On Langlands quotients of the generalized principal series isomorphic to their Aubert duals

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March 1, 2017

Abstract

We determine under which conditions is the Langlands quotient of an induced representation of the form $\delta \rtimes \sigma$, where δ is an irreducible essentially square-integrable representation of a general linear group and σ is a discrete series representation of the classical p-adic group, isomorphic to its Aubert dual.

1 Introduction

Let F denote a non-archimedean local field and let G_n stand for the symplectic or (full) orthogonal group having split rank n. The involution on the Grothendieck group of the smooth finite-length representations of a reductive group has been intensively studied by many authors, and we use an involution defined for general reductive p-adic groups in [2] and [18]. This involution is known as the Aubert involution and the image of a representation under this involution is called the Aubert dual of a representation. In this paper we regard the Aubert dual of an admissible finite length representation as a genuine representation, taking the $+$ or $-$ sign in such way that we obtain the positive element in the appropriate Grothendieck group.

MSC2000: primary 22E35; secondary 22E50, 11F70
Keywords: discrete series, classical p-adic groups, Aubert involution

The Aubert involution has a number of prominent applications in the representation theory of classical p-adic groups, and one would also like to gain a deeper knowledge on the explicit structure of the Aubert duals of irreducible representations.

In our previous work ([11, 12]), we obtained an explicit description of the Aubert duals of certain classes of discrete series representations of G_n, and in this paper we use developed methods to identify certain classes of irreducible representations which are fixed by the Aubert involution, i.e., which are isomorphic to their Aubert duals. We tackle this problem for the Langlands quotients of the generalized principal series of the group G_n. We note that the generalized principal series is an induced representation of the form $\delta \rtimes \sigma$, obtained by the parabolic induction with respect to the maximal parabolic subgroup, where the inducing representation $\delta \otimes \sigma$ has an irreducible essentially square-integrable representation on the general linear group part and an irreducible square-integrable representation of the classical group part. If $\nu^x \delta$ is unitarizable for $x < 0$, where $\nu = |\det|_F$, then the generalized principal series $\delta \rtimes \sigma$ has a unique irreducible (Langlands) quotient, which is also isomorphic to the unique irreducible subrepresentation of $\widetilde{\delta} \rtimes \sigma$. Such irreducible non-tempered representations can be observed as the first step in the Langlands classification of the non-unitary dual of G_n.

To obtain the necessary conditions under which the Langlands quotient of the generalized principal series is isomorphic to its Aubert dual, we use the Jacquet modules method and some elementary properties of the Aubert involution, together with descriptions of the Jacquet modules of certain discrete series representations, obtained in [7, 10].

Afterwards, we explicitly determine the Aubert duals of Langlands quotients satisfying the obtained necessary conditions, using methods introduced in [11], and further developed in [12]. Perhaps a bit surprisingly, an important role in such a procedure is, in the considered case, played by the composition factors of the generalized principal series $\delta \rtimes \sigma$ with a strongly positive σ, obtained in [17] and [9, Proposition 3.2]. Such a description of the composition factors enables us to control the Jacquet modules of the investigated non-tempered representations, similarly as in [8].

We summarize our main results in the following theorem.

Theorem 1.1. *The Langlands quotient of the generalized principal series $\delta([\nu^x \rho, \nu^y \rho]) \rtimes \sigma$, $x + y > 0$, is isomorphic to its Aubert dual if and only if one of the following holds:*

(1) The discrete series representation σ is cuspidal, $x = y$, and $\nu^x \rho \rtimes \sigma$ is irreducible.

(2) The discrete series representation σ is cuspidal, $x = 0$, $y = 1$, and $\rho \rtimes \sigma$ reduces.

(3) The cuspidal representation ρ is self-contragredient, the induced representation $\nu^\alpha \rho \rtimes \sigma_{cusp}$ reduces for $\alpha > 0$ (here σ_{cusp} stands for the partial cuspidal support of σ), $y = \alpha + 1$, and one of the following holds:

(i) x is a half-integer, $\frac{3}{2} \leq x \leq \alpha$, and σ is the unique irreducible subrepresentation of the induced representation

$$\nu^x \rho \times \nu^{x+1} \rho \times \cdots \times \nu^\alpha \rho \rtimes \sigma_{cusp},$$

(ii) x is a positive integer, $x \leq \alpha$, and σ is the unique irreducible subrepresentation of the induced representation

$$\nu^x \rho \times \nu^{x+1} \rho \times \cdots \times \nu^\alpha \rho \rtimes \sigma_{cusp},$$

(iii) $x = 0$ and σ is the unique irreducible subrepresentation of the induced representation

$$\nu \rho \times \nu^2 \rho \times \cdots \times \nu^\alpha \rho \rtimes \sigma_{cusp}.$$

We will now describe the contents of the paper in more details. In the following section we set up the notation and terminology, and prove some technical results which will be helpful in our investigation. In the third section we state and prove our main results, using a case-by-case consideration.

The author would like to thank the referee for helping to improve the presentation style and for the help with Proposition 3.2.

This work has been supported by Croatian Science Foundation under the project 9364.

2 Notation and preliminaries

Let F denote a non-archimedean local field of the characteristic different from two.

Let us first recall the definition of the Aubert involution and its basic properties.

For a connected reductive p-adic group G defined over F, let Σ denote the set of roots of G with respect to a fixed minimal parabolic subgroup and let Δ stand for a basis of Σ. For $\Theta \subseteq \Delta$, we let P_Θ be the standard parabolic subgroup of G corresponding to Θ and let M_Θ be the standard Levi factor of G corresponding to Θ.

For a parabolic subgroup P of G with the Levi factor M and a representation σ of M, we denote by $i_M(\sigma)$ a normalized parabolically induced representation of G_n induced from σ. For an admissible finite length representation σ of G, the normalized Jacquet module of σ with respect to the standard parabolic subgroup having Levi factor equal to M will be denoted by $r_M(\sigma)$. We recall the following definition and results from [2, 3]:

Theorem 2.1. *Define the operator on the Grothendieck group of admissible representations of finite length of G by*

$$D_G = \sum_{\Theta \subseteq \Delta} (-1)^{|\Theta|} i_{M_\Theta} \circ r_{M_\Theta}.$$

Operator D_G has the following properties:

(1) D_G is an involution.

(2) D_G takes irreducible representations to irreducible ones.

(3) If σ is an irreducible cuspidal representation, then $D_G(\sigma) = (-1)^{|\Delta|}\sigma$.

(4) For a standard Levi subgroup $M = M_\Theta$, we have

$$D_G \circ i_M = i_M \circ D_M.$$

(5) For a standard Levi subgroup $M = M_\Theta$, we have

$$r_M \circ D_G = Ad(w) \circ D_{w^{-1}(M)} \circ r_{w^{-1}(M)},$$

where w is the longest element of the set $\{w \in W : w^{-1}(\Theta) > 0\}$.

Let us now describe the groups that we consider. We look at the usual towers of orthogonal or symplectic groups $G_n = G(V_n)$ that are the groups of isometries of F-spaces $(V_n, (\ ,\ ))$, $n \geq 0$, where the form $(\ ,\ )$ is non-degenerate and it is skew-symmetric if the tower is symplectic and symmetric otherwise. The set of standard parabolic subgroups will be fixed in a usual way. Then the Levi factors of standard parabolic subgroups have the form $M \cong GL(n_1, F) \times GL(n_2, F) \times \cdots \times GL(n_k, F) \times G_{n'}$. If δ_i is a representation of $GL(n_i, F)$, for $i = 1, 2, \ldots, k$, and τ a representation of $G_{n'}$, the induced representation $i_M(\delta_1 \otimes \delta_2 \otimes \cdots \otimes \delta_k \otimes \tau)$ will be denoted by $\delta_1 \times \delta_2 \times \cdots \times \delta_k \rtimes \tau$. We use a similar notation to denote a parabolically induced representation of $GL(m, F)$.

If π is an irreducible representation of G_n, we denote by $\hat{\pi}$ the representation $\pm D_{G_n}(\pi)$, taking the sign $+$ or $-$ such that $\hat{\pi}$ is a positive element in the Grothendieck group of finite-length admissible representations of G_n. We call $\hat{\pi}$ the Aubert dual of π.

By $\mathrm{Irr}(G_n)$ we denote the set of all irreducible admissible representations of G_n. Furthermore, let $R(G_n)$ denote the Grothendieck group of admissible representations of finite length of G_n and define $R(G) = \oplus_{n \geq 0} R(G_n)$. Similarly, let $\mathrm{Irr}(GL(n, F))$ denote the set of all irreducible admissible representations of $GL(n, F)$, let $R(GL(n, F))$ denote the Grothendieck group of admissible representations of finite length of $GL(n, F)$ and define $R(GL) = \oplus_{n \geq 0} R(GL(n, F))$.

The generalized principal series are the induced representations of the form $\delta \rtimes \sigma$, where $\delta \in R(GL)$ is an irreducible essentially square-integrable representation and $\sigma \in R(G)$ is a discrete series representation.

There is a unique $e(\delta) \in \mathbb{R}$ such that $\nu^{-e(\delta)}\delta$ is unitarizable, where $\nu = |\det|_F$. If $e(\delta) > 0$, the generalized principal series $\delta \rtimes \sigma$ has a unique irreducible (Langlands) quotient, which is also the unique irreducible subrepresentation of $\widetilde{\delta} \rtimes \sigma$, where $\widetilde{\delta}$ denotes the contragredient of δ.

By the results of [23], such representation δ is attached to the segment and we write $\delta = \delta([\nu^a\rho, \nu^b\rho])$, where $a, b \in \mathbb{R}$ are such that $b - a$ is a nonnegative integer and $\rho \in \mathrm{Irr}(GL(n, F))$ is an unitary cuspidal representation. We recall that $\delta([\nu^a\rho, \nu^b\rho])$ is the unique irreducible subrepresentation of the induced representation $\nu^b\rho \times \nu^{b-1}\rho \times \cdots \times \nu^a\rho$.

