An efficient approximation for the optimal damping in mechanical systems

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Abstract

This paper is concerned with the efficient algorithm for damping optimization in mechanical systems with prescribed structure. Our approach is based on the minimization of the total energy of the system which is equivalent with the minimization of the trace of the corresponding Lyapunov equation. Thus, the prescribed structure in our case means that a mechanical system is close to the modally damped system. Even though our approach is very efficient (as it was expected) for the mechanical systems close to modally damped system, our experiments show that for some cases when systems are not modally damped the proposed approach provides efficient approximation of the optimal damping.

Keywords: Linear vibration system, damped-vibration, multi-variable optimization of dampers' viscosities, passive damping, Lyapunov equation, relative residual bound, linear residual bounds, quadratic residual bounds.

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1 Introduction

We consider a damped linear vibration system described by the differential equation

$$M\ddot{x} + D\dot{x} + Kx = 0\,,$$
$$x(0) = x_0, \quad \dot{x}(0) = \dot{x}_0\,,$$

where M, D, K (called mass, damping, stiffness matrix, respectively) are real, symmetric matrices of order n with M, K positive definite and $D = C_u + C$, where C_u is positive definite and presents the internal damping and C represents external damping and it is positive semidefinite. The matrix C_u is usually taken as a small multiple of the critical damping or the proportional damping. In this paper we assume that internal damping is a small multiple of mass matrix, that is $C_u = \alpha M$.

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The problem of deriving *an optimal damping* in some sense is an old and widely investigated problem which has been considered by many authors.

For example in [8] the question of placement of damping elements has been investigated, while in [6] the problem of a periodic optimal control, which maximize the energy dissipation, has been considered.

On the other hand the optimization problem which consider only viscosities' optimization, has been considered in the following papers [5], [13], [10], [11], [15] and [14].

Recently in papers [2], [3] and [1] authors considered approximations based on modal eigenvectors which provide an efficient calculation of objective functions. The case of mechanical systems with a given force was considered in the papers [12] and [7] where authors derived an explicit formulas for objective functions for the particular types of mechanical systems.

The purpose of this paper is to present a new results about approximation algorithms for deriving the optimal damping. As we will show, in some cases determination of the optimal damping can be given by an explicit formula, while in some other cases we present a numerical approach for determination of optimal damping which can be efficiently implemented.

We are going to use optimization criteria which has been considered in many papers, like [16],[5], [11], [10], [15]. This optimization criteria is given by requirement of the minimization of the total energy of the system, that is

$$\int_0^\infty E(t)\,dt \to \min\ . \tag{1.1}$$

Since, criterion (1.1) depends on the initial condition, the simplest way to correct this is to take average of (1.1) over all initial states of the unit total energy and a given frequency range. It can be shown that this average corresponds to the trace of the solution of the corresponding Lyapunov equation.

Since, up to date an efficient general algorithm for the optimization of damping does not exist, that is available algorithms optimize only viscosities of dampers, not their positions, we propose a simple and efficient approach for the overall damping optimization. With this new approach one can find optimal positions and corresponding dampers' viscosities efficiently with satisfactory accuracy.

Our approach is based on the fact that for the mechanical system which is modally damped the all tree matrices M, D and K can be simultaneously diagonalized. Thus, the main assumption here will be that we have the case where M, D and K are simultaneously diagonalizible or that they are close to the case when all tree matrices can be simultaneously diagonalized. Although, this approach has been widely used by different scientific communities, especially in the engineering, in this paper we propose a slightly different perspective, which will allow us to determine the optimal damping very efficiently for a certain structure of mechanical systems as as will be demonstrated later.

Moreover, since only the damping matrix $D(v)$ depends on parameters usual approaches for the viscosity optimization (v) assume the preprocessing based on the diagonalization of the mass and stiffness matrices, M and K. On the other hand, in this

paper we propose the new approach, which is based on the diagonalization of the damping matrix $D(v)$, and then calculation of the optimal viscosities. As we will show in this paper, this approach can be very efficient for structured systems which allow us to determine optimal viscosities, explicitly or numerically considerably faster.

For estimation of the optimal viscosity for a given dampers' positions we propose a new algorithm which is based on the simple "reduction" (truncation) of the corresponding Lyapunov equation, which usually produce speed up more than 40 times. While for the optimization of dampers' positions we propose a new heuristic. The both algorithms are based on certain heuristic and unfortunately we do not have bounds for their accuracy, but as it will be illustrated in the last section in examples with Lyapunov equation of modest dimensions ($n \leq 100$) they perform very good, thus we assume that the obtained results will be even better for bigger dimensions.

Currently, two types of algorithms are in use for the estimation of the optimal viscosity (for a given dampers' positions). The first type are the Newton-type algorithms for one - dimensional problems which use some Lyapunov solvers, and the second type are the algorithms which explicitly calculate the trace of the solution of the corresponding Lyapunov equation.

Algorithms of the second type have been presented in [16], [11] or [15] and they consider the case with one or more dampers with the same viscosity.

On the other hand, in [4] has been proposed the Newton-type algorithm for the case with $r \geq 1$ different dampers. As it has been shown in [15] the algorithm proposed in [4] can produce a poor result due to the problems with determination of the starting point.

The paper is organized as follows. In Section 2 we precisely define problem setting while in the Section 3 we present an approximation for our objective function. The problem of damping optimization with particular emphasis on structured case was studied in Section 4. The efficiency and performance of proposed approach is illustrated in Section 5.

We will use the following notation, matrices written in the simple Roman fonts, M, D or K for example, will have $\mathcal{O}(n^2)$ entries. Matrices written in the mathematical bold fonts, $\mathbf{A}$, $\mathbf{B}$ will have $\mathcal{O}(m^2)$ entries, where $m = 2n$. The symbol $\|\cdot\|$ stands for the stand 2-norm.

2 Problem definition

As it has been mentioned in the introduction, the minimization of the total energy (1.1) is equivalent to the minimization of the trace of the solution of corresponding Lyapunov equation (more details one can find in [9], [11], [13], [5], [10]).

Thus, let

$$M\ddot{x} + D\dot{x} + Kx = 0 \tag{2.2}$$

be the differential equation describing a damped linear vibration system, where M, D, K are mass, damping and stiffness matrix, respectively.

To (2.2) corresponds the eigenvalue problem

$$(\lambda^2 M + \lambda D + K)x = 0\,. \tag{2.3}$$

Just for the purpose of recapitulation of some basic properties of the eigenvalue problem (2.3) we will use the eigenvalue decomposition

$$\Phi^T K \Phi = \Omega^2\,, \quad \Phi^T M \Phi = I\,, \tag{2.4}$$

where

$$\Omega = \mathrm{diag}(\omega_1, \ldots, \omega_n), \qquad \omega_1 \leq \ldots \leq \omega_n\,. \tag{2.5}$$

By setting

$$y_1 = \Omega\,\Phi^T x \quad y_2 = \Phi^T \dot{x}\,, \tag{2.6}$$

(2.2) can be written as

$$\dot{\mathbf{y}} = \mathbf{A}\mathbf{y}\,, \tag{2.7}$$

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\,, \quad \mathbf{A} = \begin{bmatrix} 0 & \Omega \\ -\Omega & -\Phi^T D \Phi \end{bmatrix}\,, \tag{2.8}$$

(we are now in the $2n$-dimensional phase space), with the solution

$$\mathbf{y} = e^{\mathbf{A}t}\,\mathbf{y}_0\,, \quad \text{where } \mathbf{y}_0 \text{ is the initial data.} \tag{2.9}$$

It can be shown that criterion of the minimization of the total energy (1.1) is equivalent to

$$\mathrm{tr}(\mathbf{Z}\mathbf{X}_{\mathbf{\Phi}}) \to \min\,, \tag{2.10}$$

where $\mathbf{X}_{\mathbf{\Phi}}$ is solution of the following Lyapunov euqation

$$\mathbf{A}^T\mathbf{X}_{\mathbf{\Phi}} + \mathbf{X}_{\mathbf{\Phi}}\mathbf{A} = -\mathbf{I}\,.$$

and $\mathbf{Z}$ is a symmetric positive semidefinite matrix which may be normalized to have a unit trace. If we take for the measure σ the measure generated by Lebesgue measure on $\mathbb{R}^{2n}$, we obtain $\mathbf{Z} = \frac{1}{2n}\mathbf{I}$. Without loss of generality, hereinafter we omit the factor $\frac{1}{2n}$ from the definition of the matrix $\mathbf{Z}$.

