Composition factors of a class of induced representations of classical p-adic groups

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Abstract

We study induced representations of the form $\delta_1 \times \delta_2 \rtimes \sigma$, where δ_1, δ_2 are irreducible essentially square-integrable representations of general linear group and σ is a strongly positive discrete series of classical p-adic group, which naturally appear in the non-unitary dual. For $\delta_1 = \delta([\nu^a \rho_1, \nu^b \rho_1])$ and $\delta_2 = \delta([\nu^c \rho_2, \nu^d \rho_2])$ with $a \geq 1$ and $c \geq 1$, we determine composition factors of such induced representation.

1 Introduction

Let us denote by G_n either symplectic or odd special orthogonal group of rank n over a nonarchimedean local field F of characteristic different than two.

According to the Langlands classification, every irreducible admissible representation of G_n can be obtained as an irreducible subrepresentation of the induced representation of the form

$$\delta_1 \times \delta_2 \times \cdots \times \delta_k \rtimes \tau$$

where τ is a tempered representation of some $G_{n'}$, while δ_i is an irreducible essentially square-integrable representation of $GL(n_i, F)$, for $i = 1, 2, \ldots, k$, subject to certain constraints on the central exponents of representations δ_i

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(we refer the reader to Section 2, where this is discussed in more detail for the groups under consideration). By the results of [16], δ_i is of the form

$$\delta([\nu^{a_i}\rho_i, \nu^{b_i}\rho_i]),$$

for $a_i, b_i \in \mathbb{R}$ be such that $b_i - a_i \in \mathbb{Z}$, and an irreducible cuspidal representation ρ_i of $GL(n_{\rho_i}, F)$.

It is a result of Harish-Chandra that a tempered representation of $G_{n'}$ embeds in an induced representation of the form $\delta_1 \times \cdots \times \delta_{k'} \rtimes \sigma$, where $\delta_1, \ldots, \delta_{k'}$ are irreducible essentially square-integrable representations of general linear groups, and σ is an irreducible square-integrable representation of $G_{n''}$, $n'' \leq n'$.

Combining this result with the Langlands classification and more recent classification of the square-integrable unitary duals of classical groups over nonarchimedean local fields, given in [12], one can deduce that every irreducible admissible representation of G_n can be obtained as an irreducible subquotient of the induced representation of the form

$$\delta_1 \times \delta_2 \times \cdots \times \delta_{k''} \rtimes \sigma_{sp}, \tag{1}$$

for irreducible essentially square-integrable representations $\delta_1, \delta_2, \ldots, \delta_{k''}$ of general linear groups and a strongly positive discrete series σ_{sp} of $G_{n'''}$ for some $n''' \leq n$.

Consequently, strongly positive representations can be observed as the basic building blocks in known construction of the non-unitary duals of classical groups over nonarchimedean local fields, and it is of particular interest to obtain deeper insight into the explicit description of the composition series of representations of the form (1). We note that an algebraic classification of strongly positive discrete series of metaplectic groups, which also holds in the case of classical groups, is given in [4], and some further properties of strongly positive representations have been studied in [5], [6] and [10].

Composition factors of the generalized principal series $\delta([\nu^a\rho, \nu^b\rho]) \rtimes \sigma_{sp}$ have been completely determined by Muić in [13], in terms of the Mœglin-Tadić classification of discrete series. This classification relies on the Basic Assumption, which now follows from the recent work of Arthur ([1]) and [11, Théorème 3.1.1] (for more details on the Basic Assumption, we refer the reader to [12, Section 2]). There are three essentially different cases which appear in [13], and each of them has to be studied separately: $a \geq 1$, $a \leq 0$ and $a = \frac{1}{2}$.

In this paper, we go one step further by giving an explicit description of the composition factors of the induced representation of the form

$$\delta([\nu^a \rho_1, \nu^b \rho_1]) \times \delta([\nu^c \rho_2, \nu^d \rho_2]) \rtimes \sigma_{sp}, \tag{2}$$

where $a \geq 1, c \geq 1$, and σ_{sp} is a strongly positive discrete series. An advantage of the considered case is that every tempered subquotient of the induced representation (2) is strongly positive, and our description presents a natural continuation of the problem considered in [13].

We note that it is a direct consequence of the results of [13] and [9, Proposition 3.2], that the representation $\delta([\nu^a \rho, \nu^b \rho]) \rtimes \sigma_{sp}$ is multiplicity one, and can be of length one, two, three or four. On the other hand, our results show that the induced representation (2) is again multiplicity one, which can be of length one to eight, but not of length five.

The main role in our determination is played by the necessary and sufficient conditions under which the induced representation of the form (2) contains a strongly positive subquotient, which are given in [7] and based on the description of Jacquet modules of discrete series of particular type, given in [8]. Those results enable us to obtain candidates for irreducible subquotients of the induced representation (2), calculating various Jacquet modules of such induced representation. To determine whether an obtained irreducible representation is contained in the composition series of (2), we use sometimes rather involved arguments based on the combination of the intertwining operators method and the Jacquet modules method.

Let us describe the contents of the paper in more detail.

Section 2 provides in details the main notation and ingredients needed in this work. In the third section we recall some known results which will be used in the paper. Also, in that section we prove some technical results which enable us to determine all possible irreducible subquotients of the induced representation $\delta([\nu^a \rho_1, \nu^b \rho_1]) \times \delta([\nu^c \rho_2, \nu^d \rho_2]) \rtimes \sigma_{sp}$. Detailed analysis of the composition factors of the representation $\delta([\nu^a \rho_1, \nu^b \rho_1]) \times \delta([\nu^c \rho_2, \nu^d \rho_2]) \rtimes \sigma_{sp}$ in the case when $\delta([\nu^a \rho_1, \nu^b \rho_1]) \times \delta([\nu^c \rho_2, \nu^d \rho_2])$ is irreducible is provided in Section 4. Description of the composition factors in the remaining case is given in Section 5.

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2 Preliminaries

Let F denote a nonarchimedean local field of characteristic different than two. The groups we are considering are of the following form: we have a tower of symplectic or (full) orthogonal groups $G_n = G(V_n)$, which are the groups of isometries of F-vector spaces V_n endowed with the non-degenerate form which is skew-symmetric if the tower is symplectic and symmetric otherwise. Here n stands for the split rank of the group G_n, $n \geq 0$.

The set of standard parabolic subgroups will be fixed in a usual way, i.e., we fix a minimal F-parabolic subgroup in G_n consisting of upper-triangular matrices in the usual matrix realization of the classical group. Then the Levi factors of standard parabolic subgroups have the form $M = GL(n_1, F) \times \cdots \times GL(n_k, F) \times G_{n'}$, where $GL(m, F)$ denotes a general linear group of rank m over F. If $\delta_i, i = 1, 2, \ldots, k$ is a representation of $GL(n_i, F)$ and if τ a representation of G_m, then by $\delta_1 \times \cdots \times \delta_k \rtimes \tau$ we denote a normalized parabolically induced representation of the group G_n, induced from the representation by $\delta_1 \otimes \cdots \otimes \delta_k \otimes \tau$ of the standard parabolic subgroup with the Levi subgroup equal to $GL(n_1, F) \times \cdots \times GL(n_k, F) \times G_m$. Here n equals $n_1 + n_2 + \cdots + n_k + m$.

The set of all irreducible admissible representations of $GL(n, F)$ will be denoted by $\mathrm{Irr}(GL(n, F))$, and the set of all irreducible admissible representations of G_n will be denoted by $\mathrm{Irr}(G_n)$. Let $R(GL(n, F))$ denote the Grothendieck group of admissible representations of finite length of $GL(n, F)$ and define $R(GL) = \oplus_{n \geq 0} R(GL(n, F))$. Similarly, let $R(G_n)$ stand for the Grothendieck group of admissible representations of finite length of G_n and define $R(G) = \oplus_{n \geq 0} R(G_n)$.

We will denote by ν a composition of the determinant mapping with the normalized absolute value on F. Let ρ denote an irreducible cuspidal representation of $GL(k, F)$. By a segment of cuspidal representations, which will be denoted by $[\rho, \nu^m \rho]$, we mean the set $\{\rho, \nu\rho, \ldots, \nu^m \rho\}$. To each such segment we attach an irreducible essentially square-integrable representation $\delta([\rho, \nu^m \rho])$ of $GL(m \cdot k, F)$, which is a unique irreducible subrepresentation of $\nu^m \rho \times \cdots \times \nu\rho \times \rho$ (here we use a well known notation introduced in [16] for the normalized parabolic induction for the general linear groups with the usual choice of the standard parabolic subgroups). For integers x, y, $x \leq y$, we set $[x, y] = \{z \in \mathbb{Z} : x \leq z \leq y\}$. For irreducible essentially square-integrable representation δ, there is the unique $e(\delta) \in \mathbb{R}$ such that $\nu^{-e(\delta)}\delta$ is unitarizable.

In order to keep our results uniform, we put $\delta([\nu^a\rho, \nu^b\rho]) = 1$ (the one-dimensional representation of the trivial group) if $a = b-1$ and $\delta([\nu^a\rho, \nu^b\rho]) = 0$ if $a < b-1$.

Throughout the paper we prefer to use the subrepresentation version of the Langlands classification and write the non-tempered representation $\pi \in \mathrm{Irr}(G_n)$ as the unique irreducible (Langlands) subrepresentation of the induced representation of the form $\delta_1 \times \delta_2 \times \cdots \times \delta_k \rtimes \tau$, where $\tau \in \mathrm{Irr}(G_{n'})$ is a tempered representation and $\delta_1, \delta_2, \ldots, \delta_k$ are irreducible essentially square-integrable representations such that $e(\delta_1) \leq e(\delta_2) \leq \cdots \leq e(\delta_k) < 0$. In this case, we write $\pi = L(\delta_1 \times \delta_2 \times \cdots \times \delta_k \rtimes \tau)$. Also, for simplicity of the notation, if $\tau \in \mathrm{Irr}(G_n)$ is a tempered representation and $\delta_1, \delta_2, \ldots, \delta_k$ are irreducible essentially square-integrable representations such that $e(\delta_i) < 0$ for all $i = 1, 2, \ldots, k$, we define $L(\delta_1, \delta_2, \ldots, \delta_k, \tau)$ to be $L(\delta_{i_1} \times \delta_{i_2} \times \cdots \times \delta_{i_k} \rtimes \tau)$, where $\delta_{i_1}, \delta_{i_2} \ldots, \delta_{i_k}$ is a permutation of $\delta_1, \delta_2, \ldots, \delta_k$ such that $e(\delta_{i_1}) \leq e(\delta_{i_2}) \leq \cdots \leq e(\delta_{i_k})$. We note that similar notation has been used in [14].

Using Jacquet modules for the maximal standard parabolic subgroups of $GL(n, F)$ we can also define $m^*(\pi) = \sum_{k=0}^{n} \mathrm{s.s.}(r_{(k)}(\pi)) \in R(GL) \otimes R(GL)$, for an irreducible representation π of $GL(n, F)$, and then extend m^* linearly to the whole of $R(GL)$. Here $r_{(k)}(\pi)$ denotes the normalized Jacquet module of π with respect to the standard parabolic subgroup having Levi factor equal to $GL(k, F) \times GL(n-k, F)$.

We will frequently use the following equation:

$$m^*(\delta([\nu^a\rho, \nu^b\rho])) = \sum_{i=a-1}^{b} \delta([\nu^{i+1}\rho, \nu^b\rho]) \otimes \delta([\nu^a\rho, \nu^i\rho]).$$

Note that the multiplicativity of m^* implies

$$m^*(\prod_{j=1}^{n} \delta([\nu^{a_j}\rho_j, \nu^{b_j}\rho_j])) =$$
$$= \prod_{j=1}^{n} (\sum_{i_j=a_j-1}^{b_j} \delta([\nu^{i_j+1}\rho_j, \nu^{b_j}\rho_j]) \otimes \delta([\nu^{a_j}\rho_j, \nu^{i_j}\rho_j])).$$

For representation $\sigma \in R(G_n)$, the normalized Jacquet module of σ with respect to the parabolic subgroup P will be denoted by $r_P(\sigma)$. Furthermore, for $1 \leq k \leq n$ we denote by $r_{(k)}(\sigma)$ the normalized Jacquet module of σ with respect to the parabolic subgroup $P_{(k)}$ having Levi subgroup equal to $GL(k,F) \times G_{n-k}$. We identify $r_{(k)}(\sigma)$ with its semisimplification in $R(GL(k,F)) \otimes R(G_{n-k})$ and consider

$$\mu^*(\sigma) = 1 \otimes \sigma + \sum_{k=1}^{n} r_{(k)}(\sigma) \in R(GL) \otimes R(G).$$

We take a moment to state the crucial structural formula for our calculations with Jacquet modules ([15]), which is a version of the Geometrical Lemma of Bernstein and Zelevinsky ([2]).

Lemma 2.1. *Let $\rho \in Irr(GL(m,F))$ be a cuspidal representation and $k,l \in \mathbb{R}$ such that $k+l \in \mathbb{Z}_{\geq 0}$. Let σ be an admissible representation of finite length of G_n. Write $\mu^*(\sigma) = \sum_{\tau,\sigma'} \tau \otimes \sigma'$. Then the following holds:*

$$\begin{aligned} \mu^*(\delta([\nu^{-k}\rho, \nu^l\rho]) \rtimes \sigma) &= \sum_{i=-k-1}^{l} \sum_{j=i}^{l} \sum_{\tau,\sigma'} \delta([\nu^{-i}\widetilde{\rho}, \nu^k\widetilde{\rho}]) \times \delta([\nu^{j+1}\rho, \nu^l\rho]) \times \tau \\ &\otimes \quad \delta([\nu^{i+1}\rho, \nu^j\rho]) \rtimes \sigma'. \end{aligned}$$

An irreducible representation $\sigma \in R(G)$ is called strongly positive if for every embedding

$$\sigma \hookrightarrow \nu^{s_1}\rho_1 \times \nu^{s_2}\rho_2 \times \cdots \times \nu^{s_k}\rho_k \rtimes \sigma_{cusp},$$

where $\rho_i \in R(GL(n_{\rho_i}, F))$, $i = 1, 2, \ldots, k$, are cuspidal unitary representations and $\sigma_{cusp} \in R(G)$ is an irreducible cuspidal representation, we have $s_i > 0$ for each i.

Obviously, every strongly positive representation is square–integrable. Irreducible strongly positive representations are called strongly positive discrete series.

Let us recall an inductive description of non-supercuspidal strongly positive discrete series, which has been obtained in [4]. We note that the partial cuspidal support of an irreducible representation $\sigma \in R(G_n)$ is an irreducible cuspidal representation σ_{cusp} of some $R(n')$, $n' \leq n$, such that there exists a representation π of $GL(n-n',F)$ such that $\sigma \hookrightarrow \pi \rtimes \sigma_{cusp}$.

Proposition 2.2. *Suppose that $\sigma \in R(G_n)$ is an irreducible strongly positive representation and let $\rho \in Irr(GL(m,F))$ denote an irreducible cuspidal representation such that some twist of ρ appears in the cuspidal support of σ.*

We denote by σ_{cusp} *the partial cuspidal support of* σ*. Then there exist unique* $a, b \in \mathbb{R}$ *such that* $a > 0, b > 0$*,* $b - a \in \mathbb{Z}_{\geq 0}$*, and a unique irreducible strongly positive representation* σ' *without* $\nu^a\rho$ *in the cuspidal support, with the property that* σ *is a unique irreducible subrepresentation of* $\delta([\nu^a\rho, \nu^b\rho]) \rtimes \sigma'$*. Furthermore, there is a non-negative integer* l *such that* $a + l = s$*, where* $s > 0$ *is such that* $\nu^s\rho \rtimes \sigma_{cusp}$ *reduces. If* $l = 0$ *there are no twists of* ρ *appearing in the cuspidal support of* σ' *and if* $l > 0$ *there exist a unique* $b' > b$ *and a unique strongly positive discrete series* σ''*, which contains neither* $\nu^a\rho$ *nor* $\nu^{a+1}\rho$ *in its cuspidal support, such that* σ' *can be written as a unique irreducible subrepresentation of* $\delta([\nu^{a+1}\rho, \nu^{b'}\rho]) \rtimes \sigma''$*.*

We emphasize that if $\nu^x\rho$ appears in the cuspidal support of σ then ρ is selfcontragredient and $2x \in \mathbb{Z}$ ([11, 12]).

