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Densities of ruin-related quantities in the Cramér-Lundberg model with Pareto claims

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Abstract: In this paper, we consider the classical yet widely applicable Cramér-Lundberg risk model with Pareto distributed claim sizes. Building on the previously known expression for the ruin probability we derive distributions of different ruin-related quantities. The results rely on the theory of scale functions and are intended to illustrate the simplicity and effectiveness of the theory. A particular emphasis is put on the tail behavior of the distributions of ruin-related quantities and their tail index value is established. Numerical illustrations are provided to show the influence of the claim sizes distribution tail index on the tails of the ruin-related quantities distribution.

Keywords: Cramér-Lundberg model, Pareto distribution, scale function, ruin-related quantities, heavy-tailed distributions

1 Introduction

In the classical ruin theory, the central model for the surplus of the insurance company is provided by the Cramér-Lundberg process

$$X_t = ct - \sum_{i=1}^{N_t} \xi_i, \quad t \geq 0, \tag{1}$$

where $c > 0$ is a constant rate at which premiums are collected, $\{N_t, t \geq 0\}$ is a Poisson process with parameter λ modelling claim arrivals and $(\xi_i, i \in \mathbb{N})$ are nonnegative independent identically distributed random variables that represent the claim sizes. When the insurance company starts its business with some initial surplus x, we may describe the evolution of the surplus by the process $x + X = \{x + X_t, t \geq 0\}$. Heavy-tailed distribution are often adequate to model claim sizes. A distribution of random variable ξ_1 is said to be heavy-tailed if the tail distribution $x \mapsto P(\xi_1 > x)$ is regularly varying at infinity. This means that $P(\xi_1 > x) \sim L(x)x^{-\alpha}$ as $x \to \infty$, where L is a slowly varying function, that is $L(tx)/L(t) \to 1$ as $t \to \infty$ for every $x > 0$ (see Embrechts et al. (1997) for more details). The parameter $\alpha > 0$, called the tail index, governs the tail thickness of the distribution and thus the probabilities of extreme events. A typical choice for the heavy-tailed claim sizes distribution is Pareto distribution, which can be defined by the distribution function

$$F_{\alpha,\beta}(x) = 1 - \left(1 + \frac{x}{\beta}\right)^{-\alpha}, \; x > 0, \tag{2}$$

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where $\beta > 0$ is the scale parameter and $\alpha > 0$ is the tail index. This type of Pareto distribution is supported on $(0, \infty)$ and is also referred to as the Pareto type II distribution or Lomax distribution (see Arnold (2015)). Pareto distribution is a typical representative of the class of heavy-tailed distributions and can successfully model the occurrence of extreme events which is particularly important for insurance models (see for example (Embrechts et al. 1997, Example 6.2.9) for the application on the fire insurance data).

Since $X = \{X_t, t \geq 0\}$ is the difference between a linear trend and a compound Poisson process, it is a Lévy process, that is a process with stationary independent increments and càdlàg sample paths such that $P(X_0 = 0) = 1$. More specifically, X falls into the class of spectrally negative Lévy processes. These are Lévy processes with no positive jumps which do not have monotone paths. A rich and analytically tractable fluctuation theory has been developed for these processes (see e.g. Kyprianou (2006)).

Several practical questions arise from the risk model (1) given the initial surplus x of the insurance company. The main problem is computing the probability of ruin while within the so-called Gerber-Shiu risk theory, one is concerned with the distribution of quantities like deficit at ruin and surplus immediately prior to ruin ((Asmussen & Albrecher 2010, Chapter XII), Kyprianou (2013)). Although the model (1) is simple, explicit solutions to ruin problems are available only for some particular choices of the distribution of ξ_1 (see (Asmussen & Albrecher 2010, Chapter IV) and the references therein). For Pareto claims, the exact expression for the ruin probability has been obtained in Ramsay (2003), Ramsay (2007) (see also Albrecher & Kortschak (2009) for the different type of Pareto distribution). In the more general case of heavy-tailed claims, one has to rely on various approximation techniques and asymptotic estimates of ruin probability (see (Asmussen & Albrecher 2010, Chapter X) for a survey of results in this direction). Starting from the work of Gerber and Shiu (Gerber & Shiu (1997, 1998)), there has been a growing interest in the distribution of quantities related to ruin such as deficit at ruin and surplus immediately prior to ruin. The explicit results in this direction are far less common.

The goal of this paper is to address ruin problems for the Cramér-Lundberg model (1) with Pareto claims (2), especially the distribution of quantities related to ruin. The approach is based on the powerful fluctuation theory of spectrally negative Lévy processes, more specifically on the so-called scale function, an expression for which follows from the results of Ramsay (2007). Although the theory of scale functions is developed for the more general class of models, we use it here for methodological reasons to illustrate its simplicity and explicitness. Using the scale function, the expressions are derived for densities of the most common ruin-related quantities: deficit at ruin, surplus prior to ruin, last minimum of the surplus before ruin, maximum before the ruin and maximal severity of ruin. All of these expressions can be easily computed and well approximated. A particular focus is on the influence of the tail index value on the distribution of ruin quantities. We establish the tail behavior of the distributions considered. Numerical examples are provided to illustrate the results.

Section 2 provides an overview of the spectrally negative Lévy processes and basic results for the model (1). Section 3 contains various results concerning the laws at the first passage time. Finally, in Section 4 a numerical illustration is provided with plots of densities of ruin-related quantities.

2 Preliminaries

In this section we review basic facts about spectrally negative Lévy processes, scale functions and their use in exit problems following Kyprianou (2006) and Kuznetsov et al. (2013). In the second part we restrict our consideration to model (1) and derive the scale function based on the results of Ramsay (2007). It is worth mentioning that the theory of scale functions is far more powerful that it is needed to obtain the results for the classical model considered here.

Suppose $X = \{X_t, t \geq 0\}$ is a Lévy process on some filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$. Due to stationary and independent increments, there is a function $\Psi : \mathbb{R} \to \mathbb{C}$ such that $\log E\left[e^{i\zeta X_t}\right] = -t\Psi(\zeta)$ for $t \geq 0$, $\zeta \in \mathbb{R}$. By the Lévy-Khintchine formula Ψ has the following general form

$$\Psi(\zeta) = ia\zeta + \frac{\sigma^2}{2}\zeta^2 + \int_{\mathbb{R}} \left(1 - e^{i\zeta x} + i\zeta x \mathbf{1}_{\{|x|\leq 1\}}\right) \Pi(dx), \tag{3}$$

with $a \in \mathbb{R}$, $\sigma > 0$ and Π a measure on $\mathbb{R}\backslash\{0\}$ such that $\int (1 \wedge x^2)\Pi(dx) < \infty$. We refer to (a, σ, Π) as the characteristic triplet. In what follows, P_x will denote probability such that $P_x(X_0 = x) = 1$ and E_x the corresponding expectation operator, keeping the notation P for the case P_0.