While we have internal damping which is not trivial it can be shown that all eigenvalues of (2.3) lie in the left complex plane. That means that matrix A from (2.8) is asymptotically stable.

Further it is easy to show that

$$\mathrm{tr}(\mathbf{Z}\mathbf{X}_{\mathbf{\Phi}}) = \mathrm{tr}(\mathbf{Y})\,,$$

where $\mathbf{Y}$ is a solution of the so-called "dual Lyapunov equation"

$$\mathbf{A}\mathbf{Y} + \mathbf{Y}\mathbf{A}^T = -\mathbf{Z}. \tag{2.11}$$

The structure of the matrix $\mathbf{Z}$ has been studied in detail in [9] and some of this results are presented in [11].

Through this paper we will assume that the matrix $\mathbf{Z}$ has the following form

$$\mathbf{Z} = \begin{bmatrix} 0_{t_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & I_s & 0 & 0 & 0 & 0 \\ 0 & 0 & 0_{t_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0_{t_1} & 0 & 0 \\ 0 & 0 & 0 & 0 & I_s & 0 \\ 0 & 0 & 0 & 0 & 0 & 0_{t_2} \end{bmatrix}, \tag{2.12}$$

where I_s is the s-dimensional identity matrix, and 0_{t_i} is the t_i-dimensional ($i = 1, 2$) zero matrix, where t_1 and s are defined such that the eigenfrequencies from (2.5) smaller then ω_{t_1} and greater then ω_{t_1+s} are not dangerous (observe that $t_2 = n - t_1 - s$).

Note, that the solution of Lyapunov equation (2.11) is a function of several variables, dampers positions and corresponding viscosities. Thus, the simultaneous optimization of dampers' positions and viscosity can be very demanding computationally. In the following we will propose a new approach for dampers optimization.

First, in the next section we will present a new algorithm which approximate the solution (as well as its trace) of the corresponding Lyapunov equation, and after that in Section 4 we propose a new algorithm for finding optimal dampers' positions.

3 Approximation of the solution of the Lyapunov equation

It is well known that the linearization from (2.8) is not unique. Thus for our purpose we will rewrite (2.2) using the following linearization:

$$\dot{\mathbf{y}} = \mathbf{A}_* \mathbf{y}\,, \tag{3.13}$$

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \quad \mathbf{A}_* = \begin{bmatrix} 0 & K^{\frac{1}{2}} M^{-\frac{1}{2}} \\ -M^{-\frac{1}{2}} K^{\frac{1}{2}} & -M^{-\frac{1}{2}} D M^{-\frac{1}{2}} \end{bmatrix}, \tag{3.14}$$

Following the exposure from the previous section it follows that we are interested in minimizing the trace

$$\mathrm{tr}(\mathbf{ZY})\,,$$

where $\mathbf{Y}$ is a solution of the Lyapunov equation

$$\mathbf{A}_*{}^T \mathbf{Y} + \mathbf{Y} \mathbf{A}_* = -\mathbf{I}\,, \tag{3.15}$$

and $\mathbf{Z}$ is defined in (2.12).

Up to this point we did not introduce any new approaches or new ideas which we have described in the introduction. Thus, in continuation we proceed with the presentation of the new approach for approximation of the solution of Lyapunov equation (3.15) which is different than the standard ones (mostly used by engineers), which is based on the modal approximation of mechanical systems. Our approach will combine two aspects, one is a

modal approximation approach and the other is approach based on the improved error estimates e.g. see [2], [3] and [1].

For that purpose let

$$M^{-\frac{1}{2}}DM^{-\frac{1}{2}} = U_0 \Delta U_0^T\,, \qquad \Delta = \mathrm{diag}(\delta_1, \ldots, \delta_n)\,, \tag{3.16}$$

be the eigenvalue decomposition of the "damping matrix" $M^{-\frac{1}{2}}DM^{-\frac{1}{2}}$.

Let

$$\mathbf{T} = \begin{bmatrix} I & 0 \\ 0 & U_0 \end{bmatrix} \tag{3.17}$$

be the orthogonal matrix where U_0 is defined in (3.16). If one multiplies the Lyapunov equation (3.15) from the left and from right with $\mathbf{T}^T$ and $\mathbf{T}$, respectively then one gets

$$\mathbf{A}^T\mathbf{X} + \mathbf{X}\mathbf{A} = -\mathbf{I}\,, \tag{3.18}$$

where

$$\mathbf{A} = \mathbf{T}^T\mathbf{A}_*\mathbf{T} = \begin{bmatrix} 0 & B^T \\ -B & -\Delta \end{bmatrix} \tag{3.19}$$

where $B = U_0^T M^{-\frac{1}{2}} K^{\frac{1}{2}}$ and

$$\mathbf{X} = \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22} \end{bmatrix}. \tag{3.20}$$

Now equation (3.18) can be written as

$$\begin{bmatrix} 0 & -B^T \\ B & -\Delta \end{bmatrix} \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22} \end{bmatrix} + \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22} \end{bmatrix} \begin{bmatrix} 0 & B^T \\ -B & -\Delta \end{bmatrix} = - \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}, \tag{3.21}$$

where

$$\mathbf{Y} = \mathbf{T} \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22} \end{bmatrix} \mathbf{T}^T. \tag{3.22}$$

It obviously holds $\mathrm{tr}(\mathbf{Z}\mathbf{Y}) = \mathrm{tr}(\mathbf{Z}_\mathbf{U}\mathbf{X})$, where

$$\mathbf{Z}_\mathbf{U} = \mathbf{T}^T\mathbf{Z}\mathbf{T}\,. \tag{3.23}$$

Now, from (3.21) one gets

$$B^T X_{12}^T + X_{12} B = I\,, \tag{3.24}$$

$$-B^T X_{22} + X_{11} B^T - X_{12}\Delta = 0\,, \tag{3.25}$$

$$B X_{11} - X_{22} B - \Delta X_{12}^T = 0\,, \tag{3.26}$$

$$B X_{12} + X_{12}^T B^T - \Delta X_{22} - X_{22}\Delta = -I\,. \tag{3.27}$$

From (3.24) it follows

$$X_{12} = \frac{1}{2}B^{-1} + SB^{-1}\,, \quad \text{where} \quad S = -S^T\,. \tag{3.28}$$

Thus, if one knows the skew-symmetric matrix S from (3.28) then the solution $\mathbf{X}$ is known.

The new approach: As it has been described in the introduction our approach is based on some interesting properties of the modally damped system. As it is well known (for example see [17]), the modally damped system satisfies so called commuting condition

$$KD^{-1}M = MD^{-1}K\,. \tag{3.29}$$

In continuation we will show that this assumption is equivalent to the assumption on commuting X_{12} and B, that is we will show that if

$$X_{12}B = BX_{12}\,, \tag{3.30}$$

then (3.29) holds, and the mechanical system is modally damped.

