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Non-stationary abstract Friedrichs systems

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Abstract

Inspired by the results of Ern, Guermond and Caplain (2007) on the abstract theory for Friedrichs symmetric positive systems, we give the existence and uniqueness result for the initial (boundary) value problem for the non-stationary abstract Friedrichs system. Despite absence of well-posedness result for such systems, there were already attempts for their numerical treatment by Burman, Ern, Fernandez (2010) and Bui-Thanh, Demkowicz, Ghattas (2013). We use the semigroup theory approach, and prove that the operator involved satisfy the conditions of the Hille-Yosida generation theorem. We also address the semilinear problem and apply the new results to a number of examples, such as the symmetric hyperbolic system, the unsteady div-grad problem, and the wave equation. Special attention was paid to the (generalised) unsteady Maxwell system.

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1. Introduction

An overview

Symmetric positive systems of first-order linear partial differential equations were introduced by K. O. Friedrichs [21] in 1958 and today they are also known as *Friedrichs systems*. His immediate goal was the treatment of equations that change their type, like the equations modelling the transonic fluid flow.

The class of Friedrichs systems encompasses a wide variety of classical and neoclassical linear partial differential equations of continuum physics, regardless of their type, paired with various (initial) boundary conditions (Dirichlet, Neumann, Robin). This includes boundary value problems for some elliptic systems, mixed initial and boundary value problems for parabolic and hyperbolic equations, and boundary value problems for equations of mixed type, such as the Tricomi equation. For some specific examples we refer to [4, 5, 6, 15, 19, 24 27].

The inclusion of such a variety of different problems into a single framework requires all different characteristics to be included as well. This naturally imposes a number of technical difficulties, which many authors have since tried to surmount [27, 28].

The setting of symmetric positive systems appeared to be convenient for the numerical treatment of various boundary value problems as well. Already Friedrichs considered the numerical solution of such systems, by a finite difference scheme. The renewed interest in the theory of Friedrichs systems during the last decade resulted in development of new numerical schemes based on adaptations of the finite element method. Let us just mention [23], the Ph.D. thesis of Max Jensen [24] and, most recently, [6, 16, 17, 18]. As we understand, some numerical algorithms based on mixed finite element methods are more suitable for first order systems then for higher order systems, which makes the framework of Friedrichs systems more convenient for numerical treatment of some higher order equations. By applying some numerical scheme developed for Friedrichs systems to a particular equation of interest, one often gets a numerical result (for that equation) that was not known before.

Recently, Ern, Guermond and Caplain [16, 19] suggested another approach to the Friedrichs systems, which was, as we understand, inspired by their interest in the numerical treatment of Friedrichs systems. They expressed the theory in terms of operators acting in abstract Hilbert spaces and represented the boundary conditions in an intrinsic way, thus avoiding the question of traces for functions from the graph space of the considered operator. Some open questions regarding the relationship of different representations of boundary conditions in the abstract setting were answered in [2]. The precise relationship between the classical Friedrichs theory and its abstract counterpart was investigated in [3, 4], which resulted in new applications to second order equations, which were also addressed in [5] and [9]. In the first reference, the abstract theory was applied to the heat equation, while in the second the homogenisation theory of the Friedrichs systems was developed and contrasted to the known results for stationary diffusion equation and the heat equation.

Although some evolution (non-stationary) problems can be treated within the framework of abstract theory of Friedrichs systems developed in [19], by not making distinction between the time variable and space variables [5], the theory is not suitable for some standard problems, like the initial-boundary value problem for non-stationary Maxwell system, or the Cauchy problem for symmetric hyperbolic systems.

In this paper we develop an abstract theory for non-stationary Friedrichs systems that can address these problems as well. More precisely, we consider an abstract Cauchy problem in the Hilbert space, involving a time independent abstract Friedrichs operator. In order to prove the well-posedness, we use the semigroup theory approach and prove that the operator involved satisfies the conditions of the Hille-Yosida generation theorem. Then a number of well-posedness results for different notions of a solution (classical/strong/weak) can be derived from the classical results for the abstract Cauchy problem [14, 26]. The semigroup theory allows the treatment of semilinear problems [11] as well, resulting in the existence and uniqueness result for the semilinear non-stationary Friedrichs system. For some estimates on the solution of such problems we refer to [8].

Although the well-posedness result was not known, there was already an aspiration for the numerical treatment of non-stationary Friedrichs systems [10, 13]. Since a good well-posedness theorem is *desirable* for the convergence analysis of a numerical scheme, we hope that our paper will pave the way for new numerical results in this direction. For example, it would be interesting to see whether results of [22] can be extended to Friedrichs systems, as well.

The paper is organised as follows: in the rest of this introductory section we recall the abstract setting of [19]. The main results are contained in the second section, where we start by introducing the abstract non-stationary Friedrichs system, then discuss some a priori bounds, and finally prove that our operator satisfies conditions of the Hille-Yosida theorem. This enables us to give the existence and uniqueness results for different notions of solution. We also prove that the weak solution satisfies the starting equation in a certain vector valued L^1 space, and discuss the semilinear problem. In the third section we apply these result to a number of examples, namely the Cauchy problem for the symmetric hyperbolic system, the initial-boundary value problem for the unsteady Maxwell system, the initial-boundary value problem for the unsteady div-grad problem, and the initial-value problem for the wave equation.

Abstract Hilbert space formalism

We start by describing the Hilbert space formalism introduced in [19], and recalling some basic results about the abstract Friedrichs systems. By L we denote a real Hilbert space, identified with its dual L'. Let $\mathcal{D} \subseteq L$ be its dense subspace, and $\mathcal{L}, \tilde{\mathcal{L}} : \mathcal{D} \longrightarrow L$ (unbounded) linear operators satisfying

$$\text{(T1)} \qquad (\forall\, \boldsymbol{\varphi}, \boldsymbol{\psi} \in \mathcal{D}) \quad \langle\, \mathcal{L}\boldsymbol{\varphi} \mid \boldsymbol{\psi}\,\rangle_L = \langle\, \boldsymbol{\varphi} \mid \tilde{\mathcal{L}}\boldsymbol{\psi}\,\rangle_L\,,$$

$$\text{(T2)} \qquad (\exists\, c > 0)(\forall\, \boldsymbol{\varphi} \in \mathcal{D}) \quad \|(\mathcal{L} + \tilde{\mathcal{L}})\boldsymbol{\varphi}\|_L \leqslant c\|\boldsymbol{\varphi}\|_L\,.$$

These properties together will be referred as (T).

If we denote by W_0 the completion of the space $(\mathcal{D}, \langle\,\cdot \mid \cdot\,\rangle_{\mathcal{L}})$, with the graph inner product $\langle\,\cdot \mid \cdot\,\rangle_{\mathcal{L}} := \langle\,\cdot \mid \cdot\,\rangle_L + \langle\,\mathcal{L}\cdot \mid \mathcal{L}\cdot\,\rangle_L$ (the corresponding norm $\|\cdot\|_{\mathcal{L}}$ is usually called the *graph norm*), then $\mathcal{L}, \tilde{\mathcal{L}}$ can be extended by density to continuous linear operators $W_0 \longrightarrow L$, with (T) still holding for $\boldsymbol{\varphi}, \boldsymbol{\psi} \in W_0$. Having in mind the Gelfand triple (the imbeddings are dense and continuous)

$$W_0 \hookrightarrow L \equiv L' \hookrightarrow W_0'\,,$$

they can be further extended [19, 2] via adjoint operators to $\mathcal{L}, \tilde{\mathcal{L}} \in \mathcal{L}(L; W_0')$. By

$$W := \{\mathsf{u} \in L : \mathcal{L}\mathsf{u} \in L\} = \{\mathsf{u} \in L : \tilde{\mathcal{L}}\mathsf{u} \in L\}\,,$$

we denote the *graph space* which, equipped with the graph norm, is a Hilbert space [19, Lemma 2.1].

The problem of interest is to find sufficient conditions on a subspace $V \subseteq W$, such that the operator $\mathcal{L}_{|_V} : V \longrightarrow L$ is an isomorphism. In order to describe such V, we first introduce a *boundary operator* $D \in \mathcal{L}(W; W')$ defined by

$${}_{W'}\langle\, D\mathsf{u}, \mathsf{v}\,\rangle_W := \langle\, \mathcal{L}\mathsf{u} \mid \mathsf{v}\,\rangle_L - \langle\, \mathsf{u} \mid \tilde{\mathcal{L}}\mathsf{v}\,\rangle_L\,, \qquad \mathsf{u}, \mathsf{v} \in W\,,$$

which is *symmetric* [19, Lemma 2.3], and $\ker D = W_0$ [19, Lemma 2.4].

We are now ready to describe V: let V and $\widetilde{V}$ be two subspaces of W satisfying

$$(V1) \qquad \begin{aligned} &(\forall\, \mathsf{u} \in V) \qquad {}_{W'}\langle\, D\mathsf{u}, \mathsf{u}\,\rangle_W \geqslant 0\,,\\ &(\forall\, \mathsf{v} \in \widetilde{V}) \qquad {}_{W'}\langle\, D\mathsf{v}, \mathsf{v}\,\rangle_W \leqslant 0\,, \end{aligned}$$

$$(V2) \qquad V = D(\widetilde{V})^0\,, \qquad \widetilde{V} = D(V)^0\,,$$

where ^{0} stands for the annihilator. We shall refer to both (V1) and (V2) as (V). The following lemma is immediate.

Lemma 1. *If (T) and (V2) hold, then V and $\tilde V$ are closed, and* $\ker D = W_0 \subseteq V \cap \widetilde{V}$. ∎

In order to get the well-posedness result (for the stationary case) one additional assumption that ensures the coercivity property of $\mathcal{L}$ and $\tilde{\mathcal{L}}$ is needed:

$$\text{(T3)} \qquad (\exists\, \mu_0 > 0)(\forall \boldsymbol{\varphi} \in \mathcal{D}) \quad \langle\, (\mathcal{L} + \tilde{\mathcal{L}})\boldsymbol{\varphi} \mid \boldsymbol{\varphi}\,\rangle_L \geqslant 2\mu_0 \|\boldsymbol{\varphi}\|_L^2 \,.$$

It is easy to see that the above property then holds for an arbitrary $\boldsymbol{\varphi} \in L$. The operator $\mathcal{L}$ that satisfies (T) and (T3) for some $\tilde{\mathcal{L}}$, we call *the abstract Friedrichs operator*.

