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Intermittency of superpositions of Ornstein-Uhlenbeck type processes

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Abstract: The phenomenon of intermittency has been widely discussed in physics literature. This paper provides a model of intermittency based on Lévy driven Ornstein-Uhlenbeck (OU) type processes. Discrete superpositions of these processes can be constructed to incorporate non-Gaussian marginal distributions and long or short range dependence. While the partial sums of finite superpositions of OU type processes obey the central limit theorem, we show that the partial sums of a large class of infinite long range dependent superpositions are intermittent. We discuss the property of intermittency and behavior of the cumulants for the superpositions of OU type processes.

Keywords: Ornstein-Uhlenbeck type processes, intermittency , long range dependence, weak convergence.

1 Introduction

The phenomenon of intermittency has been widely discussed in physics literature (see for example Bertini & Cancrini (1995), Fujisaka (1984), Molchanov (1991), Woyczyński (1998), Zel'dovich et al. (1987) and (Frisch 1995, Chapter 8)). The term is used to describe models exhibiting high degree of variability and enormous fluctuations which escape from the scope of the usual limit theory. Terms multifractality, separation of scales, dynamo effect are often used interchangeably with intermittency. For a formal definition of intermittency appearing in the theory of stochastic partial differential equations (SPDE) we follow Carmona & Molchanov (1994) and (Khoshnevisan 2014, Chapter 7). There, a nonnegative random field $\{\psi_t(x), t \geq 0, x \in \mathbb{R}\}$ stationary in parameter x is said to be intermittent if the function $k \mapsto \gamma(k)/k$ is strictly increasing on $[2, \infty)$ where $\gamma(k)$ is the k-th moment Lyapunov exponent of ψ defined by

$$\gamma(k) = \lim_{t \to \infty} \frac{\log \mathbb{E} \left(\psi_t(x)\right)^k}{t}, \tag{1}$$

assuming the limit exists and is finite. This approach to intermittency is tailored for the analysis of SPDEs and characterizes fields with progressive growth of moments.

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To compare intermittency to a slower growth of moments, consider the sum $\phi_n = \sum_{i=1}^n \xi_i$, where ξ_i are positive independent identically distributed (iid) random variables with finite moments. The k-th moment of ϕ_n grows as $n^k(\mathbb{E}\xi_1)^k$, therefore

$$\gamma(k) = \lim_{n\to\infty} \frac{k\log n + k \log \mathbb{E}\xi_1}{n} = 0$$

for all $k \geq 1$. With the appropriate centering and norming, the classical central limit theorem holds.

In contrast, for a sequence of products of positive random variables $\psi_n = \prod_{i=1}^n \xi_i$

$$\gamma(k) = \lim_{n\to\infty} \frac{\log \mathbb{E}\psi_n^k}{n} = \log \mathbb{E}\xi_1^k.$$

If ξ_i are not constant a.s., then from Jensen's inequality it follows that for $l > k$

$$\mathbb{E}\xi_1^k < \left(\mathbb{E}\xi^l\right)^{\frac{k}{l}},$$

showing that $\gamma(k)/k$ is strictly increasing. The wild growth of moments of ψ_n provides the main heuristic argument that intermittency implies unusual limiting behavior. A formal argument showing that under some assumptions intermittency implies large peaks in the space coordinate of the random field can be found in Khoshnevisan (2014), some ideas of which will be used later in this paper.

By far the most investigated model exhibiting intermittent behavior is the parabolic Anderson model (see Gärtner & den Hollander (2006), Gärtner et al. (2010, 2007), Gärtner & Molchanov (1990)). In this paper we consider models provided by the partial sums of discrete superpositions of Lévy driven Ornstein-Uhlenbeck (OU) type processes. While models based on Lévy flights describe the position of particle, models given by OU dynamics describe the velocity of particle trapped in a field generated by quadratic potential (Eliazar & Klafter (2005)). Applications of Lévy-driven OU type processes include financial econometrics Barndorff-Nielsen & Shephard (2001), Leonenko et al. (2012*a*), Li & Linetsky (2014), fluid dynamics Solomon et al. (1993), plasma physics Chechkin et al. (2002) and biology Ricciardi & Sacerdote (1979). The stochastic model discussed in this paper provides another example of intermittency model based on the velocity (see (Frisch 1995, Section 8.5)). First, we modify the preceding definition of intermittency to tailor it to the analysis of sequences of partial sum processes. In the case of finite superpositions we show that the central limit theorem holds. In the case of infinite long range dependent superpositions, we show that the growth of cumulants is such that the partial sum process is intermittent. The appendix contains examples that fit our assumptions which cover, to our knowledge, all the examples with tractable distributions of superpositions.

2 Intermittency

For a process $\{Y(t),\, t \geq 0\}$, denote

$$\bar{q} = \sup\{q > 0 : \mathbb{E}|Y(t)|^q < \infty\ \forall t\}.$$

Our definition of intermittency is based on the version of Lyapunov exponent that replaces t in the denominator of (1) with $\log t$. For a stochastic process $\{Y(t),\, t \geq 0\}$, define the *scaling function* at point $q \in [0, \overline{q})$ as

$$\tau(q) = \lim_{t\to\infty} \frac{\log \mathbb{E}|Y(t)|^q}{\log t}, \tag{2}$$

assuming the limit exists and is finite for every $q \in [0, \overline{q})$. Objects similar to the scaling function (2) appear in the theory of multifractal processes (see e.g. Grahovac & Leonenko (2015)), however, there are some important differences Khoshnevisan et al. (2015). The following proposition gives some properties of τ.

Proposition 1. The scaling function τ defined by (2) has the following properties:

(i) τ is non-decreasing and so is $q \mapsto \tau(q)/q$;

(ii) τ is convex;

(iii) if for some $0 < p < r < \overline{q}$, $\tau(p)/p < \tau(r)/r$, then there is a $q \in (p, r)$ such that $\tau(p)/p < \tau(q)/q < \tau(r)/r$.

Proof. (i) For $0 \leq q_1 < q_2 < \overline{q}$ Jensen's inequality implies

$$\mathbb{E}|Y(t)|^{q_1} = \mathbb{E}\left(|Y(t)|^{q_2}\right)^{\frac{q_1}{q_2}} \leq \left(\mathbb{E}|Y(t)|^{q_2}\right)^{\frac{q_1}{q_2}}$$

and thus

$$\tau(q_1) \leq \frac{q_1}{q_2}\tau(q_2)$$

proving part (i).

(ii) Take $0 \leq q_1 < q_2 < \overline{q}$ and $w_1, w_2 \geq 0$ such that $w_1 + w_2 = 1$. It follows from Hölder's inequality that

$$\mathbb{E}|Y(t)|^{w_1 q_1 + w_2 q_2} \leq \left(\mathbb{E}|Y(t)|^{q_1}\right)^{w_1}\left(\mathbb{E}|Y(t)|^{q_2}\right)^{w_2}.$$

Taking logarithms, dividing by $\log t$ for $t > 1$ and letting $t \to \infty$ we have

$$\tau(w_1 q_1 + w_2 q_2) \leq w_1 \tau(q_1) + w_2 \tau(q_2).$$

(iii) This is clear since $q \mapsto \tau(q)/q$ is continuous by (ii). □

We now define intermittency for a stochastic process and for a sequence of random variables by using the corresponding partial sum process.

