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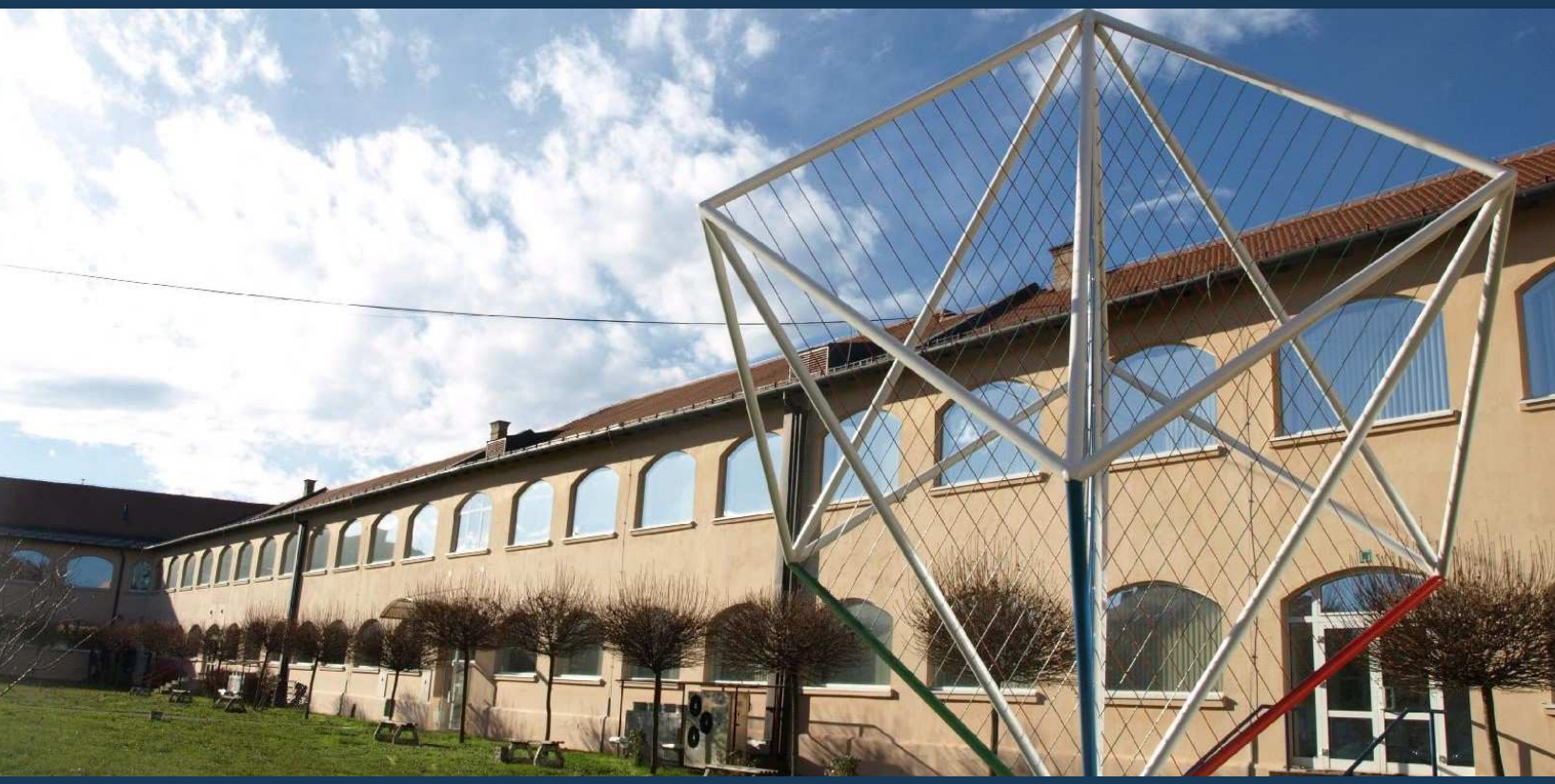

Technical report

April, 2016

Dimension reduction approach for the parameter dependent quadratic eigenvalue problem

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May 25, 2016

Abstract

This paper presents the novel approach in efficient calculation of the all or just one important part of the eigenvalues of the parameter dependent quadratic eigenvalue problem $(\lambda^2(\mathbf{v})M+\lambda(\mathbf{v})D(\mathbf{v})+K)x(\mathbf{v})=0$, where M,K are positive definite Hermitian $n\times n$ matrices and $D(\mathbf{v})$ is $n\times n$ Hermitian semidefinite matrix which depends on a parameter $\mathbf{v}=\begin{bmatrix}v_1 & \dots & v_k\end{bmatrix}\in\mathbb{R}^k_+$. With the new approach one can efficiently (and accurate enough) calculate the all (or just part of the) eigenvalues even for the case when v_i are of the modest magnitude. Moreover, for the both cases of approximations we have derived the corresponding upper bounds. The quality of the error bounds as well as the performance of the achieved eigenvalue tracking was illustrated in several numerical experiments.

Keywords: dimension reduction, parameter dependent eigenvalue problem, quadratic eigenvalue problem, tracking eigenvalues, eigenvalue error bounds.

AMS: 15A24, 15A18, 15A90

1 Introduction

The main topic of this paper is a parameter dependent quadratic eigenvalue problem (PQEP) of the form

$$(\lambda^2(\mathbf{v})M+\lambda(\mathbf{v})D(\mathbf{v})+K)x(\mathbf{v})=0\,, \tag{1.1}$$

where M,K are positive definite Hermitian $n\times n$ matrices and $D(\mathbf{v})$ is $n\times n$ Hermitian semidefinite matrix which depends on a parameter $\mathbf{v}\in\mathbb{R}^k_+$. Such a problems are connected with the second-order differential equation

$$M\ddot{x}(\mathbf{v})+D(\mathbf{v})\dot{x}(\mathbf{v})+Kx(\mathbf{v})=0\,, \tag{1.2}$$

where M and K are mass and stiffness, respectively and $D(\mathbf{v})$ is the damping matrix which can be defined in a several different ways. One of the most common ways is that

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$D(\mathbf{v}) = C_u + C_{ext}(\mathbf{v})$, where only the external damping part depends on parameters v_i for $i = 1, \ldots, k$ (called viscosities), where $\mathbf{v} = \begin{bmatrix} v_1 & \ldots & v_k \end{bmatrix}$. Moreover, external damping can be written as $C_{ext}(\mathbf{v}) = v_1 C_1 + v_2 C_2 + \cdots + v_k C_k$, where each C_i determines the geometry of the ith damper and it has a small rank, so that $C_{ext}(\mathbf{v})$ is a semidefinite matrix in general.

The main issue which we are going to addressed in this paper will be: "how one can efficiently calculate the eigenvalues $\lambda_i(\mathbf{v})$ (for all $i = 1, \ldots, n$, or just one important part of the spectrum) for the large variety of the parameters $\mathbf{v} = \begin{bmatrix} v_1 & \ldots & v_k \end{bmatrix}$".

This problem is related with the efficient solving of a various problems connected with the damped mechanical systems (1.2), such as

1. efficient calculation of approximations of eigenvalues with error bounds for all eigenvalues or the for selected "most important undamped eigenfrequencies",

2. efficient optimization of the "spectral abscissa", that is, efficient minimization of the (penalty) function $\max_i \mathrm{Re}(\lambda_i(\mathbf{v}))$,

3. the frequency isolation problem, that is re-design of a given damped mechanical systems (find appropriate $\mathbf{v}$), such that a new system does not have eigenvalues in some "dangerous interval" typically called the resonance band.

Without minimizing the importance of the omitted titles, below we list a number of papers related to each of the three above items.

More results connected with the item 1 can be found in [23, 13, 15] where authors consider approximations of eigenvalues with corresponding error bounds. On the other hand, there is a large number of papers that consider efficient eigenvalue calculation for structured eigenvalue problems, such as [1, 10, 16].

On the other hand some of the results on optimization of the "spectral abscissa", that is on the item 2, one can find in [6, 5, 9] and among the other, the problems from the item 3 have been investigated in the following papers [12, 17, 11].

This paper is mainly devoted to the problems from the item 1, that is to the problem of the efficient approximations of the eigenvalues and construction of the corresponding error bounds for PQEP from (1.1). The similar problem (the problem of efficient calculation of the eigenvalues $\lambda_i(v_1, \ldots, v_k)$) has been considered in [19]. There authors present two different approximations, one for the case when $0 \leq v_i \ll 1$ and one for $v_i \gg 1$, $i = 1, \ldots, k$.

In [24] and later in [19] it has been shown that the behavior of the eigenvalues $\lambda_i(v_1, \ldots, v_k)$ for the $0 < v_i \ll 1$ can be appropriately described (or approximated) using the results (see [24, Chapter 19]) that the spectrum is contained in the union of circles or Cassini ovals. On the other hand for the case when $1 \ll v_i$, $i = 1, \ldots, k$, in [19] the singular value decomposition of the Cholesky factor of the damping matrix D has been used to obtain an efficient approximation of the whole spectrum with corresponding Gerschorin type of bounds.

The main result of this paper will be a certain generalization of the results form [19, 24] which hold for $v_i \ll 1$ on the set of parameters of the modest magnitudes. That is we will present an approach which can be used for the efficient calculation of the whole spectrum

or just of a part of the spectrum for the problem (1.1) for a different parameters in some feasible interval, that is $0 \leq v_i \leq V_i$, for $i = 1, \ldots, k$.

The engine that drives the whole process is based mainly on the ideas from papers [2, 3] by P. Benner, Z. Tomljanović and N. Truhar where authors have considered the damping optimization based on the minimization of the total average energy.