For our purpose it is enough to consider spectrally negative Lévy processes, that is Lévy processes such that $\Pi(0, \infty) = 0$ and whose paths are not monotone. This implies that the process has no positive jumps. Except model (1), examples are Brownian motion with drift, difference of positive drift and subordinator and even processes of unbounded variation. In this case, one can work with the Laplace exponent $\psi(\theta) = \log E\left[e^{\theta X_1}\right]$, which is well defined at least for $\theta \geq 0$. General form of ψ follows from (3) by analytical extension of the characteristic exponent $\psi(\theta) = -\Psi(-i\theta)$. The value $\psi'(0+)$ determines the long term behaviour of the process since $\psi'(0+) = EX_1 \in [-\infty, \infty)$. In ruin theory, considering X as the model for surplus, the case of interest is $\psi'(0+) > 0$ when process drifts to $+\infty$ as otherwise the probability of ruin is 1. Furthermore, ψ is strictly convex and $\psi(\theta) \to \infty$ as $\theta \to \infty$, thus we can define the right continuous inverse

$$\Phi(q) = \sup\{\theta \geq 0 : \psi(\theta) = q\}. \tag{4}$$

If $\psi'(0+) \geq 0$, the equation $\psi(\theta) = q$ has a unique solution and in particular $\Phi(0) = 0$. Fluctuation theory of spectrally negative Lévy processes provides particularly nice and tractable identities. A powerful tool in this context are the scale functions defined as follows.

Definition 1. For $q \geq 0$, let $W^{(q)} : \mathbb{R} \to [0, \infty)$ be defined by $W^{(q)}(x) = 0$ for $x < 0$ and on $[0, \infty)$, $W^{(q)}$ is the unique right continuous function whose Laplace transform is given by

$$\int_0^\infty e^{-\theta x} W^{(q)}(x)dx = \frac{1}{\psi(\theta) - q}, \quad \text{for } \theta > \Phi(q).$$

We write W for $W^{(0)}$ and call it the scale function. $W^{(q)}$ is called the q-scale function.

One can show (see Kuznetsov et al. (2013)) that for every spectrally negative Lévy process, q-scale function exists for every $q \geq 0$. In the case $\psi'(0+) > 0$, scale function can be explicitly characterized as

$$W(x) = \frac{1}{\psi'(0+)} P_x(\underline{X}_\infty \geq 0), \tag{5}$$

where $\underline{X}_\infty = \inf_{t\geq 0} X_t$. The name originates from the analogy with the diffusion processes inspired by the following fluctuation identity for $q \geq 0$ and $x < a$

$$E_x\left[e^{-q\tau_a^+}\mathbf{1}_{\{\tau_a^+<\tau_0^-\}}\right] = \frac{W^{(q)}(x)}{W^{(q)}(a)}, \tag{6}$$

where for some $a \in \mathbb{R}$

$$\begin{aligned} \tau_a^+ &= \inf\{t > 0 : X_t > a\}, \\ \tau_a^- &= \inf\{t > 0 : X_t < a\}, \end{aligned}$$

denote the first passage time above and below level a, respectively. Moreover, we have that the (infinite horizon) ruin probability is

$$P_x(\tau_0^- < \infty) = \begin{cases} 1 - \psi'(0+)W(x), & \text{if } \psi'(0+) > 0, \\ 1, & \text{if } \psi'(0+) \leq 0. \end{cases} \tag{7}$$

Suppose now that $X = \{X_t,\, t \geq 0\}$ is the Cramér-Lundberg process (1) with $(\xi_i, i \in \mathbb{N})$ having Pareto distribution (2) with parameters $\alpha, \beta > 0$. We restrict our attention to the finite mean case $\alpha > 1$ and suppose that

$$m := EX_1 = c - \lambda\frac{\beta}{\alpha - 1} > 0, \tag{8}$$

that is the net profit condition is satisfied. Let $f_{\alpha,\beta}$ denote the probability density function of the Pareto distribution with parameters $\alpha, \beta > 0$, that is

$$f_{\alpha,\beta}(x) = \frac{\alpha}{\beta}\left(1 + \frac{x}{\beta}\right)^{-\alpha-1}, \quad x > 0,$$

and $f_{\alpha,\beta}(x) = 0$ for $x \leq 0$. Notice that for $x > 0$ it holds that

$$f_{\alpha-1,\beta}(x) = \frac{\alpha - 1}{\beta}\left(1 - F_{\alpha,\beta}(x)\right). \tag{9}$$

The characteristic exponent (3) of X can be written in the following form

$$\begin{aligned} \Psi(\zeta) &= -\log E\left[e^{i\zeta X_1}\right] = -\left(i\zeta c - \lambda\int_{-\infty}^{\infty}\left(1 - e^{-i\zeta x}\right) f_{\alpha,\beta}(x)dx\right) \\ &= -i\zeta c - i\zeta\int_{-1}^{1} x\lambda f(-x)dx + \int_{-\infty}^{\infty}\left(1 - e^{i\zeta x} + i\zeta x\mathbf{1}_{\{|x|\leq 1\}}\right)\lambda f_{\alpha,\beta}(-x)dx. \end{aligned}$$

Therefore, the characteristic triplet of X is $(a, 0, \Pi)$ with

$$a = -c - \lambda \int_{-1}^{0} x f_{\alpha,\beta}(-x)dx = -c - \frac{\lambda}{\alpha - 1}\left(\left(1 + \frac{1}{\beta}\right)^{-\alpha}(\alpha + \beta) - \beta\right),$$

and the Lévy measure Π is absolutely continuous with respect to Lebesgue measure with density $\pi(x) = \lambda f_{\alpha,\beta}(-x)$ supported on $(-\infty, 0)$.