Lemma 2. *([19, Lemma 3.2]) Under assumptions (T1)–(T3) and (V), the operators $\mathcal{L}$ and $\tilde{\mathcal{L}}$ are L–coercive on V and $\widetilde{V}$, respectively; in other words:*

$$\begin{aligned} &(\forall\, \mathsf{u} \in V) \qquad \langle\, \mathcal{L}\mathsf{u} \mid \mathsf{u}\,\rangle_L \geqslant \mu_0 \|\mathsf{u}\|_L^2 \,, \\ &(\forall\, \mathsf{v} \in \widetilde{V}) \qquad \langle\, \tilde{\mathcal{L}}\mathsf{v} \mid \mathsf{v}\,\rangle_L \geqslant \mu_0 \|\mathsf{v}\|_L^2 \,. \end{aligned}$$

∎

Theorem 1. *([19, Theorem 3.1]) If (T1)–(T3) and (V) hold, then the restrictions of operators $\mathcal{L}_{|_V} : V \longrightarrow L$ and $\tilde{\mathcal{L}}_{|_{\widetilde V}} : \tilde{V} \longrightarrow L$ are isomorphisms.* ∎

The importance of this well-posedness result arises from relative simplicity of geometric conditions (V), which, in the case when $\mathcal{L}$ is a partial differential operator, do not involve the notion of traces for functions in the graph space of $\mathcal{L}$.

Example. (Classical Friedrichs operator) Let $d, r \in \mathbf{N}$, and $\Omega \subseteq \mathbf{R}^d$ be an open and bounded set with a Lipschitz boundary Γ. Furthermore, assume that the matrix functions $\mathbf{A}_k \in \mathrm{W}^{1,\infty}(\Omega; \mathrm{M}_r(\mathbf{R}))$, $k \in 1..d$, satisfy

$$\text{(F1)} \qquad \mathbf{A}_k \text{ is symmetric: } \mathbf{A}_k = \mathbf{A}_k^\top \,,$$

and $\mathbf{C} \in \mathrm{L}^\infty(\Omega; \mathrm{M}_r(\mathbf{R}))$. If we denote $\mathcal{D} := \mathrm{C}_c^\infty(\Omega; \mathbf{R}^r)$, $L = \mathrm{L}^2(\Omega; \mathbf{R}^r)$, and define operators $\mathcal{L}, \tilde{\mathcal{L}} : \mathcal{D} \longrightarrow L$ by formulæ

$$\begin{aligned} \mathcal{L}\mathsf{u} &:= \sum_{k=1}^d \partial_k(\mathbf{A}_k \mathsf{u}) + \mathbf{C}\mathsf{u} \,, \\ \tilde{\mathcal{L}}\mathsf{u} &:= -\sum_{k=1}^d \partial_k(\mathbf{A}_k^\top \mathsf{u}) + (\mathbf{C}^\top + \sum_{k=1}^d \partial_k \mathbf{A}_k^\top)\mathsf{u} \,, \end{aligned}$$

then $\mathcal{L}$ and $\tilde{\mathcal{L}}$ satisfy (T). The graph space W is

$$W = \Big\{ \mathsf{u} \in \mathrm{L}^2(\Omega; \mathbf{R}^r) : \sum_{k=1}^d \partial_k(\mathbf{A}_k \mathsf{u}) + \mathbf{C}\mathsf{u} \in \mathrm{L}^2(\Omega; \mathbf{R}^r) \Big\} \,,$$

where we have to take distributional derivatives in the above formula, and the boundary operator, for $\mathsf{u}, \mathsf{v} \in \mathrm{C}_c^\infty(\mathbf{R}^d; \mathbf{R}^r)$, is given by

$${}_{W'}\langle\, D\mathsf{u}, \mathsf{v}\,\rangle_W = \int_\Gamma \mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x}) \mathsf{u}_{|_\Gamma}(\mathbf{x}) \cdot \mathsf{v}_{|_\Gamma}(\mathbf{x}) dS(\mathbf{x}) \,, \qquad \text{with} \quad \mathbf{A}_{\boldsymbol{\nu}} := \sum_{k=1}^d \nu_k \mathbf{A}_k \,,$$

where $\boldsymbol{\nu} = (\nu_1, \nu_2, \ldots, \nu_d) \in \mathrm{L}^\infty(\Gamma; \mathbf{R}^d)$ is the unit outward normal on Γ. If we additionally assume

$$\text{(F2)} \qquad (\exists\, \mu_0 > 0) \quad \mathbf{C} + \mathbf{C}^\top + \sum_{k=1}^d \partial_k \mathbf{A}_k \geqslant 2\mu_0 \mathbf{I} \qquad (\text{a.e. on } \Omega) \,,$$

then the property (T3) is also satisfied. The inequality above is meant in the sense of the order on symmetric matrices. ∎

Remark. If we start in the previous example from the beginning with the coefficients defined on the whole $\mathbf{R}^d$ instead of Ω, and consider the operators $\mathcal{L}, \tilde{\mathcal{L}} : \mathcal{D} \longrightarrow L$ defined by above formulæ, with $\mathcal{D} := \mathrm{C}_c^\infty(\mathbf{R}^d; \mathbf{R}^r)$, $L = \mathrm{L}^2(\mathbf{R}^d; \mathbf{R}^r)$, then one can easily see that $W = W_0 = \ker D$ [1, 7, 24], which implies that the boundary operator D is trivial. Therefore, we can take $V = \widetilde{V} = W$, and these spaces satisfy conditions (V), which allows us to consider the initial value problem without any other constrains in the non-stationary case. ∎

2. Non-stationary problem

The abstract Cauchy problem

In the rest of the paper we shall consider an initial-boundary value problem for a non-stationary Friedrichs system. To be specific, using the notation from the previous section, we shall consider the abstract Cauchy problem:

$$\text{(P)} \qquad \begin{cases} \mathsf{u}'(t) + \mathcal{L}\mathsf{u}(t) = \mathsf{f} \\ \mathsf{u}(0) = \mathsf{u}_0 \end{cases},$$

where $\mathsf{u} : [0, T\rangle \longrightarrow L$, $T > 0$, is the unknown function, while the right-hand side $\mathsf{f} : \langle 0, T\rangle \longrightarrow L$, the initial data $\mathsf{u}_0 \in L$ and the abstract Friedrichs operator $\mathcal{L}$ are given. More precisely, we assume that $\mathcal{L}$ does not depend on the time variable t and satisfies (T) , so that it can be extended to the continuous linear operator $L \longrightarrow W_0'$, as in the introductory section.

For more details about abstract Cauchy problems and the corresponding terminology used here we refer to [26] with one exception that for the notion of *mild* solution we use the term *weak* solution as it is the case in [11, 14].

Instead of the positivity assumption (T3), we require that $\mathcal{L}$ satisfies a weaker *nonnegativity assumption*

$$\text{(T3}'\text{)} \qquad (\forall \varphi \in \mathcal{D}) \quad \langle (\mathcal{L} + \tilde{\mathcal{L}})\varphi \mid \varphi \rangle_L \geqslant 0 ,$$

but such operator we still call the *abstract Friedrichs operator*. Similarly as it was done in the case of property (T3), one can see that (T3′) actually holds for arbitrary $\varphi \in L$.

Remark. In reference to the classical Friedrichs operator of the example above, the condition (F2) can be weakened to

$$\text{(F2}'\text{)} \qquad \mathbf{C} + \mathbf{C}^\top + \sum_{k=1}^d \partial_k \mathbf{A}_k \geqslant \mathbf{0} \qquad \text{(a.e. on } \Omega) ,$$

as this is enough to ensure the validity of (T3′).

Actually, if $\mathcal{L}$ satisfies only (F1), the positivity condition (F2) can be achieved by substituting $\mathsf{v} := e^{-\lambda t}\mathsf{u}$, for some suitable $\lambda > 0$: then the corresponding non-stationary system becomes

$$\partial_t \mathsf{v} + (\mathcal{L} + \lambda \mathbf{I})\mathsf{v} = e^{-\lambda t}\mathsf{f} ,$$

where $\mathbf{I}$ is the identity matrix, and since all matrices appearing in the above expression are bounded, we can choose λ large enough so that (F2) is satisfied. Therefore, the operator $\mathcal{L} + \lambda\mathbf{I}$ and its formal adjoint satisfy (T1)–(T3). Note also that the initial condition $\mathsf{u}(0, \cdot) = \mathsf{u}_0$ is equivalent to $\mathsf{v}(0, \cdot) = \mathsf{u}_0$ while the graph space of operator $\mathcal{L} + \lambda\mathbf{I}$ is the same as the graph space of $\mathcal{L}$, with equivalent norms. ∎

In order to fully describe problem (P), it remains to be clarified what is the domain $\mathcal{D}(\mathcal{L})$ of $\mathcal{L}$. As the equation $(\text{P})_1$ should hold in L, we can take any subspace of the graph space W. If $\mathcal{L}$ is a partial differential operator, then the choice of domain of $\mathcal{L}$ corresponds to given boundary conditions, and it is then natural to take for $\mathcal{D}(\mathcal{L})$ to be some subspace V satisfying (V). Then one can easily prove a similar result as in Lemma 2 if the weaker assumption (T3′) is taken instead of (T3).

Lemma 3. *Under assumptions (T), (T3′) and (V), the operators $\mathcal{L}$ and $\tilde{\mathcal{L}}$ are L-accretive on V and $\widetilde{V}$, respectively, i.e. they satisfy*

$$\begin{aligned} (\forall \mathsf{u} \in V) \qquad & \langle \mathcal{L}\mathsf{u} \mid \mathsf{u} \rangle_L \geqslant 0\,, \\ (\forall \mathsf{v} \in \tilde{V}) \qquad & \langle \tilde{\mathcal{L}}\mathsf{v} \mid \mathsf{v} \rangle_L \geqslant 0\,. \end{aligned}$$

∎

If u is the classical solution of (P) on $[0,T\rangle$ (see [26] for definitions of different notions of a solution), one can easily derive an a priori estimate

$$(\forall t \in [0,T]) \qquad \|\mathsf{u}(t)\|_L^2 \leqslant e^t \biggl(\|\mathsf{u}_0\|_L^2 + \int_0^t \|\mathsf{f}(s)\|_L^2 \biggr)\,,$$

from which the uniqueness of classical solution is immediate. Indeed, from $(\mathrm{P})_1$ we have

$$\langle \mathsf{u}'(t) \mid \mathsf{u}(t) \rangle_L + \langle \mathcal{L}\mathsf{u}(t) \mid \mathsf{u}(t) \rangle_L = \langle \mathsf{f}(t) \mid \mathsf{u}(t) \rangle_L\,,$$

and by (T3′) it follows

$$\frac{d}{dt}\|\mathsf{u}(t)\|_L^2 \leqslant 2\langle \mathsf{f}(t) \mid \mathsf{u}(t) \rangle_L \leqslant \|\mathsf{u}(t)\|_L^2 + \|\mathsf{f}(t)\|_L^2\,,$$

from where, after applying the differential form of the Gronwall inequality, we get the a priori estimate. Here we have assumed that the right-hand side of $(\mathrm{P})_1$ is square-integrable, i.e. $\mathsf{f} \in \mathrm{L}^2(\langle 0,T\rangle; L)$. Clearly, this is not sufficient for the existence of the classical solution and precise sufficient assumptions will be presented in the sequel, as well as a sharper a priori estimate.