For our Jacquet module considerations is more covenient to use the subrepresentation version of the Langlands classification and write a non-tempered irreducible representation π of G_n as the unique irreducible (Langlands) subrepresentation of the induced representation of the form $\delta_1 \times \delta_2 \times \cdots \times \delta_k \rtimes \tau$, where $\tau \in \mathrm{Irr}(G_m)$ is a tempered representation, $\delta_i \in \mathrm{Irr}(GL(n_i, F))$

is an essentially square-integrable representation attached to the segment $[\nu^{a_i}\rho_i, \nu^{b_i}\rho_i]$ for $i = 1, 2, \ldots, k$, and $a_1 + b_1 \leq a_2 + b_2 \leq \cdots \leq a_k + b_k < 0$. In this case, we write $\pi = L(\delta_1 \times \delta_2 \times \cdots \times \delta_k \rtimes \tau)$.

For $\sigma \in \mathrm{Irr}(G_n)$ and $1 \leq k \leq n$ we denote by $r_{(k)}(\sigma)$ the normalized Jacquet module of σ with respect to the parabolic subgroup having Levi factor equal to $GL(k, F) \times G_{n-k}$. We identify $r_{(k)}(\sigma)$ with its semisimplification in $R(GL(k, F)) \otimes R(G_{n-k})$ and consider

$$\mu^*(\sigma) = 1 \otimes \sigma + \sum_{k=1}^{n} r_{(k)}(\sigma) \in R(GL) \otimes R(G).$$

The following result, derived in [21], presents the crucial structural formula for our calculations of Jacquet modules.

Theorem 2.2. *Let ρ be an irreducible cuspidal representation of $GL(m, F)$ and $k, l \in \mathbb{R}$ such that $k + l \in \mathbb{Z}_{\geq 0}$. Let σ be an admissible representation of finite length of G_n. Write $\mu^*(\sigma) = \sum_{\tau, \sigma'} \tau \otimes \sigma'$. Then we have*

$$\begin{aligned}\mu^*(\delta([\nu^{-k}\rho, \nu^l\rho]) \rtimes \sigma) = \sum_{i=-k-1}^{l} \sum_{j=i}^{l} \sum_{\tau, \sigma'} & \delta([\nu^{-i}\widetilde{\rho}, \nu^k\widetilde{\rho}]) \times \delta([\nu^{j+1}\rho, \nu^l\rho]) \times \tau \otimes \\ & \otimes \delta([\nu^{i+1}\rho, \nu^j\rho]) \rtimes \sigma'.\end{aligned}$$

We omit $\delta([\nu^x\rho, \nu^y\rho])$ if $x > y$.

We note the following direct consequence of the previous theorem and of the Casselman square-integrability criterion.

Corollary 2.3. *Let ρ denote an irreducible self-contragedient cuspidal representation of $GL(m, F)$ and $k, l \in \mathbb{R}$ such that $k + l \in \mathbb{Z}_{\geq 0}$ and $k > 0$. If $\sigma \in Irr(G_n)$ is a discrete series representation, then $\mu^*(\delta([\nu^{-k}\rho, \nu^l\rho]) \rtimes \sigma)$ contains an irreducible constituent of the form $\nu^r\rho' \otimes \pi$, where $r \leq 0$ and ρ' is cuspidal, if and only if $l \leq 0$. Furthermore, if $l \leq 0$ and $\mu^*(\delta([\nu^{-k}\rho, \nu^l\rho]) \rtimes \sigma)$ contains an irreducible constituent of the form $\nu^r\rho' \otimes \pi$, where $r \leq 0$ and ρ' is cuspidal, then $r = l$ and $\rho' \cong \rho$.*

The following technical result will be used several times in the paper.

Lemma 2.4. *Suppose that $\pi \in Irr(G_n)$ is a subrepresentation of an induced representation of the form $\nu^{a_1}\rho_1 \times \nu^{a_2}\rho_2 \times \cdots \times \nu^{a_k}\rho_k \rtimes \pi_1$, where*

$\rho_i \in Irr(GL(m_i, F))$ is a unitary cuspidal self-contragredient representation for $i = 1, 2, \ldots, k$, and π_1 is an admissible representation of finite length. Then the Jacquet module of $\widehat{\pi}$ with respect to an appropriate parabolic subgroup contains an irreducible representation of the form $\nu^{-a_1}\rho_1 \otimes \nu^{-a_2}\rho_2 \otimes \cdots \otimes \nu^{-a_k}\rho_k \otimes \pi_2$.

Proof. Frobenius reciprocity and transitivity of Jacquet modules imply that there is an irreducible cuspidal representation π' such that the Jacquet module of π with respect to an appropriate parabolic subgroup contains the irreducible cuspidal representation $\nu^{a_1}\rho_1 \otimes \nu^{a_2}\rho_2 \otimes \cdots \otimes \nu^{a_k}\rho_k \otimes \pi'$. Using Theorem 2.1, we obtain the claim of the lemma. $\square$

We note that one can also deduce that the representation π_2 appearing in the statement of the previous lemma is in fact isomorphic to $\widehat{\pi_1}$. However, we will not use this in the sequel.

We will now recall the Mœglin-Tadić classification of discrete series for groups that we consider. Every discrete series representation of G_n is uniquely determined by three invariants: the partial cuspidal support, the Jordan block and the ϵ-function.

The partial cuspidal support of a discrete series $\sigma \in \mathrm{Irr}(G_n)$ is an irreducible cuspidal representation σ_{cusp} of some G_m such that there is an irreducible admissible representation π of $GL(n-m, F)$ such that σ is a subrepresentation of $\pi \rtimes \sigma_{cusp}$.

The Jordan block of σ, denoted by $\mathrm{Jord}(\sigma)$, is the set of all pairs (c, ρ) where ρ is an irreducible cuspidal self-contragredient representation of some $GL(n_\rho, F)$ and $c > 0$ is an integer such that the following two conditions are satisfied:

(1) c is even if and only if $L(s, \rho, r)$ has a pole at $s = 0$. The local L-function $L(s, \rho, r)$ is the one defined by Shahidi (see for instance [19], [20]), where $r = \bigwedge^2 \mathbb{C}^{n_\rho}$ is the exterior-square representation of the standard representation on $\mathbb{C}^{n_\rho}$ of $GL(n_\rho, \mathbb{C})$ if G_n is a symplectic or even-orthogonal group, and $r = \mathrm{Sym}^2\mathbb{C}^{n_\rho}$ is the symmetric-square representation of the standard representation on $\mathbb{C}^{n_\rho}$ of $GL(n_\rho, \mathbb{C})$ if G_n is an odd-orthogonal group.

(2) The induced representation $\delta([\nu^{-(c-1)/2}\rho, \nu^{(c-1)/2}\rho]) \rtimes \sigma$ is irreducible.

To explain the notion of the ϵ-function, we will first define Jordan triples. This are triples of the form $(\mathrm{Jord}, \sigma', \epsilon)$ where

- σ' is an irreducible cuspidal representation of some G_n.

- Jord is the finite set (possibly empty) of ordered pairs (c, ρ), where $\rho \in \mathrm{Irr}(GL(n_\rho, F))$ is a self-contragredient cuspidal representation, and c is a positive integer which is even if and only if $L(s, \rho, r)$ has a pole at $s = 0$ (for the local L-function as above). For an irreducible self-contragredient cuspidal representation ρ of $GL(n_\rho, F)$ we write $\mathrm{Jord}_\rho = \{c : (c, \rho) \in \mathrm{Jord}\}$. If $\mathrm{Jord}_\rho \neq \emptyset$ and $c \in \mathrm{Jord}_\rho$, we put $c_- = \max\{d \in \mathrm{Jord}_\rho : d < c\}$, if it exists.

- ϵ is the function defined on a subset of $\mathrm{Jord} \cup (\mathrm{Jord} \times \mathrm{Jord})$ and attains the values 1 and -1. If $(c, \rho) \in \mathrm{Jord}$, then $\epsilon(c, \rho)$ is not defined if and only if c is odd and $(c', \rho) \in \mathrm{Jord}(\sigma')$ for some positive integer c'. Next, ϵ is defined on a pair $((c, \rho), (c', \rho')) \in \mathrm{Jord} \times \mathrm{Jord}$ if and only if $\rho \cong \rho'$ and $c \neq c'$.

Suppose that, for the Jordan triple $(\mathrm{Jord}, \sigma', \epsilon)$, there is a $(c, \rho) \in \mathrm{Jord}$ such that $\epsilon((c_-, \rho), (c, \rho)) = 1$. If we put $\mathrm{Jord}' = \mathrm{Jord} \setminus \{(c_-, \rho), (c, \rho)\}$ and consider the restriction ϵ' of ϵ to $\mathrm{Jord}' \cup (\mathrm{Jord}' \times \mathrm{Jord}')$, we obtain a new Jordan triple $(\mathrm{Jord}', \sigma', \epsilon')$, and we say that such Jordan triple is subordinated to $(\mathrm{Jord}, \sigma', \epsilon)$.

We say that the Jordan triple $(\mathrm{Jord}, \sigma', \epsilon)$ is a triple of alternated type if $\epsilon((c_-, \rho), (c, \rho)) = -1$ whenever c_- is defined and there is an increasing bijection $\phi_\rho : \mathrm{Jord}_\rho \rightarrow \mathrm{Jord}'_\rho(\sigma')$, where $\mathrm{Jord}'_\rho(\sigma')$ equals $\mathrm{Jord}_\rho(\sigma') \cup \{0\}$ if a is even and $\epsilon(\min \mathrm{Jord}_\rho, \rho) = 1$, and $\mathrm{Jord}'_\rho(\sigma')$ equals $\mathrm{Jord}_\rho(\sigma')$ otherwise.

The Jordan triple $(\mathrm{Jord}, \sigma', \epsilon)$ dominates the Jordan triple $(\mathrm{Jord}', \sigma', \epsilon')$ if there is a sequence of Jordan triples $(\mathrm{Jord}_i, \sigma', \epsilon_i)$, $0 \leq i \leq k$, such that $(\mathrm{Jord}_0, \sigma', \epsilon_0) = (\mathrm{Jord}, \sigma', \epsilon)$, $(\mathrm{Jord}_k, \sigma', \epsilon_k) = (\mathrm{Jord}', \sigma', \epsilon')$, and $(\mathrm{Jord}_i, \sigma', \epsilon_i)$ is subordinated to $(\mathrm{Jord}_{i-1}, \sigma', \epsilon_{i-1})$ for $i \in \{1, 2, \ldots, k\}$. The Jordan triple $(\mathrm{Jord}, \sigma', \epsilon)$ is called admissible if it dominates a triple of alternated type.