Since X is a difference between a drift and a compound Poisson process it easily follows that the Laplace exponent of X is

$$\psi(\theta) = \log E\left[e^{\theta X_1}\right] = c\theta + \lambda \int_{(0,\infty)} \left(e^{-\theta x} - 1\right) dF_{\alpha,\beta}(x) = c\theta + \lambda\left(\hat{F}_{\alpha,\beta}(\theta) - 1\right),$$

with $\hat{F}_{\alpha,\beta}(\theta) = \int_{(0,\infty)} e^{-\theta x} dF_{\alpha,\beta}(x)$ denoting the Laplace transform of $F_{\alpha,\beta}$. Following Nadarajah & Kotz (2006), the Laplace transform of the Pareto distribution is given by

$$\hat{F}_{\alpha,\beta}(\theta) = \alpha(\beta\theta)^{\alpha} e^{\beta\theta}\Gamma(-\alpha, \beta\theta),$$

where $\Gamma(s, x) = \int_x^\infty t^{s-1}e^{-t}dt$ denotes the incomplete gamma function. Using the property of the incomplete gamma function $x^{-s}e^x\Gamma(s + 1, x) = sx^{-s}e^x\Gamma(s, x) + 1$ ((Abramowitz & Stegun 1964, 6.5.22)) we can rewrite

$$\begin{aligned}\psi(\theta) &= c\theta - \lambda(\beta\theta)^{\alpha}e^{\beta\theta}\Gamma(1 - \alpha, \beta\theta)\\ &= c\theta\left(1 - \frac{\lambda\beta}{c(\alpha - 1)}(\alpha - 1)(\beta\theta)^{\alpha-1}e^{\beta\theta}\Gamma(1 - \alpha, \beta\theta)\right).\end{aligned}$$

In order to find the scale function W of X we should find inverse Laplace transform of $1/\psi(\theta)$. In Ramsay (2007), the inverse Laplace transform is given for the function

$$\frac{1 - \rho}{\theta\left(1 - \rho(\alpha - 1)(\beta\theta)^{\alpha-1}e^{\beta\theta}\Gamma(1 - \alpha, \beta\theta)\right)}.$$

Using this result and putting $\rho = \frac{\lambda\beta}{c(\alpha-1)}$, we easily get the following.

Corollary 1. *For the Cramér-Lundberg model (1)-(2) such that $\alpha > 1$ and (8) holds, the scale function is given by*

$$W(x) = \frac{1}{m}\left(1 - \int_0^\infty \frac{\rho(1 - \rho)u^{\alpha-2}}{\Gamma(\alpha - 1)H(u, \alpha, \rho)}e^{-\left(1+\frac{x}{\beta}\right)u}du\right), \tag{10}$$

where

$$H(u, \alpha, \rho) = \begin{cases}\left(1 + (\alpha - 1)\rho e^{-u}\,\mathrm{Ei}_\alpha(u)\right)^2 + \left(\pi\rho\frac{u^{\alpha-1}e^{-u}}{\Gamma(\alpha-1)}\right)^2, & \text{if } \alpha = 2, 3, \dots\\ \left(1 - \rho R(u, \alpha - 1)\right)^2 + \left(\pi\rho\frac{u^{\alpha-1}e^{-u}}{\Gamma(\alpha-1)}\right)^2, & \text{if } \alpha > 1 \text{ and } \alpha \neq 2, 3, \dots\end{cases}$$

$$R(u, \alpha) = 1 + \sum_{i=1}^{\infty}\frac{u^i}{(\alpha - 1)\cdots(\alpha - i)} - \frac{\pi u^\alpha e^{-u}}{\Gamma(\alpha)}\cot(\pi\alpha), \quad \alpha \notin \mathbb{N}$$

and $\mathrm{Ei}_\alpha(u)$ *denotes the generalization of the exponential integral defined for* $u > 0$, $\alpha \in \mathbb{N}$ *by the asymptotic expansion*

$$\mathrm{Ei}_\alpha(u) = \frac{u^{\alpha-1}}{(\alpha-1)!}\left(\gamma + \ln u - \sum_{i=1}^{\alpha-1}\frac{1}{i}\right) + \sum_{i=0, i\neq\alpha-1}^{\infty}\frac{u^i}{(i-\alpha+1)i!}.$$

From (7) we get Ramsey's result for the ruin probability (Ramsay (2007)) and from (6) the solution for the two-sided exit problem. The probability of ruin given the initial surplus x is

$$P_x(\tau_0^- < \infty) = 1 - mW(x). \tag{11}$$

For $a > 0$ and $x < a$ probability of attaining a before ruin is given by

$$P_x(\tau_a^+ < \tau_0^-) = \frac{W(x)}{W(a)}.$$

See Dickson & Gray (1984) for an early work on this type of two-sided ruin problem.

3 Ruin-related quantities

Starting from the papers Gerber & Shiu (1997, 1998), there has been growing interest in quantities related to ruin. Using the process X to model the surplus of the insurance company, besides obvious interest in the time of ruin τ_0^-, we can also investigate deficit at ruin $-X_{\tau_0^-}$ and surplus immediately prior to ruin $X_{\tau_0^- -}$. The expected discounted penalty function, introduced by Gerber & Shiu (1998), provides an approach to study the joint distribution of these random variables. Let

$$\underline{X}_t = \inf_{s\leq t} X_s,$$
$$\overline{X}_t = \sup_{s\leq t} X_s.$$

In the context of spectrally negative Lévy processes, Biffis & Morales (2010) introduce the generalized expected discounted penalty function that additionally includes the last minimum of the surplus before ruin $\underline{X}_{\tau_0^- -}$. Another extension was given in Yin & Yuen (2014) by including maximum before the ruin $\overline{X}_{\tau_0^-}$. The occurrence of ruin may not necessarily mean that the insurance company will stop its business. Depending on how severe the ruin is, it may find ways to operate until recovery. The problem can be studied by investigating the maximal severity of ruin (see Picard (1994) and references therein). If η denotes the time of recovery $\eta = \inf\{t > \tau_0^- : X_t \geq 0\}$, then the severity of ruin can be analyzed through $\underline{X}_\eta$ which represents maximal deficit during the ruin, i.e. minimum of X_t over the interval $[\tau_0^-, \eta]$. For the joint laws involving $-\underline{X}_\eta$ see Yin & Yuen (2014).

To summarize, the distribution of the following ruin-related quantities will be studied in this paper:

- deficit at ruin $-X_{\tau_0^-}$

- surplus prior to ruin $X_{\tau_0^- -}$
- last minimum of the surplus before ruin $\underline{X}_{\tau_0^- -}$
- maximum before the ruin $\overline{X}_{\tau_0^-}$
- maximal deficit during the ruin $\underline{X}_\eta$

Except in solving exit problems, scale functions can be used to describe undershoot and overshoot distributions at first passage time of spectrally negative Lévy processes. Hence, they provide a powerful tool to deal with distributions of ruin-related quantities. Based on the scale function (10), the following proposition gives the joint distribution of combinations of ruin-related quantities for the model (1) with Pareto claims (2). These will be used in the next subsection to obtain the marginal distributions. The proof is based on the results of (Kyprianou 2006, Chapter 8) and Biffis & Kyprianou (2010) (see also Remark 1).