By the Mœglin-Tadić classification, strongly positive discrete series σ corresponds to so-called admissible triple of alternated type. Admissible triple corresponding to discrete series σ is an ordered triple of the form $(\mathrm{Jord}, \sigma_{cusp}, \epsilon)$, where Jord (the set Jordan blocks) is the set of pairs (c, ρ) where c is an integer of the appropriate parity and ρ an irreducible cuspidal selfcontragredient representation of general linear group, σ_{cusp} is partial cuspidal support of σ and ϵ is a function defined on the subset of $\mathrm{Jord} \times \mathrm{Jord} \cup \mathrm{Jord}$ into $\{\pm 1\}$. For irreducible cuspidal selfcontragredient representation ρ of general linear group we define $\mathrm{Jord}_\rho = \{c : (c, \rho) \in \mathrm{Jord}\}$ and for $c \in \mathrm{Jord}_\rho$ we write c_{-} for maximum of the set $\{c' \in \mathrm{Jord}_\rho : c' < c\}$ if this set is non-empty. An admissible triple $(\mathrm{Jord}, \sigma_{cusp}, \epsilon)$ is called a triple of alternated type if for every $(c, \rho) \in \mathrm{Jord}$ such that c_{-} is defined holds $\epsilon((c_{-}, \rho), (c, \rho)) = -1$. By definition of such triples, a strongly positive discrete series is completely determined by its partial cuspidal support and the set of Jordan blocks. Since all strongly positive discrete series which we study in this paper share a common partial cuspidal support we will define only the set of Jordan blocks when introducing them. Similar procedure has also been summarized in Proposition 1.2 of [13].

Through the paper we denote the set of Jordan blocks corresponding to discrete series σ by $\mathrm{Jord}(\sigma)$ and set $\mathrm{Jord}_\rho(\sigma) = \{c : (c, \rho) \in \mathrm{Jord}(\sigma)\}$.

If $\rho_1 \cong \rho_2$, we may assume $b \leq d$, since in Grothendieck group we have $\delta_1 \times \delta_2 \rtimes \sigma_{sp} = \delta_2 \times \delta_1 \rtimes \sigma_{sp}$. Also, if $\rho_1 \cong \rho_2$ and $b = d$, through the paper we additionally assume $a \leq c$.

3 Some technical results

Proposition 3.1. *Suppose that π is an irreducible subquotient of $\delta_1 \times \delta_2 \rtimes \sigma_{sp}$ and let $\pi \cong L(\delta'_1 \times \delta'_2 \times \cdots \times \delta'_k \rtimes \tau)$, where $\delta'_1, \delta'_2, \ldots, \delta'_k$ are irreducible essentially square-integrable representations such that $e(\delta'_i) < 0$ for all i and $e(\delta'_{i-1}) \leq e(\delta'_i)$ holds for $i = 2, \ldots, k$, and τ is a tempered representation of $G_{n'}$. Then $k \leq 2$ and τ is strongly positive.*

Proof. From the cuspidal support of the induced representation $\delta_1 \times \delta_2 \rtimes \sigma_{sp}$, using description of strongly positive discrete series and $a \geq 1$, $c \geq 1$, we deduce that τ has to be strongly positive.

Let us assume, on the contrary, that there is an irreducible constituent π of $\delta_1 \times \delta_2 \rtimes \sigma_{sp}$ of the form $L(\delta'_1 \times \delta'_2 \times \cdots \times \delta'_k \rtimes \tau)$, with $k \geq 3$. We write $\delta'_i = \delta([\nu^{x_i} \rho'_i, \nu^{y_i} \rho'_i])$. Obviously, we have $x_i + y_i < 0$ for $i = 1, 2, \ldots, k$ and $x_i + y_i \leq x_{i+1} + y_{i+1}$ for $i = 1, 2, \ldots, k-1$.

Frobenius reciprocity implies that $\delta'_1 \otimes \delta'_2 \otimes \cdots \otimes \delta'_k \otimes \tau$ is contained in the Jacquet module of π with respect to the appropriate parabolic subgroup. Since π is subquotient of $\delta_1 \times \delta_2 \rtimes \sigma_{sp}$, using transitivity of Jacquet modules, we deduce that there is an irreducible representation τ_1 of G_{n_1} such that $\mu^*(\delta_1 \times \delta_2 \rtimes \sigma_{sp}) \geq \delta'_1 \otimes \tau_1$. Using Lemma 2.1 twice, we deduce that there are $t_1^{(1)}$, $t_2^{(1)}$, $s_1^{(1)}$, $s_2^{(1)}$ such that $a - 1 \leq t_1^{(1)} \leq s_1^{(1)} \leq b$, $c - 1 \leq t_2^{(1)} \leq s_2^{(1)} \leq d$ and an irreducible constituent $\pi_1 \otimes \sigma_1$ of $\mu^*(\sigma_{sp})$ such that

$$\begin{aligned}\delta([\nu^{x_1} \rho'_1, \nu^{y_1} \rho'_1]) \leq & \delta([\nu^{-t_1^{(1)}} \widetilde{\rho}_1, \nu^{-a} \widetilde{\rho_1}]) \times \delta([\nu^{s_1^{(1)}+1} \rho_1, \nu^{b} \rho_1]) \times \\ & \delta([\nu^{-t_2^{(1)}} \widetilde{\rho}_2, \nu^{-c} \widetilde{\rho_2}]) \times \delta([\nu^{s_2^{(1)}+1} \rho_2, \nu^{d} \rho_1]) \times \pi_1\end{aligned}$$

and

$$\tau_1 \leq \delta([\nu^{t_1^{(1)}+1} \rho_1, \nu^{s_1^{(1)}} \rho_1]) \times \delta([\nu^{t_2^{(1)}+1} \rho_2, \nu^{s_2^{(1)}} \rho_2]) \rtimes \sigma_1.$$

Definition of strongly positive discrete series implies that if $\nu^m \rho$ appears in cuspidal support of π_1, then $m > 0$. Since $x_1 < 0$, from $-a \leq -1$, $-c \leq -1$, $s_1^{(1)} + 1 \geq 1$ and $s_2^{(1)} + 1 \geq 1$, we deduce that $s_1^{(1)} = b$, $s_2^{(1)} = d$ and $\sigma_1 = \sigma_{sp}$. Also, $(y_1, \rho'_1) \in \{(-a, \widetilde{\rho_1}), (-c, \widetilde{\rho_2})\}$.

Using transitivity of Jacquet modules again, we obtain that there is an irreducible representation τ_2 of G_{n_2} such that

$$\mu^*(\delta([\nu^{t_1^{(1)}+1} \rho_1, \nu^{b} \rho_1]) \times \delta([\nu^{t_2^{(1)}+1} \rho_2, \nu^{d} \rho_2]) \rtimes \sigma_{sp}) \geq \delta'_2 \otimes \tau_2.$$

In the same way as before, we obtain that there are $t_1^{(2)}$ and $t_2^{(2)}$ such that $t_1^{(1)} \leq t_1^{(2)} \leq b$, $t_2^{(1)} \leq t_2^{(2)} \leq d$ and

$$\delta([\nu^{x_2}\rho_2', \nu^{y_2}\rho_2']) \leq \delta([\nu^{-t_1^{(2)}}\widetilde{\rho}_1, \nu^{-t_1^{(1)}-1}\widetilde{\rho}_1]) \times \delta([\nu^{-t_2^{(2)}}\widetilde{\rho}_2, \nu^{-t_2^{(1)}-1}\widetilde{\rho}_2]).$$

Furthermore, $t_1^{(2)} \geq a$ and $t_2^{(2)} \geq c$, since otherwise we would have $y_2 = x_1 - 1$ which implies $e(\delta_1') > e(\delta_2')$, a contradiction.

Repeating the same procedure, we deduce that there are $t_1^{(3)}$ and $t_2^{(3)}$ such that $t_1^{(2)} \leq t_1^{(3)} \leq b$, $t_2^{(2)} \leq t_2^{(3)} \leq d$ and

$$\delta([\nu^{x_3}\rho_3', \nu^{y_3}\rho_3']) \leq \delta([\nu^{-t_1^{(3)}}\widetilde{\rho}_1, \nu^{-t_1^{(2)}-1}\widetilde{\rho}_1]) \times \delta([\nu^{-t_2^{(3)}}\widetilde{\rho}_2, \nu^{-t_2^{(2)}-1}\widetilde{\rho}_2]). \qquad (3)$$

Let us denote by i an element of $\{1, 2\}$ such that $x_2 = -t_i^{(2)}$. Such i is obviously unique.

If $y_2 = -t_i^{(1)} - 1$, we have $t_j^{(1)} = t_j^{(2)}$ for $j \in \{1, 2\}$, $j \neq i$. Furthermore, from $e(\delta_1') \leq e(\delta_2')$ we get $x_1 = -t_j^{(1)}$, i.e., $x_1 = -t_j^{(2)}$.

On the other hand, if $y_2 = -t_j^{(1)} - 1$ for $j \in \{1, 2\}$, $j \neq i$, we have $t_i^{(1)} = t_j^{(2)}$. Now $e(\delta_1') \leq e(\delta_2')$ implies $x_1 = -t_i^{(1)}$, i.e., $x_1 = -t_j^{(2)}$.

From (3) we obtain $y_3 \in \{-t_1^{(2)} - 1, -t_2^{(2)} - 1\}$. Since $\{-t_1^{(2)} - 1, -t_2^{(2)} - 1\} = \{-x_1 - 1, -x_2 - 1\}$, there is some $i' \in \{1, 2\}$ such that $e(\delta_3') < e(\delta_{i'}')$, which is impossible. This ends the proof. $\square$

From the proof of the previous proposition we directly obtain the following results, which will be used to determine possible irreducible subquotients of the induced representation $\delta_1 \times \delta_2 \rtimes \sigma_{sp}$.

Proposition 3.2. *Suppose that π is an irreducible subquotient of the induced representation $\delta_1 \times \delta_2 \rtimes \sigma_{sp}$. Then one of the following holds:*

(i) π is strongly positive.

(ii) $\pi \cong L(\delta([\nu^x\rho, \nu^y\rho]) \rtimes \tau)$, where $y < 0$, τ is strongly positive, and there are i and j, $a - 1 \leq i \leq b$, $c - 1 \leq j \leq d$ such that

$$\delta([\nu^x\rho, \nu^y\rho]) \leq \delta([\nu^{-i}\rho_1, \nu^{-a}\rho_1]) \times \delta([\nu^{-j}\rho_2, \nu^{-c}\rho_2])$$
$$\tau \leq \delta([\nu^{i+1}\rho_1, \nu^b\rho_1]) \times \delta([\nu^{j+1}\rho_2, \nu^d\rho_2]) \rtimes \sigma_{sp}.$$

(iii) $\pi \cong L(\delta([\nu^x\rho, \nu^y\rho]) \times \delta([\nu^{x'}\rho', \nu^{y'}\rho']) \rtimes \tau)$, *where* $y < 0$, $y' < 0$, $x + y \leq x' + y'$, τ *is strongly positive, and there are* i *and* j, $a - 1 \leq i \leq b$, $c - 1 \leq j \leq d$ *such that*

$$\delta([\nu^x\rho, \nu^y\rho]) \otimes \delta([\nu^{x'}\rho', \nu^{y'}\rho']) \leq m^*(\delta([\nu^{-i}\rho_1, \nu^{-a}\rho_1]) \times \delta([\nu^{-j}\rho_2, \nu^{-c}\rho_2]))$$
$$\tau \leq \delta([\nu^{i+1}\rho_1, \nu^b\rho_1]) \times \delta([\nu^{j+1}\rho_2, \nu^d\rho_2]) \rtimes \sigma_{sp}.$$

Proposition 3.3. *Suppose that there are* i *and* j, $a - 1 \leq i \leq b$, $c - 1 \leq j \leq d$ *such that* $(i, j) \neq (a - 1, c - 1)$ *such that the induced representation* $\delta([\nu^{i+1}\rho_1, \nu^b\rho_1]) \times \delta([\nu^{j+1}\rho_2, \nu^d\rho_2]) \rtimes \sigma_{sp}$ *contains a strongly positive irreducible subquotient* τ. *All possible irreducible subquotients of* $\delta_1 \times \delta_2 \rtimes \sigma_{sp}$ *arising from such choice of the ordered pair* (i, j) *are given as follows:*

(i) $L(\delta([\nu^{-i}\rho_1, \nu^{-a}\rho_1]) \rtimes \tau)$, *if* $j = c - 1$;

(ii) $L(\delta([\nu^{-j}\rho_2, \nu^{-c}\rho_2]) \rtimes \tau)$, *if* $i = a - 1$;

(iii) $L(\delta([\nu^{-i}\rho_1, \nu^{-a}\rho_1]), \delta([\nu^{-j}\rho_2, \nu^{-c}\rho_2]), \tau)$, *if* $i \neq a - 1$, $j \neq c - 1$ *and the induced representation* $\delta([\nu^{-i}\rho_1, \nu^{-a}\rho_1]) \times \delta([\nu^{-j}\rho_2, \nu^{-c}\rho_2])$ *is irreducible;*

(iv) $L(\delta([\nu^{-i}\rho_1, \nu^{-a}\rho_1]), \delta([\nu^{-j}\rho_2, \nu^{-c}\rho_2]), \tau)$ *and* $L(\delta([\nu^{-i}\rho_1, \nu^{-c}\rho_1]), \delta([\nu^{-j}\rho_2, \nu^{-a}\rho_2]), \tau)$, *if* $i \neq a - 1$, $j \neq c - 1$ *and the induced representation* $\delta([\nu^{-i}\rho_1, \nu^{-a}\rho_1]) \times \delta([\nu^{-j}\rho_2, \nu^{-c}\rho_2])$ *reduces.*

Proof. Let us first consider the case $j = c - 1$. The case $i = a - 1$ can be handled in the same way. Suppose that we are in the case *(iii)* of the previous proposition. Then we have $\delta([\nu^x\rho, \nu^y\rho]) \otimes \delta([\nu^{x'}\rho', \nu^{y'}\rho']) \leq m^*(\delta([\nu^{-i}\rho_1, \nu^{-a}\rho_1]))$. Using the formula for m^*, we obtain $y' = x - 1$, which is impossible. Thus, we are in the case *(ii)* of the previous proposition and $\delta([\nu^x\rho, \nu^y\rho]) = \delta([\nu^{-i}\rho_1, \nu^{-a}\rho_1])$.

It remains to consider that case $i \neq a-1$, $j \neq c-1$. Using multiplicativity of m^*, we deduce $\delta([\nu^x\rho, \nu^y\rho]) \leq \delta([\nu^{-i'}\rho_1, \nu^{-a}\rho_1]) \times \delta([\nu^{-j'}\rho_2, \nu^{-c}\rho_2])$ for i' and j' such that $a - 1 \leq i' \leq i$, $c - 1 \leq j' \leq j$. If $i' \neq a - 1$ and $j' \neq c - 1$, we have $\rho_1 \cong \rho_2$ and union of segments $[-i', a]$ and $[-j', c]$ is a segment. From $\delta([\nu^{x'}\rho, \nu^{y'}\rho]) \leq \delta([\nu^{-i}\rho_1, \nu^{-i'-1}\rho_1]) \times \delta([\nu^{-j}\rho_2, \nu^{-j'-1}\rho_2])$, using $x + y \leq x' + y'$, we get $i = i'$ or $j = j'$. Consequently, the induced representation $\delta([\nu^{-i}\rho_1, \nu^{-a}\rho_1]) \times \delta([\nu^{-j}\rho_2, \nu^{-c}\rho_2])$ reduces and we obtain possible irreducible subquotient of the form $L(\delta([\nu^{-i}\rho_1, \nu^{-c}\rho_1]), \delta([\nu^{-j}\rho_2, \nu^{-a}\rho_2]), \tau)$. We note that in the case $i = i'$ and $j = j'$ we have either

$$L(\delta([\nu^{-i}\rho_1, \nu^{-c}\rho_1]), \delta([\nu^{-j}\rho_2, \nu^{-a}\rho_2]), \tau) = L(\delta([\nu^{-i}\rho_1, \nu^{-c}\rho_1]) \rtimes \tau)$$

or

$$L(\delta([\nu^{-i}\rho_1, \nu^{-c}\rho_1]), \delta([\nu^{-j}\rho_2, \nu^{-a}\rho_2]), \tau) = L(\delta([\nu^{-j}\rho_2, \nu^{-a}\rho_2]) \rtimes \tau).$$

We will now discuss the case $i' = a - 1$. The case $j' = c - 1$ can be handled in the same way. It follows that $\delta([\nu^{x'}\rho, \nu^{y'}\rho]) \leq \delta([\nu^{-i}\rho_1, \nu^{-a}\rho_1]) \times \delta([\nu^{-j}\rho_2, \nu^{-j'-1}\rho_2])$, and we have the following two possibilities:

1. $j' = j$, which provides possible irreducible subquotient

$$L(\delta([\nu^{-i}\rho_1, \nu^{-a}\rho_1]), \delta([\nu^{-j}\rho_2, \nu^{-c}\rho_2]), \tau),$$

2. $j' < j$, and $x + y \leq x' + y'$ now implies $i = j'$ and $a < c$. Consequently, the induced representation $\delta([\nu^{-i}\rho_1, \nu^{-a}\rho_1]) \times \delta([\nu^{-j}\rho_2, \nu^{-c}\rho_2])$ reduces and we again obtain possible irreducible subquotient

$$L(\delta([\nu^{-i}\rho_1, \nu^{-c}\rho_1]), \delta([\nu^{-j}\rho_2, \nu^{-a}\rho_2]), \tau).$$

This ends the proof. □

We take a moment to recall some results from [13], which will be frequently used in the paper.

