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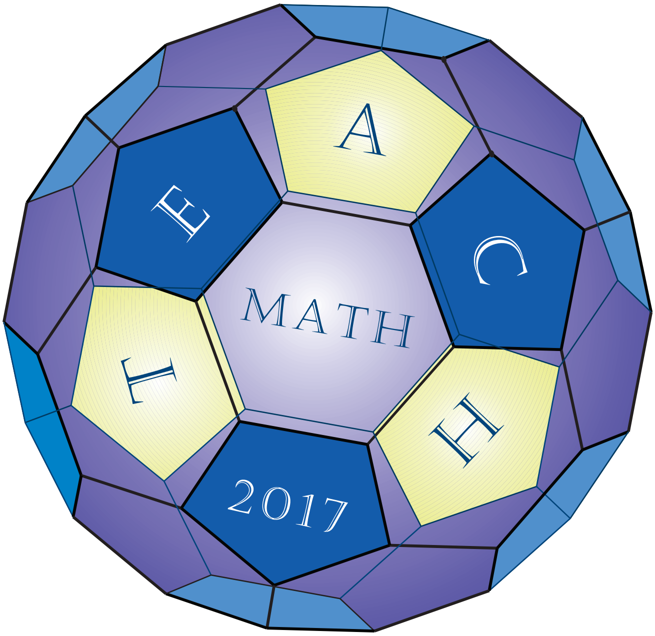

MATHEMATICS EDUCATION AS A SCIENCE AND A PROFESSION

Editors:

Zdenka Kolar-Begović
Ružica Kolar-Šuper
Ljerka Jukić Matić

2017

ELEMENT

Josip Juraj Strossmayer University of Osijek
Faculty of Education and Department of Mathematics

Sveučilište Josipa Jurja Strossmayera u Osijeku
Fakultet za odgojne i obrazovne znanosti i Odjel za matematiku

MATHEMATICS EDUCATION AS A SCIENCE AND A PROFESSION

MATEMATIČKO OBRAZOVANJE KAO ZNANOST I PROFESIJA

monograph

Editors:
Zdenka Kolar-Begović
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A word from the Editorial Board

Mathematics education as a research field experienced significant growth in the last decades. Nonetheless, we still experience the same problems over and over. The continuous problem with transition from primary to secondary education and from secondary to tertiary education is ever present. (Corriveau & Bednarz, 2017; Clark & Lovric, 2008; Thomas, 2008). It seems that mathematics teachers of each educational cycle have rather different expectations of the knowledge a student should bring from one cycle to another. The mathematics learning process is connected with doing mathematics and processes such as investigating, reflecting, reasoning, giving arguments, finding connections, etc. The research in mathematics education has been emphasizing this aspect of learning mathematics for more than twenty years. However, many teachers and learners focus on imitating procedures. What causes this discrepancy between the research and practice? The findings in mathematics education showed that the teacher had a significant role on students' achievements. Moreover, the way they conceived mathematics reflected on the learners. If the teachers saw it as a set of procedures and algorithms, they would transfer this conception of mathematics onto their students. On the other hand, if they saw it as collections of ideas, their students would have the same conception of mathematics. The student's attitudes towards mathematics also influenced their learning process. This shows that attitudes and beliefs in teaching and learning mathematics have significant impact on the outcomes in the classroom. That is why they are considered to be hidden variables in mathematics education (Leder, Pehkonen, Törner, 2002), which cannot be omitted from the didactical triangle mathematics – teacher – students.

In the primary and secondary education, mathematics is a compulsory subject in schools, but moving to the tertiary education, we find mathematics as the key component of many natural and technical sciences. Mathematics is the heart of STEM. It is important for those who will continue their profession in that direction. Moreover, a strong interaction between mathematics and technology is present: developments in technology stimulate mathematics and developments in mathematics often enhance innovations in technology (Höft, 2016). But mathematics in the form of statistics appears in humanities and social sciences. Therefore it seems that mathematics is the core science and the key feature of a successful individual. This is also supported by the definition of mathematical literacy in PISA, where the role of mathematics is described as a part of an individual's current and future private life, professional life and social life with peers and relatives (OECD; 2013).

Considering the rapid progress of technology in the world of today, we face new challenges. Living in the age of digital technologies, we feel obligated to incorporate them into our lives but also in the other areas such as the school subjects. This issue raises many still unanswered questions. Could technology improve our teaching and learning of mathematics? Should we use it constantly or occasionally? How can this technology be used efficiently? Do we design tasks in a similar manner with this technology as it was the case with pencil and paper technique?

Modern technology and its use in mathematics education has been examined in various studies for several decades. The findings of those studies showed benefits of the proper use of technology. Spreadsheets and dynamic software environments enable faster collecting of the specific cases of mathematical phenomena – values, variables, functions, shapes, locations along a graph, or the properties of geometric constructions (Aldon, Hitt, Bazzini & Gellert, 2017). This provides the opportunities for learners to consider collection of cases rather than individual instances. But Applebaum (2017) warns against jumping to conclusion that technology forces automatic generalization in mathematics; this depends on the learner, similarly as it is the case with the classic environments. Digital technology can speed up the learning process, but sometimes educators want to slow this process down. They notice that the amount of information is overwhelming for the learner. At other times, technology narrows our focus too much, or not enough (Appelbaum, 2007). Therefore, this brings us to the conclusion that the use of technology has its advantages and disadvantages which teachers should be aware of.

We addressed these issues and some other in this monograph. Some papers opened new questions for research, some showed examples of good practice and others provided more information for the earlier findings. All papers portray the complex role of mathematics education: mathematics education is a science but also a profession. Research in mathematics education is needed to investigate various phenomena, but the research has also the responsibility to inform practitioners of its findings. Mathematics teachers are a critical component in this process. Their collaboration with the researchers is necessary to achieve the shift from the traditional to contemporary teaching mathematics. This process is neither easy nor rapid. But we must be persistent in our efforts, because mathematics is inseparable from our everyday life.

References

[1] ALDON, G., HITT, F., BAZZINI, L., & GELLERT, U. (Eds.) (2017), *Mathematics and Technology. A C.I.E.A.E.M. Sourcebook*, Springer, doi. 10.1007/978-3-319-51380-5_15.

[2] APPELBAUM, P. (2007), *Children's books for grown-up teachers: Reading and writing curriculum theory*, New York: Routledge.

[3] APPELBAUM, P. (2017), in G. Aldon, F. Hitt, L. Bazzini, & U. Gellert (eds), *Mathematics and Technology. A C.I.E.A.E.M. Sourcebook*, (pp. 335–345), Springer, doi. 10.1007/978-3-319-51380-5_15.

[4] CLARK, M. & LOVRIC, M. (2008), *Suggestion for a theoretical model for secondary-tertiary transition in mathematics*, Mathematics Education Research Journal, 20(1), 25–37.

[5] CORRIVEAU, C. & BEDNARZ, N. (2017), *The secondary-tertiary transition viewed as a change in mathematical cultures: an exploration concerning symbolism and its use*, Educational Studies in Mathematics, 95(1), 1–19, doi:10.1007/s10649-016-9738-z.

[6] HÖFT, M. (2016), *IT/Mathematics: Statistical Science*, in M. Duran, M. Höft, B. Medjahed, D. Lawson & E. A. Orady (Eds.), STEM Learning, (pp. 121–152), Springer, doi. 10.1007/978-3-319-26179-9_6.

[7] LEDER, G. C., PEHKONEN, E., & TÖRNER, G. (Eds.) (2002), *Beliefs: A hidden variable in mathematics education?*, Dordrecht, The Netherlands: Kluwer Academic Publishers.

[8] OECD (2013), *PISA 2012 Assessment and Analytical Framework: Mathematics, Reading, Science, Problem Solving and Financial Literacy*, OECD Publishing, doi. 10.1787/9789264190511-en.

[9] THOMAS, M. O. J. (2008), *The transition from school to university and beyond*, Mathematics Education Research Journal, 20(1), 1–4.

Osijek, May 10, 2017

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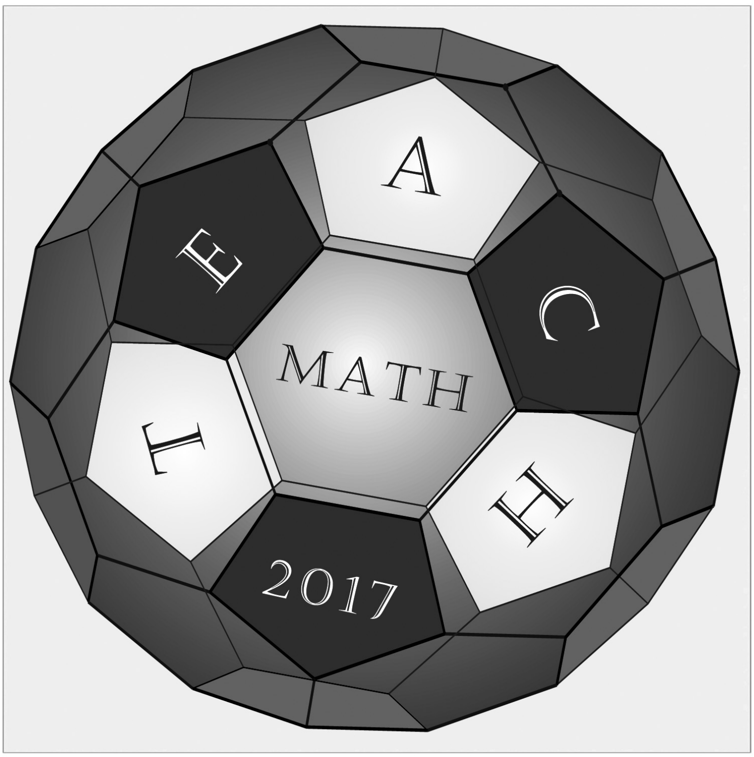
MATHEMATICS EDUCATION
AS A SCIENCE AND A PROFESSION

monograph



Preface

The papers in the Monograph addressed different topics related to teaching and learning processes of mathematics which are of great interest to both students and prospective teachers. The Monograph consists of six chapters.

In the first chapter of the Monograph the author studied the relation between the surface approach and the strategic approach on the learning outcome. This relation was obtained as the result of the conducted research with the group of university students in Denmark who were required to study mathematics, for instance calculus. In the second paper of this chapter, the author presented the results of the research with the students who were required to recognize and interpret mathematical concepts that could be interpreted from the graphs in different contexts. The third article of this chapter provides an insight into a detailed analysis of the tasks of the Croatian State Matura exam related to the mathematical domain of functions.

In the second chapter the authors examined the topic of geometric character. They discussed the topic of the importance of the spatial reasoning, as well as the effect of computer technology on geometry education. An emphasis was put on the fact that the strong links between geometry and technology are important because geometry's need for proficiency in many technical fields exceeds the traditional Euclidean space geometry. In the second paper of this chapter the authors studied regular and semiregular polyhedrons in Euclidean space. The possibility of filling the Euclidean space with the congruent copies of some polyhedrons was considered. In addition, they considered the case of hyperbolic space. The polyhedron C_{60} is specially analyzed. The authors of the third paper of this chapter gave an exploration and comparison of geometric properties of the Euclidean and hyperbolic planes.

In the third chapter the authors argued that teachers' beliefs about teaching and learning mathematics were significant in the utilization of a particular resource. They examined teachers' classroom practice and beliefs about mathematics, mathematics education, teaching mathematics and using textbooks as curriculum resources. In the second paper of this chapter the authors presented the results of a textbook analysis on asymptote and asymptotic behaviour in the two most common series of gymnasium mathematics textbooks in Croatia. The research was conducted within the theoretical framework of the Anthropological theory of the didactics. The analysis of the language of mathematics textbooks was done in the third article of this chapter.

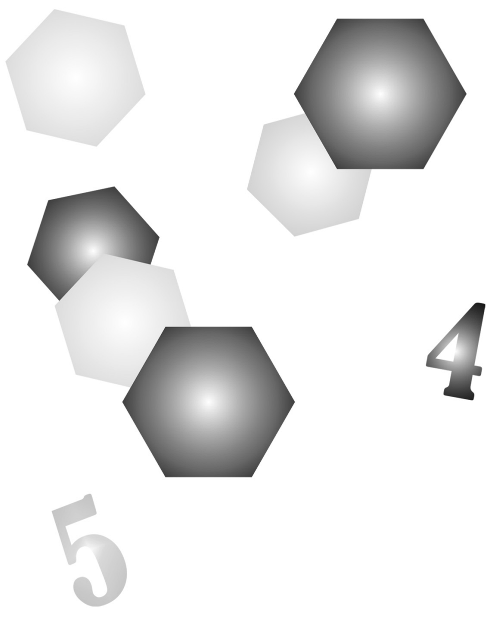
The importance of the application of ICT in teaching mathematics was discussed in the fourth chapter. The impact of using proposed computer guided discovery learning model on students' conceptual and procedural knowledge in mathematics was investigated. The illustration of the application of the program Graph and authors' experiences with the use of this program were given. The advantages of introducing three softwares to the teaching process were considered. The use of these softwares has positive effects on the motivation of the students and on improvement of their understanding and adoption of mathematical concepts.

In the first paper of the fifth chapter the author examined whether there was a connection between the attitudes towards mathematics and the performance on mathematics exams. A method for selection of a group of students which was supposed to receive additional teacher attention in order to improve their performance in the course was described in the second paper. The research on the relations between the use of mathematics tutoring services at the university level and building student profiles was presented in this chapter. The problem of mathematical anxiety was also researched in this chapter. The paper also included a discussion on the advantages and disadvantages of standardization using didactical and pedagogical approach to mathematics education. The author suggested guidelines toward upgrading learning outcomes by integrating students' abilities, needs, working habits, attitudes and beliefs with clear objectives. Inclusive approach in maths curriculum can be revealed through the presence of curriculum accommodations for pupils with disabilities. The author of the last paper in this chapter provided the content analysis of national curriculums of five European countries: Great Britain, Finland, Germany, France and Croatia.

In the last chapter of the Monograph the authors suggested some useful approaches to teaching mathematics. Learning through games increases the motivations of the students and positive attitudes towards mathematics. A practical example of using Escape Room games was presented. The frequency of using mathematical games in primary school was explored as well as the types of such games that are present in teaching mathematics. The characteristics of problem solving in mathematics education were listed in the last paper of this Monograph.

The Editorial Board

1.
Students' strategies for problem solving



The study approaches of university students in a calculus class

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Abstract. 191 US first-year university students got the ASSIST (Approaches and Study Skills Inventory for Students) questionnaire as part of a mid-term course evaluation. The students were not in any science, engineering, or mathematics study programme but took a calculus course to satisfy the university breadth requirement for mathematics. The strategic approach was the most commonly used. There was a positive correlation between the deep and the strategic approaches and a negative correlation between the surface and the strategic approaches. There was no correlation between the deep and the surface approaches except a negative correlation between the sub-scales Lack of purpose and Interest in ideas. The surface approach had a negative effect, while the strategic approach had a positive effect on learning outcome.

Keywords: learning approaches, study approaches, deep approach, strategic approach, surface approach, calculus, university students, learning outcome

1. Introduction

The focus of this paper is the group of university students who are required to study mathematics, for instance calculus, even though they do not aim at studying mathematics, engineering, or science. Many students enrol in university programmes not wishing to have mathematics courses as a compulsory part of their study programme (Guzman et al. 1998). These students often have difficulties understanding the mathematical concepts (Morgan, 1990). Jukić and Dahl (2012, 2014) investigated the long-term retention of core calculus concepts by science and engineering students and found that mostly the students had a fragile knowledge base even though they had later in their study programme encountered calculus in other courses. Abramovich and Grinshpan (2008) furthermore argue for special teaching of mathematics to non-mathematics students in engineering, business, and life sciences. Hence, more research is needed on how different kinds of students approach and learn university mathematics such as calculus.

2. Theoretical background

2.1. Deep, surface, and strategic study approaches

Generally a student's approach to studying and learning is a collection of the student's intentions and strategies which to some extent is a reflection of the context and the demands the student meets (Gadelrab, 2011). Biggs and Tang (2007), Biggs et al. (2001), Entwistle (1991), Ramsden (1979) and others argue that the study and learning approach has a crucial role in relation to the quality of learning. They describe several types of approaches. In the 'surface approach' the students use low cognitive level activities and processes such as rote learning even when higher level activities might be intended by the teacher. The students are here focused on reproducing the material, describing, performing algorithms, etc. The focus is not on creating personal meaning of the material but to avoid failure with minimal effort. Students with a 'deep approach' try to use the most appropriate cognitive activities and processes to handle the material such as focusing on the underlying meaning and learning is a pleasure for them. The students intend to understand, analyse, generalise, hypothesise, evaluate, etc. A third approach, the 'strategic', was described by researchers later than the two other approaches. This is a well-organised surface approach focused on what is required in the examination. Students may here use both deep and surface approaches as they focus on both the content of the material and the mark. Their major intention is to achieve the highest mark possible using an organised study method.

The learning approach is a function of both the individual student's characteristics and the teaching style. It is to some extent a context-dependent response to how the student perceives the learning environment. Clouder (1998) argues that the pressure for knowledge acquisition within a finite time span make university students adopt strategic approaches. Previous studies have shown a relation between approaches to learning and the quality of learning. A deep approach is associated with high quality of student learning, but surface learning was related to poor learning outcomes (Gibbs, 1020; Biggs et al., 2001). However, Darlington (2011) discusses that a surface approach of for instance rote help fully understanding a theorem or a procedure. Cano and Berbén (2009) studied first-year university students in different science subjects but all enrolled in algebra and calculus courses. They found that performance goals (which correlates with surface learning approaches) correlates negatively with achievement. There is no correlation between mastery goals (which generally correlates with deep learning approaches) and achievement.

As stated above, the learning approaches of mathematics and non-mathematics students are different and in order to get deeper into researching the non-mathematics students' approaches, we need to discuss how to measure the learning approaches.

2.2. Measuring the approaches

One way to measure students' study and learning approaches is to use ASSIST (Approaches and Study Skills Inventory for Students) which is one of several questionnaires that have been validated in several contexts. ASSIST tests general study and learning approaches. It is based on the above mentioned three approaches whereas other questionnaires (e.g. that of Biggs et al., 2001) do not include the strategic approach. ASSIST has been developed over a period of time and some editions have been made to earlier versions. For instance the strategic approach has been broadened to include aspects of metacognition ('monitoring effectiveness') and the surface approach also emphasise ineffective studying ('lack of purpose') and was therefore renamed to Surface Apathetic Approach. The deep approach requires both holistic ways of thinking ('relating ideas') and serialist ('use of evidence'). The three approaches form three scales with subsequent sub-scales each related to central aspects for the approach. ASSIST has 52 items; four items for each sub-scale. Each item is a statement on a five-point Likert scale (5=Agree, 4=Agree somewhat, 3=Unsure, 2=Disagree somewhat, 1=Disagree) (Tait et al., 1998):

Deep Approach (D) with four sub-scales: 'Seeking meaning (SM)', 'Relating ideas (RI)', 'Use of evidence (UE)', 'Interest in ideas (II)'

Strategic Approach (S) with five sub-scales: 'Organised studying (OS)', 'Time management (TM)', 'Alertness to assessment demands (AA)', 'Achieving (AC)', 'Monitoring effectiveness (ME)'

Surface Apathetic Approach (A) with four sub-scales: 'Lack of purpose (LP)', 'Unrelated memorizing (UM)', 'Syllabus-boundness (SB)', 'Fear of failure (FF)'

'Alertness to assessment demands' (AA) was the last added sub-scale. 'Monitoring effectiveness' (ME) encompasses metacognition and self-regulation. It is argued that AA and ME are mainly applicable to graduate students (Gadelrab, 2011). ASSIST was used to answer the following questions:

3. Research questions

In relation to non-mathematics first-year university students who are required to take a mathematics class: What are their learning and studying approaches and how are these approaches related? How do they respond to the two sub-scales (AA, ME) more intended for graduate students? How do their approaches relate to their learning outcome?

4. Methodology

4.1. Selection of student cohort

The study took place in the autumn of 2011 at a private US university in top 10 of Times Higher Education Ranking of North American Universities 2011-12. The author was a visiting scholar. The university was chosen through purposive sampling (Robson, 2002) as the author needed a university where students not in science, technology, engineering, or mathematics (STEM) study programmes are nevertheless required to study mathematics. In many countries (including the author's), university students are only required to study calculus, if they enter into a STEM study programme. But it is debated if other student groups need calculus and if and how they are able to learn it. It is also frequently discussed how much mathematics, including calculus students should learn in high school. The study does not focus on high school pupils but since the study is about first year students, one can assume that it might give some indications of at least students in their last years of high school. A part of the study has been published in Dahl (2017) which focus on the results of solving calculus tasks.

The university offers two courses on introductory single variable calculus. They cover the same material but at different pace. Students are encouraged to take the slower one if they only need calculus to satisfy the university's disciplinary breadth requirement. The faster one is required for engineering, science, and economics study. This paper investigates the 191 students in the slower course. The students were divided into three cohorts for the lectures. The majority of the students had not taken the US Advanced Placement Calculus exam (high school level) or did not get a good score in it. There were two lecturers from the mathematics department and four tutors.

4.2. Use of ASSIST

ASSIST became part of the midterm course evaluation which was an online survey using google docs which the lectures were supposed to do anyway. The questions in the course evaluation where for instance: "How did you find the midterm?", "How well are you able to hear your lecturer?", "How many hours per week outside of lectures do you spend on the class?", "Do you have any specific comments for your lecturer?" etc. Adding the ASSIST items to the course evaluation was not only to accommodate the author, but the lectures also found the ASSIST items useful to get to know the students and discuss with them how to study.

Smaller changes were made to nine ASSIST items. Some changes were from British English to American English: i.e. "mark" to "grade" (S02, S28). Other changes specified that the statements were about studying calculus and not studying in general (D17, D26, D33, A51) for instance in D17 where "When I read an article or book. . . " was changed to "When I read a calculus book. . . ". Some reformulations were also made (D23, S36, A38) for instance in D23 where "Often I find myself questioning things I hear in lectures of read in books", "questioning" was changed to “pondering”. A42 was not included: “I’m not really interested in this course, but I have to take it for other reasons”. It did not fit well with the students’ circumstances. The university’s breadth requirement requires all students to take one mathematics class, whether or not they are interested in it. I.e. a student in this course could ‘disagree’ with the first part but would have to ‘agree’ with the last part. A42 belonged to the LP sub-scale, but this sub-scale included other items that investigated the students’ interest in calculus, for instance A16: “There’s not much of the work here that I find interesting or relevant”.

5. Results and discussion

The data was analysed using SPSS. 87 of 191 students answered the questionnaire after the lecturers emailed several reminders. The response rate is thus 46 %. Curtin et al. (2000) argues that the exclusion of cases that require several reminders does often not have an effect. Nulty (2008) states that online surveys achieve lower response rates than paper surveys and typically the response rates do not get above 47 %. Furthermore, the class was going to have the online mid term course evaluation survey anyway and during an in-class survey not all students are present. Krosnick (1991) also found that neutral responses are more frequent in surveys done in-class than online due to ‘satisficing’ where respondents tend to chose the middle ground for fear of judgement, the pace, or distractions. SCCE (2011) argues that a 33 % response rate is adequate for large classes (200 students). Nulty (2008) however argues that even when the response rates suggested are achieved, extrapolation of results is still to be done with care as this does not in itself secure that the survey results are representative of the whole group.

5.1. Analysis of internal consistency (reliability)

Cronbach’s alpha assesses the internal consistency and how the items are correlated with each other. The alphas were as follows: Deep Approach 0.888, Strategic Approach 0.904, and the Surface Apathetic Approach 0.809. Generally, $\alpha > 0.7$ is acceptable (Bland & Altman, 1997).

The r states the correlation between the item and the sum of the other items in the scale. If $r < 0.3$, one should consider to remove the item since it does not measure the same as the other items (de Vaus, 2002). The numbers for “α if deleted” (Table 1) estimates the scale alpha if the particular item is removed. If this number is higher than the scale alpha, one should consider removing the item to increase the internal consistency (Field, 2005). Even though ASSIST has been validated elsewhere, It was necessary to perform this analysis since changes have been made to some of the items and ASSIST is mixed with a course evaluation. Furthermore Gadelrab (2011) argues that ASSIST might not be consistent across different cultures and contexts. Different academic disciplines and contexts might also foster different student approaches. This might also be the case for this group of students required to take a mathematics class. Table 1 displays the six items where this number was higher than the scale alpha. No other item had $r < 0.3$.

Table 1. Analysis internal consistency of items that would give higher alpha if deleted.

Item	r	α if deleted
S27: I'm good at following up some of the readings suggested by lectures or tutors	0.328	0.905
S28: I keep in mind who is going to grade an assignment and what they're likely to be looking for	0.269	0.908
D26: I find that studying calculus can be quite exciting at times	0.266	0.893
A12: I tend to read very little beyond what is actually required to pass	0.133	0.819
A38: I gear my studying closely to just what seems to be required for assignments and exams	0.132	0.815
A51: I like to be told precisely what to do in assignments	0.143	0.812

Item S27 should however not be removed since alpha would only marginally increase and $r > 0.3$. The remaining five items are removed from the analysis. Although four of these five items were changed from the original ASSIST, one cannot conclude that these changes caused the low r. The five other items that were changed all had $r > 0.3$.

After removing these five items, the alphas became: 0.893 (Deep Approach), 0.908 (Strategic Approach), and 0.834 (Surface Apathetic Approach). An analysis then showed that A25 (I concentrate on learning just those bits of information I have to know to pass) had $r = 0.204$. Removing A25 changed the alpha to 0.841 with all $r > 0.3$. In all four A-items that were removed constitute the SB (syllabus-boundness) sub-scale.

S28 was the only item that was removed from the sub-scales of ME and AA. 'Alertness to assessment demands' (AA) and 'Monitoring effectiveness' (ME) are mainly applicable to graduate students (see above) and both ME and AA have a significant and strong positive correlation with the strategic scale. This indicates that the two sub-scales can be used for these first-year students, perhaps due to that it is difficult to get into the university where the study took place. Scale A finally consisted of 11 items, D had 15 items, and S had 19 items. Since some items are removed from the analysis, comparison with other uses of ASSIST is to be made cautiously.

5.2. Analysis of approaches

The mean score of each scale calculated; Surface Approach ($M = 66.11$, $SD = 12.15$), Deep Approach ($M = 47.65$, $SD = 10.25$), and Strategic Apathetic Approach ($M = 32.30$, $SD = 7.87$). In a survey of undergraduate mathematics students from University of Oxford (UK), the equivalent means were $M = 69.03$, $M = 58.69$, and $M = 45.94$ (Darlington, 2011). Although one cannot just compare the means due to slightly different versions of ASSIST, we see that both student groups (from top universities) preferred the strategic approach, then the deep, and then the surface approach. Biggs et al. (2001) argues that the prevailing approach tells something about the quality of the teaching and that the presence of a surface approach is a signal that either the teaching or the assessment is not aligned since students have been allowed to adopt this approach. Following this line of arguments, the students in this study appear to meet requirements that usually do not allow them to adopt surface approaches even though they may not be interested in calculus. Table 1 gives an overview of how the three scales relate to each other.

Table 2. Correlation between the scales.

	r	p-value
Deep – Strategic	0.518	0.000
Apathetic/surface – Strategic	−0.261	0.034
Deep – Apathetic/surface	−0.009	0.944

There is a strong positive correlation between the deep and the strategic approaches and a rather strong negative correlation between the surface and the strategic approaches. The surface-deep relationship has r close to zero, but the p-value is close to one. To get deeper into the relationship between D and A, calculation of sub-scale correlations was done. The only significant relationship ($r = -0.249$, $p = 0.033$) was between LP (Lack of purpose) and II (Interest in ideas). Correlation does not imply causation. However the latter result might indicate that the more the students feel a lack of purpose (which the students in this study might indeed feel as mathematics is required), the less they are interested in the ideas in the course; or vice versa, the less they are interested in the ideas in the course, the more lack of purpose is experienced.

5.3. Analysis of relation to perceived achievement

The author were not allowed access to the students' marks, but one of the items at the course evaluation asked the following: "Please rate yourself objectively, based on the grades you have been obtaining. How well have you been doing so far?" The scale was 1–10.

Table 3. Correlation between the scales and the students' rating of own performance.

	r	p-value
Deep	0.077	0.521
Apathetic/surface	−0.584	0.000
Strategic	0.292	0.013

The surface approach strongly correlates negatively with the (perceived) learning outcome. This fits the result mentioned above where the mean of the surface approach was quite low. The students probably did not often adopt this strategy as it was not useful. The strategic approach quite strongly correlates with the (perceived) learning outcome. The result for the deep approach is not significant. Gadelrab (2011) found a positive correlation between both the deep and the strategic approaches and academic success. He argues that many students combine focusing on understanding the material with achieving the highest possible marks. The students in this study do not appear to use such a combination.

6. Conclusions

There are eight main findings concerning the non-mathematics first-year university students who are required to study mathematics: (1) The students mainly adopt a strategic approach. (2) There is a positive correlation between the deep and the strategic approaches, (3) a negative correlation between the surface and the strategic approaches, (4) and no significant correlation between the surface and the deep approaches except between the sub-scales Lack of purpose and Interest in ideas that are negatively correlated. (5) The surface approach correlates negatively with the perceived learning outcome, (6) the strategic approach correlates positively with the perceived learning outcome, (7) there is no significant correlation between the deep approach and learning outcome. (8) The students have an alertness of assessment demands (AA) and are able to monitor effectiveness (ME) even though the students are not yet in graduate study. A limitation is that although an adequate response rate was achieved, one cannot be sure that the 46 % who answered the survey are representative of all the students in the class.

The findings 1–3, 5–6 are similar to findings by others (see above) which indicate that the students in this study to some extent behave like other students even though they are taking a course they would otherwise not have chosen. However, the analysis also showed that the deep approach was not much adopted and the deep approach did not positively correlate with the perceived learning outcome or negatively correlate with a surface approach – as seen elsewhere. The students thus mainly adopt a strategic approach and they find that they are successful using this approach even though their interest is low which is also related to that they feel lack of purpose. They have experienced that the surface approach is not useful in learning the material which to some extent might also be a reflection of how the course is being taught, i.e. the students cannot achieve success in this course by only using a surface approach.

7. Acknowledgement

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References

[1] ABRAMOVICH, S. AND GRINSHPAN, A. Z. (2008), *Teaching Mathematics to Non-Mathematics Majors Through Applications*, PRIMUS, 18(5), 411–428.

[2] BIGGS, J. B., KEMBER, D. AND LEUNG, D. Y. P. (2001), *The Revised Two Factor Study Process Questionnaire: R-SPQ-2F*, British Journal of Educational Psychology, 71, 133–149.

[3] BIGGS, J. AND TANG, C. (2007), *Teaching for Quality Learning at University*, Maidenhead: Open University Press.

[4] BLAND, J. AND ALTMAN, D. (1997), *Statistics notes: Cronbach's alpha*, BMJ (British Medical Journal), 314, 572.