The efficient calculation of eigenvalues are of our interest, since in the problems similar to those above listed in items 1-3, one usually has a several parameters which impose a large number of eigendecompositions calculations. In particular, for fixed number of different dampers we have to investigate a large number of different position configurations and for each of them we need to consider viscosities over the feasible time interval $0 \leq v_i \leq V_i$. As we will illustrate in the preliminaries, even for moderate dimensions we end up with billions eigenvalue problems that need to be solved.

The paper is organized as follows. In the 2'nd Section we provide preliminary results that clarifies the problem and notation that we will use in this paper. Mainly we would like to consider two very important cases for which we derive an efficient approximations of eigenvectors and eigenvalues and corresponding error bounds. Within the first case, presented in Section 3, we consider an eigenvalue behavior for a selected undamped eigenfrequencies. This approximations follow an idea from the paper [2] where authors suggest a certain types of approximation of the PGEP (2.4), thus the smaller part of the spectrum is approximated "directly", and represents the important part for the considered spectra, while the second part of the spectrum is neglected since is not of our interest. For this approximation in Section 3.1 we give two error bounds where the first one is based on results from [15], while the second one is based on Gerschgorin bounds. Within the second case, presented in Section 4, we study an eigenvalue behavior for all eigenvalues. In this case we can approximate all eigenvalues using the main approach from the paper [3]. This means that the approximation of the eigenvalues of the PGEP (2.4) will be obtained by the block diagonal matrices, where the upper diagonal block contains all important information which includes the external damping, while the other diagonal blocks contains two by two matrices similar to the so called modally damped systems. For that part one can calculate an eigenvalue approximation by a simple formula. The corresponding error bounds, for this case are given in Section 4.1. In Section 5 we provide two examples with numerical illustration of performance of our approximation and the quality of the given bounds.

2 Preliminaries

In this section we will present some preliminaries which will be used in the next two sections in our approach for the efficient eigenvalue approximations.

Instead of analyzing the PQEP (1.1) we will consider the corresponding linearized parameter dependent generalized eigenvalue problem (PGEP). For that purpose let Φ be a matrix that simultaneously diagonalizes M and K, i.e.,

$$\Phi^T K \Phi = \Omega^2 = \operatorname{diag}(\omega_1^2, \ldots, \omega_n^2) \quad \text{and} \quad \Phi^T M \Phi = I. \tag{2.3}$$

Then the linearized parameter dependent generalized eigenvalue problem (PGEP) which corresponds to (1.1) is given by:

$$A(\mathbf{v})y(\mathbf{v}) = \lambda(\mathbf{v})Jy(\mathbf{v})\,, \tag{2.4}$$

where

$$A(\mathbf{v}) = \begin{bmatrix} 0 & \Omega \\ \Omega & \Phi^T D(\mathbf{v})\Phi \end{bmatrix}\,, \qquad J = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}\,, \tag{2.5}$$

and where

$$y = \begin{bmatrix} \Omega x \\ \lambda x \end{bmatrix}\,.$$

Since the block $\Phi^T D(\mathbf{v})\Phi$ depends on the structure of the damping we will use the general form

$$D(\mathbf{v}) = C_u + C_{ext}(\mathbf{v})\,,$$

where in the case of critical damping we have

$$C_u = \alpha M^{1/2}\sqrt{M^{-1/2}KM^{-1/2}}M^{1/2}, \tag{2.6}$$

and then $\Phi^T C_u \Phi = \alpha\Omega$, while in the case of proportional damping $C_u = \alpha M + \beta K$ it holds $\Phi^T C_u \Phi = \alpha I + \beta\Omega^2$. In the following text we will be focused on internal damping defined by (2.6), but our results can be easily applied also for other cases of internal damping.

The above linearization has been used in many books and papers, in already mentioned P. Benner, Z. Tomljanović, N. Truhar papers [2, 3] as well as in [4, 18, 21, 22, 24, 14].

As it has been emphasized in many occasions (in many different papers) the problem of calculating eigenvalues of parameter dependent generalized eigenvalue problem (PGEP) (2.4) could be extremely numerically demanding even for the problems of the modest magnitudes (with large dimensions additional problems arise even within viscosity optimization) due to the combinatorial complexity. For example, let assume that one needs to calculate the whole (or just a part of the) spectrum of the matrix $A(\mathbf{v}) \in \mathbb{R}^{2n\times 2n}$ and $n = 1000$, for $\mathbf{v} = \begin{bmatrix} v_1 & \dots & v_{20} \end{bmatrix}$, and $0 \le v_i \le 10^6$, $i = 1, \dots, 20$. Obviously, even for the case if one needs to calculate 5% of all eigenvalues for various viscosities in set $\begin{bmatrix} 0 & 10^6 \end{bmatrix}^{20}$ and various damping positions (in general case $\binom{1000}{20}$ different configurations) it will be a very demanding problem even for the standard algorithms, due to the fact that we need to calculate eigenvalues for an extremely large number of different settings.

As it has been described in the introduction we will concentrate ourself on two very important cases. In the first we will consider efficient approximation for the eigenvalues for selected part of undamped spectrum. Within the second case we will consider efficient approximation for all eigenvalues. The procedures for approximations and corresponding error bounds will be given in the next two sections.

3 The approximation of the selected eigenvalues

Before we present the main result from this section we will briefly present the results from [3] which will allow us to efficiently approximate selected eigenvalues of the parameter dependent generalized eigenvalue problem (PGEP).

Hereinafter, we will assume that the internal damping is a small multiple of the critical damping, which is not serious constrain, since our approach can be easily extended to the case where the internal damping is any matrix which is diagonalized by the matrix Φ defined in (2.3), which is an usual assumption on properties of internal damping.

Thus, let P be the perfect shuffle permutation, then instead PGEP from (2.4) we will consider permuted one of the form

$$A_P(\mathbf{v})y_P(\mathbf{v}) = \lambda(\mathbf{v})J_P y_P(\mathbf{v})\,, \tag{3.7}$$

where $A_P(\mathbf{v}) = P^T A(\mathbf{v})P$, $J_P = P^T JP$ and $y_P = Py$.

The given damper positions and corresponding viscosities are included in the matrix $C = \Phi^T C_{ext}\Phi$, where Φ is given in (2.3).

As we have mentioned in the previous section, our approach is based on dimension reduction of the parameter dependent generalized eigenvalue problem (PGEP) (3.7). For that purpose we construct an approximation of the matrix $A_P(\mathbf{v})$ by

$$\widetilde{A}_P(\mathbf{v}) = \begin{bmatrix} \widetilde{A}_{11}(\mathbf{v}) & 0 \\ 0 & \widetilde{A}_{22}(\mathbf{v}) \end{bmatrix}, \quad A_{11}(\mathbf{v}) \in \mathbb{R}^{2r\times 2r}\,, \tag{3.8}$$

while the matrix J_P from (3.7) will be decomposed as

$$J_P = P^T JP = J_{11} \oplus J_{22},\,, \quad J_{11} \in \mathbb{R}^{2r\times 2r}\,, \tag{3.9}$$

with J given by (2.5). To obtain an appropriate accuracy by this approximation we need to ensure that the off diagonal parts, are small up to a given tolerance. In order to achieve this we will need to employ additional permutation that takes the magnitude of the elements of matrix $\Phi^T C_{ext}\Phi$ into account. More details on this we provide in this section as well as the error for such an eigenvalue approximation.

Thus, instead of the calculating all eigenvalues $\lambda_i(\mathbf{v})$, where $\mathbf{v} = \begin{bmatrix} v_1 & \dots & v_k \end{bmatrix}$ of the PGEP (2.4) for a different parameters v_j, $j = 1, \dots, k$ we will calculate the approximations $\widetilde{\lambda}_i(\mathbf{v})$, with $i = 1, \dots, 2r$. These r eigenvalues contains the eigenvalues that have more influence on our system. More influence here means for example that these eigenfrequencies have to be damped, or they have to be excluded from a certain interval, or something similar. Thus, we end up with the two eigenvalue problems

$$\widetilde{A}_{11}(\mathbf{v})y_{11} = \lambda J_{11}y_{11}\,, \qquad \widetilde{A}_{22}(\mathbf{v})y_{22} = \lambda J_{22}y_{22}\,,$$

where $y_p = [y_{11}, y_{22}]$ is decomposed respectively with $\widetilde{A}_P$ and the first one contains the relevant part of the spectrum.

Now we will describe the construction of the matrix $\widetilde{A}_P$, that is of $\widetilde{A}_{11}$, $\widetilde{A}_{22}$ in more details. For this, we will need the additional permutation matrix $\widehat{P}$ which will bring up the relevant part of the damping matrix C to the upper block diagonal part.

Let the vectors $p \in \mathbb{N}^r$ and $\overline{p} \in \mathbb{N}^{n-r}$ be chosen such that the following conditions hold:

i) $p \cup \overline{p} = \{1, 2, \ldots, n\}$.

ii) p is the vector of indices of dimension s, where the first s correspond to the eigenfrequencies of our interest (for example eigenfrequencies which have to be damped).

iii) $\overline{p}$ and p are index vectors such that $\max_{ij} |C(\overline{p}(i), p(j))| \leq tol$, for a given tolerance tol.

The vectors $p \in \mathbb{N}^r$ and $\overline{p} \in \mathbb{N}^{n-r}$ should be chosen such that r is as small as possible for given parameters s and tol. A strategy for determining $p, \overline{p}$ will be discussed below.

Now we define a vector $w \in \mathbb{N}^n$ by $w(i) = p(i)$ for $i = 1, \ldots, r$ and $w(i) = \overline{p}(i - r)$ for $i = r + 1, \ldots, n$.