Lemma 3.4. *Let ρ denote a supercuspidal element of $Irr(GL(n_\rho, F))$ and $a, b \in \mathbb{R}$ such that $a \leq b$ and $1 \leq a$. Let $\sigma_{sp} \in Irr(G_n)$ stand for the strongly positive discrete series. Then the induced representation $\delta([\nu^a\rho, \nu^b\rho]) \rtimes \sigma_{sp}$ reduces if and only if ρ is selfcontragredient, $[2a-1, 2b-1] \cap Jord_\rho(\sigma_{sp}) \neq \emptyset$ and $2b+1 \notin Jord_\rho(\sigma_{sp})$. If $\delta([\nu^a\rho, \nu^b\rho]) \rtimes \sigma_{sp}$ reduces, in $R(G)$ we have*

$$\delta([\nu^a\rho, \nu^b\rho]) \rtimes \sigma_{sp} = L(\delta([\nu^{-b}\rho, \nu^{-a}\rho]) \rtimes \sigma_{sp}) + L(\delta([\nu^{-x}\rho, \nu^{-a}\rho]) \rtimes \sigma_{sp}^{(1)}),$$

for x such that $2x+1 = \max(Jord_\rho(\sigma_{sp}) \cap [2a-1, 2b-1])$ and strongly positive discrete series $\sigma_{sp}^{(1)}$ such that $Jord(\sigma_{sp}^{(1)}) = Jord(\sigma_{sp}) \setminus \{(2x+1, \rho)\} \cup \{(2a-1, \rho)\}$. In particular, the induced representation $\delta([\nu^a\rho, \nu^b\rho]) \rtimes \sigma_{sp}$ contains an irreducible strongly positive subquotient if and only if ρ is selfcontragredient and $\max([2a-1, 2b-1] \cap Jord_\rho(\sigma_{sp})) = 2a-1$.

In the following two lemmas we recall necessary and sufficient conditions under which the representation $\delta_1 \times \delta_2 \rtimes \sigma_{sp}$ contains a strongly positive discrete series, which have been obtained in [7].

Lemma 3.5. *The induced representation $\delta_1 \times \delta_2 \rtimes \sigma_{sp}$, where $\rho_1 \not\cong \rho_2$, $a \geq 1$, $c \geq 1$, and $\sigma_{sp} \in Irr(G_n)$ is a strongly positive discrete series, contains a strongly positive irreducible subquotient if and only if ρ_i is selfcontragredient for $i = 1, 2$, $[2a-1, 2b+1] \cap Jord_{\rho_1}(\sigma_{sp}) = \{2a-1\}$ and $[2c-1, 2d+1] \cap Jord_{\rho_2}(\sigma_{sp}) = \{2c-1\}$.*

Lemma 3.6. *The induced representation $\delta([\nu^a\rho, \nu^b\rho]) \times \delta([\nu^c\rho, \nu^d\rho]) \rtimes \sigma_{sp}$, where $a \geq 1$, $c \geq 1$, $b \leq d$, and $\sigma_{sp} \in Irr(G_n)$ is a strongly positive discrete series, contains a strongly positive irreducible subquotient if and only if ρ is selfcontragredient and one of the following holds:*

(i) $[2c-1, 2d+1] \cap Jord_\rho(\sigma_{sp}) = \{2c-1\}$ *and* $[2a-1, 2b+1] \cap Jord_\rho(\pi) = \{2a-1\}$, *for the unique irreducible strongly positive subquotient π of* $\delta([\nu^c\rho, \nu^d\rho]) \rtimes \sigma_{sp}$.

(ii) $[2a-1, 2b+1] \cap Jord_\rho(\sigma_{sp}) = \{2a-1\}$ *and* $[2c-1, 2d+1] \cap Jord_\rho(\pi) = \{2c-1\}$, *for the unique irreducible strongly positive subquotient π of* $\delta([\nu^a\rho, \nu^b\rho]) \rtimes \sigma_{sp}$.

(iii) $c = b+1$, $2a-1 \in Jord_\rho(\sigma_{sp})$, $2d+1 \notin Jord_\rho(\sigma_{sp})$ *and if there is an x such that in $Jord_\rho(\sigma_{sp})$ holds $(2x+1)_- = 2a-1$ then $d < x$.*

(iv) $c = b+1$, $2a-1 \in Jord_\rho(\sigma_{sp})$, $2d+1 \notin Jord_\rho(\sigma_{sp})$, *there is an x such that in $Jord_\rho(\sigma_{sp})$ holds $(2x+1)_- = 2a-1$ and $[2c-1, 2d+1] \cap Jord_\rho(\sigma_{sp}) = \{2x+1\}$.*

Let us also note a consequence of previous two lemmas.

Corollary 3.7. *Suppose that there are i and j, $a-1 \leq i \leq b$ and $c-1 \leq j \leq d$, such that the induced representation*

$$\delta([\nu^a\rho_1, \nu^i\rho_1]) \times \delta([\nu^c\rho_2, \nu^j\rho_2]) \rtimes \sigma_{sp}$$

contains a strongly positive irreducible subquotient nonisomorphic to σ_{sp}. Then at least one of the induced representations $\delta_1 \rtimes \sigma_{sp}$ and $\delta_2 \rtimes \sigma_{sp}$ reduces.

The following result is well known.

Lemma 3.8. *Let $\sigma_1, \sigma_2, \sigma_3$ denote admissible representations of finite length of G_n, such that σ_1 is irreducible subquotient of σ_3 and σ_2 is subquotient of σ_3. If there is an irreducible representation π such that π appears with multiplicity m_1 in $r_P(\sigma_1)$, with multiplicity m_2 in $r_P(\sigma_2)$ and with multiplicity m_3 in $r_P(\sigma_3)$, for the appropriate parabolic subgroup P, and $1 \leq m_1 \leq m_2 = m_3$, then σ_1 is an irreducible subquotient of σ_2.*

4 The case of irreducible $\delta_1 \times \delta_2$

In this section we determine the composition factors of the induced representation $\delta_1 \times \delta_2 \rtimes \sigma_{sp}$ in the first series of subcases. Through this section we assume that $\delta_1 \times \delta_2$ is irreducible. We note that in the case $\rho_1 \not\cong \rho_2$, the results follow immediately from [13] and [3].

For simplicity of the notation, if $\mathrm{Jord}_{\rho_1}(\sigma_{sp}) \cap [2a-1, 2b-1] \neq \emptyset$, let $2x+1 = \max(\mathrm{Jord}_{\rho_1}(\sigma_{sp}) \cap [2a-1, 2b-1])$. Also, if $\mathrm{Jord}_{\rho_2}(\sigma_{sp}) \cap [2c-1, 2d-1] \neq \emptyset$, let $2y+1 = \max(\mathrm{Jord}_{\rho_2}(\sigma_{sp}) \cap [2c-1, 2d-1])$.

The following proposition is well known:

Proposition 4.1. *Suppose that both induced representations $\delta_1 \rtimes \sigma_{sp}$ and $\delta_2 \rtimes \sigma_{sp}$ are irreducible. Then the induced representation $\delta_1 \times \delta_2 \rtimes \sigma_{sp}$ is irreducible.*

In the following two propositions we deal with the case when exactly one representations $\delta_1 \rtimes \sigma_{sp}$ and $\delta_2 \rtimes \sigma_{sp}$ reduces.

Proposition 4.2. *Suppose that the induced representation $\delta_1 \rtimes \sigma_{sp}$ reduces and that the induced representation $\delta_2 \rtimes \sigma_{sp}$ is irreducible. Let $\sigma_{sp}^{(1)}$ stand for the strongly positive discrete series such that $Jord(\sigma_{sp}^{(1)}) = Jord(\sigma_{sp}) \setminus \{(2x+1, \rho_1)\} \cup \{(2b+1, \rho_1)\}$. In $R(G)$ we have*

$$\delta_1 \times \delta_2 \rtimes \sigma_{sp} = L(\widetilde{\delta_1}, \widetilde{\delta_2}, \sigma_{sp}) + L(\delta([\nu^{-x}\rho_1, \nu^{-a}\rho_1]), \widetilde{\delta_2}, \sigma_{sp}^{(1)}).$$

Proof. To determine possible irreducible subquotients of $\delta_1 \times \delta_2 \rtimes \sigma_{sp}$ we use the approach introduced in Proposition 3.2, and determine all i and j, $a-1 \leq i \leq b$, $c-1 \leq j \leq d$ such that the induced representation

$$\delta([\nu^{i+1}\rho_1, \nu^b\rho_1]) \times \delta([\nu^{j+1}\rho_2, \nu^d\rho_2]) \rtimes \sigma_{sp}$$

contains a strongly positive irreducible subquotient. Let us first prove that $j = d$. If $i = b$, it follows directly from Lemma 3.4 that $\delta([\nu^{j+1}\rho_2, \nu^d\rho_2]) \rtimes$

σ_{sp} is irreducible for $j < d$ since $\delta_2 \rtimes \sigma_{sp}$ is irreducible, and in this case there are no strongly positive subquotients. Now, let $i < b$. Irreducibility of $\delta([\nu^{j+1}\rho_2, \nu^d\rho_2]) \rtimes \sigma_{sp}$, together with Lemma 3.4, show that situations described in Lemma 3.5 and in parts (i) and (ii) of Lemma 3.6 can not happen. Assume that we are in the case (iii) or (iv) of Lemma 3.6. Then $\rho_1 \cong \rho_2$ and, since $b \leq d$, $j = b$ and $2d+1 \notin \mathrm{Jord}_{\rho_2}(\sigma_{sp})$. This forces $c \leq b+1$ and irreducibility of $\delta_1 \times \delta_2$ gives $c \leq a$. But, reducibility of $\delta_1 \rtimes \sigma_{sp}$ and Lemma 3.4 show that $\delta_2 \rtimes \sigma_{sp}$ reduces, a contradiction.

Consequently, $j = d$ and it can be directly verified that $\delta([\nu^{i+1}\rho_1, \nu^b\rho_1]) \rtimes \sigma_{sp}$ contains a strongly positive subquotient only if $i \in \{b, x\}$, for x such that $2x + 1 = \max(\mathrm{Jord}_{\rho_1}(\sigma_{sp}) \cap [2a-1, 2b-1])$.

Since $\delta_1 \times \delta_2$ is irreducible, by Proposition 3.3 implies that $i = b$ provides the irreducible subquotient $L(\widetilde{\delta_1}, \widetilde{\delta_2}, \sigma_{sp})$ of $\delta_1 \times \delta_2 \rtimes \sigma_{sp}$.

It remains to examine the case $i = x$. If $x = a-1$, we are in the case (i) of Proposition 3.3, which provides the irreducible representation $L(\widetilde{\delta_2} \rtimes \sigma_{sp}^{(1)})$. If $x \geq a$, Proposition 3.3 provides the irreducible representation $L(\delta([\nu^{-x}\rho_1, \nu^{-a}\rho_1]), \widetilde{\delta_2}, \sigma_{sp}^{(1)})$.

It remains to prove that the obtained irreducible subquotient is contained in $\delta_1 \times \delta_2 \rtimes \sigma_{sp}$. If $x = a-1$, $L(\widetilde{\delta_2} \rtimes \sigma_{sp}^{(1)})$ is a subrepresentation of $\widetilde{\delta_2} \rtimes \sigma_{sp}^{(1)}$, which is contained in $\widetilde{\delta_2} \times \delta_1 \rtimes \sigma_{sp}$. If $x \geq a$, $\widetilde{\delta_2} \times \delta([\nu^{-x}\rho_1, \nu^{-a}\rho_1])$ is irreducible since $\delta_1 \times \delta_2$ is irreducible. Thus,

$$\widetilde{\delta_2} \times \delta([\nu^{-x}\rho_1, \nu^{-a}\rho_1]) \rtimes \sigma_{sp}^{(1)} \cong \delta([\nu^{-x}\rho_1, \nu^{-a}\rho_1]) \times \widetilde{\delta_2} \rtimes \sigma_{sp}^{(1)}$$

and this induced representation has the unique irreducible subrepresentation $L(\delta([\nu^{-x}\rho_1, \nu^{-a}\rho_1]), \widetilde{\delta_2}, \sigma_{sp}^{(1)})$. Since $\widetilde{\delta_2} \rtimes L(\delta([\nu^{-x}\rho_1, \nu^{-a}\rho_1]) \rtimes \sigma_{sp}^{(1)})$ is also a subrepresentation of $\widetilde{\delta_2} \times \delta([\nu^{-x}\rho_1, \nu^{-a}\rho_1]) \rtimes \sigma_{sp}^{(1)}$, we have

$$L(\delta([\nu^{-x}\rho_1, \nu^{-a}\rho_1]), \widetilde{\delta_2}, \sigma_{sp}^{(1)}) \hookrightarrow \widetilde{\delta_2} \rtimes L(\delta([\nu^{-x}\rho_1, \nu^{-a}\rho_1]) \rtimes \sigma_{sp}^{(1)}) \leq \widetilde{\delta_2} \times \delta_1 \rtimes \sigma_{sp}.$$

The equality $\widetilde{\delta_2} \times \delta_1 \rtimes \sigma_{sp} = \delta_1 \times \delta_2 \rtimes \sigma_{sp}$ gives the desired conclusion.

It is well known that $L(\widetilde{\delta_1}, \widetilde{\delta_2}, \sigma_{sp})$ is contained with multiplicity one in $\delta_1 \times \delta_2 \rtimes \sigma_{sp}$. Also, it is not hard to see that, if $x = a-1$, $\widetilde{\delta_2} \otimes \sigma_{sp}^{(1)}$ appears in $\mu^*(\delta_1 \times \delta_2 \rtimes \sigma_{sp})$ with multiplicity one, so $L(\widetilde{\delta_2} \otimes \sigma_{sp}^{(1)})$ is contained in $\delta_1 \times \delta_2 \rtimes \sigma_{sp}$ with multiplicity one. Similarly, if $x > a-1$, it follows directly from Lemma 2.1 that $\mu^*(\delta_1 \times \delta_2 \rtimes \sigma_{sp})$ contains the irreducible constituent $\delta([\nu^{-x}\rho_1, \nu^{-a}\rho_1]) \times \widetilde{\delta_2} \otimes \sigma_{sp}^{(1)}$ with multiplicity one. This ends the proof. $\square$

Proposition 4.3. *Suppose that the induced representation $\delta_2 \rtimes \sigma_{sp}$ reduces and that the induced representation $\delta_1 \rtimes \sigma_{sp}$ is irreducible. Let $\sigma_{sp}^{(1)}$ stand for the strongly positive discrete series such that $Jord(\sigma_{sp}^{(1)}) = Jord(\sigma_{sp}) \setminus \{(2y + 1, \rho_2)\} \cup \{(2d+1, \rho_2)\}$. In $R(G)$ we have*

$$\delta_1 \times \delta_2 \rtimes \sigma_{sp} = L(\widetilde{\delta_1}, \widetilde{\delta_2}, \sigma_{sp}) + L(\widetilde{\delta_1}, \delta([\nu^{-y}\rho_2, \nu^{-c}\rho_2]), \sigma_{sp}^{(1)}).$$

Proof. Again we start the determination of possible irreducible subquotients of $\delta_1 \times \delta_2 \rtimes \sigma_{sp}$ by determining all i and j, $a-1 \leq i \leq b$, $c-1 \leq j \leq d$ such that the induced representation

$$\delta([\nu^{i+1}\rho_1, \nu^b\rho_1]) \times \delta([\nu^{j+1}\rho_2, \nu^d\rho_2]) \rtimes \sigma_{sp} \tag{4}$$

contains a strongly positive irreducible subquotient. If $\rho_1 \not\cong \rho_2$, in the same way as in the proof of the previous proposition we get $i = b$. Let us now consider the case $\rho_1 \cong \rho_2$. Using irreducibility of $\delta_1 \rtimes \sigma_{sp}$, together with Lemma 3.4, we obtain that cases (i), (ii) and (iii) from Lemma 3.6 do not appear. The case (iv) of that lemma appears if $b < d$, $j = b$, $2b+1 \in \mathrm{Jord}_{\rho_1}(\sigma_{sp})$ and $2i+1 = \max(\mathrm{Jord}_{\rho_1}(\sigma_{sp}) \cap [2a-1, 2b-1])$ (we note that $2d+1 \notin \mathrm{Jord}_{\rho_2}(\sigma_{sp})$ since $\delta_2 \rtimes \sigma_{sp}$ reduces and irreducibility of $\delta_1 \rtimes \sigma_{sp}$ implies $2b+1 \in \mathrm{Jord}_{\rho_1}(\sigma_{sp})$). We denote the corresponding irreducible strongly positive subquotient of $\delta([\nu^{i+1}\rho_1, \nu^b\rho_1]) \times \delta([\nu^{b+1}\rho_2, \nu^d\rho_2]) \rtimes \sigma_{sp}$ by σ'_{sp}. Obviously, $\mathrm{Jord}(\sigma'_{sp}) = \mathrm{Jord}(\sigma_{sp}) \setminus \{(2i+1, \rho_1)\} \cup \{(2d+1, \rho_2)\}$.

