they exhaust all irreducible unitary genuine representations of $\widetilde{Sp(2)}$ which are supported in the minimal parabolic subgroup:

    (1) *irreducible tempered representations supported in the minimal parabolic subgroup,*

    (2) $\langle \chi_{V,\psi}\nu^s\chi, \chi_{V,\psi}\nu^s\chi^{-1}; \omega_0 \rangle$, $0 < s \leq \frac{1}{2}$,

    (3) $\langle \chi_{V,\psi}\nu^{s_1}\zeta_1, \chi_{V,\psi}\nu^{s_2}\zeta_2; \omega_0 \rangle$, $s_2 \leq s_1$, $0 < s_1 \leq \frac{1}{2}$,

    (4) $\langle \chi_{V,\psi}\nu^s\zeta; \chi_{V,\psi}\chi \rtimes \omega_0 \rangle$, $0 < s \leq \frac{1}{2}$,

    (5) $\langle \chi_{V,\psi}\nu^s\zeta; \omega^+_{\psi_a,1} \rangle$, $s \leq \frac{1}{2}$,

    (6) $\omega^+_{\psi_a,2}$.

Proof. The proof follows the same lines as in [15], Theorem 3.5, as long as we have the following key ingredients:

- Recall that in R we have:

$$\chi_{V,\psi}\nu^{\frac{3}{2}}\zeta \times \chi_{V,\psi}\nu^{\frac{1}{2}}\zeta \rtimes \omega_0 \;=\; \langle \chi_{V,\psi}\nu^{\frac{3}{2}}\zeta, \chi_{V,\psi}\nu^{\frac{1}{2}}\zeta; \omega_0 \rangle + \omega^+_{\psi_a,2} +$$
$$\langle \chi_{V,\psi}\nu^{\frac{3}{2}}\zeta; \omega^+_{\psi_a,1} \rangle + \langle \chi_{V,\psi}\nu\zeta 1_{GL(2)}; \omega_0 \rangle,$$

  where $\langle \chi_{V,\psi}\nu^{\frac{3}{2}}\zeta, \chi_{V,\psi}\nu^{\frac{1}{2}}\zeta; \omega_0 \rangle$ and $\omega^+_{\psi_a,2}$ are unitarizable. Observe that the representation $\langle \chi_{V,\psi}\nu^{\frac{3}{2}}\zeta; \omega^+_{\psi_a,1} \rangle$ (respectively, $\langle \chi_{V,\psi}\nu\zeta 1_{GL(2)}; \omega_0 \rangle$) has Jacquet modules analogous to those of the representation $L(\delta([\nu^{\frac{1}{2}}, \nu^{\frac{3}{2}}]), 1)$ (respectively, $L(\nu^{\frac{3}{2}}, St_{SO(3)})$) of the group $SO(5)$. Hence, non-unitarizability of these two representations can be proved in an analogous manner as non-unitarizability of the representations $L(\delta([\nu^{\frac{1}{2}}, \nu^{\frac{3}{2}}]), 1)$ and $L(\nu^{\frac{3}{2}}, St_{SO(3)})$, which is a special case of Propositions 4.1 and 4.6 of [10]. Namely, the arguments used there rely on the Jacquet modules method which are also applicable for the group $\widetilde{Sp(2)}$, and a simple fact that every unitary representation is also semi-simple.

