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JACQUET MODULES OF STRONGLY POSITIVE REPRESENTATIONS OF THE METAPLECTIC GROUP $\widetilde{Sp(n)}$

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Abstract. Strongly positive discrete series represent a particularly important class of irreducible square-integrable representations of p-adic groups. Indeed, these representations are used as basic building blocks in known constructions of general discrete series. In this paper, we explicitly describe Jacquet modules of strongly positive discrete series. The obtained description of Jacquet modules, which relies on the classification of strongly positive discrete series given in our earlier paper on metaplectic groups, is valid in both the classical and the metaplectic cases. We expect that our results, besides being interesting by themselves, should be relevant to some potential applications in the theory of automorphic forms, where both representations of metaplectic groups and the structure of Jacquet modules play an important part.

1. Introduction

Square-integrable representations occupy an especially important place in unitary duals of reductive groups. A complete classification of irreducible square-integrable representations (modulo cuspidal representations), so-called discrete series, for classical p-adic groups, has been given by the work of Mœglin and Tadić in papers [11] and [12]. In their classification, which relies on certain conjectures, a prominent role is played by strongly positive discrete series, which serve as a cornerstone for the construction of general discrete series. Thus, it is of interest to obtain further information about this class of representations. Recently, we have classified strongly positive discrete series of metaplectic groups ([10]). Our classification involves no additional assumptions or conjectures and it is also valid in a classical group case. The purpose of this paper is to investigate and describe Jacquet modules of strongly positive discrete series. In this way, we extend results related to Jacquet modules of regular discrete series of classical groups considered in [20]. The methods used to obtain the above-mentioned classification of strongly positive discrete series are motivated by [14], where the structure of Jacquet modules of irreducible unramified representations was also investigated.

Our approach is based on the detailed analysis of certain embeddings of strongly positive discrete series (which have been obtained in [10] and recalled in this paper) by the Geometric Lemma ([1], [18]) and Bernstein-Zelevinsky theory ([1], [21]), both written for metaplectic groups in [5]. We choose to work with symplectic groups first, and then extend our results to the metaplectic group case.

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For the convenience of the reader, we recall the definition of the strongly positive discrete series. Let σ denote an irreducible representation of the symplectic group $Sp(n)$. Such a representation σ is said to be a strongly positive discrete series if for each embedding of the form $\sigma \hookrightarrow \nu^{s_1}\rho_1 \times \cdots \times \nu^{s_m}\rho_m \rtimes \sigma_{cusp}$, where $\rho_1, \ldots, \rho_m$ are irreducible cuspidal representations of $GL(n_1, F), \ldots, GL(n_m, F)$ and σ_{cusp} is an irreducible cuspidal representation of the symplectic group $Sp(n')$, $n' = n - \sum_{i=1}^{m} n_i$, we have $s_i > 0$ for $i = 1, \ldots, m$.

Our main interest is to derive Jacquet modules of strongly positive discrete series with respect to the maximal parabolic subgroups. Iterating these results and combining them with the results of Jantzen ([7]), Jacquet modules with respect to the other parabolic subgroups may be obtained. Our results show that Jacquet modules of strongly positive discrete series are rather similar to those of generalized Steinberg representations, which have been described in the paper [20].

We expect applications of our results in the classification of general discrete series of metaplectic groups and in the description of Θ-lifts of strongly positive discrete series. Also, our results may be used for investigating reducibilities of certain Jacquet modules or of some generalized principal series, as has been done in the case of classical groups in [13]. Further, one may use them to derive various examples of strongly positive discrete series, regarding irreducibility of certain Jacquet modules.

We now describe the contents of the paper in more detail. In the next section we set up notation and terminology, while in the third section we prove some technical lemmas which are used later in the paper. The fourth section is devoted to the proof of our main results in a case of strongly positive discrete series of symplectic groups whose cuspidal support on the general-linear-group side consists only of the twists of one irreducible self-contragredient cuspidal representation. Proofs made in this case may be almost directly generalized to the case of arbitrary strongly positive discrete series of symplectic groups, and this sort of approach enables us to avoid many additional technicalities and shorten some proofs. A generalization of this case is made in the fifth section, where our main results are stated and proved. The methods of our proofs carry over without any change to the strongly positive discrete series of the special odd-orthogonal groups. In the sixth section we extend our results to the metaplectic groups.

2. Preliminaries

We will denote by F a non-Archimedean local field and write $GL(n, F)$ for the general linear group of type $n \times n$ with entries in F. Let $J_n \in GL(n, F)$ denote the $n \times n$ matrix having 1's on the second diagonal and all other entries 0. The symplectic group of rank n, $n \geq 1$ is defined as follows:

$$Sp(n) = \left\{ g \in GL(2n, F) : g \cdot \begin{pmatrix} 0 & J_n \\ -J_n & 0 \end{pmatrix} \cdot g^t = \begin{pmatrix} 0 & J_n \\ -J_n & 0 \end{pmatrix} \right\},$$

where g^t denotes the transposed matrix of g.

There is a well-known bijective correspondence between the set of standard parabolic subgroups of the group $Sp(n)$ and the set of all ordered partitions of positive integers less than or equal to n, which is described in detail in Section 1 of [4]. For an ordered partition $s = (n_1, n_2, \ldots, n_k)$ of some $m \leq n$, we denote by P_s a standard parabolic subgroup of $Sp(n, F)$ (consisting of block upper-triangular matrices), whose Levi factor M_s equals $GL(n_1, F) \times GL(n_2, F) \times \cdots \times GL(n_k, F) \times Sp(n-|s|, F)$, where $|s| = m$.

The representation of $Sp(n)$ that is parabolically induced from the representation $\pi_1 \otimes \cdots \otimes \pi_k \otimes \sigma$ of M_s will be denoted by $\pi_1 \times \cdots \times \pi_k \rtimes \sigma$.

Let $\mathcal{R}(n)$ be the Grothendieck group of the category of all admissible representations of finite length of $Sp(n)$ (i.e., a free abelian group over the set of all irreducible representations of $Sp(n)$), where we identify an irreducible representation with its isomorphism class, and define $\mathcal{R} = \bigoplus_{n \geq 0} \mathcal{R}(n)$. Also, let $\mathcal{G} = \bigoplus_n \mathcal{G}(n)$, where $\mathcal{G}(n)$ denotes the Grothendieck group of smooth representations of finite length of $GL(n, F)$.

By ν we mean a character of $GL(n, F)$ defined by $|\det|_F$, where $|\ |_F$ denotes the normalized absolute value on F. For an irreducible cuspidal representation ρ of the group $GL(n, F)$, the set of representations $\Delta = \{\nu^a\rho, \nu^{a+1}\rho, \ldots, \nu^{a+k}\rho\}$ is called a segment ($k \in \mathbb{Z}_{\geq 0}$). Here and subsequently, we use the abbreviation $\Delta = [\nu^a\rho, \nu^{a+k}\rho]$. Further, we denote by $\delta(\Delta)$ an essentially square-integrable representation, which is obtained as the unique irreducible subrepresentation of $\nu^{a+k}\rho \times \nu^{a+k-1}\rho \times \cdots \times \nu^a\rho$.

For every irreducible cuspidal representation ρ of $GL(n, F)$ there exists a unique $e(\rho) \in \mathbb{R}$ such that $\nu^{-e(\rho)}\rho$ is a unitary cuspidal representation. In the sequel, let $e([\nu^a\rho, \nu^b\rho]) = \frac{a+b}{2}$.

An irreducible representation ρ of some $GL(n, F)$ is called self-contragredient if $\rho \simeq \widetilde{\rho}$. Let $\rho_1, \ldots, \rho_k$ denote irreducible cuspidal representations of groups $GL(n_1, F), \ldots, GL(n_k, F)$, respectively, and let σ_{cusp} denote an irreducible cuspidal representation of $Sp(n')$. We say that the representation σ of $Sp(n)$ belongs to the set $D(\rho_1, \ldots, \rho_k; \sigma_{cusp})$ if the cuspidal support of σ is contained in the set $\{\nu^x\rho_1, \ldots, \nu^x\rho_k, \sigma_{cusp} : x \in \mathbb{R}\}$.

For an irreducible representation σ of $Sp(n)$ there exist an ordered partition $s = (n_1, n_2, \ldots, n_k)$ of some $m \leq n$, irreducible cuspidal representations π_i of $GL(n_i, F))$, $i = 1, 2, \ldots, k$, and an irreducible cuspidal representation σ_{cusp} of $Sp(n-m)$ such that σ is an irreducible subquotient of the induced representation $\pi_1 \times \pi_2 \times \cdots \times \pi_k \rtimes \sigma_{cusp}$. The proof of this fact can be found in [3, Theorem 5.1.2].

If σ is a discrete series, it is a classical result, which can be deduced from [19], that every representation π_i may be written in the unique way as $\nu^{x_i}\rho_i$, where ρ_i is an irreducible self-contragredient cuspidal unitarizable representation of $GL(n_i, F)$, for $i = 1, 2, \ldots, k$. Following [8], we write

$$[\sigma] = [\nu^{x_1}\rho_1, \nu^{x_2}\rho_2, \ldots, \nu^{x_k}\rho_k, \sigma_{cusp}].$$

In this way, we attach to an irreducible representation σ a multiset $\{\nu^{x_1}\rho_1, \nu^{x_2}\rho_2, \ldots, \nu^{x_k}\rho_k\}$, which is unique up to replacing some $\nu^{x_i}\rho_i$ by $\nu^{-x_i}\widetilde{\rho_i}$. Consequently, when saying $[\sigma_1] = [\sigma_2]$ we shall mean that $[\sigma_2]$ can be obtained by taking contragredients of some irreducible representations of the general linear group appearing in $[\sigma_1]$.

Next, the special odd-orthogonal group of rank n is defined by

$$SO(2n+1) = \{g \in SL(2n+1, F) : g \cdot J_{2n+1} \cdot g^t = J_{2n+1}\},$$

where $SL(n, F)$ denotes a special linear group consisting of all elements of $GL(n, F)$ with determinant equal to 1.

Standard parabolic subgroups of the group $SO(2n+1)$ can be described in pretty much the same way as for the symplectic group $Sp(n)$; for a fuller treatment we refer the reader to [4]. Moreover, the aforementioned facts about the representation theory of symplectic groups are also valid in the case of special odd-orthogonal groups.

Now we shall fix notation related to the metaplectic groups. Unlike in the symplectic case, here we assume that F has characteristic different than two.

The metaplectic group $\widetilde{Sp(n)}$ is given as the unique non-trivial two-fold central extension

$$1 \to \mu_2 \to \widetilde{Sp(n)} \to Sp(n) \to 1,$$

where $\mu_2 = \{1, -1\}$ and the cocycle involved is Rao's cocycle ([16]). More on the topology of the group $\widetilde{Sp(n)}$ and its structural theory can be found in [5], [9] and [16].

Let $\widetilde{GL(n,F)}$ be a double cover of $GL(n,F)$, where the multiplication is given by $(g_1, \epsilon_1)(g_2, \epsilon_2) = (g_1 g_2, \epsilon_1 \epsilon_2 (\det g_1, \det g_2)_F)$. Here $\epsilon_i \in \mu_2$, $i = 1, 2$ and $(\cdot, \cdot)_F$ denotes the Hilbert symbol of the field F. From now on, α denotes the character of $\widetilde{GL(n,F)}$ given by $\alpha(g) = (\det g, \det g)_F = (\det g, -1)_F$.

Let $s = (n_1, n_2, \ldots, n_k)$ denote an ordered partition of some $m \leq n$. Then the standard parabolic subgroup $\widetilde{P}_s$ of $\widetilde{Sp(n)}$ is the preimage of the standard parabolic subgroup P_s in $\widetilde{Sp(n)}$. Let us denote by $\widetilde{M}_s$ the Levi factor of the parabolic subgroup $\widetilde{P}_s$. There is an epimorphism with finite kernel

$$\phi : \widetilde{GL(n_1,F)} \times \widetilde{GL(n_2,F)} \times \cdots \times \widetilde{GL(n_k,F)} \times \widetilde{Sp(n-|s|)} \to \widetilde{M}_s.$$

So, an irreducible representation π of $\widetilde{M}_s$ may be considered as a representation $\pi_1 \otimes \cdots \otimes \pi_k \otimes \sigma$, where $\pi_1, \ldots, \pi_k, \sigma$ are irreducible representations that are all trivial or all non-trivial when restricted on μ_2. The representation of $\widetilde{Sp(n)}$ that is parabolically induced from the representation $\pi_1 \otimes \cdots \otimes \pi_k \otimes \sigma$ will again be denoted by $\pi_1 \times \cdots \times \pi_k \rtimes \sigma$.

In this paper we are interested only in genuine representations of $\widetilde{Sp(n)}$ (i.e., those which do not factor through μ_2). So, let $\mathcal{S}(n)$ be the Grothendieck group of the category of all admissible genuine representations of finite length of $\widetilde{Sp(n)}$ and define $\mathcal{S} = \bigoplus_{n \geq 0} \mathcal{S}(n)$. Further, we define $\mathcal{G}^{gen} = \bigoplus_n \mathcal{G}^{gen}(\widetilde{GL(n,F)})$, where $\mathcal{G}^{gen}(\widetilde{GL(n,F)})$ denotes the Grothendieck group of smooth genuine representations of finite length of $\widetilde{GL(n,F)}$.

For an irreducible genuine cuspidal representation ρ of the group $\widetilde{GL(n,F)}$, we say that $\Delta = \{\nu^a \rho, \nu^{a+1} \rho, \ldots, \nu^{a+k} \rho\}$ is a genuine segment ($k \in \mathbb{Z}_{\geq 0}$). Again, we use the abbreviation $\Delta = [\nu^a \rho, \nu^{a+k} \rho]$ and denote by $\delta(\Delta)$ the unique irreducible subrepresentation of $\nu^{a+k} \rho \times \nu^{a+k-1} \rho \times \cdots \times \nu^a \rho$. By [4], Proposition 4.2, $\delta(\Delta)$ is a genuine essentially square-integrable representation attached to Δ. Let $e([\nu^a \rho, \nu^b \rho]) = \frac{a+b}{2}$, and note that $e([\nu^a \rho, \nu^b \rho]) = e(\delta([\nu^a \rho, \nu^b \rho]))$, by Section 4 of [4].

An irreducible genuine representation ρ of some $\widetilde{GL(n,F)}$ is called self-dual if $\rho \simeq \alpha\widetilde{\rho}$. For an irreducible cuspidal genuine self-dual representation ρ of $\widetilde{GL(n,F)}$ and an irreducible cuspidal genuine representation σ of $\widetilde{Sp(n')}$, it is shown in [6] that there exists a unique $s \geq 0$ such that the induced representation $\nu^s\rho \rtimes \sigma$ reduces.