Classification given in [14] and [16] states that there is a one-to-one correspondence between the set of all discrete series in $\mathrm{Irr}(G)$ and the set of all admissible triples $(\mathrm{Jord}, \sigma', \epsilon)$ given by $\sigma = \sigma_{(\mathrm{Jord}, \sigma', \epsilon)}$, such that $\sigma_{cusp} = \sigma'$ and $\mathrm{Jord}(\sigma) = \mathrm{Jord}$. Furthermore, if $(c, \rho) \in \mathrm{Jord}$ is such that $\epsilon((c_-, \rho), (c, \rho)) = 1$, we set $\mathrm{Jord}' = \mathrm{Jord} \setminus \{(c_-, \rho), (c, \rho)\}$ and consider the restriction ϵ' of ϵ to $\mathrm{Jord}' \cup (\mathrm{Jord}' \times \mathrm{Jord}')$. Then $(\mathrm{Jord}', \sigma', \epsilon')$ is an admissible triple and σ is a subrepresentation of $\delta([\nu^{-(c_- - 1)/2}\rho, \nu^{(c-1)/2}\rho]) \rtimes \sigma_{(\mathrm{Jord}', \sigma', \epsilon')}$.

An irreducible representation $\sigma \in R(G)$ is called strongly positive if for every embedding

$$\sigma \hookrightarrow \nu^{s_1}\rho_1 \times \nu^{s_2}\rho_2 \times \cdots \times \nu^{s_k}\rho_k \rtimes \sigma_{cusp},$$

where $\rho_i \in R$, $i = 1, 2, \ldots, k$, are irreducible cuspidal unitary representations and $\sigma_{cusp} \in R(G)$ is an irreducible cuspidal representation, we have $s_i > 0$ for $i = 1, 2, \ldots, k$.

It has been shown in [14, Proposition 5.3] and [16, Proposition 7.1] that triples of alternated type correspond to strongly positive discrete series. Let us recall an inductive description of the non-cuspidal strongly positive discrete series, obtained in [5, Theorem 5.1], which also holds in the classical group case.

Proposition 2.5. *Suppose that $\sigma_{sp} \in R(G)$ is an irreducible strongly positive representation and let $\rho \in Irr(GL(m, F))$ denote an irreducible cuspidal representation such that some twist of ρ appears in the cuspidal support of σ_{sp}. We denote by σ_{cusp} the partial cuspidal support of σ_{sp}. Then there exist unique $a, b \in \mathbb{R}$ such that $a > 0, b > 0$, $b - a$ is a non-negative integer, and a unique irreducible strongly positive representation $\sigma_{sp}^{(1)}$ without $\nu^a\rho$ in the cuspidal support, with the property that σ_{sp} is a unique irreducible subrepresentation of $\delta([\nu^a\rho, \nu^b\rho]) \rtimes \sigma_{sp}^{(1)}$. Furthermore, there is a non-negative integer l such that $\alpha := a + l > 0$ and $\nu^\alpha\rho \rtimes \sigma_{cusp}$ reduces. If $l = 0$ there are no twists of ρ appearing in the cuspidal support of $\sigma_{sp}^{(1)}$, and if $l > 0$ there exist a unique $b' > b$ and a unique strongly positive discrete series $\sigma_{sp}^{(2)}$, which contains neither $\nu^a\rho$ nor $\nu^{a+1}\rho$ in its cuspidal support, such that $\sigma_{sp}^{(1)}$ can be written as the unique irreducible subrepresentation of $\delta([\nu^{a+1}\rho, \nu^{b'}\rho]) \rtimes \sigma_{sp}^{(2)}$.*

We say that a representation $\sigma \in \mathrm{Irr}(G_n)$ belongs to the set $D(\rho_1, \ldots, \rho_m; \sigma_{cusp})$ if every element of the cuspidal support of σ belongs to the set $\{\nu^x\rho_1, \ldots, \nu^x\rho_m, \sigma_{cusp} : x \in \mathbb{R}\}$, where $\rho_1, \ldots, \rho_m$ are mutually non-isomorphic irreducible cuspidal representations of general linear groups and σ_{cusp} is an irreducible cuspidal representation of $G_{n'}$, for some $n' \leq n$.

We note that for a self-contragredient cuspidal $\rho \in \mathrm{Irr}(GL(m, F))$ and a cuspidal $\sigma_{cusp} \in \mathrm{Irr}(G_n)$, there is a unique non-negative α such that the induced representation $\nu^\alpha\rho \rtimes \sigma_{cusp}$ reduces, and it follows from [1] and [15, Théorème 3.1.1] that α is a half-integer.

Directly from the previous proposition we obtain

Proposition 2.6. *Let $\sigma_{sp} \in Irr(G_n)$ denote a strongly positive representation and suppose that $\sigma_{sp} \in D(\rho; \sigma_{cusp})$, for an irreducible cuspidal self-contragredient representation ρ. Let α stand for the unique non-negative half-integer such that $\nu^{\alpha}\rho \rtimes \sigma_{cusp}$ reduces, and let $k = \lceil \alpha \rceil$, the smallest integer which is not smaller than α. If $k = 0$, then $\sigma_{sp} \cong \sigma_{cusp}$. Otherwise, there exists a unique k-tuple $(a_1, a_2, \ldots, a_k)$ such that $a_i - \alpha \in \mathbb{Z}$ for $i = 1, 2, \ldots, k$, $-1 < a_1 < a_2 < \ldots < a_k$, and σ_{sp} is the unique irreducible subrepresentation of the induced representation*

$$\delta([\nu^{\alpha-k+1}\rho, \nu^{a_1}\rho]) \times \delta([\nu^{\alpha-k+2}\rho, \nu^{a_2}\rho]) \times \cdots \times \delta([\nu^{\alpha}\rho, \nu^{a_k}\rho]) \rtimes \sigma_{cusp}.$$

3 Langlands quotients fixed by the Aubert involution

In this section we describe all Langlands quotients of the generalized principal series $\delta \rtimes \sigma$ which are isomorphic to their Aubert duals, using a case-by-case considerations. We write $\delta = \delta([\nu^x\rho, \nu^y\rho])$, for x, y such that $x + y > 0$. The induced representation $\delta \rtimes \sigma$ then contains a unique irreducible (Langlands) quotient, which is also the unique irreducible subrepresentation of the induced representation $\delta([\nu^{-y}\widetilde{\rho}, \nu^{-x}\widetilde{\rho}]) \rtimes \sigma$, and in what follows will be denoted by π, i.e., let $\pi = L(\widetilde{\delta} \rtimes \sigma)$.

Lemma 3.1. *If π is isomorphic to $\widehat{\pi}$, then $x \geq 0$.*

Proof. Since $x + y > 0$, we obviously have $y > 0$. Suppose that $x < 0$. From the embedding $\pi \hookrightarrow \nu^{-x}\widetilde{\rho} \times \delta([\nu^{-y}\widetilde{\rho}, \nu^{-x-1}\widetilde{\rho}]) \rtimes \sigma$ and transtivity of Jacquet modules, in the same way as in the proof of Lemma 2.4 we obtain that $\mu^*(\widehat{\pi})$ contains an irreducible constituent of the form $\nu^x\rho \otimes \pi'$. Since $y \neq x$, it follows directly from the structural formula that $\mu^*(\delta([\nu^{-y}\widetilde{\rho}, \nu^{-x}\widetilde{\rho}]) \rtimes \sigma)$ does not contain such an irreducible constituent, so $\widehat{\pi}$ is not isomorphic to π, a contradiction. □

Let us first consider the case of cuspidal σ.

Proposition 3.2. *Suppose that $\sigma \in Irr(G_n)$ is a cuspidal representation. Then the representation π is isomorphic to its Aubert dual if and only if one of the following holds:*

(1) $x = y > 0$ and the induced representation $\nu^x\rho \rtimes \sigma$ is irreducible,

(2) $(x, y) = (0, 1)$ *and the induced representation* $\rho \rtimes \sigma$ *reduces.*

Proof. We have already seen that if $\pi \cong \widehat{\pi}$ then $x \geq 0$. In the same way as in the proof of the previous lemma we deduce that $\mu^*(\widehat{\pi}) \geq \nu^{-x}\widetilde{\rho} \otimes \pi'$, for some irreducible representation π'. From the structural formula we see that this is possible only if either $x = y$ or $(x, \widetilde{\rho}) = (0, \rho)$.

Let us first consider the case $x = y$. Note that then we have $x > 0$. Furthermore, if $\nu^x\rho \rtimes \sigma$ reduces, it follows from [17, Proposition 3.1(i)] that $\mu^*(\pi)$ does not contain an irreducible constituent of the form $\nu^{-x}\widetilde{\rho} \otimes \pi'$. Consequently, if $\pi \cong \widehat{\pi}$ and $x = y$, then $\nu^x\rho \rtimes \sigma$ is irreducible.

Conversely, if the induced representation $\nu^x\rho \rtimes \sigma$ is irreducible then $\pi \cong \nu^x\rho \rtimes \sigma$ and from the part (4) of Theorem 2.1 we have $\widehat{\pi} \leq \nu^x\rho \rtimes \sigma \cong \pi$, so π is isomorphic to its Aubert dual.

Let us now assume that $x = 0$ and $\rho \cong \widetilde{\rho}$. Let s denote a unique non-negative half-integer such that $\nu^s\rho \rtimes \sigma$ reduces. We obviously have $y > 0$ and there are two possibilities to consider.

First, suppose that $y = s$ and let σ_{sp} stand for a unique irreducible subrepresentation of the induced representation $\nu\rho \times \nu^2\rho \times \cdots \times \nu^y\rho \rtimes \sigma$. It follows from [5, Theorem 4.6] that σ_{sp} is a strongly positive discrete series and one can see directly from [16, Proposition 2.1] that the induced representation $\rho \rtimes \sigma_{sp}$ reduces. By [22, Section 4], there is a unique irreducible subrepresentation τ of $\rho \rtimes \sigma_{sp}$ such that $\mu^*(\tau)$ does not contain an irreducible constituent of the form $\nu\rho \otimes \pi'$. Furthermore, if $\mu^*(\tau)$ contains an irreducible constituent of the form $\nu^a\rho \otimes \pi_1'$, then $a = 0$. Thus, if $\mu^*(\widehat{\tau})$ contains an irreducible constituent of the form $\nu^a\rho \otimes \pi_2'$, then $a = 0$. Since τ is a subrepresentation of $\rho \times \nu\rho \times \nu^2\rho \times \cdots \times \nu^y\rho \rtimes \sigma$, using Lemma 2.4 we deduce that $\widehat{\tau}$ is a subrepresentation of $\rho \times \nu^{-1}\rho \times \nu^{-2}\rho \times \cdots \times \nu^{-y}\rho \rtimes \sigma$ and, in the same way as in the proof of [11, Lemma 3.4], we deduce that $\widehat{\tau}$ is a subrepresentation of $\delta([\nu^{-y}\rho, \rho]) \rtimes \sigma$. Consequently, $\widehat{\tau} \cong \pi$ and $\widehat{\pi} \ncong \pi$.