If X_{12} and B commute, that is if (3.30) holds than (3.27) and (3.26) imply that

$$\Delta X_{22} + X_{22}\Delta = 2I \tag{3.31}$$

$$X_{11} = B^{-1}X_{22}B + B^{-1}\Delta X_{12}^T\,. \tag{3.32}$$

Here we have used the fact that if (3.30) holds, then $BX_{12} + X_{12}^T B^T = I$.

On the other hand the assumption that (3.30) holds implies that S from (3.28) is zero matrix.

The main idea: In continuation we do not assume that (3.30) holds, that is our mechanical system is no longer modally damped.

But if it is still "good in a some sense", or "close" to a modally damped system we can use the above conclusions to approximate solution $\mathbf{X}$ of the Lyapunov equation (3.18).

For that purpose, first we will approximate X_{12} from (3.28) with

$$\widetilde{X}_{12} = \frac{1}{2}B^{-1}\,. \tag{3.33}$$

Further, from (3.31) follows that

$$\widetilde{X}_{22} = \Delta^{-1}\,. \tag{3.34}$$

Once we have derived $\widetilde{X}_{22}$ it is easy to derive the last unknown approximation $\widetilde{X}_{11}$. Indeed, from (3.32) it follows

$$\widetilde{X}_{11} = B^{-1}\widetilde{X}_{22}B + \frac{1}{2}B^{-1}\Delta B^{-T}\,. \tag{3.35}$$

In the next theorem we will present the residual error

$$R_{er} = \|\mathbf{A}^T\widetilde{\mathbf{X}} + \widetilde{\mathbf{X}}\mathbf{A} + \mathbf{I}\| \tag{3.36}$$

made by the approximation

$$\widetilde{\mathbf{X}} = \begin{bmatrix} \widetilde{X}_{11} & \widetilde{X}_{12} \\ \widetilde{X}_{12}^T & \widetilde{X}_{22} \end{bmatrix}\,, \tag{3.37}$$

which is equivalent as to insert approximations $\widetilde{X}_{11}$, $\widetilde{X}_{22}$ and $\widetilde{X}_{12}$ into (3.25).

The following theoretical results will be used for the error estimation which we have made by the above approximation of the solution of the Lyapunov equation. Also, it will be used for the a priori estimation, whether the considered mechanical system be approximated with modally damped one.

Theorem 3.1. *Let $\widetilde{\mathbf{X}}$ be the approximation of the solution (3.20) of the Lyapunov equation (3.21). Then the residual error R_{er} is given by*

$$R_{er} = \|B^T\Delta^{-1} - B^{-1}\Delta^{-1}BB^T\|\,. \tag{3.38}$$

Proof. The proof simply follows by inserting $\widetilde{\mathbf{X}}$ in (3.21). Indeed, from (3.37) and (3.21) one gets

$$\begin{bmatrix} 0 & -B^T \\ B & -\Delta \end{bmatrix}\begin{bmatrix} \widetilde{X}_{11} & \widetilde{X}_{12} \\ \widetilde{X}_{12}^T & \widetilde{X}_{22} \end{bmatrix} + \begin{bmatrix} \widetilde{X}_{11} & \widetilde{X}_{12} \\ \widetilde{X}_{12}^T & \widetilde{X}_{22} \end{bmatrix}\begin{bmatrix} 0 & B^T \\ -B & -\Delta \end{bmatrix} = -\begin{bmatrix} I & Err \\ 0 & I \end{bmatrix}, \tag{3.39}$$

where

$$-Err = -B^T\widetilde{X}_{22} + B^{-1}\widetilde{X}_{22}BB^T + \frac{1}{2}\left(-B^{-1}\Delta + B^{-1}\Delta\right).$$

Now, since

$$R_{er} = \|Err\|\,,$$

the (3.38) holds, which ends the proof. ∎

Lemma 3.1. *Let $B = U_0^T M^{-\frac{1}{2}}K^{\frac{1}{2}}$ and $\Delta = U_0^T M^{-\frac{1}{2}}DM^{-\frac{1}{2}}U_0$. Then $BB^T\Delta^{-1} = \Delta^{-1}BB^T$ if and only if the mechanical system from (2.2) is modally damped, that is, equality (3.29) holds.*

Proof. Since $B = U_0^T M^{-\frac{1}{2}}K^{\frac{1}{2}}$ and $\Delta = U_0^T M^{-\frac{1}{2}}DM^{-\frac{1}{2}}U_0$ it follows that

$$BB^T = U_0^T M^{-\frac{1}{2}}KM^{-\frac{1}{2}}U_0 \quad \text{and} \quad \Delta^{-1} = U_0^T M^{\frac{1}{2}}D^{-1}M^{\frac{1}{2}}U_0\,.$$

Now, simple multiplication gives

$$\begin{aligned} BB^T\Delta^{-1} &= U_0^T M^{-\frac{1}{2}}KD^{-1}M^{\frac{1}{2}}U_0, \\ \Delta^{-1}BB^T &= U_0^T M^{\frac{1}{2}}D^{-1}KM^{-\frac{1}{2}}U_0\,, \end{aligned}$$

which together imply that if $BB^T\Delta^{-1} = \Delta^{-1}BB^T$ that

$$KD^{-1}M = MD^{-1}K\,.$$

∎

As the consequence of the above theorem we have the following corollary.

Corollary 3.1. *Let the assumptions of theorem 3.1 hold, that is*

$$BB^T\Delta^{-1} = \Delta^{-1}BB^T\,,$$

holds if and only (3.29) holds. Then $\widetilde{\mathbf{X}} = \mathbf{X}$ *are the solutions of the Lyapunov equation (3.21).*

Proof. If $BB^T\Delta^{-1} = \Delta^{-1}BB^T$ then from (3.38) follows that $R_{er} = 0$, which implies that $\widetilde{\mathbf{X}} = \mathbf{X}$. ∎

4 The damping optimization

Using the approximation from the previous section here we will present a new approach to the damping optimization. Thus, we assume that considered mechanical system is close to the perturbed modally damped system, that is further on we assume that the residual error R_{er} from (3.38) is small enough, which means that $KD^{-1}M \approx MD^{-1}K$ in some sense.

For that purpose, let

$$M = U_M\Lambda_M U_M^T, \quad U_M = \begin{bmatrix} u_1 & \dots & u_n \end{bmatrix}, \quad \Lambda_M = \begin{bmatrix} \mu_1 & \dots & \mu_n \end{bmatrix},$$

be the eigenvalue decomposition of the the mass matrix M.

We will distinguish two different cases. In the first one we assume that the damping matrix D has the same eigenvector structure as the mass matrix M, that is we will assume that

$$D_I = \nu_1 u_1 u_1^T + \nu_2 u_2 u_2^T + \ldots + \nu_n u_n u_n^T\,, \tag{4.40}$$

where $\nu_i = v_i + \alpha$, $i = 1, \ldots, n$.

For the damping matrix which is close to D_I, and in the case when the number of dampers is equal to the dimension, that is when $r = n$, we will be able to derive the explicit formula for the optimal damping's viscosities v_i, $i = 1, \ldots, n$. On the other hand, for the case when the number of dampers is less then dimension or some viscosities are the same, we will present a formula that covers these cases in more general setting.

Thus, back to the first case, we will assume that the damping matrix D is close to D_I from (4.40), that is

$$D \approx (v_1 + \alpha)u_1 u_1^T + (v_2 + \alpha)u_2 u_2^T + \ldots + (v_n + \alpha)u_n u_n^T\,. \tag{4.41}$$

Below we will derive a simple formula for calculation of the optimal viscosities $v_1, \ldots, v_n$, for which the trace of the approximation $\widetilde{\mathbf{X}}$ from (3.37) is minimal.