Proposition 1. Suppose $\{X_t,\, t \geq 0\}$ is the Cramér-Lundberg process (1)-(2) such that $\alpha > 1$ and (8) holds.

(i) The distribution of the random vector $\left(-X_{\tau_0^-}, X_{\tau_0^- -}\right)$ conditionally on $\{\tau_0^- < \infty\}$ is absolutely continuous with density given by

$$k(y,z) = \begin{cases} \frac{1}{1-mW(x)}\lambda\left(W(x) - W(x-z)\right) f_{\alpha,\beta}(y+z), & y>0,\ 0<z\leq x,\\ \frac{1}{1-mW(x)}\lambda W(x) f_{\alpha,\beta}(y+z), & y>0,\ z>x,\\ 0, & \text{otherwise.}\end{cases}$$

(ii) The distribution of the random vector $\left(-X_{\tau_0^-}, X_{\tau_0^- -}, \underline{X}_{\tau_0^- -}\right)$ conditionally on $\{\tau_0^- < \infty\}$ is

$$\bar{k}(dy,dz,dw) = \begin{cases} \frac{1}{1-mW(x)}\lambda W'(x-w) f_{\alpha,\beta}(y+z)dydzdw, & y>0,\ 0<z\leq x,\ 0<w\leq z,\\ \frac{1}{1-mW(x)}\Big(\lambda W'(x-w) f_{\alpha,\beta}(y+z)dydzdw \\ \qquad + \lambda W(0) f_{\alpha,\beta}(y+z)dydz\delta_x(dw)\Big), & y>0,\ z>x,\ 0<w\leq x,\\ 0, & \text{otherwise.}\end{cases}$$

Proof. (i) By (Kyprianou 2006, Corollary 8.8.), for any spectrally negative Lévy process X we have for any $q \geq 0$ and $y, z > 0$ that

$$E_x\left[e^{-q\tau_0^-}; -X_{\tau_0^-} \in dy, X_{\tau_0^- -} \in dz, \tau_0^- < \infty\right] = \left(e^{-\Phi(q)z}W^{(q)}(x) - W^{(q)}(x-z)\right)\Pi(-dy-z)dz,$$

where Φ is defined in (4), Π is the Lévy measure and $W^{(q)}$ is the corresponding q-scale function. Since we have assumed $m = EX_1 > 0$, then $\Phi(0) = 0$ and taking $q = 0$ in the preceding equation we obtain

$$P_x\left(-X_{\tau_0^-} \in dy, X_{\tau_0^- -} \in dz, \tau_0^- < \infty\right) = \left(W(x) - W(x-z)\right)\Pi(-dy - z)dz.$$

Lévy measure is absolutely continuous with density $\pi(x) = \lambda f_{\alpha,\beta}(-x)$. Using the expression (11) we get the density of $P_x\left(-X_{\tau_0^-} \in dy, X_{\tau_0^- -} \in dz | \tau_0^- < \infty\right)$.

(ii) The proof follows from the expression for the generalized Gerber-Shiu measure given in Biffis & Kyprianou (2010) (see Remark 1). Given any spectrally negative Lévy process with positive mean, it holds for $q, x \geq 0$ and $y, z, w \geq 0$ that

$$\begin{aligned} &E_x\left[e^{-q\tau_0^-}; -X_{\tau_0^-} \in dy, X_{\tau_0^- -} \in dz, \underline{X}_{\tau_0^- -} \in dw, \tau_0^- < \infty\right] \\ &\quad = \mathbf{1}_{\{0<w\leq z\wedge x, y>0\}} e^{-\Phi(q)(z-w)} \times \\ &\quad \times \left((W^{(q)})'(x-w) - \Phi(q)W^{(q)}(x-w)\right)\Pi(-dy - z)dzdw \\ &\quad + \mathbf{1}_{\{x<z, y>0\}} e^{-\Phi(q)(z-x)} W^{(q)}(0)\Pi(-dy-z)dz\delta_x(dw) \\ &\quad + \frac{\sigma^2}{2}\left((W^{(q)})'(x) - \Phi(q)W^{(q)}(x)\right)\delta_{(0,0,0)}(dy, dz, dw). \end{aligned} \tag{12}$$

where δ denotes the Dirac measure. Since $\Phi(0) = 0$ and in the model considered there is no Gaussian component ($\sigma = 0$), it follows that

$$\begin{aligned} &P_x\left(-X_{\tau_0^-} \in dy, X_{\tau_0^- -} \in dz, \underline{X}_{\tau_0^- -} \in dw, \tau_0^- < \infty\right) \\ &\quad = \mathbf{1}_{\{0<w\leq z\wedge x, y>0\}} W'(x-w)\Pi(-dy-z)dzdw \\ &\quad + \mathbf{1}_{\{x<z, y>0\}} W(0)\Pi(-dy-z)dz\delta_x(dw) \end{aligned}$$

The derivative W' is well defined because the function $x \mapsto \Pi(-\infty, -x)$ is continuous and therefore $W \in C^1(0, \infty)$ ((Kuznetsov et al. 2013, Lemma 2.4)). □

Remark 1. The second term on the right-hand side of (12) is missing in Biffis & Kyprianou (2010). It accounts for the fact that $\underline{X}_{\tau_0^- -}$ has a point mass at x due to the event that the ruin occurs with surplus never dropping below level x, which can happen only in the case $X_{\tau_0^- -} > x$. Assuming additionally that $\sigma = 0$, the following identity similar to (12) has been proved in (Yin & Yuen 2014, Eq. 3.3.)

$$\begin{aligned} &E_x\left[e^{-q\tau_0^-}; -X_{\tau_0^-} \in dy, X_{\tau_0^- -} \in dz, \underline{X}_{\tau_0^- -} > b, \tau_0^- < \infty\right] \\ &\quad = \mathbf{1}_{\{0<b<z\wedge x, y>0\}}\left(e^{-\Phi(q)(z-b)}W^{(q)}(x-b) - W^{(q)}(x-z)\right)\Pi(-dy-z)dz. \end{aligned}$$

From here the point mass is clearly visible as $b \to x$. The derivative of the scale function in Biffis & Kyprianou (2010) should be understood formally as the "density" of the measure $W^{(q)}(x - dw)$ which has absolutely continuous and discrete part with support at point 0.