The matrix P is the perfect shuffle permutation matrix and $\widehat{P} = I(:, w) \otimes I_2$. Now for these permutations it holds

$$\widehat{A}_P = \begin{bmatrix} 0 & \omega_{w(1)} & \cdots & 0 & 0 \\ \omega_{w(1)} & \alpha\omega_{w(1)} + c_{w(1)w(1)} & \cdots & 0 & c_{w(1)w(n)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \omega_{w(n)} \\ 0 & c_{w(1)w(n)} & \cdots & \omega_{w(n)} & \alpha\omega_{w(n)} + c_{w(n)w(n)} \end{bmatrix}, \tag{3.10}$$

where $\widehat{A}_P = \widehat{P}^T P^T A P \widehat{P}$.

We are interested in the dimension reduction which will allow us to approximate eigenvalues efficiently, thus we define approximation matrix with

$$\widetilde{A}_P = \begin{bmatrix} \widetilde{A}_{11} & 0 \\ 0 & \widetilde{A}_{22} \end{bmatrix}$$

where

$$\widetilde{A}_{11} = \widehat{A}_P(1 : 2r, 1 : 2r) \quad \text{and} \quad \widetilde{A}_{22} = \widehat{A}_P(2r + 1 : 2n, 2r + 1 : 2n) \tag{3.11}$$

for $\widehat{A}_P$ as given in (3.10).

Since by this approach we reduce dimension from n to r, the parameter r will be called the reduced dimension.

Note that, more elements of C small by magnitude, means that the reduced dimension will be smaller. The elements of the matrix C are obtained from the corresponding rows of the matrix Φ from (2.3). That is, due to the physical properties of the considered systems it is a very likely that there will be difference in the magnitude of elements in the matrix $\Phi^T C_{ext} \Phi$ (more details can be found in [2, 3, 7]) from which we make direct use in our approach.

Now we will present Algorithm 1 for the construction of the vectors p and $\overline{p}$.

In Algorithm 1, the indices $u_1, u_2, \ldots, u_s$, are included in the vector p based on the important (significant) undamped eigenfrequencies of our interest (for example those which have to be damped). Usually these indices corresponds to the the certain percentage of

Algorithm 1 (construction of p and $\overline{p}$)

Require: tol;
v_i, C_i, $i = 1, \ldots, k$ – viscosity and matrix that determines the geometry of the ith damper;
$u_1, u_2, \ldots, u_s$, – s indices of those eigenfrequencies which have to be damped;

Ensure: p, $\overline{p}$

1: $p = [\, u_1, u_2, \ldots, u_s]$
2: Determine vector $\overline{p}$ such that $p \cup \overline{p} = \{1, 2, \ldots, n\}$.
3: T=1
4: $C = \Phi^T(v_1 C_1 + v_2 C_2 + \cdots + v_k C_k)\Phi$
5: **while** T=1 **do**
6: $\widehat{C} = C(p, \overline{p})$ and $M = \max_{ij} |\widehat{C}_{i,j}|$.
7: **if** $M > tol$ **then**
8: Determine indices i_0, j_0 such that $M = |C_{i_0,j_0}|$ (ensuring that j_0 is not used before and i_0 is index which is element of p).
9: $p = [\, p,\, j_0\,]$.
10: Determine the vector $\overline{p}$ such that $p \cup \overline{p} = \{1, 2, \ldots, n\}$.
11: **else**
12: $T = 0$
13: **end if**
14: **end while**

the undamped spectra, e.g. the certain percentage of the smallest undamped eigenfrequencies or undamped eigenfrequencies within some interval. In order to achieve efficient approximation we ensure that all elements of the matrix C to be omitted are smaller up to chosen tolerance that is we have that $\max_{i,j} |C(p(i), \overline{p}(j))| < tol$ which will have direct impact in the corresponding error bound.

Once we have obtained the vectors p and $\overline{p}$, we can introduce Algorithm 2 for calculating the eigenvalue approximations.

The following section provides two error bounds for the approximations obtained by Algorithm 2.

3.1 Error bound for approximation of selected eigenfrequencies

In this section we will present the error bounds for the eigenvalue approximations made by Algorithm 2. The bounds can be derived using a several different approaches usually used for error estimation in the eigenvalue approximation. One approach is based on the standard Gerschorin type of bounds, like in e.g. [8, 20]. The another approach is based on perturbation bounds from [15].

First we will present a bound based on the bound from Theorem 4.1 from [15]. Just

Algorithm 2 Approximation of selected eigenfrequencies

Require: α, Φ – such that $\Phi^T K\Phi = \Omega^2 = \mathrm{diag}(\omega_1^2, \ldots, \omega_n^2)$ and $\Phi^T M\Phi = I$;
v_i, C_i, $i = 1, \ldots, k$ – viscosity and position matrix for the ith damper;
$p(1), p(2), \ldots, p(s)$ – indices of the eigenfrequencies which have to be damped;
tol – tolerance needed for Algorithm 1.

Ensure: eigenvalues of $(A_p(\mathbf{v}), J_P)$

1: Determine vectors $\overline{p} \in \mathbb{N}^{n-r}$ and $p \in \mathbb{N}^r$ using Algorithm 1.
2: $\Omega_r = \mathrm{diag}(\omega_{p(1)}, \omega_{p(2)}, \ldots, \omega_{p(k)})$
3: $C = \Phi^T(v_1C_1 + v_2C_2 + \cdots + v_kC_k)\Phi$
4: Calculate all eigenvalues of the pair $(\widetilde{A}_{11}, J_{11})$, where

$$\widetilde{A}_{11} = \begin{bmatrix} 0 & \Omega_r \\ \Omega_r & \alpha\Omega_r + C(p,p) \end{bmatrix} \quad \text{and} \quad J_{11} \text{ given by (3.9).}$$

for the sake of completeness we will rewrite this theorem. Thus, let

$$\widetilde{A}_P = \begin{array}{c} \\ 2r \\ 2n-r \end{array} \begin{array}{c} \begin{array}{cc} 2r & 2n-2r \end{array} \\ \begin{pmatrix} A_{11} & \\ & A_{22} \end{pmatrix} \end{array}, \quad \widetilde{J} = \begin{array}{c} \\ 2r \\ 2n-2r \end{array} \begin{array}{c} \begin{array}{cc} 2r & 2n-2r \end{array} \\ \begin{pmatrix} J_{11} & \\ & J_{22} \end{pmatrix} \end{array}, \tag{3.12a}$$

$$\widehat{A}_P = \begin{pmatrix} A_{11} & E_{12} \\ E_{21} & A_{22} \end{pmatrix}, \qquad \widehat{J} = \begin{pmatrix} J_{11} & F_{12} \\ F_{21} & J_{22} \end{pmatrix} \tag{3.12b}$$

be non-Hermitian matrices. Since, $\widetilde{J}$ is nonsingular this means that $\widetilde{A}_P - \lambda\widetilde{J}$ is diagonalizable, so that $\widetilde{A}_P - \lambda\widetilde{J}$ has only finite eigenvalues, and there exist nonsingular matrices $X = \mathrm{diag}(X_1, X_2)$ and $Y = \mathrm{diag}(Y_1, Y_2)$ such that $Y\widetilde{A}_P X = \Lambda = \mathrm{diag}(\Lambda_1, \Lambda_2)$ and $YJX = I$, where X, Y and Λ are m-by-m and Λ is the diagonal matrix of eigenvalues.

We will establish a bound on $|\mu - \widetilde{\mu}|$, where μ is an eigenvalue of $\widehat{A}_P - \lambda\widehat{J}$ and $\widetilde{\mu}$ is an eigenvalue of $\widetilde{A}_P - \lambda\widetilde{J}$.

Theorem 3.1. *Let $\widehat{A}_P, \widehat{J}, \widetilde{A}_P, \widetilde{J}$ be as in* (3.12a) *and* (3.12b)*. Suppose that there exist nonsingular matrices $X = \mathrm{diag}(X_1, X_2)$ and $Y = \mathrm{diag}(Y_1, Y_2)$ such that $YAX = \Lambda$ is diagonal and $YJX = I$. If $\widetilde{\mu}$ is an eigenvalue of $\widehat{A}_P - \lambda\widehat{J}$ such that*

$$\eta_i \stackrel{\mathrm{def}}{=} \min_{\mu \in \mathsf{eig}(\widetilde{A}_{ii}, \widetilde{J}_{ii})} |\widetilde{\mu} - \mu| > 0 \tag{3.13}$$

for $i = 1$ or 2, then $\widetilde{A}_P - \lambda\widetilde{J}$ has an eigenvalue μ such that

$$|\widetilde{\mu} - \mu| \le \|X\|_2\|Y\|_2\|E_{12} - \widetilde{\mu}F_{12}\|_2\|E_{21} - \widetilde{\mu}F_{21}\|_2\|(A_{ii} - \widetilde{\mu}J_{ii})^{-1}\|_2 \tag{3.14a}$$

$$\le \frac{\kappa_2(X)\kappa_2(Y)\|E_{12} - \widetilde{\mu}F_{12}\|_2\|E_{21} - \widetilde{\mu}F_{21}\|_2}{\eta_i}. \tag{3.14b}$$

Now using Theorem 3.1 we can obtained the error bound for the approximations λ_i, $i = 1, \ldots, r$ calculated by Algorithm 2.