Definition 1. A stochastic process $\{Y(t),\, t \geq 0\}$ is *intermittent* if there exist $p, r \in (0, \overline{q})$ such that

$$\frac{\tau(p)}{p} < \frac{\tau(r)}{r}.$$

Later in the paper, we will investigate intermittency of a stationary sequence of random variables $\{Y_i,\, i \in \mathbb{N}\}$ with finite mean. In this sense, intermittency will be considered as intermittency of the centered partial sum process

$$S(t) = \sum_{i=1}^{\lfloor t \rfloor} Y_i - \sum_{i=1}^{\lfloor t \rfloor} \mathbb{E} Y_i, \;\; t \geq 0.$$

Proposition 1(i) shows that the function $q \mapsto \tau(q)/q$ is always non-decreasing. What makes the process intermittent is the existence of points of strict increase. In section 5, we connect this property to the limiting behavior of cumulants of partial sums of superpositions of Ornstein-Uhlenbeck type processes. We show that while the partial sums of finite superpositions obey the central limit theorem, partial sums of infinite long-range dependent superpositions provide examples of intermittent processes.

3 Ornstein-Uhlenbeck type processes

An Ornstein-Uhlenbeck (OU) type process is the solution of the stochastic differential equation

$$dX(t) = -\lambda X(t)dt + dZ(\lambda t), \quad t \geq 0, \tag{3}$$

where $\lambda > 0$, and $Z(t), t \geq 0$ is a Lévy process, and the inital condition $X(0)$ is taken to be independent of the process Z. The process Z is termed the background driving Lévy process (BDLP) corresponding to the process X. The strong stationary solution of this equation exists if and only if

$$\mathbb{E} \log \left(1 + |Z\left(1\right)|\right) < \infty.$$

See Sato (1999) for a detailed discussion of OU type processes driven by Lévy noise and their properties. The solution of (3) is given by

$$X(t) = e^{-\lambda t} X(0) + \int_0^t e^{-\lambda(t-s)} dZ(\lambda s). \tag{4}$$

Equation (4) specifies the unique (up to indistinguishability) strong solution of equation (3) Sato (1999). The meaning of the stochastic integral in (4) was detailed in (Applebaum 2009, p.214).

The scaling in equation (3) is such that the marginal distribution of the solution does not depend on λ, and the law of the Lévy process is determined uniquely by the distribution of X through the relation of the cumulant transforms. Let

$$\kappa(z) = C\left\{z; X\right\} = \log \mathbb{E} \exp\left\{izX\right\}, \quad z \in \mathbb{R}$$

be the cumulant transform of a random variable X, and

$$\kappa_m = (-i)^m \frac{d^m}{dz^m} \kappa(z)|_{z=0}, \;\; m \geq 1$$

be the cumulant of order m of X.

The cumulant transforms of $X(t)$ and $Z(1)$ are related by

$$C\{z;X\} = \int_0^\infty C\left\{e^{-s}z;Z(1)\right\}ds = \int_0^z C\{\xi;Z(1)\}\frac{d\xi}{\xi}$$

and

$$C\{z;Z(1)\} = z\frac{\partial C\{z;X\}}{\partial z}.$$

By specifying the appropriate BDLP, OU type processes with given self-decomposable marginal distributions can be obtained. These distributions include normal, Gamma, inverse Gaussian, Student's t, and many others. If the second moment is finite, the correlation function is exponential:

$$\mathrm{corr}(X(t),X(s)) = e^{-\lambda(t-s)},\ t \geq s \geq 0.$$

4 Discrete superpositions of Ornstein-Uhlenbeck type processes

Superpositions of OU type processes, or supOU processes for short, were introduced in Barndorff-Nielsen (1997, 2001), Barndorff-Nielsen & Stelzer (2011), see also Barndorff-Nielsen & Leonenko (2005*b*), Barndorff-Nielsen & Shephard (2001), Fasen & Klüppelberg (2007), among others. We define the superpositions under the following condition:

(A) Let $X^{(k)}(t)$, $k \geq 1$ be the sequence of independent stationary processes such that each $X^{(k)}(t)$ is the stationary solution of the equation

$$dX^{(k)}(t) = -\lambda_k X^{(k)}(t)dt + dZ^{(k)}(\lambda_k t), \quad t \geq 0, \tag{5}$$

in which the Lévy processes $Z^{(k)}$ are independent, and $\lambda_k > 0$ for all $k \geq 1$. Assume that the self decomposable distribution of $X^{(k)}$ has finite moments of order $p \geq 2$ and that for $m = 2, \ldots, p$, the m-th order cumulant of $X^{(k)}$ is of the form $C_m \delta_k$ where C_m is a constant that does not depend on k, and δ_k is a parameter of the distribution of $X^{(k)}$.

Define the superposition of OU processes, either finite for an integer $K \geq 1$

$$X_K(t) = \sum_{k=1}^{K} X^{(k)}(t),\ t \in \mathbb{R} \tag{6}$$

or infinite

$$X_\infty(t) = \sum_{k=1}^{\infty} X^{(k)}(t),\ t \in \mathbb{R}. \tag{7}$$

The construction with infinite superposition is well-defined in the sense of mean-square or almost-sure convergence provided that the following condition holds:

$$\textbf{(B)} \quad \sum_{k=1}^{\infty} \mathbb{E}X^{(k)}(t) < \infty \text{ and } \sum_{k=1}^{\infty.} VarX^{(k)}(t) < \infty.$$

Although assumption (A) may seem restrictive, it is satisfied for many examples with tractable distributions of superpositions. The appendix provides a number of examples where both assumptions (A) and (B) are satisfied. These examples include Gamma, inverse Gaussian and other well known distributions. Their superpositions have the marginal distributions that belong to the same class as the marginal distributions of the components of superposition.

In the case of finite superposition, the covariance function of the resulting process is

$$R_{X_K}(t) = \mathrm{Cov}(X_K(0), X_K(t)) = \sum_{k=1}^{K} \mathrm{Var}(X^{(k)}(t)) e^{-\lambda_k t},$$

and the finite superposition is a short-range dependent process since the correlation function is integrable.

In the case of infinite superposition, the covariance function is

$$R_{X_\infty}(t) = \mathrm{Cov}(X_\infty(0), X_\infty(t)) = \sum_{k=1}^{\infty} \mathrm{Var}(X^{(k)}(t)) e^{-\lambda_k t},$$

and under the condition (A) the variance of $X^{(k)}(t)$ is proportional to δ_k, that is

$$\mathrm{Var}(X^{(k)}(t)) = \delta_k C_2,$$

where constant C_2 does not depend on k and reflects parameters of the marginal distribution of $X^{(k)}$. If one chooses

$$\delta_k = k^{-(1+2(1-H))}, \ \frac{1}{2} < H < 1, \quad \lambda_k = \lambda/k$$

for some $\lambda > 0$, then

$$R_{X_\infty}(t) = C_2 \sum_{k=1}^{\infty} \frac{1}{k^{1+2(1-H)}} e^{-\lambda t/k}. \tag{8}$$

The lemma below shows that the correlation function (8) is not integrable for the chosen parameters δ_k and λ_k, thus the process obtained via infinite superposition exhibits long-range dependence.

Lemma 1. *For the infinite superposition (7) of OU type processes that satisfy condition (A) with $p = 2$ and condition (B), the covariance function of $X_\infty(t)$ given by (8) with $\lambda^{(k)} = \lambda/k$ and $\delta_k = k^{-(1+2(1-H))}$, $\frac{1}{2} < H < 1$, can be written as*

$$R_{X_\infty}(t) = \frac{L(t)}{t^{2(1-H)}}, \quad t > 0$$

where L is a slowly varying at infinity function.