Since $b < d$ and $c \leq b+1$, irreducibility of $\delta_1 \times \delta_2$ gives $c \leq a$. Two possibilities will be studied separately.

- $i = a-1$. Now we are in the case (ii) of Proposition 3.2 and we obtain a candidate $L(\delta([\nu^{-b}\rho_2, \nu^{-c}\rho_2]) \rtimes \sigma'_{sp})$ for an irreducible constituent of $\delta_1 \times \delta_2 \rtimes \sigma_{sp}$. Obviously, $L(\delta([\nu^{-b}\rho_2, \nu^{-c}\rho_2]) \rtimes \sigma'_{sp})$ is a subrepresentation of $\delta([\nu^{-b}\rho_2, \nu^{-c}\rho_2]) \rtimes \sigma'_{sp}$ and

  $$\delta([\nu^{-b}\rho_2, \nu^{-c}\rho_2]) \rtimes \sigma'_{sp} \cong \delta([\nu^{c}\rho_2, \nu^{b}\rho_2]) \rtimes \sigma'_{sp},$$

  since, by Lemma 3.4, $2b+1 \in \mathrm{Jord}_{\rho_2}(\sigma'_{sp})$. Thus, $\mu^*(L(\delta([\nu^{-b}\rho_2, \nu^{-c}\rho_2]) \rtimes \sigma'_{sp})) \geq \delta([\nu^{c}\rho_2, \nu^{b}\rho_2]) \otimes \sigma'_{sp}$. Consequently, if $L(\delta([\nu^{-b}\rho_2, \nu^{-c}\rho_2]) \rtimes \sigma'_{sp})$ is an irreducible subquotient of $\delta_1 \times \delta_2 \rtimes \sigma_{sp}$, then $\mu^*(\delta_1 \times \delta_2 \rtimes \sigma_{sp})$ contains $\delta([\nu^{c}\rho_2, \nu^{b}\rho_2]) \otimes \sigma'_{sp}$. We will analyze the last inequality using Lemma 2.1. There are $a-1 \leq i_1 \leq j_1 \leq b$, $c-1 \leq i_2 \leq j_2 \leq d$ and an irreducible constituent $\delta \otimes \pi_1$ of $\mu^*(\sigma_{sp})$ such that

$$\begin{aligned}\delta([\nu^c\rho_2, \nu^b\rho_2]) \leq &\delta([\nu^{-i_1}\rho_1, \nu^{-a}\rho_1]) \times \delta([\nu^{j_1+1}\rho_1, \nu^b\rho_1]) \times \\ &\delta([\nu^{-i_2}\rho_2, \nu^{-c}\rho_2]) \times \delta([\nu^{j_2+1}\rho_2, \nu^d\rho_2]) \times \delta\end{aligned}$$

and

$$\sigma'_{sp} \leq \delta([\nu^{i_1+1}\rho_1, \nu^{j_1}\rho_1]) \times \delta([\nu^{i_2+1}\rho_2, \nu^{j_2}\rho_2]) \rtimes \pi_1.$$

We see at once that $i_1 = a-1$, $i_2 = c-1$, $j_2 = d$. From $c \leq a$ and [6, Theorem 4.6] we deduce that $j_1 = a-1$ and $\delta = \delta([\nu^c\rho_2, \nu^{a-1}\rho_2])$. [6, Theorem 4.6] also gives that π_1 is strongly positive discrete series such that $2b+1 \in \mathrm{Jord}_{\rho_1}(\pi_1)$. Consequently, $\sigma'_{sp} \leq \delta([\nu^c\rho_2, \nu^d\rho_2]) \rtimes \pi_1$, which is impossible since the induced representation $\delta([\nu^c\rho_2, \nu^d\rho_2]) \rtimes \pi_1$ does not contain irreducible strongly positive subquotient by Lemma 3.4.

- $i > a-1$. Since union of the segments $[-i, -a]$ and $[-b, -c]$ is not a segment, we are in the case (iii) of Proposition 3.3 and we obtain a candidate $L(\delta([\nu^{-i}\rho_1, \nu^{-a}\rho_1]), \delta([\nu^{-b}\rho_2, \nu^{-c}\rho_2]), \sigma'_{sp})$ for an irreducible subquotient of $\delta_1 \times \delta_2 \rtimes \sigma_{sp}$.

  In the same way as in the previously considered case we obtain that the irreducible representation $\delta([\nu^c\rho_2, \nu^b\rho_2]) \otimes \delta([\nu^{-i}\rho_1, \nu^{-a}\rho_1]) \otimes \sigma'_{sp}$ is contained in the Jacquet module of $L(\delta([\nu^{-i}\rho_1, \nu^{-a}\rho_1]), \delta([\nu^{-b}\rho_2, \nu^{-c}\rho_2]), \sigma'_{sp})$ with respect to the appropriate parabolic subgroup. Thus, if $L(\delta([\nu^{-i}\rho_1, \nu^{-a}\rho_1]), \delta([\nu^{-b}\rho_2, \nu^{-c}\rho_2]), \sigma'_{sp})$ is an irreducible subquotient of $\delta_1 \times \delta_2 \rtimes \sigma_{sp}$, using Lemma 2.1 and transitivity of Jacquet modules, we deduce that there are i_1, j_1, i_2, j_2 such that $a-1 \leq i_1 \leq j_1 \leq b$, $c-1 \leq i_2 \leq j_2 \leq d$, and an irreducible constituent $\delta \otimes \pi_1$ of $\mu^*(\sigma_{sp})$ such that

$$\begin{aligned}\delta([\nu^c\rho_2, \nu^b\rho_2]) \leq &\delta([\nu^{-i_1}\rho_1, \nu^{-a}\rho_1]) \times \delta([\nu^{j_1+1}\rho_1, \nu^b\rho_1]) \times \\ &\delta([\nu^{-i_2}\rho_2, \nu^{-c}\rho_2]) \times \delta([\nu^{j_2+1}\rho_2, \nu^d\rho_2]) \times \delta\end{aligned}$$

  and

$$\delta([\nu^{-i}\rho_1, \nu^{-a}\rho_1]) \otimes \sigma'_{sp} \leq \mu^*(\delta([\nu^{i_1+1}\rho_1, \nu^{j_1}\rho_1]) \times \delta([\nu^{i_2+1}\rho_2, \nu^{j_2}\rho_2]) \rtimes \pi_1).$$

It is not hard to deduce that $i_1 = a-1$, $i_2 = c-1$, $j_2 = d$. Also, by [6, Theorem 4.6], either $\pi_1 = \sigma_{sp}$ or $\delta = \delta([\nu^c\rho_2, \nu^t\rho_2])$ for $2t+1 = \min\{2z+1 \in \mathrm{Jord}_{\rho_2}(\sigma_{sp}) : c \le z\}$. Since $c \le a$ and $2i+1 = \max(\mathrm{Jord}_{\rho_2}(\sigma_{sp}) \cap [2a-1, 2b-1])$, it follows $t \le i$.

If $c < a$, then $\delta = \delta([\nu^c\rho_2, \nu^t\rho_2])$ and $j_1 = t$. From $t \le i$ we get $2b+1 \in \mathrm{Jord}_{\rho_2}(\pi_1)$. It follows that $\delta([\nu^{-i}\rho_1, \nu^{-a}\rho_1]) \otimes \sigma'_{sp}$ is an irreducible constituent of $\mu^*(\delta([\nu^a\rho_1, \nu^t\rho_1]) \times \delta([\nu^c\rho_2, \nu^d\rho_2]) \rtimes \pi_1)$. Using the structural formula for μ^* again, we see that there are i_3, j_3, i_4, j_4 such that $a-1 \le i_3 \le j_3 \le t$, $c-1 \le i_4 \le j_4 \le d$, and an irreducible constituent $\delta' \otimes \pi_2$ of $\mu^*(\pi_1)$ such that

$$\begin{aligned}\delta([\nu^{-i}\rho_1, \nu^{-a}\rho_1]) \le &\delta([\nu^{-i_3}\rho_1, \nu^{-a}\rho_1]) \times \delta([\nu^{j_3+1}\rho_1, \nu^t\rho_1]) \times \\ &\delta([\nu^{-i_4}\rho_2, \nu^{-c}\rho_2]) \times \delta([\nu^{j_4+1}\rho_2, \nu^d\rho_2]) \times \delta'\end{aligned}$$

and

$$\sigma'_{sp} \le \delta([\nu^{i_3+1}\rho_1, \nu^{j_3}\rho_1]) \times \delta([\nu^{i_4+1}\rho_2, \nu^{j_4}\rho_2]) \rtimes \pi_2.$$

Since $a > 0$ and $c < a$, it directly follows that $j_3 = t$, $i_4 = c-1$, $j_4 = d$ and $\pi_2 \cong \pi_1$. If $t < i$, it follows that $L(\delta([\nu^{-i}\rho_1, \nu^{-a}\rho_1]), \delta([\nu^{-b}\rho_2, \nu^{-c}\rho_2]), \sigma'_{sp})$ is not an irreducible subquotient of $\delta_1 \times \delta_2 \rtimes \sigma_{sp}$. Otherwise, we have $i_3 = t$ and $\sigma'_{sp} \le \delta([\nu^c\rho_2, \nu^d\rho_2]) \rtimes \pi_1$, which is impossible because it is a consequence of Lemma 3.4 that $\delta([\nu^c\rho_2, \nu^d\rho_2]) \rtimes \pi_1$ does not contain an irreducible strongly positive subquotient.

If $c = a$ and $\delta = \delta([\nu^c\rho_2, \nu^t\rho_2])$, then we again have $\delta([\nu^{-i}\rho_1, \nu^{-a}\rho_1]) \otimes \sigma'_{sp} \le \mu^*(\delta([\nu^a\rho_1, \nu^t\rho_1]) \times \delta([\nu^c\rho_2, \nu^d\rho_2]) \rtimes \pi_1)$. In the same way as in the case $c < a$, we obtain that σ'_{sp} is contained either in $\delta([\nu^c\rho_2, \nu^d\rho_2]) \rtimes \pi_1$ (this can happen only if $t = i$) or in $\delta([\nu^a\rho_1, \nu^t\rho_1]) \times \delta([\nu^{i+1}\rho_2, \nu^d\rho_2]) \rtimes \pi_1$, but neither of these induced representations contains an irreducible strongly positive subquotient, by Lemma 3.4 and Lemma 3.6.

If $c = a$ and $\pi_1 = \sigma_{sp}$, then $j_1 = a-1$ and $\delta([\nu^{-i}\rho_1, \nu^{-a}\rho_1]) \otimes \sigma'_{sp} \le \mu^*(\delta([\nu^c\rho_2, \nu^d\rho_2]) \rtimes \sigma_{sp})$. Following the same procedure as before we obtain $\sigma'_{sp} \le \delta([\nu^{i+1}\rho_2, \nu^d\rho_2]) \rtimes \sigma_{sp}$, which is impossible since Lemma 3.4 shows that the induced representation $\delta([\nu^c\rho_2, \nu^d\rho_2]) \rtimes \pi_1$ does not contain an irreducible strongly positive subquotient.

Consequently, we have proved that in (4) we must have $i = b$. Now the rest of the proof follows the same lines as in the proof of Proposition 4.2. □

In the rest of this section we determine composition factors of $\delta_1 \times \delta_2 \rtimes \sigma_{sp}$ when $\delta_1 \times \delta_2$ is irreducible and both representations $\delta_1 \rtimes \sigma_{sp}$ and $\delta_2 \rtimes \sigma_{sp}$ reduce. In this case, a description of the composition factors of $\delta_1 \times \delta_2 \rtimes \sigma_{sp}$ will be divided in a sequence of lemmas.

Lemma 4.4. *Suppose that $\rho_1 \not\cong \rho_2$ or $\rho_1 \cong \rho_2$ and $x < y$. Let $\sigma_{sp}^{(1)}$ denote the strongly positive discrete series such that $Jord(\sigma_{sp}^{(1)}) = Jord(\sigma_{sp}) \setminus \{(2x+1, \rho_1)\} \cup \{(2b+1, \rho_1)\}$, $\sigma_{sp}^{(2)}$ denote the strongly positive discrete series such that $Jord(\sigma_{sp}^{(2)}) = Jord(\sigma_{sp}) \setminus \{(2y+1, \rho_2)\} \cup \{(2d+1, \rho_2)\}$ and $\sigma_{sp}^{(3)}$ denote the strongly positive discrete series such that $Jord(\sigma_{sp}^{(3)}) = Jord(\sigma_{sp}) \setminus \{(2x+1, \rho_1), (2y+1, \rho_2)\} \cup \{(2b+1, \rho_1), (2d+1, \rho_2)\}$. In $R(G)$ we have*

$$\begin{aligned}\delta_1 \times \delta_2 \rtimes \sigma_{sp} = & L(\widetilde{\delta_1}, \widetilde{\delta_2}, \sigma_{sp}) + L(\delta([\nu^{-x}\rho_1, \nu^{-a}\rho_1]), \widetilde{\delta_2}, \sigma_{sp}^{(1)}) \\ & + L(\widetilde{\delta_1}, \delta([\nu^{-y}\rho_2, \nu^{-c}\rho_2]), \sigma_{sp}^{(2)}) \\ & + L(\delta([\nu^{-x}\rho_1, \nu^{-a}\rho_1]), \delta([\nu^{-y}\rho_2, \nu^{-c}\rho_2]), \sigma_{sp}^{(3)}).\end{aligned}$$

Proof. It can be directly seen that

$$\delta([\nu^{i+1}\rho_1, \nu^{b}\rho_1]) \times \delta([\nu^{j+1}\rho_2, \nu^{d}\rho_2]) \rtimes \sigma_{sp}$$

contains a strongly positive discrete series for i and j such that $a-1 \leq i \leq b$, $c-1 \leq j \leq d$ if and only if $(i,j) \in \{(b,d), (x,d), (b,y), (x,y)\}$. If $(i,j) = (x,y)$ and $(x,y) = (a-1, b-1)$, we are in the case (i) of Proposition 3.2 and $\delta_1 \times \delta_2 \rtimes \sigma_{sp}$ contains an irreducible strongly positive subquotient. If $(i,j) \in \{(a-1, t_1), (t_2, c-1)\}$, for $t_1 \neq c-1$ and $t_2 \neq a-1$, we are in the case (ii) of Proposition 3.2, and in all other cases we are in the case (iii) of the same proposition.