[5] CANO, F. AND BERBÉN, A. B. G. (2009), *University students' achievement goals and approaches to learning in mathematics*, British Journal of Educational Psychology, 79, 131–153.

[6] CLOUDER, L. (1998), *Getting the 'Right Answers': Student evaluation as a reflection of intellectual development?*, Teaching in Higher Education, 3(2), 185–195.

[7] CURTIN, R., PRESSER, S., & SINGER, E. (2000), *The effects of response rate changes on the index of consumer sentiment*, Public Opinion Quarterly, 64(4), 413–428.

[8] DARLINGTON, E. (2011), *Approaches to Learning of Undergraduate Mathematicians*, In C. Smith (Ed.), Proc. of the British Society for Research into Learning Mathematics, 31(3), 8.

[9] DAHL, B. (2017), *First-year non-STEM majors' use of definitions to solve calculus tasks: Benefits of using concept image over concept definition?*, International Journal of Science and Mathematics Education, Accepted for publication, doi: 10.1007/s10763-016-9751-9.

[10] DE VAUS, D. (2002), *Surveys in Social Research*, London: Routledge.

[11] ENTWISTLE, N. J. (1991), *Approaches to learning and perception of the learning environment: Introduction to the Special Issue*, Higher Education, 22, 201–204.

[12] FIELD, A. P. (2005), *Discovering statistics using SPSS* (2^{nd} ed.), London: Sage.

[13] GADELRAB, H. F. (2011), *Factorial Structure and Predictive Validity of Approaches and Study Skills Inventory for Students (ASSIST) in Egypt: A Confirmatory Factor Analysis Approach*, Electronic Journal of Research in Educational Psychology, 9(3), 1197–1218.

[14] GIBBS, G. (2010), *Dimensions of Quality*, Heslington: The Higher Education Academy.

[15] GUZMAN, M., HODGSON, B. R., ROBERT, A. AND VILLANI, V. (1998), *Difficulties in the passage from secondary to tertiary education*, In G. Fischer and U. Rehmann (Eds.), Proc. of the International Congress of Mathematicians (Documenta Mathematica, Extra Volume ICM 98, Vol. III, pp. 747–762), Berlin: ICM.

[16] JUKIĆ MATIĆ, LJ. AND DAHL, B. (2014), *Retention of Differential and Integral Calculus: A Case Study of a University Student in Physical Chemistry*, International Journal of Mathematical Education in Science and Technology, 45(8), 1167–1187.

[17] JUKIĆ MATIĆ, LJ. AND DAHL, B. (2012), *University students' retention of derivative concepts 14 months after the course: Influences of 'met-befores' and 'met-afters'*, International Journal of Mathematical Education in Science and Technology, 43(6), 749–764.

[18] KROSNICK, J. A. (1991), *Response Strategies for Coping with the Cognitive Demands of Attitude Measures in Surveys*, Applied Cognitive Psychology, 5(3), 213–236.

[19] MORGAN, A. T. (1990), *A study of difficulties experienced with mathematics students in higher education*, International Journal of Mathematics Education in Science and Technology, 21(6), 975–988.

[20] NULTY, D. D. (2008), *The adequacy of response rates to online and paper surveys: What can be done?*, Assessment & Evaluation in Higher Education, 33(3), 301–314.

[21] RAMSDEN, P. (1979), *Student learning and perception of the academic environment*, Higher Education, 8(4), 411–427.

[22] ROBSON, C. (2002), *Real World Research*, Oxford: Blackwell.

[23] Senate Committee on Course Evaluation [SCCE] (2011), *Review of the Course Evaluation Questionnaire (CEQ)*, A report submitted to Senate by the Senate Committee on Course Evaluation, January 2012, 7 December 2011, Memorial University, Retrieved 3 January 2013 at `http://www.mun.ca/ceq/CEQReview/SCCE_Review_Report_7Dec2011.pdf`.

[24] TAIT, H., ENTWISTLE, N. J. AND MCCUNE, V. (1998), *ASSIST: a reconceptualisation of the Approaches to Studying Inventory*, In C. Rust (Ed.), Improving students as learners, Oxford: Oxford Brookes University, The Oxford Centre for Staff and Learning Development, see also: `http://www.etl.tla.ed.ac.uk/questionnaires/ASSIST.pdf`.

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Universitetsstuderendes studiestrategier på et calculushold

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Abstrakt. 191 førsteårsstuderende fra USA fik ASSIST (Approaches and Study Skills Inventory for Students) spørgeskemaet som del af en midtvejsevaluering på et kursus. Ingen af de studerende læste til ingeniør eller var indskrevet på naturvidenskab eller matematik, men alle tog calculus for at opfylde universitetets krav om bredde i uddannelse, herunder matematik. Det var den strategiske fremgangsmåde som blev brugt oftest. Der var en positiv korrelation mellem den dybdegående og den strategiske fremgangsmåde og en negativ korrelation mellem den overfladiske og den strategiske fremgangsmåde. Der var ingen korrelation mellem den dybdegående og den strategiske fremgangsmåde på nær på under-skalaerne: 'Lack of purpose' og 'Interest in ideas'. Den overfladiske fremgangsmåde havde en negative effekt på læringsudbyttet, mens den strategiske fremgangsmåde have en positive effekt.

Søgeord: læringsstil, studiefremgangsmåde, dyb fremgangsmåde, strategisk fremgangsmåde, overfladisk fremgangsmåde, calculus, universitetsstuderende, læringsudbytte

Searching for a common ground in mathematics and physics education: the case of integral

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Abstract. In this paper we present the overview of the results of our study of university students' graph interpretation strategies and difficulties in three different domains: mathematics free of context, mathematics with (a real-life) context and physics (kinematics). We focus mostly on the interpretation of the total change of a quantity, i.e. its integral. Additionally, we also present results of a questionnaire involving students of mathematics and physics teacher programme when asked to explain the meaning of the area under a graph in different contexts.

Keywords: interdisciplinary field, mathematics and physics education, slope, area under a graph, linear graphs

1. Introduction

In mathematics, as well as in physics, students are often invited to extract information from visual representations of quantities, in particular from graphs showing their functional dependence. Thereby, they are required to recognize and interpret mathematical concepts that can be read off or interpreted from graphs in different contexts. In this paper we address the question of pre-calculus students' understanding of the concepts of derivate and integral of quantities presented by functional graphs in the context of mathematics and physics. These concepts have been studied in mathematics and physics education separately and there is already some evidence of students' difficulties (Leinhart 1990, McDermont 1987). However not many researches attempted to compare and relate these difficulties in both domains.

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In our study, beside the (abstract) mathematics context parallel to the physics (kinematics) context, we also introduced the mathematical context with graphs from real-life situations. It is a context where the same mathematical concepts were studied, but with no special conceptual knowledge required. Our study addressed the high school knowledge of mathematics, that is, pre-calculus knowledge of high school students and first-year university students at the very beginning of their study. In this way, the studied concepts of derivate and definite integral appeared in graphical representation as slopes and areas under linear graphs.

2. Results of studies on graphs in mathematics and physics

Recently, understanding of mathematical concepts in graphs both in mathematics and physics contexts were addressed in several studies with the population of university students during or after their calculus course. Thus, the study of Wemnyss & van Kampen, 2013 investigated approaches taken by first-year university students when determining a speed value from linear distance-time graphs, water level versus time graphs and context free graphs. All graphs did not go through the origin. The results suggested that many students realized context as an integral part of the question thus making the taken approaches as context dependent. The success rate of determining the value of speed was found to be low. The first study was followed by an intervention whose aim was to verify the validity of the next study. This post-test analysis showed a great improvement in mathematics context in determining the value of the slope. Still, improved mathematics was insufficient for students to successfully determine a value of the speed in physics context. Questions on water level versus time graphs were answered better than physics questions, before and after instruction.

In the study of Bajracharya & Thompson, 2014 students' understanding of the Fundamental theorem of calculus in graphical representations in mathematics and physics context was investigated. The participating students were mathematics and physics students. The study was motivated by common expectations from physics students to relate the rate of change (derivative) and the accumulation (definite integral) of a physical quantity. Quantitively, on written questions about half of students gave correct answers. Still, the study identified various students' difficulties, such as attempts to evaluate antiderivatives at individual points, confusing antiderivative by a considered function and using slope rather than area to determine the integral.

The study of Jones, 2015a investigated various conceptualizations of the definite integral concept by university students in USA. The study showed that Riemann-based conception, i.e. multiplicatively based summation, proved to be more productive in contextualized questions than area and antiderivative conceptions. At the same time the two latter conceptions showed a high prevalence when compared to Riemann-based conceptions. However, findings did not imply that the area and anti-derivative aspects are less important, but that it was at least equally important for a student to have accessible summation (i.e. accumulation) conception of the definite integral.

A study conducted in a field of business was carried out by Veldhuis & Korzilius, 2016 and it involved students of a university business school. The researchers linked interpretation of graphs in physical context of kinematics to dynamic systems of the behavioural patterns of stocks – their increase when the inflow exceeds the outflow or decrease otherwise. However, the study also investigated how individual's inference of accumulation interpreted from graphs was related to their spatial abilities. Understanding of accumulation was investigated by the Department Store Task (basic understanding of graphical presentation), Bathtub and Cash flow task, whereas spatial ability by Paper folding test (spatial visualization) and Mental rotation test (spatial relations). Results showed a positive relation of dynamic system analysis to spatial visualization.

In our previous studies (Planinic et al. 2012, Planinic et al. 2013, Ivanjek et al. 2016) we compared choices of students' strategies when solving parallel problems in different domains, as well as their main difficulties in each domain and how they are related. Our first study was motivated by the physics teachers' beliefs that high school students lack the mathematical knowledge required for successfully solving physical problems. However, the results of the study suggested that, contrary to teachers' beliefs, mathematics was not the main obstacle for solving physics problems. Our next study addressed university students. It was conducted by a questionnaire with eight parallel question in each context, mathematics without a context, mathematics with a real-life context (i.e. mathematics in contexts other than physics) and physics – kinematics. Questions were parallel in a sense that they required use of the same mathematical procedure to be solved. Mathematics questions directly investigated the concepts – calculation of slope, its properties and comparison for two different line graphs, and calculation of the area under a linear graph. Real-life and physics context question required also understanding of concepts of rate of change or total amount together with their interpretation from graphs.

In our study a natural question of transfer of knowledge was posed. Transfer of knowledge could be defined as the ability to extend concepts from one context to a new context. However, as suggested by Hammer et al., 2005, it would be more appropriate to speak about so-called activation of resources. Concepts are complex entities, made of many elements, and could not be transferred as intact units. Certain aspects of a concept can be activated in a particular context, while other aspects can remain aside. Besides, transfer of knowledge can be considered through a dynamical perspective called preparation for future learning (PFL) which considers student's possible learning and reconstructing of knowledge while transferring.

Our most dominant finding was that the strategies used by students were context depended and domain specific. Students generally did not use the same strategy on two or all three parallel questions. This suggested that students did not recognize that they are using the same concept but in different domains. Therefore, there was no direct evidence of transfer of knowledge between domains on the large scale. It was confirmed that each domain carries its own characteristics and discipline conventions thus provoking use of different, often learnt, procedures. For physics questions, the results showed that the preferred strategy was the use of formulas which relate kinematics concepts. On real-life context problems students often used creative and self-developed strategies.

In our study students' difficulties with graphs were identified with similar expressions in all domains. Very common difficulties for the concept of slope were confusion of a height of a graph with its slope and the interval-point confusion. However, these difficulties were more frequent in physics than in mathematics (even twice more frequent). In mathematics, the most difficult problem for students was the mere calculation of the slope of a line given by its graphical representation.

For the concept of area under a graph, the performed overall Rasch analysis showed that it has lower difficulty than the concept of slope in mathematics. However, the Rasch analysis also showed that this difficulty was significantly lower in mathematics domain than in other two domains, thus suggesting that additional context increased the difficulties of domains. On mathematics questions students were directly asked to calculate area under a linear graph. Although this simple procedure also revealed students' difficulties, even though the questions did not include usually problematic notion of "area under the x-axis", mathematics questions lacked interpretation of the meaning of the area under a graph. We may state that it is not surprising that area under a graph questions showed lower difficulty than slope questions since the concept of area is an intuitive, familiar concept developed from early years of mathematics education compared to, results suggest, rather vague concept of slope.

As already stated, in physics context and mathematics with real-life context, questions on area under a graph required additional interpretation, which increased the difficulties of the domains. The requirement turned out to be very difficult for students, surprisingly it seemed even more difficult in physics than in a real-life context. In physics context, when calculating a distance travelled from v vs. t graph or speed from a vs. t graphs, students often used learnt or ready-made strategy to find these values from the area under a graph. In a real-life context in order to decide which feature of a graph represents the required information, students used creative strategies, showing some understanding of the concept of the total amount of a quantity. For example, there was evidence that students used the idea of an accumulated quantity, of "adding-up pieces", while also some of them used dimensional analysis of quantities. The last idea was used three times more often in real-life context than in physics, which could be an evidence of transfer of knowledge from physics to a real-life context.

3. Results of the present study

Whereas mathematical concept of area under a graph is a familiar, intuitively well understood concept, its interpretation, that is, why it gives the required information, is very difficult for students. The process of finding a distance travelled from a v vs. t graph includes "reversion" in relation to the process of finding a speed from s vs. t graph, together with its graphical interpretation as an area under a graph. Speaking in the terms of calculus, the process revokes identification of antiderivative as well as calculation of the definite integral. It is a students' difficulty described as "how are the concepts "tangent" and "area" related?" or "finding the slope of the tangent line and calculating the area are inverse operations" (Kirsch 2014) in the context of the Fundamental theorem of calculus.

In high school physics curriculum, geometrical interpretation of the definite integral or total amount of a quantity as the area under a graph is introduced already with linear graphs for velocity in a simple uniform motion or uniformly accelerated motion. Very often it is justified by means of corresponding formulas from physics context and formulas for area and becomes a learnt strategy which is readily applied.

However, in our opinion a more comprehensible and coherent understanding of the idea behind area under a graph is built by introducing also the idea of accumulation (summation) or "adding-up pieces". Results of our previous study underpin this idea by evidence obtained from students' answers on mathematics questions in a real-life context. The studies of Jones 2015a, Jones 2015b also support this idea for university students with integrals in contextualized domains, in particular in physics and applied sciences.

Our study was motivated by the lack of the mathematics high school curriculum support for the various aspects of the concept of the definite integral for linear graphs. Even though the concept of derivative, introduced geometrically as a slope, appears much later in mathematics education than the concept of area in general, it is discussed much more often throughout education than the concept of integral, geometrically seen as the area under a graph. Even when a concept of the definite integral in mathematics is introduced by means of Riemann sums, it is mostly used for calculation of irregular figures areas which again readily connects summation process to the area.

Context-free mathematics usually does not provide an intuitive explanation why to use area under the graph, whereas adding a context does, as stated in Kouropatov, Dreyfus, 2013:

"The idea of calculus in general and the idea of the integral in particular were born from attempts to understand the world, from applications by Newton and his followers. In some way, the integral is the application. Therefore, in our eyes, *there is no way to understand integrals without understanding the strong connection between the mathematical concept and its applications*. The heart of this connection is the idea of accumulation. That is why implementations of accumulation in different contexts are completing our journey."

Our research question was formulated as follows: To what extend preservice teachers of mathematics and physics recognize, use and transfer different aspects of the definite integral in mathematics with real-life context and in physics context, particularly the intuitive idea of accumulation, when applied to linear graphs?

The study addressed the whole cohort ($n = 7$) of graduate students of mathematics and physics in teacher programme at the Faculty of Science, University of Zagreb. Students completed a five-questions questionnaire. Three questions determined the understanding of the concept of the total amount of the quantity by asking about the total amount of water and cash, including “the negative area” idea, comparison of different quantities and the moment in which the maximal amount was realized. The fourth question investigated the transfer of the observed concepts into a physical context by student’s creation of a possible physics counterpart of previous questions. The last question asked for students’ explanations of the chosen strategies as high school teachers.

On questions from mathematics with real-life context results of the study can be summarized as follows:

- Correct answers: Most of the students’ answers were correct. Not all students recognized the intersection of the graph with the x-axis as the maximal water quantity. Some pointed out the possible learner’s difficulties with such graphs because “the rate of change is positive but *decreasing* whereas the total amount of water is *increasing*” and “the negative inflow should be considered as outflow”.
- Strategies: Almost all students explained their answers by ready-made use of the area under a graph strategy, showing that they recognized the concept and did not consider it necessary to provide any further elaboration. Two students also performed dimensional analysis (Figure 1). Most of the students gave a correct graph of the total water amount in a bathtub (Sweeney, 2000). Two students very efficiently answered the question by graphically considering differences of areas (Figure 2). Other students performed calculation showing some elements of accumulation i.e. “adding-up pieces” strategy.

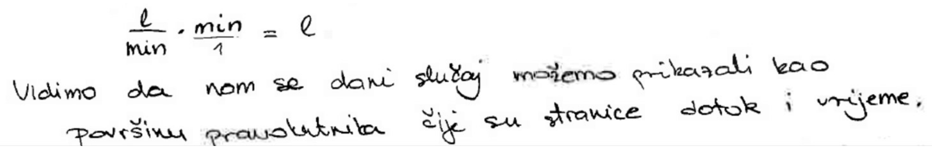

Figure 1. Dimensional analysis as a solution in a real-life context.

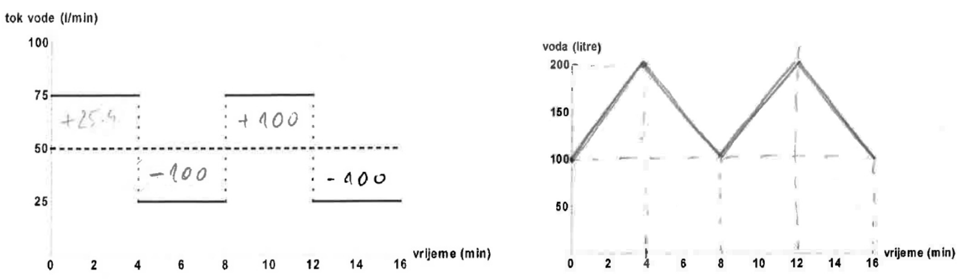

Figure 2. Differences of areas as the change of water amount.

Students’ explanation of area under a graph strategy in the last question was mostly done either by using a detailed dimensional analysis (Figure 3) or by elaborating in words that are meaningful in the context: “If we bring into the bathtub more water per unit of time than we take out, then the total amount of the water is increasing” or “one should add all cash amounts per months”. One student suggested how to calculate various areas under a graph, starting from a rectangular, followed by a triangle and a trapezium, thus connecting formula for areas to the formulas needed in the physics context.

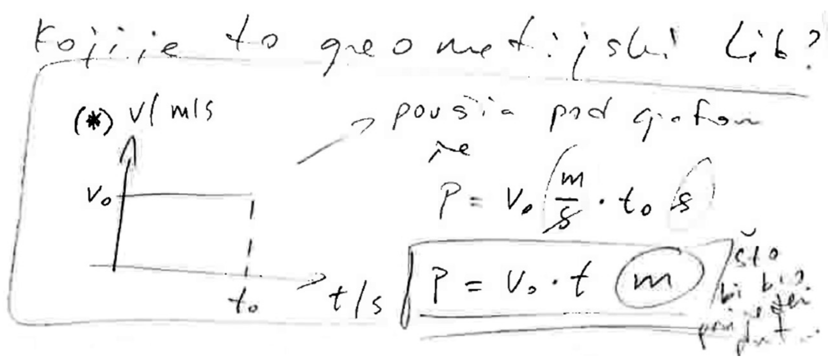

Figure 3. Dimensional analysis as an explanation of the area under a graph strategy.

Not all students were successful on creating similar questions in physics context. Those who were, created questions that involved various physical quantities, like force and distance, acceleration and spring elongation, energy and distance. However, most of the ideas were from kinematics with a speed of a motion and distance travelled. The following example is a creative physics counterpart of the bathtub problem:

“Consider a car with four-wheel drive. Two front wheels pull the car forward, while the other two pull in the opposite direction. If their speeds are shown by the graph, determine the direction and total distance travelled by the car if it is initially 100 m far from the start line.”

4. Conclusion and implications for high school mathematics curriculum

The study conducted with the graduate students of mathematics and physics in teacher programme, on the aspects of the definite integral in mathematics and physics context with linear graphs, showed the high prevalence of the area under a graph strategy as a ready-made strategy. The use of this strategy was mostly elaborated by dimensional analysis of involved quantities. Whereas these ideas are relevant and meaningful for the problems, students’ choices reflect typical ideas and methods used in their education. However, the idea of accumulation, as underlying idea of the area under a graph strategy, would possibly provide a more coherent concept construction and activation.

We promote a student’s perspective on education and hope that it is on larger scale organized as a meaningful and connected whole. The findings of our and other studies in mathematics and physics education indicate certain compartmentalization and ask for strengthening of connections within various themes and fields. The following recommendation could be useful for deeper conceptual understanding of derivative (rate of change) and the definite integral (total amount of a quantity) in high school curriculum:

- Use of good intuitive examples from real world as well as kinematics examples with their graphical representation might provoke use of multiple strategies and comprehensible understanding;
- Calculation and understanding of the concept of slope could be strengthened by use of a “rise over run triangle” in line graphs throughout mathematics education, as e.g. *Steigungsdreieck* in mathematics in German. The prevalence of the use of the formula $v = s/t$, where speed is interpreted simply as “distance over time” as a description of a very simple kinematics model, can be an obstruction for understanding of concepts of velocity and the rate of change of a quantity. “*Average speed equals change in distance over change in time*”.
- The total amount of a quantity deserves a more coherent attention throughout the mathematics school curriculum with emphases on its aspects as the area under a graph and accumulation or “adding-up pieces”, and later as the antiderivative. Interpretation of the concept only by the area under a graph as a ready-made strategy provokes difficulties which remain present even after introducing the concept of the definite integral.

The concept of the total amount is an important concept in mathematics. At the end, once again we recall its everyday presence by a description from Veldhuis & Korzilius 2016:

“The flow of materials ... is present in all facets of life; it determines what your bank statement looks like at the end of the month or whether you overfill your coffee cup in the morning.”

References

[1] BAJRACHARYA, R. R., THOMPSON, J. R. (2014), *Student understanding of the Fundamental theorem of calculus at the mathematics-physics interface*, Research in Undergraduate Mathematics education RUME 17, Denver.

[2] HAMMER, D., ELBY, A., SCHERR, R. E., AND REDISH, E. F. (2005), *Resources, framing and transfer*, in Transfer of Learning from a Modern Multidisciplinary Perspective, edited by J. Mestre, Information Age Publishing, Greenwich, CT, 89–120.

[3] IVANJEK, L., SUSAC, A., PLANINIC, M., ANDRASEVIC, A. & MILIN SIPUS, Z. (2016), *Student reasoning about graphs in different contexts*, Physical review special topics – Physics education research 12, 010106.

[4] JONES, S. R. (2015a), *The prevalence of area-under-a-curve and anti-derivative conceptions over Riemann sum-based conceptions in students' explanations of definite integrals*, International Journal of Mathematical Education in Science and Technology, 46(5), 721–736, `http://dx.doi.org/10.1080/0020739X.2014.1001454`.

[5] JONES, S. R. (2015b), *Areas, anti-derivatives, and adding up pieces: Definite integrals in pure mathematics and applied science contexts*, Journal of Mathematical Behavior 38, 9–28.

[6] KIRSCH, A. (2014), *The fundamental theorem of calculus: visually?*, ZDM Mathematics Education 46:691–695, doi: 10.1007/s11858-014-0608-9.

[7] KOUROPATOV, A., DREYFUS T. (2013), *Constructing the integral concept on the basis of the idea of accumulation: suggestion for a high school curriculum*, International Journal of Mathematical Education in Science and Technology, `http://dx.doi.org/10.1080/0020739X.2013.798875`.

[8] LEINHART, G., ZASLAVSKY, O., STEIN, M. K. (1990), *Functions, graphs, and graphing: Tasks, learning, and teaching*, Review of Educational Research, 60(1), 1–64.

[9] MCDERMONT, L. C., ROSENQUIST, M. L., VAN ZEE, E. H. (1987), *Student difficulties in connecting graphs and physics: Examples from kinematics*, American Journal of Physics 55, 503, doi: `http://dx.doi.org/10.1119/1.15104`.

[10] PLANINIC, M., MILIN SIPUS, Z., KATIC, H., SUSAC, A. & IVANJEK L. (2012), *Comparison of student understanding of line graph slope in physics and mathematics*, International Journal of Science and Mathematics Education 10(6), 1393–1414.

[11] PLANINIC, M., SUSAC, A., IVANJEK, L. & MILIN SIPUS, Z. (2013), *Comparison of university students' understanding of graphs in different contexts*, Physical review special topics – Physics education research 9, 020103.

[12] SWEENEY, L. B., STERMAN. J. (2000), *Bathtub dynamics: Initial Results of a System Thinking Inventory*, System Dynamics Review, 16(4), 249–286, doi: 10.1002/sdr.198.

[13] VELDHUIS, G. A., KORZILIUS H. (2016), *Seeing with the Mind: The Relationship Between Spatial Ability and Inferring Dynamic Behaviour from Graphs*, System Research and Behavioural Science, Willey Online Library, doi: 10.1002/sres.2404.

[14] WEMNYSS, T., VAN KAMPEN, P. (2013), *Categorization of first-year students' interpretation of numerical linear distance-time graphs*, Physical review special topics – Physics education research 9, 020107.

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Određeni integral u nastavi matematike i fizike

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Sažetak. U ovom radu prikazan je pregled provedenih istraživanja autorica o strategijama i poteškoćama studenata pri interpretaciji matematičkih i fizikalnih veličina iz grafičkih prikaza. Grafički prikazane situacije pripadale su trima različitim domenama: matematici bez konteksta, matematici s kontekstom iz stvarnih životnih situacija, te fizici (kinematici). U ovom radu, naglasak je stavljen na interpretaciju ukupne promjene veličine, dakle, na njezin određeni integral koji se geometrijski interpretira kao površina ispod grafa. Uz navedeno, prikazani su i rezultati istraživanja provedenog među studentima nastavničkog smjera matematike i fizike o strategijama kojima se koriste pri određivanju ukupne promjene veličine zadane u kontekstu stvarnih situacija, te njihovom obrazloženju.

Ključne riječi: interdisciplinarnost, nastava matematike i fizike, nagib, površina ispod grafa, graf linearne funkcije

Functions in the 2015 and 2016 Croatian State Matura in higher level Mathematics

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Abstract. This paper presents an analysis of Croatian State Matura exam tasks relating to the mathematical domain of functions. The analysis is conducted on the 2015 and the 2016 State Matura, both higher-level mathematics exams. Requirements of the tasks are explored and classified based on the instrument designed for this purpose. The instrument is conceptually based on the theory of "basic ideas" (originally: Grundvorstellungen) and Mathematical Proficiency theory. The objectives of the research were to explore the possibility of using such an instrument for task analysis, to determine the characteristics of the tasks in the Croatian State Matura, and to use student achievement to value their efficiency in the mathematical domain of functions in terms of specific classification provided by the instrument.

Keywords: state matura, Grundvorstellungen, mathematical proficiency, function, task analysis

1. Introduction

For the last seven years, gymnasium students and all those who wish to continue their education at university in the Republic of Croatia have taken the Croatian State Matura exam, the secondary school exit examination the results of which are the basis for entry into higher education institutions. Since the State Matura is a form of external assessment and is conducted on a great number of students, it seems worthwhile to explore it; both in terms of the different aspects of requirements and how well students deal with them. Mathematics is one of the three exams that are mandatory for all participating students. Students can choose between two levels: *higher* (A) and *standard* (B). According to the examinational catalogue (NCEEE, 2014), the document that stipulates fields of examination, educational outcomes and the technical structure of the exam, the level required in the higher-level exam corresponds to the gymnasium syllabus. For this reason, only the higher-level State Matura exam will be analyzed in this paper.

The higher-level exam score is 60, of which 20 points are from multiple choice questions (a choice of 4 answers), 26 points from short answer questions (only correct answers are scored), and 14 points from long answer questions (answers and the task solving process are scored). Five mathematical domains are tested in the Croatian State Matura: Numbers and Algebra, Functions, Equations and Inequalities, Geometry and Mathematical Modeling. This paper will focus on the mathematical domain of functions in the State Matura. The proportion of functions in the exam is 25 % of the score share. Taking into consideration a possible deviation of ± 5 %, functions are in the range of 12 to 18 points. The content of educational outcomes for the mathematical domain of functions refer to the concept of functions in general (different functional representations, and operations with functions), special examples (linear function, quadratic function, absolute value function, square root function, polynomials and rational functions, exponential function, logarithmic function, and trigonometric functions), series and derivatives. A detailed list of educational outcomes can be found in the Examination Catalogue (NCEEE, 2014).

Two key ideas from the didactic of mathematics will be presented in this paper under the heading "Conceptual base". This is in order to lay down a theoretical base underscoring the task analyses instrument. The instrument for the task analyses will be introduced in the "Instrument" section. Through a small number of examples it will be shown how the instrument is to be used as a theoretical criterion for task analysis. Classification of the higher-level 2015 and 2016 State Matura function tasks, according to the instrument, will be given in "Results and discussion". Also, student achievement, in general and in relation to the given classification, will be considered[1].

2. Conceptual base

There are many important ideas, in the form of "mental representations", behind mathematical content, that are important not only for understanding mathematical content but also for improving general education (Malle, 2004). To be able to answer the question: "What mental representations students have while thinking about mathematical concepts" vom Hofe (1995) developed the idea of *"Grundvorstellungen*[2]". This is the concept that describes the relationship between mathematical structures, individual processes and subject-related contexts (in brief: the relationship between mathematics, the individual and reality). In a way, GVs serve as translators between reality and mathematics. There are three aspects of GVs: the *normative* aspect is the one intended from the teacher, it expresses how students should be taught to achieve conceptual understanding; the *descriptive* aspect explains how students actually represent given concepts; and the *constructive* aspect

[1] This data is courtesy of NCEEE Center in Zagreb. The data given in this paper fully respects the NCEEE values and confidentiality.

[2] *Grundvorestellungen* in English language literature is sometimes referred as "basic ideas", but authors generally refrain from using it to avoid possible misunderstandings due to the implications that may arise from that term. This paper will use the more common term in the form of the abbreviation GV.

deals with the generation of GVs. In the past two decades, a lot of work has been done in order to identify relevant GVs. Most of the work relates to primary school education, but GVs for percentage, angle concept, function concept, and the concepts of derivative and integral are also listed (vom Hofe, Blum, 2016).

In this paper, special attention will be given to Grundvorestllungen in relation to the concept of function. According to Vollrath (1989) there are three different GVs that can be activated:

[GV1] *Mapping* – one quantity is uniquely assigned to another. Functions describe connections between quantities: one quantity is seen as dependent on the other. It is closely connected to a sense of a function as a table of mutually depended values.