Now, suppose that $y \neq s$. If $y \geq 2$, we have the following embedding and isomorphism:

$$\pi \hookrightarrow \delta([\nu^{-y+1}\rho, \rho]) \times \nu^{-y}\rho \rtimes \sigma \cong \nu^y\rho \times \delta([\nu^{-y+1}\rho, \rho]) \rtimes \sigma.$$

Lemma 2.4 implies that $\mu^*(\widehat{\pi})$ contains an irreducible constituent of the form $\nu^{-y}\rho \otimes \pi'$ and it follows directly from the structural formula that $\widehat{\pi} \ncong \pi$. Thus, we can assume that $y = 1$. If $s > 0$, then $s \neq y$ and [17, Theorem 4.1(i)] imply that $\delta([\nu^{-1}\rho, \rho]) \rtimes \sigma$ is irreducible and $\pi \cong \delta([\rho, \nu\rho]) \rtimes \sigma$. Consequently, if $s > 0$ then $\mu^*(\widehat{\pi})$ contains an irreducible constituent of the form $\nu^{-1}\rho \otimes \pi'$, and it follows directly from the structural formula that $\widehat{\pi} \not\cong \pi$.

It remains to consider the case $s = 0$. According to [17, Theorem 2.1], in $R(G)$ we have

$$\delta([\rho, \nu\rho]) \rtimes \sigma = \pi + \sigma_{ds}^{(1)} + \sigma_{ds}^{(2)},$$

where $\sigma_{ds}^{(1)}$ and $\sigma_{ds}^{(2)}$ are mutually non-isomorphic discrete series subrepresentations of $\delta([\rho, \nu\rho]) \rtimes \sigma$. Frobenius reciprocity implies that both $\mu^*(\sigma_{ds}^{(1)})$ and $\mu^*(\sigma_{ds}^{(2)})$ contain irreducible constituents of the form $\nu\rho \otimes \pi'$. It follows from the structural formula that only irreducible constituents of the form $\nu\rho \otimes \pi'$ appearing in $\mu^*(\delta([\rho, \nu\rho]) \rtimes \sigma)$ are $\nu\rho \otimes \tau_1$ and $\nu\rho \otimes \tau_{-1}$, where τ_1 and τ_{-1} are irreducible mutually non-isomorphic tempered representations such that in $R(G)$ we have $\rho \rtimes \sigma = \tau_1 + \tau_{-1}$. Furthermore, both $\nu\rho \otimes \tau_1$ and $\nu\rho \otimes \tau_{-1}$ appear in $\mu^*(\delta([\rho, \nu\rho]) \rtimes \sigma)$ with multiplicity one. Thus, $\mu^*(\pi)$ does not contain an irreducible constituent of the form $\nu\rho \otimes \pi'$ and, consequently, $\mu^*(\widehat{\pi})$ does not contain an irreducible constituent of the form $\nu^{-1}\rho \otimes \pi''$. Since π is a subrepresentation of $\rho \times \nu\rho \rtimes \sigma$, using Lemma 2.4 we obtain that $\widehat{\pi}$ is a subrepresentation of $\rho \times \nu^{-1}\rho \rtimes \sigma$ and it follows that $\widehat{\pi}$ is a unique irreducible subrepresentation of $\delta([\nu^{-1}\rho, \rho]) \rtimes \sigma$, i.e., $\pi \cong \widehat{\pi}$. This completes the proof. □

In the rest of this section we assume that σ is a non-cuspidal discrete series representation, and let σ_{cusp} denote the partial cuspidal support of σ.

Lemma 3.3. *If π is isomorphic to $\widehat{\pi}$, then $\sigma \in D(\rho; \sigma_{cusp})$. In particular, ρ is self-contragredient.*

Proof. Suppose that $\sigma \notin D(\rho; \sigma_{cusp})$. Then there is an embedding of the form $\sigma \hookrightarrow \nu^a\rho' \rtimes \sigma'$, such that $a > 0$, ρ' is an irreducible self-contragredient cuspidal representation which is not isomorphic to ρ, and σ' is irreducible. We have $\pi \hookrightarrow \widetilde{\delta} \times \nu^a\rho' \rtimes \sigma' \cong \nu^a\rho' \times \widetilde{\delta} \rtimes \sigma'$, and Lemma 2.4, together with transitivity of Jacquet modules, implies that Jacquet module of $\widehat{\pi}$ with respect to an appropriate parabolic subgroup contains an irreducible representation of the form $\nu^{-a}\rho' \otimes \sigma''$. Since σ is square-integrable, it follows that $\mu^*(\delta \rtimes \sigma)$ does not contain an irreducible constituent of the form $\nu^{-a}\rho' \otimes \sigma''$. Thus, π is not isomorphic to $\widehat{\pi}$, a contradiction. □

According to the previous lemma, in what follows we can assume that ρ is a self-contragredient representation and that $\sigma \in D(\rho; \sigma_{cusp})$. We denote by α a unique non-negative half-integer s such that $\nu^s\rho \rtimes \sigma_{cusp}$ reduces.

The following result presents the crucial step towards our description.

Theorem 3.4. *If π is isomorphic to $\widehat{\pi}$, then σ is a strongly positive discrete series. In particular, $\alpha > 0$.*

Proof. Suppose, on the contrary, that σ is not a strongly positive representation and let $(\mathrm{Jord}(\sigma), \sigma_{cusp}, \epsilon_\sigma)$ denote the corresponding Jordan triple. Since we have $\sigma \in D(\rho; \sigma_{cusp})$, there is $c \in \mathrm{Jord}_\rho(\sigma)$ such that $c_{\_}$ is defined and $\epsilon_\sigma((c_{\_}, \rho), (c, \rho)) = 1$. Also, σ is a subrepresentation of an induced representation of the form $\delta([\nu^{-\frac{c_{\_}-1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho]) \rtimes \sigma'$, for an appropriate discrete series σ'. Using [16, Lemma 3.2], we deduce that σ is a subrepresentation of an induced representation of the form $\nu^{\frac{c-1}{2}}\rho \rtimes \pi_1$, for some irreducible π_1. Since $\frac{c-1}{2} \geq 1$ and $-x \leq 0$, if $(c, x) \neq (3, 0)$ we obtain an embedding $\pi \hookrightarrow \nu^{\frac{c-1}{2}}\rho \times \delta([\nu^{-y}\rho, \nu^{-x}\rho]) \rtimes \pi_1$. Lemma 2.4 implies that $\mu^*(\widehat{\pi})$ contains an irreducible constituent of the form $\nu^{-\frac{c-1}{2}}\rho \otimes \pi_2$, and Corollary 2.3 implies that $x = \frac{c-1}{2}$.

If $c > 3$, we also have $\frac{c-3}{2} > -\frac{c-1}{2} + 1$, which gives the following embeddings and isomorphisms:

$$\begin{aligned}
\pi &\hookrightarrow \delta([\nu^{-y}\rho, \nu^{-\frac{c-1}{2}}\rho]) \times \delta([\nu^{-\frac{c_{_}-1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho]) \rtimes \sigma' \\
&\hookrightarrow \delta([\nu^{-y}\rho, \nu^{-\frac{c-1}{2}}\rho]) \times \nu^{\frac{c-1}{2}}\rho \times \nu^{\frac{c-3}{2}}\rho \times \delta([\nu^{-\frac{c_{_}-1}{2}}\rho, \nu^{\frac{c-5}{2}}\rho]) \rtimes \sigma' \\
&\cong \nu^{\frac{c-1}{2}}\rho \times \delta([\nu^{-y}\rho, \nu^{-\frac{c-1}{2}}\rho]) \times \nu^{\frac{c-3}{2}}\rho \times \delta([\nu^{-\frac{c_{_}-1}{2}}\rho, \nu^{\frac{c-5}{2}}\rho]) \rtimes \sigma' \\
&\cong \nu^{\frac{c-1}{2}}\rho \times \nu^{\frac{c-3}{2}}\rho \times \delta([\nu^{-y}\rho, \nu^{-\frac{c-1}{2}}\rho]) \times \delta([\nu^{-\frac{c_{_}-1}{2}}\rho, \nu^{\frac{c-5}{2}}\rho]) \rtimes \sigma'.
\end{aligned}$$

Since $\pi \cong \widehat{\pi}$, Lemma 2.4 and transitivity of Jacquet modules imply that the Jacquet module of π with respect to an appropriate parabolic subgroup contains an irreducible representation of the form $\nu^{-\frac{c-1}{2}}\rho \otimes \nu^{-\frac{c-3}{2}}\rho \otimes \pi_3$.

From $\pi \hookrightarrow \delta([\nu^{-y}\rho, \nu^{-\frac{c-1}{2}}\rho]) \rtimes \sigma$, using the structural formula recalled in Theorem 2.2, we obtain that $\nu^{-\frac{c-3}{2}}\rho \otimes \pi_3 \leq \mu^*(\delta([\nu^{-y}\rho, \nu^{-\frac{c+1}{2}}\rho]) \rtimes \sigma)$, which is impossible.

It remains to consider the case $c = 3$. This directly implies that $c_{\_} = 1$ and $x \in \{0, 1\}$. In other words, σ is a subrepresentation of an induced representation of the form $\delta([\rho, \nu\rho]) \rtimes \sigma'$, where σ' is a discrete series such that 1 and 3 do not appear in $\mathrm{Jord}_\rho(\sigma')$, and it follows that σ' is a strongly positive representation, since otherwise we can apply the same arguments as before to deduce that π is not isomorphic to $\widehat{\pi}$.

Let us first assume that $\mathrm{Jord}_\rho(\sigma') \neq \emptyset$. Then, as in [5, Section 4] and [16, Proposition 2.1], we see that there is an $a \geq 2$ such that σ' is a subrepresentation of $\nu^a\rho \rtimes \sigma''$, for an appropriate strongly positive representation σ''.