Semigroup approach

In order to fit our problem (P) in the setting of the semigroup theory, let us take a subspace V of W satisfying (V), and define an operator $\mathcal{A} : V \longrightarrow L$ by $\mathcal{A} := -\mathcal{L}_{|V}$. We can now rewrite our problem (P) in terms of operator $\mathcal{A}$:

$$(\mathrm{P}') \qquad \begin{cases} \mathsf{u}'(t) - \mathcal{A}\mathsf{u}(t) = \mathsf{f} \\ \qquad\quad \mathsf{u}(0) = \mathsf{u}_0 \end{cases},$$

and since the domain V of $\mathcal{A}$ is equipped with the topology inherited from L (and not the graph norm), $\mathcal{A}$ is unbounded in general.

Our goal is to show that $\mathcal{A}$ is an infinitesimal generator of a strongly continuous semigroup (C_0-semigroup) on L. If we succeed, then our problem (P′) (and thus also (P)) can be fitted in the setting of abstract Cauchy problems of the semigroup theory. The following theorem is the main result of this paper.

Theorem 2. *$\mathcal{A}$ is the infinitesimal generator of a contraction C_0-semigroup $(T(t))_{t\geqslant 0}$ on L.*

Dem. According to the Hille-Yosida generation theorem (see [26, Theorem 3.1 on p. 8]) it is necessary and sufficient to show that

a) V is dense in L,
b) $\mathcal{A}$ is closed,
c) $\rho(\mathcal{A}) \supseteq \langle 0,\infty\rangle$ and $\|R_\lambda(\mathcal{A})\| \leqslant \frac{1}{\lambda}$, for every $\lambda > 0$, where $\rho(\mathcal{A})$ is the resolvent set of $\mathcal{A}$, while $R_\lambda(\mathcal{A})$ is the resolvent.

Density of V is a direct consequence of Lemma 1. In order to prove (b), let us take a sequence (u_n) in V such that $\mathsf{u}_n \longrightarrow \mathsf{u}$ and $\mathcal{A}\mathsf{u}_n \longrightarrow \mathsf{f}$, both in L. We need to show that $\mathsf{u} \in V$ and $\mathsf{f} = \mathcal{A}\mathsf{u}$. It is easy to show that (u_n) is a Cauchy sequence in the complete graph space W. Indeed,

$$\begin{aligned} \|\mathsf{u}_n - \mathsf{u}_m\|_W^2 &= \|\mathsf{u}_n - \mathsf{u}_m\|_L^2 + \|\mathcal{L}\mathsf{u}_n - \mathcal{L}\mathsf{u}_m\|_L^2 \\ &= \|\mathsf{u}_n - \mathsf{u}_m\|_L^2 + \|\mathcal{A}\mathsf{u}_n - \mathcal{A}\mathsf{u}_m\|_L^2\,, \end{aligned}$$

and the claim follows by the assumption that (u_n) and $(\mathcal{A}\mathsf{u}_n)$ are convergent (therefore Cauchy) sequences in L. Thus (u_n) is convergent in W and, because V is closed in W, we additionally have that the limit v is in V. The convergence in the graph norm gives us $\mathsf{u}_n \longrightarrow \mathsf{v}$ and $\mathcal{A}\mathsf{u}_n \longrightarrow \mathcal{A}\mathsf{v}$ in L, which implies $\mathsf{u} = \mathsf{v} \in V$ and $\mathsf{f} = \mathcal{A}\mathsf{v} = \mathcal{A}\mathsf{u}$.

In order to prove the last statement, let us define $\mathcal{L}_\lambda := \lambda I + \mathcal{L}$ and a corresponding $\tilde{\mathcal{L}}_\lambda := \lambda I + \tilde{\mathcal{L}}$, for an arbitrary $\lambda > 0$, and let us show that $\mathcal{L}_\lambda$ is an isomorphism from V to L. The idea of proof is to show that $\mathcal{L}_\lambda$ and $\tilde{\mathcal{L}}_\lambda$ satisfy (T1)–(T3) and then apply Theorem 1. The condition (T1) is trivially satisfied, (T2) is satisfied if we take $c + 2\lambda$ as the constant, and since $\mathcal{L}$ and $\tilde{\mathcal{L}}$ satisfy (T3′) we have

$$\langle\, (\mathcal{L}_\lambda + \tilde{\mathcal{L}}_\lambda)\mathsf{u} \mid \mathsf{u}\,\rangle_L = \langle\, (\mathcal{L} + \tilde{\mathcal{L}})\mathsf{u} \mid \mathsf{u}\,\rangle_L + 2\lambda\langle\, \mathsf{u} \mid \mathsf{u}\,\rangle_L \geqslant 2\lambda\|\mathsf{u}\|_L^2\,,$$

for every $\mathsf{u} \in L$, which is exactly (T3). Before applying Theorem 1, it is important to notice that the corresponding graph space W_λ and the boundary operator D_λ of operator $\mathcal{L}_\lambda$ do not depend on λ. Indeed, one can verify that for every $\lambda > 0$ we have $W_\lambda = W$ and $D_\lambda = D$, which implies that subspace V satisfy conditions (V) in the graph space of operator $\mathcal{L}_\lambda$. Now we have that ${\mathcal{L}_\lambda}_{|_V} : V \longrightarrow L$ is an isomorphism, and therefore $\rho(\mathcal{A}) \supseteq \langle 0, \infty\rangle$. Finally, Lemma 2 gives us

$$\|{\mathcal{L}_\lambda}_{|_V}\| \geqslant \lambda \quad \Longrightarrow \quad \|R_\lambda(\mathcal{A})\| \leqslant \frac{1}{\lambda}\,,$$

because $R_\lambda(\mathcal{A}) = ({\mathcal{L}_\lambda}_{|_V})^{-1}$.

Q.E.D.

Remark. Instead of using the Hille-Yosida theorem in the proof of the previous theorem, we could alternatively (like in [14, p. 412] and [26, p. 14]) prove that $\mathcal{A}$ is maximal dissipative, i.e. $\mathcal{A}$ is dissipative (which is a direct consequence of (T3′)) and $\mathrm{im}\,(I - \mathcal{A}) = L$, and then apply the Lumer-Phillips theorem. ∎

After we have established that $\mathcal{A}$ generates a contraction C_0-semigroup we can use some classical results given in [11, 26] to derive various conclusions about solvability of (P), and one of them we summarise in the following corollary.

Corollary 1. *Let $\mathcal{L}$ be an operator that satisfies (T) and (T3′), V a subspace of its graph space satisfying (V), and* $\mathsf{f} \in \mathrm{L}^1(\langle 0, T\rangle; L)$.

a) Then for every $\mathsf{u}_0 \in L$ *the problem (P) has the unique weak solution* $\mathsf{u} \in \mathrm{C}([0,T]; L)$ *given by*

$$\mathsf{u}(t) = T(t)\mathsf{u}_0 + \int_0^t T(t-s)\mathsf{f}(s)ds\,, \qquad t \in [0,T]\,,$$

where $(T(t))_{t\geqslant 0}$ *is as in Theorem 2.*

b) If additionally $\mathsf{u}_0 \in V$ *and* $\mathsf{f} \in \mathrm{W}^{1,1}(\langle 0,T\rangle; L) \cup \Big(\mathrm{C}([0,T]; L) \cap \mathrm{L}^1(\langle 0,T\rangle; V)\Big)$*, with V equipped with the graph norm, then the above weak solution is the classical solution of (P) on* $[0, T\rangle$.

∎

Remark. From the formula for the solution one can easily get the estimate

$$(\forall\, t \in [0,T]) \qquad \|\mathsf{u}(t)\|_L \leqslant \|\mathsf{u}_0\|_L + \int_0^t \|\mathsf{f}(s)\|_L ds\,.$$

∎

By the definition of weak solution it is clear that it satisfies the initial condition, while the equation is satisfied in the meaning of [26, Theorem 2.7 on p. 108]. However, in our setting we can show that the weak solution satisfies the equation in a certain space (a similar observation has been done in [14, p. 406]).

Theorem 3. *Let* $\mathsf{u}_0 \in L$, $\mathsf{f} \in \mathrm{L}^1(\langle 0,T\rangle; L)$ *and let* u *be the weak solution of (P). Then* $\mathsf{u}', \mathcal{L}\mathsf{u}, \mathsf{f} \in \mathrm{L}^1(\langle 0,T\rangle; W_0')$ *and*

$$\mathsf{u}' + \mathcal{L}\mathsf{u} = \mathsf{f}\,,$$

in $\mathrm{L}^1(\langle 0,T\rangle; W_0')$.

Dem. According to [26, Theorem 2.7 on p. 108], there exists a sequence of the classical solutions u_n to (P′) with right hand sides $\mathsf{f}_n \in \mathrm{C}^1([0,T]; L)$ and initial conditions $\mathsf{u}_{0n} \in V$, such that $\mathsf{f}_n \longrightarrow \mathsf{f}$ in $\mathrm{L}^1(\langle 0,T\rangle; L)$, $\mathsf{u}_{0n} \longrightarrow \mathsf{u}_0$ in L, and $\mathsf{u}_n \longrightarrow \mathsf{u}$ in $\mathrm{C}([0,T]; L)$.