[GV2] *Covariation* – characteristic change of one quantity when another one changes. Functions determine what effect the change of a variable has on the dependent variable. It is closely connected to a sense of a function as the equation for evaluation. This GV is especially valuable when it is necessary to identify an important example of function from a situation based on the text. For example, in the sentence "The number of bacteria enlarges about 45 % per hour", the information about the same percentage of (bacteria) growth in the same time interval indicates characteristic change of exponential growth (Malle, 2004).

[GV3] *Object* – function as an "object" that can be described as whole, and not just as a pair of dependent values. It is closely connected to a sense of a function as a graph. Typical assignments associated with this GV are tasks relating to a graph of the function (when observed in general, not just a particular point), operations with functions (e.g. composition of functions), or tasks considering important functional properties (e.g. parity, periodicity). Also, when one of the important examples of functions is given in the task, with its name or equation, GV3 needs to be activated.

In their PISA 2000 task analyses investigation, with the aim of explaining the difficulties of the tasks, Blum at al. (2004) used the theory of Grundvorestellungen as a normative and empirical criterion. By determining which GVs are necessary for task solving, the authors grouped tasks into four levels, thereby defining the *"GV intensity"* variable. Levels go from Level 0 "no GVs required" to Level 3 "complex GV or non-trivial combination of Extended GV's required[3]". Using this scheme, both "*Solve the equation* $2x - 7 = 9$.", and *"The quadratic function* $f(x) = 2x^2 + 5x - 3$ *is given. Write* $f(x)$ *in a vertex form.""* (Blum et al., 2004) are Level 0 tasks, since no GVs are required to solve them. GVs are not the only factor in solving mathematical tasks successfully; requirements are extended to other competencies, knowledge and skills (vom Hofe, Blum, 2016). It is obvious

[3] Three kinds of GVs are distinguished: *Elementary GVs* are those whose perception is based on real object and actions (for instance GVs for elementary operations), *Extended GV* are non-trivial combinations of Elementary GVs (such as idea of increase of price which is the result of combining two other ideas: adding and percentage) or GVs whose perception is not related to the real, but more to imagined action in relation to other mathematical objects (functions), and *Complex GVs* are non-trivial combination of Extended GVs (concept of derivative as a combination of concepts of limit and gradient).

that to solve these two tasks students have to have different levels of mathematical understanding. While for the first task the student only needs to have technical skills for (simple) algebraic manipulations, for the second one the student has to understand the terminology behind the statement "*vertex form of quadratic function*" and has to have procedural skills for calculating parameters that are required (if we assume that the formula for coordinates of vertex of parabola is available to the student). The fact that this type of classification enables tasks that require a broad range of mathematical understanding to end up in the same category led me to think that it would be valuable to expand investigation criterions to include other important, from a mathematical point of view, competencies.

The framework for this could be provided by Kilpatricks' definition of mathematical proficiency, a comprehensive view of successful mathematics learning (Kilpatrick, 2001). According to this theory five equally important and mutually interdependent components, or strands, are defined: *conceptual understanding is a comprehension of mathematical concepts, operations and relations; procedural fluency is skill in carrying out procedures flexibly, accurately, efficiently and appropriately; strategic competence is the ability to formulate, represent, and solve mathematical problems; adaptive reasoning is the capacity for logical thought, reflection, explanation, and justification; and productive disposition is defined as habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy.*

Since the authors of this theory stress that these five strands are responsible for student success in mathematics, it seems to be a good framework for normative criterion for task analysis. In other words, it seems valuable to detect which of these strands and to what extent are being activated in Croatian State Matura exam tasks concerning functions. Since task analysis is at the center of this research, and there is no direct contact with students, *productive disposition* will be excluded from the criteria.

3. Instrument

In this section an instrument for task analysis will be introduced. In the section "Examples from the exam" task analysis on three tasks from the 2015 and 2016 higher level Matura exams will be performed.

Normative criterion for task analysis is a two-dimensional instrument with the objective to identify and describe task characteristics. The first component is the *conceptual characteristics* of the task and the second component is the *skills and activities* that need to be available to solve the Matura tasks, relating to functions, successfully.

Conceptual characteristics are correlated with Kilpatricks conceptual understanding and interpreted to feature two aspects: existence of GVs related to the concept of function and the use of mathematical language. The second component, *skills and activities*, unites three important aspects and describes their involvement in the process of task solving. These three aspects encompass: procedural fluency, strategic competence and adaptive reasoning. The two components will be introduced in more detail in the following part.

The first aspect refers to identification which of the three (if any) function related GVs are activated in the task. This is related to [GV1], [GV2], [GV3] described in the section "Conceptual base". There is a demand stating that, when building concept or solving mathematical modeling task, one should not separate these three GVs, but build one upon another (Leuders, Prediger, 2005). This should definitely be the methodical imperative while teaching functions, but this is not necessarily the situation when testing students' knowledge, especially if we consider the characteristics of the exam. The tasks are mainly closed tasks: for 77 % of the exam only the correct answer to the task is evaluated and students' task solving strategies or partial solutions are taken into consideration for just 23 % of the exam. Also, only 10 % of the exam is tasks of mathematical modeling, meaning that there are even less concerning functions. Therefore, separating GVs activated in the Matura tasks could give interesting results.

The National Council of Teachers of Mathematics has placed *mathematical language* as an important factor in mathematical proficiency in the "adaptive reasoning" strand because of the importance of communication and reasoning through mathematical language in the process of explanation and justification (Riccomini et al. 2015). Again, because of the characteristics of the Croatian State Matura, this aspect of the importance of mathematical language is not of interest. The exam does not include an oral component, and some form of explanation and justification is only needed in the long answer questions. For this reason another aspect of mathematical language will be considered. To obtain the meaning and requirements of the task, as a precondition for task solving, students need to identify mathematical terminology (mathematically distinctive words and symbols representing mathematical concepts or relations) but also recognize individual concepts, and relationships between those concepts (Lott Adams, 2003). An example of such process can be seen in the section "Examples from the exam" (Example 1). Although every mathematical task inevitably contains mathematical language, tasks can call for a different level of understanding behind a mathematical word or mathematical statement. That is why tasks will be classified as "using simple", "using advanced" or "using complex" mathematical language (notation ML1, ML2, ML3, respectively).

In mathematics, often one has the "idea" how to solve the task, but is not successful in carrying out all the procedures and calculations. The process of solving sometimes requires *procedural fluency* that is not on a basic level. This aspect will be classified as "not required for solving" or "required for solving" (notation PF0, PF1, respectively). Examples of such classification can be seen in the section "Examples from the exam" (Examples 1 and 2).

The aspect of *strategic competence* refers to the detection of the need for problem solving competencies. This aspect will be classified as "not required for solving" or "required for solving" (notation SC0, SC1, respectively). Although it is directly implied that this aspect is referring to mathematical modeling tasks, it is not the only facet. It also includes situations when to answer the problem set in the task multiple steps need to be executed. This situation requires the student to plan a "task solving strategy" in advance.

In the third aspect the need for mathematically distinctive *reasoning* will be detected. Cognitive demands always exist while dealing with mathematical tasks, but some tasks are of a more technical nature (Example 2) and some tasks need constant justification, or explanation (Example 1). In this aspect tasks will be classified as "require simple" or "require complex" adaptive reasoning (notation AR0, AR1, respectively).

4. Examples from the exam

In this section, three tasks will be interpreted according to the instrument described above, i.e. shown in Table 1.

Table 1. Two-dimensional instrument.

Conceptual characteristics						**Skills and activities**					
GVs			Mathematical language			Procedural fluency		Strategic competence		Adaptive reasoning	
GV1	GV2	GV3	ML1	ML2	ML3	PF0	PF1	SC0	SC1	AR0	AR1

Example 1. (NCEEE, 2015)

"Continuous function, defined for all real numbers, has exactly two points of local minimum $A(-1,2)$*,* $B(4,-3)$ *and only one local maximum point* $C(1,3)$*. Find interval/intervals of* **increase** *of given function."*

To solve this task mathematical terminology, including the meaning of corresponding concepts, needs to be available: *"continuous function defined for all real numbers"*, *"local minimum and maximum"* and *"interval of increase"*. Since behind the underlined words lies a complex mathematical concept, this task is classified as "using complex" mathematical language.

Once the terminology is understood, strategic planning is needed. An example of a thought process might be: *"I need to determine interval/s of increase. To do that I need to have a sketch of a graph of this function. Once I have the graph I can "see" where the function is increasing and write it down in an interval form"*. This would imply that strategic competencies are "required" in order to solve this task.

Adaptive reasoning is crucial during a task solving process, because each step taken needs to be justified by confirmation of relationships of concepts lying behind the given terminology. The first step would be to plot given points into a Cartesian coordinate system (Figure 1).

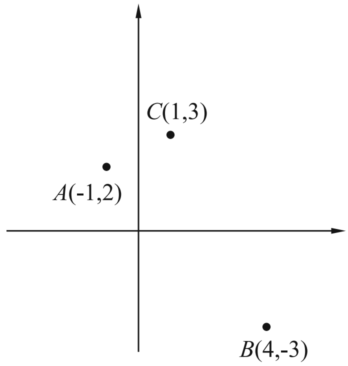

Figure 1. First step in solving the task from Example 1.

From the terms: "local minimum" and "local maximum" one makes the conclusion presented in Figure 2.

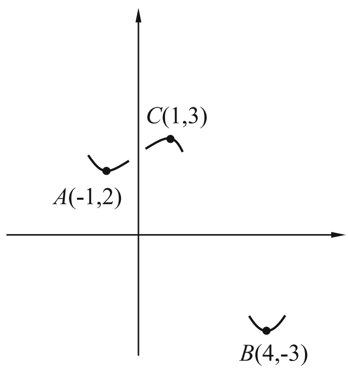

Figure 2. Example of adoptive reasoning in Example 1.

From the term "continuous and defined for all real numbers" it can be known that a continuous curve must be drawn from the left to the right side of the coordinate system in order to respect conclusion acquired from Figure 2.

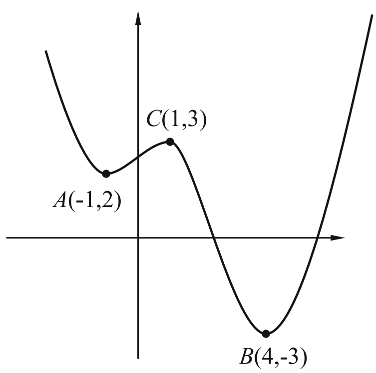

Figure 3. Final step in the reasoning in solving the task in Example 1.

It is clear from all the steps stated above that function was considered on its entire domain, not just individual pairs of values (Figure 3). Therefore, it can be concluded that GV3 was activated.

There was no need for procedural fluency. Even in drawing a graph, a well known procedure is not followed, but rationalizes every step.

To conclude, this task is classified as: GV3, ML3, PF0, SP1, AR1.

Example 2. (NCEEE, 2016)

"Function $f(x) = 3x^2 + 10$ *is given. Evaluate expression* $f(2) - f'(3)$*, where* f' *is derivative of function* f*."*

To solve this task the student needs to understand the meaning of the term "derivate of function f", and be familiar with the symbolic notation $f(2)$. Behind these words lies an advanced mathematical concept. Therefore, the task is classified as "using advanced" mathematical language.

By activation of GV1 one needs to translate from notation $f(2)$ to the question *"Which value is assigned to argument* $x = 2$*?"*.

No strategic competency or adaptive reasoning is necessary, because all the required steps are familiar, procedural and should be automatized. This is also the reason why this task requires procedural fluency. Firstly, for the accurate determination of function derivative. Then for calculating the value of function f for argument 2, and value of function f' for argument 3. And lastly, to make the translation from the task "evaluate $f(2) - f'(3)$" to the task "calculate $22 - 18$".

To conclude, this task is classified as: GV1, ML2, PF1, SP0, AR0.

Example 3. (NCEEE, 2016)

"Moss covers $1.3\ m^2$ *of the bark of the tree. At the end of each week the area of the moss covering the tree enlarges by* 5 % *in relation to the area of the moss at the end of the previous week. What area of the tree will moss cover after* 8 *weeks?"*

This task could be solved using geometric sequences or exponential function. Classification of the task could differ depending on the chosen approach. For this purpose, mathematical modeling using exponential function will be considered.

Modeling tasks call for strategic competence and adaptive reasoning in order to be able to interpret from contextual problem to mathematical problem. Real-life problems are usually not written using mathematical terms or math symbols. In this case, the mathematical words and terms appearing are "1.3 m^2", "5 %" and "area" (understanding of this term is not required to solve the task successfully) due to which this task is classified as "using simple mathematical language".

For interpretation of a real-life problem GV2 needs to be activated as this is a typical example of modeling with exponential function. The phrase: "*... enlarges by* 5 %*...*" is considered to be a trigger word for characteristic change of exponential function. In other words, it should be recognized that when one quantity (number of weeks) changes, the other quantity (area of the moss covering the tree) changes in a way that is characteristic of exponential function.

Since in the tasks concerning modeling with functions the corresponding parameters always need to be interpreted and determined, GV1 also needs to be activated.

From the conclusion that the situation is modeled with exponential function, and by activation of GV1, the situation needs to be described algebraically which is activation of GV3. To be able to efficiently create an equation of the function the student needs to have developed procedural fluency in this type of task. Otherwise it could happen that too much time is spent calculating the area of the moss behavior week by week, or an incorrect rule could be formulated e.g. $f(t) = 1.3 \cdot 0.05^t$.

5. Results and discussion

From the task analyses conducted on the higher level 2015 and 2016 Croatian State Matura, some tasks that are originally classified as functional tasks were excluded. These are tasks concerning series and tasks in which, when it comes to functional aspects, only algebraic properties of functions, such as logarithm rules or basic trigonometric identities, are being used. There are 22 out of 92 items that respect the given criterion. This corresponds to 29 out of 120 points, i.e. a 24.17 % score share. Of these 29 points, 12 are from multiple choice questions, 13 from short answer questions and 4 are from long answer questions.

The mean for these tasks is 51 % for all participants, i.e. 56.4 % for gymnasium students and 37.5 % for vocational school students. The tasks were classified, considering their p-values, into five categories: *very hard task* $0 < p < 20$, *hard task* $20 < p < 40$, *medium hard task* $40 < p < 60$, *easy task* $60 < p < 80$ and *very easy task* $80 < p < 100$ (Kleijne, Schuring, 1993). The structure of the tasks in terms of this classification is given in Table 2.

Table 2. Categorization of researched tasks according to their p-values.

Task difficulty category	**Very hard**	**Hard**	**Medium**	**Easy**	**Very easy**
Number of items (overall)	3	3	8	6	2
Number of items (gymnasium)	2	2	8	6	4
Number of items (vocational)	5	7	5	5	0

Figure 4 shows the construction of the higher level 2015 and 2016 Matura exams in terms of classification from the research instrument. It shows that when it comes to GV criterion, most of the tasks require GV3 activation. In 17 out of 22 tasks activation of GV3, alone or in combination with other GVs, is required. Also, there are no tasks requiring only GV2 activation. There are only 2 out of 22 tasks using basic mathematical language. Most of the tasks, 13 of them, use advanced mathematical language. There is no significant difference in the number of tasks regarding procedural fluency. Strategic competence is required in 13 out of 22 tasks, and there are only 6 tasks which do not require adaptive reasoning.

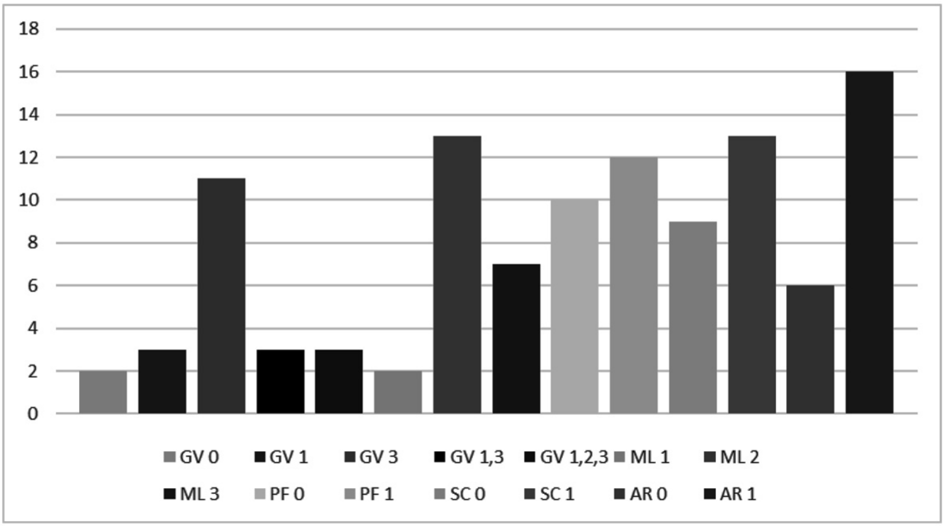

Figure 4. Number of tasks considering characteristics from research instrument.

Analysis has shown that these five characteristics appear in 17 different combinations. There are 5 combinations that appear twice. When the p-values of tasks with the same combination of characteristics are compared, in 4 out of 5 tasks p-value significantly differs. The average difference in 3 out of 4 tasks is 32.51 %. This indicates that there are more than these five characteristics that need to be considered in order to determine task difficulty. For example, it could be important to consider the type of task. This could be the case in the tasks showing p-value difference of 72.29 %. The task with an extremely low p-value is a long answer question, and the task with a high p-value is a multiple-choice task.

Figure 5 shows that students have high achievements in tasks that require activation of GV1. With the p-value of 81.17 %, these tasks have on average 33.59 % higher p-value than tasks in which some other GV, or a combination of GVs, is activated.

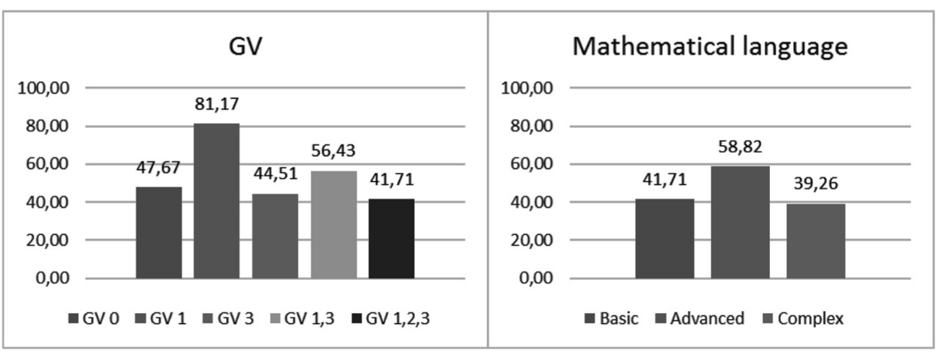

Figure 5. Task p-values in relation to GVs and mathematical language.

It is not surprising that tasks using complex mathematical language have the lowest p-value. Also, there is a significant difference, 19.56 %, in the solvability of tasks using advanced relative to complex mathematical language. But it could be supposed that tasks using basic mathematical language would have the best solvability rate, since there is no “difficult” mathematical language to get in the way of students understanding of the task. Figure 5 shows different results, there is not a significant difference in the p-value of tasks using basic or complex mathematical language. It should be taken into consideration that these two tasks using basic mathematical language are both mathematical modeling tasks. It would be safe to conclude that mathematical language was not a crucial factor influencing the low solvability rate of these two tasks.

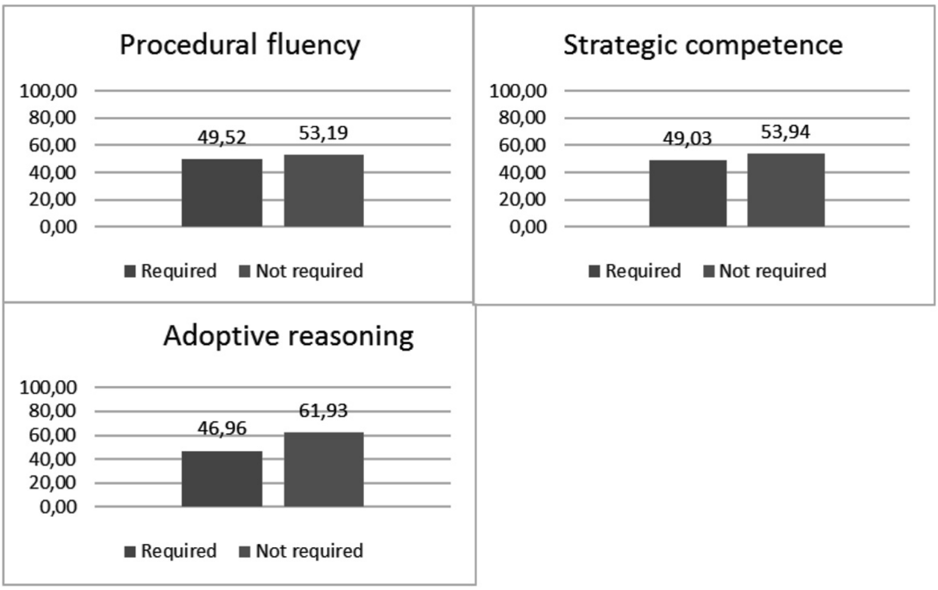

Figure 6. Tasks p-values in respect to procedural fluency, strategic competence and adaptive reasoning.

Figure 6 shows that when considering the skills and activities involved in the task solving process, significant difference can be noticed in the aspect of adaptive reasoning. There is 14.97 % of difference in the p-values of tasks that do not require adaptive reasoning relative to those that do.

6. Conclusions

According to my research instrument the higher-level 2015 and 2016 Croatian State Matura, relating to functions, shows some interesting characteristics. It is shown that 77 % of tasks are classified as activating GV3, alone or in combination with other GVs. There is a noticeable lack of tasks activating GV1 (14 %) or GV2 (none). Almost 91 % of these tasks use either advanced or complex mathematical language. This indicates that students are required to proficiently use mathematical language regarding functions. It is shown that 73 % of tasks require adaptive reasoning, and 59 % of tasks require strategic competence. This indicates that the Croatian Matura, in the domain of function, encourages students to plan their solution and rationalize, rather than follow a predetermined procedure.

Considering the complex construction of the Matura tasks and the small number of tasks examined, it is not possible to draw a conclusion about the correlation between the characteristics of a task and its p-value. To be able to establish correlation between tasks p-value and examined tasks characteristics (GVs, Mathematical language, Procedural fluency, Strategic competence and Adaptive reasoning) it would be necessary to analyze more tasks featuring previously stated characteristics. In addition, the need to take into consideration other task characteristics arose. For example, it would be worthwhile to analyze tasks in terms of task type: whether they are multiple-choice, closed answer, open answer or mathematical modeling tasks.

References

[1] BLUME, W., VOM HOFE, R., JORDAN, A., KLEINE, M. (2004), *Grundvorstellunfen als aufgabenanalytisches und diagnostisches Instrument bei PISA*, In Neubrand, M. Mathematische Kompetenzen von Schulerinnen und Schulern in Deutschland – Vertiefeude Analysen im Rahmen von PISA 2000 (pp. 145–157), Weisbaden: Leske & Budrich.

[2] KILPATRICK, J. (2001), *The strands of mathematical proficiency*, In Kilpatrick, J., Swafford, J., Findell, B. (Eds.): Adding it up: Helping children learn mathematics (pp. 115–156). Mathematics Learning Study Committee, National Research Council, Washington DC: National Academy Press.

[3] KLEIJNE, W. SCHURING, H. (1993), *Assessment of Examinations in the Netherlands*, In Niss Mogens (Ed.), Cases of Assessment in Mathematics Education, An ICMI Study. Norwood, MA: Kluwer Academic Publishers.

[4] LEUDERS, T. AND PREDIGER, S. (2005), *Funktioniert's? – Denken in Funktionen*, PM, 2, 1–7.

[5] LOTT ADAMS, T. (2003), *Reading mathematics: More than words can say*, The Reading Teacher, 56 (8), 786–795.

[6] MALLE, G. (2004), *Grundvorstellungen im Mathematikunterricht*, Innovations in mathematics, science and technology teaching. In: IMST Newsletter, `https://www.imst.ac.at/imst-wiki/images/8/83/Langfassung_Grundbildung_Malle.pdf` [Accessed 12/01/2017].

[7] MZOS (2012), *Pravilnik o polaganju Državne mature*, `http://www.azoo.hr/images/razno/Pravilnik_drzavna_matura.pdf` [Accessed 28/12/2016].

[8] NCEEE (2014), *Ispitni katalog za državnu mature u školskoj godini 2015./2016. Matematika*, `https://www.ncvvo.hr/kategorija/drzavna-matura/ispitni-katalozi/` [Accessed 29/12/2016].

[9] NCEEE (2015), *Mathematics higher-level State Exam*, https://www.ncvvo.hr/dm-2014-2015-ljetni-rok/ [Accessed 28/12/2016].

[10] NCEEE (2016), *Mathematics higher-level State Exam*, https://www.ncvvo.hr/drzavna-matura-2015-2016-ljetni-rok/ [Accessed 28/12/2016].

[11] RICCOMINI, P.-J., SMITH, G.-W., HUGHES, E.-M., FRIES, K.-M. (2015), *The language of Mathematics: The Importance of Teaching and Learning Mathematical Vocabulary*, Reading & Writing Quarterly, 31, 235–252.

[12] VOLLRATH, H.-J. (1989), *Funktionales Denken*, Jurnal fur Mathematik-Didaktik, 10 (1), 3–37.

[13] VOM HOFE, R. AND BLUM, W. (2016), *"Grundvorstellungen" as a Category of Subject-Matter Didactics*, Journal für Mathematik-Didaktik, 37 (1), 225–254.

[14] VOM HOFE, R. (1995), *Grundvorstellungen mathematischer Inhalet* Heidelberg Berlin Oxford: Spektrum.

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Funkcije na višoj razini Hrvatske državne mature provedene 2015. i 2016. godine

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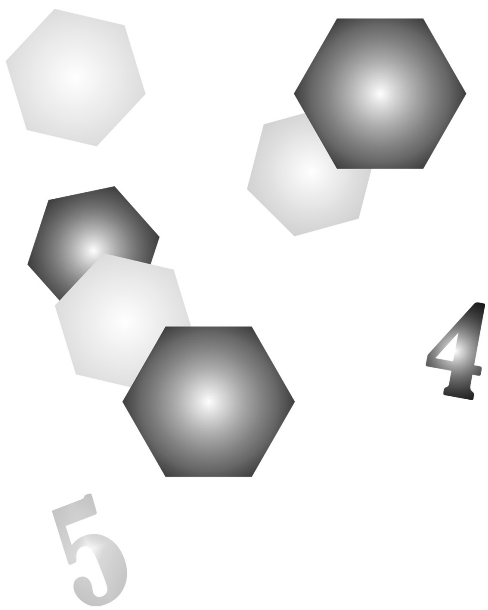
Sažetak. Rad prikazuje analizu zadataka iz područja funkcija, na višoj razini hrvatske Državne mature, provedene na ljetnom ispitnom roku 2015. i 2016. godine. Analizi zadataka prethodi uvođenje teoretskog okvira na temelju kojeg su se klasificirali zahtjevi zadataka iz područja funkcija. Razvijeni teoretski okvir baziran je na teoriji "osnovnih ideja" (Grundvorstellungen) i "matematičkih vještina" (Mathematical Proficiency). Ciljevi rada bili su istraživanje mogućnosti korištenja ovako definiranog teorijskog okvira kao kriterija za analizu i određivanje karakteristika zadataka iz područja funkcija na hrvatskoj Državnoj maturi te mogućnosti evaluacije specifičnih sposobnosti iz područja funkcija na temelju uspješnosti rješavanja zadataka i njihove klasifikacije prema definiranom teoretskom okviru.

Ključne riječi: državna matura, Grundvorstellungen, matematičke vještine, funkcije, analiza zadataka

2.
Fostering geometric thinking



Spatial reasoning in mathematics

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Abstract. With the growing interest in spatial reasoning, stimulated by the development of powerful computer-based geometry and visualization packages, it is important to be clear about what is meant by spatial reasoning in mathematics. Starting from the point of various math educators, learning spatial thinking in mathematics has different aims than learning spatial thinking in other sciences.

Hence, although spatial skills may be intellectually interesting in themselves, the focus in this paper is placed on its relationship with teaching and learning geometry at the technical faculties. Furthermore, the course *Descriptive geometry with computer graphics*, which has evolved at the Faculty of Mining, Geology and Petroleum Engineering in Zagreb in conjunction with the recent developments in the modern geometry education, is described in detail. On the basis of the classical geometrical representation methods, the course focuses not only on the uprising of graphic-visual communication and developing learners' spatial visualization skills, which play a crucial role in engineering educations, but likewise on the development of learners' capacity with deductive reasoning and making use of aids and tools in mathematics education. Also, the effect of computer technology on geometry education is discussed according to the results of the SEFI – Mathematics Working Group (SEFI – stands for "European Society for Engineering Education"). The examples of student exercises will be given to show a large range of options offered within the course to make teaching of space mathematics innovating, more interactive and at the same time applicable to specific students' interests.

Keywords: spatial reasoning, teaching tools, computer graphics, higher education, e-learning

1. Introduction

As many educators emphasize, spatial reasoning, or spatial thinking, together with verbal reasoning, is one of particularly common modalities of human thoughts (Newcombe, 2010; K-12, 2014; JMC, 2001). While the verbal reasoning is the process of forming ideas by assembling symbols into meaningful sequences, spatial reasoning may be described as the process of forming ideas through the spatial relationship between objects (JMC, 2001, p. 55; Kovačević, 2016;).

Although many point out that spatial reasoning has always been a vital capacity for human action and thought (Sorby, 2009; Newcombe, 2010), some argue how it has not always been adequately supported in formal education (Jones and Tzekaki 2016; Davis, B. et al., 2015; Clements and Sarama, 2011). Fortunately, in recent years the situation is changing. But we may note that the starting research interest is growing outside the milieu of the mathematical community (Leopold, 2015; Sorby, 2009; Uttal and Cohen, 2012). Namely, the results of the transdisciplinary studies have found many evidences that spatial reasoning plays a vital role not only in schooling, across all grades and within most academic STEM subjects, and also beyond it: in later careers, as a support to key learning (Cheng and Mix, 2004; Davis B. et al., 2015; Newcombe, 2010; Uttal and Cohen, 2012). For example, Uttal and Cohen (2012) carried out a systematic meta-analysis of the most recent 25 years of research on spatial training and showed the malleability of spatial abilities and their effects.

Therefore, with the growth interest in spatial reasoning, it is important to be clear about what is meant by it in mathematics. Particularly now when many math educators see spatial reasoning as a vital component of learners' successful mathematical thinking and problem solving, and when the development of spatial ability is declared as one of the key goals of mathematics education all around the world, from pre-school to university level (Cheng and Mix, 2014; Davis et al., 2015, p. 3; Jones and Tzekaki, 2016; JMC, 2001; ICME-13, 2016; Newcombe, 2010; K-12, 2014; Milin Šipuš and Čižmešija, 2012).

However, in recent times, there is a considerable on-going debate among researches, teachers and educators on what spatial reasoning in mathematics is (also what it is not), aiming mostly at casual use of the various notions. Therefore, in the following section we shortly clear the path for specifying what is meant by it in this paper, mainly focusing on the geometry education level of technical engineers, and its role in connection to the use of ICT technology in math education.