If $a > 2$, we have $\sigma \hookrightarrow \nu^a\rho \times \delta([\rho, \nu\rho]) \rtimes \sigma''$. Since $x \in \{0, 1\}$, in the same way as before we deduce that $\mu^*(\pi) \geq \nu^{-a}\rho \otimes \pi'$, for an irreducible representation π', which is impossible.

If $a = 2$, the irreducible representation σ' is also a subrepresentation of an induced representation of the form $\nu^2\rho \times \nu\rho \rtimes \sigma_1$. If $x = 0$, we have the following embeddings:

$$\begin{aligned}\pi &\hookrightarrow \delta([\nu^{-y}\rho, \rho]) \times \delta([\rho, \nu\rho]) \times \nu^2\rho \times \nu\rho \rtimes \sigma_1 \\ &\hookrightarrow \rho \times \nu\rho \times \nu^2\rho \times \delta([\nu^{-y}\rho, \nu^{-1}\rho]) \times \rho \times \nu\rho \rtimes \sigma_1 \\ &\hookrightarrow \rho \times \nu\rho \times \nu^2\rho \times \nu^{-1}\rho \times \rho \times \nu\rho \times \delta([\nu^{-y}\rho, \nu^{-2}\rho]) \rtimes \sigma_1.\end{aligned}$$

Using Lemma 2.4 and transitivity of Jacquet modules, we obtain that the Jacquet module of π with respect to an appropriate parabolic subgroup contains an irreducible representation of the form $\rho \otimes \nu^{-1}\rho \otimes \nu^{-2}\rho \otimes \nu\rho \otimes \rho \otimes \nu^{-1}\rho \otimes \pi_4$. Since σ is a discrete series representation, applying the structural formula several times, we deduce that $y \geq 2$ and that $\nu\rho \otimes \rho \otimes \nu^{-1}\rho \otimes \pi_4$ is contained in the Jacquet module of $\delta([\nu^{-y}\rho, \nu^{-3}\rho]) \rtimes \sigma$ with respect to an appropriate parabolic subgroup. This directly implies that the Jacquet module of σ with respect to an appropriate parabolic subgroup contains an irreducible representation of the form $\nu\rho \otimes \rho \otimes \nu^{-1}\rho \otimes \pi_5$, contradicting the square-integrability criterion. The case $x = 1$ can be handled in the same way.

Let us now assume that $\mathrm{Jord}_\rho(\sigma') = \emptyset$. This implies that σ' is a cuspidal representation and $\rho \rtimes \sigma'$ reduces. As in the proof of Proposition 3.2, in $R(G)$ we have $\rho \rtimes \sigma' = \tau_1 + \tau_{-1}$ and there is a unique $i \in \{1, -1\}$ such that σ is the unique irreducible subrepresentation of $\nu\rho \rtimes \tau_i$ or, equivalently, such that $\mu^*(\sigma) \geq \nu\rho \otimes \tau_i$. It follows from [12, Theorem 5.1] that $\widehat{\sigma} \cong L(\nu^{-1}\rho \rtimes \tau_{-i})$, and if an irreducible constituent of the form $\nu^z\rho \otimes \pi'$ appears in $\mu^*(\widehat{\sigma})$, then $z = -1$.

Again, we comment only the case $x = 0$, since the case $x = 1$ can be handled in the same way.

We have the following embeddings:

$$\begin{aligned}\pi &\hookrightarrow \delta([\nu^{-y}\rho, \rho]) \rtimes \sigma \hookrightarrow \rho \times \nu^{-1}\rho \times \cdots \times \nu^{-y}\rho \rtimes \sigma \\ &\hookrightarrow \rho \times \nu^{-1}\rho \times \cdots \times \nu^{-y}\rho \times \nu\rho \rtimes \tau_i.\end{aligned} \tag{1}$$

Frobenius reciprocity implies that the Jacquet module of π with respect to an appropriate parabolic subgroup contains the irreducible representation

$$\rho \otimes \nu^{-1}\rho \otimes \cdots \otimes \nu^{-y}\rho \otimes \nu\rho \otimes \tau_i. \tag{2}$$

Since $\pi \cong \widehat{\pi}$, applying Theorem 2.1, part (4), to the induced representation appearing in (1), we deduce that π is an irreducible subquotient of

$$\rho \times \nu^{-1}\rho \times \cdots \times \nu^{-y}\rho \rtimes L(\nu^{-1}\rho \rtimes \tau_{-i}). \tag{3}$$

Using a repeated application of the structural formula and $\tau_i \ncong \tau_{-i}$, one can show that the irreducible representation (2) is not contained in the Jacquet module of the induced representation (3) with respect to an appropriate parabolic subgroup, a contradiction. This completes the proof. $\square$

We denote $\lceil \alpha \rceil$ by k, and let $(a_1, a_2, \ldots, a_k)$ denote a unique ordered k-tuple such that $a_i - \alpha \in \mathbb{Z}$ for $i = 1, 2, \ldots, k$, $-1 < a_1 < a_2 < \ldots < a_k$, and such that σ is the unique irreducible subrepresentation of

$$\delta([\nu^{\alpha-k+1}\rho, \nu^{a_1}\rho]) \times \delta([\nu^{\alpha-k+2}\rho, \nu^{a_2}\rho]) \times \cdots \times \delta([\nu^{\alpha}\rho, \nu^{a_k}\rho]) \rtimes \sigma_{cusp}.$$

We note that such a k-tuple exists by Proposition 2.6. Since σ is non-cuspidal, there is an $i \in \{1, 2, \ldots, k\}$ such that $a_i \geq \alpha - k + i$. Let us denote the minimal such i by $i_{\min}$.

Lemma 3.5. *If π is isomorphic to $\widehat{\pi}$, then $a_{i_{\min}} = \alpha - k + i_{\min}$ and for $j = i_{\min}, i_{\min}+1, \ldots, k-1$ we have $a_{j+1} = a_j + 1$.*

Proof. It follows from [7, Theorem 4.6], or from [13, Section 8], that σ is a subrepresentation of an induced representation of the form $\nu^{a_{i_{\min}}}\rho \rtimes \sigma_{sp}$, where σ_{sp} is a strongly positive representation. If $-x \neq a_{i_{\min}} - 1$, we have an embedding $\pi \hookrightarrow \nu^{a_{i_{\min}}}\rho \times \delta([\nu^{-y}\rho, \nu^{-x}\rho]) \rtimes \sigma_{sp}$, and an application of Corollary 2.3 and Lemma 2.4 gives $x = a_{i_{\min}}$.

If $-x = a_{i_{\min}} - 1$, since $x \geq 0$ and $a_{i_{\min}} \geq \frac{1}{2}$, it follows that $x \in \{0, \frac{1}{2}\}$. Thus, if $-x = a_{i_{\min}} - 1$ then $i_{\min} = 1$ and $a_{i_{\min}} = \alpha - k + 1$.

Let us now assume that $-x \neq a_{i_{\min}} - 1$ and $a_{i_{\min}} \geq \alpha - k + i_{\min} + 1$. It follows that $a_{i_{\min}} \geq \frac{3}{2}$, so $x = a_{i_{\min}}$ and $-x < a_{i_{\min}} - 1$. There is a strongly positive representation σ'_{sp} such that σ is a subrepresentation of $\nu^{a_{i_{\min}}}\rho \times \nu^{a_{i_{\min}}-1}\rho \rtimes \sigma'_{sp}$, so we have an embedding $\pi \hookrightarrow \nu^{a_{i_{\min}}}\rho \times \nu^{a_{i_{\min}}-1}\rho \times \delta([\nu^{-y}\rho, \nu^{-x}\rho]) \rtimes \sigma'_{sp}$. Since $\pi \cong \widehat{\pi}$, it follows that the Jacquet module of

π with respect to an appropriate parabolic subgroup contains an irreducible representation of the form $\nu^{-a_{i_{\min}}}\rho \otimes \nu^{-a_{i_{\min}}+1}\rho \otimes \pi'$. From the structural formula we directly obtain that then $\mu^*(\delta([\nu^{-y}\rho, \nu^{-a_{i_{\min}}-1}\rho]) \rtimes \sigma)$ contains $\nu^{-a_{i_{\min}}+1}\rho \otimes \pi'$, a contradiction. Consequently, $a_{i_{\min}} = \alpha - k + i_{\min}$.

Now we assume that there is a $j \in \{i_{\min}, i_{\min}+1, \ldots, k-1\}$ such that $a_{j+1} \neq a_j + 1$. It follows from [5, Section 4] that $a_{j+1} \geq a_j + 2$ and π is a subrepresentation of an induced representation of the form $\nu^{a_{j+1}}\rho \rtimes \sigma'_{sp}$, for a strongly positive representation σ'_{sp}. Since we obviously have that $a_{j+1} > -x+1$, there is an embedding $\pi \hookrightarrow \nu^{a_{j+1}}\rho \times \delta([\nu^{-y}\rho, \nu^{-x}\rho]) \rtimes \sigma'_{sp}$. This leads to $\mu^*(\pi) \geq \nu^{-a_{j+1}}\rho \otimes \pi'$, for some irreducible π', contradicting Corollary 2.3, since $x < a_{j+1}$. This completes the proof. □

In the following theorem we state our first main result.

Theorem 3.6. *Suppose that α is a half-integer. Then π is isomorphic to $\widehat{\pi}$ if and only if $\alpha \geq \frac{3}{2}$, $a_{i_{\min}} \geq \frac{3}{2}$, $x = a_{i_{\min}}$, and $y = \alpha + 1$.*

Theorem 3.6 follows from the following two propositions:

Proposition 3.7. *Suppose that α is a half-integer and $\pi \cong \widehat{\pi}$. Then $\alpha \geq \frac{3}{2}$, $a_{i_{\min}} \geq \frac{3}{2}$, $x = a_{i_{\min}}$, and $y = \alpha + 1$.*

Proof. Let us first show that $x = a_{i_{\min}}$. We have an embedding $\sigma \hookrightarrow \nu^{a_{i_{\min}}}\rho \rtimes \sigma_{sp}$, for some strongly positive representation σ_{sp}. Since $x \geq 0$ and $a_{i_{\min}} > 0$, if $(x, a_{i_{\min}}) \neq (\frac{1}{2}, \frac{1}{2})$, we obtain an embedding $\pi \hookrightarrow \nu^{a_{i_{\min}}}\rho \times \delta([\nu^{-y}\rho, \nu^{-x}\rho]) \rtimes \sigma_{sp}$, which implies that $\mu^*(\pi)$ contains an irreducible constituent of the form $\nu^{-a_{i_{\min}}}\rho \otimes \pi_1$, since $\pi \cong \widehat{\pi}$. This is possible only if $x = a_{i_{\min}}$. Thus, in any case we have $x = a_{i_{\min}}$.