Since $\mathcal{L} \in \mathcal{L}(L; W_0')$, we have $\mathcal{L}\mathsf{u}_n \longrightarrow \mathcal{L}\mathsf{u}$ in $\mathrm{C}([0,T]; W_0')$, which implies the convergence in the space $\mathrm{L}^1(\langle 0,T\rangle; W_0')$. As L is continuously embedded in W_0', we also have $\mathsf{f}_n \longrightarrow \mathsf{f}$ in $\mathrm{L}^1(\langle 0,T\rangle; W_0')$. Thus

$$\mathsf{u}_n' = -\mathcal{L}\mathsf{u}_n + \mathsf{f}_n \longrightarrow -\mathcal{L}\mathsf{u} + \mathsf{f}\,,$$

in $\mathrm{L}^1(\langle 0,T\rangle; W_0')$. On the other hand $\mathsf{u}_n \longrightarrow \mathsf{u}$ in the space of vector valued distributions $\mathcal{D}'(\langle 0,T\rangle; W_0')$, which implies $\mathsf{u}_n' \longrightarrow \mathsf{u}'$ in $\mathcal{D}'(\langle 0,T\rangle; W_0')$ (see [14, p. 218]). The uniqueness of the limit in the space of vector valued distributions gives us

$$\mathsf{u}' + \mathcal{L}\mathsf{u} = \mathsf{f}$$

in $\mathcal{D}'(\langle 0,T\rangle; W_0')$, and since $\mathcal{L}\mathsf{u}, \mathsf{f} \in \mathrm{L}^1(\langle 0,T\rangle; W_0')$ and $\mathrm{L}^1(\langle 0,T\rangle; W_0')$ is continuously embedded in $\mathcal{D}'(\langle 0,T\rangle; W_0')$, we also have $\mathsf{u}' \in \mathrm{L}^1(\langle 0,T\rangle; W_0')$.

Q.E.D.

Remark. As it is well known, semilinear problems can also be treated via the semigroup theory [11, 26]. Therefore, we can get the existence and uniqueness result for the abstract Cauchy problem

$$\begin{cases} \mathsf{u}'(t) + \mathcal{L}\mathsf{u}(t) = \mathsf{f}(t, \mathsf{u}(t)) \\ \mathsf{u}(0) = \mathsf{u}_0 \end{cases},$$

where $\mathcal{L}$ is an *abstract Friedrichs operator*, and the right-hand side $\mathsf{f} : \langle 0,T\rangle \times L \longrightarrow L$ depends also on the unknown function. Usual assumptions on f that ensure existence and uniqueness of the weak solution are continuity and some kind of Lipschitz continuity in the last variable (see [11, Ch. 4] and [26, Ch. 6]). For some estimates on the solution see [8]. ∎

3. Examples

We shall now prove that some well-known examples fit in our setting of non-stationary Friedrichs systems, and apply the results of the previous section.

Initial problem for symmetric hyperbolic system

We consider the initial problem for first-order system

$$\text{(HS)} \qquad \begin{cases} \partial_t \mathsf{u} + \sum_{k=1}^{d} \partial_k(\mathbf{A}_k \mathsf{u}) + \mathbf{C}\mathsf{u} = \mathsf{f} \quad \text{in } \langle 0,T\rangle \times \mathbf{R}^d \\ \mathsf{u}(0,\cdot) = \mathsf{u}_0 \end{cases},$$

where $T > 0$ is some given time, and the coefficients $\mathbf{A}_k \in \mathrm{W}^{1,\infty}(\mathbf{R}^d; \mathrm{M}_d(\mathbf{R}))$, $k \in 1..n$, $\mathbf{C} \in \mathrm{L}^\infty(\mathbf{R}^d; \mathrm{M}_d(\mathbf{R}))$ do not depend on time. For the right-hand side and the initial datum we suppose $\mathsf{f} \in \mathrm{L}^1(\langle 0,T\rangle; \mathrm{L}^2(\mathbf{R}^d; \mathbf{R}^d))$ and $\mathsf{u}_0 \in \mathrm{L}^2(\mathbf{R}^d; \mathbf{R}^d)$, while $\mathsf{u} : [0,T\rangle \times \mathbf{R}^d \longrightarrow \mathbf{R}^d$ is the unknown function. We also suppose that each $\mathbf{A}_k$ is symmetric a.e. on $\mathbf{R}^d$.

One can easily see that (HS) fits in our framework of non-stationary Friedrichs systems with the operator $\mathcal{L}$ taken to be

$$\mathcal{L}\mathsf{u} := \sum_{k=1}^{d} \partial_k(\mathbf{A}_k \mathsf{u}) + \mathbf{C}\mathsf{u}\,.$$

Indeed, the symmetry condition (F1) is trivially satisfied, while the positivity condition (F2) can be obtained by using the substitution $\mathsf{v} := e^{-\lambda t}\mathsf{u}$, for some $\lambda > 0$ large enough, as remarked before. Then our Friedrichs system reads

$$\partial_t \mathsf{v} + (\mathcal{L} + \lambda I)\mathsf{v} = e^{-\lambda t}\mathsf{f}\,,$$

the initial condition remains $\mathsf{v}(0,\cdot) = \mathsf{u}_0$, and the graph space of the operator $\mathcal{L} + \lambda\mathbf{I}$ is the same as the graph space of $\mathcal{L}$, with equivalent norms: $W = \{\mathsf{w} \in \mathrm{L}^2(\mathbf{R}^d;\mathbf{R}^d) : \mathcal{L}\mathsf{w} \in \mathrm{L}^2(\mathbf{R}^d;\mathbf{R}^d)\}$. Now, as a consequence of the remark closing the introductory section and Corollary 1, we have the result of existence and uniqueness of the solution to (HS).

Theorem 4. *Let* $\mathsf{f} \in \mathrm{W}^{1,1}(\langle 0,T\rangle;\mathrm{L}^2(\mathbf{R}^d;\mathbf{R}^d)) \cup \Big(\mathrm{C}([0,T];\mathrm{L}^2(\mathbf{R}^d;\mathbf{R}^d)) \cap \mathrm{L}^1(\langle 0,T\rangle;W)\Big)$ *and* $\mathsf{u}_0 \in W$. *Then the abstract initial-value problem*

$$\begin{cases} \mathsf{u}' + \displaystyle\sum_{k=1}^{d} \partial_k(\mathbf{A}_k\mathsf{u}) + \mathbf{C}\mathsf{u} = \mathsf{f} \\ \qquad\qquad\qquad\qquad \mathsf{u}(0) = \mathsf{u}_0 \end{cases}$$

has the unique classical solution given by

$$\mathsf{u}(t) = e^{\lambda t}\mathbf{T}(t)\mathsf{u}_0 + \int_0^t e^{\lambda(t-s)}\mathbf{T}(t-s)\mathsf{f}(s)ds\,, \quad t \in [0,T]\,,$$

where $(\mathbf{T}(t))_{t\geqslant 0}$ *is the contraction* C_0*-semigroup generated by* $-\mathcal{L} - \lambda I$. ∎

Remark. The above formula could have been written in terms of the C_0-semigroup $\mathbf{S}(t) := e^{\lambda t}\mathbf{T}(t)$, with the infinitesimal generator $-\mathcal{L}$ [26, p. 12]. ∎

Remark. A similar existence and uniqueness result for the Cauchy problem for the symmetric hyperbolic systems, under some additional assumptions on coefficients can be found in [20]. ∎

Time-dependent Maxwell system

Let $\Omega \subseteq \mathbf{R}^3$ be open and bounded with a Lipschitz boundary Γ, $T > 0$, $\mathbf{\Sigma}_{ij} \in \mathrm{L}^\infty(\Omega;\mathrm{M}_3(\mathbf{R}))$, $i,j \in \{1,2\}$, $\mathsf{f}_1,\mathsf{f}_2 \in \mathrm{L}^1(\langle 0,T\rangle;\mathrm{L}^2(\Omega;\mathbf{R}^3))$, $\boldsymbol{\mu},\boldsymbol{\varepsilon} \in \mathrm{W}^{1,\infty}(\Omega;\mathrm{M}_3(\mathbf{R}))$ and constants $\mu_0,\varepsilon_0 \in \mathbf{R}^+$ such that for every $\mathbf{x} \in \Omega$, $\boldsymbol{\mu}(\mathbf{x})$ and $\boldsymbol{\varepsilon}(\mathbf{x})$ are symmetric, and $\boldsymbol{\mu} \geqslant \mu_0\mathbf{I}$, $\boldsymbol{\varepsilon} \geqslant \varepsilon_0\mathbf{I}$. We consider the generalised time-dependent Maxwell system:

$$\text{(MS)} \qquad \begin{cases} \boldsymbol{\mu}\partial_t\mathsf{H} + \operatorname{rot}\mathsf{E} + \mathbf{\Sigma}_{11}H + \mathbf{\Sigma}_{12}E = \mathsf{f}_1 \\ \boldsymbol{\varepsilon}\partial_t\mathsf{E} - \operatorname{rot}\mathsf{H} + \mathbf{\Sigma}_{21}H + \mathbf{\Sigma}_{22}E = \mathsf{f}_2 \end{cases} \quad \text{in } \langle 0,T\rangle \times \Omega\,,$$

where $\mathsf{E},\mathsf{H} : [0,T\rangle \times \Omega \longrightarrow \mathbf{R}^3$ are the unknown functions.

Remark. If we take $\mathsf{f}_1 \equiv 0$, $\mathsf{f}_2 = \mathsf{J}$, $\mathbf{\Sigma}_{11} = \mathbf{\Sigma}_{12} = \mathbf{\Sigma}_{21} \equiv \mathbf{0}$ we will get the (standard) Maxwell system in a linear nonhomogeneous anisotropic medium. In that case H, E, $\boldsymbol{\varepsilon}$, $\mathbf{\Sigma}_{22}$, $\boldsymbol{\mu}$, J represent the magnetic and electric fields, the electric permeability, the conductivity of the medium, the magnetic susceptibility of the material in Ω and the applied current density, respectively. ∎

The principal square roots $\boldsymbol{\mu}(\mathbf{x})^{\frac12}$ and $\boldsymbol{\varepsilon}(\mathbf{x})^{\frac12}$ of $\boldsymbol{\mu}(\mathbf{x})$ and $\boldsymbol{\varepsilon}(\mathbf{x})$ are well-defined for every $\mathbf{x} \in \Omega$ because $\boldsymbol{\mu}(\mathbf{x})$ and $\boldsymbol{\varepsilon}(\mathbf{x})$ are positive-definite matrices, which justifies definitions $\boldsymbol{\mu}^{\frac12}(\mathbf{x}) := \boldsymbol{\mu}(\mathbf{x})^{\frac12}$, $\boldsymbol{\varepsilon}^{\frac12}(\mathbf{x}) := \boldsymbol{\varepsilon}(\mathbf{x})^{\frac12}$. Moreover, we have $\boldsymbol{\mu}^{\frac12},\boldsymbol{\varepsilon}^{\frac12} \in \mathrm{W}^{1,\infty}(\Omega;\mathrm{M}_3(\mathbf{R}))$, both $\boldsymbol{\mu}^{\frac12}(\mathbf{x})$ and $\boldsymbol{\varepsilon}^{\frac12}(\mathbf{x})$ are symmetric, for every $\mathbf{x} \in \Omega$, with $\boldsymbol{\mu}^{\frac12} \geqslant {\mu_0}^{\frac12}\mathbf{I}$, $\boldsymbol{\varepsilon}^{\frac12} \geqslant {\varepsilon_0}^{\frac12}\mathbf{I}$. The last property implies that $\boldsymbol{\mu}^{\frac12}(\mathbf{x})$ and $\boldsymbol{\varepsilon}^{\frac12}(\mathbf{x})$ are regular, for every $\mathbf{x}$, and moreover $\boldsymbol{\mu}^{-\frac12},\boldsymbol{\varepsilon}^{-\frac12} \in \mathrm{W}^{1,\infty}(\Omega;\mathrm{M}_3(\mathbf{R}))$, where $\boldsymbol{\mu}^{-\frac12}(\mathbf{x}) := \boldsymbol{\mu}^{\frac12}(\mathbf{x})^{-1}$ and $\boldsymbol{\varepsilon}^{-\frac12}(\mathbf{x}) := \boldsymbol{\varepsilon}^{\frac12}(\mathbf{x})^{-1}$.