As in the symplectic case, we say that the genuine representation σ of $\widetilde{Sp(n)}$ belongs to the set $D(\rho_1, \ldots, \rho_k; \sigma_{cusp})$ if the cuspidal support of σ is contained in the set $\{\nu^x\rho_1, \ldots, \nu^x\rho_k, \sigma_{cusp} : x \in \mathbb{R}\}$, where $\rho_1, \ldots, \rho_k$ are irreducible genuine cuspidal representations of the groups $\widetilde{GL(n_1,F)}, \ldots, \widetilde{GL(n_k,F)}$ and σ_{cusp} is an irreducible genuine cuspidal representation of $\widetilde{Sp(n')}$.

Let σ denote an irreducible genuine representation $\widetilde{Sp(n)}$, for some n. By Proposition 4.4 from [5] there exists an ordered partition $s = (n_1, n_2, \ldots, n_k)$ of some $m \leq n$, irreducible genuine cuspidal representations π_i of $\widetilde{GL(n_i,F)}$, $i = 1, 2, \ldots, k$, and an irreducible genuine cuspidal representation σ_{cusp} of $\widetilde{Sp(n-m)}$ such that σ is an irreducible subrepresentation of the induced representation $\pi_1 \times \pi_2 \times \cdots \times \pi_k \rtimes \sigma_{cusp}$. If σ is a discrete series representation, it is a direct consequence of [5] and [10] that every representation π_i may be written in the unique way as $\nu^{x_i}\rho_i$, where ρ_i is an irreducible genuine self-dual cuspidal unitarizable representation of $\widetilde{GL(n_i,F)}$, for $i = 1, 2, \ldots, k$. By [8, page 236], the multiset $\{\pi_1, \pi_2, \ldots, \pi_k\}$ is unique up to replacing some π_i by $\alpha\widetilde{\pi}_i = \nu^{-x_i}\rho_i$.

Again, we write

$$[\sigma] = [\nu^{x_1}\rho_1, \nu^{x_2}\rho_2, \ldots, \nu^{x_k}\rho_k, \sigma_{cusp}],$$

and when saying $[\sigma_1] = [\sigma_2]$, for irreducible genuine representations σ_1 and σ_2 of $\widetilde{Sp(n)}$, we shall mean that $[\sigma_2]$ can be obtained by multiplying some irreducible genuine representations of two-fold covers of general linear groups appearing in $[\sigma_1]$ with the character α after taking their contragredients.

An irreducible representation σ of $Sp(n)$ is called strongly positive if for each representation $\nu^{s_1}\rho_1 \times \nu^{s_2}\rho_2 \times \cdots \times \nu^{s_k}\rho_k \rtimes \sigma_{cusp}$, where ρ_i, $i = 1, 2, \ldots, k$, are irreducible cuspidal unitary representations, σ_{cusp} an irreducible cuspidal representation of $Sp(n')$ and $s_i \in \mathbb{R}$, $i = 1, 2, \ldots, k$, such that

$$\sigma \hookrightarrow \nu^{s_1}\rho_1 \times \nu^{s_2}\rho_2 \times \cdots \times \nu^{s_k}\rho_k \rtimes \sigma_{cusp},$$

we have $s_i > 0$ for each i. Strongly positive representations of metaplectic groups and of other classical groups are defined in a completely analogous way. Observe that every strongly positive representation is square-integrable.

Irreducible strongly positive representations are often called strongly positive discrete series.

In [10] we have shown that every strongly positive discrete series representation can be realized in a unique way (up to a certain permutation) as a unique irreducible subrepresentation of the induced representation

$$(\prod_{i=1}^{m}\prod_{j=1}^{k_i} \delta([\nu^{a_{\rho_i}-k_i+j}\rho_i, \nu^{b_j^{(i)}}\rho_i])) \rtimes \sigma_{cusp}, \tag{2.1}$$

where

- $\rho_1, \ldots, \rho_m$ are non-isomorphic irreducible cuspidal representations of groups $GL(n_1, F), \ldots, GL(n_m, F)$ and σ_{cusp} is an irreducible cuspidal representation of $Sp(n')$,
- $a_{\rho_i} > 0$, such that $\nu^{a_{\rho_i}} \rho_i \rtimes \sigma_{cusp}$ reduces,
- $k_i = \lceil a_{\rho_i} \rceil$, where $\lceil a_{\rho_i} \rceil$ denotes the smallest integer which is not smaller than a_{ρ_i},
- $b_j^{(i)} > -1$ such that $b_j^{(i)} - a_{\rho_i} \in \mathbb{Z}_{\geq 0}$, for $i = 1, \ldots, m$, $j = 1, \ldots, k_i$,
- $b_j^{(i)} < b_{j+1}^{(i)}$ for $1 \leq j \leq k_i - 1$.

We omit $\delta([\nu^x \rho, \nu^y \rho])$ if $x > y$.

It is important to note that a completely analogous classification holds for special odd-orthogonal groups and for the metaplectic ones (of course, if σ is a genuine representation of $\widetilde{Sp(n)}$, representations $\rho_1, \ldots, \rho_m$ and σ_{cusp} should also be genuine representations of corresponding two-fold covers).

For the convenience of the reader we recall both the classical and the metaplectic versions of the useful Tadić's structure formula (Theorem 5.4 from [18] and Proposition 4.5 from [5]), which enable us to calculate Jacquet modules of an induced representation. We denote by m the linear extension to $\mathcal{G} \otimes \mathcal{G}$ of parabolic induction from a maximal parabolic subgroup. Let σ denote an irreducible representation of $Sp(n)$. Then $r_{(k)}(\sigma)$ (the normalized Jacquet module of σ with respect to the standard maximal parabolic subgroup $P_{(k)}$) can be interpreted as a representation of $GL(k, F) \times Sp(n-k)$, i.e., is an element of $\mathcal{G} \otimes \mathcal{R}$. For such a σ we can introduce $\mu^*(\sigma) \in \mathcal{G} \otimes \mathcal{R}$ by

$$\mu^*(\sigma) = \sum_{k=0}^{n} \text{s.s.}(r_{(k)}(\sigma))$$

(s.s. denotes the semisimplification) and extend μ^* linearly to the whole of $\mathcal{R}$.

Using Jacquet modules for the maximal parabolic subgroups of $GL(n, F)$ we can also define $m^*(\pi) = \sum_{k=0}^{n} \text{s.s.}(r_k(\pi)) \in \mathcal{R} \otimes \mathcal{R}$, for an irreducible representation π of $GL(n, F)$, and then extend m^* linearly to the whole of $\mathcal{R}$. Here $r_k(\pi)$ denotes the Jacquet module of the representation π with respect to the parabolic subgroup whose Levi factor is $GL(k, F) \times GL(n-k, F)$. We define $\kappa : \mathcal{R} \otimes \mathcal{R} \to \mathcal{R} \otimes \mathcal{R}$ by $\kappa(x \otimes y) = y \otimes x$ and extend the contragredient $\tilde{}$ to an automorphism of $\mathcal{R}$ in the natural way. Let $M^* : \mathcal{R} \to \mathcal{R} \otimes \mathcal{R}$ be defined by

$$M^* = (m \otimes id) \circ (\tilde{} \otimes m^*) \circ \kappa \circ m^*.$$

The following theorem presents a crucial formula for our calculations with Jacquet modules:

Theorem 2.1. *For $\pi \in \mathcal{G}$ and $\sigma \in \mathcal{R}$, the following structure formula holds:*

$$\mu^*(\pi \rtimes \sigma) = M^*(\pi) \rtimes \mu^*(\sigma).$$

Using the previous theorem, we obtain:

Lemma 2.2. *Let ρ be an irreducible cuspidal representation of $GL(n, F)$ and $a, b \in \mathbb{R}$ such that $a + b \in \mathbb{Z}_{\geq 0}$. Let σ be an admissible representation of finite length of*

$Sp(m)$. Write $\mu^*(\sigma)=\sum_{\tau,\sigma'}\tau\otimes\sigma'$. Then the following hold:

$$M^*(\delta([\nu^{-a}\rho,\nu^b\rho]))=\sum_{i=-a-1}^{b}\sum_{j=i}^{b}\delta([\nu^{-i}\widetilde{\rho},\nu^a\widetilde{\rho}])\times\delta([\nu^{j+1}\rho,\nu^b\rho])\otimes\delta([\nu^{i+1}\rho,\nu^j\rho]),$$

$$\mu^*(\delta([\nu^{-a}\rho,\nu^b\rho])\rtimes\sigma)=\sum_{i=-a-1}^{b}\sum_{j=i}^{b}\sum_{\tau,\sigma'}\delta([\nu^{-i}\widetilde{\rho},\nu^a\widetilde{\rho}])\times\delta([\nu^{j+1}\rho,\nu^b\rho])\times\tau \otimes\delta([\nu^{i+1}\rho,\nu^j\rho])\rtimes\sigma'.$$

We omit $\delta([\nu^x\rho,\nu^y\rho])$ if $x>y$.

Let us briefly describe the extension of the stated structure formula to the metaplectic groups. We mainly follow the notation introduced for symplectic groups.

For an irreducible genuine representation σ of $\widetilde{Sp(n)}$ we define $\mu_1^*(\sigma)\in\mathcal{G}^{gen}\otimes\mathcal{S}$ by

$$\mu_1^*(\sigma)=\sum_{k=0}^{n}\text{s.s.}(r_{(k)}(\sigma)),$$

where $r_{(k)}(\sigma)$ now stands for the normalized Jacquet module of σ with respect to the maximal parabolic subgroup $\widetilde{P}_{(k)}$, and extend μ_1^* linearly to the whole of $\mathcal{S}$.

For an irreducible genuine representation π of the group $\widetilde{GL(n,F)}$, set $m_1^*(\pi)=\sum_{k=0}^{n}\text{s.s.}(r_k(\pi))\in\mathcal{G}^{gen}\otimes\mathcal{G}^{gen}$ and extend m_1^* linearly to the whole of $\mathcal{G}^{gen}$ (here $r_k(\pi)$ denotes the Jacquet module of the representation π with respect to the parabolic subgroup whose Levi factor is $\widetilde{GL(k,F)}\times\widetilde{GL(n-k,F)}$).

We denote by m_1 the linear extension to $\mathcal{G}^{gen}\otimes\mathcal{G}^{gen}$ of parabolic induction from a maximal parabolic subgroup. By κ_1 we will denote the mapping of $\mathcal{G}^{gen}\otimes\mathcal{G}^{gen}$ into $\mathcal{G}^{gen}\otimes\mathcal{G}^{gen}$ defined by $\kappa_1(x\otimes y)=y\otimes x$, and we extend the contragredient $\widetilde{\ }$ to an automorphism of $\mathcal{G}^{gen}$ in the natural way.

Finally, we define $M_1^*:\mathcal{G}^{gen}\to\mathcal{G}^{gen}\otimes\mathcal{G}^{gen}$ by

$$M_1^*=(m_1\otimes id)\circ(\widetilde{\ }\alpha\otimes m_1^*)\circ\kappa_1\circ m_1^*,$$

where $\widetilde{\ }\alpha$ means taking the contragredient of the representation and then multiplying by the character α.

The structure formula for genuine representations of metaplectic groups is given by the following theorem:

Theorem 2.3. *For $\pi\in\mathcal{G}^{gen}$ and $\sigma\in\mathcal{S}$, the following structure formula holds:*

$$\mu_1^*(\pi\rtimes\sigma)=M_1^*(\pi)\rtimes\mu_1^*(\sigma).$$

Further, let ρ denote an irreducible cuspidal genuine representation of $\widetilde{GL(n,F)}$ and let $a,b\in\mathbb{R}$ such that $a+b\in\mathbb{Z}_{\geq 0}$. Let σ stand for an admissible genuine representation of finite length of $\widetilde{Sp(m)}$ and write $\mu_1^*(\sigma)=\sum_{\tau,\sigma'}\tau\otimes\sigma'$. Then the following holds:

$$M_1^*(\delta([\nu^{-a}\rho,\nu^b\rho]))=\sum_{i=-a-1}^{b}\sum_{j=i}^{b}\delta([\nu^{-i}\alpha\widetilde{\rho},\nu^a\alpha\widetilde{\rho}])\times\delta([\nu^{j+1}\rho,\nu^b\rho]) \otimes\delta([\nu^{i+1}\rho,\nu^j\rho]),$$

where we omit $\delta([\nu^x\rho,\nu^y\rho])$ if $x>y$.

The following fact, which is proved in [4, Theorem 2.1], will also be used: for an irreducible representation π of $GL(k,F)$ and an irreducible representation σ of $Sp(n)$, in $\mathcal{R}$ we have

$$\pi \rtimes \sigma = \widetilde{\pi} \rtimes \sigma.$$

A similar result for metaplectic groups follows directly from [5] (or by using the geometric construction of the intertwining operators from [15]): if π is an irreducible genuine representation of $\widetilde{GL(k,F)}$ and σ an irreducible genuine representation of $\widetilde{Sp(n)}$, then the following equality

$$\pi \rtimes \sigma = \widetilde{\pi}\alpha \rtimes \sigma$$

holds in $\mathcal{S}$.

We also use the following equation:

$$m^*(\delta([\nu^a\rho, \nu^b\rho])) = \sum_{i=a-1}^{b} \delta([\nu^{i+1}\rho, \nu^b\rho]) \otimes \delta([\nu^a\rho, \nu^i\rho]).$$

Note that multiplicativity of m^* implies

$$\begin{aligned} &m^*(\prod_{j=1}^{n} \delta([\nu^{a_j}\rho_j, \nu^{b_j}\rho_j])) \\ &= \prod_{j=1}^{n}(\sum_{i_j=a_j-1}^{b_j} \delta([\nu^{i_j+1}\rho_j, \nu^{b_j}\rho_j]) \otimes \delta([\nu^{a_j}\rho_j, \nu^{i_j}\rho_j])). \end{aligned} \tag{2.2}$$

It is clear that the mapping m_1^* has completely analogous properties.

We take a moment to recall the Langlands classification for representation of general linear groups. As in [7], we favor the subrepresentation version of this classification over the quotient version. The main advantage of this version is that it enables us to recover some interesting representations from certain members of their Jacquet modules.