2. What is spatial reasoning in mathematics?

Connecting spatial reasoning to math is not of recent date and it is not an intention of the author in this paper to describe in details any of its threads, but only to point out the myriad of approaches focusing onto the same topic of interest, spatial reasoning in mathematics (Davis et al. 2015; ICME-13, 2016) or, more precisely in this case, geometrical reasoning (Bishop, 1980; Clements and Battista, 1992; Kovačević, 2016; Stachel, 2015; JMC, 2001; K-12, 2014).

Let us start with one, perhaps the most important fact for this paper. Namely, according to the well-known Clements and Battista's *Handbook of research on mathematics teaching and learning: Geometry and spatial reasoning*, back in the 20^{th} century most mathematicians and mathematics educators seemed to include spatial reasoning directly as a part of a geometry curriculum, emphasizing in that way a strong relations spatial reasoning and school geometry had/have (Clements and Battista, 1992, p. 420; Bishop, 1980). Importantly though, the results of various studies highlight significant variation across the European countries in the historical design of mathematics curricula, and spatial geometry curricula in particular (Bishop, 1980; Davis et al. 2015, p. 48; JMC, 2001 p. 33; Lawrence, 2003). The main differences in treating and teaching spatial problem tasks in European countries are shown in Figure 1 within the comparison of the systems of graphical communication educations back in the 19^{th} century in France, Germany and Great Britain, taken from Lawrence (2002, p. 1278).

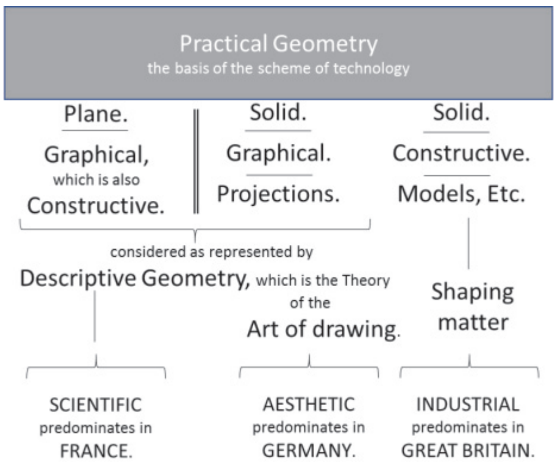


Figure 1. The history of spatial curricula in the 19^{th} century in some European countries.

Hence, following these historical variations in educational trends, back in 20^{th} century many European countries put a strong emphasis in geometry curricula on the traditional Euclidean geometry (JMC, p. 31). Conversely, Croatia, as a historical part of Austro-Hungarian Empire, mostly followed the central European approach in spatial geometry within the subject descriptive geometry. Thus, as Stachel (2015) points out for the central European countries, descriptive geometry as a subject in the hierarchy of sciences, is placed somewhere within or next to the field of Mathematics, but also near to Architecture, Mechanical Engineering and Engineering Graphics (Stachel, 2015). Even today its specificity at many technical faculties in Croatia is focused on making mathematics more applicable to engineering education through the promotion of the spatial reasoning and its graphic representation within the area of projective and synthetic geometry (Horvatić-Baldasar and Hozjan, 2010; Weiss, 2015; Stachel, 2015; Weiss, 2015).

Furthermore, in connection to the historical development of the spatial reasoning in mathematics, an examination of the history of higher education in 20^{th} century revealed, as Horton (1955) interestingly puts it, that often the Euclidean geometry has been treated as a prerequisite to collegiate matriculation aiming at three basic needs that Euclidean geometry filled at the time: the necessity for higher education, the screening device of the unfit for higher education, and the development of a way of reasoning. Thereat often only the Euclidean plane geometry was studied. Naturally, some of the above-mentioned educational outcomes were questioned over the past decades, either by the mathematicians and math educators, either by the psychologists or some other educators, and there have been substantial changes in geometry education in the second half of the twentieth century over the countries (Davis et al., 2015 p. 48; JMC, 2001, p. 31; Lawrence, 2003; Weiss, 2015; Stachel, 2015). Unfortunately, the overall geometry content changes in primary and secondary education are being less visible in the 21^{st} century mostly because of the new "outcomes to competence oriented curriculum" (Horvatić-Baldazar and Hozjan, 2010; Kovačević, 2016). But, the *Working group on the teaching and learning geometry 11–19* in UK states for example that basic changes are mostly regarding the increasing emphasis on the "applicable" geometrical content embodied in coordinate geometry, vectors and transformations, at the expense of the "purer" mathematics of classical Euclid (JMC, 2001 p. 31).

Furthermore, it is important to note that nowadays, in connection to mathematics education, the importance of spatial reasoning is recognized beyond the limits of geometry, and the existing literature provides a firm basis for a conclusion that spatial ability and mathematics share cognitive processes beginning early in development (Cheng and Mix, 2014 p. 3; Davis et al. 2015; Jones and Tzekaki 2016; K-12, 2014, p. 3). So, spatial reasoning seems to become crucial at the very beginning of the math education.

On the other hand, it seems that the influence of the non-mathematical researches becomes larger in some areas of the mathematics, given the continuing expansion of the important role of mathematical education in science and contemporary society. Some argues that that may again become a misfortune for teaching and learning geometry at the higher education levels (Jones and Tzekaki 2016; Kovačević, 2016; Stachel, 2015; Weiss, 2015). Also, there are still some who follow the viewport of studies going after a seemingly paradoxical hypothesis: *even though spatial abilities are highly correlated with entry into a STEM field, they actually tend to become less important as a student progresses to mastery and ultimately expertise* (Uttal and Cohen, p. 157). In other words, some believe that spatial reasoning is of less importance as progress in a STEM field increases. But, even if the mentioned assumption turns out to be true for some science disciplines (or some areas of mathematics), there are still areas strongly relying onto spatial and visual abilities in their reasoning processes even in their expertise level (Weiss, 2015; Stachel, 2015; Gorjanc and Jurkin, 2015). Furthermore, it remains questionable, whether (and when) one should be focusing in mathematics on the development of the spatial ability per se, and when on the spatial reasoning, or on spatial thinking.

There are many didactical and cognitive problems in connection to the role of spatial reasoning in math that are still waiting to be solved (Jones and Tzekaki 2016; Davis et al. 2015).

For example, how learners' mathematical/spatial reasoning is influenced by the ways in which geometric objects are represented? Or, how can one analyze the spatial/reasoning processes involved in mathematical activity? Does mathematical activity require only one common cognitive process, or, indeed, certain very specific cognitive structures whose development must be taken care of in teaching? Namely, as Tartre (1990) pointed out in her study of the role of spatial orientation skill in the solution of mathematics problems, it is questionable whether any attempt to verbalize the processes involved in spatial thinking ceases to be spatial thinking. Also, a French psychologist R. Duval discovered many didactical problems when analyzing the cognitive model of mathematical, and particularly geometrical reasoning, as well as the use of, today inevitable, graphical representation in mathematics (Duval, 2002). He studied how visualization works towards understanding in mathematics, aiming thereby at the important fact pointed out by Sorby (2009) that the graphical expression in engineering field is both a form of communication and a means for analysis and synthesis. Duval further claims that representation in mathematics becomes usable only when it involves physical things or concrete situations (Duval, 2002, p. 333).

To conclude, although many new results regarding malleability of spatial reasoning are encouraging (Davis et al., 2015, p. 85), and the fact that spatial abilities can be improved through education and experience may suggest that spatial ability training can improve math performance (K-12, 2014, p. 6), our focus in this paper is not onto spatial ability per se but on the applicable geometry, or as some refer to it as "vision guided spatial reasoning", i.e. descriptive geometry (Stachel, 2015; Gorjanc and Jurkin, 2015; Kovačević, 2016).

Furthermore, in this paper the reader may spot the author's often mixing the use of terms "spatial reasoning", "spatial thinking" and "spatial ability", purely because of the recent review of the research literature. Therefore, only for the purpose of this paper, in the rest of the section we will briefly clarify these notions, primarily emphasizing their inevitable interrelation. Namely, while some papers explicitly distinguished the terms in question, others did not. Also, some researchers suggest how this tendency of "mixing notions" is particularly prominent in areas of the mathematics sciences associated with geometry whereas geometry is being marginalized in many mathematics curricula unlike 3D geometry and associated spatial reasoning that is, according to various researches, widespread over a number of applied areas (Clements and Sarama 2011; Davis et al. 2015, p. 12; Jones and Tzekekaki, 2016; Kovačević, 2016).

2.1. Spatial reasoning, spatial ability or spatial thinking

For example, Clements and Battista (1992) used the first notion, "spatial reasoning" purely in connection to the specific *set of cognitive processes by which mental representation for spatial objects, relationships and transformations are constructed and manipulated* (p. 420). Interestingly, they further described the "school geometry" as the *study of those spatial objects, relationships and transformations that have been formalized (mathematized) and the axiomatic mathematical systems that have been constructed to represent them* (p. 420), mainly pointing at the traditional Euclidean geometry that was, for a long time, synonym for the school geometry in many countries (JMC, 2001, p. 31).

However, in connection to the spatial reasoning mentioned in the title of this paper, Clements and Battista further distinguished the use of the term "spatial thinking" in connection to the scientific mode of thought used to represent and manipulate information in learning and problem solving (p. 442). They probably aimed at the suggestions of some researches that *spatial ability* and *visual imagery* play vital roles in mathematical thinking (p. 443). Namely, spatial thinking was often perceived as one of different modes of thinking in mathematics. Its importance is recognized and emphasized also in the lifelong education in the definition of the mathematical competence as one of its eights key competences (EFQ, 2006; Kovačević, 2016). Some even argued, following Einstein's comments on thinking in images, that much of the thinking required in higher mathematics is spatial in nature (Duval, 2002; JMC 2001, p. 55; Newcombe 1980). But, as we have already pointed out, researchers had, and still have, their own different descriptions or subtle distinctions.

For example, Duval (2002) sees reasoning only as a part of visualization process, claiming further that representation and visualization are at the core of understanding in mathematics thinking (p. 312). But, he argues that representation becomes usable in mathematics only when it involves physical things or concrete situations (p. 333).

On the other hand, Jones and Tzekaki (2016) also emphasize the inevitable overlapping of geometrical visualization and spatial reasoning, whereby they take visualization to be *the capacity to represent, transform, generate, communicate, document and reflect on visual information* (p. 114), and they associate the process of geometrical reasoning to the deductive reasoning and proof (p. 124). Furthermore, regarding visualization, they have pointed out in their comprehensive review of recent research in geometry education that visualization is indispensable in proving and problem solving, but visual representations or processes they develop are not always effective in solving or proving relevant tasks (p. 117).

Newcombe (2010) is more focused on purely psychological aspect of the spatial thinking and in her studies spatial thinking is defined by the four tests (3D spatial visualization, 2D spatial visualization, mechanical reasoning and abstract reasoning) used to assess it (p. 31).

Thus, nowadays in some papers/studies various terms are used interchangeably, or with subtle distinction demands, aiming sometimes at "spatial reasoning", as a thinking process particularly important in the development of mathematical competence (EFQ, 2006; JMC, 2001; K-12, 2014; Kovačević, 2016; Stachel, 2015), and sometimes aiming at "spatial reasoning" as spatial skills (Clements and Battista, 1992; Sorby, 2009; Milin Šipuš and Čižmešija, 2012). Or sometimes even using the terms "thinking" and "reasoning" interchangeably (for example Davis et al., 2014, p. 5 or K-12, 2014, p. 3).

It is also important to note, from the mathematical point of view and in connection to the teaching and learning processes, that mathematical educators sometimes distinguish between the competences "thinking mathematically" and "reasoning mathematically", whereby the first competence includes *the recognition of mathematical concepts and an understanding of their scope and limitations* (Alpers et al., 2013, p. 13), and the second one *includes the constructions of chains of logical arguments and of transforming heuristic reasoning into proofs* (for details on general mathematical competencies for engineers see Alpers et al., 2013).

In this paper, the focus is on the development of the mathematical competence as a whole, and the term "spatial reasoning" is used merely to emphasize the spatial aspect of higher cognitive mode of thinking particularly significant in the teaching and learning of geometry at the technical faculties, more precisely in our case, of descriptive geometry, the mathematical subject in question discussed in the third section of this paper. Also, in this paper we shall continue to take "spatial reasoning in mathematics" to be the "geometrical reasoning" (Bishop, 1986; Jones and Tzekaki, 2016; Kovačević, 2016) aiming thereby not just on Euclidean spatial geometry but also on projective geometry (Lawrence, 2003; Stachel, 2015; Weiss, 2015) that deals with three-dimensional objects and their plane representations.

3. Descriptive geometry with computer graphics at the Faculty of Mining, Geology and Petroleum Engineering

Descriptive geometry has been a part of applicable geometry dealing with methods which aim to study 3D geometry and providing an important theoretical basis on which all the modern graphical communication was built. It enables insight into geometrical structure and metrical properties of spatial objects, processes and principles. Typical for it is the interplay between the 3D situation and its 2D representation, and between intuitive grasping and rigorous logical reasoning.

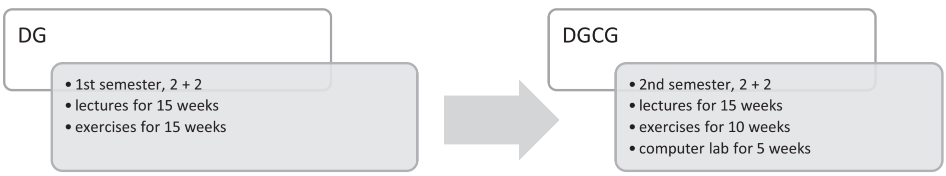


Figure 2. Descriptive geometry curriculum at the Faculty of Mining, Geology and Petroleum Engineering.

Descriptive geometry at the Faculty of Mining, Geology and Petroleum Engineering is focused on the developing a set of learning outcomes of basic knowledge of natural sciences and technical fields important for the scientific fields of mining, petroleum and geological engineering. It is currently taught within two obligatory courses: *Descriptive geometry* (DG) and *Descriptive geometry with computer graphics (DGCG)*, each within one semester (for about 180 students per each course) as it is shown in Figure 2.

3.1. Subject contents and methods of teaching

The last content changes within the courses DG and DGCG were made in 2013/2014. From the content point of view, there has been no substantial changes from the traditional subject content of descriptive geometry besides reducing the scope of course geometrical content primarily regarding the more complex geometrical structures (Horvatić-Baldazar and Hozjan, 2010). However, today emphasis is not being placed on the education of practical techniques, but on teaching the theory behind the specific techniques and the development of associated mathematical concepts. The subject is also responsible for establishing the foundation of mathematical representational systems and the use of various drawing tools important in graphical communication of engineers.

Although both courses are taught with hand drawing (sketching as well), commercial graphic processing software is being used by students only in DGCG. Namely, after basics of geometry of projection (extended Euclidean space objects – affine and projective transformations) and of two- and three-dimensional objects (basic plane and space curves, surfaces, solids) are introduced in the DG, together with some basic descriptive geometry relations and constructive principles (perpendicular relationship, piercing points, plane intersections, intersection of two solids), the experience of Computer Aided Design (CAD) software is introduced as one of the educational objectives in DGCG.

However since the focus is on geometry, CAD software is used through geometric problem solving and modelling. In doing so, problem solving in descriptive geometry involves the planning and implementation of the 2D representations of 3D objects and the corresponding relations, both in the plane and in the space, using appropriate tools, methods and principles. Furthermore, modelling in descriptive geometry means transferring previously analyzed data in a simplified and idealized geometric shape. An example of such activities is decomposition of complex structures from the actual context into geometrical objects and recognition of relationship between objects used within the computer lab exercises.

There are some content overlapping in both courses (DG and DGCG) that allow simultaneous approach to the same problem situation using various descriptive geometry methods. During the teaching process, this overlap enable constant comparison of advantages and disadvantages of different methods and principles used within the subject.

It should also be noted that the future professionals of mining, geological and oil profile, in contrast to, for example mechanical engineer, require knowledge not only of classical orthogonal projections and axonometric, unavoidable for computer graphic, but of other methods of descriptive geometry, particularly the projections with elevation which is used for solving various mining and topographic problems in relation to engineering profession.

3.1.1. Exercises in Computer Lab

Most of the educational e-materials used within the computer lab exercises were made during the year 2012 on the joint project of four technical faculties of the University of Zagreb with twelve participating teachers. Within the project the repository consisting of about 50 five-min videos helping to learn basics of *Rhinoceros*

3D modelling was produced (for details on the project see Gorjanc and Jurkin, 2015). Further developments were made during the years 2015 and 2016. Namely, because of the implementations of higher standards of qualifications and occupations in mining and geology in accordance to the Croatian Qualification Framework (CROQF), additional course implementations were made within the project TARGET by raising the level of e-learning technology in both DG and DGCG.

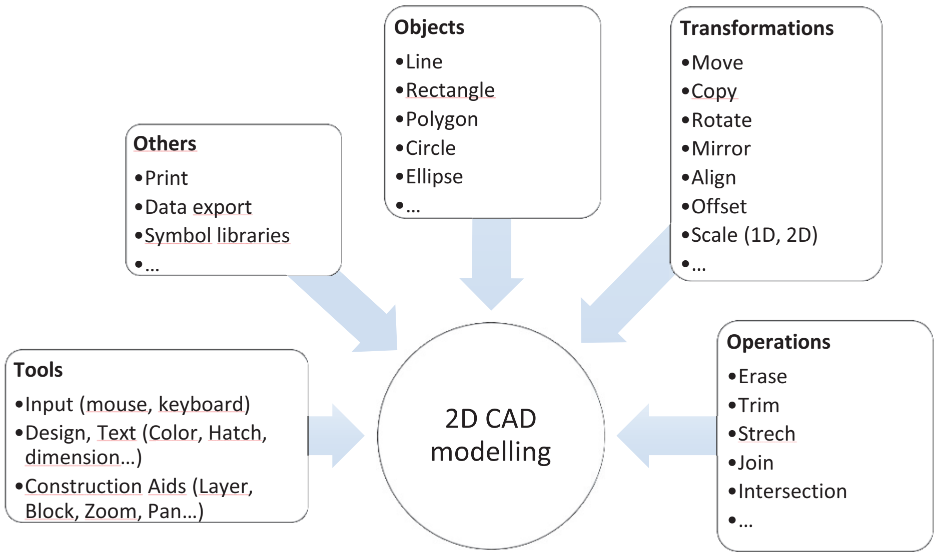


Figure 3. CAD areas for 2D modelling.

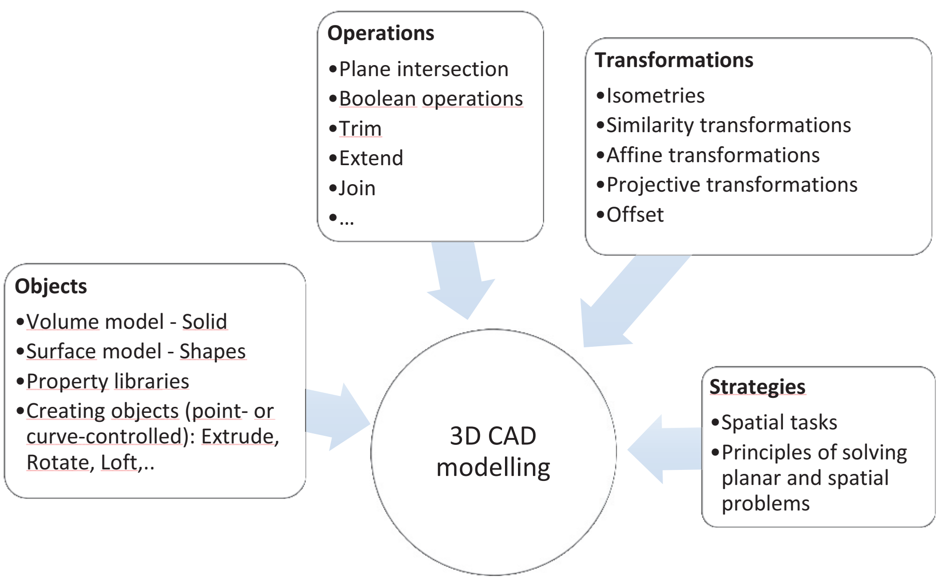


Figure 4. CAD areas for 3D modelling.

For the purpose of the courses DG and DGCG, and with a view to facilitating the initial work with commercial CAD programs, following mostly modern methodical principles regarding the teaching of descriptive geometry, 2D-CAD and 3D-CAD parts are divided into areas shown in Figure 3 and Figure 4. Only some of the mentioned topics were covered within the courses, and the overall subject contents regarding computer graphic in DGCG part are shown in Table 1.

Table 1. Subject contents of CAD modelling within DGCG at the Faculty of Mining, Geology and Petroleum Engineering

1^{st} time-block		**2^{nd} time-block**	
	Guidance, Interface of Rhinoceros 3D	1^{st} week	3D – CAD modelling: Operations, Transformations, Extrusion Working planes and working axes
1^{st} week	2D – CAD modelling: Tools, Editing operation Coordinate systems: WCS, UCS, Objects, Transformations	2^{nd} week	Application of descriptive geometry procedures to solve spatial problems (Regular polyhedral, Shortest distance problems, Terrain and layer)
2^{nd} week	3D – CAD modelling: Guidance, Viewports, View control, Objects, Editing operations	3^{rd} week	Testing – individual tasks (overall assignments)

Hence, exercises in computer lab (90 min time per slot per 5 week) were divided into two time-blocks $(2+3$ weeks). The basic guidance information about the interface of *Rhinoceros 3D*, views, viewports and work with layers are provided through e-materials by using e-learning platform *Merlin*, the system based on the learning management system *Moodle*. The students were encouraged to learn interface basics on their own by watching videos at home. After that, in each time-block, for each unit, first brief lectures are given concerning related topic and then the operational methods of *Rhinoceros 3D*, as a representative of Computer Aided Design (CAD) software, are taught using simple examples. During the course, and particularly in between the time-blocks, students were given time to improve their abilities with ICT on their own by further watching videos at home or in the computer lab, under teacher assistance if needed.

Thus, in the 2^{nd} 3-week teaching block, students had to use their knowledge to solve problems typical for descriptive geometry on their own, after solving one or two similar tasks on the spot with teacher. Details of the assignments are shown in the following section.

Importantly, from 2005 – 2012 *AutoCAD* was used as the main graphic processing software, but since 2013 *Rhinoceros* is being introduced as a CAD representative. These software changes within the course were made mostly for two reasons. Firstly, to facilitate the teaching and learning process in the course in accordance with recent changes in the undergraduate study programs. These changes allowed, in the same working time, more accent to be placed onto basic knowledge of mathematics and mathematical principles in problem solving, and not on the computer work with more and more complex commercial software at the very beginning of the study. Namely, rapid changes in commercial software industry cause constant changes in software interfaces and increase of the complexity of application programming tools. Hence, there is an inevitable move of educational focus from purely developing the students' mathematical competence also to the teaching and learning specific software-version techniques and procedures. In order to facilitate the synergy between various educational goals, a more geometry friendly commercial software *Rhinoceros 3D* was chosen as a CAD representative. And secondly, as a result of harmonization of educational standards on various technical faculties at the University of Zagreb within the previously mentioned project, a basic repository providing adequate e-materials was made using *Rhinoceros 3D*. This repository facilitates the educational process for both, teachers and students. For a number of teachers, the joined database of preparatory materials is available for use, and for students, a large amount of e-materials, including video-lessons, allowed them to individually choose the time and place for qualitative learning.

Before reporting on the activities used within the course DGCG, let us discuss some didactical principles of importance for the chosen activities.

3.1.2. Some didactical principles

Firstly, it is important to note that there is a growing number of mathematicians and mathematics educators that find well-known thinking frameworks like the Van Hiele's or Piaget's ones helpful only in the first access to geometry by young children (Davis et al., 2014), but unfit when it comes to teaching geometry at higher educational level, such as high school or university levels, (ICME-13, 2016; Kuzniak et al., 2007).

Furthermore, although both courses, DG and DGCG at the Faculty of Mininig, Geology and Petroleum Engineering combine different didactical principles in the teaching process, for the purpose of this paper and in connection to the teaching descriptive geometry, particularly interesting is the Duval's theory on figural apprehension in mathematical reasoning, especially in geometrical reasoning and work with geometric drawings and computers (Duval, 2002; ICME-13, 2016; Jones, 1998; Kuzniak and Richard, 2014).

Duval also proposes the synergy of three cognitive process necessary for proficiency in geometry which fulfil specific epistemological functions. Those are: visualization, construction and reasoning (see Figure 5, from Jones, 1998). His work on cognitive process level, important for geometry and mathematics as well, was further adapted by Kuzniak and Richard (2014).

All three processes are included in geometrical reasoning and can be performed separately. However, Duval emphasizes, visualization doesn't necessarily depend on construction and it doesn't always help reasoning. The reasoning process, on the other hand, can be developed in an independent way of two other processes included. In the Figure 5 below, each arrow represents a direction in which one kind of cognitive process can support another kind in any geometrical activity. As it can be seen, the construction process, that is in the focus when it comes to application and practice, depends only on connections between relevant mathematical properties and the constraints of the tools being used and cannot be directly supported by visualization.

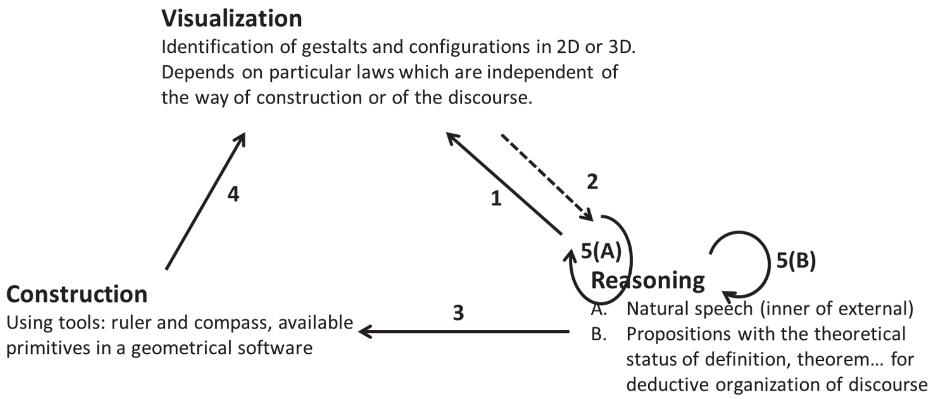


Figure 5. The underlying cognitive interactions involved in geometrical activity.

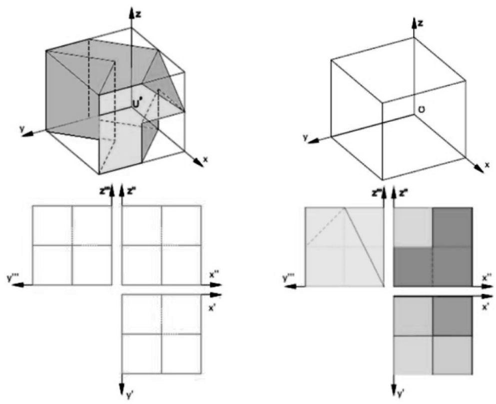
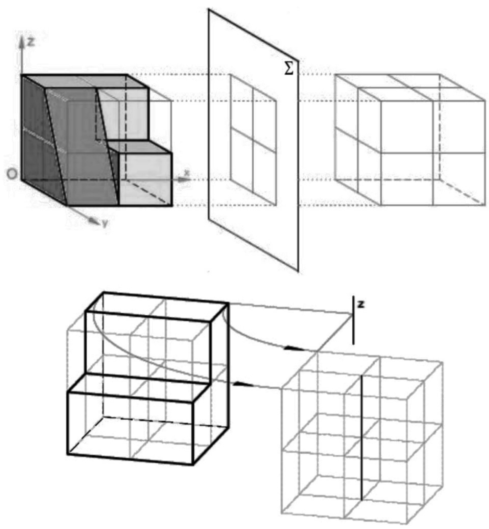
Hence, in order to achieve an ultimate goal of mathematical education of engineers, which is according to SEFI group to make engineers mathematically competent (Alpers et al., 2013, p. 65), activities focusing on a particular cognitive process are often included in the course, whether students are to work individually (at home or in the class) or in pairs, in parts of lessons with individual or mixed interaction.

3.1.3. Example of activities

Practicing visualization tasks

Over the years we have noticed a growing number of students at the Faculty of Mining, Geology and Petroleum Engineering having trouble with simple visualization of basic geometric objects based on the given data. Since these visuospatial abilities are prerequisite for their further study in technical fields, we have considered a set of different visualization tasks that are offered to students in the class at the very beginning of the course, lasting 15 to 20 minutes. These tasks mostly serve students to detect, if there are, their basic visuospatial problems and to encourage them to work on it. Further visualization tasks, aimed at further improvement of student visuospatial skills, are offered to students individually for home-based practice. Examples of visualization tasks are given in Table 2.

Table 2. Examples of visualization tasks.

	**Orthogonal projection** a) Given the object, draw its principal views (top, front, left side) in the prepared grid (left image). b) Given three principal views of an object, determine its image in the prepared coordinate system (right image).
	**Congruent transformations** a) Given the mirror plane Σ, draw a mirror image of the given object in the prepared grid. b) Rotate the object around the z-axis and draw its image in the prepared grid.

Positioning and metrical tasks

Since geometry originated from practical needs, the geometric courses necessarily combines not only mathematical content but, in our case, mining and topographic content specific to geology, mining and petroleum engineering. Many of the problems included are based on construction tasks and tasks on sets (loci) of points with certain properties. Within each course, DG and DGCG, three individual geometrical problems are given to each student. Four of them are focused purely onto mathematical content, and only two combine specific geological contents with mathematical concepts.

Examples of two individual mathematical tasks were given in Table 3 were hand drawings are made by students. Both examples require students first to transform some spatial problems into a graphical one. They should think of representing spatial objects by a two dimensional drawings by means of some pictorial overview (representing mathematical entities). The requirement of the use of point coordinates emphases the mathematical understanding of spatial relations as well as the understanding of projection method, which in this case is the Monge projection. During the course the same construction problems are solved by applying different projection methods and by CAD.

Table 3. Examples of student individual mathematical tasks by using Monge projection.