After introducing the substitutions (using the previous notation)

$$\mathsf{u} := \begin{bmatrix} \mathsf{u}_1 \\ \mathsf{u}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{\mu}^{\frac{1}{2}}\mathsf{H} \\ \boldsymbol{\varepsilon}^{\frac{1}{2}}\mathsf{E} \end{bmatrix},$$

and multiplying the first three equations by $\boldsymbol{\mu}^{-\frac{1}{2}}$, and the last three by $\boldsymbol{\varepsilon}^{-\frac{1}{2}}$, we can write our system as

$$\partial_t \mathsf{u} + \mathcal{L}\mathsf{u} = \mathsf{F},$$

where

$$\mathcal{L}\mathsf{u} := \begin{bmatrix} \boldsymbol{\mu}^{-\frac{1}{2}}\operatorname{rot}(\boldsymbol{\varepsilon}^{-\frac{1}{2}}\mathsf{u}_2) \\ -\boldsymbol{\varepsilon}^{-\frac{1}{2}}\operatorname{rot}(\boldsymbol{\mu}^{-\frac{1}{2}}\mathsf{u}_1) \end{bmatrix} + \begin{bmatrix} \boldsymbol{\mu}^{-\frac{1}{2}}\boldsymbol{\Sigma}_{11}\boldsymbol{\mu}^{-\frac{1}{2}} & \boldsymbol{\mu}^{-\frac{1}{2}}\boldsymbol{\Sigma}_{12}\boldsymbol{\varepsilon}^{-\frac{1}{2}} \\ \boldsymbol{\varepsilon}^{-\frac{1}{2}}\boldsymbol{\Sigma}_{21}\boldsymbol{\mu}^{-\frac{1}{2}} & \boldsymbol{\varepsilon}^{-\frac{1}{2}}\boldsymbol{\Sigma}_{22}\boldsymbol{\varepsilon}^{-\frac{1}{2}} \end{bmatrix}\begin{bmatrix} \mathsf{u}_1 \\ \mathsf{u}_2 \end{bmatrix}, \qquad \mathsf{F} = \begin{bmatrix} \boldsymbol{\mu}^{-\frac{1}{2}}\mathsf{f}_1 \\ \boldsymbol{\varepsilon}^{-\frac{1}{2}}\mathsf{f}_2 \end{bmatrix}.$$

To conclude that $\mathcal{L}$ is a classical Friedrichs operator we still need to do some computations:

$$\begin{aligned} \boldsymbol{\mu}^{-\frac{1}{2}}\operatorname{rot}(\boldsymbol{\varepsilon}^{-\frac{1}{2}}\mathsf{u}_2) &= \boldsymbol{\mu}^{-\frac{1}{2}}\sum_{k=1}^{3}\partial_k(B_k\boldsymbol{\varepsilon}^{-\frac{1}{2}}\mathsf{u}_2) \\ &= \sum_{k=1}^{3}\partial_k(\boldsymbol{\mu}^{-\frac{1}{2}}B_k\boldsymbol{\varepsilon}^{-\frac{1}{2}}\mathsf{u}_2) - \Big(\sum_{k=1}^{3}(\partial_k\boldsymbol{\mu}^{-\frac{1}{2}})B_k\boldsymbol{\varepsilon}^{-\frac{1}{2}}\Big)\mathsf{u}_2, \end{aligned}$$

where

$$B_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \qquad B_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

are matrices corresponding to the differential operator rot. We can analogously rewrite the last three components $-\boldsymbol{\varepsilon}^{-\frac{1}{2}}\operatorname{rot}(\boldsymbol{\mu}^{-\frac{1}{2}}\mathsf{u}_1)$ and get:

$$\begin{aligned} \begin{bmatrix} \boldsymbol{\mu}^{-\frac{1}{2}}\operatorname{rot}(\boldsymbol{\varepsilon}^{-\frac{1}{2}}\mathsf{u}_2) \\ -\boldsymbol{\varepsilon}^{-\frac{1}{2}}\operatorname{rot}(\boldsymbol{\mu}^{-\frac{1}{2}}\mathsf{u}_1) \end{bmatrix} &= \sum_{k=1}^{3}\partial_k\bigg(\begin{bmatrix} \mathbf{0} & \boldsymbol{\mu}^{-\frac{1}{2}}B_k\boldsymbol{\varepsilon}^{-\frac{1}{2}} \\ \boldsymbol{\varepsilon}^{-\frac{1}{2}}B_k^\top\boldsymbol{\mu}^{-\frac{1}{2}} & \mathbf{0} \end{bmatrix}\mathsf{u}\bigg) \\ &\quad + \bigg(\sum_{k=1}^{3}\begin{bmatrix} \mathbf{0} & (\partial_k\boldsymbol{\mu}^{-\frac{1}{2}})B_k^\top\boldsymbol{\varepsilon}^{-\frac{1}{2}} \\ (\partial_k\boldsymbol{\varepsilon}^{-\frac{1}{2}})B_k\boldsymbol{\mu}^{-\frac{1}{2}} & \mathbf{0} \end{bmatrix}\bigg)\mathsf{u}. \end{aligned}$$

If we define

$$\begin{aligned} \mathbf{A}_k &:= \begin{bmatrix} \mathbf{0} & \boldsymbol{\mu}^{-\frac{1}{2}}B_k\boldsymbol{\varepsilon}^{-\frac{1}{2}} \\ \boldsymbol{\varepsilon}^{-\frac{1}{2}}B_k^\top\boldsymbol{\mu}^{-\frac{1}{2}} & \mathbf{0} \end{bmatrix}, \quad k \in 1..3, \\ \mathbf{C} &:= \sum_{k=1}^{3}\begin{bmatrix} \mathbf{0} & (\partial_k\boldsymbol{\mu}^{-\frac{1}{2}})B_k^\top\boldsymbol{\varepsilon}^{-\frac{1}{2}} \\ (\partial_k\boldsymbol{\varepsilon}^{-\frac{1}{2}})B_k\boldsymbol{\mu}^{-\frac{1}{2}} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \boldsymbol{\mu}^{-\frac{1}{2}}\boldsymbol{\Sigma}_{11}\boldsymbol{\mu}^{-\frac{1}{2}} & \boldsymbol{\mu}^{-\frac{1}{2}}\boldsymbol{\Sigma}_{12}\boldsymbol{\varepsilon}^{-\frac{1}{2}} \\ \boldsymbol{\varepsilon}^{-\frac{1}{2}}\boldsymbol{\Sigma}_{21}\boldsymbol{\mu}^{-\frac{1}{2}} & \boldsymbol{\varepsilon}^{-\frac{1}{2}}\boldsymbol{\Sigma}_{22}\boldsymbol{\varepsilon}^{-\frac{1}{2}} \end{bmatrix}, \end{aligned}$$

then one can easily see that $\mathcal{L}$ takes the form

$$\mathcal{L}\mathsf{u} = \sum_{k=1}^{3}\partial_k(\mathbf{A}_k\mathsf{u}) + \mathbf{C}\mathsf{u}$$

of the classical Friedrichs partial differential operator. Indeed, the symmetry condition (F1) is trivially satisfied, while the positivity condition (F2) can be obtained using the substitution $\mathsf{v} := e^{-\lambda t}\mathsf{u}$, as in the previous example. Therefore, without loss of generality, let us assume that our operator $\mathcal{L}$ satisfies the positivity condition (F2).

Remark. As the system above is a symmetric hyperbolic system, the Cauchy problem on whole $\mathbf{R}^3$ is contained in the setting of the previous example, so the existence and uniqueness result is immediate. Using that, we can get the existence and uniqueness result for the Cauchy problem of the starting Maxwell system (MS). ∎

Obviously, the spaces involved are

$$
\begin{aligned}
L &= \mathrm{L}^2(\Omega;\mathbf{R}^3)\times\mathrm{L}^2(\Omega;\mathbf{R}^3)\,,\\
W &= \left\{\mathsf{u}=\begin{bmatrix}\mathsf{u}_1\\ \mathsf{u}_2\end{bmatrix}\in L : \begin{bmatrix}\mathsf{rot}\,(\boldsymbol{\mu}^{-\frac12}\mathsf{u}_1)\\ \mathsf{rot}\,(\boldsymbol{\varepsilon}^{-\frac12}\mathsf{u}_2)\end{bmatrix}\in L\right\}\\
&= \left\{\mathsf{u}=\begin{bmatrix}\mathsf{u}_1\\ \mathsf{u}_2\end{bmatrix}\in L : \begin{bmatrix}\boldsymbol{\mu}^{-\frac12}\mathsf{u}_1\\ \boldsymbol{\varepsilon}^{-\frac12}\mathsf{u}_2\end{bmatrix}\in \mathrm{L}^2_{\mathrm{rot}}(\Omega;\mathbf{R}^3)\times\mathrm{L}^2_{\mathrm{rot}}(\Omega;\mathbf{R}^3)\right\},
\end{aligned}
$$

where $\mathrm{L}^2_{\mathrm{rot}}(\Omega;\mathbf{R}^3)$ stands for the graph space of differential operator rot. Because $\boldsymbol{\mu}^{-\frac12}$ and $\boldsymbol{\varepsilon}^{-\frac12}$ are uniformly bounded from below and above the graph norm $\|\cdot\|_{\mathcal{L}}$ is equivalent with the norm

$$
\|\mathsf{u}\| := \left\|\begin{bmatrix}\boldsymbol{\mu}^{-\frac12}\mathsf{u}_1\\ \boldsymbol{\varepsilon}^{-\frac12}\mathsf{u}_2\end{bmatrix}\right\|_{\mathrm{L}^2_{\mathrm{rot}}(\Omega;\mathbf{R}^3)\times\mathrm{L}^2_{\mathrm{rot}}(\Omega;\mathbf{R}^3)}
$$

on W.