- The existence of the intertwining operator

  $$A(s_1, s_2, \zeta_1, \zeta_2, w) : \chi_{V,\psi}\nu^{s_1}\zeta_1 \times \chi_{V,\psi}\nu^{s_2}\zeta_2 \rtimes 1 \rightarrow \chi_{V,\psi}\nu^{-s_1}\zeta_1 \times \chi_{V,\psi}\nu^{-s_2}\zeta_2 \rtimes 1$$

  follows from [18] (we just note that the integral used in definition of the intertwining operator $A(s_1, s_2, \zeta_1, \zeta_2, w)$ is taken over $\widetilde{K}$, where $\widetilde{K}$ denotes the inverse image of the maximal good compact subgroup K of $Sp(n)$)

$\square$

5 Unitary dual supported in maximal parabolic subgroups

5.1 The Siegel case

Using [9], Corollary 5.2, and [15], Proposition 4.1, we directly get the following:

Proposition 5.1. *Let ρ be an irreducible cuspidal representation of $GL(2, F)$. There is at most one $s \geq 0$ such that $\chi_{V,\psi}\nu^s\rho \rtimes \omega_0$ reduces. One of the following holds:*

1. *If ρ is not self-dual, then $\chi_{V,\psi}\rho \rtimes \omega_0$ is irreducible and unitarizable. Also, the representations $\chi_{V,\psi}\nu^s\rho \rtimes \omega_0$, $s > 0$ are irreducible and non-unitarizable.*

2. *If ρ is self-dual and $\omega_\rho = 1$, where ω_ρ denotes the central character of ρ, then the representation $\chi_{V,\psi}\rho \rtimes \omega_0$ reduces while all of the representations $\chi_{V,\psi}\nu^s\rho \rtimes \omega_0$, $s > 0$, are non-unitarizable.*

3. *If ρ is self-dual and $\omega_\rho \neq 1$, then the unique $s \geq 0$ such that $\chi_{V,\psi}\nu^s\rho \rtimes \omega_0$ reduces, equals $\frac{1}{2}$. For $0 \leq s \leq \frac{1}{2}$, all of the representations $\chi_{V,\psi}\nu^s\rho \rtimes \omega_0$ are unitarizable, while for $s > \frac{1}{2}$, all of the representations $\chi_{V,\psi}\nu^s\rho \rtimes \omega_0$ are non-unitarizable. All irreducible subquotients of $\chi_{V,\psi}\nu^{\frac{1}{2}}\rho \rtimes \omega_0$ are unitarizable.*

5.2 The non-Siegel case

In Section 5.2 of [9] the reducibility points of the representations $\chi_{V,\psi}\nu^s\zeta \rtimes \pi$, for $s \in \mathbb{R}$, $\zeta \in \widehat{F^\times}$ and irreducible cuspidal representation π of $\widetilde{Sp(1)}$ are determined. After determining the reducibility points, the unitarizability of the induced representations and irreducible subquotients follow in the same way as in Proposition 5.1. For the convenience of the reader, we write down all the results.

To the fixed quadratic character χ_V we attach, as in [12], Chapter V, two odd-orthogonal towers, +-tower and −-tower. We denote by $\Theta^+(\pi)$ (resepectively, $\Theta^-(\pi)$) the first appearance of the representation $\Theta(\pi)$ in the +-tower (respectively, the −-tower). Analogously, for $r \geq 0$, we denote by

$\Theta^+(\pi, r)$ (resepectively, $\Theta^-(\pi, r)$) the lift of the representation π to the r-th level of the +-tower (respectively, of the −-tower).

Since the representation $\chi_{V,\psi}\nu^s\zeta \rtimes \pi$ is irreducible for $\zeta^2 \neq 1$, we suppose $\zeta^2 = 1$ and consider following two cases:

(a) $\zeta \neq 1$

Applying Theorem 3.5 from [9] we obtain:
$\chi_{V,\psi}\nu^s\zeta \rtimes \pi$ reduces if and only if $\zeta\nu^s \rtimes \Theta^+(\pi)$ reduces in (+-tower) if and only if $\zeta\nu^s \rtimes \Theta^-(\pi)$ reduces in (−-tower).