Proof. The proof of this lemma is essentially the same as the proofs presented for particular cases of superpositions of OU processes in Leonenko et al. (2012*b*), Leonenko &

Taufer (2005). We provide it here for completeness and for the remark that follows. The remark will be used for proofs later in the paper. Let

$$L(t) = C_2 t^{2(1-H)} \sum_{k=1}^{\infty} \frac{1}{k^{1+2(1-H)}} e^{-\lambda t/k}.$$

Estimate the sum appearing in the expression for L as follows:

$$\int_1^{\infty} \frac{e^{-\lambda t/u}}{u^{1+2(1-H)}} du \leq \sum_{k=1}^{\infty} \frac{1}{k^{1+2(1-H)}} e^{-\lambda t/k} \leq \int_1^{\infty} \frac{e^{-\lambda t/u}}{u^{1+2(1-H)}} du + e^{-\lambda t}.$$

Transform the variables $\lambda t/u = s$ to get

$$\frac{C_2}{\lambda^{2(1-H)}} \int_0^{\lambda t} e^{-s} s^{2(1-H)-1} ds \leq L(t) \leq \frac{C_2}{\lambda^{2(1-H)}} \int_0^{\lambda t} e^{-s} s^{2(1-H)-1} ds + C_2 e^{-\lambda t} t^{2(1-H)}.$$

Since

$$\int_0^{\lambda t} e^{-s} s^{2(1-H)-1} ds \to \Gamma(2(1-H))$$

as $t \to \infty$, it follows that $\lim_{t\to\infty} L(tv)/L(t) = 1$ for any fixed $v > 0$.

□

Remark 1. From proof of Lemma 1

$$\begin{aligned} L([Nt]) &\leq \frac{C_2}{\lambda^{2(1-H)}} \int_0^{\lambda[Nt]} e^{-s} s^{2(1-H)-1} ds + C_2 e^{-\lambda[Nt]} [Nt]^{2(1-H)} \\ &\leq \frac{C_2}{\lambda^{2(1-H)}} \Gamma(2(1-H)) + C_2 e^{-2(1-H)} \left(\frac{2(1-H)}{\lambda} \right)^{2(1-H)} \end{aligned}$$

for all $N \geq 1$ and $t \in [0,1]$ since the function $x^{2(1-H)} e^{-x}$ is bounded (attains its maximum at $x = 2(1-H)$). Also from the proof of Lemma 1

$$L(N) \geq \frac{C_2}{\lambda^{2(1-H)}} \int_0^{\lambda N} e^{-s} s^{2(1-H)-1} ds \geq \frac{C_2}{\lambda^{2(1-H)}} \int_0^{\lambda} e^{-s} s^{2(1-H)-1} ds$$

for all $N \geq 1$. Also note that $L(0) = 0$. Therefore the ratio $L([Nt])/L(N)$ is bounded uniformly in $N \geq 1$ and $t \geq 0$.

5 Limit distributions of partial sums of superpositions of supOU processes

For $t > 0$, consider partial sum processes

$$S_K(t) = \sum_{i=1}^{[t]} X_K(i) \tag{9}$$

and

$$S_\infty(t) = \sum_{i=1}^{[t]} X_\infty(i). \tag{10}$$

We begin with the limit distribution of the partial sum process for the finite superposition. The asymptotic normality in this case is easy to prove using the strong mixing property of OU processes established in Jongbloed et al. (2005), Masuda (2004). Previously asymptotic normality of partial sums was reported for inverse Gaussian and gamma finite superpositions Leonenko et al. (2012*a*). The result below is a straightforward generalization to a more general class of processes.

Theorem 1. *For a fixed integer $K \geq 1$, let X_K be defined by (6), where the stationary OU type processes $\{X^{(k)}, k = 1, \dots, K\}$ defined by (5) are independent and $\mathbb{E}|X^{(k)}|^{2+d} < \infty$ for some $d > 0$ and all $k = 1, \dots, K$. Then the partial sums process (9), centered and appropriately normed, converges to the Brownian motion*

$$\frac{1}{c_K N^{1/2}}\Big(S_K([Nt]) - \mathbb{E}S_K([Nt])\Big) \to B(t), \quad t \in [0,1],$$

as $N \to \infty$ in the sense of weak convergence in Skorokhod space $D[0,1]$. The norming constant c_K is given by

$$c_K = \left(\sum_{k=1}^{K} \operatorname{Var}\left(X^{(k)}\right) \frac{1 - e^{-\lambda^{(k)}}}{1 + e^{-\lambda^{(k)}}}\right)^{1/2}.$$

Proof. Since each OU process in the superposition has a finite second moment, β-mixing (absolute regularity) for each OU process holds with the exponential rate. Namely, there exists $a_k > 0$ such that the mixing coefficient $\beta_{X^{(k)}}(t) = O(e^{-a_k t})$ (Masuda 2004, Theorem 4.3). Denote by $\alpha^{(k)}(t)$ the strong mixing coefficient of the process $X^{(k)}$, then from Bradley (2005), $2\alpha^{(k)}(t) \leq \beta^{(k)}(t) \leq D_k e^{-a_k t}$ for a constant D_k, for each $k = 1, \dots, m$. A finite sum of α-mixing processes with exponentially decaying mixing coefficients is also α-mixing with exponentially decaying mixing coefficient, therefore weak convergence of partial sums of the process X_K in $D[0,1]$ follows from (Davydov 1968, Theorem 4.2). □

We now proceed with the limit distribution of the partial sum process for the infinite superposition (7). The variance of this process has been computed in (Leonenko & Taufer 2005, Equation (5.3)), however the result on the asymptotic normality of the partial sum process (Leonenko & Taufer 2005, Theorem 3) was not correct. Also incorrect was statement (30) of (Barndorff-Nielsen & Leonenko 2005*a*, Theorem 5). Here we provide the derivation of the variance and correct the result on the limit distribution.

Lemma 2. *For the infinite superposition (7) of OU type processes that satisfy condition (A) with $p = 2$ and condition (B), set $\lambda^{(k)} = \lambda/k$ and $\delta_k = k^{-(1+2(1-H))}$, $\frac{1}{2} < H < 1$. Then*

$$\operatorname{Var}\left(S_\infty([Nt])\right) = \frac{L(N)[Nt]^{2H}}{H(2H-1)}\left(1 + o(1)\right) \text{ as } N \to \infty, \tag{11}$$

where L is a slowly varying at infinity function.