Let us now prove that $a_{i_{\min}} \geq \frac{3}{2}$. Assume, on the contrary, that $a_{i_{\min}} = \frac{1}{2}$. Using Lemma 3.5 and [17, Theorem 5.1], we obtain that $\nu^z\rho \rtimes \sigma$ is irreducible for $z \notin \{\frac{1}{2}, \alpha+1\}$. If $y \notin \{\frac{1}{2}, \alpha+1\}$, we have the following embedding and isomorphism:

$$\pi \hookrightarrow \delta([\nu^{-y+1}\rho, \nu^{-\frac{1}{2}}\rho]) \times \nu^{-y}\rho \rtimes \sigma \cong \nu^{y}\rho \times \delta([\nu^{-y+1}\rho, \nu^{-\frac{1}{2}}\rho]) \rtimes \sigma.$$

In the same way as before we conclude that $\mu^*(\pi)$ contains an irreducible constituent of the form $\nu^{-y}\rho \otimes \pi_1$, which is impossible unless $y = \frac{1}{2}$. Thus, $y \in \{\frac{1}{2}, \alpha+1\}$.

It follows at once that π is a subrepresentation of $\nu^{-\frac{1}{2}}\rho \times \delta([\nu^{-y}\rho, \nu^{-\frac{3}{2}}\rho]) \rtimes \sigma$, and Lemma 2.4, together with transitivity of Jacquet modules, shows that

$\mu^*(\pi)$ contains an irreducible constituent of the form $\nu^{\frac{1}{2}}\rho \otimes \pi_1$. We will show that this is impossible, implying $a_{i_{\min}} \geq \frac{3}{2}$. Note that $\epsilon_\sigma(\rho, 2) = 1$, where $(\mathrm{Jord}(\sigma), \sigma_{cusp}, \epsilon_\sigma)$ stands for the Jordan triple corresponding to σ, so we use [17, Theorem 5.1(ii)]. Only the case $y = \alpha + 1$ and $\alpha \geq \frac{3}{2}$ will be described in detail, since other cases can be obtained in the same way and the case $(y, \alpha) = (\frac{3}{2}, \frac{1}{2})$ is also, in the split case, discussed in [4]. The following equality holds in $R(G)$:

$$\delta([\nu^{\frac{1}{2}}\rho, \nu^{\alpha+1}\rho]) \rtimes \sigma = \pi + L(\delta([\nu^{-\alpha-1}\rho, \nu^{\frac{1}{2}}\rho]) \rtimes \sigma_{sp}^{(1)}) + \\ + L(\delta([\nu^{-\alpha}\rho, \nu^{-\frac{1}{2}}\rho]) \rtimes \sigma_{sp}^{(2)}) + L(\delta([\nu^{-\alpha}\rho, \nu^{\frac{1}{2}}\rho]) \rtimes \sigma_{sp}^{(3)}),$$

where $\sigma_{sp}^{(1)}$ is the unique irreducible subrepresentation of $\nu^{\frac{3}{2}}\rho \times \nu^{\frac{5}{2}}\rho \times \cdots \times \nu^{\alpha}\rho \rtimes \sigma_{cusp}$, $\sigma_{sp}^{(2)}$ is the unique irreducible subrepresentation of $\nu^{\frac{1}{2}}\rho \times \nu^{\frac{3}{2}}\rho \times \cdots \times \nu^{\alpha-1}\rho \times \delta([\nu^{\alpha}\rho, \nu^{\alpha+1}\rho]) \rtimes \sigma_{cusp}$, and $\sigma_{sp}^{(3)}$ is the unique irreducible subrepresentation of $\nu^{\frac{3}{2}}\rho \times \nu^{\frac{5}{2}}\rho \times \cdots \times \nu^{\alpha-1}\rho \times \delta([\nu^{\alpha}\rho, \nu^{\alpha+1}\rho]) \rtimes \sigma_{cusp}$. We note that $\sigma_{sp}^{(i)}$ is strongly positive, for $i = 1, 2, 3$.

Using the structural formula, we obtain that if $\nu^{\frac{1}{2}}\rho \otimes \pi_1$ is an irreducible constituent of $\mu^*(\delta([\nu^{\frac{1}{2}}\rho, \nu^{\alpha+1}\rho]) \rtimes \sigma)$, then π_1 is an irreducible subquotient of $\delta([\nu^{\frac{1}{2}}\rho, \nu^{\alpha+1}\rho]) \rtimes \sigma_{sp}^{(1)}$. By [17, Theorem 5.1(i)], in $R(G)$ we have:

$$\delta([\nu^{\frac{1}{2}}\rho, \nu^{\alpha+1}\rho]) \rtimes \sigma_{sp}^{(1)} = L(\delta([\nu^{-\alpha-1}\rho, \nu^{-\frac{1}{2}}\rho]) \rtimes \sigma_{sp}^{(1)}) + L(\delta([\nu^{-\alpha}\rho, \nu^{-\frac{1}{2}}\rho]) \rtimes \sigma_{sp}^{(3)}).$$

Furthermore, both irreducible constituents $\nu^{\frac{1}{2}}\rho \otimes L(\delta([\nu^{-\alpha-1}\rho, \nu^{-\frac{1}{2}}\rho]) \rtimes \sigma_{sp}^{(1)})$ and $\nu^{\frac{1}{2}}\rho \otimes L(\delta([\nu^{-\alpha}\rho, \nu^{-\frac{1}{2}}\rho]) \rtimes \sigma_{sp}^{(3)})$ appear in $\mu^*(\delta([\nu^{\frac{1}{2}}\rho, \nu^{\alpha+1}\rho]) \rtimes \sigma)$ with multiplicity one, and Frobenius reciprocity implies that both $\mu^*(L(\delta([\nu^{-\alpha-1}\rho, \nu^{\frac{1}{2}}\rho]) \rtimes \sigma_{sp}^{(1)}))$ and $\mu^*(L(\delta([\nu^{-\alpha}\rho, \nu^{\frac{1}{2}}\rho]) \rtimes \sigma_{sp}^{(3)}))$ contain irreducible constituents of the form $\nu^{\frac{1}{2}}\rho \otimes \pi_1$, so $\mu^*(\pi)$ does not contain such an irreducible constituent.

Since $a_{i_{\min}} \geq \frac{3}{2}$, Lemma 3.5 implies that $\alpha \geq \frac{3}{2}$. From $y \geq a_{i_{\min}}$, using [17, Proposition 3.1], we obtain that $\nu^{y}\rho \rtimes \sigma$ is irreducible if $y \neq \alpha + 1$. Suppose that $y \neq \alpha + 1$. Then we have the following embedding and isomorphism:

$$\pi \hookrightarrow \delta([\nu^{-y+1}\rho, \nu^{-x}\rho]) \times \nu^{-y}\rho \rtimes \sigma \cong \nu^{y}\rho \times \delta([\nu^{-y+1}\rho, \nu^{-x}\rho]) \rtimes \sigma.$$

In the same way as before we conclude that $\mu^*(\pi)$ contains an irreducible constituent of the form $\nu^{-y}\rho \otimes \pi_1$, which is impossible unless $x = y$. Thus, $y = a_{i_{\min}}$. It follows from Lemma 3.5 and [17, Proposition 3.1] that $\nu^{-a_{i_{\min}}}\rho \rtimes \sigma \cong \nu^{a_{i_{\min}}}\rho \rtimes \sigma$, so π is an irreducible subrepresentation of $\nu^{a_{i_{\min}}}\rho \times \nu^{a_{i_{\min}}}\rho \rtimes \sigma_{sp}$.

Consequently, Lemma 2.4 and transitivity of Jacquet modules imply that the Jacquet module of π with respect to an appropriate parabolic subgroup contains an irreducible representation of the form $\nu^{-a_{i_{\min}}}\rho \otimes \nu^{-a_{i_{\min}}}\rho \otimes \pi_2$, and an easy application of Theorem 2.2 implies the Jacquet module of $\delta([\nu^{-y}\rho, \nu^{-x}\rho]) \rtimes \sigma$ with respect to an appropriate parabolic subgroup does not contain such a representation. Thus, $y = \alpha + 1$, and the proposition is proved. $\square$

Proposition 3.8. *Suppose that α is a half-integer, $\alpha \geq \frac{3}{2}$, $a_{i_{\min}} \geq \frac{3}{2}$, $x = a_{i_{\min}}$, and $y = \alpha + 1$. Then π is isomorphic to $\widehat{\pi}$.*

Proof. Let us first prove that if an irreducible constituent of the form $\nu^z\rho \otimes \pi_1$, with $z \geq 0$, appears in $\mu^*(\pi)$, then $z = a_{i_{\min}}$. Using the structural formula and [7, Theorem 4.6], we deduce that if an irreducible constituent of the form $\nu^z\rho \otimes \pi_1$, with $z \geq 0$, appears in $\mu^*(\delta([\nu^{-\alpha-1}\rho, \nu^{-a_{i_{\min}}}\rho]) \rtimes \sigma)$, then $z \in \{a_{i_{\min}}, \alpha + 1\}$.