Let $\widetilde{A}_P$ be perturbed matrix obtained by Algorithm 2, that is after the perfect shuffle permutation $\widetilde{A}_P$ can be written as

$$\widetilde{A}_P = \begin{array}{c} \\ 2r \\ 2n-2r \end{array} \begin{array}{c} \begin{array}{cc} 2r & 2n-2r \end{array} \\ \begin{pmatrix} \widetilde{A}_{11} & \\ & \widetilde{A}_{22} \end{pmatrix} \end{array}, \qquad J = \begin{array}{c} \\ 2r \\ 2n-2r \end{array} \begin{array}{c} \begin{array}{cc} 2r & 2n-r \end{array} \\ \begin{pmatrix} J_{11} & \\ & J_{22} \end{pmatrix} \end{array}. \tag{3.15}$$

Further, let X and $Y = X^{-1}J$ be the non singular matrices which diagonalize matrix $\widetilde{A}$, that is $Y\widetilde{A}_P X = \Lambda$. Obviously, due to the block structure of $\widetilde{A}_P$, the matrices X and Y have the similar block structure, i.e.

$$X = \begin{array}{c} \\ 2r \\ 2n-2r \end{array} \begin{array}{c} \begin{array}{cc} 2r & 2n-2r \end{array} \\ \begin{pmatrix} X_{11} & \\ & X_{12} \end{pmatrix} \end{array}, \qquad Y = \begin{array}{c} \\ 2r \\ 2n-2r \end{array} \begin{array}{c} \begin{array}{cc} 2r & 2n-2r \end{array} \\ \begin{pmatrix} Y_{11} & \\ & Y_{22} \end{pmatrix} \end{array}.$$

On the other hand, let the unperturbed matrix (the original matrix) be given as

$$\widehat{A}_P = \begin{array}{c} \\ 2r \\ 2n-2r \end{array} \begin{array}{c} \begin{array}{cc} 2r & 2n-2r \end{array} \\ \begin{pmatrix} \widetilde{A}_{11} & \widetilde{A}_{12} \\ \widetilde{A}_{21} & \widetilde{A}_{22} \end{pmatrix} \end{array}, \tag{3.16}$$

then we have the following result.

Corollary 3.1. *Let $\widetilde{A}_P$ and $\widehat{A}_P$ be as in* (3.15) *and* (3.16) *(that is system matrix is given by Algorithm 2). If $\widetilde{\lambda}_i$ is an eigenvalue of $\widetilde{A}_{11}$, and if*

$$\eta_2(\widetilde{\lambda}_i) \stackrel{\text{def}}{=} \min_{\mu \in \mathsf{eig}(\widetilde{A}_{22}, J_{22})} |\widetilde{\lambda}_i - \mu| > 0\,, \tag{3.17}$$

then $\widehat{A}_P$ has an (exact) eigenvalue $\lambda_{\pi(i)}$ such that

$$|\widetilde{\lambda}_i - \lambda_{\pi(i)}| \le \|X\|\|Y\|\|\widetilde{A}_{12}\|_2\|\widetilde{A}_{21}\|_2\|(\widetilde{A}_{22} - \widetilde{\lambda}_i J_{22})^{-1}\|_2 \tag{3.18a}$$

$$\le \kappa_2(X)\kappa_2(Y)\frac{\|\widetilde{A}_{12}\|_2\|\widetilde{A}_{21}\|_2}{\eta_2(\widetilde{\lambda}_i)}. \tag{3.18b}$$

Proof. The proof simple follows from Theorem 3.1. ∎

The application of the bounds (3.18a) and (3.18b) have sense if $\kappa_2(X)$ and $\kappa_2(Y)$ have the modest magnitudes and if gap $\eta_2(\widetilde{\lambda}_i)$ can be calculated efficiently and is not too small. For the purpose of clarifying this, we will present the following remark.

Remark **3.1.** *Since, trough this section we are interested in calculation of the approximation of the part of the spectrum ($\widetilde{\lambda}_i$, $i = 1, \ldots, r$), we can assume that X and $Y = X^{-1}J$ can be written as*

$$X = \begin{array}{c} \\ 2r \\ 2n-2r \end{array} \begin{array}{c} \begin{array}{cc} 2r & 2n-2r \end{array} \\ \begin{pmatrix} X_{11} & \\ & I \end{pmatrix} \end{array}, \qquad Y = \begin{array}{c} \\ 2r \\ 2n-2r \end{array} \begin{array}{c} \begin{array}{cc} 2r & 2n-2r \end{array} \\ \begin{pmatrix} Y_{11} & \\ & I \end{pmatrix} \end{array},$$

where $Y_{11} = X_{11}^{-1} J_{11}$. *Then from Corollary 3.1, (that is from (3.18a and 3.18b) it follows that*

$$|\widetilde{\lambda}_i - \lambda_{\pi(i)}| \leq \|X_{11}\|\|Y_{11}\|\|\widetilde{A}_{12}\|_2\|\widetilde{A}_{21}\|_2\|(\widetilde{A}_{22} - \widetilde{\lambda}_i J_{22})^{-1}\|_2 \tag{3.19a}$$

$$\leq \kappa_2(X_{11})\kappa_2(Y_{11})\frac{\|\widetilde{A}_{12}\|_2\|\widetilde{A}_{21}\|_2}{\eta_2(\widetilde{\lambda}_i)}\,, \tag{3.19b}$$

where

$$\eta_2(\widetilde{\lambda}_i) = \min_{\mu \in \mathsf{eig}(\widetilde{A}_{22}, J_{22})} |\widetilde{\lambda}_i - \mu| > 0\,.$$

On the other hand, one can notice that in general calculation of the gap, $\eta_2(\widetilde{\lambda}_i)$ *can be demanding, especially if* $r \ll n$, *that is when* $\widetilde{A}_{22}$ *has "significant" dimension.*

If that is a case, the bound (3.19b) or (3.19a) can be applied using the simple estimation for the $\eta_2(\widetilde{\lambda}_i)$, *based on the approximation of the spectrum of the pair* $(\widetilde{A}_{22}, J_{22})$, *which can be obtained easily. As we will explain in more details in the next section, one example of the matrix with the above property is* $\widetilde{A}_{22}$ *of the following form*

$$\widetilde{A}_{22} = \bigoplus_{i=r+1}^{n} \Psi_{w(i)} + E_{22}, \quad \textit{where} \quad \Psi_i = \begin{bmatrix} 0 & \omega_i \\ \omega_i & \gamma_i + C_{ii} \end{bmatrix}$$

and where $\|E_{22}\|$ *has a modest magnitude. The spectrum of the pair* $(\widetilde{A}_{22}, J_{22})$ *can be easily approximated with eigenvalues of the pairs* (Ψ_i, Υ_i), *where*

$$\Upsilon_i = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad i = 1, \ldots, n-r.$$

As we have elaborated above the bounds (3.19a) and (3.19b) can be useful if gap $\eta_2(\widetilde{\lambda}_i)$ can be efficiently and accurately estimated which in general will not be possible. Thus, in continuation we would like to present an error bound in the terms of Gershgorin eigenvalue bounds.

Similarly as the above, we assume that we have determined index vectors $\overline{p}$ and p such that $\max_{ij} |C(\overline{p}(i), p(j))| \leq tol$ for a given tolerance tol. Then, in order to apply Gerschgorin bound we will multiply the matrix $\widehat{A}_P$ given by (3.10) from the left-hand side with J, where J is given by (3.9) (recall that $J^T = J$ and $J^2 = I$).

Then, our matrix of interest can be written in the block diagonal form as:

$$\widehat{A}_P^J = J\widehat{A}_P = \begin{bmatrix} \widetilde{A}_{11} & \widetilde{A}_{12} \\ \widetilde{A}_{21} & \widetilde{A}_{22} \end{bmatrix},$$

with

$$\widetilde{A}_{11} = \widehat{A}_P^J(1:2r, 1:2r) \quad \text{and} \quad \widetilde{A}_{22} = \widehat{A}_P^J(2r+1:2n, 2r+1:2n), \tag{3.20}$$

$$\widetilde{A}_{21} = \widehat{A}_P^J(2r+1:2n, 1:2r) \quad \text{and} \quad \widetilde{A}_{12} = \widetilde{A}_{12}^T. \tag{3.21}$$

Now, in order to apply Gerschgorin bound we will diagonalize block $\widetilde{A}_{11}$, and for this we assume that all eigenvalues from the block $\widetilde{A}_{11}$ are simple. In the most cases this will be the truth, while if is not than we still can obtain the approximations, but without an efficient error bound. Let X_{11} be such that

$$\widetilde{A}_{11} = X_{11}\Lambda_{11}X_{11}^{-1}, \tag{3.22}$$

where $\widetilde{A}_{11}$ is given by (3.20). Here diagonal elements of the matrix $\Lambda_{11} = \mathrm{diag}(\widetilde{\lambda}_1, \ldots, \widetilde{\lambda}_{2r})$ provide an approximations of desired eigenvalues given by Algorithm 2 .

Then, using the block diagonal matrix

$$\widehat{X}_1 = \begin{bmatrix} X_{11} & 0 \\ 0 & I \end{bmatrix}$$

we obtain

$$\widehat{X}_1^{-1}\widehat{A}_P\widehat{X}_1 = \begin{bmatrix} \Lambda_{11} & X_{11}^{-1}\widetilde{A}_{12} \\ \widetilde{A}_{21}X_{11} & \widetilde{A}_{22} \end{bmatrix}.$$

In order to obtain an error bound that separately provides an error for each eigenvalue we will apply Gerschgorin theorem (e.g. see [8, 20]). Here we will use a row version of Gershgorin bound, thus, for each i there exists an index $\pi(i)$ such that the following bound holds:

$$|\widetilde{\lambda}_i - \lambda_{\pi(i)}(\widehat{A}_P)| \leq \sum_{j=1}^{2n-2r} |(X_{11}^{-1}\widetilde{A}_{12})_{ij}|\,, \tag{3.23}$$

for $i = 1, \ldots, 2r$ where $\widetilde{A}_{12}$ and X_{11} are given by (3.21) and (3.22), respectively.

Now, once we have the two error bounds, the legitimate question is which of the bounds, (3.19b) or (3.23) is better? As we will see in the section with numerical examples, sometimes is better bound (3.19b) and sometimes (3.23), thus the best option would be to take the minimum of both derived bounds for the given eigenvalue approximation.