First, for every irreducible essentially square-integrable representation δ of the group $GL(n,F)$, there exists an $e(\delta) \in \mathbb{R}$ such that the representation $\nu^{-e(\delta)}\delta$ is unitarizable. Suppose $\delta_1, \delta_2, \ldots, \delta_k$ are irreducible, essentially square-integrable representations of $GL(n_1,F), GL(n_2,F), \ldots, GL(n_k,F)$ with $e(\delta_1) \leq e(\delta_2) \leq \ldots \leq e(\delta_k)$. Then the induced representation $\delta_1 \times \delta_2 \times \cdots \times \delta_k$ has a unique irreducible subrepresentation, which we denote by $L(\delta_1, \delta_2, \ldots, \delta_k)$. This irreducible subrepresentation is called the Langlands subrepresentation, and it appears with multiplicity one in $\delta_1 \times \delta_2 \times \cdots \times \delta_k$. Every irreducible representation π of $GL(n,F)$ is isomorphic to some $L(\delta_1, \delta_2, \ldots, \delta_k)$. Given π, the representations $\delta_1, \delta_2, \ldots, \delta_k$ are unique up to a permutation. If $i_1, i_2, \ldots, i_k$ is a permutation of $1, 2, \ldots, k$ such that the representations $\delta_{i_1} \times \delta_{i_2} \times \cdots \times \delta_{i_k}$ and $\delta_1 \times \delta_2 \times \cdots \times \delta_k$ are isomorphic, we also write $L(\delta_{i_1}, \delta_{i_2}, \ldots, \delta_{i_k})$ for $L(\delta_1, \delta_2, \ldots, \delta_k)$.

It is important to note that a completely analogous classification holds for irreducible genuine representations of two-fold covers of general linear groups. This version, which will be used in the final section of this paper, can be obtained using Lemma 3.1 (i) from [10] and part 3 of the Proposition 4.2 from [5].

3. Some technical results

In this section we collect some technical facts that will be used throughout the paper.

In the sequel, we shall say that an irreducible representation π_1 is contained in the representation π_2, or $\pi_1 \leq \pi_2$, if π_1 is an irreducible subquotient of π_2.

First, we shall prove a lemma which will be needed for the determination of the GL-parts of Jacquet modules of strongly positive discrete series (this is Lemma 1.3.1 from [7], but we were unable to find the convenient reference for the proof, so, for the sake of completeness, we give one).

Lemma 3.1. *Let ρ denote an irreducible unitarizable cuspidal representation of the group $GL(n, F)$. Let π be an irreducible subquotient of the induced representation $\delta([\nu^{a_1}\rho, \nu^{b_1}\rho]) \times \delta([\nu^{a_2}\rho, \nu^{b_2}\rho]) \times \cdots \times \delta([\nu^{a_n}\rho, \nu^{b_n}\rho])$, where $b_1 \leq b_2 \leq \ldots \leq b_n$ and $a_i \leq b_i$ for $i = 1, 2, \ldots, n$. Then,*

$$\pi = L(\delta([\nu^{a'_1}\rho, \nu^{b_1}\rho]), \delta([\nu^{a'_2}\rho, \nu^{b_2}\rho]), \ldots, \delta([\nu^{a'_n}\rho, \nu^{b_n}\rho]))$$

for some permutation $a'_1, a'_2, \ldots, a'_n$ of $a_1, a_2, \ldots, a_n$. (Here we allow case $a'_i > b_i$ for some i; i.e., some segments in the Langlands subrepresentation may be empty.)

Proof. In the proof of this lemma we use the following technical claim:

Claim 3.2. Let $b_1 \leq b_2 \leq \ldots \leq b_n$. Suppose that π is an irreducible subquotient of the induced representation $\delta([\nu^{a_1}\rho, \nu^{b_1}\rho]) \times \delta([\nu^{a_2}\rho, \nu^{b_2}\rho]) \times \cdots \times \delta([\nu^{a_n}\rho, \nu^{b_n}\rho])$, where $a_i \leq b_i$ for $i = 1, 2, \ldots, n$. If the representation $\delta([\nu^{a'_1}\rho, \nu^{b'_1}\rho]) \otimes \delta([\nu^{a'_2}\rho, \nu^{b'_2}\rho]) \otimes \cdots \otimes \delta([\nu^{a'_k}\rho, \nu^{b'_k}\rho])$, where $b'_1 \leq b'_2 \leq \ldots \leq b'_k$ and $a'_i \leq b'_i$ for $i = 1, 2, \ldots, k$, is contained in the Jacquet module of π, then $k \leq n$. Also, multisets $\{a'_1, a'_2, \ldots, a'_k\}$ and $\{b'_1, b'_2, \ldots, b'_k\}$ are contained in multisets $\{a_1, a_2, \ldots, a_n\}$ and $\{b_1, b_2, \ldots, b_n\}$, respectively. Further, if $k = n$, then $b'_i = b_i$ for $i = 1, 2, \ldots, n$ and $a'_1, a'_2, \ldots, a'_n$ is a permutation of $a_1, a_2, \ldots, a_n$.

Proof of the Claim 3.2. The proof is by induction over n.

For $n = 1$ the claim obviously holds.

We prove the claim for $n = 2$. Let $\pi \leq \delta([\nu^{a_1}\rho, \nu^{b_1}\rho]) \times \delta([\nu^{a_2}\rho, \nu^{b_2}\rho])$, where $b_1 \leq b_2$. If the representation $\delta([\nu^{a_1}\rho, \nu^{b_1}\rho]) \times \delta([\nu^{a_2}\rho, \nu^{b_2}\rho])$ is irreducible, then $\pi = L(\delta([\nu^{a_1}\rho, \nu^{b_1}\rho]), \delta([\nu^{a_2}\rho, \nu^{b_2}\rho]))$ and the claim follows directly from (2.2). The details are left to the reader. If the representation $\delta([\nu^{a_1}\rho, \nu^{b_1}\rho]) \times \delta([\nu^{a_2}\rho, \nu^{b_2}\rho])$ reduces, we have two possibilities:

- $\pi = L(\delta([\nu^{a_1}\rho, \nu^{b_1}\rho]), \delta([\nu^{a_2}\rho, \nu^{b_2}\rho]))$, and the claim follows in the same way as in the previous case;
- $\pi = \delta([\nu^{a_2}\rho, \nu^{b_1}\rho]) \times \delta([\nu^{a_1}\rho, \nu^{b_2}\rho])$, and the claim is again implied by the formula (2.2). Observe that the first segment appears to be empty for $a_2 = b_1 + 1$.

Suppose that the claim holds for all $m \leq n - 1$. We prove it for $m = n$.

Since $\pi \leq \delta([\nu^{a_1}\rho, \nu^{b_1}\rho]) \times \delta([\nu^{a_2}\rho, \nu^{b_2}\rho]) \times \cdots \times \delta([\nu^{a_n}\rho, \nu^{b_n}\rho])$, transitivity of Jacquet modules and formula (2.2) imply that there exist $i_1, i_2, \ldots, i_n$, satisfying $a_j - 1 \leq i_j \leq b_j$, such that $\delta([\nu^{a'_k}\rho, \nu^{b'_k}\rho])$ is an irreducible subquotient of the representation $\delta([\nu^{a_1}\rho, \nu^{i_1}\rho]) \times \delta([\nu^{a_2}\rho, \nu^{i_2}\rho]) \times \cdots \times \delta([\nu^{a_n}\rho, \nu^{i_n}\rho])$ and $\delta([\nu^{i_1+1}\rho, \nu^{b_1}\rho]) \times \delta([\nu^{i_2+1}\rho, \nu^{b_2}\rho]) \times \cdots \times \delta([\nu^{i_n+1}\rho, \nu^{b_n}\rho])$ contains irreducible representation $\delta([\nu^{a'_1}\rho, \nu^{b'_1}\rho]) \otimes \delta([\nu^{a'_2}\rho, \nu^{b'_2}\rho]) \otimes \cdots \otimes \delta([\nu^{a'_{k-1}}\rho, \nu^{b'_{k-1}}\rho])$ in its Jacquet module.

Thus, we obtain the following equality of sets:

$$[\nu^{a'_k}\rho, \nu^{b'_k}\rho] = \{\nu^{a_1}\rho, \nu^{a_1+1}\rho, \dots, \nu^{i_1}\rho, \nu^{a_2}\rho, \nu^{a_2+1}\rho, \dots, \nu^{i_2}\rho, \dots, \nu^{a_n}\rho, \nu^{a_n+1}\rho, \dots, \nu^{i_n}\rho\}.$$

The maximality of b'_k implies $b'_k = b_n$, so there is some j_k such that $i_{j_k} = b_{j_k} = b_n$. Also, if $a'_k < a_{j_k}$, then there is some j'_k, $1 \le j'_k \le n$, such that $i_{j'_k}$ equals $a_{j_k} - 1$. We continue in the same fashion to obtain that $i_1, i_2, \dots, i_{j_k-1}, i_{j_k+1}, \dots, i_n$ is a permutation of $a_1 - 1, a_2 - 1, \dots, a_{j_k-1} - 1, a_{j_k+1} - 1, \dots, a_n - 1$, while $i_{j_k} = b_{j_k}$ (which is equal to b_n). Obviously, $a'_k = a_j$, for some $j \in \{1, 2, \dots, n\}$.

Further, the induced representation

$$\delta([\nu^{i_1+1}\rho, \nu^{b_1}\rho]) \times \cdots \times \delta([\nu^{i_{j_k-1}+1}\rho, \nu^{b_{j_k-1}}\rho]) \times\delta([\nu^{i_{j_k+1}+1}\rho, \nu^{b_{j_k+1}}\rho]) \times \cdots \times \delta([\nu^{i_n+1}\rho, \nu^{b_n}\rho])$$

has to contain $\delta([\nu^{a'_1}\rho, \nu^{b'_1}\rho]) \otimes \delta([\nu^{a'_2}\rho, \nu^{b'_2}\rho]) \otimes \cdots \otimes \delta([\nu^{a'_{k-1}}\rho, \nu^{b'_{k-1}}\rho])$ in its Jacquet module.

Since this representation is a product of $n-1$ irreducible essentially square-integrable representations, $b_1 \le b_2 \le \dots \le b_{j_k-1} \le b_{j_k+1} = b_{j_k+2} = \cdots = b_n$ and $i_1+1, i_2+1, \dots, i_{j_k-1}+1, i_{j_k+1}+1, \dots, i_n$ is a permutation of $a_1, a_2, \dots, a_{j_k-1}, a_{j_k+1}, \dots, a_n$, by the inductive assumption we get $k-1 \le n-1$, $\{a'_1, a'_2, \dots, a'_{k-1}\}$ is contained in the multiset $\{a_1, a_2, \dots, a_{j_k-1}, a_{j_k+1}, \dots, a_n\}$, and $\{b'_1, b'_2, \dots, b'_{j_k-1}, b'_{j_k+1}, \dots b'_k\}$ is contained in the multiset $\{b_1, b_2, \dots, b_{j_k-1}, b_n, \dots, b_n\}$. This proves the claim. □

We proceed with the proof of Lemma 3.1.

By the Langlands classification, π is isomorphic to $L(\delta_1, \delta_2, \dots, \delta_k)$, where each δ_i, $i = 1, 2, \dots, k$, is an irreducible, essentially square-integrable representation of $GL(n_i, F)$ and $e(\delta_1) \le e(\delta_2) \le \dots \le e(\delta_k)$. Write $\delta_i = \delta([\nu^{a'_i}\rho, \nu^{b'_i}\rho])$.

Let us denote by A, B the multisets $\{a_1, a_2, \dots, a_n\}$, $\{b_1, b_2, \dots, b_n\}$, respectively. Also, we denote by A', B' the multisets $\{a'_1, a'_2, \dots, a'_k\}$, $\{b'_1, b'_2, \dots, b'_k\}$, respectively.

If $b'_i > b'_j$ for $i < j$, then the inequality $e(\delta_i) \le e(\delta_j)$ yields $a'_i < a'_j$. So, the segment $[\nu^{a'_i}\rho, \nu^{b'_i}\rho]$ contains the segment $[\nu^{a'_j}\rho, \nu^{b'_j}\rho]$ and the induced representations $\delta([\nu^{a'_i}\rho, \nu^{b'_i}\rho]) \times \delta([\nu^{a'_j}\rho, \nu^{b'_j}\rho])$ and $\delta([\nu^{a'_j}\rho, \nu^{b'_j}\rho]) \times \delta([\nu^{a'_i}\rho, \nu^{b'_i}\rho])$ are isomorphic. Therefore, we may suppose that $b'_1 \le b'_2 \le \dots \le b'_k$.

Since π is the (unique irreducible) subrepresentation of the induced representation $\delta([\nu^{a'_1}\rho, \nu^{b'_1}\rho]) \times \delta([\nu^{a'_2}\rho, \nu^{b'_2}\rho]) \times \cdots \times \delta([\nu^{a'_k}\rho, \nu^{b'_k}\rho])$, Frobenius reciprocity implies that the Jacquet module of π contains $\delta([\nu^{a'_1}\rho, \nu^{b'_1}\rho]) \otimes \delta([\nu^{a'_2}\rho, \nu^{b'_2}\rho]) \otimes \cdots \otimes \delta([\nu^{a'_k}\rho, \nu^{b'_k}\rho])$ as an irreducible subquotient.

Now Claim 3.2 implies $k \le n$. Further, the multiset A' is contained in the multiset A, while the multiset B' is contained in the multiset B.

Suppose $k < n$. Then we can write $A \backslash A' = \{a''_1, a''_2, \dots, a''_{n-k}\}$ and $B \backslash B' = \{b''_1, b''_2, \dots, b''_{n-k}\}$. We assume $a''_i \le a''_j$ and $b''_i \le b''_j$ for $i \le j$. It may be easily concluded from the proof of the Claim 3.2 that $a''_i > b''_i$, for $i = 1, 2, \dots, n-k$.

Thus, after adding $n-k$ empty segments, we may write

$$\pi = L(\delta([\nu^{a'''_1}\rho, \nu^{b_1}\rho]), \delta([\nu^{a'''_2}\rho, \nu^{b_2}\rho]), \dots, \delta([\nu^{a'''_n}\rho, \nu^{b_n}\rho])),$$

where $a'''_1, a'''_2, \dots, a'''_n$ is a permutation of $a_1, a_2, \dots, a_n$. This completes the proof. □

The following result may be proved in much the same way as Lemma 3.1:

Lemma 3.3. *Let ρ denote an irreducible unitarizable cuspidal representation of the group $GL(n, F)$. Let π be an irreducible subquotient of the induced representation $\delta([\nu^{a_1}\rho, \nu^{b_1}\rho]) \times \delta([\nu^{a_2}\rho, \nu^{b_2}\rho]) \times \cdots \times \delta([\nu^{a_n}\rho, \nu^{b_n}\rho])$, where $a_1 \leq a_2 \leq \ldots \leq a_n$ and $a_i \leq b_i$ for $i = 1, 2, \ldots, n$. Then,*

$$\pi = L(\delta([\nu^{a_1}\rho, \nu^{b'_1}\rho]), \delta([\nu^{a_2}\rho, \nu^{b'_2}\rho]), \ldots, \delta([\nu^{a_n}\rho, \nu^{b'_n}\rho]))$$

for some permutation $b'_1, b'_2, \ldots, b'_n$ of $b_1, b_2, \ldots, b_n$. (Here we allow the case $a_i > b'_i$ for some i; i.e., some segments in the Langlands subrepresentation may be empty.)