3.1 Distributions of ruin-related quantities

The following theorem summarizes distributions of ruin-related quantities studied in this paper.

Theorem 1. *Suppose* $\{X_t,\, t \geq 0\}$ *is the Cramér-Lundberg process* (1)-(2) *such that* $\alpha > 1$ *and* (8) *holds.*

(i) The density of $-X_{\tau_0^-}$ *conditionally on* $\{\tau_0^- < \infty\}$ *is*

$$k_1(y) = \frac{1}{1 - mW(x)} \Big((c-m) W(x) f_{\alpha-1,\beta}(y) \\ - \lambda \int_0^x W(x-z) f_{\alpha,\beta}(y+z) dz \Big) \mathbf{1}_{\{y>0\}}.$$

(ii) The density of $X_{\tau_0^- -}$ *conditionally on* $\{\tau_0^- < \infty\}$ *is*

$$k_2(z) = \begin{cases} \frac{c-m}{1-mW(x)} \left(W(x) - W(x-z) \right) f_{\alpha-1,\beta}(z), & 0 < z \leq x, \\ \frac{c-m}{1-mW(x)} W(x) f_{\alpha-1,\beta}(z), & z > x, \\ 0, & z \leq 0. \end{cases}$$

(iii) The distribution of $\underline{X}_{\tau_0^- -}$ *conditionally on* $\{\tau_0^- < \infty\}$ *is*

$$k_3(dw) = \frac{c-m}{1 - mW(x)} \Big(W'(x-w) \left(1 - F_{\alpha-1,\beta}(w)\right) dw \\ + W(0) \left(1 - F_{\alpha-1,\beta}(x)\right) \delta_x(dw) \Big) \mathbf{1}_{\{0<w\leq x\}}.$$

(iv) The density of $\overline{X}_{\tau_0^-}$ *conditionally on* $\{\tau_0^- < \infty\}$ *is*

$$k_4(v) = \frac{W(x)}{(1 - mW(x))} \frac{W'(v)}{W(v)^2} \mathbf{1}_{\{x \leq v\}}.$$

(v) The density of $-\underline{X}_\eta$ *conditionally on* $\{\tau_0^- < \infty\}$ *is*

$$k_5(u) = \frac{1}{(1 - mW(x))W(u)^2} \Big(W'(x+u) W(u) \\ - \left(W(x+u) - W(x) \right) W'(u) \Big) \mathbf{1}_{\{u>0\}}.$$

Proof. (i) For $y>0$ we have from Proposition 1(i) by using (9):

$$\begin{aligned}
k_1(y) &= \int_0^\infty k(y,z)dz \\
&= \frac{1}{1-mW(x)}\Bigg(\lambda \int_0^x \left(W(x)-W(x-z)\right) f_{\alpha,\beta}(y+z)dz \\
&\qquad\qquad\qquad\qquad + \lambda W(x)\int_x^\infty f_{\alpha,\beta}(y+z)dz\Bigg) \\
&= \frac{1}{1-mW(x)}\left(\lambda W(x)\int_0^\infty f_{\alpha,\beta}(y+z)dz - \lambda\int_0^x W(x-z)f_{\alpha,\beta}(y+z)dz\right) \\
&= \frac{1}{1-mW(x)}\left((c-m)W(x)f_{\alpha-1,\beta}(y) - \lambda\int_0^x W(x-z)f_{\alpha,\beta}(y+z)dz\right).
\end{aligned}$$

(ii) From Proposition 1(i) for $0<z\le x$ it follows that

$$\begin{aligned}
k_2(z) &= \int_0^\infty k(y,z)dy \\
&= \frac{1}{1-mW(x)}\lambda\left(W(x)-W(x-z)\right)\int_0^\infty f_{\alpha,\beta}(y+z)dy \\
&= \frac{1}{1-mW(x)}\lambda\left(W(x)-W(x-z)\right)\left(1-F_{\alpha,\beta}(z)\right) \\
&= \frac{c-m}{1-mW(x)}\left(W(x)-W(x-z)\right)f_{\alpha-1,\beta}(z),
\end{aligned}$$

while for $z>x$

$$\begin{aligned}
k_2(z) &= \frac{1}{1-mW(x)}\lambda W(x)\int_0^\infty f_{\alpha,\beta}(y+z)dy \\
&= \frac{c-m}{1-mW(x)}W(x)f_{\alpha-1,\beta}(z).
\end{aligned}$$

(iii) Proposition 1(ii) gives that for $w>0$

$$k_3(dw) = \int_0^x\int_0^\infty \bar{k}(dy,dz,dw) + \int_x^\infty\int_0^\infty \bar{k}(dy,dz,dw) =: I_1 + I_2.$$

Using (9) we get

$$\begin{aligned}
I_1 &= \frac{\lambda W'(x-w)}{1-mW(x)}\int_0^x \left(1-F_{\alpha,\beta}(z)\right)\mathbf{1}_{\{0<w\le z\}}dzdw \\
&= \frac{c-m}{1-mW(x)}W'(x-w)\int_w^x f_{\alpha-1,\beta}(z)dzdw \\
&= \frac{c-m}{1-mW(x)}W'(x-w)\left(F_{\alpha-1,\beta}(x)-F_{\alpha-1,\beta}(w)\right)dw
\end{aligned}$$

and

$$
\begin{aligned}
I_2 &= \frac{1}{1-mW(x)}\Bigg(\lambda W'(x-w)\int_x^\infty (1-F_{\alpha,\beta}(z))\,dzdw \\
&\qquad\qquad + \lambda W(0)\int_x^\infty (1-F_{\alpha,\beta}(z))\,dz\delta_x(dw)\Bigg) \\
&= \frac{c-m}{1-mW(x)}\left(W'(x-w)\int_x^\infty f_{\alpha-1,\beta}(z)dzdw + W(0)\int_x^\infty f_{\alpha-1,\beta}(z)dz\delta_x(dw)\right) \\
&= \frac{c-m}{1-mW(x)}\left(W'(x-w)\left(1-F_{\alpha-1,\beta}(x)\right)dw + W(0)\left(1-F_{\alpha-1,\beta}(x)\right)\delta_x(dw)\right).
\end{aligned}
$$

By combining the two expressions, the statement follows.