By [17, Proposition 3.1(i)], in $R(G)$ we have

$$\delta([\nu^{a_{i_{\min}}}\rho, \nu^{\alpha+1}\rho]) \rtimes \sigma = \pi + L(\delta([\nu^{-\alpha}\rho, \nu^{-a_{i_{\min}}}\rho]) \rtimes \sigma_{sp}),$$

where σ_{sp} denotes the unique irreducible subrepresentation of

$$\nu^{a_{i_{\min}}}\rho \times \nu^{a_{i_{\min}}+1}\rho \times \cdots \times \nu^{\alpha-1}\rho \times \delta([\nu^{\alpha}\rho, \nu^{\alpha+1}\rho]) \rtimes \sigma_{cusp}.$$

We note that σ_{sp} is a strongly positive representation. Using [7, Theorem 4.6], Frobenius reciprocity, and transitivity of Jacquet modules, we obtain that $\mu^*(L(\delta([\nu^{-\alpha}\rho, \nu^{-a_{i_{\min}}}\rho]) \rtimes \sigma_{sp}))$ contains an irreducible constituent of the form $\nu^{\alpha+1}\rho \otimes \pi'$. The induced representation $\delta([\nu^{a_{i_{\min}}}\rho, \nu^{\alpha}\rho]) \rtimes \sigma$ is irreducible (by [17, Proposition 3.1(ii)]), so the only such irreducible constituent appearing in $\mu^*(\delta([\nu^{a_{i_{\min}}}\rho, \nu^{\alpha+1}\rho]) \rtimes \sigma)$ is $\nu^{\alpha+1}\rho \otimes \delta([\nu^{a_{i_{\min}}}\rho, \nu^{\alpha}\rho]) \rtimes \sigma$, which appears there with multiplicity one. Therefore, there is no irreducible constituent of the form $\nu^{\alpha+1}\rho \otimes \pi_1$ appearing in $\mu^*(\pi)$. Furthermore, Lemma 2.4 implies that if an irreducible constituent of the form $\nu^z\rho \otimes \pi_1$, with $z \leq 0$, appears in $\mu^*(\widehat{\pi})$, then $z = -a_{i_{\min}}$.

Since π is a subrepresentation of $\delta([\nu^{-\alpha-1}\rho, \nu^{-a_{i_{\min}}}\rho]) \rtimes \sigma$ and $a_{i_{\min}} \geq \frac{3}{2}$, we have the following embedding and isomorphisms:

$$\begin{aligned}
\pi &\hookrightarrow \nu^{a_{i_{\min}}}\rho \times \cdots \times \nu^{\alpha}\rho \times \nu^{-a_{i_{\min}}}\rho \times \cdots \times \nu^{-\alpha}\rho \times \nu^{-\alpha-1}\rho \rtimes \sigma_{cusp} \\
&\cong \nu^{a_{i_{\min}}}\rho \times \cdots \times \nu^{\alpha}\rho \times \nu^{-a_{i_{\min}}}\rho \times \cdots \times \nu^{-\alpha}\rho \times \nu^{\alpha+1}\rho \rtimes \sigma_{cusp} \\
&\cong \nu^{a_{i_{\min}}}\rho \times \cdots \times \nu^{\alpha}\rho \times \nu^{\alpha+1}\rho \times \nu^{-a_{i_{\min}}}\rho \times \times \cdots \times \nu^{-\alpha}\rho \rtimes \sigma_{cusp}.
\end{aligned}$$

Using Lemma 2.4, Theorem 2.1, and [16, Lemma 3.1], we obtain that $\widehat{\pi}$ is a subrepresentation of the induced representation

$$\nu^{-a_{i_{\min}}}\rho \times \cdots \times \nu^{-\alpha-1}\rho \times \nu^{a_{i_{\min}}}\rho \times \cdots \times \nu^{\alpha}\rho \rtimes \sigma_{cusp}.$$

It follows from [16, Lemma 3.2] that there is an irreducible subquotient π_1 of $\nu^{-a_{i_{\min}}}\rho \times \cdots \times \nu^{-\alpha-1}\rho$ such that $\widehat{\pi}$ is a subrepresentation of $\pi_1 \times \times \nu^{a_{i_{\min}}}\rho \times \cdots \times \nu^{\alpha}\rho \rtimes \sigma_{cusp}$. Since $\mu^*(\widehat{\pi})$ does not contain an irreducible constituent of the form $\nu^z\rho \otimes \pi_1$ for $z \leq 0$ and $z \neq -a_{i_{\min}}$, we deduce that $\pi_1 \cong \delta([\nu^{-\alpha-1}\rho, \nu^{-a_{i_{\min}}}\rho])$.

By [16, Lemma 3.2], there is an irreducible representation π' such that $\widehat{\pi}$ is a subrepresentation of $\delta([\nu^{-\alpha-1}\rho, \nu^{-a_{i_{\min}}}\rho]) \rtimes \pi'$ and, obviously, the cuspidal support of π' equals $\{\nu^{a_{i_{\min}}}\rho, \nu^{a_{i_{\min}}+1}\rho, \ldots, \nu^{\alpha}\rho, \sigma_{cusp}\}$.

Let us first suppose that π' is a non-tempered representation and write $\pi' \cong L(\delta_1 \times \delta_2 \times \cdots \times \delta_k \rtimes \tau)$, where $\delta_i \in \mathrm{Irr}(GL(n_i, F))$ is an irreducible essentially square-integrable representation, for $i = 1, 2, \ldots, k$, $e(\delta_i) \leq e(\delta_{i+1}) < 0$ for $i = 1, 2, \ldots, k-1$, and $\tau \in \mathrm{Irr}(G_{n'})$ is a irreducible tempered representation. Write $\delta_i = \delta([\nu^{a_i}\rho, \nu^{b_i}\rho])$. From $a_i + b_i < 0$ and from the description of the cuspidal support of π' follows that $b_i \leq -a_{i_{\min}}$, for $i = 1, 2, \ldots, k$. It directly follows that π' is a subrepresentation of an induced representation of the form $\nu^{b_1}\rho \rtimes \pi''$ and, since $-\alpha \leq b_1$, we have the following embedding and isomorphism:

$$\widehat{\pi} \hookrightarrow \delta([\nu^{-\alpha-1}\rho, \nu^{-a_{i_{\min}}}\rho]) \times \nu^{b_1}\rho \rtimes \pi'' \cong \nu^{b_1}\rho \times \delta([\nu^{-\alpha-1}\rho, \nu^{-a_{i_{\min}}}\rho]) \rtimes \pi'',$$

and it follows that $\mu^*(\widehat{\pi})$ contains an irreducible constituent of the form $\nu^{b_1}\rho \otimes \pi_2$, which is impossible unless $b_1 = -a_{i_{\min}}$. If this is the case, we have an embedding

$$\widehat{\pi} \hookrightarrow \nu^{-a_{i_{\min}}}\rho \times \nu^{-a_{i_{\min}}}\rho \times \delta([\nu^{-\alpha-1}\rho, \nu^{-a_{i_{\min}}-1}\rho]) \rtimes \pi''$$

and Lemma 2.4 and transitivity of Jacquet modules imply that the Jacquet module of π with respect to an appropriate parabolic subgroup contains an irreducible representation of the form $\nu^{a_{i_{\min}}}\rho \otimes \nu^{a_{i_{\min}}}\rho \otimes \pi_2$. Using the structural formula, [7, Theorem 4.6], and the fact that $\alpha + 1 > a_{i_{\min}}$, we deduce that a representation of the form $\nu^{a_{i_{\min}}}\rho \otimes \nu^{a_{i_{\min}}}\rho \otimes \pi_2$ does not appear in the Jacquet module of the induced representation $\delta([\nu^{-\alpha-1}\rho, \nu^{-a_{i_{\min}}}\rho]) \rtimes \sigma$ with respect to an appropriate parabolic subgroup, a contradiction.

Consequently, π' is a tempered representation and, using the description of its cuspidal support and [6, Theorem 3.5], we conclude that π' is strongly positive. Since the strongly positive representation is completely determined by its cuspidal support ([7, Lemma 3.5]), it follows at once that π' is isomorphic to σ. Thus, $\widehat{\pi}$ is an irreducible subrepresentation of $\delta([\nu^{-\alpha-1}\rho, \nu^{-a_{i_{\min}}}\rho]) \rtimes \sigma$, leading to $\widehat{\pi} \cong \pi$. This completes the proof. $\square$

Now we state our second main result.

Theorem 3.9. *Suppose that α is an integer. Then π is isomorphic to $\widehat{\pi}$ if and only if $y = \alpha + 1$ and either $x = a_{i_{\min}}$ or $(a_{i_{\min}}, x) = (1, 0)$.*

Theorem 3.9 follows from the following two propositions:

Proposition 3.10. *Suppose that α is an integer and $\pi \cong \widehat{\pi}$. Then $y = \alpha + 1$ and either $x = a_{i_{\min}}$ or $(a_{i_{\min}}, x) = (1, 0)$.*

Proof. If $a_{i_{\min}} \geq 2$, in the same way as in the proof of Proposition 3.7, we deduce that $(x, y) = (a_{i_{\min}}, \alpha + 1)$.

Let us now assume that $a_{i_{\min}} = 1$. Then σ is a subrepresentation of an induced representation of the form $\nu\rho \rtimes \sigma'$ and if $x > 0$ we have an embedding $\pi \hookrightarrow \nu\rho \times \delta([\nu^{-y}\rho, \nu^{-x}\rho]) \rtimes \sigma'$. In the same way as in the proof of Proposition 3.7, we get that $x \in \{0, 1\}$. Note that $y > x$ if $x = 0$. Let us prove that $y = \alpha + 1$. Suppose, contrary to our assumption, that $y \neq \alpha + 1$. Since $y \geq x$, it follows from [17, Proposition 3.1] that $\nu^y\rho \rtimes \sigma$ is irreducible. We have

$$\pi \hookrightarrow \delta([\nu^{-y+1}\rho, \nu^{-x}\rho]) \times \nu^{-y}\rho \rtimes \sigma \cong \delta([\nu^{-y+1}\rho, \nu^{-x}\rho]) \times \nu^{y}\rho \rtimes \sigma,$$

and if $y \neq -x+1$ one obtains a contradiction in the same way as in the proof of Proposition 3.7. Since $y \geq x$ and $x \geq 0$, we see that $y = -x + 1$ holds only if $(x, y) = (0, 1)$. In that case, we have $\pi \hookrightarrow \rho \times \nu\rho \times \nu\rho \rtimes \sigma'$. Using Lemma 2.4 and transitivity of Jacquet modules we get that $r_M(\pi)$ contains an irreducible representation of the form $\rho \otimes \nu^{-1}\rho \otimes \nu^{-1}\rho \otimes \sigma''$, where M denotes the Levi factor of an appropriate parabolic subgroup. But, since σ is strongly positive, $r_M(\delta([\rho, \nu\rho]) \rtimes \sigma)$ does not contain an irreducible representation of the form $\rho \otimes \nu^{-1}\rho \otimes \nu^{-1}\rho \otimes \sigma'$, a contradiction. This completes the proof. $\square$

Proposition 3.11. *Suppose that α is an integer. If $y = \alpha + 1$ and either $x = a_{i_{\min}}$ or $(a_{i_{\min}}, x) = (1, 0)$, then π is isomorphic to $\widehat{\pi}$.*