The following lemma is important for further calculations:

Lemma 3.4. *Let σ denote a strongly positive discrete series representation of $Sp(n, F)$. Suppose that τ is an irreducible representation of $GL(t, F)$ and σ' an irreducible representation of $Sp(n-t, F)$ such that $\tau \otimes \sigma'$ is an irreducible subquotient of $r_{(t)}(\sigma)$. Then σ' is a strongly positive discrete series.*

Proof. Suppose that σ' is not a strongly positive discrete series. Then there is an embedding $\sigma' \hookrightarrow \nu^{a_1}\rho_1 \times \cdots \times \nu^{a_{l_1}}\rho_{l_1} \rtimes \sigma_{cusp}$ such that $a_i \leq 0$ for some $i \in \{1, \ldots, l_1\}$. Frobenius reciprocity implies that the Jacquet module of σ' with respect to the appropriate parabolic subgroup contains $\nu^{a_1}\rho_1 \otimes \cdots \otimes \nu^{a_{l_1}}\rho_{l_1} \otimes \sigma_{cusp}$.

Further, we fix an embedding $\tau \hookrightarrow \nu^{a_{l_1+1}}\rho_{l_1+1} \times \cdots \times \nu^{a_{l_2}}\rho_{l_2}$, where $\nu^{a_i}\rho_i$ is an irreducible cuspidal representation of $GL(n_i, F)$, for $i = l_1+1, \ldots, l_2$. Then the Jacquet module of τ with respect to the appropriate parabolic subgroup contains $\nu^{a_{l_1+1}}\rho_{l_1+1} \otimes \cdots \otimes \nu^{a_{l_2}}\rho_{l_2}$. Using transitivity of Jacquet modules we conclude that the representation $\nu^{a_{l_1+1}}\rho_{l_1+1} \otimes \cdots \otimes \nu^{a_{l_2}}\rho_{l_2} \otimes \nu^{a_1}\rho_1 \otimes \cdots \otimes \nu^{a_{l_1}}\rho_{l_1} \otimes \sigma_{cusp}$ is an irreducible subquotient of the Jacquet module of σ with respect to the appropriate parabolic subgroup. Since this representation is cuspidal, [2], Lemma 26, implies that it is a quotient. Consequently, σ is a subrepresentation of $\nu^{a_{l_1+1}}\rho_{l_1+1} \times \cdots \times \nu^{a_i}\rho_i \times \cdots \times \nu^{a_{l_1}}\rho_{l_1} \rtimes \sigma_{cusp}$. Since $a_i \leq 0$, that contradicts the strong positivity of σ and thus the lemma is proved.

Notice that we have also proved $a_i > 0$, for $l_1 + 1 \leq i \leq l_2$. □

Finally, we show some specific and useful properties of strongly positive discrete series.

Lemma 3.5. *Let σ be a strongly positive discrete series representation of $Sp(n)$. Then σ is uniquely determined by $[\sigma]$.*

Proof. Write $[\sigma] = [\pi_1, \pi_2, \ldots, \pi_l, \sigma_{cusp}]$ and denote by M the multiset $\{\pi_1, \pi_2, \ldots, \pi_l\}$. Results of [10] enable us to assume that every element π_i is of the form $\nu^x\rho$, where ρ is an irreducible unitarizable self-contragredient cuspidal representation and $x > 0$. Representation σ may be written as a unique irreducible subrepresentation of the induced representation of the form (2.1).

Then σ is uniquely determined by the cuspidal representation σ_{cusp} and sequence

$$(b_1^{(1)}, b_2^{(1)}, \ldots, b_{k_1}^{(1)}, b_1^{(2)}, \ldots, b_{k_2}^{(2)}, \ldots, b_1^{(m)}, \ldots, b_{k_m}^{(m)}).$$

Let $i \in \{1, 2, \ldots, m\}$ be arbitrary, but fixed. We denote by M_i the submultiset consisting of elements of M of the form $\nu^x\rho_i$ (where every element is taken with the same multiplicity as in M).

Define $x_1 = \max\{x : \nu^x\rho_i \in M_i\}$. Theorem 5.1 of [10] implies that $\nu^{x_1}\rho_i$ appears in M_i with multiplicity one. Thus, $b_{k_i}^{(i)} = x_1$.

Now, define $M_i^{(1)} = M_i \setminus \{\nu^{a_{\rho_i}}\rho_i, \nu^{a_{\rho_i}+1}\rho_i, \ldots, \nu^{b_{k_i}^{(i)}}\rho_i\}$. If the multiset $M_i^{(1)}$ is non-empty, it may be concluded in the same way as before that $b_{k_i-1}^{(i)}$ equals $\max\{x : \nu^x\rho_i \in M_i^{(1)}\}$; otherwise $b_{k_i-1}^{(i)} = a_{\rho_i} - 2$.

We continue in this fashion to obtain that the sequence $(b_1^{(i)}, b_2^{(i)}, \ldots, b_{k_i}^{(i)})$ is uniquely determined by $[\sigma]$, for every $i = 1, 2, \ldots, m$. This shows that σ is uniquely determined by $[\sigma]$, which is the desired conclusion. □

Lemma 3.6. *Let σ_1, σ_2 denote representations in discrete series of $Sp(n)$, and suppose that $[\sigma_1]$ and $[\sigma_2]$ are equal. If one of these representations is strongly positive, then $\sigma_1 \simeq \sigma_2$.*

Proof. Write $[\sigma_1] = [\sigma_2] = [\nu^{s_1}\rho_1, \nu^{s_2}\rho_2, \ldots, \nu^{s_t}\rho_t, \sigma_{cusp}]$, where ρ_i is an irreducible self-contragredient cuspidal unitarizable representation, for $i = 1, 2, \ldots, t$.

We may suppose that the representation σ_1 is strongly positive and realize it as a unique irreducible subrepresentation of the representation of the form (2.1). Thus, we may write

$$\sigma_1 \hookrightarrow (\prod_{i=1}^{m}\prod_{j=1}^{k_i} \delta([\nu^{a_{\rho_i}-k_i+j}\rho_i, \nu^{b_j^{(i)}}\rho_i])) \rtimes \sigma_{cusp},$$

with m minimal and k_i minimal, for $i = 1, 2, \ldots, m$ (this just allows us to drop out all perhaps empty segments appearing in (2.1)). Obviously,

$$[\sigma_1] = [\nu^{a_{\rho_1}-k_1+1}\rho_1, \ldots, \nu^{b_1^{(1)}}\rho_1, \nu^{a_{\rho_1}-k_1+2}\rho_1, \ldots, \nu^{b_2^{(1)}}\rho_1, \ldots, \nu^{b_{k_m}^{(m)}}\rho_m, \sigma_{cusp}].$$

Let us first show that σ_2 also has to be a strongly positive representation. On the contrary, suppose that there exists some embedding

$$\sigma_2 \hookrightarrow \nu^{x_1}\rho_{i_1} \times \cdots \times \nu^{x_r}\rho_{i_r} \times \cdots \times \nu^{x_t}\rho_{i_t} \rtimes \sigma_{cusp},$$

where $x_r \leq 0$. Define $y = \min\{r : x_r \leq 0\}$. If $y = 1$, we get a contradiction with the square-integrability of the representation σ_2. Suppose $y \geq 2$.

Equality $[\sigma_1] = [\sigma_2]$ implies that $x_i \neq 0$, for $i = 1, 2, \ldots, t$. This yields $x_y < 0$. We have the following possibilities:

- $x_y \leq -1$: For $j < y$ we have $x_j > 0$. Hence, representation $\nu^{x_j}\rho_{i_j} \times \nu^{x_y}\rho_{i_y}$ is irreducible for $j < y$ and thus isomorphic to $\nu^{x_y}\rho_{i_y} \times \nu^{x_j}\rho_{i_j}$. We obtain the following isomorphisms:

$$\begin{aligned}
\sigma_2 &\hookrightarrow \nu^{x_1}\rho_{i_1} \times \cdots \times \nu^{x_{y-1}}\rho_{i_{y-1}} \times \nu^{x_y}\rho_{i_y} \times \cdots \times \nu^{x_t}\rho_{i_t} \rtimes \sigma_{cusp} \\
&\simeq \nu^{x_1}\rho_{i_1} \times \cdots \times \nu^{x_y}\rho_{i_y} \times \nu^{x_{y-1}}\rho_{i_{y-1}} \times \cdots \times \nu^{x_t}\rho_{i_t} \rtimes \sigma_{cusp} \\
&\vdots \\
&\simeq \nu^{x_y}\rho_{i_y} \times \nu^{x_1}\rho_{i_1} \times \cdots \times \nu^{x_t}\rho_{i_t} \rtimes \sigma_{cusp},
\end{aligned}$$

  contradicting square-integrability of σ_2.
- $-1 < x_y$: Inspecting the embedding (2.1) more precisely, it is not hard to see that for each $i = 1, 2, \ldots, m$ there exists at most one representation of the form $\nu^{z_i}\rho_i$, with $0 < z_i < 1$, appearing in $[\sigma_1]$. Moreover, if such a representation appears in $[\sigma_1]$ it must be equal to $\nu^{a_{\rho_i}-k_i+1}\rho_i$, implying that x_y equals $k_i - a_{\rho_i} - 1$ for some $i \in \{1, 2, \ldots, m\}$. Since $[\sigma_1] = [\sigma_2]$, the representations $\nu^{x_y}\rho_{i_y}$ and $\nu^{-x_y}\rho_{i_y}$ can't both appear in $[\sigma_2]$. It follows that the representation $\nu^{x_j}\rho_{i_j} \times \nu^{x_y}\rho_{i_y}$ is irreducible for $j < y$. Now we get

the contradiction with square-integrability of σ_2 in the same way as in the previous case.

Since σ_1 and σ_2 are strongly positive representations such that $[\sigma_1] = [\sigma_2]$, the previous lemma completes the proof. $\square$

4. Jacquet modules of strongly positive representations of the symplectic group: $D(\rho, \sigma_{cusp})$ case

In this section, we explicitly describe Jacquet modules of strongly positive representations contained in the set $D(\rho, \sigma_{cusp})$, where ρ is an irreducible cuspidal self-contragredient representation of $GL(n_\rho, F)$ and σ_{cusp} is an irreducible cuspidal representation of $Sp(n')$. In the following section we discuss Jacquet modules in the general case.

If the representation $\rho \rtimes \sigma_{cusp}$ reduces, then the only strongly positive discrete series in $D(\rho, \sigma_{cusp})$ is the representation σ_{cusp}, so in the rest of this section we assume that the representation $\nu^a \rho \rtimes \sigma_{cusp}$ reduces for $a > 0$ (uniqueness of such an a was proved in [17]). Let $\sigma \in D(\rho, \sigma_{cusp})$ denote the strongly positive discrete series representation of $Sp(n)$, which will be fixed throughout this section. We may assume that σ is not equal to σ_{cusp}.

We analyze the Jacquet modules of the representation σ using the classification obtained in Section 4 of [10]. Theorems 4.4 and 4.5 of that paper assert that there exists a unique increasing sequence of real numbers $0 < b_1 < b_2 < \cdots < b_k$, $b_i - a + k - i \in \mathbb{Z}_{\geq 0}$ for $1 \leq i \leq k$, such that σ is the unique irreducible subrepresentation of

$$\delta([\nu^{a-k+1}\rho, \nu^{b_1}\rho]) \times \cdots \times \delta([\nu^{a}\rho, \nu^{b_k}\rho]) \rtimes \sigma_{cusp}. \tag{4.1}$$

We note that $k \leq \lceil a \rceil$.

Using Lemma 2.2 and strong positivity of the representation σ, (4.1) gives

$$\mu^*(\sigma) \leq \prod_{j=1}^{k} \Big(\sum_{i_j = a-k+j-1}^{b_j} \delta([\nu^{i_j+1}\rho, \nu^{b_j}\rho]) \otimes \delta([\nu^{a-k+j}\rho, \nu^{i_j}\rho]) \Big) \rtimes (1 \otimes \sigma_{cusp}). \tag{4.2}$$

Let $\tau \otimes \sigma'$ denote an irreducible subquotient of $r_{(t)}(\sigma)$, for some t, where τ is an irreducible representation of $GL(t, F)$ and σ' an irreducible representation of $Sp(n-t)$. From Lemma 3.4 we know that σ' is strongly positive, while from (4.2) we conclude that there are $i_1, i_2, \ldots, i_k$, $a - k + j - 1 \leq i_j \leq b_j$, $i_j - a \in \mathbb{Z}$, for $j = 1, 2, \ldots, k$, such that $\tau \otimes \sigma'$ is a subquotient of

$$\begin{gathered} \delta([\nu^{i_1+1}\rho, \nu^{b_1}\rho]) \times \cdots \times \delta([\nu^{i_k+1}\rho, \nu^{b_k}\rho]) \otimes \\ \delta([\nu^{a-k+1}\rho, \nu^{i_1}\rho]) \times \cdots \times \delta([\nu^{a}\rho, \nu^{i_k}\rho]) \rtimes \sigma_{cusp}. \end{gathered}$$

In the following proposition we determine possible situations when σ' may appear.

Proposition 4.1. *Suppose that there is some strongly positive irreducible subquotient σ' of the representation $\delta([\nu^{a-k+1}\rho, \nu^{i_1}\rho]) \times \delta([\nu^{a-k+2}\rho, \nu^{i_2}\rho]) \times \cdots \times \delta([\nu^{a}\rho, \nu^{i_k}\rho]) \rtimes \sigma_{cusp}$, where $i_j - a + k - j \geq 0$. Then $i_1 < i_2 < \cdots < i_k$ and σ' is the unique irreducible subrepresentation of the above representation. Moreover, σ' is the unique strongly positive irreducible subquotient of the above representation.*

Proof. Obviously, $\sigma' \in D(\rho, \sigma_{cusp})$. Since σ' is strongly positive, results of [10] imply that there is an increasing sequence of real numbers $b'_1 < b'_2 < \cdots < b'_{k'}$, where $k' \leq \lceil a \rceil$ and $b'_j - a + k' - j \in \mathbb{Z}_{\geq 0}$ for $1 \leq j \leq k'$, such that σ' is the unique irreducible subrepresentation of

$$\delta([\nu^{a-k'+1}\rho, \nu^{b'_1}\rho]) \times \delta([\nu^{a-k'+2}\rho, \nu^{b'_2}\rho]) \times \cdots \times \delta([\nu^{a}\rho, \nu^{b'_{k'}}\rho]) \rtimes \sigma_{cusp}.$$

We claim that $k = k'$ and $b'_j = b_j$ for $1 \leq j \leq k$.