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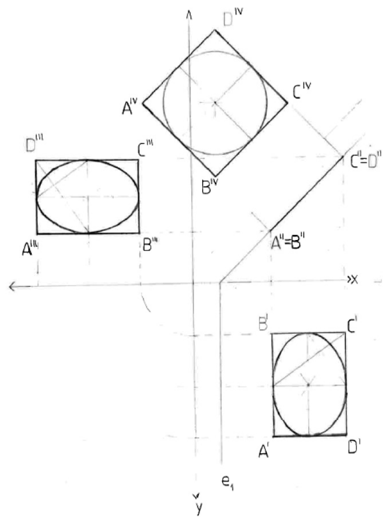
<td>
</td>
<td>Example 1.
Draw principal views (top, front ad left side) of the square $ABCD$ $\lceil a = 4 \rceil$ and its inscribed circle if it is perpendicular to plane π_2, makes an angle of 45° with π_1, and its two sides are perpendicular to π_2 having its bottom foreground vertex $A(3,6,2)$. Choose any profile plane π_4 so that the square $ABCD$ will be projected on that plane in its true size.</td>
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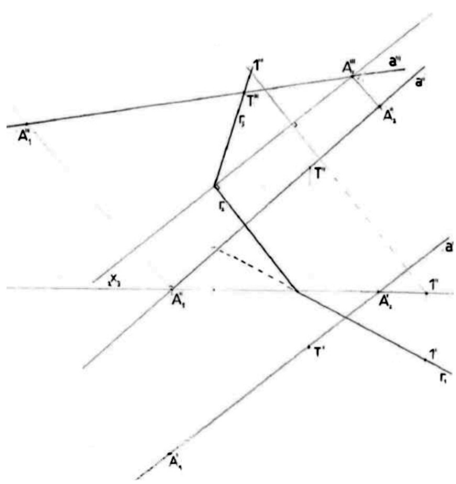
<td></td>
<td>Example 2.
There are given plane $P(2,-1,2.5)$ and line $a \equiv A_1A_2[A_1(-1,4,0), A_2(4,0,4.5)]$. Using the Monge projection determine the intersection of line a with plane P.</td>
</tr>
</table>

Reading the first example, students have to reason about the properties of the square and corresponding properties of the projection method (mathematical thinking) in order to solve the problem, thinking of various possibilities of square position in space (modelling mathematically). In the second example, after graphically representing the given objects, to solve the problem student must understand the spatial relations among objects and then choose one of the constructive principles taught within the course to determine the piercing point. The task is solved by adding additional profile view in which the given plane appears as a line.

Student Self-Assessment Test

Within the e-learning platform *Merlin* students were given three tests as a student self-assessments tasks, available for them to be taken anywhere, at any time. The tests were created by a randomized selection of questions out of a larger question bank so that students could do each test many times.

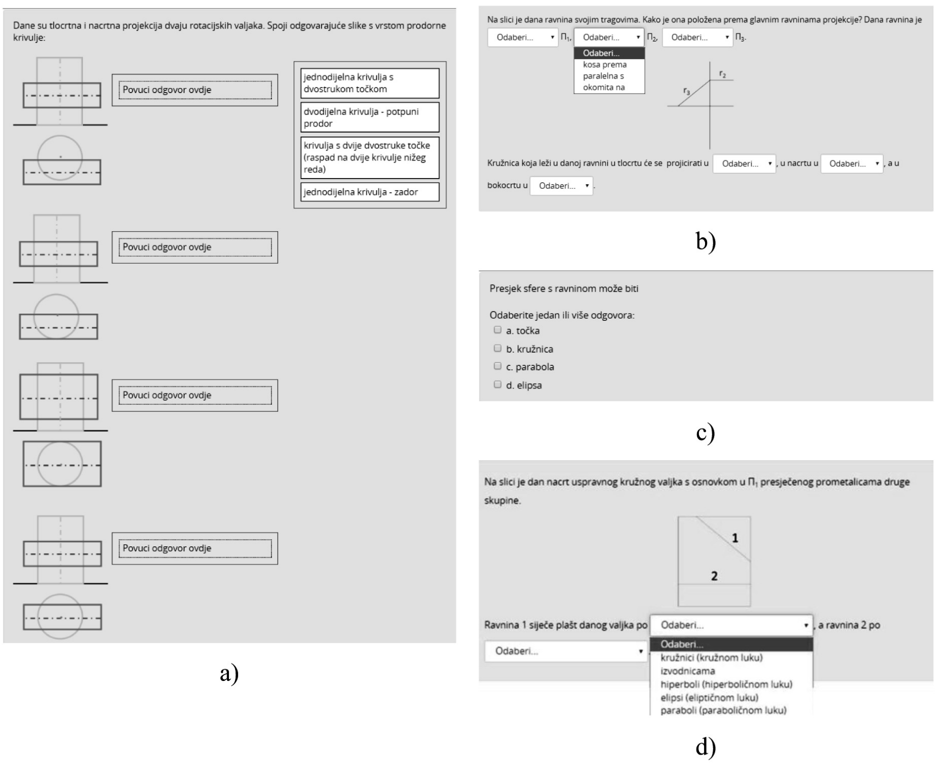


Figure 6. Some task examples of different self-assessment tests.

The tests were optional and their results were not used for students' grading, but to give students an opportunity to identify where their knowledge was weak and to revise their work. Due to the specific course interdisciplinary learning outcomes, computer-supported on-line assessments were very simple and short, mostly consisting of a number of multiple-choice questions some of which are shown in Figure 6, combining graphic representation and applying mathematical knowledge within concrete graphical situation.

Also, some of the given tasks required students' "pure" factual mathematical knowledge demonstration. The last year results pointed out that students were less successful when the formulation of questions was similar, but not exact, to those in the textbook. However, the number of students taking the tests was not representative, since the test was at this phase optional. Once a suitable set of questions are imported in the system, the system will provide students continuous on-line sport.

We may also note that teachers can also benefit from using such tests. For example, the overall students' weak spots could be detected, due to the specific concept-oriented tests, further providing teachers with valuable sources where to put additional focus when teaching.

Computer Lab Assessments

The subject descriptive geometry at the Faculty of Mining, Geology and Petroleum Engineering is organized so as to follow ideas of two sets of learning outcomes (CROQF proposal – level 6) of what student is supposed to acquire:

- basic knowledge of the natural sciences in the area of mining, geology and petroleum engineering
- technical knowledge in the area of mining, geology and petroleum engineering.

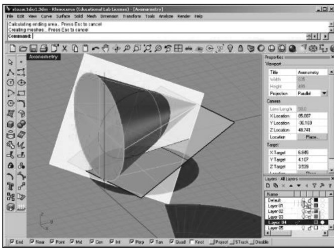
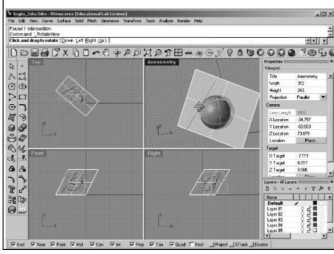
The 1^{st} time-block tasks in the computer lab were more or less similar to those taught at various engineering drawing course aiming at the development of the ICT skills, i.e. in this case CAD modelling skills, important for the technical engineers in their future jobs. Those tasks were mostly focused onto the development of the construction (using tools) and visualization processes, already highlighted in the Duval's cognitive model of geometrical reasoning.

However, the 2^{nd} time-block tasks are more problem-oriented, focusing onto reasoning process. In other words, the simple tasks were chosen with a specific goal: to emphasize the importance of mathematical knowledge in the problem solving activities. To solve this problems, student first had to understand that mathematics can do the job. Only after that some computer modelling, using tools, should take place.

Some of the tasks examples are given in the following table 2. Sometimes the same problem tasks were solved by various descriptive geometry method; either using Monge's projection, or projection with height, or axonometric projection, or by using CAD program. The tasks given in Table 4 contribute to the achievement of two course learning outcomes of what student is supposed to be able to demonstrate after he/she has completed the course DGCG:

- to apply basic mathematical knowledge in solving spatial problems
- to use appropriate software to address the technical and mathematical problems.

Table 4. Examples of learning activities in the 2^{nd} time-block in the computer laboratory.

	**Zadatak 4:** • Konstruirajte presjek rotacijskog stošca [os SV, S(6,0,6), V(6,10,6), r=5, osnovica u Π_2] ravninom P(4,-2,3). • Konstruirajte tangentu u točki T presječne krivulje, ako ta točka leži na donjem dijelu stošca, a drugo probodište njezine izvodnice je točka N_2(7,0,–). Tangentu konstruirajte kao presječnicu tangencijalne ravnine stošca i ravnine presjeka. • Razvijte plašt stošca s presječnom krivuljom.
	**Zadatak 7:** • Konstruirajte presjek kugle [S(-4,4,4), r=3] ravninom P(-6.5,-5.5,5.5). • Odredite tangencijalnu ravninu kugle i tangentu presječne krivulje u točki T(–,–,5.5) na lijevoj strani plohe.

To successfully solve them, student should be able to:

- understand the use of the basic orthographic views of elementary solids
- understand the basic concepts of geometric congruence transformations in space
- recognize examples of an affine transformations (oblique parallel projection)
- recognize and use the plane-intersections of elementary solids
- understand and distinguish the concepts of tangent line and tangential plane
- understand the basics of 3D co-ordinate geometry.

4. Conclusion

With the expansion of higher education, the number of pupils entering technical faculties in Croatia that have had limited experience in their formal education in spatial activities (be that the spatial abilities or the spatial reasoning/spatial thinking process) has been growing. Also, the number of pupils that have finished gymnasiums programs and have entered various technical faculties is at the moment increasing in Croatia. At the Faculty of Mining, Geology and Petroleum Engineering there is already up to 40 % of gymnasium graduates. Consequently, many undergraduates are not properly equipped to deal with a large amount of spatial content used within their scientific courses. The author's many years of experience in teaching geometry at tertiary level have shown that many students often lack "spatial experience" not only in the case when dealing with basic 3D objects and relations, but even with 2D objects when they are placed in space. And naturally, when students lack experience it is hard to sort their knowledge into any system of a knowledge in a logical order.

Hence, although there is a never ending discussion what comes first, theory or practice, author firmly follows conviction that the basis for learning practical geometry at the tertiary level should be clarifying and fixing in mind basic geometrical concepts and principles, and only then applying these “knowledge-tool” for solving specific engineering construction problems. Geometry education still can provide *both*: a means of developing learners’ spatial visualization skills *and* a vehicle for developing their capacity with deductive reasoning and proving.

Namely, much of basic science in technical fields requires good mathematical knowledge and skills, not only in numeracy, but also in dealing with spatial reasoning, intimately related to geometry. A broader geometrical education, including knowledge of various curves and solid/surface shapes (and their visual 2D representatives), projection methods (orthogonal projection, parallel projection, central projection. . .) and different congruence and non-congruence 3D transformations, is needed to provide some of the foundations upon which mathematical understanding could be built.

Furthermore, visual aspect of geometry also underpins much of information technology and lately relies a lot on computer graphics demanding higher ICT skills of both, teachers and students. These strong links between geometry and technology are also important because geometry needed for proficiency in many technical fields exceeds far beyond traditional Euclidean space geometry and deeply enters the area of affine and projective geometry, which in general is not taught at many technical faculties in Croatia.

Thus, regarding the spatial reasoning in mathematics we may conclude, accompanying numerous educators, the following:

- one unified and wide accepted definition of spatial reasoning does not exist
- there is a converging agreement on the importance and malleability of (visuo)spatial reasoning among researchers in various scientific fields (psychology, mathematics, technology, engineering, didactics. . .) for it can support learning and communication
- and most importantly, regarding its connection to mathematics, we may follow Kuzniak who said that “*. . . it appeared that rather than focusing on thinking first, it would be more efficient to define and study* ***what kind of geometrical work was at stake*** *in geometry teaching and learning. In this trend, studying geometrical thinking remains a basic and fundamental problem but drawn by geometry understanding in a school context rather than in a laboratory environment*.” (Kuzniak et al., 2007, p. 956)

References

[1] Alpers, B. A., Demlova, M., Fant, C.-H., Gustafsson, T., Lawson, D., Mustoe, L., Olsen-Lehtonen, Brita, Robinskon, C. L. and Velichova, D. (2013), *A Framework for mathematics curricula in Engineering Education: a report of the mathematics working group*, European Society for Engineering Education, SEFI.

[2] BISHOP, A. (1980), *Spatial Abilities and Mathematics Education: A Review*, Educational Studies in Mathematics, Vol. 11 (3), pp. 257–269.

[3] CHENG, Y. AND MIX, K. S. (2014), *Spatial Training Improves Children's mathematics Ability*, Journal of Cognition and Development 15 (1), pp. 2–11.

[4] CLEMENTS, D. H. AND BATTISTA, M. T. (1992), *Geometry and spatial reasoning*, In Grouws, Douglas A. (Ed.) Handbook of research on mathematics teaching and learning: A project of the National Council of Teachers of Mathematics (pp. 420–464), New York, NY, England Macmillan Publisching Co.

[5] CLEMENTS, D. H. AND SARAMA, J. (2011), *Early childhood teacher education: the case of geometry*, In Journal of Mathematics Teacher Education 14 (2), pp. 133–148.

[6] DAVIS, B. and the Spatial Reasoning Study Group (2015), *Spatial Reasoning in the early years: Principles, Assertions and Speculations*, New York: Routledge.

[7] DUVAL, R. (2002), *Representation, Vision and Visualization: Cognitive Functions in Mathematical thinking*, In Representations and Mathematics Visualization, Hitt, F. (Ed.), North American Chapter of the International Group for the Psychology of Mathematics Education, (pp. 311–336), Cinvestav – IPN.

[8] EFQ (2006), *Key competences for lifelong learning*, `http://eur-lex.europa.eu/legal-content/EN/TXT/PDF/?uri=CELEX:32006H0962&from=EN` [Accessed 15/12/2016].

[9] GORJANC, S. AND JURKIN, E. (2015), *Introducing 3D Modeling into Geometry Education at Technical Colleges*, Cocchiarella, L. (Ed.) The Visual Language of Technique Vol. 3 – Heritage and Expectations in Education, (pp. 57–68), Springer, Milano.

[10] HORTON, R. E. (1955), *Geometry in General Education*, Mathematics Magazine, Vol. 29 (1) pp. 41–44.

[11] HORVATIĆ-BALDASAR, K. AND HOZJAN, S. (2010), *Nastava nacrtne geometrije na Sveučilištu u Zagrebu*, [Teaching descriptive geometry at the University of Zagreb], Glasnik Matematički 45 (65), pp. 613–639.

[12] SINCLAIR, N., BARTOLLINI BUSSI, M. G., DE VILLERS, M., JONES, K, KORTENKAMP, U., LEUNG, A. AND OWENS, K. (2016), *Survey paper: Recent research on geometry education: an ICME-13 survey team report*, the International Group for the Psychology of Mathematics Education, ZDM Mathematics Education, pp. 311–336.

[13] JMC (2001), *English national document*, Teaching and learning geometry 11–19, Report of a Royal Society, Joint Mathematical Council working group, `https://royalsociety.org/topics-policy/publications/2001/teaching-geometry/` [Accessed 05/12/2016].

[14] JONES, K. (1998), *Theoretical Frameworks for the Learning of Geometrical Reasoning*, In Proceedings of the British Society for Research into Learning Mathematics (18, 1 & 2.), (pp. 29–34).

[15] JONES, K AND TZEKAKI, M. (2016), *Research on the teaching and learning geometry*, In, Gutierrez, Angel, Leder, Gilah and Boero, Paulo (Eds.), The second Handbook of Research on the Psychology of Mathematics Education (pp. 109–149), Rotterdam, NL, Sense Publishers.

[16] K-12 (2014), *Canadian national document*, Paying Attention to Spatial Reasoning, K-12: Support Document for Paying Attention to Mathematics Education, An Ontario Ministry of Education, `http://www.edu.gov.on.ca/eng/literacynumeracy/LNSPayingAttention.pdf` [Accessed 30/11/2016].

[17] Kovačević, N. (2016), *Prostorno mišljenje i geometrija prostora*, [Spatial reasoning and space geometry], Matematika i škola, pp. 196–204.

[18] Kuzniak, A., Gagatsis, A., Ludwig, M. and Marchini, C. (2007), *WG7 Report: From geometrical thinking to geometrical work*, In: Proceedings of the 5^{th} Congress of the European Society for Research in Mathematics Education (pp. 955–962), Larnaca, Cyprus 22 – 26 February 2007.

[19] Kuzniak, A. and Richard, R. (2014), *Spaces for Mathematical work: Viewpoints and perspectives*, Latin American Journal for Research in Mathematics Education, Vol. 17 (3), pp. 17–27.

[20] Lawrence, S. (2003), *History of Descriptive Geometry in England*, Huerta S. (Ed.), In Proceedings of the 1^{st} International Congress on Construction History, (pp. 1269–1281), Madrid.

[21] Leopold, C. (2015), *Visual Representations for Spatial Thinking*, Cocchiarella, L. (Ed.), The Visual Language of Technique Vol. 3 – Heritage and Expectations in Education, (pp. 37–56), Springer, Milano.

[22] Milin Šipuš, Ž. and Čižmešija, A. (2012), *Spatial ability of students of mathematics education in Croatia evaluated by the Mental Cutting Test*, Annales Mathematicae et Informaticae 40, pp. 203–216.

[23] Newcombe, N. S. (2010), *Picture this: Increasing Math and Science Learning by Improving Spatial Thinking*, American Educator, pp. 29–43.

[24] Sorby, S. A. (2009), *Educational research in developing 3D spatial skills for Engineering Students*, International Journal of Science Education, Vol. 31 (3), pp. 459–480.

[25] Stachel, H. (2015), *Descriptive Geometry – Vision Guided Spatial Reasoning*, Cocchiarella, L. (Ed.), The Visual Language of Technique Vol. 1 – History and Epistemology, (pp. 71–84), Springer, Milano.

[26] Uttal, D. H. and Cohen, C. A. (2012), *Spatial thinking and STEM Education: When, Why, and How?*, In Psychology of Learning and Motivation, Vol. 57, Chp. IV. (pp. 147–181), Elsevier Inc.

[27] Tartre, L. A. (1990), *Spatial Orientation Skill and Mathematical Problem Solving*, Journal for Research in Mathematics Education, Vol. 21 (3), pp. 216–229.

[28] Weiss, G. (2015), *Geometry – Daughter of Art, Mother of Mathematics*, Cocchiarella, L. (Ed.), The Visual Language of Technique Vol. 1 – History and Epistemology, (pp. 41–58), Springer, Milano.

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Prostorno rasuđivanje u matematici

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Sažetak. Uz sve veći interes za prostornim rasuđivanjem, potaknut razvojem snažnih vizualizacijskih programa i računalne geometrije, važno je razjasniti što se podrazumijeva pod prostornim rasuđivanjem u matematici. Polazeći od gledišta različitih matematičkih edukatora, ostvarenje istog obrazovnog cilja *razvijanje prostornog rasuđivanja* ne ostvaruje se uvijek na isti način u matematici kao i u drugim znanostima.

Dakle, premda prostorne sposobnosti mogu biti same po sebi intelektualno zanimljive, u ovom se radu fokus stavlja na njihovu povezanost s poučavanjem i učenjem geometrije na tehničkim fakultetima. Nadalje, detaljno će se opisati kolegij *Nacrtna geometrija s računalnom grafikom* koji je nastao na Rudarsko-geološko-naftnom fakultetu u Zagrebu slijedeći suvremene trendove razvoja geometrijskog obrazovanja. Koristeći između ostalog i tradicionalne geometrijske metode reprezentacije, kolegij se ne usmjerava samo na podizanje razine grafičke i vizualne komunikacije i razvijanje prostornih sposobnosti pojedinca, koje imaju ključnu ulogu u obrazovanju inženjera, već i na razvoj sposobnosti deduktivnog rasuđivanja te korištenje različitih alata i pomagala u matematičkom obrazovanju inženjera. U radu se raspravlja i o utjecaju računalne tehnologije na geometrijsko obrazovanje slijedeći smjernice SEFI – matematičke radne skupine. Dani su i različiti primjeri studentskih vježbi kako bi se prikazale brojne mogućnosti koje se nude studentima kroz razvoj inovativnih i interaktivnih obrazovnih metoda istovremeno primjenjivih u učenju matematike prostora i usklađenih sa specifičnim studentskim interesima.

Ključne riječi: prostorno rasuđivanje, nastavna pomagala, računalna grafika, visoko obrazovanje, e-učenje

The football {5, 6, 6} and its geometries: from a sport tool to fullerens and further

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Abstract. This presentation starts with the regular polygons, of course, then with the Platonic and Archimedean solids. The latter ones are whose symmetry groups are transitive on the vertices, and in addition, whose faces are regular polygons (see only I. Prok's home page [11] for them). Then there come these symmetry groups themselves (starting with the cube and octahedron, of course, then icosahedron and dodecahedron). Then come the space filling properties: Namely the cube is a space filler for the Euclidean space $\mathbf{E}^3$. Then we jump for the other regular solids that cannot fil $\mathbf{E}^3$, but can hyperbolic space $\mathbf{H}^3$, a new space. These can be understood better if we start regular polygons, of course, that cannot fil $\mathbf{E}^2$ in general, but can fil the new plane $\mathbf{H}^2$, as hyperbolic or Bolyai-Lobachevsky plane. Now it raises the problem, whether a football polyhedron – with its congruent copies – fil a space. It turns out that $\mathbf{E}^3$ is excluded (it remains an open problem – for you, of course, in other aspects), but $\mathbf{H}^3$ can be filled as a schematic construction show this (Fig. 5), far from elementary. Then we mention some stories on Buckminster Fuller, an architect, who imagined first time fullerens as such crystal structures. Many problems remain open, of course, we are just in the middle of living science.

Keywords: Mathematics teacher as popularizer of science, Platonic and Archimedean solid, tiling, Euclidean and non-Euclidean manifold, crystal structure, fulleren

Mathematics Subject Classification 2010: *57M07, 57M60, 52C17.*

1. Introduction

After the above abstract we start with the regular n-sided polygons and its symmetry groups, generated by two line reflections $\mathbf{a}$ and $\mathbf{b}$ with relations $\mathbf{1} = \mathbf{a}^2 = \mathbf{b}^2 =$

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$(\mathbf{ab})^n$, i.e. the edge bisector line a and angle bisector b intersect each other in the centre O of the n-gon *in angle* π/n. Then come the 5 regular Platonic solids with symbols (p,q) : $(3,3)$ = tetrahedron (self dual), $(3,4)$ = octahedron, $(4,3)$ = cube, $(3,5)$ = icosahedron, $(5,3)$ = dodecahedron.

These can also be derived by an elementary observation, which asserts that the angle of a regular Euclidean p-gon is $(p-2)\pi/p$ (the angle sum of a (say convex) p-gon is $(p-2)\pi$, since it can be divided by $p-3$ diagonals, from any vertex, into $p-2$ triangles, each having an angle sum $\pi = 180°$). So q pieces of them meet at a vertex (with non-plane vertex figure), if

$$\frac{q(p-2)\pi}{p} < 2\pi \Leftrightarrow \text{(equivalent to)}\ \frac{1}{2} < \frac{1}{p} + \frac{1}{q},$$

a necessary condition. This will be sufficient if we think of a spherical triangle (so-called *characteristic triangle*) with angles $\pi/2$, π/p and π/q (with angle sum larger than π, generating the above vertex figure.

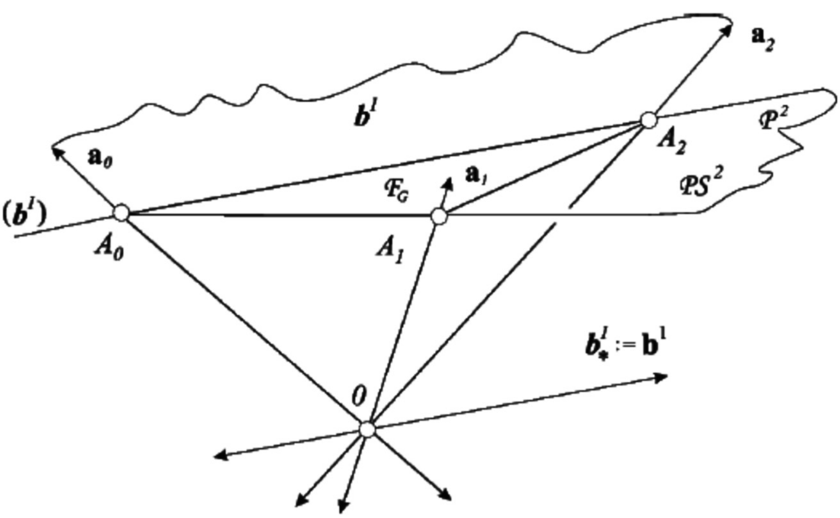


Figure 1. A projective coordinate triangle to our fundamental triangle $A_0\,A_1\,A_2$. The plane by form class $\boldsymbol{b}^1$ (e.g.) describes the line $A_0\,A_2$. The point by polar vector $\boldsymbol{b}^1_* = \mathbf{b}^1 = b^{10}\mathbf{a}_0 + b^{11}\mathbf{a}_1 + b^{12}\mathbf{a}_2 =: b^{1j}\mathbf{a}_j$ is its pole.

In more general extension, (p,q) and (q,p) are dual pairs: as the octahedron and cube, icosahedron and dodecahedron, above. To the symmetry group $\mathbf{G}$ we have 3 generating plane reflections, denoted by $\boldsymbol{b}^0, \boldsymbol{b}^1, \boldsymbol{b}^2$ (with the corresponding planes b^i, $i \in \{0,1,2\}$), and defining relations: $\mathbf{1} = \boldsymbol{b}^0\boldsymbol{b}^0 = \boldsymbol{b}^1\boldsymbol{b}^1 = \boldsymbol{b}^2\boldsymbol{b}^2 = (\boldsymbol{b}^0\boldsymbol{b}^1)^p = (\boldsymbol{b}^0\boldsymbol{b}^2)^2 = (\boldsymbol{b}^1\boldsymbol{b}^2)^q$, as the Coxeter-Schläfli diagram

$$\overset{0}{\circ}\overset{p}{\text{————}}\overset{1}{\circ}\overset{q}{\text{————}}\overset{2}{\circ} \tag{1.1}$$

indicates this with nodes o for the reflection planes, and the *branches* p and q for the relations above. If two nodes are not connected then their reflections commute (with their orthogonal planes, the product order is 2 then). Thus the tetrahedron group is of order 24, as the symmetry group $\mathbf{G}$ of the platonic solid are in general of order $|\mathbf{G}| = 4\pi/(\pi/p+\pi/q-\pi/2)$. Namely, the surface area $4\pi/R^2$ of the sphere $\mathbf{S}^2$ of radius R is divided by the area (with angle excess $R^2[(\pi/p+\pi/q+\pi/2)-\pi]$) of the fundamental triangle of $\mathbf{G}$ (see the above elementary observations as well). For octahedron and cube we have 48 symmetry elements, for icosahedron and dodecahedron we have 120 elements for the symmetry group $\mathbf{G}$.

At the same time we can introduce the so-called Coxeter-Schläfli matrix

$$(b^{ij}) = \langle \boldsymbol{b}^i, \boldsymbol{b}^j \rangle = \begin{bmatrix} 1 & -\cos\left(\frac{\pi}{p}\right) & 0 \\ -\cos\left(\frac{\pi}{p}\right) & 1 & -\cos\left(\frac{\pi}{q}\right) \\ 0 & -\cos\left(\frac{\pi}{q}\right) & 1 \end{bmatrix} = (cos(\pi-\beta^{ij})). \tag{1.2}$$

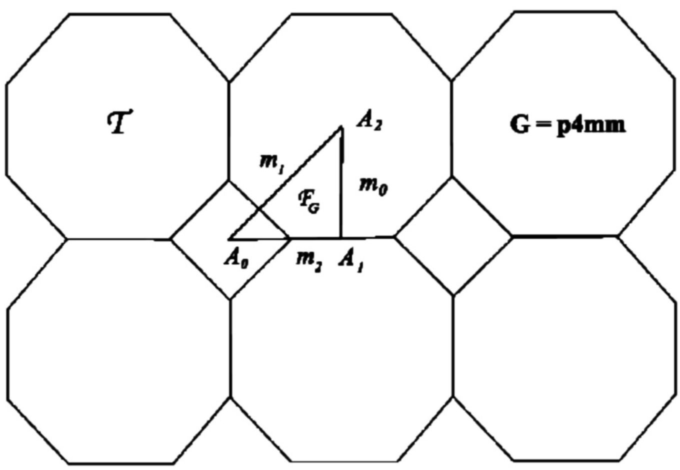


Figure 2. A well-known Archimedean tiling in the Euclidean plane with its fundamental triangle.

This is derived from the formal scalar products of the linear forms (normal unit vectors) $\boldsymbol{b}^i$, ordered to the side lines b^i of the characteristic triangle $A_0A_1A_2$ with angles $\beta^{01} = \pi/p$, $\beta^{02} = \pi/2$, $\beta^{12} = \pi/q$. By convention, any side line b^i above closes angle π with itself, it lies opposite to vertex A_i.

Fig. 1 shows symbolically a $(2+1)$-(dimensional picture to the projective triangle (simplex) coordinate system, also for the later higher dimensional analogue in a $(d+1)$-(dimensional vector space $\mathbf{V}^{d+1}$ and its dual $\boldsymbol{V}_{d+1}$. Here O denotes the origin from where vectors $\mathbf{a}_i = OA_i(i \in \{0,1,2\}$ point to the vertices of triangle $A_0A_1A_2$, forms as normal vectors $\mathbf{b}^j$ are placed *to the side plane b^j*. See also the later Fig. 2 for an *Archimedean tiling* in $\mathbf{E}^2$. Thus

$$\mathbf{a}_i\mathbf{b}^j = \delta_i^j$$

(the *Kronecker symbol*, $i,j \in \{0,1,2\}$ indicates the incidence relations. By the way, the inverse matrix $(a_{ij}) = (b^{ij})^{-1}$ of the above Coxeter-Schläfli matrix just serves the distance metrics of the triangle $A_0A_1A_2$. The side length A_iA_j can be expressed from spherical angle by

$$\cos\left(\frac{A_iA_j}{R}\right) = \frac{a_{ij}}{\sqrt{a_{ii}a_{jj}}}. \tag{1.3}$$

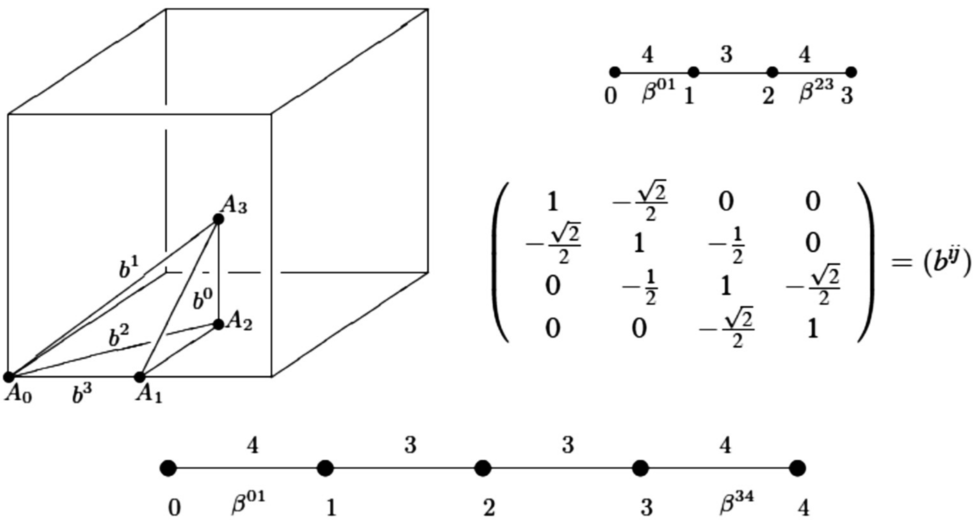


Figure 3. Cube tiling in $\mathbf{E}^3$ and symbols for it.
Coxeter-Schläfli diagram for the $\mathbf{E}^4$ cube tiling.