Proof. First we suppose that $(x, y) = (a_{i_{\min}}, \alpha+1)$. Let us prove that if $\mu^*(\pi)$ contains an irreducible constituent of the form $\nu^z\rho\otimes\pi'$, with $z \geq 0$, then $z = a_{i_{\min}}$. It follows from the structural formula that if $\mu^*(\delta([\nu^{a_{i_{\min}}}\rho, \nu^{\alpha+1}\rho]) \rtimes \sigma)$ contains an irreducible constituent of the form $\nu^z\rho \otimes \pi'$, with $z \geq 0$, then $z \in \{a_{i_{\min}}, \alpha + 1\}$. Also, if $z = \alpha + 1$, then π' is an irreducible subquotient of $\delta([\nu^{a_{i_{\min}}}\rho, \nu^{\alpha}\rho]) \rtimes \sigma$. By [17, Proposition 3.1], $\delta([\nu^{a_{i_{\min}}}\rho, \nu^{\alpha}\rho]) \rtimes \sigma$ is irreducible and $\nu^{\alpha+1}\rho\otimes\delta([\nu^{a_{i_{\min}}}\rho, \nu^{\alpha}\rho])\rtimes\sigma$ is contained in $\mu^*(\delta([\nu^{a_{i_{\min}}}\rho, \nu^{\alpha+1}\rho])\rtimes \sigma)$ with multiplicity one. Using [17, Proposition 3.1(i)], we deduce that in $R(G)$ holds $\delta([\nu^{a_{i_{\min}}}\rho, \nu^{\alpha+1}\rho]) \rtimes \sigma = \pi + L(\delta([\nu^{-\alpha}\rho, \nu^{-a_{i_{\min}}}\rho]) \rtimes \sigma_{sp})$, where σ_{sp} denotes the unique irreducible subrepresentation of

$$\nu^{a_{i_{\min}}}\rho \times \nu^{a_{i_{\min}}+1}\rho \times \cdots \times \nu^{\alpha-1}\rho \times \delta([\nu^{\alpha}\rho, \nu^{\alpha+1}\rho]) \rtimes \sigma_{cusp}.$$

We note that σ_{sp} is a strongly positive representation. It is now easy to conclude, using Frobenius reciprocity and irreducibility of $\nu^x\rho\times\nu^{\alpha+1}\rho$ for $x < \alpha$, that $\mu^*(L(\delta([\nu^{-\alpha}\rho, \nu^{-a_{i_{\min}}}\rho]) \rtimes \sigma_{sp}))$ contains an irreducible constituent of the form $\nu^{\alpha+1}\rho\otimes\pi'$, so $\mu^*(\pi)$ does not contain such an irreducible constituent. Now, in the same way as in the proof of Proposition 3.8, we obtain that $\widehat{\pi}$ is a subrepresentation of $\delta([\nu^{-\alpha-1}\rho, \nu^{-a_{i_{\min}}}\rho]) \rtimes \sigma$, i.e., $\pi \cong \widehat{\pi}$.

Now we turn our attention to the case $(a_{i_{\min}}, x, y) = (1, 0, \alpha + 1)$. In this case, we have the following embedding and isomorphisms:

$$\begin{aligned}
\pi &\hookrightarrow \rho \times \nu^{-1}\rho \times \cdots \times \nu^{-\alpha-1}\rho \times \nu\rho \times \nu^2\rho \times \cdots \times \nu^{\alpha}\rho \rtimes \sigma_{cusp} \\
&\cong \rho \times \nu\rho \times \nu^2\rho \times \cdots \times \nu^{\alpha}\rho \times \nu^{-1}\rho \times \cdots \times \nu^{-\alpha}\rho \times \nu^{-\alpha-1}\rho \rtimes \sigma_{cusp} \\
&\cong \rho \times \nu\rho \times \nu^2\rho \times \cdots \times \nu^{\alpha}\rho \times \nu^{-1}\rho \times \cdots \times \nu^{-\alpha}\rho \times \nu^{\alpha+1}\rho \rtimes \sigma_{cusp} \\
&\cong \rho \times \nu\rho \times \nu^2\rho \times \cdots \times \nu^{\alpha}\rho \times \nu^{\alpha+1}\rho \times \nu^{-1}\rho \times \cdots \times \nu^{-\alpha}\rho \rtimes \sigma_{cusp}.
\end{aligned}$$

In the same way as before, we obtain that $\widehat{\pi}$ is a subrepresentation of the induced representation

$$\rho \times \nu^{-1}\rho \times \nu^{-2}\rho \times \cdots \times \nu^{-\alpha}\rho \times \nu^{-\alpha-1}\rho \times \nu\rho \times \cdots \times \nu^{\alpha}\rho \rtimes \sigma_{cusp}.$$

We will show that if $\mu^*(\pi)$ contains an irreducible constituent of the form $\nu^z\rho \otimes \pi_1$, with $z \geq 0$, then $z = 0$. The rest of the proof then follows the same lines as in the proof of Proposition 3.8.

Note that if an irreducible constituent of the form $\nu^z\rho \otimes \pi_1$, with $z \geq 0$, appears in $\mu^*(\delta([\rho, \nu^{\alpha+1}\rho]) \rtimes \sigma)$, then $z \in \{0, 1, \alpha + 1\}$. We will comment only the case $\alpha \geq 2$, since the case $\alpha = 1$ can be handled in the same way but more easily, and in the split case it can also be obtained using [4].

According to [17, Theorem 4.1(iv)], in $R(G)$ we have

$$\delta([\rho,\nu^{\alpha+1}\rho]) \rtimes \sigma = \pi + L(\delta([\nu^{-\alpha-1}\rho,\nu\rho]) \rtimes \sigma_{sp}^{(1)}) + \\ + L(\delta([\nu^{-\alpha}\rho,\rho]) \rtimes \sigma_{sp}^{(2)}) + L(\delta([\nu^{-\alpha}\rho,\nu\rho]) \rtimes \sigma_{sp}^{(3)}),$$

where $\sigma_{sp}^{(1)}$ is the unique irreducible subrepresentation of $\nu^2\rho \times \cdots \times \nu^{\alpha}\rho \rtimes \sigma_{cusp}$, $\sigma_{sp}^{(2)}$ is the unique irreducible subrepresentation of $\nu\rho \times \cdots \times \nu^{\alpha-1}\rho \times \delta([\nu^{\alpha}\rho,\nu^{\alpha+1}\rho]) \rtimes \sigma_{cusp}$, and $\sigma_{sp}^{(3)}$ is the unique irreducible subrepresentation of $\nu^2\rho \times \cdots \times \nu^{\alpha-1}\rho \times \delta([\nu^{\alpha}\rho,\nu^{\alpha+1}\rho]) \rtimes \sigma_{cusp}$. We note that $\sigma_{sp}^{(i)}$ is strongly positive for $i = 1, 2, 3$.

If $\mu^*(\delta([\rho,\nu^{\alpha+1}\rho]) \rtimes \sigma)$ contains an irreducible constituent of the form $\nu\rho \otimes \pi_1$, then π_1 is an irreducible subquotient of $\delta([\rho,\nu^{\alpha+1}\rho]) \rtimes \sigma_{sp}^{(1)}$. By [17, Theorem 4.1(ii)], in $R(G)$ we have

$$\delta([\rho,\nu^{\alpha+1}\rho]) \rtimes \sigma_{sp}^{(1)} = L(\delta([\nu^{-\alpha-1}\rho,\rho]) \rtimes \sigma_{sp}^{(1)}) + L(\delta([\nu^{-\alpha}\rho,\rho]) \rtimes \sigma_{sp}^{(3)}).$$

Also, $\nu\rho \otimes L(\delta([\nu^{-\alpha-1}\rho,\rho]) \rtimes \sigma_{sp}^{(1)})$ and $\nu\rho \otimes L(\delta([\nu^{-\alpha}\rho,\rho]) \rtimes \sigma_{sp}^{(3)})$ appear in $\mu^*(\delta([\rho,\nu^{\alpha+1}\rho]) \rtimes \sigma)$ with multiplicity one and are obviously contained in $\mu^*(L(\delta([\nu^{-\alpha-1}\rho,\nu\rho]) \rtimes \sigma_{sp}^{(1)}))$ and in $\mu^*(L(\delta([\nu^{-\alpha}\rho,\nu\rho]) \rtimes \sigma_{sp}^{(3)}))$. Thus, there are no irreducible constituents of the form $\nu\rho \otimes \pi_1$ appearing in $\mu^*(\pi)$.

Similarly, if $\mu^*(\delta([\rho,\nu^{\alpha+1}\rho]) \rtimes \sigma)$ contains an irreducible constituent of the form $\nu^{\alpha+1}\rho \otimes \pi_1$, then π_1 is an irreducible subquotient of $\delta([\rho,\nu^{\alpha}\rho]) \rtimes \sigma$. By [17, Theorem 4.1(iii)], in $R(G)$ we have

$$\delta([\rho,\nu^{\alpha}\rho]) \rtimes \sigma = L(\delta([\nu^{-\alpha}\rho,\rho]) \rtimes \sigma) + L(\delta([\nu^{-\alpha}\rho,\nu\rho]) \rtimes \sigma_{sp}^{(1)}).$$

Also, $\nu^{\alpha+1}\rho \otimes L(\delta([\nu^{-\alpha}\rho,\rho]) \rtimes \sigma_{sp})$ and $\nu^{\alpha+1}\rho \otimes L(\delta([\nu^{-\alpha}\rho,\nu\rho]) \rtimes \sigma_{sp}^{(1)})$ appear in $\mu^*(\delta([\rho,\nu^{\alpha+1}\rho]) \rtimes \sigma)$ with multiplicity one and obviously appear in $\mu^*(L(\delta([\nu^{-\alpha}\rho,\rho]) \rtimes \sigma_{sp}^{(2)}))$ and in $\mu^*(L(\delta([\nu^{-\alpha}\rho,\nu\rho]) \rtimes \sigma_{sp}^{(3)}))$. Thus, $\mu^*(\pi)$ does not contain irreducible constituents of the form $\nu^{\alpha+1}\rho \otimes \pi_1$. Consequently, $\mu^*(\pi)$ does not contain an irreducible constituent of the form $\nu^{z}\rho \otimes \pi_1$ with $z > 0$, and the proposition is proved. □

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