(iv) Since the event $\{\overline{X}_{\tau_0^-} \le v\}$ simply means the ruin happens before level v is reached, we have for $v \ge x$ from (6)

$$
P_x\left(\overline{X}_{\tau_0^-} \le v, \tau_0^- < \infty\right) = P_x\left(\tau_v^+ > \tau_0^-, \tau_0^- < \infty\right) = 1 - \frac{W(x)}{W(v)}. \tag{13}
$$

By the same argument as in the proof of Proposition 1(ii), we have that W is differentiable and the distribution of $\overline{X}_{\tau_0^-}$ is absolutely continuous. Taking derivative with respect to v in (13) we get

$$
P_x\left(\overline{X}_{\tau_0^-} \in dv, \tau_0^- < \infty\right) = \mathbf{1}_{\{x \le v\}} W(x)\frac{W'(v)}{W(v)^2}.
$$

Dividing by $P_x\left(\tau_0^- < \infty\right) = 1 - mW(x)$ gives (iv).

(v) The distribution of $-\underline{X}_\eta$ has a very simple expression in terms of the scale function. In the compound Poisson case considered here, for $u > 0$ it holds that (Picard (1994), (Asmussen & Albrecher 2010, Proposition 2.15); see also Remark 2)

$$
P_x\left(-\underline{X}_\eta \le u, \tau_0^- < \infty\right) = \frac{W(x+u) - W(x)}{W(u)}. \tag{14}
$$

Since the distribution is absolutely continuous taking derivative of this expression yields the statement.

□

Remark 2. By adapting the argument of Picard (1994) (see also (Asmussen & Albrecher 2010, Proposition 2.15)), (14) can be extended to any spectrally negative Lévy process. Indeed, suppose X is a spectrally negative Lévy process such that $EX_1 > 0$. First, since X drifts to ∞ it follows that $\eta < \infty$ on the event $\{\tau_0^- < \infty\}$. The event $\{-\underline{X}_\eta \le u\}$ means the ruin occurs with deficit $-X_{\tau_0^-} \le u$ and starting from $X_{\tau_0^-}$ the recovery is reached before level $-u$, that is by the strong Markov property and (6) we have

$$\begin{aligned} P_x\left(-\underline{X}_\eta \le u, \tau_0^- < \infty\right) &= \int_0^\infty P_x\left(-X_{\tau_0^-} \in dy, -\underline{X}_\eta \le u, \tau_0^- < \infty\right) \\ &= \int_0^u P_x\left(-X_{\tau_0^-} \in dy, \tau_0^- < \infty\right) P_{-y}\left(\tau_0^+ < \tau_{-u}^-\right) \\ &= \int_0^u P_x\left(-X_{\tau_0^-} \in dy, \tau_0^- < \infty\right) P_{u-y}\left(\tau_u^+ < \tau_0^-\right) \\ &= \int_0^u P_x\left(-X_{\tau_0^-} \in dy, \tau_0^- < \infty\right) \frac{W(u-y)}{W(u)}. \end{aligned} \tag{15}$$

Next, using spatial homogeneity, strong Markov property and (7) we get

$$\begin{aligned} &1-\psi'(0+)W(x+u) = P_{x+u}\left(\tau_0^- < \infty\right) \\ &= \int_0^u P_{x+u}\left(u - X_{\tau_u^-} \in dy, \tau_0^- < \infty\right) + \int_u^\infty P_{x+u}\left(u - X_{\tau_u^-} \in dy, \tau_0^- < \infty\right) \qquad (16) \\ &= \int_0^u P_{x+u}\left(u - X_{\tau_u^-} \in dy, \tau_u^- < \infty\right) P_{u-y}\left(\tau_0^- < \infty\right) \\ &\qquad\qquad + \int_u^\infty P_x\left(-X_{\tau_0^-} \in dy, \tau_0^- < \infty\right) \\ &= \int_0^u P_x\left(-X_{\tau_0^-} \in dy, \tau_0^- < \infty\right)\left(1-\psi'(0+)W(u-y)\right) \\ &\qquad\qquad + \int_u^\infty P_x\left(-X_{\tau_0^-} \in dy, \tau_0^- < \infty\right) \\ &= \int_0^\infty P_x\left(-X_{\tau_0^-} \in dy, \tau_0^- < \infty\right) - \psi'(0+)\int_0^u P_x\left(-X_{\tau_0^-} \in dy, \tau_0^- < \infty\right) W(u-y) \\ &= 1-\psi'(0+)W(x) - \psi'(0+)\int_0^u P_x\left(-X_{\tau_0^-} \in dy, \tau_0^- < \infty\right) W(u-y). \end{aligned}$$

In words, the two terms in (16) account for the case the ruin does not occur with the first crossing bellow level u and the case the ruin occurs exactly then. Again, by using (7) it follows that

$$\int_0^u P_x\left(-X_{\tau_0^-} \in dy, \tau_0^- < \infty\right) W(u-y) = W(x+u) - W(x).$$

By combining with (15), the statement follows.

It should be emphasized that although the expressions for densities in Theorem 1 seem complicated, they all except k_1 involve only the scale function, its derivative and density and distribution function of different Pareto distributions. Both W and W' can be well approximated by using numerical integration and the functions under the integral in (10) can be approximated to a high precision with computer algebra systems like e.g. *Mathematica*. Since the integrand in (10) is non-oscillating (Ramsay (2007)), even standard methods of numerical integration work well. By the dominated convergence theorem, W' can be expressed using the notation from Corollary 1 as

$$W'(x) = \frac{1}{m}\int_0^\infty \frac{\rho(1-\rho)u^{\alpha-2}}{\Gamma(\alpha-1)H(u,\alpha,\rho)} e^{-\left(1+\frac{x}{\beta}\right)u}\frac{u}{\beta}du,$$

which again can be computed as in the case of the scale function W. Further quantities in Theorem 1 are straightforward to compute, except the integral appearing in (i). However, using Fubini's theorem and (10) we get the following

$$\int_0^x W(x-z)f_{\alpha,\beta}(y+z)dz = \frac{1}{m}\left(F_{\alpha,\beta}(x+y)-F_{\alpha,\beta}(y)\right) - \frac{1}{m}\int_0^\infty \frac{\rho(1-\rho)u^{\alpha-2}}{\Gamma(\alpha-1)H(u,\alpha,\rho)}e^{-(1+\frac{x}{\beta})u}J_{\alpha,\beta}(x,y,u)du,$$

where

$$J_{\alpha,\beta}(x,y,u) = \int_0^x e^{\frac{zu}{\beta}} f_{\alpha,\beta}(y+z)dz.$$

As we illustrate in Section 4, this integral can also be approximated successfully.