4 The approximation of all eigenvalues

Apart from the above mentioned case where we had interest in the important (significant) part of the spectrum (such as damping or exclusion of the part of the spectrum), in continuation we would like to consider the eigenvalue behavior of all eigenvalues, that is behavior of all undamped eigenfrequencies.

Similarly as above we will apply the perfect shuffle permutation matrix P and add an additional permutation that will allow us to approximate all eigenvalues. In order to achieve that we need to determine all elements from the matrix $C = \Phi^T C_{ext}\Phi$ with absolute value larger than a given tolerance. Thus, apart from the permutations in the above section, here we determine the vectors $p \in \mathbb{N}^r$ and $\overline{p} \in \mathbb{N}^{n-r}$ chosen such that the following conditions hold:

i) $p \cup \overline{p} = \{1, 2, \ldots, n\}$.

ii) $\overline{p}$ and p are index vectors such that $\max_{ij} |C(\overline{p}(i), p(j))| \leq tol$ for a given tolerance tol.

iii) $\overline{p}$ and p are index vectors such that $\max_{ij} |C(\overline{p}(i), \overline{p}(j)) - \mathrm{diag}(C(\overline{p}(i), \overline{p}(j)))| \leq tol$ for a given tolerance tol.

Here we will also try to obtained vectors $p \in \mathbb{N}^r$ and $\overline{p} \in \mathbb{N}^{n-r}$, such that r is as small as possible, but on the other hand the elements which have to be neglected, will be much bigger than bound in the previous section, thus usually we will ends with a bigger reduced dimension r than in the case of approximation of the important (significant) part of the spectrum.

Moreover, since we have more demands in determination of the first r indices, it is more likely that the final reduced dimension will be larger than in the previous section. On the other hand, we will be able to approximate all eigenvalues, so we also expect that the bigger reduced dimension is needed.

Now, similarly as in the previous section we define a vector $w \in \mathbb{N}^n$ by $w(i) = p(i)$ for $i = 1, \ldots, r$ and $w(i) = \overline{p}(i - r)$ for $i = r + 1, \ldots, n$. With this permutation our aim is to approximate all eigenvalues of the matrix $\widehat{A}_P^J = J\widehat{A}_P$ which has the same form as in (3.10).

In this case we approximate matrix $\widehat{A}_P^J$ by the matrix

$$\overline{A}_P = \begin{bmatrix} \overline{A}_{11} & 0 \\ 0 & \overline{A}_{22} \end{bmatrix},$$

with

$$\overline{A}_{11} = \widehat{A}_P^J(1 : 2r, 1 : 2r)\,, \tag{4.24}$$

$$\overline{A}_{22} = \bigoplus_{i=r+1}^{n} \Psi_{w(i)}^J\,, \quad \text{where} \quad \Psi_i^J = \begin{bmatrix} 0 & \omega_i \\ -\omega_i & -\gamma_i - C_{ii} \end{bmatrix}. \tag{4.25}$$

Due to the block structure of our approximated matrix $\overline{A}_P$, in order to efficiently determine approximation of all eigenvalues, we need to determine eigenvalues of the matrix $\overline{A}_{11}$ while eigenvalues of the matrix $\overline{A}_{22}$ can be calculated by a formula. That is, it holds that the matrix

$$\Psi_i^J = \begin{bmatrix} 0 & \omega_i \\ -\omega_i & -\gamma_i - C_{ii} \end{bmatrix}, \tag{4.26}$$

has eigenvalues

$$\widetilde{\lambda}_{2i-1} = \frac{-\gamma_i - C_{ii} - \sqrt{(\gamma_i + C_{ii})^2 - 4\omega_i^2}}{2} \quad \text{for } i = r + 1, \ldots, n, \tag{4.27}$$

$$\widetilde{\lambda}_{2i} = \frac{-\gamma_i - C_{ii} + \sqrt{(\gamma_i + C_{ii})^2 - 4\omega_i^2}}{2} \quad \text{for } i = r + 1, \ldots, n, \tag{4.28}$$

where $C_{ii} = \sum_i^k v_i \Phi(:,i)^T C_i \Phi(:,i)$ and γ_i depends on internal damping. For example if internal damping is a small multiple of the critical damping given in (2.6) then $\gamma_i = \alpha\omega_i$.

All above stated we will summarize in Algorithm 3 which calculates approximation of all eigenvalues.

Algorithm 3 Approximation of all eigenvalues

Require: α, Φ – such that $\Phi^T K \Phi = \Omega^2 = \mathrm{diag}(\omega_1^2, \ldots, \omega_n^2)$ and $\Phi^T M \Phi = I$;
v_i, C_i, $i = 1, \ldots, k$ – viscosity and position matrix for the ith damper;
tol – tolerance needed for determination in vector w.

Ensure: eigenvalues of $(A_p(\mathbf{v}), J_P)$

1: indices (i,j), $i \neq j$ of elements in C such that $|C_{i,j}| > tol$. This determines vectors $\overline{p} \in \mathbb{N}^{n-r}$ and $p \in \mathbb{N}^r$.
2: $\Omega_r = \mathrm{diag}(\omega_{p(1)}, \omega_{p(2)}, \ldots, \omega_{p(k)})$
3: $C = \Phi^T(v_1 C_1 + v_2 C_2 + \cdots + v_k C_k)\Phi$
4: Calculate all eigenvalues of the pair $(\widetilde{A}_{11}, J_{11})$, where

$$\widetilde{A}_{11} = P^T \begin{bmatrix} 0 & \Omega_r \\ \Omega_r & \alpha\Omega_r + C(p,p) \end{bmatrix} P,$$

where P is perfect shuffle permutation and J_{11} is given by (3.9).
5: Calculate other eigenvalues using formulae (4.27)-(4.28).

In the following subsection we will provide an error bound for the approximation given by Algorithm 3.

We would like to emphasized that in general the new approach (summarized in Algorithm 3) improves the approximation technique which was studied in [19, 24] (derived specifically for a small viscosity $v \ll 1$). More precisely, using Algorithm 3 with tolerance tol large enough one can obtain approximations of the same quality as in [19, 24]. On the other hand, the smaller tolerance tol, will result with the reduced dimension (obtain by Algorithm 3) larger than zero, that is $r > 0$, which further insures a better approximation than the one from [19, 24], as we will illustrate in the numerical experiments.

The benefits of the new approximation technique over the approach which has been derived specifically for a small viscosity v, in [19, 24], will be illustrated in the section Numerical experiments, especially the Figure 5.2 illustrates eigenvalue behavior obtained by formulas $(4.27 - 4.28)$ as well as the approximations obtained also by Algorithm 3 for the tolerance tol large enough.

4.1 Error bound for approximation of all eigenvalues

In this section we will present the error bounds for the eigenvalue approximations made by Algorithm 3. Here we will present a corresponding error bound in the sense of Gershgorin. We assume that for the fixed viscosity $\mathbf{v}$ we have determined index vectors $\overline{p}$ and p as in Algorithm 3.

Now, similarly as in the previous section using (3.10) and J is given by (3.9), our

matrix of interest can be written in the block diagonal form as:

$$\widehat{A}_P^J = J\widehat{A}_P = \begin{bmatrix} \overline{A}_{11} & \overline{A}_{12} \\ \overline{A}_{21} & \overline{A}_{22} \end{bmatrix}$$

where $\widetilde{A}_{11}$ is a matrix of dimension $2r \times 2r$. Now, our approximation corresponds to a matrix

$$\overline{A}_P = \begin{bmatrix} \overline{A}_{11} & 0 \\ 0 & \oplus_{i=k+1}^{n} \Psi_{w(i)}^J \end{bmatrix}$$

where

$$\Psi_i^J = \begin{bmatrix} 0 & \omega_i \\ -\omega_i & -\gamma_i - C_{ii} \end{bmatrix}. \tag{4.29}$$

As in the previous section here similarly assume that all eigenvalues from the matrix $\overline{A}_P$ are simple.

In order to apply Gerschgorin bound we need to diagonalize all diagonal blocks of the matrix $\overline{A}_P$. Thus, first we will diagonalize the block $\overline{A}_{11}$, that is, we calculate the matrix X_{11} such that $\overline{A}_{11} = X_{11}\mathrm{diag}(\widetilde{\lambda}_1, \ldots, \widetilde{\lambda}_{2r})X_{11}^{-1}$ where elements $\widetilde{\lambda}_1, \ldots, \widetilde{\lambda}_{2r}$ provide the approximations of our eigenvalues.

Moreover, we need to diagonalize all two by two matrices $\Psi_{w(i)}^J$ for $i = r+1, \ldots, n$. For that we assume that $(\gamma_w(i) + C_{w(i)w(i)})^2 - 4\omega_{w(i)} \neq 0$, for $i = r+1, \ldots, n$, thus there exists matrices $Y_{r+1,r+1}, \ldots, Y_{n,n}$ such that

$$\Psi_{w(i)}^J = Y_{ii}\mathrm{diag}(\lambda_{2i-1}, \lambda_{2i})Y_{ii}^{-1}, \quad \forall i = r+1, \ldots, n,$$

where $\Psi_{w(i)}^J$ and λ_{2i-1}, λ_{2i} are given in (4.26), (4.27),(4.28), respectively.