Proof. Using the expression for the covariance function of the infinite superposition from Lemma 1, write

$$\begin{aligned}\operatorname{Var}\left(S_\infty([Nt])\right) &= \sum_{m,\,n=1}^{[Nt]} \operatorname{Cov}\left(X_\infty(m), X_\infty(n)\right)\\ &= [Nt]\operatorname{Var}\left(X_\infty(0)\right) + 2\sum_{m,\,n=1,\,m>n}^{[Nt]} \frac{L(m-n)}{(m-n)^{2(1-H)}}\\ &= C_2[Nt]\zeta(1+2(1-H)) + 2\sum_{j=1}^{[Nt]-1}\left([Nt]-j\right)\frac{L(j)}{j^{2(1-H)}},\end{aligned}$$

where $\zeta(\cdot)$ is Riemann's zeta function. The sum appearing in the expression for the variance

$$\sum_{j=1}^{[Nt]-1}\left([Nt]-j\right)\frac{L(j)}{j^{2(1-H)}}$$

is a Riemann sum for the following integral:

$$\int_0^1 ([Nt]-[Nt]u)\frac{L([Nt]u)}{([Nt]u)^{2(1-H)}}[Nt]du = [Nt]^{2H}\int_0^1 (1-u)u^{2H-2}L([Nt]u)du.$$

Consider the integral

$$\int_0^1 u^{2H-2}L([Nt]u)du = \frac{1}{[Nt]^{2H-1}}\int_0^{[Nt]} v^{2H-2}L(v)dv,$$

and apply Karamata's theorem (Resnick 2007, Theorem 2.1) to get

$$\int_0^{[Nt]} v^{2H-2}L(v)dv = \frac{L(N)[Nt]^{2H-1}}{2H-1}(1+o(1))$$

as $N\to\infty$. Similarly,

$$\int_0^1 u^{2H-1}L([Nt]u)du = \frac{L(N)}{2H}(1+o(1))$$

as $N\to\infty$, and therefore

$$\int_0^1 ([Nt]-[Nt]u)\frac{L([Nt]u)}{([Nt]u)^{2(1-H)}}[Nt]du = \frac{L(N)[Nt]^{2H}}{2H(2H-1)}(1+o(1)).$$

For $\frac{1}{2}<H<1$, the second term in the expression for the variance of $S_\infty([Nt])$ dominates the first, and (11) follows.

□

In order to characterize the limit distribution of the partial sums of the infinite superpositions, we use the representation of the discretized stationary OU process as a first order autoregressive sequence

$$X^{(k)}(i) = e^{-\lambda_k}X^{(k)}(i-1) + W^{(k)}(i), \tag{12}$$

where $W^{(k)}(i)$ is independent of $X^{(k)}(j)$ for all $j < i$. Denote by $\rho_k = e^{-\lambda_k}$. The following lemma provides a useful representation of the partial sum process for the infinite superposition.

Lemma 3. *The centered partial sum of the superposition of processes that satisfy condition (A) with $p = 2$ and condition (B) with $\lambda^{(k)} = \lambda/k$ and $\delta_k = k^{-(1+2(1-H))}$, $\frac{1}{2} < H < 1$, can be written as*

$$S_\infty([Nt]) - \mathbb{E}S_\infty([Nt]) = \sum_{k=1}^{\infty} b_{[Nt]}^{(k)} \tau^{(k)}(0) + \sum_{j=1}^{[Nt]} \sum_{k=1}^{\infty} a_{[Nt]-j}^{(k)} V^{(k)}(j), \tag{13}$$

where $\tau^{(k)}(0)$, $V^{(k)}(j)$ are independent for different k, for each k $V^{(k)}(j)$ are independent for different j and also independent of $\tau^{(k)}(0)$. The series in (13) converges almost surely, and the coefficients are given by

$$b_{[Nt]}^{(k)} = \sum_{i=1}^{[Nt]} \rho_k^i = \frac{\rho_k(1-\rho_k^{[Nt]})}{1-\rho_k}, \tag{14}$$

and

$$a_{[Nt]-j}^{(k)} = \sum_{i=0}^{[Nt]-j} \rho_k^i = \frac{1-\rho_k^{[Nt]-j+1}}{1-\rho_k}. \tag{15}$$

Proof. Introduce the centered random variables

$$\tau^{(k)}(i) = X^{(k)}(i) - \mathbb{E}X^{(k)}(i), \quad V^{(k)}(i) = W^{(k)}(i) - \mathbb{E}W^{(k)}(i)$$

to arrive at centered version of (12)

$$\tau^{(k)}(i) = \rho_k \tau^{(k)}(i-1) + V^{(k)}(i). \tag{16}$$

Iterate (16) to obtain

$$\tau^{(k)}(i) = \rho_k^i \tau^{(k)}(0) + \sum_{j=1}^{i} \rho_k^{i-j} V^{(k)}(j).$$

Now the partial sum of $\tau^{(k)}$ can be written

$$\begin{aligned}
\sum_{i=1}^{[Nt]} \tau^{(k)}(i) &= \tau^{(k)}(0) \sum_{i=1}^{[Nt]} \rho_k^i + \sum_{i=1}^{[Nt]} \sum_{j=1}^{i} \rho_k^{i-j} V^{(k)}(j) \\
&= \tau^{(k)}(0) \sum_{i=1}^{[Nt]} \rho_k^i + \sum_{j=1}^{[Nt]} V^{(k)}(j) \sum_{i=j}^{[Nt]} \rho_k^{i-j} \\
&= \tau^{(k)}(0) \sum_{i=1}^{[Nt]} \rho_k^i + \sum_{j=1}^{[Nt]} V^{(k)}(j) \sum_{m=0}^{[Nt]-j} \rho_k^m \\
&= b_{[Nt]}^{(k)} \tau^{(k)}(0) + \sum_{j=1}^{[Nt]} a_{[Nt]-j}^{(k)} V^{(k)}(j),
\end{aligned}$$

where the coefficients are given by (14) and (15). Note that for different j, $V^{(k)}(j)$ are independent due to (12), and they are also independent of $\tau^{(k)}(0)$. For different k, independence follows from the independence of OU type processes $X^{(k)}$. Summing with respect to k completes the derivation of (13), provided that the series in (13) converge almost surely. Series convergence holds because the terms have zero mean, and the series of second moments converge. The latter is shown as follows. Series of the second moments for the first term series in (13) is

$$\sum_{k=1}^{\infty}(b_{[Nt]}^{(k)})^2\mathbb{E}(\tau^{(k)}(0))^2 = C_2\sum_{k=1}^{\infty}(b_{[Nt]}^{(k)})^2\delta_k$$
$$= C_2\sum_{k=1}^{\infty}\sum_{j,i=1}^{[Nt]}\rho_k^{i+j}\delta_k = \sum_{j,i=1}^{[Nt]}\frac{L(i+j)}{(i+j)^{2(1-H)}}.$$

The sum can be viewed as a Riemann sum for the double integral:

$$\frac{1}{[Nt]^2}\sum_{j,i=1}^{[Nt]}\frac{L(i+j)}{(i+j)^{2(1-H)}} = \int_0^1\int_0^1\frac{L([Nt](x+y))}{([Nt](x+y))^{2(1-H)}}dxdy(1+o(1))$$
$$= \frac{L(N)}{[Nt]^{2(1-H)}}\int_0^1\int_0^1\frac{dx\,dy}{(x+y)^{2(1-H)}}(1+(o(1))$$

as $N\to\infty$. The last equality is justified using Karamata's theorem as in Lemma 2, or by considering

$$L(N)\int_{x=\epsilon}^1\int_{y=0}^1\frac{L([Nt](x+y))}{L(N)}\frac{dx\,dy}{(x+y)^{2(1-H)}}$$

and using Remark 1 and the dominated convergence theorem. Therefore the variance of the first series in (13) is of the order $L(N)N^{2H}$.