3.2 Tail behaviour

The tail index represents the main parameter of the claims distribution that governs the size of the claims. It is therefore interesting to analyze how the heavy-tails of the claims affect the ruin distributions. Asymptotic ruin probabilities as the initial surplus tends to infinity have attracted a lot of attention, both for small and large claims and for a range of different risk models. For the classical model, in (Asmussen & Albrecher 2010, Chapter X) a survey of the main results can be found with an overview of the important references. These results have been extended to Lévy based risk process with particular attention to models incorporating large claims through subexponential and convolution equivalent distributions (Klüppelberg et al. (2004)). When it comes to ruin-related quantities, most references are investigating asymptotic behaviour when the initial surplus tends to infinity and are restricted to deficit at ruin and surplus prior to ruin. For the classical model see e.g. Willmot & Lin (1998) and Schmidli (1999), while for the Lévy risk process see Klüppelberg & Kyprianou (2006) and Griffin et al. (2012).

Here we establish a simple characterization of the tails of the ruin distributions considered. We write that a random variable $Y \in RV(-\alpha)$ if it is heavy-tailed with index α, that is the tail distribution is $-\alpha$-regularly varying function:

$$\lim_{t\to\infty} \frac{P(Y>tx)}{P(Y>t)} = x^{-\alpha}, \qquad \text{for every } x>0.$$

As the next theorem shows, when the claim distribution is Pareto with tail index α, the ruin-related quantities are also heavy-tailed but with lower value of the tail index $\alpha-1$.

Theorem 2. *If* $\{X_t,\, t\geq 0\}$ *is the Cramér-Lundberg process* (1)-(2) *such that* $\alpha>1$ *and* (8) *holds, then*

(i) $-X_{\tau_0^-} \in RV(-\alpha+1)$,

(ii) $X_{\tau_0^- -} \in RV(-\alpha+1)$,

(iii) $\overline{X}_{\tau_0^-} \in RV(-\alpha+1)$,

(iv) $-\underline{X}_\eta \in RV(-\alpha+1)$.

Proof. (i) First, notice that we can write

$$\frac{\int_0^x W(x-z)f_{\alpha,\beta}(ty+z)dz}{\int_0^x W(x-z)f_{\alpha,\beta}(t+z)dz} = \frac{f_{\alpha,\beta}(ty)\int_0^x W(x-z)\frac{f_{\alpha,\beta}(ty+z)}{f_{\alpha,\beta}(ty)}dz}{f_{\alpha,\beta}(t)\int_0^x W(x-z)\frac{f_{\alpha,\beta}(t+z)}{f_{\alpha,\beta}(t)}dz},$$

and that

$$\frac{f_{\alpha,\beta}(ty+z)}{f_{\alpha,\beta}(ty)} \to 1 \text{ as } t\to\infty.$$

Since $f_{\alpha,\beta}$ is $(-\alpha-1)$-regularly varying, it follows that so is $y\mapsto \lambda\int_0^x W(x-z)f_{\alpha,\beta}(y+z)dz$. From Theorem 1(i), k_1 is then a difference between $-\alpha$-regularly varying function $f_{\alpha-1,\beta}$ and $(-\alpha-1)$-regularly varying function. It is easy to see that k_1 is then $-\alpha$-regularly varying and by Karamata's theorem (see e.g. (Embrechts et al. 1997, Appendix A3)) it follows that $-X_{\tau_0^-}\in RV(-\alpha+1)$.

(ii) is obvious from Theorem 1(ii).

(iii) From (13) we have that the tail distribution function of $\overline{X}_{\tau_0^-}$ is

$$P_x\left(\overline{X}_{\tau_0^-} > v|\tau_0^- < \infty\right) = 1-\frac{1}{1-mW(x)}\left(1-\frac{W(x)}{W(v)}\right) = \frac{W(x)}{1-mW(x)}\frac{1-mW(v)}{W(v)}.$$

Since $W(v) = \left(1-P_v(\tau_0^-<\infty)\right)/m \to 1/m$ as $v\to\infty$, the tail behaviour of $\overline{X}_{\tau_0^-}$ is the same as for the ruin probability $1-mW(v)$. By (Asmussen & Albrecher 2010, Theorem 2.1), the ruin probability for heavy-tailed claims is $(-\alpha+1)$-regularly varying and the statement follows.

(iv) Directly from (14) we have

$$\begin{aligned} P_x\left(-\underline{X}_\eta > u|\tau_0^- < \infty\right) &= 1-\frac{1}{1-mW(x)}\frac{W(x+u)-W(x)}{W(u)} \\ &= \frac{W(x)}{1-mW(x)}\frac{1-mW(u)}{W(u)} - \frac{W(x+u)-W(u)}{W(u)}. \end{aligned}$$

Since

$$\frac{W(x+u)-W(u)}{W(u)} \to 0,$$

as $u\to\infty$, the argument is the same as in (iii). □

Remark 3. The argument used in the proof of (iii) and (iv) of Theorem 2 can be applied to a more general situation than considered in this paper. It requires only the ruin probability to be slowly varying.

4 Numerical illustration

In this section we plot the ruin probability and the densities of various ruin-related quantities obtained in Theorem 1. We are interested in observing how the ruin probability and distributions change with the tail heaviness of the claim sizes distribution. Therefore, for each object we provide several plots for a range of α values ($\alpha = 1.5,\ 2,\ 2.5,\ 3,\ 4$). Other parameters are kept fixed: $\beta = 1$, $\lambda = 1$, $c = 5$ and so the net profit condition (8) is satisfied for every value of α considered. The computation and corresponding plots are generated using *Mathematica* (Wolfram Research Inc. (2014)). Numerical integration is preformed using the built-in modules based on adaptive sampling strategies. All the code is available from the author upon request.

We start with the probability of ruin, that is we plot the function $x \mapsto P_x(\tau_0^- < \infty)$ for a range of α values (Fig. 1a). Some values are shown in Table 1 for specific values of the initial surplus x. One can notice how heavier tails of the claim sizes produce slower decay of the probability of ruin as the initial surplus increases. This illustrates how extreme events constitute a major risk for the insurance company and how the value of the tail index of the claims distribution is a good indicator of the riskiness of the insurer's portfolio.