If we denote by $\boldsymbol{\nu}=(\nu_1,\nu_2,\nu_3)\in\mathrm{L}^\infty(\Gamma;\mathbf{R}^3)$ the unit outward normal on Γ, matrix field $\mathbf{A}_{\boldsymbol{\nu}}$ can be written as a block matrix

$$
\mathbf{A}_{\boldsymbol{\nu}} = \begin{bmatrix}\mathbf{0} & \boldsymbol{\mu}^{-\frac12}\mathbf{A}^{\mathsf{rot}}_{\boldsymbol{\nu}}\,\boldsymbol{\varepsilon}^{-\frac12}\\ -\boldsymbol{\varepsilon}^{-\frac12}\mathbf{A}^{\mathsf{rot}}_{\boldsymbol{\nu}}\,\boldsymbol{\mu}^{-\frac12} & \mathbf{0}\end{bmatrix},
$$

where

$$
\mathbf{A}^{\mathsf{rot}}_{\boldsymbol{\nu}} = \begin{bmatrix}0 & -\nu_3 & \nu_2\\ \nu_3 & 0 & -\nu_1\\ -\nu_2 & \nu_1 & 0\end{bmatrix}
$$

is a matrix corresponding to the operator rot. We can describe the boundary operator D in terms of the trace operator $\mathcal{T}_{\mathrm{H}^1}:\mathrm{H}^1(\Omega;\mathbf{R}^3)\longrightarrow\mathrm{H}^{\frac12}(\Gamma;\mathbf{R}^3)$ and the tangential trace operator (see [1], [7]) $\mathcal{T}_{\mathrm{rot}}:\mathrm{L}^2_{\mathrm{rot}}(\Omega;\mathbf{R}^3)\longrightarrow\mathrm{H}^{-\frac12}(\Gamma;\mathbf{R}^3)$. Indeed, for $\mathsf{u},\mathsf{v}\in\mathrm{C}^\infty_c(\mathbf{R}^3;\mathbf{R}^6)$, we have

$$
\begin{aligned}
{}_{W'}\langle\, D\mathsf{u},\mathsf{v}\,\rangle_W &= \int_\Gamma \mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})\mathsf{u}_{|_\Gamma}(\mathbf{x})\cdot\mathsf{v}_{|_\Gamma}(\mathbf{x})dS(\mathbf{x})\\
&= {}_{\mathrm{H}^{-\frac12}}\langle\,\boldsymbol{\mu}^{-\frac12}\mathcal{T}_{\mathrm{rot}}(\boldsymbol{\varepsilon}^{-\frac12}\mathsf{u}_2),\mathcal{T}_{\mathrm{H}^1}\mathsf{v}_1\,\rangle_{\mathrm{H}^{\frac12}} - {}_{\mathrm{H}^{-\frac12}}\langle\,\boldsymbol{\varepsilon}^{-\frac12}\mathcal{T}_{\mathrm{rot}}(\boldsymbol{\mu}^{-\frac12}\mathsf{u}_1),\mathcal{T}_{\mathrm{H}^1}\mathsf{v}_2\,\rangle_{\mathrm{H}^{\frac12}}\\
&= {}_{\mathrm{H}^{-\frac12}}\langle\,\boldsymbol{\mu}^{-\frac12}\mathcal{T}_{\mathrm{rot}}(\boldsymbol{\varepsilon}^{-\frac12}\mathsf{v}_2),\mathcal{T}_{\mathrm{H}^1}\mathsf{u}_1\,\rangle_{\mathrm{H}^{\frac12}} - {}_{\mathrm{H}^{-\frac12}}\langle\,\boldsymbol{\varepsilon}^{-\frac12}\mathcal{T}_{\mathrm{rot}}(\boldsymbol{\mu}^{-\frac12}\mathsf{v}_1),\mathcal{T}_{\mathrm{H}^1}\mathsf{u}_2\,\rangle_{\mathrm{H}^{\frac12}}\,,
\end{aligned}
$$

where the second equation can be extended by density to $(\mathsf{u},\mathsf{v})\in W\times\mathrm{H}^1(\Omega;\mathbf{R}^6)$, and the third one to $(\mathsf{u},\mathsf{v})\in\mathrm{H}^1(\Omega;\mathbf{R}^6)\times W$. Also note that in the above expression we have multiplication of a functional from $\mathrm{H}^{-\frac12}(\Gamma;\mathbf{R}^3)$ by matrix Lipschitz functions $\boldsymbol{\mu}^{-\frac12}$ and $\boldsymbol{\varepsilon}^{-\frac12}$, which is well-defined as follows

$$
{}_{\mathrm{H}^{-\frac12}}\langle\,\boldsymbol{\mu}^{-\frac12}\mathsf{g},\mathsf{z}\,\rangle_{\mathrm{H}^{\frac12}} := {}_{\mathrm{H}^{-\frac12}}\langle\,\mathsf{g},\boldsymbol{\mu}^{-\frac12}_{|_\Gamma}\mathsf{z}\,\rangle_{\mathrm{H}^{\frac12}}\,,\qquad \mathsf{z}\in\mathrm{H}^{\frac12}(\Gamma;\mathbf{R}^3)\,,\quad \mathsf{g}\in\mathrm{H}^{-\frac12}(\Gamma;\mathbf{R}^3)\,,
$$

and analogously for $\boldsymbol{\varepsilon}^{-\frac12}$.

Let us take the subspace V that corresponds to the boundary condition $\boldsymbol{\nu}\times\mathsf{E}_{|_\Gamma}=0$ (in the case when the corresponding function is not smooth, we interpret this notion in terms of the trace operators):

$$
\begin{aligned}
V = \tilde{V} &= \{\mathsf{u}\in W : \mathcal{T}_{\mathrm{rot}}(\boldsymbol{\varepsilon}^{-\frac12}\mathsf{u}_2)=0\}\\
&= \{\mathsf{u}\in W : \mathcal{T}_{\mathrm{rot}}\mathsf{E}=0\}\,,
\end{aligned}
$$

and verify the conditions (V1) and (V2). Note that $\mathsf{u} \in V$ if and only if $[\mathsf{H}\ \mathsf{E}]^\top \in \mathrm{L}^2_{\mathrm{rot}}(\Omega;\mathbf{R}^3) \times \mathrm{L}^2_{\mathrm{rot},0}(\Omega;\mathbf{R}^3)$, where $\mathrm{L}^2_{\mathrm{rot},0}(\Omega;\mathbf{R}^3) := \mathsf{Cl}_{\,\mathrm{L}^2_{\mathrm{rot}}(\Omega;\mathbf{R}^3)}\mathrm{C}^\infty_c(\Omega;\mathbf{R}^3)$.

To prove (V1) we need to show

$$(\forall\, \mathsf{u} \in V) \qquad {}_{W'}\langle\, D\mathsf{u}, \mathsf{u}\,\rangle_W = 0\,.$$

Since $\mathsf{u} = [\mathsf{u}_1\ \mathsf{u}_2]^\top \in V$ implies $\boldsymbol{\mu}^{-\frac{1}{2}}\mathsf{u}_1 \in \mathrm{L}^2_{\mathrm{rot}}(\Omega;\mathbf{R}^3)$ and $\boldsymbol{\varepsilon}^{-\frac{1}{2}}\mathsf{u}_2 \in \mathrm{L}^2_{\mathrm{rot},0}(\Omega;\mathbf{R}^3)$, there is a sequence $(\mathsf{u}^n) = ([\mathsf{u}_1^n\ \mathsf{u}_2^n]^\top)$ in $\mathrm{H}^1(\Omega;\mathbf{R}^3) \times \mathrm{C}^\infty_c(\Omega;\mathbf{R}^3)$ such that $\mathsf{u}^n \longrightarrow \mathsf{u}$ in W. As $\mathcal{T}_{\mathrm{H}^1}\mathsf{u}_2^n = 0$ and $\mathcal{T}_{\mathrm{rot}}(\boldsymbol{\varepsilon}^{-\frac{1}{2}}\mathsf{u}_2^n) = \boldsymbol{\nu} \times \mathcal{T}_{\mathrm{H}^1}(\boldsymbol{\varepsilon}^{-\frac{1}{2}}\mathsf{u}_2^n) = 0$, using computed identity for D and its continuity, we get

$${}_{W'}\langle\, D\mathsf{u}, \mathsf{u}\,\rangle_W = \lim_n {}_{W'}\langle\, D\mathsf{u}^n, \mathsf{u}^n\,\rangle_W = \lim_n 0 = 0\,,$$

which proves (V1).

In order to prove (V2), let us first take arbitrary $\mathsf{u}, \mathsf{v} \in V$ and the corresponding approximating sequences (u^n), (v^n) from $\mathrm{H}^1(\Omega;\mathbf{R}^3) \times \mathrm{C}^\infty_c(\Omega;\mathbf{R}^3)$, as in the previous case. Similarly like before, we have

$${}_{W'}\langle\, D\mathsf{v}, \mathsf{u}\,\rangle_W = \lim_n {}_{W'}\langle\, D\mathsf{v}^n, \mathsf{u}^n\,\rangle_W = \lim_n 0 = 0\,,$$

as $\mathcal{T}_{\mathrm{rot}}(\boldsymbol{\varepsilon}^{-\frac{1}{2}}\mathsf{u}_2^n) = \mathcal{T}_{\mathrm{H}^1}\mathsf{v}_2^n = 0$, which gives us $V \subseteq D(V)^0$. For the opposite inclusion we need to show that for an arbitrary $\mathsf{u} = [\mathsf{u}_1\ \mathsf{u}_2]^\top \in W$, condition

$$(\forall\, \mathsf{v} \in V) \qquad {}_{W'}\langle\, D\mathsf{v}, \mathsf{u}\,\rangle_W = 0\,,$$

implies $\mathcal{T}_{\mathrm{rot}}(\boldsymbol{\varepsilon}^{-\frac{1}{2}}\mathsf{u}_2) = 0$. If we put in the above equality $\mathsf{v} = [\mathsf{v}_1\ 0]^\top$, $\mathsf{v}_1 \in \mathrm{H}^1(\Omega;\mathbf{R}^3)$, we get

$${}_{W'}\langle\, D\mathsf{v}, \mathsf{u}\,\rangle_W = {}_{\mathrm{H}^{-\frac{1}{2}}}\langle\, \boldsymbol{\mu}^{-\frac{1}{2}}\mathcal{T}_{\mathrm{rot}}(\boldsymbol{\varepsilon}^{-\frac{1}{2}}\mathsf{u}_2), \mathcal{T}_{\mathrm{H}^1}\mathsf{v}_1\,\rangle_{\mathrm{H}^{\frac{1}{2}}} = 0\,.$$

Since $\mathsf{v}_1 \in \mathrm{H}^1(\Omega;\mathbf{R}^3)$ is arbitrary, $\mathcal{T}_{\mathrm{H}^1}$ is surjective and the mapping $\mathsf{z} \mapsto \boldsymbol{\mu}^{-\frac{1}{2}}\mathsf{z}$ is from $\mathrm{H}^{\frac{1}{2}}(\Gamma;\mathbf{R}^3)$ onto $\mathrm{H}^{\frac{1}{2}}(\Gamma;\mathbf{R}^3)$, we get, $\mathcal{T}_{\mathrm{rot}}(\boldsymbol{\varepsilon}^{-\frac{1}{2}}\mathsf{u}_2) = 0$, and (V2) is obtained.