For the second term in (13), the series of second moments is

$$\sum_{j=1}^{[Nt]}\sum_{k=1}^{\infty}\left(a_{[Nt]-j}^{(k)}\right)^2\mathbb{E}\left(V^{(k)}(j)\right)^2 = \sum_{j=1}^{[Nt]}\sum_{k=1}^{\infty}\left(\sum_{i=0}^{[Nt]-j}\rho_k^i\right)^2(1-\rho_k^2)C_2\delta_k,$$

since $\mathbb{E}(V^{(k)}(j))^2 = (1-\rho_k^2)\mathbb{E}(\tau^{(k)})^2$. The series of second moments becomes

$$\sum_{j=1}^{[Nt]}\sum_{k=1}^{\infty}\sum_{i_1,i_2=0}^{[Nt]-j}\rho_k^{i_1+i_2}(1-\rho_k^2)C_2\delta_k$$
$$= \sum_{j=1}^{[Nt]}\sum_{i_1,i_2=0}^{[Nt]-j}\left(\frac{L(i_1+i_2)}{(i_1+i_2)^{2(1-H)}} - \frac{L(i_1+i_2+2)}{(i_1+i_2+2)^{2(1-H)}}\right) = \frac{[Nt]^3}{\lambda^{2(1-H)}}\times$$
$$\int_{x=0}^1\int_{y=0}^{1-x}\int_{z=0}^{1-x}\left(\frac{L([Nt](y+z))}{([Nt](y+z))^{2(1-H)}} - \frac{L([Nt](y+z)+2)}{([Nt](y+z)+2)^{2(1-H)}}\right)dxdydz$$
$$\times(1+o(1)).$$

Arguing in the same way as for the first term in (13), we have

$$\sum_{j=1}^{[Nt]}\sum_{k=1}^{\infty}\sum_{i_1,i_2=0}^{[Nt]-j}\rho_k^{i_1+i_2}(1-\rho_k^2)C_2\delta_k = [Nt]^{2H+1}L(N)$$
$$\times\int_{x=0}^{1}\int_{y=0}^{1-x}\int_{z=0}^{1-x}\left(\frac{1}{(y+z)^{2(1-H)}}-\frac{1}{((y+z)+2/[Nt])^{2(1-H)}}\right)dxdydz$$
$$=\frac{[Nt]^{2H}L(N)}{(2H-1)}\times\int_{x=0}^{1}\int_{y=0}^{1-x}[Nt]\Big((y+2/[Nt])^{2H-1}-y^{2H-1}$$
$$-\left((y+1-x+2/[Nt])^{2H-1}-(y+1-x)^{2H-1}\right)\Big)dxdy(1+o(1)).$$

It is not hard to see that as $[Nt]\to\infty$ the integrand converges to

$$2(2H-1)\left(y^{2H-2}-(y+1-x)^{2H-2}\right).$$

Also, by the mean value theorem

$$[Nt]\left((y+2/[Nt])^{2H-1}-y^{2H-1}\right)=\frac{2((2H-1)}{\theta^{2(1-H)}}\le\frac{2(2H-1)}{y^{2(1-H)}},$$

for some $\theta\in(y,y+2/[Nt])$. Similar integrable bound holds for the second difference in the integrand above, and the dominated convergence theorem yields

$$\sum_{j=1}^{[Nt]}\sum_{k=1}^{\infty}\sum_{i_1,i_2=0}^{[Nt]-j}\rho_k^{i_1+i_2}(1-\rho_k^2)C_2\delta_k$$
$$=2[Nt]^{2H}L(N)\int_{x=0}^{1}\int_{y=0}^{1-x}\left(y^{2H-2}-(y+1-x)^{2H-2}\right)dxdy(1+o(1)),$$

which shows that the series in the second term converges almost surely, and that the variance of the second term has the same order as the variance of the first term, namely $L(N)[Nt]^{2H}$. □

The next theorem gives the asymptotic behavior of the cumulants of the partial sum process.

Theorem 2. *The m-th cumulant of the centered partial sum of the superposition of processes that satisfy condition (A) for all $p\ge 2$, condition (B), and has $\lambda^{(k)}=\lambda/k$ and $\delta_k=k^{-(1+2(1-H))}$, $\frac{1}{2}<H<1$, has the following asymptotic behavior*

$$\kappa_{m,N}=D_mL(N)[Nt]^{m-2(1-H)}(1+o(1))$$

as $N\to\infty$, where the $D_m=C_mK$ for some positive constant K.

Proof. Using (13), the logarithm of the characteristic function of the partial sum process can be written as

$$\log\mathbb{E}\exp\left\{iu(S_\infty([Nt])-\mathbb{E}S_\infty([Nt])\right\}$$
$$=\sum_{k=1}^{\infty}\log\mathbb{E}\exp\left\{ib_{[Nt]}^{(k)}u\tau^{(k)}(0)\right\}+\sum_{j=1}^{[Nt]}\sum_{k=1}^{\infty}\log\mathbb{E}\exp\left\{ia_{[Nt]-j}^{(k)}uV^{(k)}(j)\right\}.$$

Under this theorem's assumptions, the logarithm of the characteristic function of $\tau^{(k)}(0)$ can be expanded

$$\log \mathbb{E} \exp\left\{iu\tau^{(k)}(0)\right\} = \sum_{m=2}^{\infty} \frac{(iu)^m}{m!} C_m \delta_k,$$

where the summation is from $m = 2$ due to centering. From (16), the logarithm of the characteristic function of $V^{(k)}(j)$ can also be expanded as follows:

$$\begin{aligned}\log \mathbb{E} \exp\left\{iuV^{(k)}(j)\right\} &= \log\ E \exp\left\{iu\tau^{(k)}(i)\right\} - \log \mathbb{E} \exp\left\{iu\rho_k\tau^{(k)}(i-1)\right\} \\ &= \sum_{m=2}^{\infty} \frac{(iu)^m}{m!} C_m \delta_k - \sum_{m=2}^{\infty} \frac{(iu\rho_k)^m}{m!} C_m \delta_k \\ &= \sum_{m=2}^{\infty} \frac{(iu)^m}{m!} C_m (1-\rho_k^m)\delta_k.\end{aligned}$$

Therefore the m-th cumulant of the centered partial sum process is

$$\kappa_{m,N} = C_m \sum_{k=1}^{\infty} \left(b_{[Nt]}^{(k)}\right)^m \delta_k + C_m \sum_{j=1}^{[Nt]} \sum_{k=1}^{\infty} \left(a_{[Nt]-j}^{(k)}\right)^m (1-\rho_k^m)\delta_k = I + II.$$

Consider the first term:

$$\begin{aligned} I &= C_m \sum_{k=1}^{\infty} \left(b_{[Nt]}^{(k)}\right)^m \delta_k = C_m \sum_{k=1}^{\infty} \delta_k \left(\sum_{i=1}^{[Nt]} \rho_k^i\right)^m \\ &= C_m \sum_{i_1,\dots,i_m=1}^{[Nt]} \sum_{k=1}^{\infty} \delta_k \rho_k^{i_1+\cdots+i_m} = \frac{C_m}{C_2} \sum_{i_1,\dots,i_m=1}^{[Nt]} \frac{L(i_1+\cdots+i_m)}{(i_1+\cdots+i_m)^{2(1-H)}} \\ &= \frac{C_m[Nt]^m}{C_2} \int_0^1 \cdots \int_0^1 \frac{L([Nt](x_1+\cdots+x_m))}{([Nt](x_1+\cdots+x_m))^{2(1-H)}} dx_1 \dots dx_m \, (1+o(1)) \\ &= \frac{C_m[Nt]^m L(N)}{C_2[Nt]^{(2(1-H)}} \int_0^1 \cdots \int_0^1 \frac{dx_1 \dots dx_m}{(x_1+\cdots+x_m)^{2(1-H)}} \, (1+o(1)) \, ,\end{aligned}$$

where we used Remark 1 and the dominated convergence argument for the slowly varying function. This shows that the first part of the expression for the m-th cumulant behaves like $L(N)[Nt]^{m-2(1-H)}$ multiplied by a constant

$$D_{m,I} = \frac{C_m}{C_2} \int_0^1 \cdots \int_0^1 \frac{dx_1 \dots dx_m}{(x_1+\cdots+x_m)^{2(1-H)}}.$$