If we are interested in damping the s undamped frequencies, then using the matrix Z from (2.12) we obtain that for the matrix Z_U from (3.23), can be written as

$$\mathbf{Z_U} \doteq \begin{bmatrix} Z_1 & 0 \\ 0 & Z_2 \end{bmatrix},$$

where

$$Z_1 = \operatorname{diag}(0_{t_1}, I_s, 0_{t_2})\,, \tag{4.42}$$

$$Z_2 = U_0^T \operatorname{diag}(0_{t_1}, I_s, 0_{t_2}) U_0. \tag{4.43}$$

Since our penalty function is a trace of the solution of the corresponding Lyapunov equation, note that for the approximation of the trace holds

$$\begin{aligned}\operatorname{tr}(\mathbf{Z_U X}) \approx \operatorname{tr}(\mathbf{Z_U}\widetilde{\mathbf{X}}) &= \operatorname{tr}(Z_1\widetilde{X}_{11}) + \operatorname{tr}(Z_2\widetilde{X}_{22}) \\ &= \operatorname{tr}(Z_1\widetilde{X}_{22}) + \operatorname{tr}(Z_2\widetilde{X}_{22}) + \frac{1}{2}\operatorname{tr}(B^{-1}Z_1\Delta B^{-T})\,,\end{aligned}$$

that is

$$\operatorname{tr}(\mathbf{Z_U X}) \approx \operatorname{tr}(\mathbf{Z_U}\widetilde{\mathbf{X}}) = \operatorname{tr}(Z_1\Delta^{-1}) + \operatorname{tr}(Z_2\Delta^{-1}) + \frac{1}{2}\operatorname{tr}(Z_1\Delta B^{-T}B^{-1})\,. \tag{4.44}$$

The approximation (4.44) will be our starting point which will allow us to derive approximation for optimal $v_1^*, \ldots, v_n^*$.

Note that from (4.41) and (3.16) follows that

$$M^{-\frac{1}{2}} D M^{-\frac{1}{2}} = U_0 \Delta U_0^T, \quad \text{where } \Delta = \operatorname{diag}(v_1+\alpha, v_2+\alpha, \ldots, v_n+\alpha)\,. \tag{4.45}$$

Now from (4.44) and (4.45) one gets

$$\operatorname{tr}(\mathbf{Z_U}\widetilde{\mathbf{X}}(v_1,\ldots,v_n)) = \sum_{i=1}^{n} \frac{(Z_1)_{ii} + (Z_2)_{ii}}{v_i+\alpha} + \frac{1}{2}\sum_{i=1}^{n}(v_i+\alpha)(Z_1)_{ii}b_i\,, \tag{4.46}$$

where $b_i = \|T(:,i)\|^2\,, \quad T = B^{-1}$ for $i = 1, \ldots, n$.

Using the fact that all quantities in (4.46) are nonnegative, simply using the partial derivatives,

$$\frac{\delta}{\delta v_i}\operatorname{tr}(\mathbf{Z_U}\widetilde{\mathbf{X}}(v_1,\ldots,v_n)) = -\frac{(Z_1)_{ii} + (Z_2)_{ii}}{(v_i+\alpha)^2} + \frac{1}{2}(Z_1)_{ii}b_i\,, \quad i = 1, \ldots, n\,, \tag{4.47}$$

and equalizing to zeros, one gets that

$$v_i^* = \sqrt{\frac{2(Z_1)_{ii} + 2(Z_2)_{ii}}{(Z_1)_{ii}b_i}} - \alpha\,, \qquad i = 1, \ldots, n\,.$$

are minimum for the trace, that is

$$(v_1^*, \ldots, v_n^*) = \arg\min \operatorname{tr}(\mathbf{Z_U}\widetilde{\mathbf{X}}(v_1,\ldots,v_n)).$$

The setting given by (4.40) was just a motivation for a more general case that we will consider in the next section.

4.1 The damping optimization for the structured case

Trough this section we assume that the eigenvalue decomposition of the matrix $M^{-\frac{1}{2}}DM^{-\frac{1}{2}}$ is given by

$$M^{-\frac{1}{2}}DM^{-\frac{1}{2}} = U_0\Delta U_0^T\,, \quad \Delta = v_1D_1 \oplus v_2D_2 \oplus \cdots \oplus v_dD_d \tag{4.48}$$

where each matrix D_i, $i = 1, \ldots, d$ is diagonal matrix and it has dimension d_i, $i = 1, \ldots, d$, respectively, with $\sum_{i=1}^{d} d_i = n$.

The above assumption, means that the matrix Δ is direct sum of smaller matrices that correspond to the same viscosities and it arises from the fact that very often the damping matrix D can have blocks of dampers which have the same viscosities. Moreover, in assumed setting the damping blocks with a different viscosities do not interlace with each other.

Note that setting included in (4.40) is also covered by (4.48), since we can use this approach also in the case when $d_i = 1$, $\forall i = 1, \ldots, n$ considering that all viscosities are different. On the other hand, we would like to emphasize that form given in (4.48) generalizes motivation setting (from previous sections where we assumed all different viscosities) on the case when some viscosities can be the same. Moreover, it also includes more general cases in which the damping matrix D is permutation-similar to the block diagonal matrix where each block corresponds to the damping parts with its own viscosity parameter.

Similarly as the above if we are interested in damping the first s most important eigenfrequencies, then the matrix $\mathbf{Z_U}$ from (3.23), can be written as

$$\mathbf{Z_U} = \begin{bmatrix} Z_1 & 0 \\ 0 & Z_2 \end{bmatrix},$$

where Z_1 and Z_2 are given by (4.42-4.43).

Also, for the approximation of the trace it holds

$$\operatorname{tr}(\mathbf{Z_U}\mathbf{X}) \approx \operatorname{tr}(\mathbf{Z_U}\widetilde{\mathbf{X}}) = \operatorname{tr}(Z_1\widetilde{X}_{22}) + \operatorname{tr}(Z_2\widetilde{X}_{22}) + \frac{1}{2}\operatorname{tr}(B^{-1}Z_1\Delta B^{-T})\,,$$

that is

$$\operatorname{tr}(\mathbf{Z_U}\mathbf{X}) \approx \operatorname{tr}(\mathbf{Z_U}\widetilde{\mathbf{X}}) = \operatorname{tr}(Z_1\Delta^{-1}) + \operatorname{tr}(Z_2\Delta^{-1}) + \frac{1}{2}\operatorname{tr}(Z_1\Delta B^{-T}B^{-1})\,. \tag{4.49}$$

Now using the approximation (4.49) together with (4.48) we can derive approximate optimal parameters $v_1^*, \ldots, v_d^*$.

In particular, from (4.48) and (4.49) one gets

$$\begin{aligned}\operatorname{tr}(\mathbf{Z_U}\widetilde{\mathbf{X}}(v_1, \ldots, v_d)) = &\sum_{i=1}^{d} \sum_{j=d_1+\cdots+d_{i-1}+1}^{d_1+\cdots+d_i} \frac{(Z_1)_{jj} + (Z_2)_{jj}}{v_i(D_i)_{k_jk_j} + \alpha} \\ &+ \frac{1}{2}\sum_{i=1}^{d} \sum_{j=d_1+\cdots+d_{i-1}+1}^{d_1+\cdots+d_i} (Z_1)_{jj}b_j(v_i(D_i)_{k_jk_j} + \alpha)\,,\end{aligned} \tag{4.50}$$

where k_j determines index that depend on j and it holds that $k_j = j - (d_1 + \cdots + d_{j-1})$. Moreover, b_i is the 2-norm of the column of the matrix $T = B^{-1}$, that is $b_i = \|T(:,i)\|^2$.