This is related now to the spherical geometry of the Platonic solid above, where

$$\det(b^{ij}) = 1 - cos^2(\pi/p) - cos^2(\pi/q) > 0. \tag{1.4}$$

E.g. for the cube $(p,q) = (4,3)$ we get $\det(b^{ij}) = 1 - \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$. This is more critical for the dodecahedron (or icosahedron): $\det(b^{ij}) = 1 - \cos^2(\pi/5) - \frac{1}{4} = \frac{3-\sqrt{5}}{8} \approx 0.09549 > 0$.

2. Euclidean and non-Euclidean mosaics

If above $\det(b^{ij}) = 0$ in formula (1.4), i.e. $(p,q) = (3,6), (6,3), (4,4)$, then we get mosaic or tiling in Euclidean plane $\mathbf{E}^2$. The latter one is the well-known squared paper. These follow also by the above elementary observations.

But we can assume in formula (1.4) also $\det(b^{ij}) < 0$, i.e. then we get an infinite series for $(p,q) = (3,7), (7,3), (3,8), (8,3), \ldots, (4,5), (5,4), (4,6), (6,4), \ldots$ Above, we can imagine characteristic triangles with angles π/p, π/q, $\pi/2$, their sum is smaller than $\pi = 180°$ as it holds for Euclidean plane $\mathbf{E}^2$. We have obtained a new geometry, the so-called hyperbolic or Bolyai-Lobachevsky plane, denoted by $\mathbf{H}^2$. This geometry was discovered and elaborated first, approximately in the same time and independently, by the Hungarian János Bolyai and the Russian Nikolai Ivanović Lobačevskii in 1820's years.

The area of a triangle with angles α, β, γ is equal (proportional) to the defect $\pi - (\alpha + \beta + \gamma)$ in $\mathbf{H}^2$. (This was observed also by the fore-runners of the new geometry, so by Carl Friedrich Gauss, who played so important role in the life of Farkas (Wolgang) and János Bolyai, see e.g. [9].) Thus the starting tilings $(3,7)$, $(7,3)$ have just a minimal characteristic triangle with area $\pi/2 - \pi/3 - \pi/7 = \pi/42$.

It turned out, that the formulas of spherical geometry $\mathbf{S}^2$ become true formulas in $\mathbf{H}^2$ if we substitute imaginary radius $ki = R$ into the spherical formulas, e.g. into (1.1) above ($i = \sqrt{-1}$ is the imaginary complex unit as usual). Thus, so-called hyperbolic functions come into the play, etc. So we get a unifying concept, *absolute geometry* in the sense of János Bolyai, a joint kernel of geometries $\mathbf{E}^2$, $\mathbf{S}^2$, $\mathbf{H}^2$, and later in any dimension d for $\mathbf{E}^d$, $\mathbf{S}^d$, $\mathbf{H}^d$, where the analogy does not remain so easy, and we have many open problems.

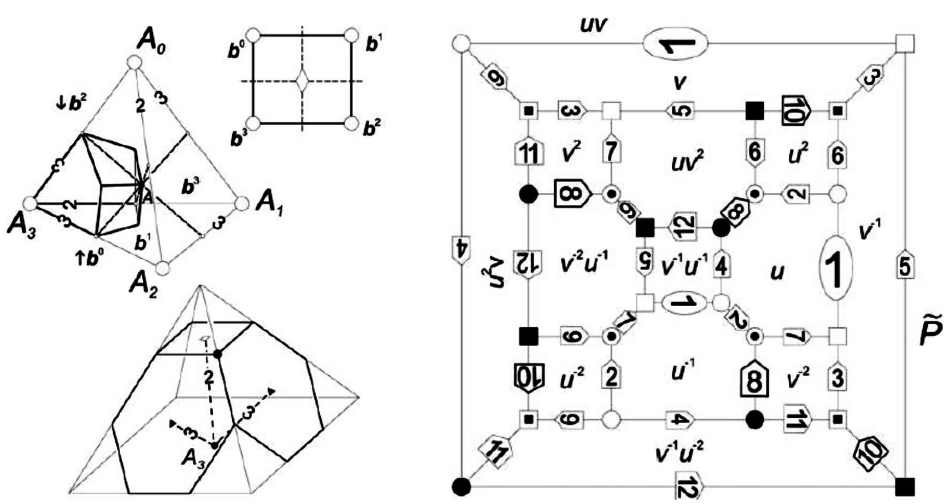


Figure 4. The Archimedean solid, $\{4,6,6\}$ as truncated octahedron and its Euclidean manifold $\mathbf{E}^3/\mathbf{P}2_12_12_1 = \tilde{P}$.

However, for the above regular tilings (p, q) we have a unified theory (see e.g. [2] and our Sect. 5), where reflection groups (so-called Coxeter groups, as above in formulas $(1.1-4)$), play important roles.

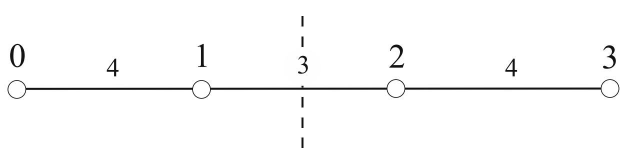
An interesting topic is the so-called *Archimedean or vertex transitive tilings* (mosaics, see Fig. 2) by polygons, where the symmetry group $\mathbf{G}$ of the tiling T acts *transitively* on the vertices (i.e. all vertices form one equivalence class under the symmetry group $\mathbf{G}(T)$. In addition, for nice pictures, we assume that the tiles of T are regular polygons (they are possibly non-congruent, of course).

The above series of Coxeter reflection groups, denoted by $[2, p, q]$ with fundamental triangle of angles $\pi/2$, π/p, π/q, each provides nice Archimedean tiling. E.g. $\{4, 6, 6\}$, where square-hexagon-hexagon meet at every vertex, is the truncated octahedron as a space filler polyhedron of Euclidean space $\mathbf{E}^3$. Analogously $\{4, 8, 8\}$, where square-octagon-octagon meet at every vertex, is a very popular pavement of our Euclidean streets (Fig. 2, 4). Moreover, $\{4, 10, 10\}$ will be a hyperbolic tiling in $\mathbf{H}^2$, see also [4].

Special interest deserves the football polyhedron $\{5, 6, 6\}$, where pentagon-hexagon-hexagon meet at every vertex (Fig. 5). Thus, we obtain 12 pentagons and 20 hexagons with 60 vertices at all. This polyhedron can be obtained from the regular icosahedron (with group $[2, 3, 5]$ above) by truncating its 12 vertices each derives a regular pentagon, so that the 20 triangles become 20 hexagons. This football "plays important role in our life", of course, and it turned out, this can play new roles in the structure of new materials as fullerens in crystallography, may be in non-Euclidean crystallography as follows in the next sections.

3. Space filler polyhedra in Euclidean and non-Euclidean spaces

We know that the cube is a space filler polyhedron, i.e. we can fil Euclidean space $\mathbf{E}^3$ with its congruent copies, face-to-face without gaps and overlaps. Namely, this can be derived by the arguments close to that of the introduction at the Coxeter-Schläfli diagram and matrix, now especially with $(p, q, r) = (4, 3, 4)$ with some extension:



(3.1)

$$(b^{ij}) = \langle \boldsymbol{b}^i, \boldsymbol{b}^j \rangle = \begin{bmatrix} 1 & -\cos\left(\frac{\pi}{p}\right) & 0 & 0 \\ -\cos\left(\frac{\pi}{p}\right) & 1 & -\cos\left(\frac{\pi}{q}\right) & 0 \\ 0 & -\cos\left(\frac{\pi}{q}\right) & 1 & -\cos\left(\frac{\pi}{r}\right) \\ 0 & 0 & -\cos\left(\frac{\pi}{r}\right) & 1 \end{bmatrix} \tag{3.2}$$
$$= (cos(\pi - \beta^{ij})).$$

That means (Fig. 3), we have a characteristic tetrahedron (simplex) $A_0A_1A_2A_3 = b^0b^1b^2b^3$ with the cube centre A_3, with a face centre A_2, with midpoint A_1 of an edge incident to the previous face, and a vertex A_0 of the previous edge. Furthermore, $b^i = A_jA_kA_l$ with $\{i,j,k,l\} = \{0,1,2,3\}$ hold. We apply real (left) vector space $\mathbf{V}^4$ for points and its (right) dual space $\boldsymbol{V}_4$ for planes up to projective equivalences $\sim$ as usual. Vector coefficients are written in rows on the left to column basis; dual forms with row basis have column coefficients on the right. Forms act on the right on vectors.

Now the characteristic simplex has a half-turn symmetry $0 \leftrightarrow 3$, $1 \leftrightarrow 2$ as indicated in the extended diagram (3.1). The simple reflection diagram describes the crystallographic space group 221. **Pm3m** with so-called primitive cube lattice [3]. The (with half-turn) extended diagram describes the space group 229. **Im3m** with body centred (*innenzentriert* in German) cube lattice. The last assertions indicate that our topic has important applications in crystallography, describing atomic or molecular structures, etc. You find [3] in the Internet for free download. We will extend these concepts to non-Euclidean geometry as well.

4. A Euclidean space form, as typical example

For the later generalization, we analyse Fig. 4 with the truncated octahedron $\{4,6,6\}$ as Archimedean solid that can fil Euclidean space $\mathbf{E}^4$ with its congruent copies. To this a famous space filler tetrahedron (simplex) $A_0A_1A_2A_3 = b^0b^1b^2b^3$, the so-called *sphenoid*, with extended diagram in Fig. 4 (to the space group 224. **Pn3m**) and Coxeter-Schläfli matrix

$$(b^{ij}) = (\langle \boldsymbol{b}^i, \boldsymbol{b}^j \rangle) = \begin{bmatrix} 1 & -\frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 1 & -\frac{1}{2} \\ -\frac{1}{2} & 0 & -\frac{1}{2} & 1 \end{bmatrix} = (\cos(\pi - \beta^{ij})) \tag{4.1}$$

play important roles. That means, the simplex has two opposite rectangles $\beta^{02} = \beta^{13} = \pi/2$ as dihedral angles, the remaining four dihedral angles are $\pi/3$. Moreover, the simplex has 3 half-turns, their axes connect the corresponding opposite edge midpoints and meet at the simplex centre A. The images of A will just form the Archimedean vertex class equivalent under the 24 regular tetrahedron symmetries around A_3. Thus, we get the truncated octahedron $\{4,6,6\}$ denoted by P and its so-called Schlegel diagram in Fig. 4 right in $\mathbf{E}^2$.

Now we organize new face pairing isometries and new space filler tiling with $\tilde{P} = \{4,6,6\}$ as fundamental polyhedron under a new fixed-point-free space group (finally it will be 19. $\mathbf{P2_12_12_1}$ by [3]).

Our method will be very general, and we apply it in the following.

Look at the sphenoid in Fig. 4 left that 3 truncated octahedron meet at any edge of the $\{4,6,6\}$ tiling, along square-hexagon, hexagon-hexagon, hexagon-square. Thus in Fig. 4 right, we want to construct a new group with fixed-point-free action, where any point will have a ball-like neighbourhood without its other image point in that ball. We start a directed edge class consisting of three edges, denoted say by 1. The first 1 is defined on the boundary of a square and a hexagon denoted by u^{-1}, then the second 1 is on the **u-**image hexagon u, while hexagon v^{-1} follows on its other side, then the third 1 is placed on the **v**-image hexagon v with a square on its other side. Now the first square will be denoted by $(uv)^{-1} = v^{-1}u^{-1}$, the last square (now with the outer "infinite" face) by uv because the side pairing mapping with usual conventions:

$$\mathbf{uv}{:}(uv)^{-1}{=}v^{-1}u^{-1} \leftrightarrow uv \text{ and } (\mathbf{uv})^{-1}{=}\mathbf{v}^{-1}\mathbf{u}^{-1}{:}uv \leftarrow (uv)^{-1}{=}v^{-1}u^{-1}. \quad (4.2)$$

Imagine our new tiling as representing the elements of our new group $\mathbf{G}(= \mathbf{P2_12_12_1})$ by the images of a starting fundamental domain $\tilde{P}^{\mathbf{I}}$ (the identity domain). The **u**-image domain $\tilde{P}^{\mathbf{u}}$ lies just besides face u with its face $(u^{-1})^u$. Similarly, the $\mathbf{u}^{-1}$-image domain $\tilde{P}^{u^{-1}}$ lies besides face u^{-1} with face $(u)^{\mathbf{u}-1}$. Thus, to the first edge 1 and face u^{-1} of $\tilde{P}$, we find three edge-domains each between two faces:

$$\begin{aligned} (v^{-1}u^{-1})|\tilde{P}|(u^{-1}) \ \text{ then } \ & (u)^{\mathbf{u}-1}|(\tilde{P})^{\mathbf{u}-1}|(v^{-1})^{\mathbf{u}-1} \\ \text{then } \ & (v)^{\mathbf{v}-1\mathbf{u}-1}|(\tilde{P})^{\mathbf{v}-1\mathbf{u}-1}|(uv)^{\mathbf{v}-1\mathbf{u}-1} \end{aligned} \quad (4.3)$$

then cyclically comes the first identity edge domain, now with a formal general rule, as **uv**-image of $(\tilde{P})^{\mathbf{v}-1\mathbf{u}-1}$, i.e. $[(\tilde{P})^{\mathbf{v}-1\mathbf{u}-1}]^{\mathbf{uv}} = \tilde{P}$.

Consider the edge between u and u^{-1}, numbered by 2 as an arrow. The mappings **u** and $\mathbf{u}^{-1}$ fixed before, carry this edge 2 into two other edges denoted also by 2: first to the edge between $u$and a square face, denoted by u^2, second to that edge between u^{-1} and the square u^{-2}. So, the face pairing isometries to the edge class 2:

$$\mathbf{u}^2 : u^{-2} \leftarrow u^2 \text{ and } \mathbf{u}^{-2} : u^2 \leftarrow u^{-2} \quad (4.4)$$

have also been introduced.

Then we obtain a straightforward procedure with 12 edge classes: either we get a new face pairing isometry expressed by generators $\mathbf{u}$ and $\mathbf{v}$, mapping $\tilde{P}$ onto an adjacent image; or we get a trivial relation, going around an edge in the tiling, e.g. at 6: $\mathbf{u}^2\mathbf{v}\mathbf{v}^{-1}\mathbf{u}^{-2} = \mathbf{1}$, as consequence; or we get a non-trivial so-called defining relation, e.g. at 8: $\mathbf{v}^2\mathbf{u}\mathbf{v}^2\mathbf{u}^{-1} = \mathbf{1}$; or we get a consequence of the former defining relations.

Now it turns out, as a lucky situation, that the procedure goes smoothly to the end, without contradiction. Every edge class has 3 edges with the same rules of adjacencies, and two generators are sufficient. So we obtain a presentation of our space group named 19. $\mathbf{P2_12_12_1}$ in [3] (see also author's further papers in references, especially [5], [7], [10]):

$$\mathbf{G} = \mathbf{P2_12_12_1} = (\mathbf{u}, \mathbf{v} \mid \mathbf{v}^2\mathbf{u}\mathbf{v}^2\mathbf{u}^{-1} = \mathbf{u}^2\mathbf{v}^{-1}\mathbf{u}^2\mathbf{v} = 1). \tag{4.5}$$

The vertices of $\tilde{P}$ are divided into 6 equivalence classes, four vertices in each class. To every vertex classes join 4 edge classes (as chemical bounds between atoms) as the next formula shows:

$$\circ\,(1, 2, 4, 6); \qquad \square\,(1, 3, 5, 7); \qquad \bullet\,(4, 8, 11, 12);$$

$$\blacksquare\,(5, 9, 10, 12); \qquad \odot\,(2; 7; 8; 9); \qquad \boxed{\bullet}\,(3; 6; 10; 11)$$

Table 1. The truncated octahedron, as fundamental domain for the following Euclidean space groups [3], with its face pairings given up to its symmetries (computer classification by István Prok and Zsanett Szuda (BME Math MSc student)).

No.	Name	#
1	**P1**	1
2	$\mathbf{P\bar{1}}$	2
4	$\mathbf{P2_1}$	2
5	**C2**	3
7	**Pc**	2
9	**Cc**	3
13	**P2/c**	3
14	$\mathbf{P2_1/c}$	10
15	**C2/c**	8
16	**P222**	1
18	$\mathbf{P2_12_12}$	2
19	$\mathbf{P2_12_12_1}$	2
20	$\mathbf{C222_1}$	2
23	**I222**	2
27	**Pcc2**	2
29	$\mathbf{Pca2_1}$	3
32	**Pba2**	1
33	$\mathbf{Pna2_1}$	4

No.	Name	#
34	**Pnn2**	1
41	**Aea2**	3
43	**Fdd2**	1
45	**Iba2**	3
50	**Pban**	3
54	**Pcca**	3
56	**Pccn**	3
60	**Pbcn**	6
61	**Pbca**	3
68	**Ccce**	2
70	**Fddd**	1
76	$\mathbf{P4_1}$	1
78	$\mathbf{P4_3}$	
77	$\mathbf{P4_2}$	1
81	$\mathbf{P\bar{4}}$	1
82	$\mathbf{I\bar{4}}$	1
86	$\mathbf{P4_2/n}$	1
88	$\mathbf{I4_1/a}$	2

No.	Name	#
91	$\mathbf{P4_122}$	1
95	$\mathbf{P4_322}$	
92	$\mathbf{P4_12_12}$	2
96	$\mathbf{P4_32_12}$	
94	$\mathbf{P4_22_12}$	1
98	$\mathbf{I4_122}$	1
106	$\mathbf{P4_2bc}$	1
110	$\mathbf{I4_1cd}$	1
112	$\mathbf{P\bar{4}2c}$	1
114	$\mathbf{P\bar{4}2_1c}$	2
117	$\mathbf{P\bar{4}b2}$	1
120	$\mathbf{I\bar{4}c2}$	1
122	$\mathbf{I\bar{4}2d}$	1
144	$\mathbf{P3_1}$	1
145	$\mathbf{P3_2}$	
152	$\mathbf{P3_121}$	1
154	$\mathbf{P3_221}$	

With the scalar product $\langle \boldsymbol{b}^i, \boldsymbol{b}^j \rangle = b^{ij}$ of matrix (4.1) we can formally define a quadratic form (with Einstein-Schouten index convention for summing):

$$\begin{aligned} w_i b^{ij} w_j &= w_0w_0 - w_0w_1 - w_0w_3 + w_1w_1 - w_1w_2 + w_2w_2 - w_2w_3 + w_3w_3 \\ &= (w_0 - \frac{1}{2}w_1 - \frac{1}{2}w_3)^2 + \frac{3}{4}(w_1 - \frac{2}{3}w_2 - \frac{1}{3}w_3)^2 + \frac{2}{3}(w_2 - w_3)^2 \end{aligned} \tag{4.7}$$

with sum of three positive squares. We say that the scalar product of matrix (4.1) is of signature $(+,+,+,0)$, and this is characteristic for Euclidean geometry $\mathbf{E}^3$. The determinant of matrix (4.1) equals to 0 (zero). But take $(p,q,r) = (5,3,5)$ in matrix (3.2) instead of diagram (3.1). Then the determinant will be negative and we would obtain a negative square summand besides of the three positive square summand in the quadratic form, analogue to (4.7) (not uniquely).

This will be the topic in the next main sections.

However, before that, we add Table 1, as complete computer classification of all the space groups which can have the truncated octahedron as fundamental domain. Its face pairings are given up to its symmetries.

5. The hyperbolic space $\mathbf{H}^3$ to the football {5, 6, 6}

We start with our Fig. 5, the plane figure of the real football (again, the so-called Schlegel diagram). Maybe surprisingly, the Coxeter-Schläfli matrix and diagram provide us with all the theoretical tools, as above. Now it will be in the hyperbolic space $\mathbf{H}^3$, modelled in the projec-tive metric space $\mathcal{P}^3(\mathbf{V}^4, \boldsymbol{V}_4, \sim, \langle\, ,\rangle\,)$ with a scalar product (or polarity) of above signature $(+,+,+,-)$. For the Coxeter-Schläfli diagram in Fig. 5 we formally introduce the matrix

$$(b^{ij}) = \begin{bmatrix} 1 & -\cos\left(\frac{\pi}{5}\right) & 0 & 0 \\ -\cos\left(\frac{\pi}{5}\right) & 1 & -\cos\left(\frac{\pi}{3}\right) & 0 \\ 0 & -\cos\left(\frac{\pi}{3}\right) & 1 & -\cos\left(\frac{\pi}{5}\right) \\ 0 & 0 & -\cos\left(\frac{\pi}{5}\right) & 1 \end{bmatrix}, \tag{5.1}$$

and fix basis $\boldsymbol{b}^i$ of (the dual form space) $\boldsymbol{V}_4$ to the side faces of simplex $A_0A_1A_2A_3 = b^0b^1b^2b^3$ with the scalar product

$$\langle\, ,\rangle : \boldsymbol{V}_4 \times \boldsymbol{V}_4 \to \mathbf{R} \text{ (the real field) by } \langle \boldsymbol{b}^i, \boldsymbol{b}^j \rangle = b^{ij} \text{ with the matrix (5/1).} \tag{5.2}$$

This scalar product has the signature $(+,+,+,-)$ now, since

$$B := \det\,(b^{ij}) = \left(1 - \cos^2\frac{\pi}{5}\right)^2 - \cos^2\frac{\pi}{3} < 0, \tag{5.3}$$

however, all the principal minors are positive in the chain

$$b^{00}=1,\begin{vmatrix} b^{00} & b^{01} \\ b^{10} & b^{11} \end{vmatrix}=1-\cos^2\frac{\pi}{5},\begin{vmatrix} b^{00} & b^{01} & b^{02} \\ b^{10} & b^{11} & b^{12} \\ b^{20} & b^{21} & b^{22} \end{vmatrix}=1-\cos^2\frac{\pi}{5}-\cos^2\frac{\pi}{3}>0. \tag{5.4}$$

The inverse matrix (a_{ij}) of (b^{ij}), with $b^{ij}\cdot a_{jk}=\delta^i_k$, has great importance, since it induces the scalar product on $\mathbf{V}^4$ by linear extension

$$\langle\,,\rangle:\mathbf{V}^4\times\mathbf{V}^4\to\mathbf{R}\text{ by }\langle\,\mathbf{a}_i,\mathbf{a}_j\rangle\ =a_{ij}, \tag{5.5}$$

where $\{\mathbf{a}_j\}$ is just the dual basis to $\{\boldsymbol{b}^i\}$ defined by $\mathbf{a}_j\mathbf{b}^i=\delta^i_j$. Geometrically, the vectors $\mathbf{a}_j$ represent the vertices A_j of the Lanner simplex $F_{\mathbf{L}}$ whose side planes m_i are described by forms $\boldsymbol{b}^i$. In general, the vectors $\mathbf{x}$ in the cone

$$C=\{x\in\mathbf{V}^4:\langle\,\mathbf{x};\mathbf{x}\rangle\ <0\} \tag{5.6}$$

define the proper points $(\mathbf{x})$ of the hyperbolic space $\mathbf{H}^3$ embedded in $\mathcal{P}^3$. If $(\mathbf{x})$ and $(\mathbf{y})$ are proper points, with $\langle\,\mathbf{x}\,;\,\mathbf{y}\rangle\ <0$, then their distance $d(\mathbf{x},\,\mathbf{y})$ is defined by

$$\cos h\frac{d}{k}=\frac{-\langle\,\mathbf{x};\mathbf{y}\rangle}{\sqrt{\langle\mathbf{x};\mathbf{x}\rangle\,\langle\,\mathbf{y};\mathbf{y}\rangle}}\geq 1, \tag{5.7}$$

where $k=\sqrt{-\dfrac{1}{K}}$ is the metric constant of $\mathbf{H}^3$ (of constant negative curvature K). The forms $\boldsymbol{u}$ in the complementary cone

$$C^*=\{\boldsymbol{u}\in\boldsymbol{V}_4\langle\,\boldsymbol{u};\boldsymbol{u}\rangle\ >0\} \tag{5.8}$$

define the proper planes $(\boldsymbol{u})$ of $\mathbf{H}^3$. Suppose that $(\boldsymbol{u})$ and $(\boldsymbol{v})$ are proper planes. They intersect in a proper straight line, iff $\langle\,\boldsymbol{u};\boldsymbol{u}\rangle\,\langle\,\boldsymbol{v};\boldsymbol{v}\rangle\,-\langle\,\boldsymbol{u};\boldsymbol{v}\rangle^{\,2}>0$. One of their angles $\alpha(\boldsymbol{u};\boldsymbol{v})$ can be defined by

$$\cos\alpha=\frac{\langle\,\boldsymbol{u};\,\upsilon\rangle}{\sqrt{\langle\,\boldsymbol{u};\boldsymbol{u}\rangle\,\langle\,\upsilon;\,\upsilon\rangle}},\ 0<\alpha<\pi. \tag{5.9}$$

The scalar products introduced in $\boldsymbol{V}_4$ and $\mathbf{V}^4$, respectively, allow us to define a bijective polarity between vectors and forms:

$$\begin{aligned}
&({}_*):\boldsymbol{V}_4\to\mathbf{V}^4;\ \boldsymbol{u}\to\boldsymbol{u}_*=:\mathbf{u}\text{ by}\\
&\mathbf{u}\upsilon:=\langle\,\mathbf{u};\,\upsilon\rangle\ \text{ for every }\upsilon\in\boldsymbol{V}_4,\\
&({}_*):V_4\to\boldsymbol{V}^4;\ \mathbf{x}\to\mathbf{x}^*=:\boldsymbol{x}\text{ by}\\
&\boldsymbol{y}\boldsymbol{x}:=\langle\,\mathbf{y};\mathbf{x}\rangle\ \text{ for every }\mathbf{y}\in\mathbf{V}^4.
\end{aligned} \tag{5.10}$$

So we get a geometric polarity between points and planes: proper points have improper polars, proper planes have improper poles. A polar $(\boldsymbol{u})$ and its pole $(\mathbf{u})$ are incident iff

$$0=\boldsymbol{u}\mathbf{u}=\langle\,\boldsymbol{u};\boldsymbol{u}\rangle\ =\langle\,\mathbf{u};\mathbf{u}\rangle \tag{5.11}$$

Such a point $(\mathbf{u})$ is called end of $\mathbf{H}^3$ (or point on the absolute quadratic or quadric), its polar $(\boldsymbol{u})$ is a boundary plane (tangent to the absolute at the end). They form points and planes of a quadric of type $(+, +, +, -)$.

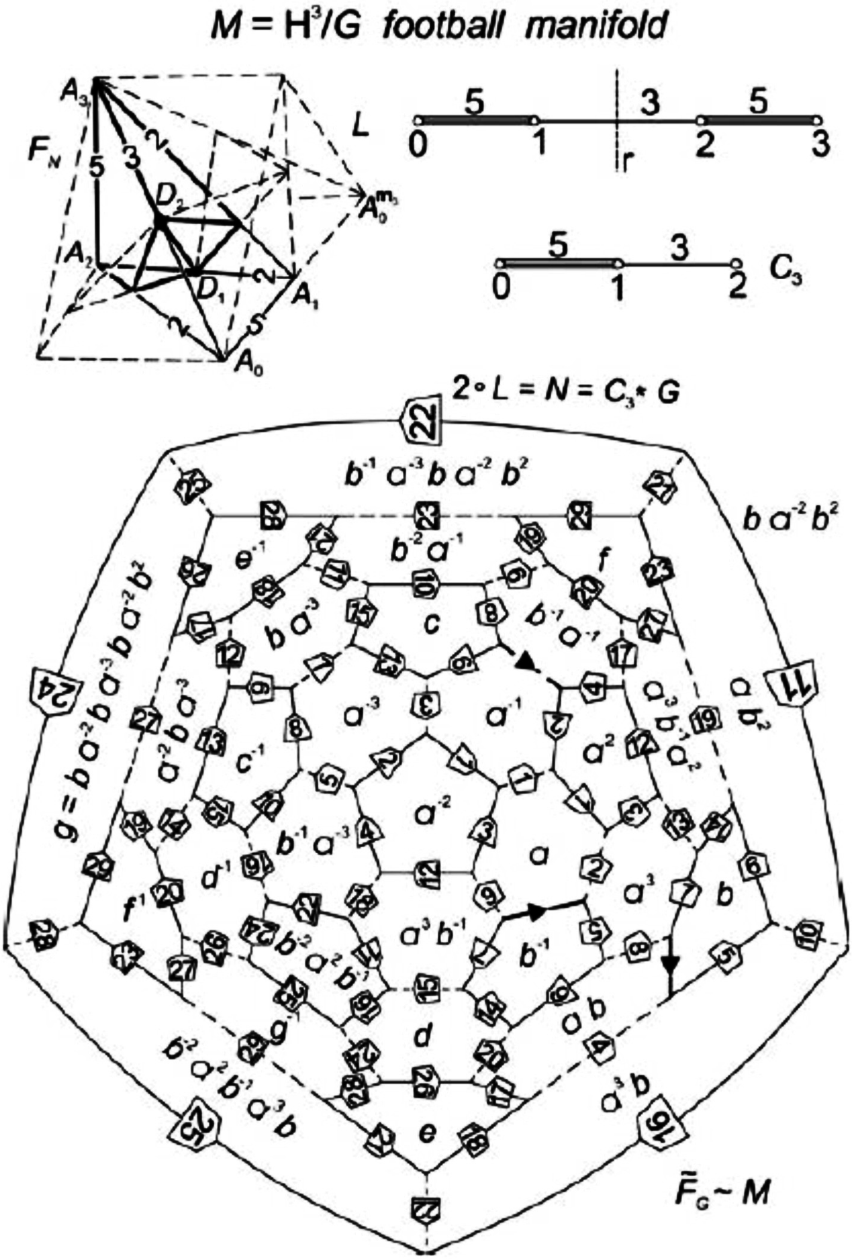


Figure 5. The hyperbolic football manifold for the Archimedean solid $\{5, 6, 6\}$.