Using our assumptions on δ_1 and δ_2, together with Proposition 3.3, we deduce that each of four possibilities for (i,j) provides one candidate for the irreducible subquotient of $\delta_1 \times \delta_2 \rtimes \sigma_{sp}$, and all obtained candidates are mutually non-isomorphic. To show that the obtained candidate π is an irreducible subquotient of $\delta_1 \times \delta_2 \rtimes \sigma_{sp}$ which appears with multiplicity one, we will discuss only the case $x > a-1$, $y > b-1$ and $\pi \cong L(\delta([\nu^{-x}\rho_1, \nu^{-a}\rho_1]), \delta([\nu^{-y}\rho_2, \nu^{-c}\rho_2]), \sigma_{sp}^{(3)})$. Other cases can be handled in an analogous way. We have

$$\pi \leq \delta([\nu^{-x}\rho_1, \nu^{-a}\rho_1]) \times \delta([\nu^{-y}\rho_2, \nu^{-c}\rho_2]) \rtimes \sigma_{sp}^{(3)}$$

and
$$\sigma_{sp}^{(3)} \hookrightarrow \delta([\nu^{x+1}\rho_1, \nu^b\rho_1]) \times \delta([\nu^{y+1}\rho_2, \nu^d\rho_2]) \rtimes \sigma_{sp},$$
and it can be easily seen that both representations π and $\delta_1 \times \delta_2 \rtimes \sigma_{sp}$ are contained in the induced representation
$$\pi' = \delta([\nu^a\rho_1, \nu^x\rho_1]) \times \delta([\nu^{x+1}\rho_1, \nu^b\rho_1]) \times \delta([\nu^c\rho_2, \nu^y\rho_2]) \times \delta([\nu^{y+1}\rho_2, \nu^d\rho_2]) \rtimes \sigma_{sp}.$$
Furthermore, Frobenius reciprocity shows that
$$\delta([\nu^{-x}\rho_1, \nu^{-a}\rho_1]) \otimes \delta([\nu^{x+1}\rho_1, \nu^b\rho_1]) \otimes \delta([\nu^{-y}\rho_2, \nu^{-c}\rho_2]) \otimes \delta([\nu^{y+1}\rho_2, \nu^d\rho_2]) \otimes \sigma_{sp}$$
appears in the Jacquet module of π with respect to the appropriate parabolic subgroup, and structural formula can be used to obtain that the same irreducible representation appears with multiplicity one in the Jacquet module of $\delta_1 \times \delta_2 \rtimes \sigma_{sp}$ with respect to the appropriate parabolic subgroup and in the Jacquet modules of π' with respect to the appropriate parabolic subgroup. Lemma 3.8 shows that π is an irreducible subquotient of $\delta_1 \times \delta_2 \rtimes \sigma_{sp}$ and we have also proved that it appears with multiplicity one. □

In the rest of this section we suppose $\rho_1 \cong \rho_2$ and, for simplicity of the notation, we write ρ instead of ρ_1 and ρ_2.

Lemma 4.5. *Suppose that $b = d$ and let $\sigma_{sp}^{(1)}$ denote the strongly positive discrete series such that $Jord(\sigma_{sp}^{(1)}) = Jord(\sigma_{sp}) \setminus \{(2x+1, \rho)\} \cup \{(2b+1, \rho)\}$. If $a = c$, in $R(G)$ we have*
$$\delta_1 \times \delta_2 \rtimes \sigma_{sp} = L(\widetilde{\delta_1}, \widetilde{\delta_2}, \sigma_{sp}) + L(\widetilde{\delta_1}, \delta([\nu^{-x}\rho, \nu^{-a}\rho]), \sigma_{sp}^{(1)}).$$
If $a \neq c$, we can assume $a < c$. Then in $R(G)$ we have
$$\begin{aligned}\delta_1 \times \delta_2 \rtimes \sigma_{sp} = {} & L(\widetilde{\delta_1}, \widetilde{\delta_2}, \sigma_{sp}) + L(\delta([\nu^{-x}\rho, \nu^{-a}\rho]), \widetilde{\delta_2}, \sigma_{sp}^{(1)}) \\ & + L(\widetilde{\delta_1}, \delta([\nu^{-x}\rho, \nu^{-c}\rho]), \sigma_{sp}^{(1)}).\end{aligned}$$

Proof. Obviously, $x = y$. Induced representation
$$\delta([\nu^{i+1}\rho, \nu^b\rho]) \times \delta([\nu^{j+1}\rho, \nu^b\rho]) \rtimes \sigma_{sp}$$
contains a strongly positive discrete series for i and j such that $a-1 \leq i \leq b$, $c-1 \leq j \leq b$ if and only if $(i,j) \in \{(b,b), (x,b), (b,x)\}$.

We will first consider the case $a = c$.

For $(i,j) \neq (b,b)$, we obtain possible irreducible subquotient π of $\delta_1 \times \delta_2 \rtimes \sigma_{sp}$ of the form $\pi \cong L(\widetilde{\delta_1}, \delta([\nu^{-x}\rho, \nu^{-a}\rho]), \sigma_{sp}^{(1)})$ if $x > a-1$, and of the form $\pi \cong L(\widetilde{\delta_1} \rtimes \sigma_{sp}^{(1)})$, if $x = a-1$.

If $x > a-1$, then $L(\delta([\nu^{-x}\rho, \nu^{-a}\rho]) \rtimes \sigma_{sp}^{(1)})$ is a subrepresentation of $\delta([\nu^{a}\rho, \nu^{b}\rho]) \rtimes \sigma_{sp}$, and if $x = a-1$ then $\sigma_{sp}^{(1)}$ is a subrepresentation of $\delta([\nu^{a}\rho, \nu^{b}\rho]) \rtimes \sigma_{sp}$. In the same way as in the proof of Proposition 4.2 we see $\pi \leq \delta_1 \times \delta_2 \rtimes \sigma_{sp}$.

If $x > a-1$, we have the following embeddings and isomorphisms:

$$\begin{aligned}
\pi &\hookrightarrow \widetilde{\delta_1} \times \delta([\nu^{-x}\rho, \nu^{-a}\rho]) \rtimes \sigma_{sp}^{(1)} \cong \delta([\nu^{-x}\rho, \nu^{-a}\rho]) \times \widetilde{\delta_1} \rtimes \sigma_{sp}^{(1)} \\
&\cong \delta([\nu^{-x}\rho, \nu^{-a}\rho]) \times \delta_1 \rtimes \sigma_{sp}^{(1)} \cong \delta_1 \times \delta([\nu^{-x}\rho, \nu^{-a}\rho]) \rtimes \sigma_{sp}^{(1)} \\
&\hookrightarrow \delta_1 \times \delta([\nu^{-x}\rho, \nu^{-a}\rho]) \times \delta([\nu^{x+1}\rho, \nu^{b}\rho]) \rtimes \sigma_{sp} \\
&\cong \delta([\nu^{x+1}\rho, \nu^{b}\rho]) \times \delta_1 \times \delta([\nu^{-x}\rho, \nu^{-a}\rho]) \rtimes \sigma_{sp} \\
&\hookrightarrow \delta([\nu^{x+1}\rho, \nu^{b}\rho]) \times \delta([\nu^{x+1}\rho, \nu^{b}\rho]) \times \delta([\nu^{a}\rho, \nu^{x}\rho]) \times \delta([\nu^{a}\rho, \nu^{x}\rho]) \rtimes \sigma_{sp}.
\end{aligned}$$

Consequently,

$$\mu^*(\pi) \geq \delta([\nu^{x+1}\rho, \nu^{b}\rho]) \times \delta([\nu^{x+1}\rho, \nu^{b}\rho]) \otimes \delta([\nu^{a}\rho, \nu^{x}\rho]) \times \delta([\nu^{a}\rho, \nu^{x}\rho]) \rtimes \sigma_{sp},$$

and this representation is irreducible since $2x+1 \in \mathrm{Jord}_\rho(\sigma_{sp})$. Using Lemma 2.1 and [6, Theorem 4.6], we obtain that $\mu^*(\delta_1 \times \delta_2 \rtimes \sigma_{sp})$ contains $\delta([\nu^{x+1}\rho, \nu^{b}\rho]) \times \delta([\nu^{x+1}\rho, \nu^{b}\rho]) \otimes \delta([\nu^{a}\rho, \nu^{x}\rho]) \times \delta([\nu^{a}\rho, \nu^{x}\rho]) \rtimes \sigma_{sp}$ with multiplicity one and it follows that π appears in $\delta_1 \times \delta_2 \rtimes \sigma_{sp}$ with multiplicity one.

If $x = a-1$, in the similar way we obtain $\mu^*(\pi) \geq \delta([\nu^{a}\rho, \nu^{b}\rho]) \times \delta([\nu^{a}\rho, \nu^{b}\rho]) \otimes \sigma_{sp}$, and $\mu^*(\delta_1 \times \delta_2 \rtimes \sigma_{sp})$ contains $\delta([\nu^{a}\rho, \nu^{b}\rho]) \times \delta([\nu^{a}\rho, \nu^{b}\rho]) \otimes \sigma_{sp}$ with multiplicity one. Thus, in this case π again appears in $\delta_1 \times \delta_2 \rtimes \sigma_{sp}$ with multiplicity one.

Let us now consider the case $a < c$. If $(i,j) = (b,x)$ we obtain possible irreducible subquotient π_1 of $\delta_1 \times \delta_2 \rtimes \sigma_{sp}$ of the form $\pi_1 \cong L(\widetilde{\delta_1}, \delta([\nu^{-x}\rho, \nu^{-c}\rho]), \sigma_{sp}^{(1)})$ if $x > c-1$, and of the form $\pi_1 \cong L(\widetilde{\delta_1} \rtimes \sigma_{sp}^{(1)})$ if $x = c-1$ (note that the segment $[-x,-c]$ is contained in $[-b,-a]$).

It remains to examine the case $(i,j) = (x,b)$. By Proposition 3.3, this case provides two possible irreducible subquotients π_2 and π_3 of $\delta_1 \times \delta_2 \rtimes \sigma_{sp}$: $\pi_2 \cong L(\delta([\nu^{-x}\rho, \nu^{-a}\rho]), \widetilde{\delta_2}, \sigma_{sp}^{(1)})$ and $\pi_3 \cong \pi_1$. It can be proved in the same way as in the proof of Proposition 4.2 that both π_1 and π_2 appear in the composition series of $\delta_1 \times \delta_2 \rtimes \sigma_{sp}$. What is still left is to show that these irreducible constituents appear with multiplicity one.

If $x > c-1$, we denote by $2y'+1 = \min(\mathrm{Jord}_\rho(\sigma_{sp}) \cap [2c-1, 2b-1])$. Now we have the following embeddings and isomorphisms:

$$\begin{aligned}
\pi_1 &\hookrightarrow \delta([\nu^{-b}\rho, \nu^{-a}\rho]) \times \delta([\nu^{-x}\rho, \nu^{-c}\rho]) \rtimes \sigma_{sp}^{(1)} \\
&\cong \delta([\nu^{-x}\rho, \nu^{-c}\rho]) \times \delta([\nu^{-b}\rho, \nu^{-a}\rho]) \rtimes \sigma_{sp}^{(1)} \\
&\hookrightarrow \delta([\nu^{-x}\rho, \nu^{-c}\rho]) \times \delta([\nu^{-y'}\rho, \nu^{-a}\rho]) \times \delta([\nu^{-b}\rho, \nu^{-y'-1}\rho]) \rtimes \sigma_{sp}^{(1)} \\
&\cong \delta([\nu^{-x}\rho, \nu^{-c}\rho]) \times \delta([\nu^{-y'}\rho, \nu^{-a}\rho]) \times \delta([\nu^{y'+1}\rho, \nu^{b}\rho]) \rtimes \sigma_{sp}^{(1)} \\
&\cong \delta([\nu^{-x}\rho, \nu^{-c}\rho]) \times \delta([\nu^{y'+1}\rho, \nu^{b}\rho]) \times \delta([\nu^{-y'}\rho, \nu^{-a}\rho]) \rtimes \sigma_{sp}^{(1)}.
\end{aligned}$$

The induced representation $\delta([\nu^{-x}\rho, \nu^{-c}\rho]) \times \delta([\nu^{y'+1}\rho, \nu^{b}\rho])$ is obviously irreducible. Consequently, the irreducible representation $\delta([\nu^{-x}\rho, \nu^{-c}\rho]) \times \delta([\nu^{y'+1}\rho, \nu^{b}\rho]) \otimes \delta([\nu^{-y'}\rho, \nu^{-a}\rho]) \otimes \sigma_{sp}^{(1)}$ is contained in the Jacquet module of π_1 with respect to the appropriate parabolic subgroup P_1. Since $2y'+1 \in \mathrm{Jord}_\rho(\sigma_{sp})$ and $a < c$, from Lemma 2.1 and [6, Theorem 4.6] we deduce that the multiplicity of $\delta([\nu^{-x}\rho, \nu^{-c}\rho]) \times \delta([\nu^{y'+1}\rho, \nu^{b}\rho]) \otimes \delta([\nu^{-y'}\rho, \nu^{-a}\rho]) \otimes \sigma_{sp}^{(1)}$ in $r_{P_1}(\delta_1 \times \delta_2 \rtimes \sigma_{sp})$ equals the multiplicity of $\delta([\nu^{-y'}\rho, \nu^{-a}\rho]) \otimes \sigma_{sp}^{(1)}$ in $\mu^*(\delta([\nu^{a}\rho, \nu^{y'}\rho]) \times \delta([\nu^{x+1}\rho, \nu^{b}\rho]) \rtimes \sigma_{sp})$, which equals one, since $y' \leq x$ and σ_{sp} is strongly positive.

If $x = c-1$, since $2d+1 \in \mathrm{Jord}_\rho(\sigma_{sp}^{(1)})$, we have the following embeddings and isomorphisms:

$$\begin{aligned}
\pi_1 \hookrightarrow \widetilde{\delta_1} \rtimes \sigma_{sp}^{(1)} &\hookrightarrow \delta([\nu^{-c+1}\rho, \nu^{-a}\rho]) \times \delta([\nu^{-d}\rho, \nu^{-c}\rho]) \rtimes \sigma_{sp}^{(1)} \\
&\cong \delta([\nu^{-c+1}\rho, \nu^{-a}\rho]) \times \delta([\nu^{c}\rho, \nu^{d}\rho]) \rtimes \sigma_{sp}^{(1)} \\
&\cong \delta([\nu^{c}\rho, \nu^{d}\rho]) \times \delta([\nu^{-c+1}\rho, \nu^{-a}\rho]) \rtimes \sigma_{sp}^{(1)}.
\end{aligned}$$

Thus, $\delta([\nu^{c}\rho, \nu^{d}\rho]) \otimes \delta([\nu^{-c+1}\rho, \nu^{-a}\rho]) \otimes \sigma_{sp}^{(1)}$ is contained in the Jacquet module of π_2 with respect to the appropriate parabolic subgroup P_2. Since we have $2c-1 \in \mathrm{Jord}_\rho(\sigma_{sp})$ and $a < c$, it is not hard to see, using Lemma 2.1 and [6, Theorem 4.6], that the multiplicity of $\delta([\nu^{c}\rho, \nu^{d}\rho]) \otimes \delta([\nu^{-c+1}\rho, \nu^{-a}\rho]) \otimes \sigma_{sp}^{(1)}$ in $r_{P_2}(\delta_1 \times \delta_2 \rtimes \sigma_{sp})$ equals the multiplicity of $\delta([\nu^{-c+1}\rho, \nu^{-a}\rho]) \otimes \sigma_{sp}^{(1)}$ in $\mu^*(\delta_1 \rtimes \sigma_{sp})$, which equals one.

It follows that π_1 appears with multiplicity one in $\delta_1 \times \delta_2 \rtimes \sigma_{sp}$.

Representation π_2 also appears with multiplicity one in $\delta_1 \times \delta_2 \rtimes \sigma_{sp}$ since Frobenius reciprocity shows that $\widetilde{\delta_2} \otimes \delta([\nu^{-x}\rho, \nu^{-a}\rho]) \otimes \sigma_{sp}^{(1)}$ is contained in the

Jacquet module of π_2 with respect to the appropriate parabolic subgroup, and it can be seen using Lemma 2.1 that this irreducible representation appears with multiplicity one in the Jacquet module of $\delta_1 \times \delta_2 \rtimes \sigma_{sp}$ with respect to the appropriate parabolic subgroup. This ends the proof. □

Lemma 4.6. *Suppose that $b < d$, $x = y$ and $a \neq c$. Let $\sigma_{sp}^{(1)}$ denote the strongly positive discrete series such that Jord($\sigma_{sp}^{(1)}$) = Jord(σ_{sp}) $\setminus \{(2x+1, \rho)\} \cup \{(2b+1, \rho)\}$ and $\sigma_{sp}^{(2)}$ denote the strongly positive discrete series such that Jord($\sigma_{sp}^{(2)}$) = Jord(σ_{sp}) $\setminus \{(2x+1, \rho)\} \cup \{(2d+1, \rho)\}$. In $R(G)$ we have*

$$\begin{aligned}\delta_1 \times \delta_2 \rtimes \sigma_{sp} = {} & L(\widetilde{\delta_1}, \widetilde{\delta_2}, \sigma_{sp}) + L(\delta([\nu^{-x}\rho, \nu^{-a}\rho]), \widetilde{\delta_2}, \sigma_{sp}^{(1)}) \\ & + L(\widetilde{\delta_1}, \delta([\nu^{-x}\rho, \nu^{-c}\rho]), \sigma_{sp}^{(2)}) \\ & + L(\delta([\nu^{-x}\rho, \nu^{-a}\rho]), \delta([\nu^{-b}\rho, \nu^{-c}\rho]), \sigma_{sp}^{(2)}).\end{aligned}$$

Proof. Since $b < d$, $x = y$ and $\delta_1 \times \delta_2$ is irreducible, it follows that $c < a$. Furthermore, $a \neq c$ provides $x \geq c$. In the same way as before, we deduce that the induced representation

$$\delta([\nu^{i+1}\rho, \nu^{b}\rho]) \times \delta([\nu^{j+1}\rho, \nu^{d}\rho]) \rtimes \sigma_{sp}$$

contains a strongly positive discrete series for i and j such that $a-1 \leq i \leq b$, $c-1 \leq j \leq d$ if and only if $(i, j) \in \{(b, d), (x, d), (b, x)\}$.