Observe that the Jacquet module of σ' with respect to the appropriate parabolic subgroup contains

$$\delta([\nu^{a-k'+1}\rho, \nu^{b'_1}\rho]) \otimes \delta([\nu^{a-k'+2}\rho, \nu^{b'_2}\rho]) \otimes \cdots \otimes \delta([\nu^{a}\rho, \nu^{b'_{k'}}\rho]) \otimes \sigma_{cusp}.$$

Since

$$\mu^*(\sigma') \leq \mu^*(\delta([\nu^{a-k+1}\rho, \nu^{i_1}\rho]) \times \delta([\nu^{a-k+2}\rho, \nu^{i_2}\rho]) \times \cdots \times \delta([\nu^{a}\rho, \nu^{i_k}\rho]) \rtimes \sigma_{cusp}),$$

Theorem 2.1 implies that there is an irreducible subquotient π of

$$M^*(\delta([\nu^{a-k+1}\rho, \nu^{i_1}\rho]) \times \delta([\nu^{a-k+2}\rho, \nu^{i_2}\rho]) \times \cdots \times \delta([\nu^{a}\rho, \nu^{i_k}\rho]))$$

such that the Jacquet module of π with respect to the appropriate parabolic subgroup contains

$$\delta([\nu^{a-k'+1}\rho, \nu^{b'_1}\rho]) \otimes \delta([\nu^{a-k'+2}\rho, \nu^{b'_2}\rho]) \otimes \cdots \otimes \delta([\nu^{a}\rho, \nu^{b'_{k'}}\rho]). \tag{4.3}$$

Lemma 2.2 implies that there are $a - k + j - 1 \leq x_j \leq i_j$, $j = 1, 2, \ldots, k$ such that

$$\pi \leq \prod_{j=1}^{k} (\delta([\nu^{-x_j}\rho, \nu^{-a+k-j-1}\rho]) \times \delta([\nu^{x_j+1}\rho, \nu^{i_j}\rho])).$$

Condition $a - k' + 1 > 0$ forces $x_i = a - k + j - 1$. This gives

$$\pi \leq \delta([\nu^{a-k+1}\rho, \nu^{i_1}\rho]) \times \delta([\nu^{a-k+2}\rho, \nu^{i_2}\rho]) \times \cdots \times \delta([\nu^{a}\rho, \nu^{i_k}\rho]).$$

From the previous inequality and (4.3) it may be concluded that the following equality of multisets holds:

$$\sum_{j=1}^{k} [\nu^{a-k+j}\rho, \nu^{i_j}\rho] = \sum_{l=1}^{k'} [\nu^{a-k'+l}\rho, \nu^{b'_l}\rho]. \tag{4.4}$$

It follows immediately that $k = k'$. We have $b'_k \geq i_j$, for each $j = 1, 2, \ldots, k$, because b'_k is the largest exponent appearing on the right-hand side of (4.4).

To study the appearance of the representation (4.3) in the Jacquet module of the representation π, we use the formula for m^*, which combined with (2.2) implies that there are $a - k + j - 1 \leq x_j \leq i_j$, for $j = 1, 2, \ldots, k$ such that $\delta([\nu^a\rho, \nu^{b'_k}\rho]) \leq \delta([\nu^{a-k+1}\rho, \nu^{x_1}\rho]) \times \cdots \times \delta([\nu^{a}\rho, \nu^{x_k}\rho])$. The equality of sets

$$[\nu^a\rho, \nu^{b'_k}\rho] = [\nu^{a-k+1}\rho, \nu^{x_1}\rho] \cup \cdots \cup [\nu^a\rho, \nu^{x_k}\rho]$$

gives $x_j = a - k + j - 1$ for $1 \leq j \leq k - 1$ and $x_k = b'_k$. This forces $b'_k \leq i_k$. By the previous discussion, we get $b'_k = i_k$.

Using (2.2) again, we conclude that there is some irreducible subquotient π' of $\delta([\nu^{a-k+1}\rho, \nu^{i_1}\rho]) \times \cdots \times \delta([\nu^{a-1}\rho, \nu^{i_{k-1}}\rho])$ which contains the representation $\delta([\nu^{a-k+1}\rho, \nu^{b'_1}\rho]) \otimes \cdots \otimes \delta([\nu^{a-1}\rho, \nu^{b'_{k-1}}\rho])$ in its Jacquet module. Canceling (equal) segments $[\nu^a\rho, \nu^{i_k}\rho]$ and $[\nu^a\rho, \nu^{b'_k}\rho]$ in (4.4) we deduce $b'_{k-1} \geq i_j$, for $j = 1, \ldots, k-1$. Repeating the same arguments as above, we obtain $b'_{k-1} = i_j$.

Proceeding in the same way, we conclude that $i_j = b'_j$, for $j = 1, 2, \ldots, k$. Thus, we have proved $i_1 < i_2 < \cdots < i_k$.

What is left is to show that σ' is the subrepresentation of the induced representation from the statement of the proposition. Now, Theorem 4.6 from [10] shows that the unique irreducible subrepresentation of $\delta([\nu^{a-k+1}\rho, \nu^{i_1}\rho]) \times \delta([\nu^{a-k+2}\rho, \nu^{i_2}\rho]) \times \cdots \times \delta([\nu^{a}\rho, \nu^{i_k}\rho]) \rtimes \sigma_{cusp}$ is strongly positive. Let us denote this strongly positive discrete series by $\sigma_{(i_1,i_2,\ldots,i_k)}$. We show that there are no other strongly positive irreducible subquotients of the above induced representation.

Observe that $\sigma_{(i_1,i_2,\ldots,i_k)}$ is an irreducible subrepresentation of

$$L(\delta([\nu^{a-k+1}\rho, \nu^{i_1}\rho]), \ldots, \delta([\nu^{a}\rho, \nu^{i_k}\rho])) \rtimes \sigma_{cusp}.$$

This yields that $L(\delta([\nu^{a-k+1}\rho, \nu^{i_1}\rho]), \delta([\nu^{a-k+2}\rho, \nu^{i_2}\rho]), \ldots, \delta([\nu^{a}\rho, \nu^{i_k}\rho])) \otimes \sigma_{cusp}$ is an irreducible subquotient of $r_{(n-n')}(\sigma_{(i_1,i_2,\ldots,i_k)})$.

Suppose that σ'' is some strongly positive irreducible subquotient of $\delta([\nu^{a-k+1}\rho, \nu^{i_1}\rho]) \times \delta([\nu^{a-k+2}\rho, \nu^{i_2}\rho]) \times \cdots \times \delta([\nu^{a}\rho, \nu^{i_k}\rho]) \rtimes \sigma_{cusp}$ different from $\sigma_{(i_1,i_2,\ldots,i_k)}$. Suppose that $\pi \otimes \sigma_{cusp}$ is some irreducible subquotient of $r_{(n-n')}(\sigma'')$. Using Lemma 2.2 we obtain that there are indices $a - k + j - 1 \leq x_j \leq i_j$, $j = 1, 2, \ldots, k$, such that $\pi \leq \prod_{j=1}^{k} (\delta([\nu^{-x_j}\rho, \nu^{-a+k-j}\rho]) \times \delta([\nu^{x_j+1}\rho, \nu^{i_j}\rho]))$. As in the proof of Lemma 3.4, we conclude that $x_j = a - k + j - 1$ for $j = 1, 2, \ldots, k$; otherwise some $\nu^s\rho$ where $s \leq 0$ would appear in the cuspidal support of π.

Since π is an irreducible subquotient of $\prod_{j=1}^{k} \delta([\nu^{a-k+j}\rho, \nu^{i_j}\rho])$, Lemma 3.1 implies $\pi = L(\delta([\nu^{a_1}\rho, \nu^{i_1}\rho]), \delta([\nu^{a_2}\rho, \nu^{i_2}\rho]), \ldots, \delta([\nu^{a_k}\rho, \nu^{i_k}\rho]))$ for some permutation $a_1, a_2, \ldots, a_k$ of $a - k + 1, a - k + 2, \ldots, a$. Since σ'' is not isomorphic to $\sigma_{(i_1,i_2,\ldots,i_k)}$ and $L(\delta([\nu^{a-k+1}\rho, \nu^{i_1}\rho]), \ldots, \delta([\nu^{a-k+2}\rho, \nu^{i_2}\rho])$ appears with multiplicity one in $\prod_{j=1}^{k} \delta([\nu^{a-k+j}\rho, \nu^{i_j}\rho])$, there exists some m, $1 \leq m \leq k$, such that $a_m \neq a - k + m$. We choose the largest such m and denote it by m again. Obviously, $a_m < a - k + m$ and $a - k + m \leq i_m$. This fact gives us the following embeddings and isomorphisms:

$$\begin{aligned}
\pi \hookrightarrow\ & \delta([\nu^{a_1}\rho, \nu^{i_1}\rho]) \times \cdots \times \delta([\nu^{a_m}\rho, \nu^{i_m}\rho]) \times \delta([\nu^{a-k+m+1}\rho, \nu^{i_{m+1}}\rho]) \times \cdots \times \\
& \delta([\nu^{a}\rho, \nu^{i_k}\rho]) \\
\hookrightarrow\ & \delta([\nu^{a_1}\rho, \nu^{i_1}\rho]) \times \cdots \times \delta([\nu^{a_m+1}\rho, \nu^{i_m}\rho]) \times \nu^{a_m}\rho \times \\
& \delta([\nu^{a-k+m+1}\rho, \nu^{i_{m+1}}\rho]) \times \cdots \times \delta([\nu^{a}\rho, \nu^{i_k}\rho]) \\
\simeq\ & \delta([\nu^{a_1}\rho, \nu^{i_1}\rho]) \times \cdots \times \delta([\nu^{a_m+1}\rho, \nu^{i_m}\rho]) \times \delta([\nu^{a-k+m+1}\rho, \nu^{i_{m+1}}\rho]) \times \\
& \nu^{a_m}\rho \times \cdots \times \delta([\nu^{a}\rho, \nu^{i_k}\rho]) \\
& \vdots \\
\simeq\ & \delta([\nu^{a_1}\rho, \nu^{i_1}\rho]) \times \cdots \times \delta([\nu^{a_m+1}\rho, \nu^{i_m}\rho]) \times \delta([\nu^{a-k+m+1}\rho, \nu^{i_{m+1}}\rho]) \times \\
& \delta([\nu^{a-k+m+2}\rho, \nu^{i_{m+2}}\rho]) \times \cdots \times \delta([\nu^{a}\rho, \nu^{i_k}\rho]) \times \nu^{a_m}\rho \\
\hookrightarrow\ & \nu^{i_1}\rho \times \cdots \times \nu^{a_1}\rho \times \cdots \times \nu^{i_k}\rho \times \cdots \times \nu^{a}\rho \times \nu^{a_m}\rho.
\end{aligned}$$

Thus, the Jacquet module of π with respect to the appropriate parabolic subgroup contains $\nu^{i_1}\rho \otimes \cdots \otimes \nu^{a_m}\rho$. It is now easy to see that the cuspidal representation $\nu^{i_1}\rho \otimes \cdots \otimes \nu^{a_m}\rho \otimes \sigma_{cusp}$ is an irreducible subquotient of the Jacquet module of σ'' with respect to the appropriate parabolic subgroup. By Lemma 26 from [2] it must be a quotient. Using Frobenius reciprocity, we see that $\sigma'' \hookrightarrow \nu^{i_1}\rho \times \cdots \times \nu^{a_m}\rho \rtimes$

$\sigma_{cusp} \simeq \nu^{i_1}\rho \times \cdots \times \nu^{-a_m}\rho \rtimes \sigma_{cusp}$. This contradicts strong positivity of σ'' and proves the proposition. □

From the proof of the previous proposition and (4.1), it may be concluded that $r_{(n-n')}(\sigma) = L(\delta([\nu^{a-k+1}\rho, \nu^{i_1}\rho]), \ldots, \delta([\nu^{a}\rho, \nu^{i_k}\rho])) \otimes \sigma_{cusp}$. We will see that rather similar identities also hold for Jacquet modules of the representation σ with respect to the other maximal parabolic subgroups. Clearly, the above equation shows that the representation $r_{(n-n')}(\sigma)$ is irreducible, which was already noted in [12].

In the rest of this section, the unique strongly positive irreducible subquotient of $\delta([\nu^{a-k+1}\rho, \nu^{i_1}\rho]) \times \cdots \times \delta([\nu^{a}\rho, \nu^{i_k}\rho]) \rtimes \sigma_{cusp}$, where $i_1 < \cdots < i_k$ and $k < a+1$, will be denoted by $\sigma_{(i_1,i_2,\ldots,i_k)}$.

The previous proposition implies that the Sp-part of every irreducible representation which appears in $\mu^*(\sigma)$ has to be isomorphic to some $\sigma_{(i_1,i_2,\ldots,i_k)}$.

In the next proposition we characterize an irreducible strongly positive representation due to a prominent member of its Jacquet module. This result enables us to determine the Sp-parts of the irreducible members of $\mu^*(\sigma)$.

Proposition 4.2. *Let $\sigma' \in D(\rho;\sigma_{cusp})$ denote an irreducible strongly positive representation such that the Jacquet module of σ' with respect to the appropriate parabolic subgroup contains $\delta([\nu^{a-k+1}\rho, \nu^{i_1}\rho]) \otimes \delta([\nu^{a-k+2}\rho, \nu^{i_2}\rho]) \otimes \cdots \otimes \delta([\nu^{a}\rho, \nu^{i_k}\rho]) \otimes \sigma_{cusp}$, where $i_1 < i_2 < \cdots < i_k$ and $i_j - a + k - j \in \mathbb{Z}_{\geq 0}$ for $j = 1, 2, \ldots, k$. Then σ' is the unique irreducible subrepresentation of the representation $\delta([\nu^{a-k+1}\rho, \nu^{i_1}\rho]) \times \delta([\nu^{a-k+2}\rho, \nu^{i_2}\rho]) \times \cdots \times \delta([\nu^{a}\rho, \nu^{i_k}\rho]) \rtimes \sigma_{cusp}$, i.e., $\sigma' \simeq \sigma_{(i_1,i_2,\ldots,i_k)}$.*

Proof. We give two proofs of this proposition. The first one is based on the direct computation with Jacquet modules, while the second one relies on some specific properties of strongly positive discrete series which have been discussed in the previous section.