The isometry group $\mathbf{G}$ of $\mathbf{H}^3$ can be generated by plane reflections. If $(\boldsymbol{u})$ is a proper plane and $(\mathbf{u})$ is its pole, then the reflection formulas for points (vectors) and planes (forms) are

$$x \to x - \frac{2(x\boldsymbol{u})}{\langle \boldsymbol{u};\mathbf{u}\rangle}\cdot \mathbf{u};\ \upsilon \to \upsilon - \boldsymbol{u}\cdot \frac{2\langle\ \upsilon;\boldsymbol{u}\rangle}{\langle \boldsymbol{u};\boldsymbol{u}\rangle}. \tag{5.12}$$

Now, we can define the Lanner group, denoted by $\mathbf{L}$, as a group generated by reflections in the side planes of the simplex $F_L{=}A_0A_1A_2A_3$, given by the forms $(\boldsymbol{b}^j)$ and their poles. As the matrix (b^{ij}) in (5.1) shows, the face angle between m_0 and m_1 is equal to $\frac{\pi}{5} = \beta^{01}$. Analogously hold $\beta^{02} = \frac{\pi}{2} = \beta^{03}$; $\beta^{12} = \frac{\pi}{3}$; $\beta^{13} = \frac{\pi}{2}$; $\beta^{23} = \frac{\pi}{5}$ as it has just been prescribed by the Coxeter-Schläfli diagram of $\mathbf{L}$ in Fig. 5. We can check that the subgroup generated by the plane reflections m_0 and m_1, for instance, is of order $2 \times 5 = 10$. Moreover, the subgroup $\mathbf{C}_3$, stabilizing the point A_3 in $\mathbf{L}$, is just the dodecahedron group of order 120, generated by the reflections $\mathbf{m}_0$, $\mathbf{m}_1$, $\mathbf{m}_2$ in $\mathbf{H}^3$. A fundamental polyhedron of the group $\mathbf{L}$ is the simplex $A_0A_1A_2A_3 = F_{\mathbf{L}}$ itself. Its points $(\mathbf{x})$ can be characterized by

$$F_L = \left\{(\mathbf{x}) : \boldsymbol{x}\boldsymbol{b}^{\mathbf{j}} \geq 0 \textbf{ for each } \boldsymbol{b}^{\mathbf{j}},\ \mathbf{j} = 0,1,2,3\right\} \tag{5.13}$$

This is uniquely determined by the fixed point set of $\mathbf{L}$, i.e. any other one is an $\mathbf{L}$-image of F_L. From the projective model of $\mathbf{H}^3$ we can turn to other models, e.g. to the usual Cayley-Klein model in a Euclidean ball. Our figures are shown in this model (without indicating the absolute ball).

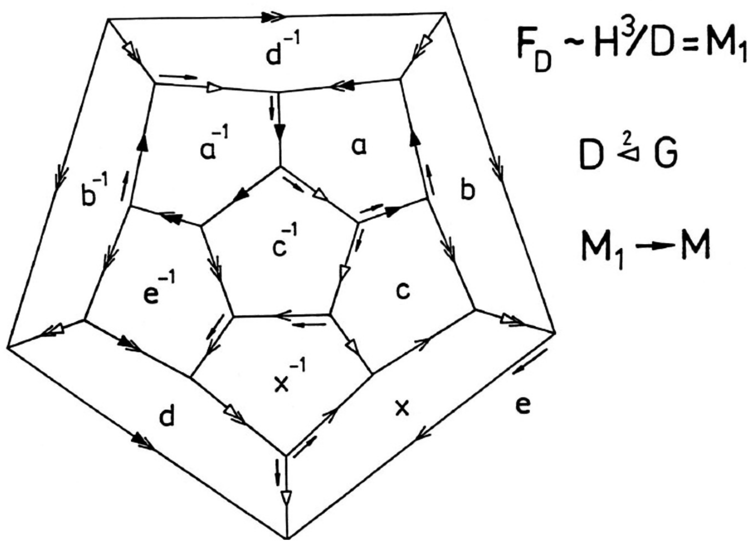


Figure 6. Twofold covering the football manifold
as a hyperbolic dodecahedron manifold (see also [10]).

6. Construction of the hyperbolic football manifold $M = \mathbf{H}^3/\mathbf{G}$

Now we shorten our discussion, because of the strict analogy of the previous example in Sect. 4. The half-turn with axis D_1D_2 with an orthogonal plane to A_0A_3 halves the simplex $(5,3,5)$. If we reflect this half simplex around vertex A_3, then the spherical icosahedral or dodecahedral group $[2,3,5]$ (by its 120 elements) just derives the football polyhedron $\{5,6,6\}$. Moreover, the half-turn extended reflection group $\mathbf{N} := \mathbf{2} \circ \mathbf{L}$ (extended *Lanner group* in the diagram of Fig. 5) fills up the space $\mathbf{H}^3$ with the congruent copies of our football.

Now we define a new fixed-point-free group (denoted again by) $\mathbf{G}$ by pairing the faces of our football $\tilde{F} = \{5,6,6\}$ as fundamental domain of $\mathbf{G}$. Again, three edges have to fall in each equivalence class, since 3 edge domains meet at every edge of the space tiling with joins pentagon-hexagon, hexagon-hexagon, hexagon-pentagon.

We start with three equivalent directed edges denoted by $\longrightarrow\!\!\!\!\!\blacktriangleright$. We choose the first one incident with the face assigned by a^{-1}. Then we introduce the screw motion $\mathbf{a}$ mapping the first edge $\longrightarrow\!\!\!\!\!\blacktriangleright$ into the second one, the a^{-1}-hexagon onto the a-hexagon and F_G onto its $\mathbf{a}$-image $F_{\mathbf{G}}^{\mathbf{a}}$ along the a-face of F_G. We define the screw motion $\mathbf{b}$, mapping the second edge $\longrightarrow\!\!\!\!\!\blacktriangleright$ onto the third one, the b^{-1} pentagon onto the b-pentagon and F_G onto its $\mathbf{b}$-image $F_{\mathbf{G}}^{\mathbf{b}}$ adjacent to $F_{\mathbf{G}}$ along its b-face. Now, we are required to associate the hexagon $b^{-1}\,a^{-1} = (ab)^{-1}$ with the hexagon ab by the product map $\mathbf{ab}$. Indeed, the first edge $\longrightarrow\!\!\!\!\!\blacktriangleright$ is surrounded in the $F_{\mathbf{G}}$-tiling as follows

> first: F_G;
>
> second: along its a^{-1}-hexagon comes $F_{\mathbf{G}}^{\mathbf{a}^{-1}}$ (the $\mathbf{a}^{-1}$-image of F_G);
>
> third: along the $\mathbf{a}^{-1}$-image of b^{-1}-pentagon comes $F_G^{b^{-1}a^{-1}}$;
>
> finally: along the $\mathbf{b}^{-1}\mathbf{a}^{-1}$-image of ab-pentagon comes $F_{\mathbf{G}}^{(\mathbf{ab})\mathbf{b}^{-1}}\mathbf{a}^{-1} = F_{\mathbf{G}}$ which is the starting polyhedron. At the same time we have explained our notations and the fixed-point-free action of $\mathbf{G}$ at inner points of the starting edge $\longrightarrow\!\!\!\!\!\blacktriangleright$ and its images.

Now, the process is straightforward. We take the common edge of the hexagons a^{-1} and a. Then the $\mathbf{a}^{-1}$-image and $\mathbf{a}$-image have been determined and the directed edge class 1 is defined. Moreover, the a^{-2}-pentagon and a^2-pentagon are assigned and the screw motion $\mathbf{a}^2$ can be given as a new identifying mapping. So, we have already guaranteed the free action of $\mathbf{G}$ at inner points of the 1-edges, etc. We pick out a new directed edge in common with faces already paired, determine its edge class, either define a new identifying mapping with its paired faces or get a relation. A relation is either trivial (e.g., at the edge classes 3 and 5), or non-trivial (e.g., at 23 and 26), or a consequence of previous relations (e.g., at $27-29$). The process in Fig. 5 is briefly written down as follows:

$$\longrightarrow: \mathbf{a},\mathbf{b},\mathbf{ab},\ 1:\mathbf{a}^2,\ 2:\mathbf{a}^3\ 3:\mathbf{a}\cdot\mathbf{a}^2\cdot\mathbf{a}^{-3}=1-,\ 4:\mathbf{a}^3\mathbf{b},\ 5:-,\ 6:\mathbf{ab}^2,$$
$$7:\mathbf{a}^3\mathbf{b}^{-1}, 8:\mathbf{a}^3\mathbf{b}^{-1}\mathbf{a}^{-1}=:\mathbf{c},\ 9:-,\ 10:-,\ 11:\mathbf{ba}^{-2}\mathbf{b}^2\ 12:\mathbf{a}^3\mathbf{b}^{-1}\mathbf{a}^2,$$
$$13:-,\ 14:\mathbf{a}^3\mathbf{b}^{-1}\mathbf{a}^2\mathbf{b}^{-1}=:\mathbf{d},\ 15:-,\ 16:-,\ 17:\mathbf{a}^3\mathbf{b}^{-1}\mathbf{a}^3\mathbf{b}=:\mathbf{e},\ 18:-,$$
$$19:\mathbf{a}^3\mathbf{b}^{-1}\mathbf{a}^2\mathbf{b}^{-2}\mathbf{a}^{-1}=:\mathbf{f},\ 20:-,\ 21:\mathbf{b}^{-2}\mathbf{a}^2\mathbf{b}^{-1}\mathbf{a}^3\mathbf{b},\ 22:-,$$
$$23:(\mathbf{a}^3\mathbf{b}^{-1}\mathbf{a}^2\mathbf{b}^{-2}\mathbf{a}^{-1})(\mathbf{b}^{-2}\mathbf{a}^{-1})(\mathbf{b}^{-2}\mathbf{a}^2\mathbf{b}^{-1}\mathbf{a}^3\mathbf{b})=1,$$
$$24:\mathbf{b}^{-2}\mathbf{ba}^{-3}\mathbf{ba}^{-2}\mathbf{b}^2=:\mathbf{g},$$
$$25:(\mathbf{b}^{-2}\mathbf{ba}^{-3}\mathbf{ba}^{-2}\mathbf{b}^2)(\mathbf{b}^{-2}\mathbf{a}^2\mathbf{b}^{-1}\mathbf{a}^3 b)(\mathbf{b}^{-2}\mathbf{a}^2\mathbf{b}^{-1})=1$$
$$26:(\mathbf{a}^3\mathbf{b}^{-1}\mathbf{a}^3 b)(\mathbf{ba}^{-2}\mathbf{ba}^{-3})(\mathbf{ba}^{-2}\mathbf{ba}^{-3}\mathbf{ba}^{-2}\mathbf{b}^2)=1,$$
$$27:(\mathbf{a}^3\mathbf{b}^{-1}\mathbf{a}^2)(\mathbf{ab}^2\mathbf{a}^{-2}\mathbf{ba}^{-3})(\mathbf{ba}^{-2}\mathbf{ba}^{-3}\mathbf{ba}^{-2}\mathbf{b}^2)\overset{26}{=}1-,$$
$$28:(\mathbf{a}^3\mathbf{b}^{-1}\mathbf{a}^3\mathbf{b})(\mathbf{ba}^{-2}\mathbf{ba}^{-3}\mathbf{ba}^{-2}\mathbf{b}^2)(\mathbf{b}^{-1}\mathbf{a}^{-3}\mathbf{ba}^{-2}\mathbf{b}^2)\overset{26}{=}1-,$$
$$29:(\mathbf{a}^3\mathbf{b}^{-1}\mathbf{a}^2\mathbf{b}^{-2}\mathbf{a}^{-1})(\mathbf{b}^{-2}\mathbf{a}^2\mathbf{b}^{-1}\mathbf{a}^3\mathbf{b})(\mathbf{ba}^{-2}\mathbf{ba}^{-3}\mathbf{ba}^{-2}\mathbf{b}^2)$$
$$=(\mathbf{b}^{-1}\mathbf{a}^{-3}\mathbf{ba}^{-2}\mathbf{b}^2\mathbf{a})(\mathbf{a}^2\mathbf{b}^{-1}\mathbf{a}^3\mathbf{b})(\mathbf{ba}^{-2}\mathbf{ba}^{-3}\mathbf{ba}^{-2}\mathbf{b}^2)\overset{26}{=}1, \tag{6.1}$$

In fact, the identifications satisfy the requirements for edge classes. It is already obvious, but we also realize that four vertices of F_G are contained in each class of $\mathbf{G}$ equivalence, four edge classes start or end with them. To summarize, we formulate

Theorem 1. *The Archimedean solid* $(5,6,6)$ *can be equipped with face identifications and locally hyperbolic metric so that it becomes an orientable compact hyperbolic space form* $M := \mathbf{H}^3/\mathbf{G} \sim \tilde{F}_\mathbf{G} := (5,6,6)$. *The fundamental group* $\mathbf{G}$ *is generated by the screw motions* $\mathbf{a}$ *and* $\mathbf{b}$ *(in Fig. 5) and the relations of* $\mathbf{G}$ *are given at edge classes 23 and 26. The first homology group of* M *is* Z_{14}, *by* $\mathbf{G}/[\mathbf{G},\mathbf{G}]$, *i.e., taking* $\mathbf{G}$ *to be commutative.*

Proof: The first part is proved by the constructions and the so-called Poincaré theorem as mentioned. The relations to edge class 23 and to 26 provide us just the presentation of $\mathbf{G}$. The first homology group is

$$H_1(M) = \mathbf{G}/[\mathbf{G},\mathbf{G}] = Z_{14} \tag{6.2}$$

the cyclic group mod 14. It can be easily calculated from the presentation by taking the commutator factor group of $\mathbf{G}$. From the relations 23 of (6.1) we get $a^8b^{-7} = 1$, from 26 of (6.1) $a^6b^7 = 1$, by so-called abelianization. That means $a^2 = 1$, $b^7 = 1$. □

Now, let us consider the super group $\mathbf{N}$ of $\mathbf{G}$ which can be written as a decomposition

$$\mathbf{N} = \mathbf{C}_3{}^*\mathbf{G} \tag{6.3}$$

by (6.4). That means, each element $\mathbf{n}$ of $\mathbf{N}$ can be uniquely written in the form $\mathbf{n} = \mathbf{c}_3 \cdot \mathbf{g}$ with $\mathbf{c}_3 \in \mathbf{C}_3$, $\mathbf{g} \in \mathbf{G}$. This makes possible to express the generators of $\mathbf{G}$ by the generators of $\mathbf{N}$ (see Fig. 5)

$$\mathbf{a}^{-1} = \mathbf{r}\mathbf{m}_0\mathbf{m}_1\mathbf{m}_2\mathbf{m}_1, \quad \mathbf{b} = \mathbf{m}_3\mathbf{m}_0\mathbf{m}_2\mathbf{m}_1\mathbf{m}_0\mathbf{m}_1. \tag{6.4}$$

Imagine the vertex A_2 of $F_{\mathbf{G}}$ in the middle of the face a and D_1 at the end of edge $\longrightarrow\!\!\!\!\!\blacktriangleright\!\!\!\longrightarrow$.

Of course, the expressions in (6.4) are not unique. We could determine the distance and angle of our generating screw motions **a**, **b**.

Remarks: We have some freedom in halving $F_{\mathbf{L}}$ to $F_{\mathbf{N}}$. So, we could obtain other (also non-convex) fundamental domains for the group **G**.

1. In [8] we computed important data for this football manifold. It has seemingly extremal inscribed ball and extremal circumscribed ball. Its maximal ball packing density is 0.77147... and minimal ball covering density is 1.36893..., respectively (the corresponding ball volume is related to the polyhedron volume Vol $(\tilde{F})$. Both are conjectured (by the authors) best possible among all ball packings and coverings, respectively, of hyperbolic space $\mathbf{H}^3$. These extrema may involve applications in the “experimental” crystallography as well. Fullerens as “idealized constructions” initiated by Buckminster Fuller, an architect and discoverer, without any knowledge on hyperbolic geometry, have got surprising realizations [5].
2. We observe that the whole Lanner simplex $(5,3,5)$, in Fig. 5 under reflections around A_3 with 120 group elements, generate a regular hyperbolic dodecahedron with dihedral angles $2\pi/5$. That fills $\mathbf{H}^3$ with its congruent copies, 5 dodecahedra meet at every edge. Imagine that we can construct a dodecahedron space form $\tilde{F}_{\mathbf{D}} = \mathbf{H}^3/\mathbf{D} = \mathbf{M}_1$ covering two-times our former football manifold, again with fixed-point-free orientable face pairing group **D**. The result is seen in Fig. 6 with the face paired dodecahedron $\tilde{F}_{\mathbf{D}}$ as fundamental domain. Observe that five edges fall in every **D**-equivalence class, indeed by the dihedral angle $2\pi/5$ above.

The second author found in [10] all the 12 analogous hyperbolic dodecahedron face pairings (up to dodecahedron symmetries) with his systematic computer program.

7. Hyperbolic Cobweb Manifolds, as infinite series of possible material structures?!

Our next section follows the former ideas in Figures 7 − 8. This is a new topic by our papers [7] and [8]. We indicate how to construct an infinite series of orientable compact hyperbolic manifolds (space forms).

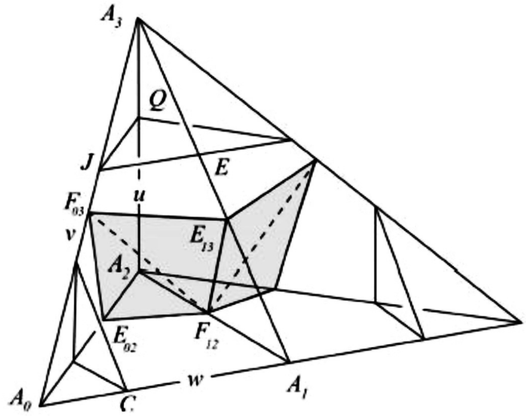


Figure 7. Fundamental domains for the half orthoscheme $W_{uvw} = W_{666}$ and for the gluing procedure at point Q for getting cobweb manifold $Cw(6;6;6)$.

The characteristic simplex (orthoscheme) of the regular dodecahedron $(5,3,5)$ and its Coxeter-Schläfli matrix in formula (5.1) can be extended, where the vertices A_3 and A_0 are outer ideal points of the hyperbolic space $\mathbf{H}^3$. This can be achieved, if we take new natural parameters (u,v,w), above, $3 \leq u,v,w \in \mathbf{N}$ (natural numbers), so that

$$\pi/u + \pi/v < \pi/2 \quad \text{and} \quad \pi/v + \pi/w < \pi/2. \tag{7.1}$$

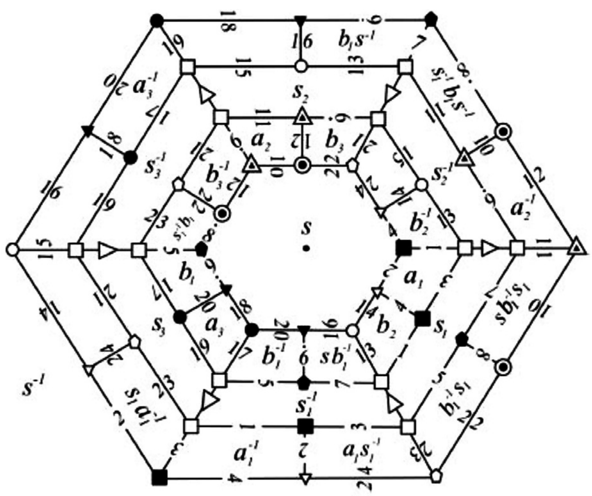


Figure 8. The hyperbolic cobweb manifold $Cw(6;6;6)$ with complete (symbolic) face pairing.

Imagine (Fig. 7) these dihedral face angles as $\pi/u = \beta^{01}$, $\pi/v = \beta^{12}$, $\pi/2 = \beta^{02}$ at vertex A_3, then the subdeterminant of (5.1) (by rows and columns 0, 1, 2) will be negative; and similarly $\pi/v = \beta^{12}$, $\pi/w = \beta^{23}$, $\pi/2 = \beta^{13}$ at vertex A_0. The complete determinant of (5.1) remains also negative:

$$B = \det(b^{ij}) = \sin^2\frac{\pi}{u}\sin^2\frac{\pi}{\omega} - \cos^2\frac{\pi}{\upsilon} < 0, \text{ i.e. } \sin\frac{\pi}{u}\sin\frac{\pi}{\omega} - \cos\frac{\pi}{\upsilon} < 0. \quad (7.2)$$

Then we cut the simplex by their polar planes a_3 and a_0, respectively, to get a truncated orthoscheme as generalization. The complete truncated orthoscheme group will be generated by these 5 plane reflections $\boldsymbol{b}^0$,$\boldsymbol{b}^1$,$\boldsymbol{b}^2$,$\boldsymbol{b}^3$, a_3, a_0, with a compact fundamental domain W_{uvw} filling the hyperbolic space $\mathbf{H}^3$. In Fig. 7 we further specialize the situation by taking

$$\begin{gathered} u = v = w = 6, \text{ or more generally} \\ u = v = w = 2 \cdot z, \text{ where } 3 \leq z \text{ is odd natural number.} \end{gathered} \quad (7.3)$$

Then a further half-turn with axis $F_{03}F_{12}$ and a halving plane $F_{03}E_{02}F_{12}E_{13}$ (with great freedom) can be introduced (analogously as at the football manifold construction).

Now, reflect this half W_{666} domain around point $Q = A_2A_3 \cup a_3$. Then we obtain a cobweb solid $Cw(6,6,6)$, as in Fig. 8, with two basis faces, 6 (or $2 \cdot z$) hexagons in the middle, furthermore 24 (or $8 \cdot z$) deltoids at the two basis faces.

Again, we shall introduce face pairing isometries, so that we get a fixed point free group $\mathbf{Cw}$ and a compact hyperbolic manifolds $Cw = \mathbf{H}^3/\mathbf{Cw}$. The result is completely seen in Fig. 8.

We emphasize only the most important arguments (by Figures 7–8).

We started by the $3(=z)$ half screws

$$\mathbf{s}_1 : s_1^{-1} \to s_1, \; \mathbf{s}_2 : s_2^{-1} \to s_2, \; \mathbf{s}_3 : s_3^{-1} \to s_3, \quad (7.4)$$

in the middle of Cw around in Fig. 8, so that the 6 edges with arrow —▷—, with dihedral angles $\pi/3$ fall into one equivalence class.

Then we concentrate on the odd numbered edge classes $1-23$ (corresponding to $F_{03}E_{02} = F_{03}E_{13}$ in Fig. 7) each containing 3 edges; moreover on the even numbered edge classes $2-24$ (corresponding to $F_{12}E_{02} = F_{12}E_{13}$ in Fig. 7) each containing 3 edges, too. This is because 3 copies of Cw have to meet at these image edges in the space filling, respectively. To this the deltoid faces should be chosen adequately. First to

$$\begin{gathered} 1 : \mathbf{a}_1 : a_1^{-1} \to a_1, \mathbf{b}_2 : b_2^{-1} \to b_2 \text{ on faces } s_1^{-1} \text{ and } s_1, \text{ then to} \\ 2 : \mathbf{s} : s^{-1} \to s, \text{ as basis face pairing} \end{gathered} \quad (7.5)$$

will be introduced. This becomes so luckily that the procedure smoothly goes to the end without contradiction.

Observe also the equivalence classes of vertices. At each vertex of $Cw(6,6,6)$ we shall have a ball-like neighbourhood, if we compare the situation with Fig. 7.

Theorem 2. *The hyperbolic Cobweb Manifold Cw =* $\mathbf{H}^3/\mathbf{Cw}$ *has been constructed.*

The cyclical symmetry (and our "fortune") will guarantee that this procedure can be extended for $2 \cdot z$ "cobweb prisms", $3 \leq z$ is odd number. This will be a next publication on the base of [7] and [8], indicated here as well.

It is an open problem, what happens for $4p$-side "cobweb prisms" $(2 \leq p \in \mathbf{N})$?

And, is there any application in "experimental crystallography"?

References

[1] CAVICCHIOLI, A. AND TELLONI, A. I. (2009), *On football manifolds of E. Molnár*, Acta Math. Hungar, 124/4 (2009), 321–332, DOI: 10.1007/s10-474-009-8196-9.

[2] COXETER, H. S. M. (1948), *Regular Polytopes*, Methuen & Co. Ltd., London.

[3] HAHN, T. (ED.) (2005), *International Tables for Crystallography*, Vol. A, Space-group symmetry, Fifth Edition, Springer.

[4] MOLNÁR, E. (2005), *Combinatorial construction of tilings by barycentric simplex orbits (D-symbols) and their realizations in Euclidean and other homogeneous spaces*, Acta Cryst., A61, 542–552, DOI:10.1107/S0108767305026760.

[5] MOLNÁR, E. (2012), *On non-Euclidean crystallography, some football manifolds*, Struct Chem, 23:1057–1069, DOI:10.1007/s11224-012-0041-z.

[6] MOLNÁR, E. AND PROK, I. (2013), *Three- and four-dimensional regular 4-solids move in the computer 2-screen*, In M. Pavleković, Z. Kolar-Begović, R. Kolar-Šuper, (Eds.), Mathematics teaching for the future, Josip Juraj Strossmayer University of Osijek Faculty of Teacher Education and Department of Mathematics (pp. 173–185), Zagreb 2013.

[7] MOLNÁR, E. AND SZIRMAI, J. (2017), *On hyperbolic cobweb manifolds*, arXiv 1701.06757v1. (to appear in Studies of the University of Žilina Mathematical Series)

[8] MOLNÁR, E. AND SZIRMAI, J. (2017), *Top dense hyperbolic ball packings and coverings for complete Coxeter orthoscheme groups*. arXiv 1612.04541v1. (submitted to Publications de l'Institut Mathematique (published by Serbian Academy of Sciences and Arts)).

[9] PRÉKOPA, A. AND MOLNÁR, E. (EDS) (2006), *Non-Euclidean Geometries*, János Bolyai Memorial Volume. Mathematics and Its Applications, Vol. 581, Springer, $14 + 506$.

[10] PROK, I. (1998), *Classification of dodecahedral space forms*. Beitr. Alg. Geom., 39, No. 2, 497–515.

[11] PROK, I. (2014), `http://www.math.bme.hu/~prok/` [Accessed 05/08/2014].

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Az {5, 6, 6} futball és geometriája: a játékszertől a fullerénekig és tovább

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Összefoglaló. Ez a "tudomány-népszerűsítő" előadás a szabályos sokszögekből indul ki. Aztán Prok I. honlapját bemutatva, a Platón-féle szabályos és az Archimédesz-féle félig szabályos testekkel, ezek szimmetriáival folytatja. A gömbi geometria síktükrözései és az úgy-nevezett alaptartományok elemzése lesz a fő eszközünk. Például a kocka középpontjában $\pi/4 = 45°$, $\pi/3 = 60°$ és $\pi/2 = 90°$ lapszögű szimmetriasíkok találkoznak. Ezekre a síkokra tükrözve, ennek a karakterisztikus gömbháromszögnek 48 példánya kövezi ki a gömböt, amit úgy is mondunk: a kocka szimmetriacsoportja 48 elemű és 3 síktükrözés generálja. A többi szabályos és félig szabályos testet, sőt síkbeli szabályos mintákat is jellemezhetünk így. A karakterisztikus háromszög szögösszegének a $\pi = 180°$-tól való eltérése jellemzi a szabályos testeket, ha az eltérés pozitív; a kockánál pl. $\pi/3 + \pi/4 + \pi/2 - \pi = \pi/12$. Az euklideszi mintáknál ez a szögkülönbség 0. De elképzelhető negatív eltérés is, ha a háromszög szögösszege kisebb 180°-nál. Ez jellemzi a Bolyai-Lobacsevszkij-féle hiperbolikus sík mintáit, kövezéseit. A tér kövezései esetében is fontos szerepet játszanak a tükrözések, melyeket a lineáris algebra eszközeivel terjeszthetünk ki L. Schläfli és H.S.M. Coxeter nyomán.

Kiderül, hogy az $\{5, 6, 6\}$ szimbólumú futball-labda, melynél minden csúcsban egy szabályos (gömbi) ötszög és két szabályos (gömbi) hatszög találkozik, poliéder – azaz síklapú test – formájában, egybevágó példányaival nem tudja kitölteni euklideszi terünket, de a Bolyai Lobacsevszkij-féle hiperbolikus teret ki tudja tölteni (ahogy ezt Molnár E. 1988-ban észrevette és publikálta. Sőt a kitöltés fixpontmentes egybevágóságokkal történhet: A futball-labdák olyanok, mintha egy véges térben lennénk, ahol minden pontnak hiperbolikus labda környezete van. Úgy tűnik, hogy az anyag tudományokban előtérbe került *fullerének*, azaz C_{60} molekulák ezt a nagyon „sűrű hiperbolikus anyagot" követik (melynek további szélső-érték tulajdonságai Szirmai J. és Molnár E. kutatásaiban is előjönnek).

Tehát Bolyai János és N.I. Lobacsevszkij hiperbolikus geometriájának kristálytani alkalmazásai is lehetnek az eddigi csillagméretű vonatkozások mellett.

Kulcsszavak: A matematika tanár, mint a tudomány népszerűsítője; platóni és archimédeszi test; követés; euklideszi és nem euklideszi sokaság; kristálystruktúra, fullerén

Mathematics Subject Classification 2010: *57M07, 57M60, 52C17.*

Holes in alien quilts

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Abstract. Mathematical developments in last two decades brought cube complexes in a spotlight of the research in theoretical mathematics. The concept of CAT(0) cube complexes emerged as a particularly elegant structure suitable for modeling a variety of problems. We will start with the definition of cube complex and then introduce the large link and extra large link conditions, two combinatorial properties of CAT(0) cube complexes. Using examples of two-dimensional CAT(0) cube complexes, we will explain how these properties affect the geometry of the complex, in particular, the divergence and the isoperimetric inequality.

Keywords: CAT(0) spaces, growth, divergence, isoperimetric inequlity

Mathematics Subject Classification 2010: *20F65, 20F67, 57M20.*

1. Introduction

In the construction of quilts by humans, there are two absolutes. The first is that a quilt is made up of squares connected edge to edge, called panels. The second is that no more than four panels could meet at a point. Our quilts are flat!

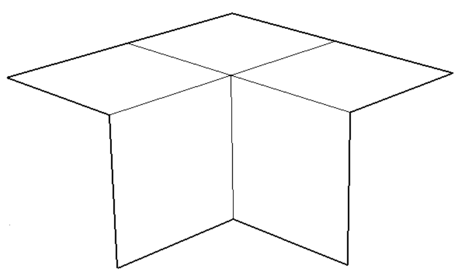

Figure 1. A three-dimensional representation of five squares intersecting at a point.

The fact that four panels is the maximum number of panels which meet at a point is merely a limit created by our minds. Figure 1 illustrates a different type of a quilt, one in which five panels meet at a point, a quilt which is not flat. This makes it seem like not quite what traditional quilt makers would call a quilt. However, it still satisfies the rules of quilts: the panels are squares connected edge to edge. It might not be a *human* quilt, but it could be an alien quilt. Or, maybe a group of creative children, whose imagination has not being constraint by rules that adults tend to obey, started playing with squares and creating such unusual shapes, unaware that they are actually constructing a space with *hyperbolic*, geometry.

This paper is envisioned as an exploration and comparison of geometric properties of the Euclidean and hyperbolic planes. Human quilts are Euclidean and the alien quilts are hyperbolic, and we hope that our elementary approach will enable teachers to, in a playful manner, illustrate to their students the riches of different geometries.

The teachers and students are familiar with the beautiful Euclidean geometry based on postulates in Euclid's *Elements*, as well as the centuries long controversy of the *Fifth Postulate*, or *the parallel postulate*, that many mathematicians tried to resolve. Euclid stated his Fifth Postulate, unlike other postulates, in a rather complex manner, so, instead of the original, Euclid's formulation, an equivalent postulate, called Playfair's axiom is often used. Playfair's axiom states that "In a plane, given a line and a point not on it, at most one line parallel to the given line can be drawn through the point" [14].