Now consider the second term

$$
\begin{aligned}
II &= C_m \sum_{j=1}^{[Nt]} \sum_{k=1}^{\infty} \left(a_{[Nt]-j}^{(k)}\right)^m (1-\rho_k^m)\delta_k \\
&= C_m \sum_{j=1}^{[Nt]} \sum_{k=1}^{\infty} \left(\sum_{i=0}^{[Nt]-j} \rho_k^i\right)^m (1-\rho_k^m)\delta_k \\
&= C_m \sum_{j=1}^{[Nt]} \sum_{k=1}^{\infty} \sum_{i_1,\dots,i_m=0}^{[Nt]-j} \rho_k^{i_1+\cdots+i_m}(1-\rho_k^m)\delta_k = \frac{C_m}{C_2} \\
&\times \sum_{j=1}^{[Nt]} \sum_{i_1,\dots,i_m=0}^{[Nt]-j} \left(\frac{L(i_1+\cdots+i_m)}{(i_1+\cdots+i_m)^{2(1-H)}} - \frac{L(i_1+\cdots+i_m+m)}{(i_1+\cdots+i_m+m)^{2(1-H)}}\right) = \frac{C_m[Nt]^{m+1}}{C_2\lambda^{2(1-H)}} \\
&\times \int_{x=0}^{1}\int_{y_1=0}^{1-x}\cdots\int_{y_m=0}^{1-x}\left(\frac{L([Nt](y_1+\cdots+y_m))}{([Nt](y_1+\cdots+y_m))^{2(1-H)}} - \frac{L([Nt](y_1+\cdots+y_m)+m)}{([Nt](y_1+\cdots+y_m)+m)^{2(1-H)}}\right) \\
&\times dy_1\dots dy_m dx\,(1+o(1))
\end{aligned}
$$

Remark 1 and the dominated convergence argument followed by integration with respect to y_m yield

$$
\begin{aligned}
II &= \frac{C_m L(N)[Nt]^{m-2(1-H)+1}}{C_2} \\
&\times \int_{x=0}^{1}\int_{y_1=0}^{1-x}\cdots\int_{y_m=0}^{1-x}\left(\frac{1}{(y_1+\cdots+y_m)^{2(1-H)}} - \frac{1}{((y_1+\cdots+y_m)+m/[Nt])^{2(1-H)}}\right) \\
&\times dy_1\dots dy_m dx\,(1+o(1)) = \frac{C_m L(N)[Nt]^{m-2(1-H)}}{C_2(2H-1)} \\
&\times \int_{x=0}^{1}\int_{y_1=0}^{1-x}\cdots\int_{y_{m-1}=0}^{1-x}[Nt]\Big((y_1+\cdots+y_{m-1}+m/[Nt])^{2H-1} - (y_1+\cdots+y_{m-1})^{2H-1} \\
&- ((y_1+\cdots+y_{m-1}+1-x+m/[Nt])^{2H-1} - (y_1+\cdots+y_{m-1}+1-x)^{2H-1})\Big) \\
&\times dy_1\dots dy_{m-1}(1+o(1)).
\end{aligned}
$$

Note as $[Nt]\to\infty$ the limit of the integrand is

$$
m(2H-1)((y_1+\cdots+y_{m-1})^{2H-2} - (y_1+\cdots+y_{m-1}+1-x)^{2H-2}).
$$

The same argument as in the proof of Lemma 3 and the dominated convergence theorem yield

$$
\begin{aligned}
II =& \frac{mC_m L(N)[Nt]^{m-2(1-H)}}{C_2}\int_{x=0}^{1}\int_{y_1=0}^{1-x}\cdots\int_{y_{m-1}=0}^{1-x}\Big((y_1+\cdots+y_{m-1})^{2H-2} \\
&- (y_1+\cdots+y_{m-1}+1-x)^{2H-2}\Big)dy_1\dots dy_{m-1}dx\,(1+o(1)) \\
&= D_{m,II}L(N)[Nt]^{m-2(1-H)}\,(1+o(1))
\end{aligned}
$$

with

$$D_{m,II} = \frac{mC_m}{C_2} \int_{x=0}^{1} \int_{y_1=0}^{1-x} \cdots \int_{y_{m-1}=0}^{1-x} \Big((y_1 + \cdots + y_{m-1})^{2H-2} - (y_1 + \cdots + y_{m-1} + 1 - x)^{2H-2} \Big)\, dy_1 \ldots dy_{m-1} dx.$$

Thus the asymptotic behavior of the second term is the same as of the first term, namely $L(N)[Nt]^{m-2(1-H)}$. □

Corollary 1. *Under the assumptions of Theorem 2, the centered partial sum process* $\{S_\infty(t) - \mathbb{E}S_\infty(t),\, t \geq 0\}$ *is intermittent.*

Proof. Let $Y(u) = S_\infty(u) - \mathbb{E}S_\infty(u)$. We show intermittency at $p = 2$ and $r = 4$. From Theorem 2, the m-th cumulant of $Y([Nt])$ equals

$$D_m L(N)[Nt]^{m-2(1-H)}(1 + o(1))$$

as $N \to \infty$. Since $L([Nt])/L(N) \to 1$ as $N \to \infty$ for any $t > 0$, the m-th cumulant of $Y(u)$, denoted by $\tilde{\kappa}_{m,u}$, equals

$$D_m L(u) u^{m-2(1-H)}(1 + o(1))$$

as $u \to \infty$.

Using the relation between moments and cumulants it follows from Theorem 2 that

$$\begin{aligned}
\mathbb{E}|Y(u)|^2 &= \tilde{\kappa}_{2,u} + \tilde{\kappa}_{1,u}^2 = D_2 L(u) u^{2H}(1 + o(1)), \\
\mathbb{E}|Y(u)|^4 &= \tilde{\kappa}_{4,u} + 3\tilde{\kappa}_{2,u}^2 = D_4 L(u) u^{2H+2}(1 + o(1)) + 3D_2^2 L(u)^2 u^{4H}(1 + o(1))
\end{aligned}$$

as $u \to \infty$. Since $H < 1$ implies $2H + 2 > 4H$, we have

$$\begin{aligned}
\tau(2) &= 2H, \\
\tau(4) &= 2H + 2,
\end{aligned}$$

and thus $\tau(2)/2 < \tau(4)/4$. □

Note that the behavior of moments shown in the proof implies that

$$\mathbb{E}Y(u)^4 / (\mathbb{E}Y(u)^2)^2$$

grows to infinity as $u \to \infty$, the behavior noted by Frisch ((Frisch 1995, Section 8.2) as a manifestation of intermittency. Other examples of unusual growth of moments are given in Sandev et al. (2015) in the context of fractional diffusion.

Similar behavior of cumulants was obtained in (Barndorff-Nielsen 2001, Example 4.1) for a case of continuous (integrated) superpositions of OU type processes and also in Iglói & Terdik (2003) in the context of proving central limit theorem type results. The authors of Iglói & Terdik (2003) noted that the existence of the limit was unlikely given the behavior of cumulants. We concur with this statement, but showing that there is no weak limit under intermittency in the usual partial sum setting remains an open problem. The consideration of why the existence of a weak limit is unlikely is as follows.