Let us summarise, using Corollary 1, all that has been shown in the following theorem.

Theorem 5. *Let $\mathsf{E}_0 \in \mathrm{L}^2_{\mathrm{rot},0}(\Omega;\mathbf{R}^3), \mathsf{H}_0 \in \mathrm{L}^2_{\mathrm{rot}}(\Omega;\mathbf{R}^3)$ and let $\mathsf{f}_1, \mathsf{f}_2 \in \mathrm{C}([0,T];\mathrm{L}^2(\Omega;\mathbf{R}^3))$ satisfy at least one of the following two conditions*

i) $\mathsf{f}_1, \mathsf{f}_2 \in \mathrm{W}^{1,1}(\langle 0,T\rangle;\mathrm{L}^2(\Omega;\mathbf{R}^3))$;

ii) $\boldsymbol{\mu}^{-1}\mathsf{f}_1 \in \mathrm{L}^1(\langle 0,T\rangle;\mathrm{L}^2_{\mathrm{rot}}(\Omega;\mathbf{R}^3))$, $\boldsymbol{\varepsilon}^{-1}\mathsf{f}_2 \in \mathrm{L}^1(\langle 0,T\rangle;\mathrm{L}^2_{\mathrm{rot},0}(\Omega;\mathbf{R}^3))$.

Then the abstract initial-boundary value problem

$$\begin{cases} \boldsymbol{\mu}\mathsf{H}' + \operatorname{rot}\mathsf{E} + \boldsymbol{\Sigma}_{11}H + \boldsymbol{\Sigma}_{12}E = \mathsf{f}_1 \\ \boldsymbol{\varepsilon}\mathsf{E}' - \operatorname{rot}\mathsf{H} + \boldsymbol{\Sigma}_{21}H + \boldsymbol{\Sigma}_{22}E = \mathsf{f}_2 \\ \mathsf{E}(0) = \mathsf{E}_0 \\ \mathsf{H}(0) = \mathsf{H}_0 \\ \boldsymbol{\nu} \times \mathsf{E}_{|_\Gamma} = 0 \end{cases}$$

has the unique classical solution given by

$$\begin{bmatrix} \mathsf{H} \\ \mathsf{E} \end{bmatrix}(t) = \begin{bmatrix} \boldsymbol{\mu}^{-\frac{1}{2}} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\varepsilon}^{-\frac{1}{2}} \end{bmatrix}\mathbf{T}(t)\begin{bmatrix} \boldsymbol{\mu}^{\frac{1}{2}}\mathsf{H}_0 \\ \boldsymbol{\varepsilon}^{\frac{1}{2}}\mathsf{E}_0 \end{bmatrix} + \begin{bmatrix} \boldsymbol{\mu}^{-\frac{1}{2}} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\varepsilon}^{-\frac{1}{2}} \end{bmatrix}\int_0^t \mathbf{T}(t-s)\begin{bmatrix} \boldsymbol{\mu}^{-\frac{1}{2}}\mathsf{f}_1(s) \\ \boldsymbol{\varepsilon}^{-\frac{1}{2}}\mathsf{f}_2(s) \end{bmatrix} ds\,, \quad t \in [0,T]\,,$$

where $(\mathbf{T}(t))_{t\geqslant 0}$ is the contraction C_0-semigroup generated by $-\mathcal{L}$. ∎

Remark. The above theorem is valid if the corresponding Friedrichs operator satisfies (F2′). As written before, if this is not the case, then the positivity condition can be obtained by a simple change of variable. The statement of the above theorem should also be changed accordingly. ∎

Remark. Note that we assume Lipschitz continuity of coefficients $\boldsymbol{\mu}$ and $\boldsymbol{\varepsilon}$, as they appear in matrices $\mathbf{A}_k$ for which Lipschitz continuity is required. This condition can be slightly relaxed [see 19] by assuming that each $\mathbf{A}_k$, as well as $\sum_k \partial_k \mathbf{A}_k$ is bounded function. However, these assumptions are still to strong to allow piecewise smooth coefficients in Maxwell system, which are often of interest. ∎

We can get a similar result if we take the boundary condition $\boldsymbol{\nu} \times \mathsf{H}_{|\Gamma} = 0$ instead of $\boldsymbol{\nu} \times \mathsf{E}_{|\Gamma} = 0$. More precisely, it can be shown that spaces

$$\begin{aligned} V = \tilde{V} &= \{\mathsf{u} \in W : \mathcal{T}_{\mathrm{rot}}(\boldsymbol{\mu}^{-\frac{1}{2}}\mathsf{u}_1) = 0\} \\ &= \{\mathsf{u} \in W : \mathcal{T}_{\mathrm{rot}}\mathsf{H} = 0\} \end{aligned}$$

satisfy (V), which gives us well-posedness of the corresponding initial-boundary value problem.

Remark. For more information regarding the Maxwell system, we refer to [12], [14]. ∎

Unsteady div-grad problem

Let $\Omega \subseteq \mathbf{R}^d$ be open and bounded with a Lipschitz boundary Γ, $\mathsf{f}_1 \in \mathrm{L}^1(\langle 0,T\rangle;\mathrm{L}^2(\Omega;\mathbf{R}^d))$, $f_2 \in \mathrm{L}^1(\langle 0,T\rangle;\mathrm{L}^2(\Omega;\mathbf{R}))$, and $c > 0$. We consider a system of equations

$$\text{(DG)} \qquad \begin{cases} \partial_t \mathsf{q} + \nabla p = \mathsf{f}_1 \\ \dfrac{1}{c^2}\partial_t p + \operatorname{div}\mathsf{q} = f_2 \end{cases} \quad \text{in } \langle 0,T\rangle \times \Omega\,,$$

with the unknowns $p : [0,T\rangle \times \Omega \longrightarrow \mathbf{R}$ and $\mathsf{q} : [0,T\rangle \times \Omega \longrightarrow \mathbf{R}^d$.

Remark. These are linearised equations for the propagation of sound in the inviscid, elastic and compressible fluid, describing small disturbances [25]. Here, the unknowns q and p correspond to the velocity of the fluid and the pressure, while the constant c stands for the speed of sound in the fluid. ∎

After introducing the substitution

$$\mathsf{u} := \begin{bmatrix} u_1 \\ \mathsf{u}_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{c}p \\ \mathsf{q} \end{bmatrix},$$

we get an evolution Friedrichs system

$$\partial_t \mathsf{u} + \mathcal{L}\mathsf{u} = \mathsf{F}\,,$$

where $\mathsf{F} = \begin{bmatrix} cf_2 \\ \mathsf{f}_1 \end{bmatrix}$ and $\mathcal{L}\mathsf{u} := \begin{bmatrix} c\operatorname{div}\mathsf{u}_2 \\ c\nabla u_1 \end{bmatrix}$ is the classical Friedrichs operator satisfying (F1) and (F2′), with $\mathbf{C} = \mathbf{0}$ and $\mathbf{A}_k = c\,\mathsf{e}_1 \otimes \mathsf{e}_{k+1} + c\,\mathsf{e}_{k+1} \otimes \mathsf{e}_1 \in \mathrm{M}_{1+d}(\mathbf{R})$.

Obviously, we have

$$\begin{aligned} W &= \mathrm{H}^1(\Omega) \times \mathrm{L}^2_{\mathrm{div}}(\Omega;\mathbf{R}^d)\,, \\ W_0 &= \mathrm{H}^1_0(\Omega) \times \mathrm{L}^2_{\mathrm{div},0}(\Omega;\mathbf{R}^d) = \mathsf{Cl}_W\,\mathrm{C}^\infty_c(\Omega;\mathbf{R}^{1+d})\,, \end{aligned}$$

where $\mathrm{L}^2_{\mathrm{div}}(\Omega;\mathbf{R}^d)$ is the graph space of differential operator div, and $\mathrm{L}^2_{\mathrm{div},0}(\Omega;\mathbf{R}^d)$ is the closure of $\mathrm{C}^\infty_c(\Omega;\mathbf{R}^d)$ in the graph norm of $\mathrm{L}^2_{\mathrm{div}}(\Omega;\mathbf{R}^d)$.

Remark. As the system above is a symmetric hyperbolic system, the Cauchy problem on whole $\mathbf{R}^d$ is contained in the setting of the first example, so the existence and uniqueness result is immediate, which also implies the existence and uniqueness result for the Cauchy problem of the unsteady div-grad problem. In this case we have $V = \tilde{V} = W = \mathrm{H}^1(\mathbf{R}^d) \times \mathrm{L}^2_{\mathrm{div}}(\mathbf{R}^d;\mathbf{R}^d)$. ∎

We shall study two possible boundary conditions, the first one is given by

$$V = \tilde{V} = \mathrm{H}^1_0(\Omega) \times \mathrm{L}^2_{\mathrm{div}}(\Omega; \mathbf{R}^d) \,,$$

and it clearly corresponds to the boundary condition $p_{|_\Gamma} = 0$ (in the case when the corresponding function is not smooth, we interpret this notion in terms of the trace operators). In [19, Lemma 5.3] it is shown that (V) is satisfyied for these V and $\tilde{V}$, hence by Corollary 1 we have the existence result.