Observe that $\Theta^+(\pi)$ is an irreducible cuspidal representation of some of the groups $O(1)$, $O(3)$ or $O(5)$. Let r denote the first occurrence of non-zero lift of representation π in odd orthogonal +-tower. We have several cases depending on r:

- If $r = 0$, i.e., if π equals $\omega_{\psi_a^-, 1}$, which is an odd part of the Weil representation attached to additive character ψ, then $\Theta^+(\pi, 0) = sgn_{O(1)}$, so the representation $\zeta\nu^s \rtimes sgn_{O(1)}$ reduces if and only if the representation $\zeta\nu^s \rtimes 1$ reduces in $SO(3)$. It is well-known that this representation reduces when $s = \frac{1}{2}$.

- If $r = 1$, the representation $\zeta\nu^s \rtimes \pi$ reduces if and only if the representation $\zeta\nu^s \rtimes \Theta^+(\pi, 1)|_{SO(3)}$ reduces. As in [15], we obtain that the unique s such that $\zeta\nu^s \rtimes \pi$ reduces equals $\frac{1}{2}$.

- If $r = 2$, the representation $\zeta\nu^s \rtimes \pi$ reduces if and only if the representation $\zeta\nu^s \rtimes \Theta^+(\pi, 2)$ reduces, and that is if and only if the representation $\zeta\nu^s \rtimes \Theta^+(\pi, 2)|_{SO(5)}$ reduces. Observe that we do not know if the representation $\Theta^+(\pi, 2)$ is generic, so we turn our attention to the representation $\zeta\nu^s \rtimes \Theta^-(\pi, 0)$, because we know that $\Theta^-(\pi, 0)$ is a non-zero representation of the group $O(1)$ (since π is cuspidal, the Dichotomy Conjecture is proved). Recall that $\zeta\nu^s \rtimes \Theta^-(\pi, 0)$ reduces for $s = 0$ if and only if $\mu(s, \zeta \otimes \Theta^-(\pi, 0)) \neq 0$ for $s = 0$ and that $\zeta\nu^s \rtimes \Theta^-(\pi, 0)$ reduces for $s_0 > 0$ if and only if $\mu(s, \zeta \otimes \Theta^-(\pi, 0))$ has a pole for $s = s_0$. In the same way as in [9], section 5.2 (the third case there), we obtain $\mu(s, \zeta \otimes \Theta^-(\pi, 0)) \cong \mu(s, \zeta \otimes JL(\Theta^-(\pi, 0)))$, where $JL(\Theta^-(\pi, 0)))$ denotes the Jacquet-Langlands lift of $\Theta^-(\pi, 0))$. Now we consider two possibilities:

  (ı) Suppose that $\Theta^-(\pi, 0)$ is not one-dimensional. Then $JL(\Theta^-(\pi, 0))$ is a cuspidal generic representation of $SO(3, F)$ and the reducibility point is $s = \frac{1}{2}$.

(II) Suppose $\Theta^-(\pi, 0) = \zeta_1 \circ \nu_D$, where ζ_1 is a quadratic character of $F^\times$, while ν_D is a reduced norm on $D^\times$ (D is a non-split quaternion algebra over F). We have $JL(\Theta^-(\pi, 0)) = \zeta_1 St_{GL(2,F)}$. If $\zeta_1 = \zeta$, then the reducibility point is $s = \frac{1}{2}$, otherwise the reducibility point is $s = \frac{3}{2}$.

(b) $\zeta = 1$

This case can be completely solved using Theorem 4.2 from [9]. We again denote by r the first occurrence of non-zero lift of representation π in odd orthogonal +-tower and consider all the possible cases:

- If $r = 0$, π equals $\omega_{\psi_a^-,1}$ and the representation $\chi_{V,\psi}\nu^s \rtimes \omega_{\psi_a^-,1}$ reduces for $s = \pm\frac{3}{2}$.

- If $r = 1$, the representation $\chi_{V,\psi}\nu^s \rtimes \pi$ reduces for $s = \frac{1}{2}$.

- If $r = 2$, the representation $\chi_{V,\psi}\nu^s \rtimes \pi$ reduces for $s = \frac{1}{2}$.

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