For the first proof, write σ' as the unique irreducible subrepresentation of the representation of the form

$$\delta([\nu^{a-k'+1}\rho, \nu^{b'_1}\rho]) \times \delta([\nu^{a-k'+2}\rho, \nu^{b'_2}\rho]) \times \cdots \times \delta([\nu^{a}\rho, \nu^{b'_{k'}}\rho]) \rtimes \sigma_{cusp}.$$

We claim that $k = k'$ and $i_j = b'_j$ for $j = 1, 2, \ldots, k$. The exactness and transitivity of Jacquet modules imply that there is an irreducible subquotient $\pi \otimes \sigma_{cusp}$ of $\mu^*(\sigma')$ such that the Jacquet module of π with respect to the appropriate parabolic subgroup contains $\delta([\nu^{a-k+1}\rho, \nu^{i_1}\rho]) \otimes \delta([\nu^{a-k+2}\rho, \nu^{i_2}\rho]) \otimes \cdots \otimes \delta([\nu^{a}\rho, \nu^{i_k}\rho])$. Combining Lemma 2.2 with the strong positivity of the representation σ' we can assert that π is an irreducible subquotient of $\delta([\nu^{a-k'+1}\rho, \nu^{b'_1}\rho]) \times \delta([\nu^{a-k'+2}\rho, \nu^{b'_2}\rho]) \times \cdots \times \delta([\nu^{a}\rho, \nu^{b'_{k'}}\rho])$. Following the same lines as in the proof of Proposition 4.1, we get desired identities $k = k'$ and $i_j = b'_j$ for $j = 1, 2, \ldots, k$. This ends the first proof.

For the second proof, observe that the transitivity of Jacquet modules yields that the cuspidal representation

$$\nu^{i_1}\rho \otimes \nu^{i_1-1}\rho \otimes \cdots \otimes \nu^{a-k+1}\rho \otimes \cdots \otimes \nu^{i_k}\rho \otimes \nu^{i_k-1} \otimes \cdots \otimes \nu^{a}\rho \otimes \sigma_{cusp}$$

is contained in the Jacquet module of σ' with respect to the appropriate parabolic subgroup. As in the proof of Lemma 3.4, we deduce that σ' is a subrepresentation of $\nu^{i_1}\rho \times \nu^{i_1-1}\rho \times \cdots \times \nu^{a-k+1}\rho \times \cdots \times \nu^{i_k}\rho \times \nu^{i_k-1} \times \cdots \times \nu^{a}\rho \rtimes \sigma_{cusp}$. It is immediate that $[\sigma'] = [\sigma_{(i_1,i_2,\ldots,i_k)}]$ and Lemma 3.5 completes the proof. □

We emphasize that the previous lemma holds more generally, i.e., for general discrete series $\sigma' \in D(\rho; \sigma_{cusp})$. In that case Lemma 3.6 should be used to finish the proof.

Definition 4.3. We call an ordered k-tuple $(i_1, i_2, \ldots, i_k)$ of real numbers acceptable if the following conditions hold:

- $i_1 < i_2 < \cdots < i_k$,
- $i_j - a \in \mathbb{Z}$, for $j = 1, 2, \ldots, k$,
- $a - k + j - 1 \leq i_j \leq b_j$, for $j = 1, 2, \ldots, k$.

Observe that, for an acceptable k-tuple $(i_1, i_2, \ldots, i_k)$, if the segment $[\nu^{a-k+j}\rho, \nu^{i_j}\rho]$ is non-empty for some $1 \leq j \leq k$, then all the segments $[\nu^{a-k+j+1}\rho, \nu^{i_{j+1}}\rho]$, $\ldots, [\nu^a\rho, \nu^{i_k}\rho]$ are also non-empty.

Using this selection, we obtain the following embeddings and isomorphisms:

$$\begin{aligned}
\sigma \quad &\hookrightarrow \quad \delta([\nu^{a-k+1}\rho, \nu^{b_1}\rho]) \times \delta([\nu^{a-k+2}\rho, \nu^{b_2}\rho]) \times \cdots \times \delta([\nu^a\rho, \nu^{b_k}\rho]) \rtimes \sigma_{cusp} \\
&\hookrightarrow \quad \delta([\nu^{i_1+1}\rho, \nu^{b_1}\rho]) \times \delta([\nu^{a-k+1}\rho, \nu^{i_1}\rho]) \times \delta([\nu^{a-k+2}\rho, \nu^{b_2}\rho]) \times \cdots \times \\
&\qquad \delta([\nu^a\rho, \nu^{b_k}\rho]) \rtimes \sigma_{cusp} \\
&\hookrightarrow \quad \delta([\nu^{i_1+1}\rho, \nu^{b_1}\rho]) \times \delta([\nu^{a-k+1}\rho, \nu^{i_1}\rho]) \times \delta([\nu^{i_2+1}\rho, \nu^{b_2}\rho]) \times \\
&\qquad \delta([\nu^{a-k+2}\rho, \nu^{i_2}\rho]) \times \cdots \times \delta([\nu^a\rho, \nu^{b_k}\rho]) \rtimes \sigma_{cusp} \\
&\simeq \quad \delta([\nu^{i_1+1}\rho, \nu^{b_1}\rho]) \times \delta([\nu^{i_2+1}\rho, \nu^{b_2}\rho]) \times \delta([\nu^{a-k+1}\rho, \nu^{i_1}\rho]) \times \\
&\qquad \delta([\nu^{a-k+2}\rho, \nu^{i_2}\rho]) \times \cdots \times \delta([\nu^a\rho, \nu^{b_k}\rho]) \rtimes \sigma_{cusp} \\
&\vdots \\
&\simeq \quad \delta([\nu^{i_1+1}\rho, \nu^{b_1}\rho]) \times \delta([\nu^{i_2+1}\rho, \nu^{b_2}\rho]) \times \cdots \times \delta([\nu^{i_k+1}\rho, \nu^{b_k}\rho]) \times \\
&\qquad \delta([\nu^{a-k+1}\rho, \nu^{i_1}\rho]) \times \delta([\nu^{a-k+2}\rho, \nu^{i_2}\rho]) \times \cdots \times \delta([\nu^a\rho, \nu^{i_k}\rho]) \rtimes \sigma_{cusp}.
\end{aligned}$$

Frobenius reciprocity now shows that, for every acceptable k-tuple $(i_1, i_2, \ldots, i_k)$, representation σ contains the representation

$$\delta([\nu^{i_1+1}\rho, \nu^{b_1}\rho]) \otimes \delta([\nu^{i_2+1}\rho, \nu^{b_2}\rho]) \otimes \cdots \otimes \delta([\nu^{i_k+1}\rho, \nu^{b_k}\rho])$$
$$\otimes \delta([\nu^{a-k+1}\rho, \nu^{i_1}\rho]) \otimes \delta([\nu^{a-k+2}\rho, \nu^{i_2}\rho]) \otimes \cdots \otimes \delta([\nu^a\rho, \nu^{i_k}\rho]) \otimes \sigma_{cusp}$$

in its Jacquet module with respect to the appropriate parabolic subgroup.

Transitivity and exactness of Jacquet modules imply that for every acceptable k-tuple $(i_1, i_2, \ldots, i_k)$, there is an irreducible subquotient $\pi \otimes \sigma'$ of $r_{(t)}(\sigma)$, for appropriate t, such that the Jacquet module of σ' with respect to the appropriate parabolic subgroup contains $\delta([\nu^{a-k+1}\rho, \nu^{i_1}\rho]) \otimes \delta([\nu^{a-k+2}\rho, \nu^{i_2}\rho]) \otimes \cdots \otimes \delta([\nu^a\rho, \nu^{i_k}\rho]) \otimes \sigma_{cusp}$. Proposition 4.2 and Lemma 3.4 force $\sigma \simeq \sigma_{(i_1, i_2, \ldots, i_k)}$.

In the following, we determine the GL-parts of the irreducible representations appearing in $\mu^*(\sigma)$.

Lemma 4.4. *Let us denote by τ the induced representation $\delta([\nu^{i_1+1}\rho, \nu^{b_1}\rho]) \times \delta([\nu^{i_2+1}\rho, \nu^{b_2}\rho]) \times \cdots \times \delta([\nu^{i_k+1}\rho, \nu^{b_k}\rho])$. If $i_1 < i_2 < \ldots < i_k$, then there exists a unique irreducible subquotient τ' of τ such that the Jacquet module of τ' contains $\delta([\nu^{i_1+1}\rho, \nu^{b_1}\rho]) \otimes \delta([\nu^{i_2+1}\rho, \nu^{b_2}\rho]) \otimes \cdots \otimes \delta([\nu^{i_k+1}\rho, \nu^{b_k}\rho])$. Also, $\tau' \hookrightarrow \tau$ and*

$$\tau' = L(\delta([\nu^{i_1+1}\rho, \nu^{b_1}\rho]), \delta([\nu^{i_2+1}\rho, \nu^{b_2}\rho]), \ldots, \delta([\nu^{i_k+1}\rho, \nu^{b_k}\rho])).$$

Proof. Clearly, it is sufficient to show that the representation $\delta([\nu^{i_1+1}\rho, \nu^{b_1}\rho]) \otimes \delta([\nu^{i_2+1}\rho, \nu^{b_2}\rho]) \otimes \cdots \otimes \delta([\nu^{i_k+1}\rho, \nu^{b_k}\rho])$ appears with multiplicity one in the Jacquet module of τ with respect to the appropriate parabolic subgroup. We prove this using formula (2.2), by induction on k.

If $k=1$, the claim trivially holds. Suppose that the claim holds for all numbers less than k. We prove it for k. Formula (2.2) yields that there exist $x_1, x_2, \ldots, x_k$, $i_j \leq x_j \leq b_j$, $j=1,2,\ldots,k$, such that $\delta([\nu^{i_k+1}\rho, \nu^{b_k}\rho])$ is an irreducible subquotient of $\prod_{j=1}^{k} \delta([\nu^{i_j+1}\rho, \nu^{x_j}\rho])$ and some irreducible subquotient of $\prod_{j=1}^{k} \delta([\nu^{x_j+1}\rho, \nu^{b_j}\rho])$ contains $\delta([\nu^{i_1+1}\rho, \nu^{b_1}\rho]) \otimes \cdots \otimes \delta([\nu^{i_{k-1}+1}\rho, \nu^{b_{k-1}}\rho])$ in its Jacquet module.

In this way, we have obtained the following equality of the sets:

$$[\nu^{i_k+1}\rho, \nu^{b_k}\rho] = \bigcup_{j=1}^{k} [\nu^{i_j+1}\rho, \nu^{x_j}\rho].$$

Since $i_j < i_k$ and $x_j < b_k$ for $1 \leq j \leq k-1$, we deduce $x_k = b_k$ and $x_j = i_j$ for $1 \leq j \leq k-1$. Hence, multiplicity of $\delta([\nu^{i_1+1}\rho, \nu^{b_1}\rho]) \otimes \cdots \otimes \delta([\nu^{i_k+1}\rho, \nu^{b_k}\rho])$ in the Jacquet module of τ with respect to the appropriate parabolic subgroup equals the multiplicity of $\delta([\nu^{i_1+1}\rho, \nu^{b_1}\rho]) \otimes \cdots \otimes \delta([\nu^{i_{k-1}+1}\rho, \nu^{b_{k-1}}\rho])$ in the Jacquet module of the representation $\delta([\nu^{i_1+1}\rho, \nu^{b_1}\rho]) \times \cdots \times \delta([\nu^{i_{k-1}+1}\rho, \nu^{b_{k-1}}\rho])$ with respect to the appropriate parabolic subgroup. The inductive assumption finishes the proof. $\square$

From the previous discussion and Lemma 4.4 we conclude that

$$L(\delta([\nu^{i_1+1}\rho, \nu^{b_1}\rho]), \delta([\nu^{i_2+1}\rho, \nu^{b_2}\rho]), \ldots, \delta([\nu^{i_k+1}\rho, \nu^{b_k}\rho])) \otimes \sigma_{(i_1, i_2, \ldots, i_k)}$$

appears in $\mu^*(\sigma)$ for every acceptable k-tuple $(i_1, i_2, \ldots, i_k)$.

We are now ready to complete our description of GL-parts of irreducible representations appearing in $\mu^*(\sigma)$.

Suppose that $\tau \otimes \sigma'$ is an irreducible subquotient of $r_{(t)}(\sigma)$, for some t. From Lemma 3.4 and Proposition 4.1 it follows that $\sigma' = \sigma_{(i_1, i_2, \ldots, i_k)}$, for some acceptable k-tuple $(i_1, i_2, \ldots, i_k)$. It remains to describe τ.

Proposition 4.5. *Suppose that $\tau \otimes \sigma_{(i_1, i_2, \ldots, i_k)}$ is an irreducible subquotient of $r_{(t)}(\sigma)$, for appropriate t, where $(i_1, i_2, \ldots, i_k)$ is an acceptable k-tuple. Then*

$$\tau = L(\delta([\nu^{i_1+1}\rho, \nu^{b_1}\rho]), \delta([\nu^{i_2+1}\rho, \nu^{b_2}\rho]), \ldots, \delta([\nu^{i_k+1}\rho, \nu^{b_k}\rho])).$$

Proof. Since τ is evidently an irreducible subquotient of the representation

$$\delta([\nu^{i_1+1}\rho, \nu^{b_1}\rho]) \times \delta([\nu^{i_2+1}\rho, \nu^{b_2}\rho]) \times \cdots \times \delta([\nu^{i_k+1}\rho, \nu^{b_k}\rho]),$$

Lemma 3.1 implies that τ is isomorphic to $L(\delta([\nu^{a'_1}\rho, \nu^{b_1}\rho]), \delta([\nu^{a'_2}\rho, \nu^{b_2}\rho]), \ldots, \delta([\nu^{a'_k}\rho, \nu^{b_k}\rho]))$, for some permutation $a'_1, a'_2, \ldots, a'_k$ of $i_1+1, i_2+1, \ldots, i_k+1$.

Suppose that τ is not isomorphic to the representation

$$L(\delta([\nu^{i_1+1}\rho, \nu^{b_1}\rho]), \delta([\nu^{i_2+1}\rho, \nu^{b_2}\rho]), \ldots, \delta([\nu^{i_k+1}\rho, \nu^{b_k}\rho]));$$

i.e., suppose that there is some j, $1 \leq j \leq k$, such that $a'_j \neq i_j + 1$.