Then, using block diagonal matrix

$$\widehat{X} = \begin{bmatrix} X_{11} & 0 \\ 0 & \bigoplus_{i=1}^{n-k} Y_{ii} \end{bmatrix}$$

we obtain

$$\widehat{X}^{-1}\overline{A}_P\widehat{X} = \begin{bmatrix} \Lambda_{11} & X_{11}^{-1}\overline{A}_{12}\bigoplus_{i=1}^{n-k} Y_{ii} \\ \left(\bigoplus_{i=r+1}^{n} Y_{ii}\right)^{-1}\overline{A}_{21}X_{11} & A_Y \end{bmatrix},$$

with

$$A_Y = \left(\bigoplus_{i=r+1}^{n} Y_{ii}\right)^{-1} \overline{A}_{22} \bigoplus_{i=r+1}^{n} Y_{ii} \tag{4.30}$$

Similarly as in the previous section here we will apply Gerschgorin theorem.

For the upper and the lower diagonal block we apply a row version of Gershgorin bound, thus, for each $i \in \{1, 2, \ldots, 2n\}$ there exists permutation π of the set $\{1, 2, \ldots, 2n\}$

such that the following bound holds:

$$
\begin{aligned}
|\widetilde{\lambda}_i - \lambda_{\pi(i)}(\widehat{A}_P)| &\le \sum_{j=r+1}^{2n} \left| (X_{11}^{-1}\widetilde{A}_{12} \oplus_{l=r+1}^{n} Y_{ll})_{ij} \right|, \quad i = 1, \ldots, 2r, \\
|\widetilde{\lambda}_{2i-1} - \lambda_{\pi(i)}(\widehat{A}_P)| &\le \sum_{j=1}^{2r} \left| \left(\left(\oplus_{l=r+1}^{n} Y_{ll} \right)^{-1} \overline{A}_{21} X_{11} \right)_{2i-2r-1,j} \right| \\
&\quad + \sum_{\substack{j=1, \\ j \neq 2i-2r-1}}^{n-2r} |(A_Y)_{2i-2r-1,j}|, \quad i = r+1, \ldots, n, \\
|\widetilde{\lambda}_{2i} - \lambda_{\pi(i)}(\widehat{A}_P)| &\le \sum_{j=1}^{2r} \left| \left(\left(\oplus_{l=r+1}^{n} Y_{ll} \right)^{-1} \overline{A}_{21} X_{11} \right)_{2i-2r,j} \right| \\
&\quad + \sum_{\substack{j=1 \\ j \neq 2i-2r}}^{n-2r} |(A_Y)_{2i-2r,j}|, \quad i = r+1, \ldots, n,
\end{aligned}
$$

where r is number of elements in vector p and A_Y is given in (4.30).

5 Numerical experiments

In this section we will present two examples. In these examples the corresponding eigenvalue problems have been solved by Matlab's function `eig` in double precision.

Example 5.1. In the first example we will consider an n-mass oscillator or oscillator ladder with two dampers, which describes the mechanical system of n masses and $n+1$ springs. Similar models were considered e.g. in [2, 3, 22, 24]. For this mechanical system the mathematical model is given by (1.1), where the mass and stiffness matrices are

$$
\begin{aligned}
M &= \operatorname{diag}(m_1, m_2, \ldots, m_n), \\
K &= \begin{pmatrix}
k_1 + k_2 & -k_2 & & & \\
-k_2 & k_2 + k_3 & -k_3 & & \\
& \ddots & \ddots & \ddots & \\
& & -k_{n-1} & k_{n-1} + k_n & -k_n \\
& & & -k_n & k_n + k_{n+1}
\end{pmatrix}.
\end{aligned}
$$

The damping matrix is $D(\mathbf{v}) = C_u + C_{ext}(\mathbf{v})$, where the internal damping C_u is defined as in (2.6) with $\alpha_0 = 0.04$.

Mass and stiffness will be given by the following configuration

$$
n = 1000; \quad k_i = 1, \quad \forall i; \qquad\qquad m_i = \begin{cases} 1200 - 2i, & i = 1, \ldots, 200, \\ 4i, & i = 201, \ldots, n. \end{cases}
$$

We will consider two dampers of different viscosities, that is $\mathbf{v} = \begin{bmatrix} v_1, & v_2 \end{bmatrix}$, thus the external damping is defined by $C_{ext} = v_1 e_i e_i^T + v_2 e_j e_j^T$, where $1 \leq i < j \leq n$. We can consider different damping positions, but in order to illustrate benefits of our approximation technique while we vary viscosities, we will fix damping positions to: $i = 600$ and $j = 900$ with viscosities which will vary over the feasible interval.

In this example we would like to consider eigenvalue behaviour for all eigenvalues which we have considered in Section 4. For the purpose of easier illustration of eigenvalue behaviour we will consider the following configurations of viscosities:

$$(v_1, v_2) = (\frac{v}{4}, v), \quad v = 1, 2, \ldots, 100. \tag{5.31}$$

For the tolerance needed in Algorithm 3 we have used $tol = 10^{-5}$.

Following figures illustrate the quality of the obtained approximation and the eigenvalue behavior while we vary viscosity parameters.

First, on Figure 5.1 we illustrate eigenvalue behavior when parameter v given by configuration (5.31) varies from 1 to 100 with step 1. On this figure the exact eigenvalues are calculated directly by Matlab and are denoted by red circles. On the other hand the approximations given by formulas (4.27-4.28) are denoted by black x-es, while the approximation obtained by solving a small eigenvalue problem defined by matrix $\overline{A}_{11}$ from (4.24) are denoted by blue dots. Here we should note that we plot the eigenvalues behavior only for a small parts of considered eigenvalues, since otherwise it would be hard to follow all eigenvalues on one figure.

On Figure 5.2 we would like to illustrate the benefits of our approximation technique over the approach which is derived specifically for a small viscosity v, which was studied in [19, 24]. On this figure we plot eigenvalues behavior obtained by using only formulas $(4.27 - 4.28)$ for approximation. The following approximation can be obtained also by Algorithm 3 by setting the tolerance tol to be large enough.

Figure 5.3 shows the relative errors for the eigenvalues and the upper bounds for the relative errors for the viscosity $v = 10$. For this viscosity the reduced dimension for the above given tolerance was equal to $r = 416$. On this figure we plot all relative errors larger than 10^{-12}.

In order to illustrate the magnitude of the reduced dimension while we vary parameter v as in (5.31), we have shown reduced dimension r on Figure 5.4. Here the reduced dimension r varies from $r = 0$ (meaning that we use only formulas $(4.27 - 4.28)$) to $r = 733$ which is 73.3% of the full dimension.

In the previous example we have considered eigenvalue behavior of all eigenvalues. Thus, in the next example we will illustrate our approach on the case when one is interested in calculation of a part of the spectrum.

Example 5.2. In this example we consider the mechanical system shown in Figure 5.5 with three rows of d masses which gives $3d+1$ masses and $3d+4$ springs. Here we consider three dampers of the same viscosity where each damper is placed on corresponding row of masses. Each row has springs of the same stiffness equal to k_1, k_2, k_3, respectively. The last mass (m_{3d+1}) is connected to the fixed base with the spring with stiffness k_4.