Theorem 6. *Let* $\mathsf{q}_0 \in \mathrm{L}^2_{\mathrm{div}}(\Omega; \mathbf{R}^d)$, $p_0 \in \mathrm{H}^1_0(\Omega)$ *and let functions* $\mathsf{f}_1 \in \mathrm{C}([0,T]; \mathrm{L}^2(\Omega; \mathbf{R}^d))$ *and* $f_2 \in \mathrm{C}([0,T]; \mathrm{L}^2(\Omega))$ *satisfy at least one of the following two conditions*
i) $\mathsf{f}_1 \in \mathrm{W}^{1,1}(\langle 0,T\rangle; \mathrm{L}^2(\Omega; \mathbf{R}^d))$, $f_2 \in \mathrm{W}^{1,1}(\langle 0,T\rangle; \mathrm{L}^2(\Omega))$*;*
ii) $\mathsf{f}_1 \in \mathrm{L}^1(\langle 0,T\rangle; \mathrm{L}^2_{\mathrm{div}}(\Omega; \mathbf{R}^d))$, $f_2 \in \mathrm{L}^1(\langle 0,T\rangle; \mathrm{H}^1_0(\Omega))$.
Then the abstract initial-boundary-value problem

$$\begin{cases} q' + \nabla p = \mathsf{f}_1 \\ \dfrac{1}{c^2} p' + \operatorname{div} \mathsf{q} = f_2 \\ \mathsf{q}(0) = \mathsf{q}_0 \\ p(0) = p_0 \\ p_{|_\Gamma} = 0 \end{cases}$$

has the unique classical solution given by

$$\begin{bmatrix} p \\ \mathsf{q} \end{bmatrix} = \begin{bmatrix} c & \mathsf{0}^\top \\ \mathsf{0} & \mathbf{I} \end{bmatrix} \mathbf{T}(t) \begin{bmatrix} \frac{1}{c} p_0 \\ \mathsf{q}_0 \end{bmatrix} + \begin{bmatrix} c & \mathsf{0}^\top \\ \mathsf{0} & \mathbf{I} \end{bmatrix} \int_0^t \mathbf{T}(t-s) \begin{bmatrix} c f_2 \\ \mathsf{f}_1 \end{bmatrix} ds \,, \quad t \in [0,T] \,,$$

where $(\mathbf{T}(t))_{t \geqslant 0}$ *is the contraction* C_0*-semigroup generated by* $-\mathcal{L}$. ∎

We can get a similar result for the another possible boundary condition $\boldsymbol{\nu} \cdot \mathsf{q} = 0_{|_\Gamma}$, i.e. for the subspaces

$$V = \tilde{V} = \mathrm{H}^1(\Omega) \times \mathrm{L}^2_{\mathrm{div},0}(\Omega; \mathbf{R}^d) \,.$$

Initial-value problem for the wave equation

Let us now consider the initial-value problem

$$\text{(WE)} \qquad \begin{cases} \partial_{tt} u - c^2 \triangle u = f & \text{in } \langle 0,T\rangle \times \mathbf{R}^d \\ u(0,\cdot) = u_0 \\ \partial_t u(0,\cdot) = u_0^1 \end{cases} ,$$

where $T > 0$ is a given constant, $c > 0$ represents the propagation speed of the wave, $f \in \mathrm{L}^1(\langle 0,T\rangle; \mathrm{L}^2(\mathbf{R}^d))$ is the external force, $u_0 \in \mathrm{H}^1(\mathbf{R}^d)$ and $u_0^1 \in \mathrm{L}^2(\mathbf{R}^d)$ are the initial position and velocity respectively, while $u : [0,T\rangle \times \mathbf{R}^d \longrightarrow \mathbf{R}$ is the unknown function which represents the displacement.

After a change of variable

$$\mathsf{u} := \begin{bmatrix} u_1 \\ \mathsf{u}_2 \end{bmatrix} = \begin{bmatrix} \partial_t u \\ -c \nabla u \end{bmatrix} ,$$

we get the same system $\partial_t \mathsf{u} + \mathcal{L}\mathsf{u} = \mathsf{F}$ as in the previous example with $\mathsf{f}_1 \equiv \mathsf{0}$ and $f_2 = \frac{1}{c} f$. Note that the last d equations of this system are actually the Schwarz symmetry relations, while the first one results from the wave equation we started with. The initial condition transforms to

$$\mathsf{u}(0) = \begin{bmatrix} u_1(0) \\ \mathsf{u}_2(0) \end{bmatrix} = \begin{bmatrix} u_0^1 \\ -c \nabla u_0 \end{bmatrix} =: \mathsf{u}_0 \,.$$

Our problem fits in the framework of the Cauchy problem for the unsteady div-grad problem (which is remarked in the previous example), so for $V = \tilde{V} = W = \mathrm{H}^1(\mathbf{R}^d) \times \mathrm{L}^2_{\rm div}(\mathbf{R}^d;\mathbf{R}^d)$ we have the existence result.

However, due to our change of variable, we only have information about the derivatives of the original unknown. Therefore, in order to find u one must solve the problem

$$\text{(odeP)} \qquad \begin{cases} u'(t) = u_1(t) \\ u(0) = u_0 \end{cases} .$$

Let us illustrate this in the case of the classical solution; if we additionally assume $f \in \mathrm{W}^{1,1}(\langle 0,T\rangle;\mathrm{L}^2(\mathbf{R}^d)) \cup \Big(\mathrm{C}([0,T];\mathrm{L}^2(\mathbf{R}^d)) \cap \mathrm{L}^1(\langle 0,T\rangle;\mathrm{H}^1(\mathbf{R}^d))\Big)$, $u_0^1 \in \mathrm{H}^1(\mathbf{R}^d)$ and $\triangle u_0 \in \mathrm{L}^2(\mathbf{R}^d)$, then (by Corollary 1) the abstract Cauchy problem

$$\text{(P)} \qquad \begin{cases} \mathsf{u}'(t) + \mathcal{L}\mathsf{u}(t) = \mathsf{F} \\ \mathsf{u}(0) = \mathsf{u}_0 \end{cases}$$

has the classical solution $\mathsf{u} \in \mathrm{C}^1(\langle 0,T\rangle;\mathrm{L}^2(\mathbf{R}^d;\mathbf{R}^{d+1})) \cap \mathrm{C}([0,T];\mathrm{L}^2(\mathbf{R}^d;\mathbf{R}^{d+1})) \cap \mathrm{C}(\langle 0,T\rangle;W)$. Therefore

$$\begin{aligned} &u_1 \in \mathrm{C}^1(\langle 0,T\rangle;\mathrm{L}^2(\mathbf{R}^d)) \cap \mathrm{C}([0,T];\mathrm{L}^2(\mathbf{R}^d)) \cap \mathrm{C}(\langle 0,T\rangle;\mathrm{H}^1(\mathbf{R}^d)) \,, \\ &\mathsf{u}_2 \in \mathrm{C}^1(\langle 0,T\rangle;\mathrm{L}^2(\mathbf{R}^d;\mathbf{R}^d)) \cap \mathrm{C}([0,T];\mathrm{L}^2(\mathbf{R}^d;\mathbf{R}^d)) \cap \mathrm{C}(\langle 0,T\rangle;\mathrm{L}^2_{\rm div}(\mathbf{R}^d;\mathbf{R}^d)) \,, \end{aligned}$$

which ensures that the problem (odeP) has the unique solution u belonging to

$$\mathrm{C}^2(\langle 0,T\rangle;\mathrm{L}^2(\mathbf{R}^d)) \cap \mathrm{C}^1(\langle 0,T\rangle;\mathrm{H}^1(\mathbf{R}^d)) \cap \mathrm{C}([0,T];\mathrm{H}^1(\mathbf{R}^d)) \,.$$

Note that this u already satisfies the first initial condition in (WE). Let us show that it also satisfies the wave equation in $\mathrm{C}(\langle 0,T\rangle;\mathrm{L}^2(\mathbf{R}^d))$.

Since $\nabla : \mathrm{H}^1(\mathbf{R}^d) \longrightarrow \mathrm{L}^2(\mathbf{R}^d)$ is a continuous linear operator, it follows that the operator (which we denote the same) defined by $(\nabla u)(t) := \nabla(u(t))$ is a linear continuous operator $\mathrm{C}^1(\langle 0,T\rangle;\mathrm{H}^1(\mathbf{R}^d)) \longrightarrow \mathrm{C}^1(\langle 0,T\rangle;\mathrm{L}^2(\mathbf{R}^d))$ and $\mathrm{C}(\langle 0,T\rangle;\mathrm{H}^1(\mathbf{R}^d)) \longrightarrow \mathrm{C}(\langle 0,T\rangle;\mathrm{L}^2(\mathbf{R}^d))$, which *commutes with the time derivative*: $\nabla(u') = (\nabla u)'$. Using this, the last d equations in (P) and $(\text{odeP})_1$, we get $\mathsf{u}_2' = (-c\nabla u)'$ in the space $\mathrm{C}(\langle 0,T\rangle;\mathrm{L}^2(\mathbf{R}^d;\mathbf{R}^d))$, which together with the initial conditions for u_2 and u imply $\mathsf{u}_2 = -c\nabla u$. Note that, as an additional consequence we have $\nabla u \in \mathrm{C}(\langle 0,T\rangle;\mathrm{L}^2_{\rm div}(\mathbf{R}^d))$. Substituting u_1 and u_2 in the first equation in (P) we get (in $\mathrm{C}(\langle 0,T\rangle;\mathrm{L}^2(\mathbf{R}^d))$)

$$u'' = u_1' = -c\,\mathsf{div}\,\mathsf{u}_2 + f = -c\,\mathsf{div}\,(-c\nabla u) + f = c^2\triangle u + f \,.$$

It remains to prove that u satisfies the second initial condition in (WE), which easily follows from $u' = u_1$ in $\mathrm{C}(\langle 0,T\rangle;\mathrm{L}^2(\mathbf{R}^d))$, $u_1 \in \mathrm{C}([0,T];\mathrm{L}^2(\mathbf{R}^d))$ and $u_1(0) = u_0^1$.

Remark. A similar existence and uniqueness result for the initial-value problem for the wave equation within the framework of a semigroup theory can be found in [26]. ∎

Remark. Using results of the unsteady div-grad problem, we can get the existence and uniqueness results for the initial-boundary value problem for the wave equation with two possible boundary conditions:

$$\partial_t u_{|_\Gamma} = 0 \,,$$

which corresponds to the Dirichlet boundary conditions, or

$$\boldsymbol{\nu} \cdot \nabla u_{|_\Gamma} = 0 \,.$$

which is the homogeneous Neumann boundary condition. ∎

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