In general, for this function we are not able to derive an explicit formula form optimal viscosities. But since in this case where, the matrix which diagonalizes the matrix $M^{-\frac{1}{2}} D M^{-\frac{1}{2}}$ is the same for all viscosities, we can determine optimal viscosities efficiently by numerical optimization procedure which will be described in the next section.

Additionally, we are also able to derive an explicit formula for global minimum if $\sum_{i=1}^{d} \operatorname{rank}(D_i) = n$ and $\alpha = 0$. In that case our objective function has the following form:

$$\begin{aligned}\operatorname{tr}(\mathbf{Z_U}\widetilde{\mathbf{X}}(v_1,\ldots,v_d)) = &\sum_{i=1}^{d} \frac{1}{v_i} \sum_{j=d_1+\cdots+d_{i-1}+1}^{d_1+\cdots+d_i} \frac{(Z_1)_{jj} + (Z_2)_{jj}}{(D_i)_{k_j k_j}} \\ &+ \frac{1}{2} \sum_{i=1}^{d} v_i \sum_{j=d_1+\cdots+d_{i-1}+1}^{d_1+\cdots+d_i} (Z_1)_{jj} b_j (D_i)_{k_j k_j} \,. \end{aligned} \tag{4.51}$$

Using the fact that all quantities in (4.50) are nonnegative one easily obtains the partial derivatives

$$\frac{\delta}{\delta v_i} \operatorname{tr}(\mathbf{Z_U}\widetilde{\mathbf{X}}(v_1,\ldots,v_d)) = -\frac{\sum_{j=d_1+\cdots+d_{i-1}+1}^{d_1+\cdots+d_i} \frac{(Z_1)_{jj}+(Z_2)_{jj}}{(D_i)_{k_j k_j}}}{v_i^2} + \frac{1}{2} \sum_{j=d_1+\cdots+d_{i-1}}^{d_1+\cdots+d_i} (Z_1)_{jj} b_j (D_i)_{k_j k_j} \,, \tag{4.52}$$

for $i = 1, \ldots, d$. Now, by equalizing above derivations with zero, one gets that

$$v_i^* = \sqrt{\frac{2\sum_{j=d_1+\cdots+d_{i-1}+1}^{d_1+\cdots+d_i} \frac{(Z_1)_{jj}+(Z_2)_{jj}}{(D_i)_{k_j k_j}}}{\sum_{j=d_1+\cdots+d_{i-1}}^{d_1+\cdots+d_i} (Z_1)_{jj} b_j (D_i)_{k_j k_j}}} \,, \qquad i = 1, \ldots, d \,, \tag{4.53}$$

are optimal viscosities, that is

$$(v_1^*, \ldots, v_d^*) = \arg\min \operatorname{tr}(\mathbf{Z_U}\widetilde{\mathbf{X}}(v_1,\ldots,v_d)) \,.$$

REMARK **4.1.** Objective function given by (4.50) for parameter $\alpha \neq 0$ can be efficiently optimized using numerical optimization procedure. In particular, in this case we are dealing with minimization of d function where ith function f_i is given by

$$f_i(v_i) = \sum_{j=d_1+\cdots+d_{i-1}+1}^{d_1+\cdots+d_i} \frac{(Z_1)_{jj} + (Z_2)_{jj}}{v_i (D_i)_{k_j k_j} + \alpha} + \sum_{j=d_1+\cdots+d_{i-1}+1}^{d_1+\cdots+d_i} (Z_1)_{jj} b_j (v_i (D_i)_{k_j k_j} + \alpha) \,, \tag{4.54}$$

for $i = 1, \ldots, d$. Here function f_i is strictly convex function with global minima v_i^*, for $i = 1, \ldots, d$ respectively, where minima v_i^*, for $i = 1, \ldots, d$ can be efficiently determined using iterative solvers. By this approach we are able to determine optimal parameters v_i^*, for $i = 1, \ldots, d$ that minimizes $\operatorname{tr}(\mathbf{Z_U}\widetilde{\mathbf{X}}(v_1,\ldots,v_d))$.

4.2 The damping optimization for the general case

In this section we consider a more general case, then the two from the previous sections, but still we assume that our system corresponds to the configuration where (3.38) is small enough, or that approximation $KD^{-1}M \approx MD^{-1}K$ holds in a a certain sense.

Since, M is a positive definite and D is a positive semidefinite matrix, there exists an orthogonal matrix U such that

$$M^{-\frac{1}{2}}DM^{-\frac{1}{2}} = U\Delta U^T\,, \quad \Delta = \mathrm{diag}(\delta_1, \ldots, \delta_n). \tag{4.55}$$

Apart from the previous cases where we are able to derive an explicit formula for the global minima, in this section we will present a numerical approach for calculation of an approximation of the optimal viscosities. The main problem within this *general case* is that the matrix U which diagonalizes the matrix $M^{-\frac{1}{2}}DM^{-\frac{1}{2}}$ depends on viscosities that determine the damping matrix D, contrary the cases from previous sections.

Thus, let us assume that is a given d dampers with corresponding viscosities v_i, $i = 1, \ldots, d$, which determine our external damping matrix $C(v_1, \ldots, v_d)$, that is damping matrix is given by $D(v_1, \ldots, v_d) = C_u + C(v_1, \ldots, v_d)$. Since in general the matrix U from (4.55) depends on viscosities, let U_0 be the unitary matrix which diagonalize $D(v_1^0, \ldots, v_d^0)$ for the initial viscosities $(v_1^0, \ldots, v_d^0)$.

Now similarly as in the beginning of this section, we can calculate approximation of the trace of the solution of the corresponding Lyapunov equation for the given viscosities $(v_1^0, \ldots, v_d^0)$. That is, similarly as the above we can show that

$$\mathrm{tr}(\mathbf{Z_U}\widetilde{\mathbf{X}}(v_1^0, \ldots, v_d^0)) = \sum_{i=1}^{n} \frac{(Z_2)_{ii} + (Z_1)_{ii}}{\delta_i} + \frac{1}{2}\sum_{i=1}^{n} \delta_i (Z_1)_{ii} b_i\,, \tag{4.56}$$

where Z_1 and Z_2 are given by (4.42-4.43) and

$$M^{-\frac{1}{2}}(D(v_1^0, \ldots, v_d^0))M^{-\frac{1}{2}} = U_0\Delta U_0^T\,, \quad \Delta = \mathrm{diag}(\delta_1, \ldots, \delta_n)\,, \tag{4.57}$$

$$b_i = \|T_0(:, i)\|^2\,, \quad T_0 = B^{-1}\,, \quad i = 1, \ldots, n. \tag{4.58}$$

Now, we do not have an explicit formula for the optimal viscosities, thus we propose the following numerical approach for viscosity optimization.

During the optimization process, the next iteration (for viscosities) $(v_1^1, \ldots, v_d^1)$ can be calculated using corresponding matrix U_1, given as

$$M^{-\frac{1}{2}}(D(v_1^1, \ldots, v_d^1))M^{-\frac{1}{2}} = U_1\Delta U_1^T\,, \quad \Delta = \mathrm{diag}(\delta_1^1, \ldots, \delta_n^1)\,,$$

which insures the corresponding trace approximation. Here is important to notice, that very often, during the optimization process, the same subspace U is also good for a several iteration steps (that is for a several viscosity updates). Thus, during the optimization process we first check if the same subspace is good enough, meaning that the residual error

$$er_U = \|M^U - \mathrm{diag}(M_{11}^U, M_{22}^U, \ldots, M_{nn}^U)\| < tol_U \tag{4.59}$$

where $M^U = U_0^T M^{-\frac{1}{2}}(C_u + C(v_1^1, \ldots, v_d^1))M^{-\frac{1}{2}}U_0$ and tol_U is some given tolerance.