We will first consider the case $x > a-1$. The choice $(i, j) = (b, d)$ provides the candidate $\pi_1 = L(\widetilde{\delta_1}, \widetilde{\delta_2}, \sigma_{sp})$ for an irreducible subquotient of $\delta_1 \times \delta_2 \rtimes \sigma_{sp}$, the choice $(i, j) = (x, d)$ provides the candidate $\pi_2 = L(\delta([\nu^{-x}\rho, \nu^{-a}\rho]), \widetilde{\delta_2}, \sigma_{sp}^{(1)})$, while the choice $(i, j) = (b, x)$ provides candidates $\pi_3 = L(\widetilde{\delta_1}, \delta([\nu^{-x}\rho, \nu^{-c}\rho]), \sigma_{sp}^{(2)})$ and $\pi_4 = L(\delta([\nu^{-x}\rho, \nu^{-a}\rho]), \delta([\nu^{-b}\rho, \nu^{-c}\rho]), \sigma_{sp}^{(2)})$. In the same way as before we see that π_1, π_2 and π_3 are irreducible subquotients of $\delta_1 \times \delta_2 \rtimes \sigma_{sp}$, and each of them appears with multiplicity one. It remains to discuss π_4.

From $x < b$ and $c < a$, we deduce that $\delta([\nu^{-b}\rho, \nu^{-c}\rho]) \times \delta([\nu^{-x}\rho, \nu^{-a}\rho])$ is irreducible, and π_4 is the unique irreducible subrepresentation of

$$\delta([\nu^{-b}\rho, \nu^{-c}\rho]) \times \delta([\nu^{-x}\rho, \nu^{-a}\rho]) \rtimes \sigma_{sp}^{(2)},$$

and, consequently,

$$\pi_4 \hookrightarrow \delta([\nu^{-b}\rho, \nu^{-c}\rho]) \rtimes L(\delta([\nu^{-x}\rho, \nu^{-a}\rho]) \rtimes \sigma_{sp}^{(2)}). \tag{5}$$

Since $x = y$, the strongly positive discrete series $\sigma_{sp}^{(2)}$ is a subrepresentation of $\delta([\nu^{b+1}\rho, \nu^d\rho]) \rtimes \sigma_{sp}^{(1)}$. It follows that $L(\delta([\nu^{-x}\rho, \nu^{-a}\rho]) \rtimes \sigma_{sp}^{(2)})$ is an irreducible subrepresentation of the induced representation

$$\delta([\nu^{b+1}\rho, \nu^d\rho]) \times \delta([\nu^{-x}\rho, \nu^{-a}\rho]) \rtimes \sigma_{sp}^{(1)},$$

and, since $2d+1 \notin \mathrm{Jord}_\rho(\sigma_{sp}^{(1)})$, [6, Theorem 4.6] can be used the determine that $\delta([\nu^{b+1}\rho, \nu^d\rho]) \otimes \delta([\nu^{-x}\rho, \nu^{-a}\rho]) \otimes \sigma_{sp}^{(1)}$ appears with multiplicity one in the Jacquet module of $\delta([\nu^{b+1}\rho, \nu^d\rho]) \times \delta([\nu^{-x}\rho, \nu^{-a}\rho]) \rtimes \sigma_{sp}^{(1)}$ with respect to the appropriate parabolic subgroup. Thus, $\delta([\nu^{b+1}\rho, \nu^d\rho]) \times \delta([\nu^{-x}\rho, \nu^{-a}\rho]) \rtimes \sigma_{sp}^{(1)}$ has the unique irreducible subrepresentation and we have

$$L(\delta([\nu^{-x}\rho, \nu^{-a}\rho]) \rtimes \sigma_{sp}^{(2)}) \hookrightarrow \delta([\nu^{b+1}\rho, \nu^d\rho]) \rtimes L(\delta([\nu^{-x}\rho, \nu^{-a}\rho]) \rtimes \sigma_{sp}^{(1)}).$$

This implies

$$\pi_4 \hookrightarrow \delta([\nu^{-b}\rho, \nu^{-c}\rho]) \times \delta([\nu^{b+1}\rho, \nu^d\rho]) \rtimes L(\delta([\nu^{-x}\rho, \nu^{-a}\rho]) \rtimes \sigma_{sp}^{(1)})$$

and

$$\pi_4 \leq \delta([\nu^c\rho, \nu^b\rho]) \times \delta([\nu^{b+1}\rho, \nu^d\rho]) \rtimes L(\delta([\nu^{-x}\rho, \nu^{-a}\rho]) \rtimes \sigma_{sp}^{(1)}).$$

From (5) we get that $\delta([\nu^{-b}\rho, \nu^{-c}\rho]) \otimes \delta([\nu^{-x}\rho, \nu^{-a}\rho]) \otimes \sigma_{sp}^{(2)}$ is contained in the Jacquet module of π_4 with respect to the appropriate parabolic subgroup P_1. Using Lemma 2.1, together with $x < b$, we obtain that the multiplicity of $\delta([\nu^{-b}\rho, \nu^{-c}\rho]) \otimes \delta([\nu^{-x}\rho, \nu^{-a}\rho]) \otimes \sigma_{sp}^{(2)}$ in $r_{P_1}(\delta([\nu^c\rho, \nu^b\rho]) \times \delta([\nu^{b+1}\rho, \nu^d\rho]) \rtimes L(\delta([\nu^{-x}\rho, \nu^{-a}\rho]) \rtimes \sigma_{sp}^{(1)}))$ equals the multiplicity of $\delta([\nu^{-x}\rho, \nu^{-a}\rho]) \otimes \sigma_{sp}^{(2)}$ in $\mu^*(\delta([\nu^{b+1}\rho, \nu^d\rho]) \rtimes L(\delta([\nu^{-x}\rho, \nu^{-a}\rho]) \rtimes \sigma_{sp}^{(1)}))$ and it can be easily seen that this multiplicity equals one.

The induced representation $\delta([\nu^c\rho, \nu^d\rho]) \rtimes L(\delta([\nu^{-x}\rho, \nu^{-a}\rho]) \rtimes \sigma_{sp}^{(1)})$ is obviously a subquotient of $\delta([\nu^c\rho, \nu^b\rho]) \times \delta([\nu^{b+1}\rho, \nu^d\rho]) \rtimes L(\delta([\nu^{-x}\rho, \nu^{-a}\rho]) \rtimes \sigma_{sp}^{(1)})$ and it follows directly from the structural formula for μ^* that $r_{P_1}(\delta([\nu^c\rho, \nu^d\rho]) \rtimes L(\delta([\nu^{-x}\rho, \nu^{-a}\rho]) \rtimes \sigma_{sp}^{(1)}))$ contains $\delta([\nu^{-b}\rho, \nu^{-c}\rho]) \otimes \delta([\nu^{-x}\rho, \nu^{-a}\rho]) \otimes \sigma_{sp}^{(2)}$. Consequently,

$$\pi_4 \leq \delta([\nu^c\rho, \nu^d\rho]) \rtimes L(\delta([\nu^{-x}\rho, \nu^{-a}\rho]) \rtimes \sigma_{sp}^{(1)})$$

and, by Lemma 3.4,

$$\pi_4 \leq \delta([\nu^c\rho, \nu^d\rho]) \times \delta([\nu^a\rho, \nu^b\rho]) \rtimes \sigma_{sp} = \delta_1 \times \delta_2 \rtimes \sigma_{sp}.$$

Furthermore, using the definition of $\sigma_{sp}^{(2)}$ we obtain the following embeddings and isomorphisms:

$$\begin{aligned}
\pi_4 &\hookrightarrow \delta([\nu^{-x}\rho, \nu^{-a}\rho]) \times \delta([\nu^{-b}\rho, \nu^{-c}\rho]) \rtimes \sigma_{sp}^{(2)} \\
&\hookrightarrow \delta([\nu^{-x}\rho, \nu^{-a}\rho]) \times \delta([\nu^{-x}\rho, \nu^{-c}\rho]) \times \delta([\nu^{-b}\rho, \nu^{-x-1}\rho]) \rtimes \sigma_{sp}^{(2)} \\
&\cong \delta([\nu^{-x}\rho, \nu^{-a}\rho]) \times \delta([\nu^{-x}\rho, \nu^{-c}\rho]) \times \delta([\nu^{x+1}\rho, \nu^{b}\rho]) \rtimes \sigma_{sp}^{(2)} \\
&\cong \delta([\nu^{x+1}\rho, \nu^{b}\rho]) \times \delta([\nu^{-x}\rho, \nu^{-a}\rho]) \times \delta([\nu^{-x}\rho, \nu^{-c}\rho]) \rtimes \sigma_{sp}^{(2)} \\
&\hookrightarrow \delta([\nu^{x+1}\rho, \nu^{b}\rho]) \times \delta([\nu^{-x}\rho, \nu^{-a}\rho]) \times \delta([\nu^{-x}\rho, \nu^{-c}\rho]) \times \delta([\nu^{x+1}\rho, \nu^{d}\rho]) \rtimes \sigma_{sp} \\
&\cong \delta([\nu^{x+1}\rho, \nu^{b}\rho]) \times \delta([\nu^{x+1}\rho, \nu^{d}\rho]) \times \delta([\nu^{-x}\rho, \nu^{-a}\rho]) \times \delta([\nu^{-x}\rho, \nu^{-c}\rho]) \rtimes \sigma_{sp}.
\end{aligned}$$

Frobenius reciprocity shows that the Jacquet module of π_4 with respect to the appropriate parabolic subgroup P_2 contains the irreducible representation

$$\pi = \delta([\nu^{x+1}\rho, \nu^{b}\rho]) \times \delta([\nu^{x+1}\rho, \nu^{d}\rho]) \otimes \delta([\nu^{-x}\rho, \nu^{-a}\rho]) \times \delta([\nu^{-x}\rho, \nu^{-c}\rho]) \otimes \sigma_{sp}.$$

Since $2x+1 \in \mathrm{Jord}_\rho(\sigma_{sp})$, it follows that the multiplicity of π in $r_{P_2}(\delta_1 \times \delta_2 \rtimes \sigma_{sp})$ equals the multiplicity of $\delta([\nu^{-x}\rho, \nu^{-a}\rho]) \times \delta([\nu^{-x}\rho, \nu^{-c}\rho]) \otimes \sigma_{sp}$ in $\mu^*(\delta([\nu^{a}\rho, \nu^{x}\rho]) \times \delta([\nu^{c}\rho, \nu^{x}\rho]) \rtimes \sigma_{sp})$, which obviously equals one.

Thus, π_4 appears in the composition series of the induced representation $\delta_1 \times \delta_2 \rtimes \sigma_{sp}$ with multiplicity one.

Now we consider the case $x = a-1$. The choice $(i,j) = (b,d)$ provides the candidate $\pi_5 = L(\widetilde{\delta_1}, \widetilde{\delta_2}, \sigma_{sp})$ for an irreducible subquotient of $\delta_1 \times \delta_2 \rtimes \sigma_{sp}$, the choice $(i,j) = (x,d)$ provides the candidate $\pi_6 = L(\widetilde{\delta_2} \rtimes \sigma_{sp}^{(1)})$, while the choice $(i,j) = (b,x)$ provides candidates $\pi_7 = L(\widetilde{\delta_1}, \delta([\nu^{-x}\rho, \nu^{-c}\rho]), \sigma_{sp}^{(2)})$ and $\pi_8 = L(\delta([\nu^{-b}\rho, \nu^{-c}\rho]) \rtimes \sigma_{sp}^{(2)})$. Similarly as before, we deduce that π_5, π_6 and π_7 are irreducible subquotients of $\delta_1 \times \delta_2 \rtimes \sigma_{sp}$, and each of them appears with the multiplicity one in the composition series. We discuss the representation π_8. We have the following embeddings and isomorphisms:

$$\begin{aligned}
\pi_8 &\hookrightarrow \delta([\nu^{-b}\rho, \nu^{-c}\rho]) \rtimes \sigma_{sp}^{(2)} \hookrightarrow \delta([\nu^{-x}\rho, \nu^{-c}\rho]) \times \delta([\nu^{-b}\rho, \nu^{-x-1}\rho]) \rtimes \sigma_{sp}^{(2)} \\
&\cong \delta([\nu^{-x}\rho, \nu^{-c}\rho]) \times \delta([\nu^{x+1}\rho, \nu^{b}\rho]) \rtimes \sigma_{sp}^{(2)} \\
&\cong \delta([\nu^{x+1}\rho, \nu^{b}\rho]) \times \delta([\nu^{-x}\rho, \nu^{-c}\rho]) \rtimes \sigma_{sp}^{(2)}.
\end{aligned}$$

Thus, π_8 is an irreducible subquotient of $\delta([\nu^{x+1}\rho, \nu^{b}\rho]) \times \delta([\nu^{c}\rho, \nu^{x}\rho]) \rtimes \sigma_{sp}^{(2)}$, and $\delta([\nu^{x+1}\rho, \nu^{b}\rho]) \otimes \delta([\nu^{-x}\rho, \nu^{-c}\rho]) \otimes \sigma_{sp}^{(2)}$ is contained in the Jacquet module of π_8 with respect to the appropriate parabolic subgroup P_3. Consequently, there is some irreducible subquotient π'' of $\delta([\nu^c\rho, \nu^x\rho]) \rtimes \sigma_{sp}^{(2)}$ such that $\pi_8 \leq \delta([\nu^{x+1}\rho, \nu^b\rho]) \rtimes \pi''$. Since $2b+1 \not\in \mathrm{Jord}_\rho(\sigma_{sp}^{(2)})$, using Lemma 2.1 and [6, Theorem 4.6] we get that $\mu^*(\pi'') \geq \delta([\nu^{-x}\rho, \nu^{-c}\rho]) \otimes \sigma_{sp}^{(2)}$. Since $\delta([\nu^{-x}\rho, \nu^{-c}\rho]) \otimes \sigma_{sp}^{(2)}$ appears with multiplicity one in $\mu^*(\delta([\nu^c\rho, \nu^x\rho]) \rtimes \sigma_{sp}^{(2)})$ and $\mu^*(L(\delta([\nu^{-x}\rho, \nu^{-c}\rho]) \rtimes \sigma_{sp}^{(2)})) \geq \delta([\nu^{-x}\rho, \nu^{-c}\rho]) \otimes \sigma_{sp}^{(2)}$, we get

$$\pi_8 \leq \delta([\nu^{x+1}\rho, \nu^b\rho]) \rtimes L(\delta([\nu^{-x}\rho, \nu^{-c}\rho]) \rtimes \sigma_{sp}^{(2)}).$$

Lemma 3.4 now gives

$$\pi_8 \leq \delta([\nu^a\rho, \nu^b\rho]) \times \delta([\nu^c\rho, \nu^d\rho]) \rtimes \sigma_{sp}.$$

Also, from $a > c$ and $2a-1 \in \mathrm{Jord}_\rho(\sigma_{sp})$, we deduce that multiplicity of $\delta([\nu^a\rho, \nu^b\rho]) \otimes \delta([\nu^{-a+1}\rho, \nu^{-c}\rho]) \otimes \sigma_{sp}^{(2)}$ in $r_{P_3}(\delta_1 \times \delta_2 \rtimes \sigma_{sp})$ equals the multiplicity of $\delta([\nu^{-a+1}\rho,\ \nu^{-c}\rho]) \otimes \sigma_{sp}^{(2)}$ in $\mu^*(\delta_2 \rtimes \sigma_{sp})$, which equals one. Consequently, π_8 appears with multiplicity one in $\delta_1 \times \delta_2 \rtimes \sigma_{sp}$. $\square$

The following lemma can be proved applying the same arguments as in the proof of Lemma 4.6, details being left to the reader.