If the limit of the partial sum process for the infinite superposition existed in the sense of convergence of all finite dimensional distribution, then by the Lamperti's theorem (see, for example, (Embrechts & Maejima 2002, Theorem 2.1.1)), the norming had to be a regularly varying function of N. That is, the weak convergence would hold for

$$N^a L_1(N) \left(S_\infty([Nt]) - \mathbb{E}S_\infty([Nt])\right)$$

for some $a \in \mathbb{R}$ and a slowly varying at infinity function L_1. However, no matter what $a \in \mathbb{R}$ is chosen, all cumulants of the centered and normed partial sum cannot converge. This is because the m-th cumulant of $N^a \left(S_\infty([Nt]) - \mathbb{E}S_\infty([Nt])\right)$ behaves like $N^{m(a+1)-2(1-H)}$.

Also note that for even q, the scaling function defined in (2) in this case is $\tau(q) = q - 2(1-H)$, and

$$\frac{\tau(q)}{q} = 1 - \frac{2(1-H)}{q}$$

is strictly increasing in q. The term $-2(1-H)$ in the exponent of the asymptotic behavior of the cumulants

$$\kappa_{q,N} = D_q L(N)[Nt]^{q-2(1-H)}(1+o(1))$$

gives the reason for both the increasing behavior of $\tau(q)/q$ and for the lack of norming that would make cumulants converge. Of course, convergence of cumulants provides a sufficient means for proving the existence of the limit by showing the convergence of the characteristic function. The formal link between intermittency and lack of the limit theorems needs to be further developed for the partial sums and other sequences of stochastic processes.

6 Appendix

The examples in this section have been discussed in Barndorff-Nielsen (2001), Leonenko & Taufer (2005). We briefly present them to illustrate that conditions (A) and (B) are satisfied for a number of OU type processes.

Example 1. The stationary Gamma OU type process $\{X(t), t \geq 0\}$ with gamma marginal distribution has the cumulant generating function

$$\kappa(\zeta) = \log \mathbb{E} \exp\{i\zeta X(t)\} = -\alpha \log\left(1 - \frac{i\zeta}{\beta}\right) = \sum_{m=1}^{\infty} \frac{\alpha (i\zeta)^m}{m\beta^m}, \tag{17}$$

$\alpha > 0$, $\beta > 0$, $\zeta < \beta$. If $\{X^{(k)}(t), t \geq 0\}$, $k \geq 1$ are independent stationary Gamma OU type processes with marginal distributions $\Gamma(\alpha_k, \beta)$, $k \geq 1$ where

$$\alpha_k = \alpha k^{-(1+2(1-H))}, \quad \frac{1}{2} < H < 1,$$

then condition (A) is satisfied with $\delta_k = \alpha_k$, and if $\sum_{k=1}^{\infty} \alpha_k < \infty$, condition (B) is satisfied as well. The supOU process $X_\infty(t) = \sum_{k=1}^{\infty} X^{(k)}(t)$, $t \geq 0$ has a marginal $\Gamma(\sum_{k=1}^{\infty} \alpha_k, \beta)$ distribution.

Example 2. The stationary inverse Gaussian OU type process has the cumulant generating function

$$\kappa(\zeta) = \log \mathbb{E} \exp \{i\zeta X(t)\} = \delta \left(\gamma - \sqrt{\gamma^2 - 2i\zeta}\right) = \sum_{m=1}^{\infty} \frac{\delta(2m)!(i\zeta)^m}{(2m-1)(m!)^2 2^m \gamma^{2m-1}},$$

$\gamma > 0$, $\delta > 0$. It follows that independent stationary OU type processes $\{X^{(k)}(t), t \geq 0\}$, $k \geq 1$ with marginals $IG(\delta_k, \gamma)$, $k \geq 1$ where

$$\delta_k = \delta k^{-(1+2(1-H))}, \quad \frac{1}{2} < H < 1,$$

satisfy conditions (A) and (B), and we obtain inverse Gaussian supOU process

$$X_\infty(t) = \sum_{k=1}^{\infty} X^{(k)}(t), \quad t \geq 0,$$

with marginal $IG(\sum_{k=1}^{\infty} \delta_k, \gamma)$ distribution.

Example 3. The stationary Variance Gamma OU type process has the the cumulant generating function

$$\kappa(\zeta) = \log \mathbb{E} \exp \{i\zeta X(t)\} = i\mu\zeta + 2\kappa \log \left(\frac{\gamma}{\alpha^2 - (\beta + i\zeta)^2}\right),$$

$\kappa > 0$, $\alpha > |\beta| > 0$, $\mu \in \mathbb{R}$, $\gamma^2 = \alpha^2 - \beta^2$, $|\beta + \zeta| < \alpha$. It follows that $VG(\kappa, \alpha, \beta, \mu)$ distribution is closed under convolution with respect to the parameters κ and μ. Independent stationary OU type processes $\{X^{(k)}(t), t \geq 0\}$, $k \geq 1$ with marginals $VG(\kappa_k, \alpha, \beta, \mu_k)$, $k \geq 1$ where $\sum_{k=1}^{\infty} \mu_k$ converges and

$$\kappa_k = \kappa k^{-(1+2(1-H))}, \quad \frac{1}{2} < H < 1,$$

satisfy conditions (A) and (B), and we obtain variance gamma supOU process

$$X_\infty(t) = \sum_{k=1}^{\infty} X^{(k)}(t), \quad t \geq 0,$$

with marginal $VG(\sum_{k=1}^{\infty} \kappa_k, \alpha, \beta, \sum_{k=1}^{\infty} \mu_k)$ distribution.

Example 4. The stationary normal inverse Gaussian OU type process has cumulant generating function

$$\kappa(\zeta) = \log \mathbb{E} \exp \{i\zeta X(t)\} = i\mu\zeta + \delta \left(\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + i\zeta)^2}\right),$$

$\alpha \geq |\beta| \geq 0$, $\delta > 0$, $\mu \in \mathbb{R}$, $|\beta + \zeta| < \alpha$. It follows that $NIG(\alpha, \beta, \delta, \mu)$ distribution is closed under convolution with respect to the parameters δ and μ. Independent stationary

OU type processes $\{X^{(k)}(t), t \geq 0\}$, $k \geq 1$ with marginals $NIG(\alpha, \beta, \delta_k, \mu_k)$, $k \geq 1$ with convergent $\sum_{k=1}^{\infty} \mu_k$,

$$\delta_k = \delta k^{-(1+2(1-H))}, \quad \frac{1}{2} < H < 1,$$

satisfy conditions (A) and (B), and we obtain normal inverse Gaussian supOU process

$$X_\infty(t) = \sum_{k=1}^{\infty} X^{(k)}(t), \quad t \geq 0,$$

with marginal $NIG(\alpha, \beta, \sum_{k=1}^{\infty} \delta_k, \sum_{k=1}^{\infty} \mu_k)$ distribution.

Example 5. The stationary positive tempered stable OU type process has the cumulant generating function

$$\kappa(\zeta) = \log \mathbb{E} \exp\{i\zeta X(t)\} = \delta\gamma - \delta \left(\gamma^{\frac{1}{\kappa}} - 2i\zeta\right)^{\kappa},$$

$\kappa \in (0,1)$, $\delta > 0$, $\gamma > 0$, $0 < \zeta < \frac{\gamma^{1/\kappa}}{2}$. Thus the $TS(\kappa, \delta, \gamma)$ distribution is closed under convolution with respect to the parameter δ. Independent stationary OU type processes $\{X^{(k)}(t), t \geq 0\}$, $k \geq 1$ with marginals $TS(\kappa, \delta_k, \gamma)$, $k \geq 1$ where

$$\delta_k = \delta k^{-(1+2(1-H))}, \quad \frac{1}{2} < H < 1,$$

satisfy conditions (A) and (B), and we obtain tempered stable supOU process

$$X_\infty(t) = \sum_{k=1}^{\infty} X^{(k)}(t), \quad t \geq 0,$$

with marginal $TS(\kappa, \sum_{k=1}^{\infty} \delta_k, \gamma)$ distribution.