This means that in optimization process if er_U defined in (4.59), for viscosities $(v_1^1, \ldots, v_d^1)$ is smaller than the tolerance tol_U we will use the unitary matrix U_0 instead of U_1 for approximation of the trace $\mathrm{tr}(\mathbf{Z_U}\widetilde{\mathbf{X}}(v_1^1, \ldots, v_d^1))$.

The algorithm for the optimization of the viscosities is summarized in the Algorithm 1.

We would like to emphasize that the main cost in the trace approximation (4.56) belongs to the calculation of the matrix U, thus, in Algorithm 1 by using the residual tolerance tol_U we can avoid calculation of the matrix U for some viscosities $(v_1^i, \ldots, v_d^i)$ (for some i's) which significantly accelerates the optimization process. Moreover, as one could expect, the matrix U does not need to be calculated (up to tolerance tol_U) in each step of iterations (for the various viscosities $(v_1^i, \ldots, v_d^i)$) if mechanical system has some special structure or if the changes in viscosities are small (which often appears during the optimization process).

REMARK **4.2.** Note that using Algorithm 1 we can also minimize objective functions given by (4.50). In that case the err_U will be zero (up to machine tolerance) and we will be able to calculate approximation of the objective function without calculating the matrix U at each step. On the other hand, since in this case the objective function consists of d independent functions it is even more efficient to use an approach described in Remark 4.1.

5 Numerical experiments

In this section we will illustrate the performance of the new approach on two examples which consider the mechanical system, so called the *n-mass oscillator*. In all examples we will take $\mathbf{Z_U} = \mathbf{I}$.

Example 5.1. *In this example we consider the system from (2.2) with dimension $n = 20$, where the mass and the stiffness matrices are defined as:*

$$M = \mathrm{diag}(m_1, m_2, \ldots, m_n)\,,$$

$$m_i = \begin{cases} 200 - 20(i-1) & , i = 1, \ldots, 10 \\ 201 + 20(i-11) & , i = 11, \ldots, 20 \end{cases}$$

$$K = \begin{bmatrix} 4 & -1 & -1 & & & \\ -1 & 4 & -1 & -1 & & \\ -1 & -1 & 4 & -1 & -1 & \\ & \ddots & \ddots & \ddots & \ddots & \ddots \\ & -1 & -1 & 4 & -1 & -1 \\ & & -1 & -1 & 4 & -1 \\ & & & -1 & -1 & 4 \end{bmatrix}.$$

Structure of masses and stiffness is shown on Figure 5.1. Further, the damping matrix

Algorithm 1 Computation of optimal viscosities

Require: System matrices; tolerance tol_U for updating eigensubspace U; starting viscosities $(v_1^0, \ldots, v_d^0)$.

Ensure: Approximation of optimal viscosities.

1: Calculate approximation of the trace given in (4.56) and U_0 given in (4.57). Set $U = U_0$.

2: Find optimal viscosities by using an appropriate optimization procedure (e.g. the Nelder-Mead algorithm). Evaluate the function value using trace approximation at the given viscosities $(v_1^i, \ldots, v_d^i)$ as in steps 3 to 8:

3: Calculate the error for the subspace U from

$$er_U = \|M^U - \mathrm{diag}(M_{11}^U, M_{22}^U, \ldots, M_{nn}^U)\|$$

where $M^U = U^T M^{-\frac{1}{2}}(C_u + C(v_1^i, \ldots, v_d^i))M^{-\frac{1}{2}}U$.

4: **if** $err_U < tol_U$ **then**

5: Compute function value at viscosities $(v_1^i, \ldots, v_d^i)$ using

$$\mathrm{tr}(\mathbf{Z_U}\widetilde{\mathbf{X}}(v_1^i, \ldots, v_d^i)) = \sum_{i=1}^{n} \frac{(Z_2)_{ii} + (Z_1)_{ii}}{\delta_i} + \frac{1}{2}\sum_{i=1}^{n} \delta_i (Z_1)_{ii} b_i \,,$$

where where Z_1 and Z_2 are given by (4.42-4.43) and

$$\Delta = \mathrm{diag}(\delta_1, \ldots, \delta_n), \quad \delta_i = (U^T M^{-\frac{1}{2}}(C_u + C(v_1^i, \ldots, v_d^i))M^{-\frac{1}{2}}U)_{ii} \,,\ i = 1, \ldots, n.$$

6: **else**

7: Compute new U and Δ, such that

$$M^{-\frac{1}{2}} D M^{\frac{1}{2}} = U \Delta U^T \,, \quad \Delta = \mathrm{diag}(\delta_1, \ldots, \delta_n) \,.$$

Compute function value at viscosities $(v_1^i, \ldots, v_d^i)$ using formula

$$\mathrm{tr}(\mathbf{Z_U}\widetilde{\mathbf{X}}(v_1^i, \ldots, v_d^i)) = \sum_{i=1}^{n} \frac{(Z_2)_{ii} + (Z_1)_{ii}}{\delta_i} + \frac{1}{2}\sum_{i=1}^{n} \delta_i (Z_1)_{ii} b_i \,,$$

where $b_i = \|T_0(:, i)\|^2 \,, \quad T_0 = B^{-1}, \quad i = 1, \ldots, n$ and Z_1, Z_2 are given by (4.42-4.43).

8: **end if**

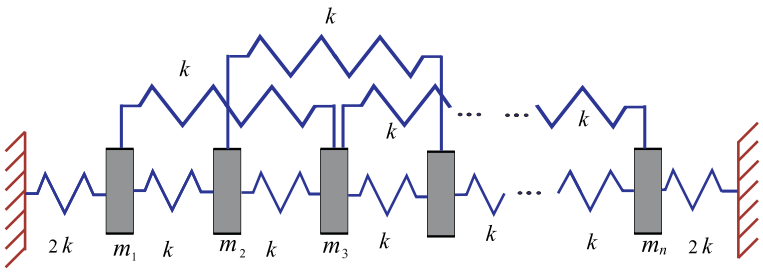

Figure 5.1: n-mass oscillator

has the block diagonal structure $D = diag(D_1, D_2, \ldots, D_{10})$ *where each block has its own viscosity* v_j *for* $j = 1, \ldots, 10$. *The block diagonal structure of the matrix* D *is shown at the Figure 5.2.*

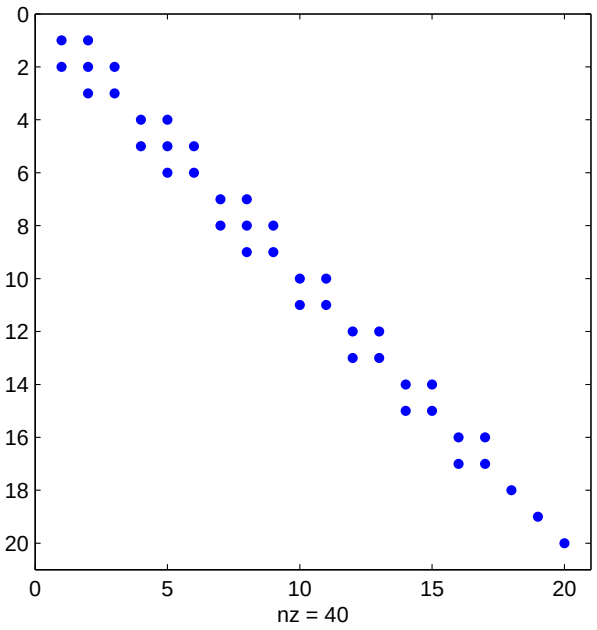

Figure 5.2: Block diagonal structure of matrix D

The blocks are defined as:

$$\begin{aligned} D_i &= \begin{bmatrix} v_i + v_i p & -v_i p & 0 \\ -v_i p & v_i + 2v_i p & -v_i p \\ 0 & -v_i p & v_i + v_i p \end{bmatrix}, \; i = 1, 2, 3, \\ D_i &= \begin{bmatrix} v_i + v_i p & -v_i p \\ -v_i p & v_i + v_i p \end{bmatrix}, \; i = 4, 5, 6, 7, \\ D_i &= \begin{bmatrix} v_i + v_i p \end{bmatrix}, \; i = 8, 9, 10, \end{aligned}$$

where $p = 0.001$.