N.T. ne treba only approximation, dovoljno je approximation

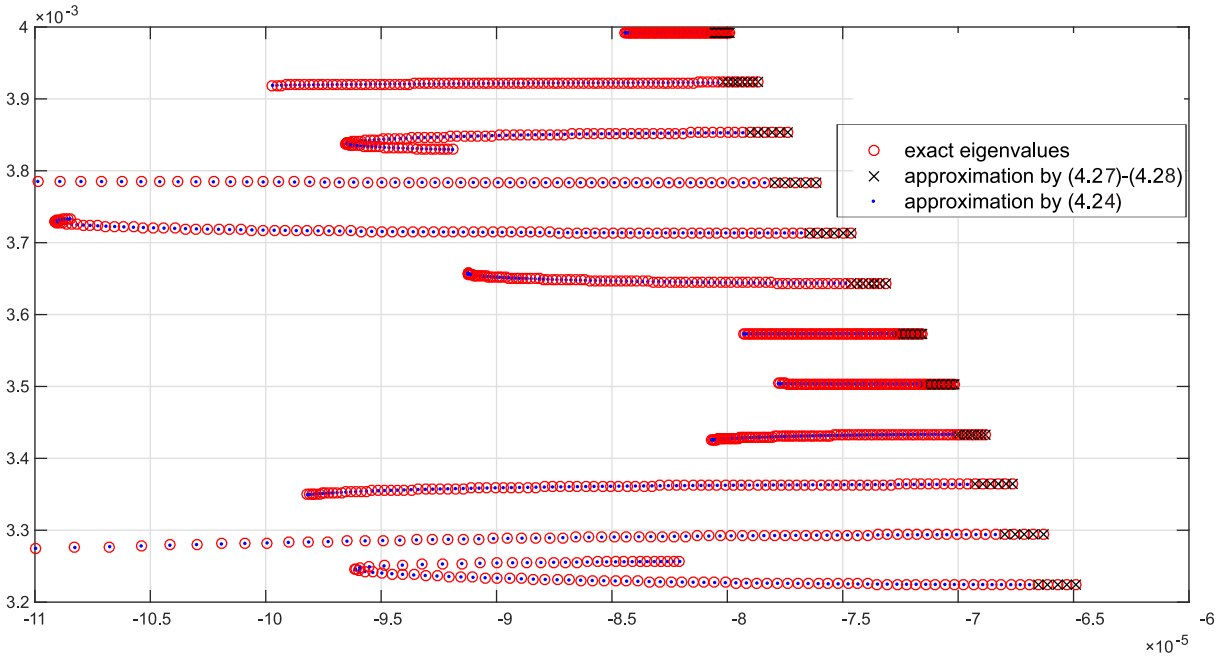

Figure 5.1: Eigenvalue behaviour for Example 5.1

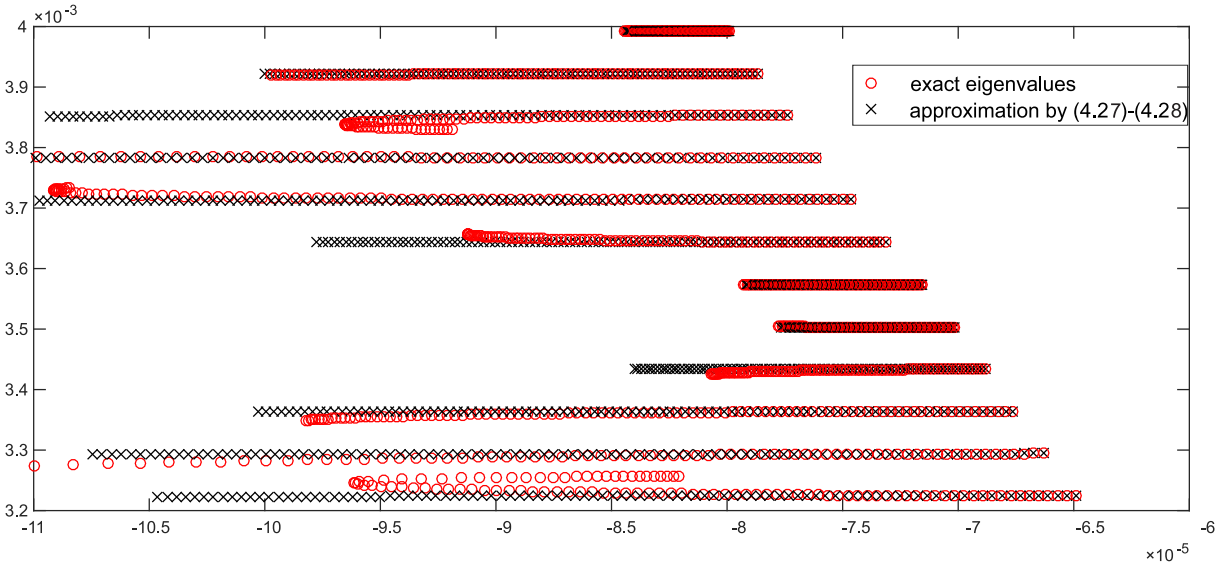

Figure 5.2: Eigenvalue behaviour for Example 5.1 for approximation obtained using only formulas $(4.27 - 4.28)$

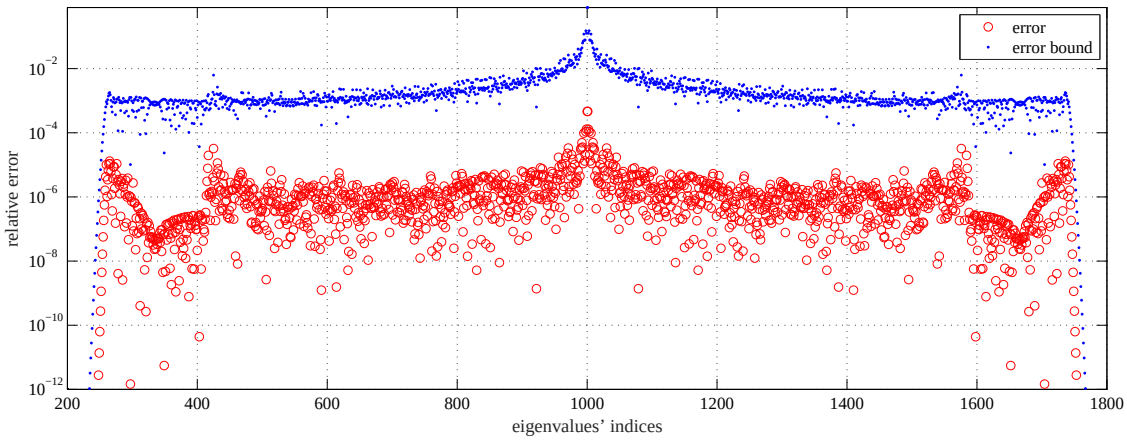

Figure 5.3: Relative error for Example 5.1 for $v = 10$, $r = 416$.

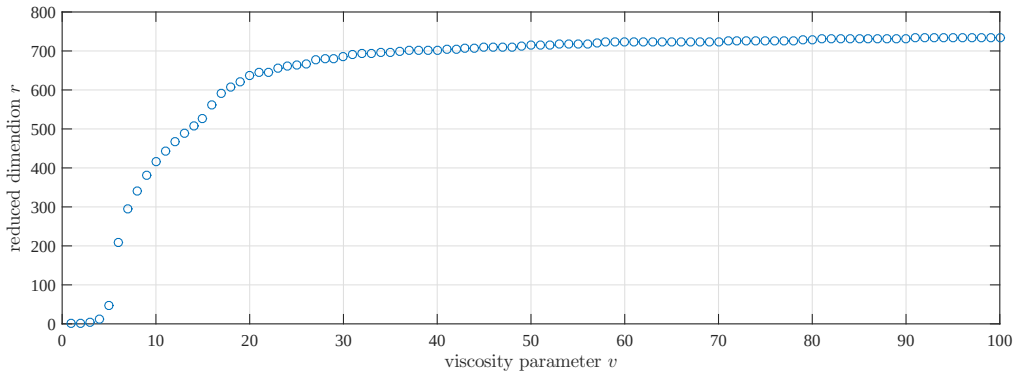

Figure 5.4: Reduction dimensions r for Example 5.1 for $v = 1, 2, 3, \ldots, 100$

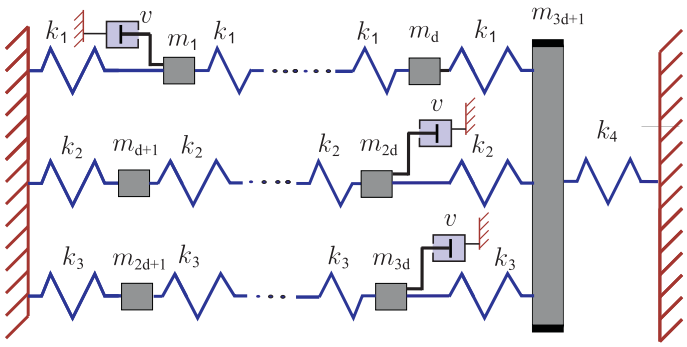

Figure 5.5: $3d + 1$ mass oscillator

The mathematical model for the considered vibrational system is given by (1.2) where the mass matrix is

$$M = \mathrm{diag}(m_1, m_2, \ldots, m_n).$$

The stiffness matrix is defined as

$$K = \begin{bmatrix} K_{11} & & & -\kappa_1 \\ & K_{22} & & -\kappa_2 \\ & & K_{33} & -\kappa_3 \\ -\kappa_1^T & -\kappa_2^T & -\kappa_3^T & k_1+k_2+k_3+k_4 \end{bmatrix},$$

where

$$K_{ii} = k_i \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}, \quad \kappa_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ k_i \end{bmatrix}, \quad i = 1, 2, 3.$$

In our example we will consider the following configuration

$$\begin{aligned} d &= 400, \quad n = 3d + 1 = 1201, \\ m_k &= k, \quad k = 1, \ldots, n, \\ k_1 &= 1, \quad k_2 = 20, \quad k_3 = 40, \quad k_4 = 50. \end{aligned}$$

Similarly as in the previous example we will fix dampers positions. Since our example has three dampers with the same viscosity v we have that damping matrix is equal to

$$D = C_u + C_{ext}, \quad \text{with,} \quad C_{ext} = ve_{350}e_{350}^T + ve_{600}e_{600}^T + ve_{1000}e_{1000}^T.$$

where for the sake of easier illustration of obtained results, we set damping positions to $350, 600, 1000$ and internal damping is given by (2.6) with $\alpha_0 = 0.002$. In this example, in Algorithm 2 we use tolerance $tol = 10^{-4}$.

We will illustrate the quality of eigenvalue approximation given by Algorithm 2 where we are interested in behavior (in damping) only of the part of the undamped eigenfrequencies larger that 0.1 and smaller than 0.11. For this example this means that we need to consider eigenvalue behavior of only 49 undamped eigenfrequencies which gives that parameter $s = 49$ in Algorithm 2. That is, undamped eigenfrequencies ω_i for $i = 335, \ldots, 383$ where ω_i from (2.3) are sorted in increasing order and there indices define vector p required in Algorithm 2.

The Figure 5.6 contains the relative errors and the bounds derived in Section 3.1. In particular, with red circles we have denoted exact error calculated by Matlab. With blue dots we show the relative errors for the bound given by (3.23) while the error bound given by (3.19b) is presented by green triangles. The figure shows the quality of the derived upper bound for the fixed viscosity $v = 2$ where Algorithm 1 returned the reduced

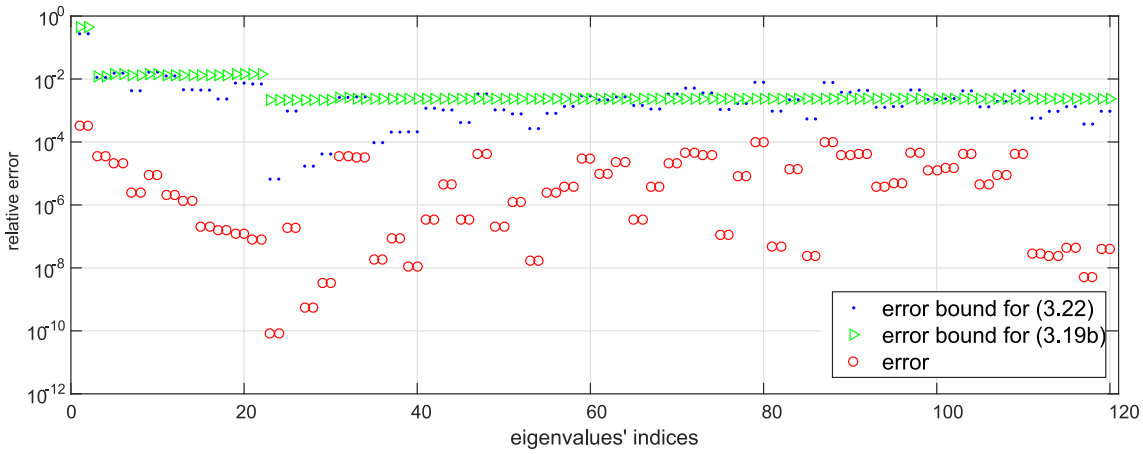

Figure 5.6: Relative error for Example 5.2 for $v = 2$, $r = 60$.

dimension is $r = 60$. Thus, instead of 98 eigenvalues that we would like to track we have obtained approximations for 120 eigenvalues.