More examples of supOU type processes satisfying Condition (A) can be derived from other distributions, for example, normal tempered stable, Euler's gamma distribution and z-distribution.

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References

Applebaum, D. (2009), *Lévy Processes and Stochastic Calculus*, Cambridge University Press, Cambridge.

Barndorff-Nielsen, O. E. (1997), 'Processes of normal inverse Gaussian type', *Finance and stochastics* **2**(1), 41–68.

Barndorff-Nielsen, O. E. (2001), 'Superposition of Ornstein–Uhlenbeck type processes', *Theory of Probability & Its Applications* **45**(2), 175–194.

Barndorff-Nielsen, O. E. & Leonenko, N. (2005*a*), 'Burgers' turbulence problem with linear or quadratic external potential', *Journal of applied probability* **42**(2), 550–565.

Barndorff-Nielsen, O. E. & Leonenko, N. (2005*b*), 'Spectral properties of superpositions of Ornstein-Uhlenbeck type processes', *Methodology and computing in applied probability* **7**(3), 335–352.

Barndorff-Nielsen, O. E. & Shephard, N. (2001), 'Non-Gaussian Ornstein–Uhlenbeck-based models and some of their uses in financial economics', *Journal of the Royal Statistical Society: Series B (Statistical Methodology)* **63**(2), 167–241.

Barndorff-Nielsen, O. E. & Stelzer, R. (2011), 'Multivariate supOU processes', *The Annals of Applied Probability* **21**(1), 140–182.

Bertini, L. & Cancrini, N. (1995), 'The stochastic heat equation: Feynman-Kac formula and intermittence', *Journal of Statistical Physics* **78**(5-6), 1377–1401.

Bradley, R. C. (2005), 'Basic properties of strong mixing conditions. a survey and some open questions', *Probability surveys* **2**(2), 107–144.

Carmona, R. & Molchanov, S. A. (1994), *Parabolic Anderson Problem and Intermittency*, Vol. 518, American Mathematical Soc.

Chechkin, A., Gonchar, V. Y., Szydl, M. et al. (2002), 'Fractional kinetics for relaxation and superdiffusion in a magnetic field', *Physics of Plasmas* **9**(1), 78–88.

Davydov, Y. A. (1968), 'Convergence of distributions generated by stationary stochastic processes', *Theory of Probability & Its Applications* **13**(4), 691–696.

Eliazar, I. & Klafter, J. (2005), 'Lévy, Ornstein–Uhlenbeck, and subordination: Spectral vs. jump description', *Journal of statistical physics* **119**(1-2), 165–196.

Embrechts, P. & Maejima, M. (2002), *Selfsimilar Processes*, Princeton University Press, Princeton, NJ.

Fasen, V. & Klüppelberg, C. (2007), Extremes of supOU processes, *in* 'Stochastic Analysis and Applications', Springer, pp. 339–359.

Frisch, U. (1995), *Turbulence: the legacy of A.N. Kolmogorov*, Cambridge University Press, Cambridge.

Fujisaka, H. (1984), 'Theory of diffusion and intermittency in chaotic systems', *Progress of theoretical physics* **71**(3), 513–523.

Gärtner, J. & den Hollander, F. (2006), 'Intermittency in a catalytic random medium', *The Annals of Probability* **34**(6), 2219–2287.

Gärtner, J., Den Hollander, F. & Maillard, G. (2010), 'Intermittency on catalysts: voter model', *The Annals of Probability* **38**(5), 2066–2102.

Gärtner, J., König, W. & Molchanov, S. (2007), 'Geometric characterization of intermittency in the parabolic Anderson model', *The Annals of Probability* **35**(2), 439–499.

Gärtner, J. & Molchanov, S. A. (1990), 'Parabolic problems for the Anderson model', *Communications in Mathematical Physics* **132**(3), 613–655.

Grahovac, D. & Leonenko, N. N. (2015), 'Bounds on the support of the multifractal spectrum of stochastic processes', *arXiv preprint arXiv:1406.2920* .

Iglói, E. & Terdik, G. (2003), 'Superposition of diffusions with linear generator and its multifractal limit process', *ESAIM: Probability and Statistics* **7**, 23–88.

Jongbloed, G., Van Der Meulen, F. H. & Van Der Vaart, A. W. (2005), 'Nonparametric inference for Lévy-driven Ornstein-Uhlenbeck processes', *Bernoulli* **11**(5), 759–791.

Khoshnevisan, D. (2014), *Analysis of stochastic partial differential equations*, Vol. 119, American Mathematical Soc.

Khoshnevisan, D., Kim, K. & Xiao, Y. (2015), 'Intermittency and multifractality: A case study via parabolic stochastic PDEs', *arXiv preprint arXiv:1503.06249* .

Leonenko, N. N., Petherick, S. & Sikorskii, A. (2012*a*), 'Fractal activity time models for risky asset with dependence and generalized hyperbolic distributions', *Stochastic Analysis and Applications* **30**(3), 476–492.

Leonenko, N. N., Petherick, S. & Sikorskii, A. (2012*b*), 'A normal inverse Gaussian model for a risky asset with dependence', *Statistics & Probability Letters* **82**(1), 109–115.

Leonenko, N. & Taufer, E. (2005), 'Convergence of integrated superpositions of Ornstein-Uhlenbeck processes to fractional Brownian motion', *Stochastics: An International Journal of Probability and Stochastics Processes* **77**(6), 477–499.

Li, L. & Linetsky, V. (2014), 'Time-changed Ornstein–Uhlenbeck processes and their applications in commodity derivative models', *Mathematical Finance* **24**(2), 289–330.

Masuda, H. (2004), 'On multidimensional Ornstein-Uhlenbeck processes driven by a general Lévy process', *Bernoulli* **10**(1), 97–120.

Molchanov, S. A. (1991), 'Ideas in the theory of random media', *Acta Applicandae Mathematica* **22**(2-3), 139–282.

Resnick, S. I. (2007), *Heavy-tail Phenomena: Probabilistic and Statistical Modeling*, Springer Science & Business Media, New York.

Ricciardi, L. M. & Sacerdote, L. (1979), 'The Ornstein-Uhlenbeck process as a model for neuronal activity', *Biological cybernetics* **35**(1), 1–9.

Sandev, T., Chechkin, A. V., Korabel, N., Kantz, H., Sokolov, I. M. & Metzler, R. (2015), 'Distributed-order diffusion equations and multifractality: Models and solutions', *Physical Review E* **92**(4), 042117.

Sato, K.-I. (1999), *Lévy Processes and Infinitely Divisible Distributions*, Cambridge University Press, Cambridge.

Solomon, T., Weeks, E. R. & Swinney, H. L. (1993), 'Observation of anomalous diffusion and lévy flights in a two-dimensional rotating flow', *Physical Review Letters* **71**(24), 3975.

Woyczyński, W. A. (1998), *Burgers-KPZ turbulence(Göttingen lectures)*, Lecture Notes in Mathematics, Springer-Verlag, Berlin.

Zel'dovich, Y. B., Molchanov, S., Ruzmaĭkin, A. & Sokolov, D. D. (1987), 'Intermittency in random media', *Soviet Physics Uspekhi* **30**(5), 353.