We will calculate the optimal viscosities for the two different cases:

Case 1. *In the first case we assume that there is no internal damping, that is* $\alpha = 0$.
For the purpose of comparison we will present the optimal viscosity and corresponding minimal trace, denoted with $(v^*, \operatorname{tr}(\widetilde{\mathbf{X}}(v^*)))$*, obtained by direct calculations using the formula (4.53) and the optimal viscosity and corresponding minimal trace denoted with* $(v, \operatorname{tr}(\mathbf{X}(v)))$*, obtained by the minimization of the trace of "dual Lyapunov equation" of equation (3.18) directly with MATLAB's function* `fminsearch`*, where we have used MATLAB's function* `lyap` *for solving Lyapunov equations.*

For $(v^*, \operatorname{tr}(\widetilde{\mathbf{X}}(v^*)))$ *and* $(v, \operatorname{tr}(\mathbf{X}(v)))$ *we have obtained the following:*

$$v^* = \begin{bmatrix} 37.9626 \\ 23.3395 \\ 14.7396 \\ 19.4686 \\ 28.6084 \\ 32.6407 \\ 38.6879 \\ 45.7553 \\ 54.7100 \\ 64.6193 \end{bmatrix}, \quad \operatorname{tr}(\widetilde{\mathbf{X}}(v^*)) = 487.4226, \quad v = \begin{bmatrix} 38.1249 \\ 23.1773 \\ 14.5789 \\ 17.4601 \\ 28.4168 \\ 32.4962 \\ 38.5573 \\ 45.6625 \\ 55.0314 \\ 65.0329 \end{bmatrix}, \quad \operatorname{tr}(\mathbf{X}(v)) = 484.8125\,.$$

Thus, the relative errors for the obtained approximations are:

$$err_v = \frac{||v - v^*||}{||v||} = 0.0171, \tag{5.1}$$

$$err_{tr} = \frac{||\operatorname{tr}(\widetilde{\mathbf{X}}(v^*)) - \operatorname{tr}(\mathbf{X}(v))||}{||\operatorname{tr}(\mathbf{X}(v))||} = 0.0054. \tag{5.2}$$

Here the residual error from (3.38) is $R_{er} = 0.3534$*. This shows that even if the consider mechanical system is not very close to the modally damped one (*R_{er} *is not significantly smaller than 1) the formula (4.53) still insures the satisfying result.*

Case 2. *Within the second case we will assume the existence of the internal damping, thus let* $\alpha = 0.01$ *be coefficient of the internal damping.*

As it has been emphasized in Remark 4.1 for the case $\alpha \neq 0$ *one can not use formula (4.53) directly. Thus we will use Newton's method for optimization of the trace approximation given by formula (4.54). Again by* $(v^*, \mathrm{tr}(\widetilde{\mathbf{X}}(v^*)))$ *we denote the obtained approximation for the optimal viscosity and corresponding minimal trace. Similarly, by* $(v, \mathrm{tr}(\mathbf{X}(v)))$ *we denote the optimal viscosity and corresponding function value obtained by the minimization of the trace of "dual Lyapunov equation" of equation (3.18) directly with MATLAB's function* `fminsearch`*, where Lyapunov equation was solved by MATLAB's function* `lyap`*.*

Here are the obtained quantities:

$$v^* = \begin{bmatrix} 36.3126 \\ 21.9638 \\ 13.9714 \\ 15.8175 \\ 26.1052 \\ 29.7869 \\ 35.4482 \\ 42.2551 \\ 51.4233 \\ 61.2265 \end{bmatrix}, \mathrm{tr}(\widetilde{\mathbf{X}}(v^*)) = 486.3990, \qquad v = \begin{bmatrix} 36.1512 \\ 22.1206 \\ 14.0986 \\ 17.6473 \\ 26.2969 \\ 29.9301 \\ 35.5781 \\ 42.3487 \\ 51.1036 \\ 60.8131 \end{bmatrix}, \mathrm{tr}(\mathbf{X}(v)) = 483.9260\,.$$

For the relative errors defined in (5.1), (5.2) here we have:

$$err_v = 0.0169, \quad err_{tr} = 0.0051.$$

In this example the residual error from (3.38) has the similar magnitude that is $R_{er} = 0.3049$.

In the second example we will consider a more general structure.

Example 5.2. *In this example we will consider the system from (2.2) with dimension* $n = 500$ *and the matrices* M *and* K *defined as:*

$$M = 10^3 \mathrm{diag}(m_1, m_2, \ldots, m_n)\,,$$

$$m_i = \begin{cases} 200 - 20(i-1) & , i = 1, \ldots, 250 \\ 201 + 20(i-11) & , i = 251, \ldots, 500 \end{cases}$$

$$K = \begin{bmatrix} 10 & -1 & & & & \\ -1 & 10 & -1 & & & \\ & -1 & 10 & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & -1 & 10 & -1 & \\ & & & -1 & 10 & \end{bmatrix}.$$

The damping matrix D has the block diagonal structure as follows

$$D = \begin{bmatrix} 0 & & & & \\ & D_1 & & & \\ & & 0 & & \\ & & & D_2 & \\ & & & & 0 \end{bmatrix} + C_u \,. \tag{5.3}$$

where $\alpha = 0.01$ and 0 represents a zero matrix of the corresponding dimension. The matrix D_i is defined as:

$$D_i = \begin{bmatrix} v_{i_1} + v_{i_1}p & -v_{i_1}p & & & \\ -v_{i_1}p & v_{i_1} + 2v_{i_1}p & -v_{i_1}p & & \\ & v_{i_1}p & v_{i_1} + (v_{i_1} + v_{i_2})p + v_{1_2} & -v_{i_2}p & \\ & & -v_{i_2}p & v_{1_2} + v_{i_2}p & \end{bmatrix},$$

where $p = 0.01$. Thus, the each block has 2 different viscosities, which means that we have 4 different viscosity parameters to optimize.

Again we will compare the approximation of the optimal viscosities obtained by our new approach proposed in section (4.2) with the optimal viscosity obtained by the minimization of the trace of of "dual Lyapunov equation" of equation (3.18) directly with MATLAB's function `fminsearch`*, based on the MATLAB's function* `lyap` *for solving Lyapunov equations.*

This comparison has been performed for the different positions of the matrices D_1, D_2, that is for in each new configuration we will change the position of matrices D_1 and D_2. The following configurations are taken into consideration:

$$\begin{aligned}(i,j) \quad \in \quad & \{(2,17),(2,67),(2,117),(2,267),(2,317),(52,67),(52,117),(52,167),(52,267),\\ & (52,317),(52,367),(52,417),(102,117),(102,217),(102,367),(152,167),(152,267),\\ & (152,317),(202,417),(252,267),(252,367),(252,417),(252,467),(302,367),(302,417)\\ & (352,417),(352,467)\}\end{aligned}$$

where i represents position of the matrix D_1 and j represents position of the matrix D_2. The Figure (5.3) shows the relative error

$$err_{tr} = \frac{||\operatorname{tr}(\mathbf{X}(v)) - \operatorname{tr}(\widetilde{\mathbf{X}}(v^*))||}{||\operatorname{tr}(\mathbf{X}(v))||},$$

for each configuration.