Lemma 4.7. *Suppose that $b < d$, $x = y$ and $a = c$. Let $\sigma_{sp}^{(1)}$ denote the strongly positive discrete series such that Jord$(\sigma_{sp}^{(1)})$ = Jord$(\sigma_{sp}) \setminus \{(2x+1, \rho)\} \cup \{(2b+1, \rho)\}$ and $\sigma_{sp}^{(2)}$ denote the strongly positive discrete series such that Jord$(\sigma_{sp}^{(2)})$ = Jord$(\sigma_{sp}) \setminus \{(2x+1, \rho)\} \cup \{(2d+1, \rho)\}$. In $R(G)$ we have*

$$\begin{aligned}\delta_1 \times \delta_2 \rtimes \sigma_{sp} = {} & L(\widetilde{\delta_1}, \widetilde{\delta_2}, \sigma_{sp}) + L(\delta([\nu^{-x}\rho, \nu^{-a}\rho]), \widetilde{\delta_2}, \sigma_{sp}^{(1)}) \\ & + L(\widetilde{\delta_1}, \delta([\nu^{-x}\rho, \nu^{-a}\rho]), \sigma_{sp}^{(2)}).\end{aligned}$$

5 The case of reducible $\delta_1 \times \delta_2$

This section is devoted to description of the composition factors of $\delta_1 \times \delta_2 \rtimes \sigma_{sp}$ when the induced representation $\delta_1 \times \delta_2$ reduces. It follows that $\rho_1 \cong \rho_2$ and, for simplicity of the notation, we write ρ instead ρ_1 and ρ_2. From $b \leq d$ and reducibility of $\delta_1 \times \delta_2$ we deduce $a < c$, $b < d$ and $c \leq b+1$. Also, we have

$$\delta([\nu^a\rho, \nu^d\rho]) \times \delta([\nu^c\rho, \nu^b\rho]) \rtimes \sigma_{sp} \leq \delta_1 \times \delta_2 \rtimes \sigma_{sp},$$

and note that, for $c \geq b$, the composition factors of $\delta([\nu^a\rho, \nu^d\rho]) \times \delta([\nu^c\rho, \nu^b\rho]) \rtimes \sigma_{sp}$ have been obtained in the previous section.

Throughout this section, let $\delta_3 = \delta([\nu^{-d}\rho, \nu^{-a}\rho])$ and $\delta_4 = \delta([\nu^{-b}\rho, \nu^{-c}\rho])$. Since in $R(G)$ we have $\delta_1 \times \delta_2 \rtimes \sigma_{sp} = \widetilde{\delta_1} \times \widetilde{\delta_2} \rtimes \sigma_{sp}$, it follows from [16, Propositon 4.6] that $L(\delta_3, \delta_4, \sigma_{sp})$ is an irreducible subquotient of $\delta_1 \times \delta_2 \rtimes \sigma_{sp}$.

If $\mathrm{Jord}_\rho(\sigma_{sp}) \cap [2a-1, 2b-1]) \neq \emptyset$, let $2x+1 = \max(\mathrm{Jord}_\rho(\sigma_{sp}) \cap [2a-1, 2b-1])$. Also, if $\mathrm{Jord}_\rho(\sigma_{sp}) \cap [2c-1, 2d-1] \neq \emptyset$, let $2y+1 = \max(\mathrm{Jord}_\rho(\sigma_{sp}) \cap [2c-1, 2d-1])$.

Proof of the following proposition is immediate:

Proposition 5.1. *Suppose that both representations $\delta_1 \rtimes \sigma_{sp}$ and $\delta_2 \rtimes \sigma_{sp}$ are irreducible. In $R(G)$ we have*

$$\delta_1 \times \delta_2 \rtimes \sigma_{sp} = L(\widetilde{\delta_1}, \widetilde{\delta_2}, \sigma_{sp}) + L(\delta_3, \delta_4, \sigma_{sp}).$$

Proposition 5.2. *Suppose that representation $\delta_1 \rtimes \sigma_{sp}$ reduces and that $\delta_2 \rtimes \sigma_{sp}$ is irreducible. We denote by $\sigma_{sp}^{(1)}$ the strongly positive discrete series such that Jord$(\sigma_{sp}^{(1)})$ = Jord$(\sigma_{sp}) \setminus \{(2x+1, \rho)\} \cup \{(2b+1, \rho)\}$ and by $\sigma_{sp}^{(2)}$ the strongly positive discrete series such that Jord$(\sigma_{sp}^{(2)})$ = Jord$(\sigma_{sp}) \setminus \{(2x+1, \rho)\} \cup \{(2d+1, \rho)\}$. If $2d+1 \notin$ Jord$_\rho(\sigma_{sp})$, in $R(G)$ we have*

$$\begin{aligned}\delta_1 \times \delta_2 \rtimes \sigma_{sp} = {} & L(\widetilde{\delta_1}, \widetilde{\delta_2}, \sigma_{sp}) + L(\delta_3, \delta_4, \sigma_{sp}) + L(\delta([\nu^{-x}\rho, \nu^{-a}\rho]), \widetilde{\delta_2}, \sigma_{sp}^{(1)}) \\ & + L(\delta([\nu^{-b}\rho, \nu^{-c}\rho]), \delta([\nu^{-x}\rho, \nu^{-a}\rho]), \sigma_{sp}^{(2)}).\end{aligned}$$

If $2d+1 \in$ Jord$_\rho(\sigma_{sp})$ and $x \geq c-1$, in $R(G)$ we have

$$\begin{aligned}\delta_1 \times \delta_2 \rtimes \sigma_{sp} = {} & L(\widetilde{\delta_1}, \widetilde{\delta_2}, \sigma_{sp}) + L(\delta_3, \delta_4, \sigma_{sp}) + L(\delta([\nu^{-x}\rho, \nu^{-a}\rho]), \widetilde{\delta_2}, \sigma_{sp}^{(1)}) \\ & + L(\delta([\nu^{-d}\rho, \nu^{-a}\rho]), \delta([\nu^{-x}\rho, \nu^{-c}\rho]), \sigma_{sp}^{(1)}).\end{aligned}$$

If $2d+1 \in$ Jord$_\rho(\sigma_{sp})$ and $x < c-1$, in $R(G)$ we have

$$\delta_1 \times \delta_2 \rtimes \sigma_{sp} = L(\widetilde{\delta_1}, \widetilde{\delta_2}, \sigma_{sp}) + L(\delta_3, \delta_4, \sigma_{sp}) + L(\delta([\nu^{-x}\rho, \nu^{-a}\rho]), \widetilde{\delta_2}, \sigma_{sp}^{(1)}).$$

Proof. Let us determine all i and j, $a-1 \leq i \leq b$, $c-1 \leq j \leq d$, such that the induced representation

$$\delta([\nu^{i+1}\rho, \nu^{b}\rho]) \times \delta([\nu^{j+1}\rho, \nu^{d}\rho]) \rtimes \sigma_{sp}$$

contains a strongly positive discrete series. We consider the following possibilities:

- $2d+1 \notin \mathrm{Jord}_\rho(\sigma_{sp})$. Since $\delta_2 \rtimes \sigma_{sp}$ is irreducible, Lemma 3.4 implies $x < c-1$. Using Lemma 3.6, we deduce $(i,j) \in \{(b,d),(x,d),(x,b)\}$. The ordered pair (b,d) provides two possible irreducible subquotients, while each of other two ordered pairs provides one possible irreducible subquotient.

- $2d+1 \in \mathrm{Jord}_\rho(\sigma_{sp})$ and $x \geq c-1$. In this case we have $(i,j) \in \{(b,d),(x,d)\}$ and, since $\delta([\nu^a\rho,\nu^x\rho]) \times \delta([\nu^c\rho,\nu^d\rho])$ reduces, Proposition 3.3 shows that in each case we have two possible irreducible subquotients.

- $2d+1 \in \mathrm{Jord}_\rho(\sigma_{sp})$ and $x < c-1$. Again, we have $(i,j) \in \{(b,d),(x,d)\}$ and, since $\delta([\nu^a\rho,\nu^x\rho]) \times \delta([\nu^c\rho,\nu^d\rho])$ is irreducible, Proposition 3.3 provides three possible irreducible subquotients.

Now the rest of the proof follows in the same way as in the proof of Proposition 4.2. $\square$

Proposition 5.3. *Suppose that the representation $\delta_1 \rtimes \sigma_{sp}$ is irreducible and that $\delta_2 \rtimes \sigma_{sp}$ reduces. We denote by $\sigma_{sp}^{(1)}$ the strongly positive discrete series such that $Jord(\sigma_{sp}^{(1)}) = Jord(\sigma_{sp}) \setminus \{(2y+1,\rho)\} \cup \{(2d+1,\rho)\}$. If $2b+1 \notin Jord_\rho(\sigma_{sp})$ or $2b+1 \in Jord_\rho(\sigma_{sp})$ and $b<y$, in $R(G)$ we have*

$$\begin{aligned}\delta_1 \times \delta_2 \rtimes \sigma_{sp} = {} & L(\widetilde{\delta_1},\widetilde{\delta_2},\sigma_{sp}) + L(\delta_3,\delta_4,\sigma_{sp}) + L(\widetilde{\delta_1},\delta([\nu^{-y}\rho,\nu^{-c}\rho]),\sigma_{sp}^{(1)}) \\ & + L(\delta([\nu^{-b}\rho,\nu^{-c}\rho]),\delta([\nu^{-y}\rho,\nu^{-a}\rho]),\sigma_{sp}^{(1)}).\end{aligned}$$

If $b=y$ and $Jord_\rho(\sigma_{sp}) \cap [2a-1,2b-1] \neq \emptyset$, in $R(G)$ we have

$$\begin{aligned}\delta_1 \times \delta_2 \rtimes \sigma_{sp} = {} & L(\widetilde{\delta_1},\widetilde{\delta_2},\sigma_{sp}) + L(\delta_3,\delta_4,\sigma_{sp}) + L(\widetilde{\delta_1},\delta([\nu^{-b}\rho,\nu^{-c}\rho]),\sigma_{sp}^{(1)}) \\ & + L(\delta([\nu^{-x}\rho,\nu^{-a}\rho]),\delta([\nu^{-b}\rho,\nu^{-c}\rho]),\sigma_{sp}^{(2)}),\end{aligned}$$

for strongly positive discrete series $\sigma_{sp}^{(2)}$ such that $Jord(\sigma_{sp}^{(2)}) = Jord(\sigma_{sp}^{(1)}) \setminus \{(2x+1,\rho)\} \cup \{(2b+1,\rho)\}$.
If $b=y$ and $Jord_\rho(\sigma_{sp}) \cap [2a-1,2b-1] = \emptyset$, in $R(G)$ we have

$$\delta_1 \times \delta_2 \rtimes \sigma_{sp} = L(\widetilde{\delta_1},\widetilde{\delta_2},\sigma_{sp}) + L(\delta_3,\delta_4,\sigma_{sp}) + L(\widetilde{\delta_1},\delta([\nu^{-b}\rho,\nu^{-c}\rho]),\sigma_{sp}^{(1)}).$$

Proof. We discuss only the case $2b+1 \in \mathrm{Jord}_\rho(\sigma_{sp})$, $2b+1 = y$ and $\mathrm{Jord}_\rho(\sigma_{sp}) \cap [2a-1,2b-1] \neq \emptyset$. Other cases can be handled in the same way, but more easily.

In this case, the induced representation

$$\delta([\nu^{i+1}\rho, \nu^b\rho]) \times \delta([\nu^{j+1}\rho, \nu^d\rho]) \rtimes \sigma_{sp}$$

contains a strongly positive discrete series for i and j such that $a-1 \leq i \leq b$, $c-1 \leq j \leq d$, if and only if $(i,j) \in \{(b,d),(b,b),(x,b)\}$.

For $(i,j) = (b,d)$, we obtain candidates $\pi_1 = L(\widetilde{\delta_1}, \widetilde{\delta_2}, \sigma_{sp})$ and $\pi_2 = L(\delta_3, \delta_4, \sigma_{sp})$ for irreducible subquotients of $\delta_1 \times \delta_2 \rtimes \sigma_{sp}$. For $(i,j) = (b,b)$, we obtain a candidate $\pi_3 = L(\widetilde{\delta_1}, \delta([\nu^{-b}\rho, \nu^{-c}\rho]), \sigma_{sp}^{(1)})$ for an irreducible subquotient of $\delta_1 \times \delta_2 \rtimes \sigma_{sp}$, and it can be shown in the same way as before that π_1, π_2 and π_3 appear with multiplicity one in composition series of $\delta_1 \times \delta_2 \rtimes \sigma_{sp}$.

If $(i,j) = (x,b)$ and $c-1 > x$, using Proposition 3.3 we obtain a candidate $\pi_4 = L(\delta([\nu^{-x}\rho, \nu^{-a}\rho], \delta([\nu^{-b}\rho, \nu^{-c}\rho]), \sigma_{sp}^{(2)})$ for irreducible subquotient of $\delta_1 \times \delta_2 \rtimes \sigma_{sp}$. If $(i,j) = (x,b)$ and $c-1 \leq x$, then the induced representation $\delta([\nu^c\rho, \nu^b\rho]) \times \delta([\nu^a\rho, \nu^x\rho])$ reduces and in this case Proposition 3.3 provides, besides π_4, an additional candidate $\pi_5 = L(\delta([\nu^{-b}\rho, \nu^{-a}\rho]), \delta([\nu^{-x}\rho, \nu^{-c}\rho]), \sigma_{sp}^{(2)})$ for irreducible subquotient of $\delta_1 \times \delta_2 \rtimes \sigma_{sp}$.

Let us prove that π_4 is contained in the composition series of $\delta_1 \times \delta_2 \rtimes \sigma_{sp}$. Several possibilities will be analyzed separately:

- $(x,b) = (a-1, c-1)$. It follows directly from Lemma 3.6 (iii) that $\pi_4 \cong \sigma_{sp}^{(2)}$ is a subquotient of $\delta_1 \times \delta_2 \rtimes \sigma_{sp}$.

- $x > a-1$ and $b = c-1$. Both representations π_4 and $\delta_1 \times \delta_2 \rtimes \sigma_{sp}$ are contained in the induced representation

$$\delta([\nu^a\rho, \nu^x\rho]) \times \delta([\nu^{x+1}\rho, \nu^b\rho]) \times \delta([\nu^c\rho, \nu^d\rho]) \rtimes \sigma_{sp}. \tag{6}$$

  Since $x < c$ and σ_{sp} is strongly positive, the structural formula for μ^* implies that the irreducible representation $\delta([\nu^{-x}\rho, \nu^{-a}\rho]) \otimes \sigma_{sp}^{(2)}$ appears with multiplicity one in the Jacquet modules of induced representations $\delta_1 \times \delta_2 \rtimes \sigma_{sp}$ and (6) with respect to the appropriate parabolic subgroup. By Frobenius reciprocity, the representation $\delta([\nu^{-x}\rho, \nu^{-a}\rho]) \otimes \sigma_{sp}^{(2)}$ appears in the Jacquet modules of π_4 with respect to the appropriate parabolic subgroup. Now Lemma 3.8 implies that π_4 is an irreducible subquotient of $\delta_1 \times \delta_2 \rtimes \sigma_{sp}$.

- $x = a-1$ and $b > c-1$. In this case, both representations π_4 and $\delta_1 \times \delta_2 \rtimes \sigma_{sp}$ are contained in the induced representation

$$\delta([\nu^a\rho, \nu^b\rho]) \times \delta([\nu^c\rho, \nu^b\rho]) \times \delta([\nu^{b+1}\rho, \nu^d\rho]) \rtimes \sigma_{sp}. \tag{7}$$

Since $a < c$ and σ_{sp} is strongly positive, it can be directly obtained that the irreducible representation $\delta([\nu^{-b}\rho, \nu^{-c}\rho]) \otimes \sigma_{sp}^{(2)}$ appears with multiplicity one in the Jacquet modules of induced representations $\delta_1 \times \delta_2 \rtimes \sigma_{sp}$ and (7) with respect to the appropriate parabolic subgroup. Since the same irreducible representation appears in the Jacquet modules of π_4 with respect to the appropriate parabolic subgroup, Lemma 3.8 shows that π_4 is an irreducible subquotient of $\delta_1 \times \delta_2 \rtimes \sigma_{sp}$.