As one can see from Figure 5.6 there exists eigenvalues for which is the bound (3.23) better, but also exist eigenvalues for which the bound (3.19b) gives better estimation for the error. In general, this can vary while we change the viscosity v, so the best option would be to take the minimum of both derived upper bounds.

The eigenvalue behavior is shown at Figure 5.7 where with blue dots we denote approximations obtained by Algorithm 2, while with red circles we denote exact eigenvalues. We would like to note that the reduced dimension was varied from $r = 60$ to $r = 131$ while v was varied from 0.05 to 3. Moreover, since we consider damping of undamped eigenfrequencies that are larger than 0.1 and smaller than 0.11 on Figure 5.7 we plot eigenvalues whose imaginary parts lay between 0.1 and 0.11. From this figure, but also from Figure 5.1 we can note that we have achieved satisfactory eigenvalue tracking even for moderate viscosities v, while we ensure an efficient error bounds for the obtained approximation.

6 Conclusion

In this paper we have shown how one can efficiently calculate the whole spectrum or just one important part of it, of the parameter dependent quadratic eigenvalue problem , for the variety of the parameters $\mathbf{v} = \begin{bmatrix} v_1 & \dots & v_k \end{bmatrix}$, $v_i \in \mathbb{R}$. Our approach generalize the results form [19, 24] which hold for $v_i \ll 1$, on the v_i of the modest magnitude. Moreover, even for the case when $v_i \ll 1$ our approximations are more accurate then those from [19, 24]. At the same time, for the both kind of approximations (the all eigenvalues or just one part of them) we have derived the corresponding upper bounds. The quality of the error bounds as well as the performance of the achieved eigenvalue tracking was illustrated in several numerical experiments.

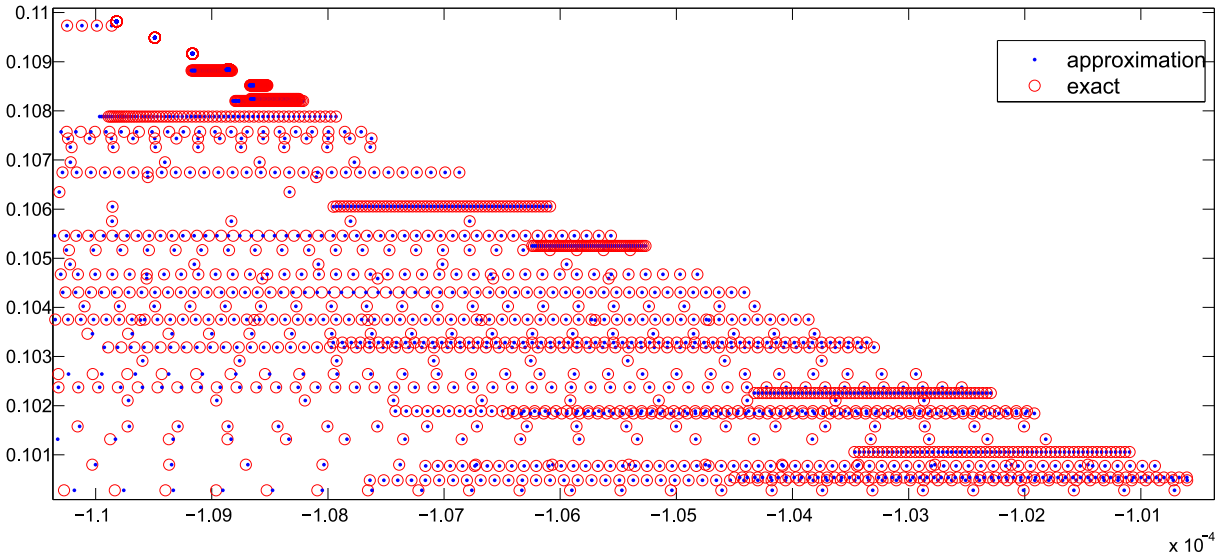

Figure 5.7: Eigenvalue behaviour for Example 5.2

7 Acknowledgements

This work has been supported in part by the Croatian Science Foundation (HRZZ) under the project Optimization of parameter dependent mechanical systems (IP-2014-09-9540), Grant Nr. 9540.

References

[1] Zhaojun Bai and Yangfeng Su. SOAR: A Second-order Arnoldi Method for the Solution of the Quadratic Eigenvalue Problem. *SIAM Journal on Matrix Analysis and Applications*, 26(3):640–659, 2003.

[2] P. Benner, Z. Tomljanović, and N. Truhar. Dimension reduction for damping optimization in linear vibrating systems. *Z. Angew. Math. Mech.*, 91(3):179–191, 2011. DOI: 10.1002/zamm.201000077.

[3] P. Benner, Z. Tomljanović, and N. Truhar. Optimal Damping of Selected Eigenfrequencies Using Dimension Reduction. *Numerical Linear Algebra with Applications*, 20(1):1–17, 2013. DOI: 10.1002/nla.833.

[4] K. Brabender. *Optimale Dämpfung von linearen Schwingungssystemen*. PhD thesis, Fernuniversität, Hagen, 1998.

[5] J.V. Bruke, A.S. Lewis, and M.L. Overton. Optimization and pseudospectra, with applications to robust stability. *SIAM J. Matrix Anal. Appl.*, 25(1):80–104, 2003.

[6] P. Freitas and P. Lancaster. The optimal value of the spectral abscissa for a system of linear oscillators. *SIAM. J. Matrix Anal. & Appl.*, 21(1):195–208, 1999.

[7] W.K. Gawronski. *Advanced Structural Dynamics and Active Control of Structures.* Springer, New York, USA, 2004.

[8] G. H. Golub and C. Van F. Loan. *Matrix Computations.* The Johns Hopkins University Press, 3rd edition, 1996.

[9] S. Grundel and M. L. Overton. Variational analyisis of the spectral abscissa at a matrix with a nongeneric multiple eigenvalue. *Set-Valued and Variational Analysis*, 22(1):1–25, 2013.

[10] Michiel E. Hochstenbacha, Andrej Muhič, and Bor Plestenjak. Jacobidavidson methods for polynomial two-parameter eigenvalue problems. *Journal of Computational and Applied Mathematics*, 288:251–263, 2015.

[11] J.C.Egana, N.M.Kuhl, and L.C.Santos. An inverse eigenvalemethod for frequency isolation in spring-mass system. *Numer. Linear Algebra Appl.*, 9:65–79, 2002.

[12] K. T. Joseph. Inverse eigenvalue problem in structural design. *Am. Inst. Aeronaut. Astronaut. J.*, 30:2890–2896, 1992.

[13] A.V. Knyazev and K. Neymeyr. Efficient solution of symmetric eigenvalue problems using multigrid preconditioners in the locally optimal block conjugate gradient method. *Electron. Trans. Numer. Anal.*, 15:38–55, 2003.

[14] I. Kuzmanović, Z. Tomljanović, and N. Truhar. Damping optimization over the arbitrary time of the excited mechanical system. *Journal of Computational and Applied Mathematics*, 304:120–129, 2016. DOI: 10.1016/j.cam.2016.03.005.

[15] R. Li, Y. Nakatsukasa, N. Truhar, and S. Xu. Perturbation of partitioned hermitian generalized eigenvalue problem. *SIAM Journal on Matrix Analysis and Applications*, 32(2):642–663, 2011.

[16] Xin Lu, Shu fang Xu, and Yun feng Cai. Partial derivatives of the eigen-triplet of the quadratic eigenvalue problem depending on several parameters. *Applied Mathematics and Computation*, 219(24):11348–11357, 2013.

[17] J. Moro and J.C.Egana. Directional algorithams for frequency isolation problem in undamped vibrational systems. *Mechanical Systems and Signal Processing*, 75:11–26, 2016.

[18] I. Nakić. *Optimal damping of vibrational systems.* PhD thesis, Fernuniversität, Hagen, 2002.

[19] I. Nakić, Z. Tomljanović, and N. Truhar. Optimal direct velocity feedback. *Applied mathematics and computation*, 225:590–600, 2013.

[20] G. W. Stewart. *Matrix Algorithms. Volume II: Eigensystems.* SIAM, Philadelphia, 2001.

[21] N. Truhar and K. Veselić. On some properties of the lyapunov equation for damped systems. *Mathematical Communications*, 9:189–197, 2004.

[22] N. Truhar and K. Veselić. An efficient method for estimating the optimal dampers' viscosity for linear vibrating systems using Lyapunov equation. *SIAM J. Matrix Anal. Appl.*, 31(1):18–39, 2009.

[23] Ninoslav Truhar and Suzana Miodragović. An efficient algorithm for damper optimization for linear vibrating systems using Lyapunov equation. *Appl. Numer. Math.*, 98:106–121, 2015.

[24] K. Veselić. *Damped Oscillations of Linear Systems.* Springer Lecture Notes in Mathematics, Springer-Verlag, Berlin, 2011.