During the optimization process using Algorithm 1, we have calculated the percentage of updates of the matrix U with the tolerance $tol_U = 10^{-5}$. The number of updates for each configuration is shown on the Figure (5.4).

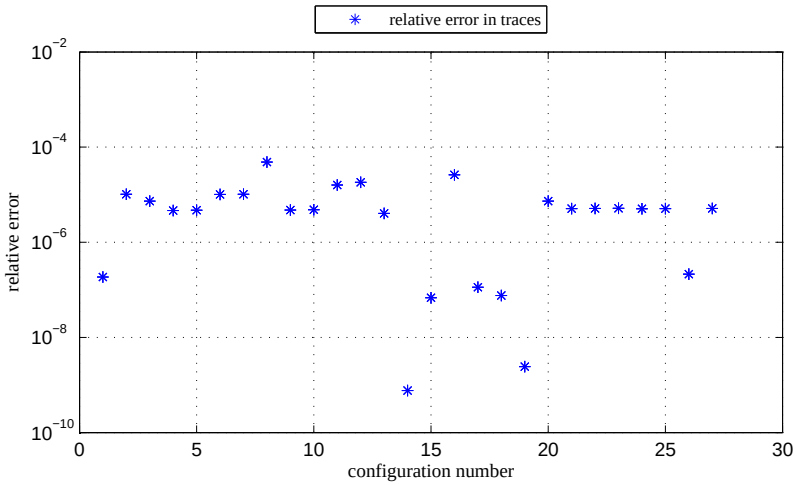

Figure 5.3: Relative errors err_{tr} for different positions of matrices D_1, D_2

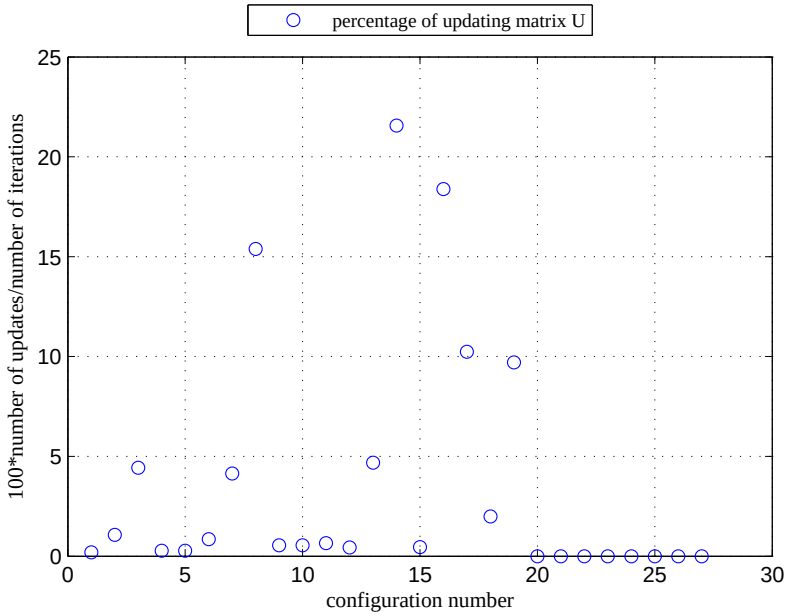

Figure 5.4: Percentage of updating matrix U for different positions of matrices D_1, D_2

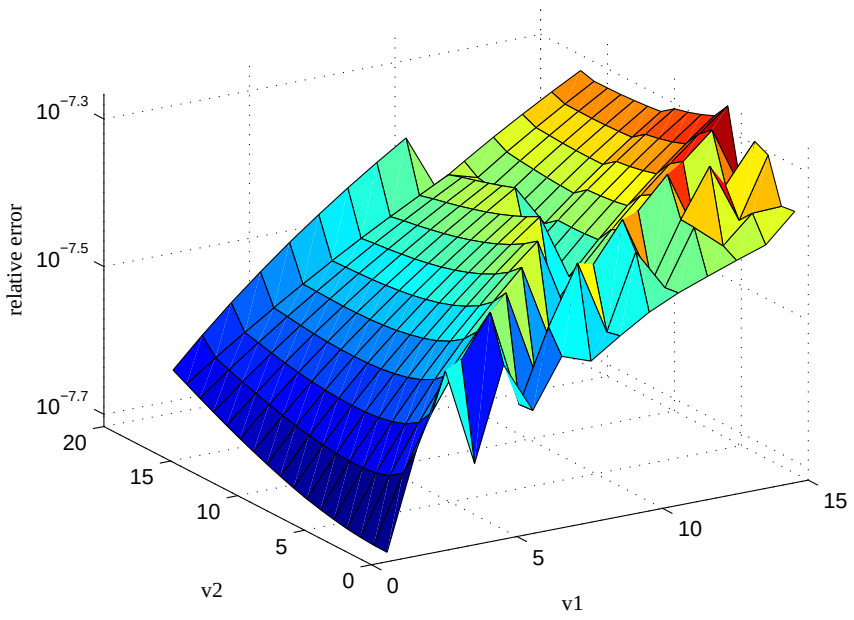

Figure 5.5: Relative error of function value

Moreover, in order to illustrate the quality of the new approach using the surface plot in prescribed viscosities, we will set $v_1 = v_3$, $v_2 = v_4$, *while* v_1 *and* v_2 *vary in segments* $[40, 200]$, $[200, 340]$, *respectively. The block with viscosities* v_1, v_2 *starts at position* 242 *and block with* v_3, v_4 *starts at* 470.

For the first step in iterations we have used the matrix U *which is defined by the optimal viscosities* $v_1 = 101.4445$ *and* $v_2 = 268.3622$, *while during the iteration process the matrix* U *has been updated with the tolerance* $tol_U = 10^{-5}$.

In Figure (5.5) we can see relative error

$$err_{rel} = \frac{||f_{lyap} - f_{aprox}||}{||f_{lyap}||},$$

in which f_{lyap} *represents the trace of the solution of the Lyapunov equation for certain viscosities* v_1, v_2 *and* f_{aprox} *represents the trace calculated by our algorithm. Relative error is less then* 10^{-7}.

Conclusion

Trough this paper we have considered the damping optimization for the mechanical system $M\ddot{x} + D(v)\dot{x} + Kx = 0$. Since only the damping matrix $D(v)$ depends on parameters, the typical (or often used, standard) approach for the viscosity optimization (v) assumes the preprocessing based on the diagonalization of the mass and stiffness matrices, M and K.

Contrary to this approach, we propose the new approach, which is based on the diagonalization of the damping matrix $D(v)$, and then calculation of the optimal viscosities. This is the main contribution of this paper, that is we have shown that slightly change in paradigm of damping optimization, for a certain structure, can significantly improve the performance of optimization methods.

Although, in general the new approach can not be more efficient then the standard one, we have shown that in the case when M, D and K are closed to the the case when all three can be simultaneously (or when the mechanical system is closed to modally damped one) we can derive optimal viscosities, explicitly or numerically very efficiently.

We have also provided the bounds which can be easily used to determine whether the considered mechanical system is suitable for applying the new approach , that are considered mechanical system close to modally damped one or not.

Our numerical examples show that with proposed approach we can obtain satisfactory approximation for optimal parameters. Moreover, we illustrate that with our approach we can significantly accelerate optimization process for the structured systems.

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