Table 1: Probability of ruin for a range of α and initial surplus x values

$\alpha \setminus x$	1	2	5	10	15	20
1.5	0.32388	0.28431	0.2232	0.17662	0.15128	0.13461
2	0.11393	0.0806	0.04283	0.02377	0.01634	0.01241
2.5	0.05345	0.03052	0.01112	0.00444	0.00251	0.00166
3	0.02822	0.01302	0.00328	0.00096	0.00045	0.00026
4	0.00932	0.00282	0.00035	0.00006	0.00002	$7.81 \cdot 10^{-6}$

Figures 1b-1f show densities obtained in Theorem 1. Here the initial surplus is kept fixed at $x = 5$. Comparing the deficit at ruin for different α values (Fig. 1b), one can notice that the lower the α is, the distribution has more mass away from the origin which accounts for the higher deficit when the ruin occurs. In this situation, ruin is due to a single large claim, which represents a large jump downwards of the surplus process below level zero. Such scenario is hazardous for the insurance company. The discontinuity at x in the density of $X_{\tau_0^-}$ (Fig. 1c) is well documented (see Dickson (1992) and (Asmussen & Albrecher 2010, Proposition 2.14)). For lower values of α, the distribution of $X_{\tau_0^-}$ still has a considerable mass away from x as the ruin can occur even when the surplus is far away from the initial value. From the insurance company point of view, such portfolios may cause ruin even when the surplus is at its high values due to the possible large claim occurrence. Figure 1d shows the absolutely continuous part (k_3^{ac}) of the distribution of $\underline{X}_{\tau_0^- -}$ supported on $(0, x)$. It also illustrates that the point mass at x is larger for the larger values of α. This means that for the less risky portfolios with lighter tails, the minimum is more likely to never drop bellow initial surplus. The maximal surplus achieved before the ruin is more likely to be near the initial surplus for higher value of α than for the lower (Fig. 1e). Finally, the ruin can be much more severe when α is small as the distribution of $-\underline{X}_\eta$ assigns more mass to higher values than in the case of larger α (Fig. 1f). This is connected to the distribution of the deficit at ruin and accounts for the fact that in the small α case ruin can occur by a single large claim.

The simulation presented here also illustrates that the expressions appearing in Theorem 1 can be well approximated. For instance, all the densities considered in this example and plotted in Figure 1, integrate numerically to one up to an error which is of order of the machine precision.

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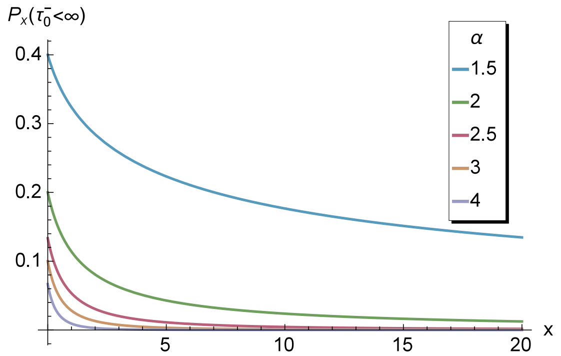
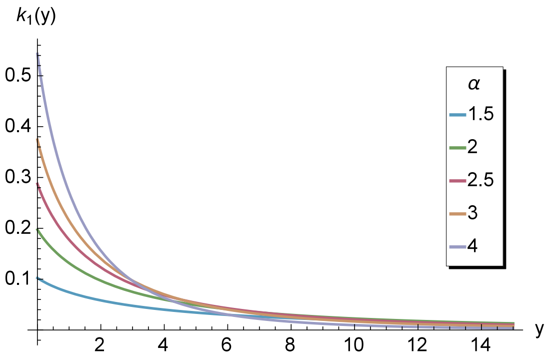
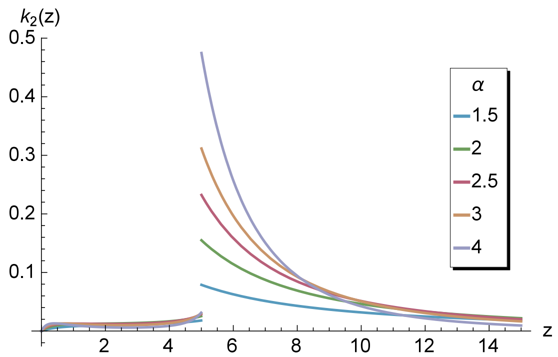


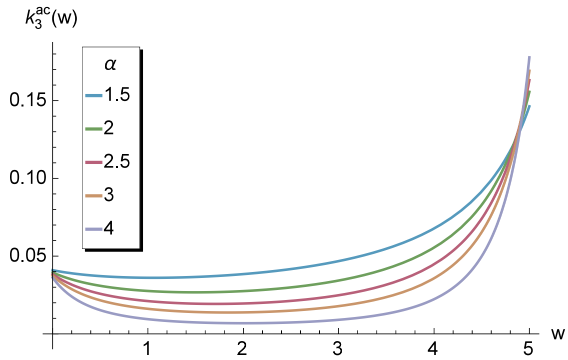
(a) Ruin probability



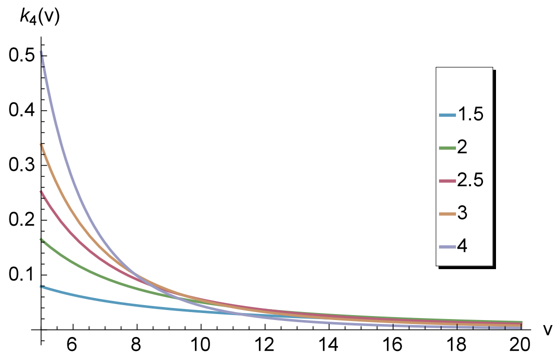
(b) Deficit at ruin



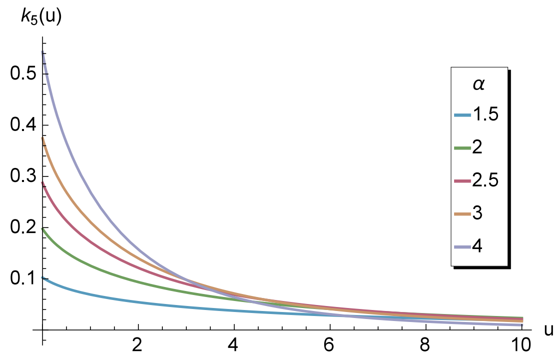
(c) Surplus prior to ruin



(d) Last minimum of the surplus before ruin



(e) Maximum before the ruin



(f) Maximal severity of ruin

Figure 1: Distributions of ruin-related quantities

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