- $x > a - 1$ and $b > c - 1$. In the same way as in the proof of Lemma 4.6 we deduce that π_4 is an irreducible subquotient of the induced representation $\delta([\nu^{c}\rho, \nu^{b}\rho]) \rtimes L(\delta([\nu^{-x}\rho, \nu^{-a}\rho]) \rtimes \sigma_{sp}^{(2)})$. Also, using the definition of x and Lemma 3.4 we obtain that $L(\delta([\nu^{-x}\rho, \nu^{-a}\rho]) \rtimes \sigma_{sp}^{(2)})$ is contained in $\delta([\nu^{a}\rho, \nu^{b}\rho]) \rtimes \sigma_{sp}^{(3)}$, where $\sigma_{sp}^{(3)}$ denotes the strongly positive discrete series such that $\mathrm{Jord}(\sigma_{sp}^{(3)}) = \mathrm{Jord}(\sigma_{sp}^{(2)}) \setminus \{(2b+1, \rho)\} \cup \{(2x+1, \rho)\}$. Consequently, we have

  $$\pi_4 \leq \delta([\nu^{c}\rho, \nu^{b}\rho]) \times \delta([\nu^{a}\rho, \nu^{b}\rho]) \rtimes \sigma_{sp}^{(3)} = \delta([\nu^{a}\rho, \nu^{b}\rho]) \times \delta([\nu^{c}\rho, \nu^{b}\rho]) \rtimes \sigma_{sp}^{(3)}.$$

  In $R(G)$ we have

  $$\delta([\nu^{c}\rho, \nu^{b}\rho]) \rtimes \sigma_{sp}^{(3)} = L(\delta([\nu^{-b}\rho, \nu^{-c}\rho]) \rtimes \sigma_{sp}^{(3)}) + L(\delta([\nu^{-x}\rho, \nu^{-c}\rho]) \rtimes \sigma_{sp}^{(2)}),$$

  so there is $\pi \in \{L(\delta([\nu^{-b}\rho, \nu^{-c}\rho]) \rtimes \sigma_{sp}^{(3)}), L(\delta([\nu^{-x}\rho, \nu^{-c}\rho]) \rtimes \sigma_{sp}^{(2)})\}$ such that $\pi_4 \leq \delta([\nu^{a}\rho, \nu^{b}\rho]) \rtimes \pi$. Since $\mu^*(\pi_4) \geq \delta([\nu^{-b}\rho, \nu^{-c}\rho]) \otimes \pi'$ for some irreducible representation π', and $a < c$, it follows that $\mu^*(\pi) \geq \delta([\nu^{-b}\rho, \nu^{-c}\rho]) \otimes \pi''$ for some irreducible representation π''. Clearly, this leads to $\pi \cong L(\delta([\nu^{-b}\rho, \nu^{-c}\rho]) \rtimes \sigma_{sp}^{(3)})$.

  Using Lemma 3.4 we deduce $L(\delta([\nu^{-b}\rho, \nu^{-c}\rho]) \rtimes \sigma_{sp}^{(3)}) \leq \delta_2 \rtimes \sigma_{sp}$, and we have

  $$\pi_4 \leq \delta([\nu^{a}\rho, \nu^{b}\rho]) \rtimes L(\delta([\nu^{-b}\rho, \nu^{-c}\rho]) \rtimes \sigma_{sp}^{(3)}) \leq \delta_1 \times \delta_2 \rtimes \sigma_{sp}.$$

It can be easily verified that the irreducible representation

$$\delta([\nu^{-x}\rho, \nu^{-a}\rho]) \otimes \delta([\nu^{-b}\rho, \nu^{-c}\rho]) \otimes \sigma_{sp}^{(2)}$$

appears in the Jacquet module of π_4 with respect to the appropriate parabolic subgroup, and appears with multiplicity one in the Jacquet module of $\delta_1 \times$

$\delta_2 \rtimes \sigma_{sp}$ with respect to the same parabolic subgroup. Thus, π_4 appears in the composition series of $\delta_1 \times \delta_2 \rtimes \sigma_{sp}$ with multiplicity one.

In the rest of the proof we will be concerned with representation π_5. Since $\delta([\nu^{-b}\rho, \nu^{-a}\rho]) \times \delta([\nu^{-x}\rho, \nu^{-c}\rho])$ is irreducible, using Lemma 3.4 we obtain following embeddings and intertwining operators:

$$\begin{aligned}
\pi_5 &\hookrightarrow \delta([\nu^{-x}\rho, \nu^{-c}\rho]) \times \delta([\nu^{-b}\rho, \nu^{-a}\rho]) \rtimes \sigma_{sp}^{(2)} \\
&\cong \delta([\nu^{-x}\rho, \nu^{-c}\rho]) \times \delta([\nu^{a}\rho, \nu^{b}\rho]) \rtimes \sigma_{sp}^{(2)} \\
&\cong \delta([\nu^{a}\rho, \nu^{b}\rho]) \times \delta([\nu^{-x}\rho, \nu^{-c}\rho]) \rtimes \sigma_{sp}^{(2)} \\
&\hookrightarrow \delta([\nu^{a}\rho, \nu^{b}\rho]) \times \delta([\nu^{-x}\rho, \nu^{-c}\rho]) \times \delta([\nu^{x+1}\rho, \nu^{b}\rho]) \rtimes \sigma_{sp}^{(3)} \\
&\cong \delta([\nu^{x+1}\rho, \nu^{b}\rho]) \times \delta([\nu^{a}\rho, \nu^{b}\rho]) \times \delta([\nu^{-x}\rho, \nu^{-c}\rho]) \rtimes \sigma_{sp}^{(3)} \\
&\hookrightarrow \delta([\nu^{x+1}\rho, \nu^{b}\rho]) \times \delta([\nu^{x+1}\rho, \nu^{b}\rho]) \times \delta([\nu^{a}\rho, \nu^{x}\rho]) \times \delta([\nu^{-x}\rho, \nu^{-c}\rho]) \rtimes \sigma_{sp}^{(3)},
\end{aligned}$$

for strongly positive discrete series $\sigma_{sp}^{(3)}$ such that $\mathrm{Jord}(\sigma_{sp}^{(3)}) = \mathrm{Jord}(\sigma_{sp}^{(2)}) \setminus \{(2b+1, \rho)\} \cup \{(2x+1, \rho)\}$.

Frobenius reciprocity and transitivity of Jacquet modules imply that there is an irreducible representation π such that

$$\mu^*(\pi_5) \geq \delta([\nu^{x+1}\rho, \nu^{b}\rho]) \times \delta([\nu^{x+1}\rho, \nu^{b}\rho]) \otimes \pi.$$

On the other hand, since $2x+1 \in \mathrm{Jord}_\rho(\sigma_{sp})$ and $b < d$, using Lemma 2.1 and [6, Theorem 4.6] we deduce that $\mu^*(\delta_1 \times \delta_2 \rtimes \sigma_{sp})$ does not contain an irreducible constituent of the form $\delta([\nu^{x+1}\rho, \nu^{b}\rho]) \times \delta([\nu^{x+1}\rho, \nu^{b}\rho]) \otimes \pi$. Consequently, π_5 is not an irreducible subquotient of $\delta_1 \times \delta_2 \rtimes \sigma_{sp}$ and proposition is proved. □

In the rest of this section we analyze the case when both induced representations $\delta_1 \rtimes \sigma_{sp}$ and $\delta_2 \rtimes \sigma_{sp}$ reduce. In what follows, let $\sigma_{sp}^{(1)}$ denote the strongly positive discrete series such that $\mathrm{Jord}(\sigma_{sp}^{(1)}) = \mathrm{Jord}(\sigma_{sp}) \setminus \{(2x+1, \rho)\} \cup \{(2b+1, \rho)\}$, let $\sigma_{sp}^{(2)}$ denote the strongly positive discrete series such that $\mathrm{Jord}(\sigma_{sp}^{(2)}) = \mathrm{Jord}(\sigma_{sp}) \setminus \{(2y+1, \rho)\} \cup \{(2d+1, \rho)\}$, and let $\sigma_{sp}^{(3)}$ denote the strongly positive discrete series such that $\mathrm{Jord}(\sigma_{sp}^{(3)}) = \mathrm{Jord}(\sigma_{sp}) \setminus \{(2x+1, \rho), (2y+1, \rho)\} \cup \{(2b+1, \rho), (2d+1, \rho)\}$.

Note that if $x < y$, then the reducibility of $\delta_1 \rtimes \sigma_{sp}$ implies $b < y$.

Description of the composition factors of $\delta_1 \times \delta_2 \rtimes \sigma_{sp}$ in the case when both induced representations $\delta_1 \rtimes \sigma_{sp}$ and $\delta_2 \rtimes \sigma_{sp}$ reduce is divided in the following sequence of propositions.

Proposition 5.4. *Suppose that $x = y$. Then in $R(G)$ we have*

$$\begin{aligned}\delta_1 \times \delta_2 \rtimes \sigma_{sp} = {} & L(\widetilde{\delta_1}, \widetilde{\delta_2}, \sigma_{sp}) + L(\delta_3, \delta_4, \sigma_{sp}) + L(\delta([\nu^{-x}\rho, \nu^{-a}\rho]), \widetilde{\delta_2}, \sigma_{sp}^{(1)}) \\ & + L(\delta([\nu^{-d}\rho, \nu^{-a}\rho]), \delta([\nu^{-x}\rho, \nu^{-c}\rho]), \sigma_{sp}^{(1)}) \\ & + L(\widetilde{\delta_1}, \delta([\nu^{-x}\rho, \nu^{-c}\rho]), \sigma_{sp}^{(2)}) \\ & + L(\delta([\nu^{-x}\rho, \nu^{-a}\rho]), \delta([\nu^{-b}\rho, \nu^{-c}\rho]), \sigma_{sp}^{(2)}).\end{aligned}$$

Proof. The induced representation

$$\delta([\nu^{i+1}\rho, \nu^{b}\rho]) \times \delta([\nu^{j+1}\rho, \nu^{d}\rho]) \rtimes \sigma_{sp}$$

contains a strongly positive discrete series for i and j such that $a - 1 \leq i \leq b$, $c - 1 \leq j \leq d$, if and only if $(i, j) \in \{(b, d), (x, d), (b, x), (x, b)\}$. Since $\delta_1 \times \delta([\nu^{c}\rho, \nu^{x}\rho])$ is irreducible, ordered pair $(i, j) = (b, x)$ provides one candidate for irreducible subquotient of $\delta_1 \times \delta_2 \rtimes \sigma_{sp}$, while all other choices provide two candidates. Also, the candidate $L(\widetilde{\delta_1}, \delta([\nu^{-x}\rho, \nu^{-c}\rho]), \sigma_{sp}^{(2)})$ is provided by both choices $(i, j) = (b, x)$ and $(i, j) = (x, b)$. The rest of the proof follows in the same way as before. □

It remains to discuss the case $x < y$. In this case, it is a direct consequence of Lemma 3.6 that the induced representation $\delta_1 \times \delta_2 \rtimes \sigma_{sp}$ contains a strongly positive discrete series subquotient if and only if $x = a - 1$, $b = c - 1$ and $x = y_{\_}$ in $\mathrm{Jord}_{\rho}(\sigma_{sp})$. Furthermore, if $\delta_1 \times \delta_2 \rtimes \sigma_{sp}$ contains a strongly positive discrete series subquotient, then the induced representation

$$\delta([\nu^{i+1}\rho, \nu^{b}\rho]) \times \delta([\nu^{j+1}\rho, \nu^{d}\rho]) \rtimes \sigma_{sp}$$

contains a strongly positive discrete series for i and j such that $a - 1 \leq i \leq b$, $c - 1 \leq j \leq d$, if and only if $(i, j) \in \{(b, d), (x, d), (b, y), (x, y), (a, c)\}$. On the other hand, if $\delta_1 \times \delta_2 \rtimes \sigma_{sp}$ does not contain a strongly positive discrete series subquotient, then the induced representation $\delta([\nu^{i+1}\rho, \nu^{b}\rho]) \times \delta([\nu^{j+1}\rho, \nu^{d}\rho]) \rtimes \sigma_{sp}$ contains a strongly positive discrete series for i and j such that $a - 1 \leq i \leq b$, $c - 1 \leq j \leq d$, if and only if $(i, j) \in \{(b, d), (x, d), (b, y), (x, y)\}$.

Choices $(i, j) = (b, d)$ and $(i, j) = (b, y)$ provide two possible irreducible subquotients of $\delta_1 \times \delta_2 \rtimes \sigma_{sp}$. If $x \geq c - 1$, the choices $(i, j) = (x, d)$ and $(i, j) = (x, y)$ also provide two possible irreducible subquotients, otherwise each of those choices provides one possible irreducible subquotient. It can be seen in the same way as before that each obtained irreducible representation appears in the composition series of $\delta_1 \times \delta_2 \rtimes \sigma_{sp}$ with multiplicity one, i.e., we have the following results:

Proposition 5.5. *Suppose that $x < y$ and $x \geq c-1$. Then in $R(G)$ we have*

$$\begin{aligned}\delta_1 \times \delta_2 \rtimes \sigma_{sp} = {} & L(\widetilde{\delta_1}, \widetilde{\delta_2}, \sigma_{sp}) + L(\delta_3, \delta_4, \sigma_{sp}) + L(\delta([\nu^{-x}\rho, \nu^{-a}\rho]), \widetilde{\delta_2}, \sigma_{sp}^{(1)}) \\ & + L(\delta([\nu^{-d}\rho, \nu^{-a}\rho]), \delta([\nu^{-x}\rho, \nu^{-c}\rho]), \sigma_{sp}^{(1)}) \\ & + L(\widetilde{\delta_1}, \delta([\nu^{-y}\rho, \nu^{-c}\rho]), \sigma_{sp}^{(2)}) \\ & + L(\delta([\nu^{-b}\rho, \nu^{-c}\rho]), \delta([\nu^{-y}\rho, \nu^{-a}\rho]), \sigma_{sp}^{(2)}) \\ & + L(\delta([\nu^{-x}\rho, \nu^{-a}\rho]), \delta([\nu^{-y}\rho, \nu^{-c}\rho]), \sigma_{sp}^{(3)}) \\ & + L(\delta([\nu^{-x}\rho, \nu^{-c}\rho]), \delta([\nu^{-y}\rho, \nu^{-a}\rho]), \sigma_{sp}^{(3)}).\end{aligned}$$

Proposition 5.6. *Suppose that $x < y$ and $x < c-1$. If $x = a-1$, $b = c-1$ and $x = y_-$ in $Jord_\rho(\sigma_{sp})$, then in $R(G)$ we have*

$$\begin{aligned}\delta_1 \times \delta_2 \rtimes \sigma_{sp} = {} & L(\widetilde{\delta_1}, \widetilde{\delta_2}, \sigma_{sp}) + L(\delta_3, \delta_4, \sigma_{sp}) + L(\delta([\nu^{-x}\rho, \nu^{-a}\rho]), \widetilde{\delta_2}, \sigma_{sp}^{(1)}) \\ & + L(\widetilde{\delta_1}, \delta([\nu^{-y}\rho, \nu^{-c}\rho]), \sigma_{sp}^{(2)}) \\ & + L(\delta([\nu^{-b}\rho, \nu^{-c}\rho]), \delta([\nu^{-y}\rho, \nu^{-a}\rho]), \sigma_{sp}^{(2)}) \\ & + L(\delta([\nu^{-x}\rho, \nu^{-a}\rho]), \delta([\nu^{-y}\rho, \nu^{-c}\rho]), \sigma_{sp}^{(3)}) + \sigma_{sp}^{(4)},\end{aligned}$$

for the strongly positive discrete series $\sigma_{sp}^{(4)}$ such that $Jord(\sigma_{sp}^{(4)}) = Jord(\sigma_{sp}) \setminus \{(2x+1, \rho)\} \cup \{(2d+1, \rho)\}$.

Proposition 5.7. *Suppose that $x < y$ and $x < c-1$. If either $x > a-1$, or $b > c-1$ or $x < y_-$ in $Jord_\rho(\sigma_{sp})$, in $R(G)$ we have*

$$\begin{aligned}\delta_1 \times \delta_2 \rtimes \sigma_{sp} = {} & L(\widetilde{\delta_1}, \widetilde{\delta_2}, \sigma_{sp}) + L(\delta_3, \delta_4, \sigma_{sp}) + L(\delta([\nu^{-x}\rho, \nu^{-a}\rho]), \widetilde{\delta_2}, \sigma_{sp}^{(1)}) \\ & + L(\widetilde{\delta_1}, \delta([\nu^{-y}\rho, \nu^{-c}\rho]), \sigma_{sp}^{(2)}) \\ & + L(\delta([\nu^{-b}\rho, \nu^{-c}\rho]), \delta([\nu^{-y}\rho, \nu^{-a}\rho]), \sigma_{sp}^{(2)}) \\ & + L(\delta([\nu^{-x}\rho, \nu^{-a}\rho]), \delta([\nu^{-y}\rho, \nu^{-c}\rho]), \sigma_{sp}^{(3)}).\end{aligned}$$

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