We fix the embedding of the representation τ as in the subrepresentation version of the Langlands classification:

$$L(\delta([\nu^{a'_1}\rho, \nu^{b_1}\rho]), \ldots, \delta([\nu^{a'_k}\rho, \nu^{b_k}\rho])) \hookrightarrow \delta([\nu^{a'_{j_1}}\rho, \nu^{b_{j_1}}\rho]) \times \cdots \times \delta([\nu^{a'_{j_k}}\rho, \nu^{b_{j_k}}\rho]),$$

where $j_1, \ldots, j_k$ denotes a permutation of $1, \ldots, k$ chosen in such way that

$$e([\nu^{a'_{j_1}}\rho, \nu^{b_{j_1}}\rho]) \leq \cdots \leq e([\nu^{a'_{j_k}}\rho, \nu^{b_{j_k}}\rho]).$$

If $e([\nu^{a'_n}\rho, \nu^{b_n}\rho]) \leq e([\nu^{a'_m}\rho, \nu^{b_m}\rho])$ for $n > m$, then $b_m < b_n$ gives $a'_n > a'_m$. In that case, the segment $[\nu^{a'_m}\rho, \nu^{b_m}\rho]$ is contained in the segment $[\nu^{a'_n}\rho, \nu^{b_n}\rho]$, so the representations $\delta([\nu^{a'_m}\rho, \nu^{b_m}\rho]) \times \delta([\nu^{a'_n}\rho, \nu^{b_n}\rho])$ and $\delta([\nu^{a'_n}\rho, \nu^{b_n}\rho]) \times \delta([\nu^{a'_m}\rho, \nu^{b_m}\rho])$ are isomorphic. This short discussion enables us to obtain the following embedding of the representation τ:

$$\tau \hookrightarrow \delta([\nu^{a'_1}\rho, \nu^{b_1}\rho]) \times \delta([\nu^{a'_2}\rho, \nu^{b_2}\rho]) \times \cdots \times \delta([\nu^{a'_k}\rho, \nu^{b_k}\rho]). \tag{4.5}$$

Let us denote by x the largest index j, $1 \leq j \leq k$, such that $a'_j \neq i_j + 1$. Observe that $a'_x < i_x + 1$. Therefore, (4.5) gives the following embeddings:

$$\begin{aligned} \tau \quad \hookrightarrow \quad & \delta([\nu^{a'_1}\rho, \nu^{b_1}\rho]) \times \cdots \times \delta([\nu^{a'_{x-1}}\rho, \nu^{b_{x-1}}\rho]) \times \delta([\nu^{a'_x}\rho, \nu^{b_x}\rho]) \times \\ & \delta([\nu^{i_{x+1}+1}\rho, \nu^{b_{x+1}}\rho]) \times \cdots \times \delta([\nu^{i_k+1}\rho, \nu^{b_k}\rho]) \\ \hookrightarrow \quad & \delta([\nu^{a'_1}\rho, \nu^{b_1}\rho]) \times \cdots \times \delta([\nu^{a'_{x-1}}\rho, \nu^{b_{x-1}}\rho]) \times \delta([\nu^{i_x+1}\rho, \nu^{b_x}\rho]) \times \\ & \delta([\nu^{a'_x}\rho, \nu^{i_x}\rho]) \times \delta([\nu^{i_{x+1}+1}\rho, \nu^{b_{x+1}}\rho]) \times \cdots \times \delta([\nu^{i_k+1}\rho, \nu^{b_k}\rho]). \end{aligned}$$

We may conclude that the Jacquet module of the representation τ with respect to the appropriate parabolic subgroup contains the irreducible representation

$$\delta([\nu^{a'_1}\rho, \nu^{b_1}\rho]) \otimes \cdots \otimes \delta([\nu^{a'_{x-1}}\rho, \nu^{b_{x-1}}\rho]) \otimes \delta([\nu^{i_x+1}\rho, \nu^{b_x}\rho])$$
$$\otimes \delta([\nu^{a'_x}\rho, \nu^{i_x}\rho]) \otimes \delta([\nu^{i_{x+1}+1}\rho, \nu^{b_{x+1}}\rho]) \otimes \cdots \otimes \delta([\nu^{i_k+1}\rho, \nu^{b_k}\rho]).$$

Transitivity of Jacquet modules implies that the Jacquet module of σ with respect to the appropriate parabolic subgroup contains

$$\delta([\nu^{a'_1}\rho, \nu^{b_1}\rho]) \otimes \cdots \otimes \delta([\nu^{i_x+1}\rho, \nu^{b_x}\rho]) \otimes \delta([\nu^{a'_x}\rho, \nu^{i_x}\rho]) \otimes \cdots \otimes \delta([\nu^{i_k+1}\rho, \nu^{b_k}\rho])$$
$$\otimes \delta([\nu^{a-k+1}\rho, \nu^{i_1}\rho]) \otimes \delta([\nu^{a-k+2}\rho, \nu^{i_2}\rho]) \otimes \cdots \otimes \delta([\nu^{a}\rho, \nu^{i_k}\rho]) \otimes \sigma_{cusp}.$$

By the exactness and transitivity of Jacquet modules, there is an irreducible subquotient $\tau_1 \otimes \sigma''$ of $r_{(t')}(\sigma)$, for appropriate t', such that the Jacquet module of σ'' with respect to the appropriate parabolic subgroup contains

$$\delta([\nu^{a'_x}\rho, \nu^{i_x}\rho]) \otimes \delta([\nu^{i_{x+1}+1}\rho, \nu^{b_{x+1}}\rho]) \otimes \cdots \otimes \delta([\nu^{i_k+1}\rho, \nu^{b_k}\rho])$$
$$\otimes \delta([\nu^{a-k+1}\rho, \nu^{i_1}\rho]) \otimes \delta([\nu^{a-k+2}\rho, \nu^{i_2}\rho]) \otimes \cdots \otimes \delta([\nu^{a}\rho, \nu^{i_k}\rho]) \otimes \sigma_{cusp}.$$

Observe that σ'' is an irreducible representation of $Sp(n-t')$. From what has already been proved, we conclude that σ'' must be isomorphic to some $\sigma_{(i'_1, i'_2, \ldots, i'_{k'})}$, where $(i'_1, i'_2, \ldots, i'_{k'})$ is an acceptable k'-tuple. It follows that σ'' is a subrepresentation of

$$\delta([\nu^{a-k'+1}\rho, \nu^{i'_1}\rho]) \times \delta([\nu^{a-k'+2}\rho, \nu^{i'_2}\rho]) \times \cdots \times \delta([\nu^{a}\rho, \nu^{i'_{k'}}\rho]) \rtimes \sigma_{cusp}.$$

It is easy to conclude that $k' = k$. We proceed by analyzing Jacquet modules of the representation σ''.

Strong positivity of the representation σ'' and Lemma 2.2 imply that

$$r_{(n-t'-n')}(\sigma'') \leq \delta([\nu^{a-k+1}\rho, \nu^{i'_1}\rho]) \times \delta([\nu^{a-k+2}\rho, \nu^{i'_2}\rho]) \times \cdots \times \delta([\nu^{a}\rho, \nu^{i'_k}\rho]) \otimes \sigma_{cusp}.$$

Thus, the exactness and transitivity of Jacquet modules yield that there is an irreducible representation $\tau_2 \otimes \delta([\nu^a\rho, \nu^{i_k}\rho])$ contained in $m^*(\delta([\nu^{a-k+1}\rho, \nu^{i'_1}\rho]) \times \cdots \times \delta([\nu^a\rho, \nu^{i'_k}\rho]))$ such that the Jacquet module of τ_2 with respect to the appropriate parabolic subgroup contains

$$\delta([\nu^{a'_x}\rho, \nu^{i_x}\rho]) \otimes \delta([\nu^{i_{x+1}+1}\rho, \nu^{b_{x+1}}\rho]) \otimes \cdots \otimes \delta([\nu^{i_k+1}\rho, \nu^{b_k}\rho])$$
$$\otimes \delta([\nu^{a-k+1}\rho, \nu^{i_1}\rho]) \otimes \delta([\nu^{a-k+2}\rho, \nu^{i_2}\rho]) \otimes \cdots \otimes \delta([\nu^{a-1}\rho, \nu^{i_k-1}\rho]).$$

Using (2.2), we obtain that there exist indices $s_1, s_2, \ldots, s_k$, $a-k+j-1 \leq s_j \leq i'_j$ for $j = 1, 2, \ldots, k$, such that $\delta([\nu^a\rho, \nu^{i_k}\rho])$ is a subquotient of $\delta([\nu^{a-k+1}\rho, \nu^{s_1}\rho]) \times \delta([\nu^{a-k+2}\rho, \nu^{s_2}\rho]) \times \cdots \times \delta([\nu^a\rho, \nu^{s_k}\rho])$. Obviously, this gives $s_j = a-k+j-1$ for $j = 1, \ldots, k-1$, and $s_k = i_k \leq i'_k$. So, τ_2 is an irreducible subquotient of the induced representation

$$\delta([\nu^{a-k+1}\rho, \nu^{i'_1}\rho]) \times \delta([\nu^{a-k+2}\rho, \nu^{i'_2}\rho]) \times \cdots \times \delta([\nu^{i_k+1}\rho, \nu^{i'_k}\rho]).$$

Proceeding in the same fashion, we obtain $i_j \leq i'_j$ for all $j = 1, 2, \ldots, k$. Also, there is an irreducible subquotient τ_3 of $\delta([\nu^{i_1+1}\rho, \nu^{i'_1}\rho]) \times \delta([\nu^{i_2+1}\rho, \nu^{i'_2}\rho]) \times \cdots \times \delta([\nu^{i_k+1}\rho, \nu^{i'_k}\rho])$ which contains

$$\delta([\nu^{a'_x}\rho, \nu^{i_x}\rho]) \otimes \delta([\nu^{i_{x+1}+1}\rho, \nu^{b_{x+1}}\rho]) \otimes \cdots \otimes \delta([\nu^{i_k+1}\rho, \nu^{b_k}\rho])$$

in its Jacquet module with respect to the appropriate parabolic subgroup.

Repeated application of the formula (2.2) enables us to conclude that $b_j \leq i'_j$ for $j = x+1, x+2, \ldots, k$. Further, $\delta([\nu^{a'_x}\rho, \nu^{i_x}\rho])$ is an irreducible subquotient of the representation

$$\delta([\nu^{i_1+1}\rho, \nu^{i'_1}\rho]) \times \cdots \times \delta([\nu^{i_x+1}\rho, \nu^{i'_x}\rho]) \times \delta([\nu^{b_{x+1}+1}\rho, \nu^{i'_{x+1}}\rho]) \times \cdots \times \delta([\nu^{b_k+1}\rho, \nu^{i'_k}\rho]).$$

Clearly, this forces $b_j \geq i'_j$ for $j = x+1, x+2, \ldots, k$ and $i_x \geq i'_x$. Besides that, there is some $1 \leq l \leq x-1$ such that $i'_l = i_x$. Therefore, $i'_l \geq i'_x$, for $l < x$, contradicting acceptability of the k-tuple $(i'_1, i'_2, \ldots, i'_k)$. This proves the theorem. □

Let us denote by $Acc(\sigma)$ the set of all acceptable k-tuples in the sense of Definition 4.3. We gather the results of this section in the following theorem:

Theorem 4.6. *Let σ denote a strongly positive discrete series of $Sp(n)$ whose cuspidal support is contained in $D(\rho, \sigma_{cusp})$. The following equality holds in $\mathcal{G} \otimes \mathcal{R}$:*

$$\mu^*(\sigma) = \sum_{(i_1, i_2, \ldots, i_k) \in Acc(\sigma)} L(\delta([\nu^{i_1+1}\rho, \nu^{b_1}\rho]), \ldots, \delta([\nu^{i_k+1}\rho, \nu^{b_k}\rho])) \otimes \sigma_{(i_1, i_2, \ldots, i_k)}.$$

5. Jacquet modules of strongly positive representations of the symplectic group: General case

In this section we prove our results in the general case. Let σ denote a strongly positive discrete series representation of $Sp(n)$. Suppose that σ is contained in $D(\rho_1, \ldots, \rho_m; \sigma_{cusp})$, with m minimal and each ρ_i a unitary self-contragredient representation. We may suppose that $m \geq 1$. Let $a_{\rho_i} > 0$ such that $\nu^{a_{\rho_i}}\rho_i \rtimes \sigma_{cusp}$ reduces. We realize σ as the unique irreducible subrepresentation of the representation of the form (2.1), i.e.,

$$\sigma \hookrightarrow (\prod_{i=1}^{m} \prod_{j=1}^{k_i} \delta([\nu^{a_{\rho_i}-k_i+j}\rho_i, \nu^{b_j^{(i)}}\rho_i])) \rtimes \sigma_{cusp},$$

with each k_i minimal, for $i = 1, 2, \ldots, m$ (in this way we again exclude all perhaps empty segments appearing in (2.1)).

We start with a generalization of the definition given in the previous section.

Definition 5.1. We call an ordered m-tuple of the form

$$((i_1^{(1)}, i_2^{(1)}, \ldots, i_{k_1}^{(1)}), (i_1^{(2)}, i_2^{(2)}, \ldots, i_{k_2}^{(2)}), \ldots, (i_1^{(m)}, i_2^{(m)}, \ldots, i_{k_m}^{(m)}))$$

acceptable if the following holds:

- $i_1^{(j)} < i_2^{(j)} < \cdots < i_{k_j}^{(j)}$, for $j = 1, 2, \ldots, m$,
- $i_{l_j}^{(j)} - a_{\rho_j} \in \mathbb{Z}$, for $j = 1, 2, \ldots, m$, $l_j = 1, 2, \ldots, k_j$,
- $a_{\rho_j} - k_j + l_j - 1 \leq i_{l_j}^{(j)} \leq b_{l_j}^{(j)}$, for $j = 1, 2, \ldots, m$, $l_j = 1, 2, \ldots, k_j$.

To shorten the notation, we sometimes write s_j instead of $(i_1^{(j)}, i_2^{(j)}, \ldots, i_{k_j}^{(j)})$, for $j = 1, 2, \ldots, m$.

Now we are ready to analyze $\mu^*(\sigma)$. We apply arguments similar to those in the previous section.

The following result is a straightforward generalization of Lemma 3.1 (we just recall that the representations $\delta([\nu^{x_1}\rho, \nu^{y_1}\rho]) \times \delta([\nu^{x_2}\rho', \nu^{y_2}\rho'])$ and $\delta([\nu^{x_2}\rho', \nu^{y_2}\rho']) \times \delta([\nu^{x_1}\rho, \nu^{y_1}\rho])$ are isomorphic for non-isomorphic ρ and ρ').

Lemma 5.2. *Let $\rho_1, \ldots, \rho_l$ denote irreducible unitarizable cuspidal representations of $GL(n_1, F), \ldots, GL(n_l, F)$. Let π be an irreducible subquotient of the representation*

$$\delta([\nu^{c_1^{(1)}}\rho_1, \nu^{d_1^{(1)}}\rho_1]) \times \cdots \times \delta([\nu^{c_{n_1}^{(1)}}\rho_1, \nu^{d_{n_1}^{(1)}}\rho_1]) \times \delta([\nu^{c_1^{(2)}}\rho_2, \nu^{d_1^{(2)}}\rho_2]) \times \cdots$$
$$\times \delta([\nu^{c_{n_2}^{(2)}}\rho_2, \nu^{d_{n_2}^{(2)}}\rho_2]) \times \cdots \times \delta([\nu^{c_1^{(l)}}\rho_l, \nu^{d_1^{(l)}}\rho_l]) \times \cdots \times \delta([\nu^{c_{n_l}^{(l)}}\rho_l, \nu^{d_{n_l}^{(l)}}\rho_l]),$$

where $d_1^{(j)} \leq d_2^{(j)} \leq \ldots \leq d_{n_j}^{(j)}$ for every $1 \leq j \leq l$. Then,

$$\begin{aligned}\pi \;=\; & L(\delta([\nu^{c_1'^{(1)}}\rho_1, \nu^{d_1^{(1)}}\rho_1]), \ldots, \delta([\nu^{c_{n_1}'^{(1)}}\rho_1, \nu^{d_{n_1}^{(1)}}\rho_1]), \delta([\nu^{c_1'^{(2)}}\rho_2, \nu^{d_1^{(2)}}\rho_2]), \ldots, \\ & \delta([\nu^{c_{n_2}'^{(2)}}\rho_2, \nu^{d_{n_2}^{(2)}}\rho_2]), \ldots, \delta([\nu^{c_1'^{(l)}}\rho_l, \nu^{d_1^{(l)}}\rho_l]), \ldots, \delta([\nu^{c_{n_l}'^{(l)}}\rho_l, \nu^{d_{n_l}^{(l)}}\rho_l])),\end{aligned}$$

where each $c_1'^{(i)}, c_2'^{(i)}, \ldots, c_{n_i}'^{(i)}$ is some permutation of $c_1^{(i)}, c_2^{(i)}, \ldots, c_{n_i}^{(i)}$, for $i = 1, 2, \ldots, l$.

Let $\tau \otimes \sigma'$ be an irreducible subquotient of $r_{(t)}(\sigma)$, for some t. Using Lemma 2.2 as before, we obtain that there exist indices $i_{l_j}^{(j)}$, $j = 1, \ldots, m$, $l_j = 1, \ldots, k_j$, with $i_{l_j}^{(j)} - a_{\rho_j} \in \mathbb{Z}$ and $a_{\rho_j} - k_j + l_j - 1 \leq i_{l_j}^{(j)} \leq b_{l_j}^{(j)}$, such that σ' is subquotient of

$$(\prod_{j=1}^{m} \prod_{l_j=1}^{k_j} \delta([\nu^{a_{\rho_j} - k_j + l}\rho_j, \nu^{i_{l_j}^{(j)}}\rho_j])) \rtimes \sigma_{cusp}. \tag{5.1}$$

Lemma 3.4 implies that σ' is a strongly positive discrete series. Analysis similar to that in the proof of Proposition 4.1 (using the results from the fifth section of [10] and Lemma 5.2), shows that the m-tuple $((i_1^{(1)}, \ldots, i_{k_1}^{(1)}), \ldots, (i_1^{(m)}, \ldots, i_{k_m}^{(m)}))$ is acceptable in the sense of Definition 5.1 and that σ' is a subrepresentation of the induced representation (5.1). We denote such a representation by $\sigma_{(s_1, \ldots, s_m)}$, where $s_j = (i_1^{(j)}, \ldots, i_{k_j}^{(j)})$, for $1 \leq j \leq m$.

Repeating the arguments from the proof of Proposition 4.2 and those after Definition 4.3, we deduce that for each acceptable m-tuple $(s_1, \ldots, s_m)$ there exists some irreducible representation τ such that $\tau \otimes \sigma_{(s_1, \ldots, s_m)}$ is an irreducible subquotient of $r_{(t)}(\sigma)$, for appropriate t.

The task is now to determine the GL-parts in $\mu^*(\sigma)$. So, suppose that $\tau \otimes \sigma_{(s_1,\dots,s_m)}$ is an irreducible representation appearing in $\mu^*(\sigma)$, where τ is an irreducible representation of the general linear group and $(s_1, \dots, s_m)$ is an acceptable m-tuple in the sense of Definition 5.1. By a reasoning completely analogous to that used in the proofs of Lemma 4.4 and Proposition 4.5, combined with Lemma 5.2, we get

$$\begin{aligned}\tau &= L(\delta([\nu^{a_{\rho_1}-k_{\rho_1}+1}\rho_1, \nu^{i_1^{(1)}}\rho_1]), \dots, \delta([\nu^{a_{\rho_1}}\rho_1, \nu^{i_{k_1}^{(1)}}\rho_1]), \dots, \\ &\quad \delta([\nu^{a_{\rho_m}-k_{\rho_m}+1}\rho_m, \nu^{i_1^{(m)}}\rho_m]), \dots, \delta([\nu^{a_{\rho_m}}\rho_m, \nu^{i_{k_m}^{(m)}}\rho_m])),\end{aligned}$$

where $s_j = (i_1^{(j)}, i_2^{(j)}, \dots, i_{k_j}^{(j)})$, $1 \leq j \leq m$. We denote such a representation τ by $L((s_1, s_2, \dots, s_k))$.

We denote by $Acc'(\sigma)$ the collection of all acceptable m-tuples in the sense of Definition 5.1. In the following theorem, which follows from the previous discussion, we give an explicit description of the Jacquet modules of strongly positive discrete series σ with respect to the maximal parabolic subgroups:

Theorem 5.3. *The following equality holds in $\mathcal{G} \otimes \mathcal{R}$:*

$$\mu^*(\sigma) = \sum_{(s_1,s_2,\dots,s_m)\in Acc'(\sigma)} L((s_1, s_2, \dots, s_m)) \otimes \sigma_{(s_1,s_2,\dots,s_m)}.$$

We emphasize that all our proofs regarding Jacquet modules of strongly positive representations of symplectic groups can be applied in an entirely analogous manner to such representations of special odd-orthogonal groups, since a completely analogous description of standard parabolic subgroups, classification of strongly positive discrete series and Tadić's structure formula ([18, Theorem 6.5]) hold for these groups.

6. Jacquet modules of strongly positive representations of the metaplectic group

The purpose of this section is to show how the results established in the previous sections can be extended to the metaplectic group $\widetilde{Sp(n)}$ over a non-Archimedean local field F of characteristic different from two.

Thus, let σ denote a strongly positive discrete series of $\widetilde{Sp(n)}$ which we realize, due to Theorem 5.3 of [10], as a unique irreducible subrepresentation of the induced representation

$$(\prod_{i=1}^{m}\prod_{j=1}^{k_i}\delta([\nu^{a_{\rho_i}-k_i+j}\rho_i, \nu^{b_j^{(i)}}\rho_i])) \rtimes \sigma_{cusp},$$

with m minimal and each k_i minimal, for $i = 1, 2, \dots, m$. Here each ρ_i, $i = 1, 2, \dots, m$, denotes an irreducible genuine unitary self-dual cuspidal representation of $\widetilde{GL(n_i, F)}$ such that the induced representation $\nu^{a_{\rho_i}}\rho_i \rtimes \sigma_{cusp}$ reduces.

Let $\tau \otimes \sigma'$ denote an irreducible representation appearing in $\mu_1^*(\sigma)$.

First we observe that σ' is a strongly positive discrete series. The main tool in the proof of this fact is Lemma 26 in [2], which states that an irreducible cuspidal subquotient is a quotient, and which can be applied in our situation, as is explained in detail in the proof of Lemma 3.1 in [6]. As soon as this is established, we can proceed with the proof similarly as in the proof of Lemma 3.4.

The arguments used in proofs of the results in Section 4 rely on the Jacquet modules method, which also applies to group $\widetilde{Sp(n)}$. Moreover, since every representation ρ_i, $i = 1, 2, \ldots, m$, is self-dual, all the calculations made in the symplectic case in Section 4 using Lemma 2.2 can be directly carried over to the metaplectic case, using Theorem 2.3. We note that the isomorphisms of the induced genuine representations of the metaplectic groups, analogous to those used in the proof of Proposition 4.1 and after Definition 4.3, follow from Proposition 4.3 of [5].

It is now easily seen that there is some m-tuple $(s_1, \ldots, s_m)$, acceptable in the sense of Definition 5.1, such that σ' is the unique irreducible subrepresentation of the induced representation of the form (5.1), where $s_j = (i_1^{(j)}, \ldots, i_{k_j}^{(j)})$, for $j = 1, 2, \ldots, m$. Following the notation introduced in the previous section, we denote such a representation by $\sigma_{(s_1,\ldots,s_m)}$ and let $Acc'(\sigma)$ denote the collection of all acceptable m-tuples in the sense of Definition 5.1. Further, analysis similar to that in the proof of Proposition 4.2 and after Definition 4.3 shows that for every $(s_1, \ldots, s_m) \in Acc'(\sigma)$ exists some irreducible genuine representation τ such that $\tau \otimes \sigma_{(s_1,\ldots,s_m)}$ appears as an irreducible subquotient of $r_{(t)}(\sigma)$, for appropriate t.

It remains to describe the $\widetilde{GL}$-parts of the irreducible members of $\mu_1^*(\sigma)$.

Since the proof of Lemma 3.1 is entirely based on algebraic techniques and the Langlands classification, which hold for representations of the two-fold covers of general linear groups, that proof carries over directly to the irreducible genuine representations of $\widetilde{GL(k, F)}$. Again, it is a simple matter to obtain an analogous statement of Lemma 5.2 for genuine representations $\rho_1, \rho_2, \ldots, \rho_m$ using the results mentioned above. This puts us in position to apply the same arguments as in the symplectic case to deduce that if an irreducible genuine representation $\tau \otimes \sigma_{(s_1,\ldots,s_m)}$ appears as an irreducible subquotient of $r_{(t)}(\sigma)$, then τ is isomorphic to $L((s_1, s_2, \ldots, s_m))$, where $L((s_1, s_2, \ldots, s_m))$ is defined as in the previous section.

Summarizing, we have the following description of Jacquet modules of strongly positive discrete series of the metaplectic group:

Theorem 6.1. *The following equality holds in $\mathcal{G}^{gen} \otimes \mathcal{S}$:*

$$\mu_1^*(\sigma) = \sum_{(s_1,s_2,\ldots,s_m)\in Acc'(\sigma)} L((s_1, s_2, \ldots, s_m)) \otimes \sigma_{(s_1,s_2,\ldots,s_m)}.$$

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References

1. I. N. Bernstein and A. V. Zelevinsky, *Induced representations of reductive p-adic groups. I*, Ann. Sci. École Norm. Sup. (4) **10** (1977), no. 4, 441–472. MR0579172 (58:28310)
2. J. Bernstein, *Draft of: Representations of p-adic groups.*, Lectures at Harvard University, 1992, written by Karl E. Rumelhart.
3. W. Casselman, *Introduction to the theory of admissible representations of p-adic reductive groups*, preprint, avaliable at http://www.math.ubc.ca/~cass/research/pdf/p-adic-book.pdf.
4. Marcela Hanzer and Goran Muić, *On an algebraic approach to the Zelevinsky classification for classical p-adic groups*, J. Algebra **320** (2008), no. 8, 3206–3231. MR2450724
5. ———, *Parabolic induction and Jacquet functors for metaplectic groups*, J. Algebra **323** (2010), no. 1, 241–260. MR2564837

6. ______, *Rank one reducibility for metaplectic groups via theta correspondence*, Canad. J. Math. **63** (2011), no. 3, 591–615. MR2828534
7. Chris Jantzen, *Jacquet modules of p-adic general linear groups*, Represent. Theory **11** (2007), 45–83. MR2306606 (2008g:22023)
8. Stephen S. Kudla, *On the local theta-correspondence*, Invent. Math. **83** (1986), no. 2, 229–255. MR818351 (87e:22037)
9. ______, *Notes on the local theta correspondence (lectures at the European School in Group Theory)*,(1996).
10. Ivan Matić, *Strongly positive representations of metaplectic groups*, J. Algebra **334** (2011), 255–274. MR2787663
11. Colette Mœglin, *Sur la classification des séries discrètes des groupes classiques p-adiques: paramètres de Langlands et exhaustivité*, J. Eur. Math. Soc. (JEMS) **4** (2002), no. 2, 143–200. MR1913095 (2003g:22021)
12. Colette Mœglin and Marko Tadić, *Construction of discrete series for classical p-adic groups*, J. Amer. Math. Soc. **15** (2002), no. 3, 715–786. MR1896238 (2003g:22020)
13. Goran Muić, *Composition series of generalized principal series; the case of strongly positive discrete series*, Israel J. Math. **140** (2004), 157–202. MR2054843 (2005e:22018)
14. ______, *On the non-unitary unramified dual for classical p-adic groups*, Trans. Amer. Math. Soc. **358** (2006), no. 10, 4653–4687. MR2231392 (2007j:22029)
15. ______, *A geometric construction of intertwining operators for reductive p-adic groups*, Manuscripta Math. **125** (2008), no. 2, 241–272. MR2373084
16. R. Ranga Rao, *On some explicit formulas in the theory of Weil representation*, Pacific J. Math. **157** (1993), no. 2, 335–371. MR1197062 (94a:22037)
17. Allan J. Silberger, *Special representations of reductive p-adic groups are not integrable*, Ann. of Math. (2) **111** (1980), no. 3, 571–587. MR577138 (82k:22015)
18. Marko Tadić, *Structure arising from induction and Jacquet modules of representations of classical p-adic groups*, J. Algebra **177** (1995), no. 1, 1–33. MR1356358 (97b:22023)
19. ______, *On reducibility of parabolic induction*, Israel J. Math. **107** (1998), 29–91. MR1658535 (2001d:22012)
20. ______, *On regular square integrable representations of p-adic groups*, Amer. J. Math. **120** (1998), no. 1, 159–210. MR1600280 (99h:22026)
21. A. V. Zelevinsky, *Induced representations of reductive p-adic groups. II. On irreducible representations of $GL(n)$*, Ann. Sci. École Norm. Sup. (4) **13** (1980), no. 2, 165–210. MR584084 (83g:22012)

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