The complexity of Euclid's original formulation lead mathematicians to believe that The Fifth Postulate is not a postulate, but a theorem that can be proved based on the remaining axioms. Many attempts to prove it failed – rightly so, since in early 19th century the Hungarian mathematician János Bolyai and the Russian mathematician Nikolai Ivanovich Lobachevsky independently discovered *hyperbolic geometry*, a geometry where for a given line l and a point P, there are multiple lines parallel to l, through P, thus proving that The Fifth is indeed an independent postulate.

While many texts and papers with beautiful treatments of the axiomatic approach to hyperbolic geometry are available, describing a model of a hyperbolic plane that can be used to picture and understand its geometric properties with elementary tools is quite a challenge. We hope that our alien quilts will provide a model that is easy to present and study without very advanced mathematical knowledge. For a reader interested in further exploration of non-euclidean geometries in intuitive manner we recommend an innovative textbook by Henderson and Taimina, [10] where one can learn how to crochet a piece of a hyperbolic plane.

In this paper we will use the human and alien quilts to study and compare several geometric properties of Euclidean and hyperbolic plane. In particular, we study the *growth* and *divergence* in infinite quilts, as well as a concept called isoperimetric inequality.

Our goal is to explain these concepts at a level accessible to high school teachers and students, but still keep the rigor of the mathematical proof. All the results presented are well known to experts in the field, but are proved in a more general context and at very advanced level.

We chose to use this setting of quilts built out of squares for two reasons. Firstly, their rigidity and structure makes them very suitable for developing elementary proofs. The second, but not any less important reason, our quilts are examples of *cube complexes*, spaces that one can think of as n-dimensional quilts, and cube complexes have recently claimed the centerstage in the proof of the last big open conjecture in the mathematics of 3 dimensional shapes, the virtual Haken conjecture, [11]. Thus, you can proudly tell your students who are pondering the alien quilts that they are playing with the same toys as the mathematicians that shape current mathematical thought.

One of the early topics discussed in geometry is that the area of a circle of radius r is πr^2. Then, later, you would learn that the volume of the ball is $\frac{4}{3}\pi r^3$. We observe that the area and the volume grow as the radius grows. If we are interested in how they grow "on a large scale" we can ignore the constants, and say that the area of the circle grows quadratically, that is as r^2, and the volume of the sphere grows cubically. We then define the *growth of space* to be a function that describes how the size of "the balls" grows as their radius grows, where a ball is a general term for the set of all points that are less than r away from a fixed point O [9, 7]. Thus a filled circle, or a disk, is a 2-dimensional ball, and in three dimensional space, this would be a literal ball, a sphere that has been filled in. One- dimensional ball is a subset of a number line consisting of all real numbers with absolute value less than or equal to 1. We will discuss the concept of growth in more detail in section §3.2 and describe spaces with various types of growth.

Divergence is, similarly, a function which assigns a number, obtained by measuring a geometric property, to the radius of a ball, [5, 6, 12]. To define divergence, we first consider the detour distance, where we are required to "detour" around balls. We consider a ball $B(O,r)$ and two points P,Q outside of $B(O,r)$, as pictured in Figure 2. We define a *detour path* connecting P and Q to be a path between P and Q that does not intersect $B(O,r)$. The *detour distance* between P,Q is the length of the shortest detour path from P to Q, and we that denote it by $\delta_r(P,Q)$.

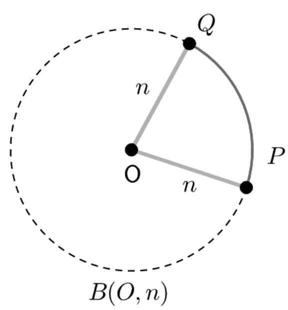

Figure 2. The ball $B(O,n)$, with points P and Q, in the Euclidean plane. The highlighted arc is the detour distance between P and Q.

We then define *the divergence of a space* to be a function f, such that $f(r)$ is the maximum of detour distances $\delta_r(P,Q)$, for all points P and Q such that $d(P,O) = d(Q,O) = r$, that is for all for all points P and Q on the sphere of radius r. As an example, consider a ball (disk) in the Euclidean plane, and let P and Q be two points on the boundary circle. The shortest detour path between P and Q is an arc of the circle that connects them, and the length of such an arc will be the longest when P and Q are two diametrically opposite points on the circle. That means that for a disk of radius r, the largest detour distance is πr. Since this function grows linearly, we say that the divergence is linear. The divergence is a more subtle property of space, and we give the full motivation and detailed discussion of this concept in Section 4.

The title of the paper, "Holes in Alien Quilts", is motivated by the definitions of growth and divergence. Growth measures how big a hole is of radius r, and divergence is how far you have to route around such hole. In the following section we the definitions of quilts and geometric concepts that we discuss in the paper, that is the *growth* of a space, and the *divergence*.

2. Preliminaries

In this section we present definitions and properties of metric spaces and functions that describe their geometric properties .

2.1. Metric spaces

A metric space (X,d_X) is set X with a distance function d_X defined for each pair of points. The metric spaces we study in this paper are *length metric spaces*, that is, spaces where we can compute lengths of paths. The distance between two points is then defined to be the length of a shortest path that connects them. Such paths are called geodesics. In general, geodesics might not exist, or there might be multiple geodesics connecting two same points, as for example, on the sphere. The geodesics that connect the North and the South pole are arcs of great circles, and there are infinitely many of them. The spaces we study in this paper are CAT(0) spaces, a well studied class of metric spaces best described as spaces where triangles are thinner than in the Euclidean plane. In CAT(0) spaces geodesics exits and are unique.

When studying metric spaces one is naturally interested in functions that preserve distance, called isometries. More precisely, an isometry between metric spaces (X,d_X) and (Y,d_Y) is a surjective function $f : X \to Y$ that satisfies $d_Y(f(x_1),f(x_2)) = d_X(x_1,x_2)$. Since distance preserving function is injective, every isometry is a bijection. Examples of isometries are Euclidean translations, rotations and reflections.

In metric space X we define an open ball $B(O,r) \subset X$ of radius r centered at O to be the set of all points in X that are strictly less than r away from O, that is,

$B(O,r) = \{x \in X, d(x,O) < r\}$, as discussed in the introduction. The sphere of radius r centered at O is the set of all points in X that are at the distance r from O, and the closed ball $\bar{B}(O,r)$ is the union of the open ball $B(O,r)$ and $S(O,r)$.

As discussed in the introduction, balls in two dimensional metric spaces are two-dimensional, and the measure of their size is the area. In three dimensional spaces the measure of the size of a ball its volume.

2.2. CAT(0) spaces

CAT(0) spaces are spaces where triangles are "thinner" than the triangles in the Euclidean plane, as we can observe in Figure 3, where the triangle on the left is a thinner than the Euclidean triangle on the right. To formalize this idea, we define a comparison triangle for a given triangle.

Let (X,d) be a metric space and let (E,d_E) be the Euclidean plane,where d and d_E are the respective metrics. A geodesic triangle $\Delta = \Delta(P,Q,R)$ in X consists of three points P, Q, $R \in X$, its vertices, and a choice of three geodesic segments γ_{PQ}, γ_{QR} and γ_{PR} joining the vertices, its sides. If the point T lies in the union of γ_{PQ}, γ_{QR} and γ_{PR}, then we write $T \in \Delta$.

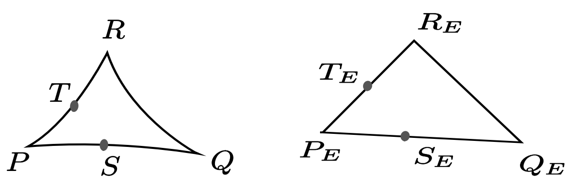

Figure 3. A comparison triangle.

A geodesic triangle $\Delta_E = \Delta(P_E,Q_E,R_E)$ in E is called a comparison triangle for the triangle $\Delta(P,Q,R)$ if $d(P,Q) = d_E(P_E,Q_E)$, $d(P,R) = d_E(P_E,R_E)$ and $d(Q,R) = d_E(Q_E,R_E)$. A point T_E on $\gamma_{P_EQ_E}$ is called a comparison point for T in γ_{PQ} if $d(P,T) = d_E(P_E,T_E)$. Comparison points for points on γ_{PR} and γ_{QR} are defined in the same way.

Definition 1. A metric space X is a CAT(0) space if it is a geodesic metric space all of whose triangles satisfy the CAT(0) inequality:

Let Δ be a geodesic triangle in X and let Δ_E be a comparison triangle in the Euclidean plane E. Then, Δ is said to satisfy the CAT(0) inequality if for all $S,T \in \Delta$ and all comparison points $S_E,T_E \in \Delta_E$, $d(S,T) \leq d_E(S_E,T_E)$.

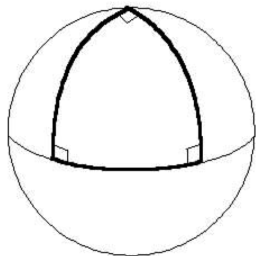

Figure 4. A triangle on the sphere. Geodesics on the sphere are arcs of great circles.

In the hyperbolic plane triangles are thin, while the triangles on the sphere are not, as it can be observed in Figure 4, thus a sphere is not a CAT(0) space.

The spaces we study in this paper are CAT(0) spaces, and thus have a number of nice properties. One such property is that geodesics between two point always exits and are unique. Another property is that hyperplanes (to be defined and discussed in the next section) separate the complex. We refer the reader to [2] for a detailed treatment CAT(0) spaces.

2.3. Equivalence of functions

When studying geometric properties of metric spaces and groups acting on them, like growth and divergence described in the introduction, we are not always interested in finding the precise function that characterizes a property. Instead, we describe a general class of functions that capture the behavior, requiring that those functions can be related by linear stretching, or shrinking, or adding constants.

More precisely, we say that $f \preceq g$ if there are constants A, B, C, D, $E > 0$ such that

$$f(x) \leq Ag(Bx + C) + Dx + E \text{ for every } x > 0.$$

We define two functions $f, g : \mathbf{R}^+ \cup 0 \longrightarrow \mathbf{R}^+ \cup \{0, \infty\}$, to be *equivalent*, $f \sim g$, if $f \preceq g$ and $g \preceq f$. This gives equivalence relation capturing the qualitative agreement of growth rates.

The main different equivalence classes are:

1. Linear functions, $f(x) = ax + b$ for some constants a, b, $a > 0$.
2. Polynomial with degree k: the equivalence class of the function if $f(x) = x^k$, where $k \neq 0$.
3. Exponential functions: the class of all functions equivalent to $f(x) = a^x$ for $a > 0$.

Since we are interested only in the equivalence class of a function $f : [0, \infty) \rightarrow [0, \infty)$ that characterize a geometric property, we will often consider only the values of the function on non-negative integers, which in our examples will fully determine the equivalence class.

3. Alien quilts are square complexes

The quilts studied in this paper are built out of Euclidean unit squares

$$E = [0,1] \times [0,1],$$

and are 2-dimensional examples of *cube complexes*. An n-dimensional cube is the set of points in $\mathbf{R}^n$ whose coordinates are between 0 and 1, that is

$$\mathrm{Cube}(n) = \{(x_1, x_2, \dots x_n) \in \mathbf{R}^n | 0 \leq x_1, x_2, \dots x_n \leq 1\}.$$

In 3-dimensional case, this is the standard unit cube. In dimension 1, it is the segment $[0,1]$. The two dimensional cube is the unit square $E = [0,1] \times [0,1]$ and thus we call our complexes square complexes.

A *face* of a cube $\mathrm{Cube}(n)$ consists of all points in $\mathrm{Cube}(n)$ where one coordinate is 0, or, where a coordinate is 1. An edge of E is its face. As expected, the square has four faces, two that are the sets of points in E where one coordinate is 0, and the other two corresponding to the sets of points where one coordinate is 1. The faces of the one dimensional cube, the segment $[0,1]$, are its endpoints.

When making a quilt, we stitch squares along their faces (edges), requiring that they line up appropriately. We formalize this property with the requirement that the quilt is *neatly stitched*. Mathematically speaking, we start with a disjoint union U of squares (our pile of not-stitched square patches), possibly infinitely many of them, and define an equivalence relation on U using the isometry $f_i : E \to S_i$ from the standard unit square E to a square $S_i \subseteq U$, which maps edges to edges isometrically. To stitch two squares S_i and S_j *neatly* we choose edges e_1, e_2 in E, (we allow that $e_1 = e_2$) and define that $f_i(x) \sim f_j(y)$ for all $x \in e_1$ and $y \in e_2$.

We call a space obtained by this construction a *square complex*. Thus square complexes correspond nicely to the notion of a quilt: the squares in our mathematical quilts are next to each other, and lined up nicely, see Figure 5.

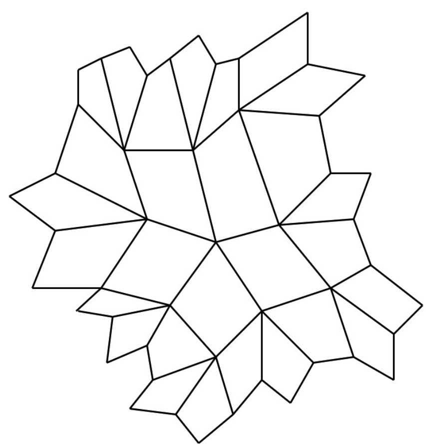

Figure 5. An alien quilt.

We will also study some properties of one-dimensional cube complexes, that is *graphs*.

We now introduce properties of square complexes that make them particularly nice objects to model different geometries.

Let S be a square. Then a midplane m of S the line segment connecting midpoints of two opposite edges. We refer to the two midpoints of the edges as the end points of the midplane. The intersection of the midplanes is called the *center* of the square. It is important to note that a midplane divides a square into two pieces, two connected components, in the sense that any path from one piece of the square to the other must cross this midplane. Since, when constructing a square complex we identify midpoints, of the corresponding edges, we can extend a midplane by stitching an endpoint to the endpoints of all other midplanes that contain that endpoint. This gives raise to the concept of *hyperplanes*.

Definition 2. Let X be a square complex, a hyperplane in X is 1-dimensional complex obtained by identifying the endpoints of the midplanes that intersect.

We note that, in general, a hyperplane need not be a line. It is possible for a hyperplane to branch, but in this paper we will only study square complexes whose hyperplanes are lines. The *star* of a hyperplane h is the union of all squares that h intersects.

The next concept we define is the link of a vertex. We first need to introduce circuits and describe a way in which we measure their size. A circuit is pretty much a circle built of line segments, but here is a more precise definition.

Definition 3. A circuit is a graph in which every vertex is contained in exactly two edges. The number of edges is referred to as the size of the circuit.

Definition 4. Let v be a vertex of a square complex X. The link of v, denoted by $\mathrm{lk}(v)$, is a graph whose vertices are midpoints of all edges that contain v. Two vertices in $\mathrm{lk}(v)$ are connected by an edge if they are both contained in a single square.

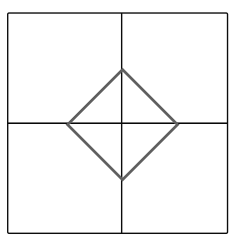

Figure 6. The link of size 4.

We observe that every circuit in the $\mathrm{lk}(v)$ encompasses a polygon centered at v whose edges are the edges of the circuit. If we multiply the size of the circuit by $\frac{\pi}{2}$, we get the total central angle of this polygon, since each square contributes exactly $\frac{\pi}{2}$ radians.

We say that a square complex has *large links*, and if each circuit in every link has size greater or equal to four, or, in the angular measure, greater or equal to 2π. A square complex has *extra large links* if each circuit in every link has size strictly greater than four (2π).

It can be easily show that square complexes with large links satisfy the requirement that links are flag complexes, a standard criterium that implies that a cube complex is a $\mathrm{CAT}(0)$ cube complex, see [2, 13]. Niblo and Reeves proved in [13] that then each hyperplane separates the complex into exactly two connected components. Thus, the hyperplanes in our square complexes separate them into exactly two components.

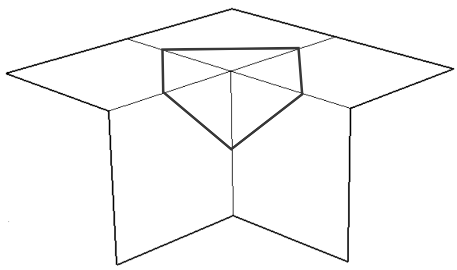

Figure 7. The link of size 5.

Since we are interested in geometric properties of spaces that resemble quilts, we will only consider square complexes with properties consistent with our intuitive notion of a quilt.

Definition 5. Let X be a square complex. We say that X has a *regular pattern*, or is *regular*, if

1. hyperplanes are (bi-infinite) lines, that is there is no branching;
2. all links are of the same size.

Our goal is to understand growth, divergence and isoperimetric inequality in regular square complexes with large links.

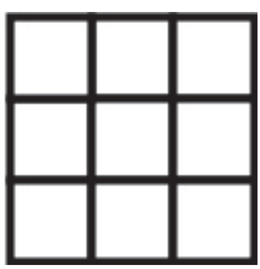

Figure 8. The link size is 4.

First we show that if X is a regular square complex with the link size four, then X is our familiar Euclidean plane $\mathbf{R} \times \mathbf{R}$. We start with one square S_0 and two hyperplanes h_1, h_2 that intersect it. Since all hyperplanes are lines, the star of of every hyperplane is isomorphic to $\mathbf{R} \times [0,1]$, and S_0 is the intersection of stars of h_1 and h_2. We can then inductively fill the four "quadrants" in $X \setminus [\mathrm{star}(h_1) \cup \mathrm{star}(h_2)]$ by adding the one missing square in the link of a vertex in each step.

4. Growth, divergence, and the isoperimetric inequality

4.1. Growth in binary trees

We first consider the growth of an infinite rooted binary tree, see Figure 9. A binary rooted tree is a graph with no circuits which we define inductively. A binary rooted tree T_1 of depth 1 consist of three vertices and two edges, where one vertex, v_0 is chosen as the root of the tree and connected to the each of the remaining vertices by an edge. The root vertex is also referred to as the parent vertex of the other two, which are called the children of v_0. The root vertex is said to be at the 0-th level of the tree, and its children are at the level 1. Binary rooted tree of depth $n+1$ for $n \geq 1$ is built from the binary rooted tree of depth n by adding two vertices v', v'' and two edges e', e'' for each vertex v in the n-level, where both edges have the initial endpoint at v and the terminal endpoints are v' and v'' respectively. Their terminal endpoints are the children of their parent vertex v, and make the $n+1$ level of the rooted tree T_{n+1}.

This inductive construction can be carried on indefinitely, and we denote the infinite rooted binary tree T. Vertices of graphs are also often called nodes.

Figure 9. The first 3 levels of the infinite rooted binary tree.

The growth of a tree could be measured by the total length of the edges contained in the ball of radius r, but it is more common to count vertices. If r is an integer, then number of edges is one less than the number of vertices (the root vertex is the extra vertex), so, for simplicity, (and to be consistent with the geometric group theory definition of the growth of a group), we will count vertices in the balls of integer radii.

We claim that the size of balls of radius n grows exponentially as n increases, that is, the growth function of the infinite binary tree is exponential. The ball of radius n centered at the root vertex consists of the vertices of T that are in the i-th level of the tree for $i \leq n$, and we denote the number of such vertices by $g(n)$.

We first compute $f(i)$ be the number of vertices of T that are in the i-th level: since $f(i+1) = 2f(i)$ and $f(1) = 2$, we, inductively, obtain that $f(i) = 2^i$. The fact that the growth is at least exponential in n follows directly since $g(n) \geq f(n) = 2^n$.

Since the number of vertices in the levels $i = 0, \ldots, n$ form a geometric series, the exact growth can be easily computed: $g(n) = 2^{n+1} - 1$.

We generalize the definition of a binary rooted tree by defining the *regular rooted k-tree* to be a rooted tree where every vertex has exactly k children. The above discussion, carried out for binary trees, generalizes in a straightforward manner to a proof that for a regular rooted k-tree the function $f(n)$ is the exponential k^n and also that $g(n) = \dfrac{1}{k-1}(k^{n+1} - 1)$.

We also prove a lemma that will be needed in the following section. Before stating the lemma, we observe that if T' is a subtree of a rooted binary tree T then all vertices have valence 1, 2 or 3. We first define a leaf of a finite tree to be a vertex of valence one.

Lemma 6 (Techinical Lemma 1). *Let T' be finite subtree of the infinite binary rooted tree T that contains the root vertex, the root vertex has valence 2, and all remaining vertices have valence 1 or 3. Let $l(T)$ be the number of leaves and $g(T)$ the total number of vertices in T. Then $g(T) \leq 2l(T) - 1$.*

Proof. We prove our statement using the strong induction on maximal depth of a vertex in the finite subtree. If T' contains vertices of depth at most 1, then the it has the root vertex and two children, thus it has a total of three vertices, two of which are leaves, and our inequality holds. Assume that the inequality holds for trees with vertices of maximal depth $i < n$, and let T' be a tree that has a vertex of depth $n > 1$. Since we require that the root vertex has valence two, there are two vertices, v_1 and v_2 at depth 1 in the tree. If one of them, say v_1, has valence one, then we obtain the total number of vertices in T' by adding two (for the root vertex and v_1) to the number of vertices in T_2, the subtree of T' rooted in v_2 and the inequality holds. If neither v_1 nor v_2 are leaves, then have valence 3 and are roots (with two children each) of the subtrees T_1 and T_2. The maximal depth of vertices in T_1 and T_2 is strictly less than n, and we have that

$$\begin{aligned} g(T') &= 1 + g(T_1) + g(T_2) \leq 1 + 2l(T_1) - 1 + 2l(T_2) - 1 \\ &= 2l(T_1) + 2l(T_2) - 1 = 2l(T') - 1, \end{aligned}$$

which concludes our proof. □

4.2. Growth in the Euclidean plane

We discussed the growth in the Euclidean plane in the introduction, concluding that the growth function of the plane is $f(r) = \pi r^2$. Since the only regular square complex with link size four is the Euclidean plane, we have the following theorem.

Theorem 7. *The growth of the Euclidean quilt space, that is the regular square complex with link size four is quadratic.*

4.3. Divergence

Recall from the introduction that we define the detour distance between P and Q to be the shortest path from P to Q, or vice versa, which does not intersect the ball $B(O,n)$ of radius n centered around the origin, and we denote this as $\delta_n(P,Q)$. We then define the divergence to be a function

$$f(n) = \max\{\delta_n(P,Q) : d(P,O) = d(Q,O) = n\}.$$

In order to visualize and understand these definitions, we first consider the detour and divergence in the Euclidean plane, that is $\mathbf{R} \times \mathbf{R}$ with standard Euclidean metric that we are familiar with from high school and collegiate geometry. We center our balls at the origin $O = (0,0)$ The ball $B(O,n)$, then, is the open disk D_n of radius n. The sphere of radius n is then the boundary of this disk, which is the circle of radius n. If P and Q on this circle, then the shortest path between them that does not intersect the disk D_n is the arc of the circle that connects them. Thus, $\delta_n(P,Q)$ is the length of the (shorter) arc between P and Q, and the maximal length of such an arc is πn, achieved when the points P and Q are diametrically opposite points. Thus, the divergence $f(n)$ of the Euclidean plane is $f(n) = \pi n$. In the case of the Euclidean plane, we are able to find an exact function. However, what is most important is not the precise expression of the function, but rather its equivalence class. In this case, $f(n) = an + b$ for $a = \pi$ and $b = 0$ and the Euclidean plane has linear divergence.

This same approach can be used to show that divergence of the three-dimensional Euclidean space $\mathbf{R}^3$ has linear divergence. In $\mathbf{R}^3$, the ball $B(O,n)$ is the standard Euclidean three dimensional open ball of radius n. The furthest two points on the sphere of radius n are the diametrically opposite points, also called antipodal points, and the shortest path between them is the great circle of length πn that connects them. In fact, we can generalize the Euclidean metric to $\mathbf{R}^k$, defining that

$$d(\mathbf{x},\mathbf{y}) = \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2 + \ldots + (y_k - x_k)^2}$$

for points $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ in $\mathbf{R}^k$, and this metric space also has linear divergence for any $k \in \mathbf{N}$.

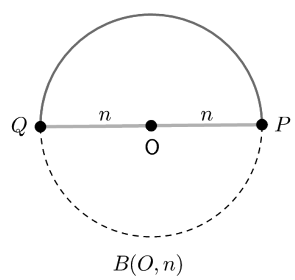

Figure 10. Diametrically opposite points on the circle have an arc distance of πn, so $\max\{\delta_n(P,Q) : d(P,O) = d(Q,O) = n\} = \pi n$.

4.4. Isoperimetric inequality

The word "isoperimetric" begins with the prefix *iso* coming from the greek word which means equal, followed by *perimetric* which refers to the word "perimeter". In two dimensions, the perimeter of planar figure is the length of its boundary curve. Consequently, isoperimetric inequality relates the area of the figure (region) to the length of its boundary [4, 7, 3, 8, 1].

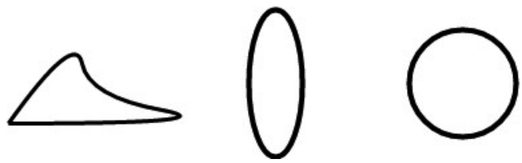

Figure 11. Loops whose enclosed areas are differ, but could potentially have the same perimeter. In the plane, the circle has the most area to perimeter of all loops.

A *loop* is a path whose initial and terminal endpoint are the same. Figure 11 depicts loops of same length that enclose regions of different areas.

For a loop α we define $f(\alpha)$ to be the area of the smallest disk that fills α. We introduce a new function f_D on the space X, called the Dehn function of X. The Dehn function f_D is a function from non-negative reals to non-negative reals defined in the following way. For a given $r \in \mathbf{R}$, $r \geq 0, f_D(r)$ is the least upper bound of all of the areas enclosed by the loop of length r. If a space X has a Dehn function that belongs to the equivalence class of a function $f(r)$, then we say that X satisfies the isoperimetric inequality for this equivalence class of functions.

Our next goal is to show that the Euclidean plane satisfies the quadratic isoperimetric inequality. Here we can simply rely on a beautiful theorem whose proof dates back into antiquity. That is for loop of a a given length $l > 0$, the figure of the largest area is can enclose is a circle. Therefore l is the circumference of some circle, whose radius is unknown. A simple equation $l = 2\pi r$ shows us that $r = \dfrac{l}{2\pi}$. This means that the area of a minimal disk that fills the loop of length l is $\dfrac{l^2}{4\pi}$. Therefore the isoperimetric inequality in the plane is quadratic.

5. Spaces with extra large links

We now turn to examining a space that is rather distinct from the plane (or Human Quilt Space). This is a space where every link is an *extra large link* instead of just a large link. It means at every vertex the squares attached create a link that looks like Figure 6, or larger. This is a space that is very hard to visualize, as it jives so strongly with our intuition. Figure 5 illustrates a small piece of such a space, where the the length of links are 5π.

However, it fits perfectly into our definition of Alien Quilts, and aliens who abide by different geometries that would create quilts like this. As we discussed growth, divergence, and isoperimetric inequality for the Euclidean plane (or Human Quilt Space), we now explore these properties in alien quilts.

We study growth, divergence and isoperimetric inequality of square complexes with extra large links using comparison with regular rooted trees. We start by showing that, if X is a regular square complex with extra large links, then we can embed into X binary rooted trees whose nodes are centers of squares in X, and whose edges are segments of the hyperplanes connecting the centers of the squares. Moreover, the centers of the squares in the n-th level of the tree are contained in the ball of radius n centered at the center of the root node, that is the square S_0. To avoid confusion, in this section we use the word nodes for the vertices of the embedded tree, and reserve the word vertex for the vertices in X. We refer to this construction as *Technical Lemma 2*.

We show how one can embed such trees in the case of a regular square complex with link size 5, to make this rather technical discussion easier to follow.

We start with a single square S_0^0 and two hyperplanes h and h' that intersect it. We pick any component C of $X \setminus (h \cup h')$ and also mark the (only) vertex v_0 of the square S_0 contained in C.

We denote by SC the smallest subcomplex containing C, and by U_0 the intersection of the star of the vertex v_0 with SC. We also denote by α_0 and β_0 the hyperplane rays that are the intersections of CS with h and h' respectively. The initial point of both α_0 and β_0 is the center of the square S_0. We say that α_0 and β_0 are highlighted hyperplane rays in SC.

Our construction consists of inductively embedding hyperplane rays that do not intersect outside of the ball of radius n centered at v_0.

We order the squares in the star of v_0 counterclockwise, in the circular manner, starting with $S_0^0 = S_0$ and index them with elements of $\mathbf{Z}_5$. Thus we have squares S_i^0 for $i = 0, 1, 2, 3$ and 4, where 0 in the superscript stands for the 0-th level of our construction. We denote the edge that is the intersection of the squares S_i^0 and S_j^0 by e_{ij}^0.

The square S_0^0 is the root node of our binary tree T_0, and the two squares S_1^0 and S_4^0 that each share an edge with S_0^0 are its children. We denote by γ_{01} and γ_{04} the hyperplane rays that originate in the centers of S_1^0 and S_4^0 and are perpendicular to α_0 and β_0 respectively, as illustrated in Figure 12. We add γ_{01} and γ_{04} to our collection of highlighted hyperplane rays. Since the link size is 5, there are also at squares S_2^0 and S_3^0 in the star of the vertex v_0. Thus, the hyperplane rays γ_{01} and γ_{04} intersect the squares S_2^0 and S_3^0 respectively, but do not intersect each other in U_0.

We observe that for the square S_1^0, a child of S_0^0, we have now constructed a pair of perpendicular hyperplane rays α_0 and γ_{01} that intersect in the center of S_1^0. We denote by v_{11} the vertex that is the endpoint of the edge e_{12} and observe that α_0

and γ_{01} intersect at the midpoint of one square in the star of v_{11}. A similar summary hold for the second child of S_0^0, the square S_4^0.

We could repeat the described construction to create the children of S_1^0 and S_4^0, if we can show that γ_{01} and γ_{04} do not intersect in the squares adjacent to S_0^2 and S_0^3. To show that is indeed the case, we first observe that the intersection of γ_{01} and γ_{04} with S_2^0 and S_3^0 are parallel to the edge e_{23}. The initial endpoint of e_{23} is v_0 and we denote the terminal endpoint by v_{14}, as the vertex "between" the hyperplanes γ_{01} and γ_{04}. Since the link size is 5, γ_{01} and γ_{04} cannot intersect in the link of v_{14}.

We denote by V_1 the union of the stars of vertices in U_0 and by U_1 the intersection of V_1 and SC. We already proved that γ_{01} and γ_{04} do not intersect in U_1. We can now order the highlighted hyperplanes. We start with α_0, followed by γ_{01}, then γ_4, and, finally, β_0. We have established that no two consecutive pairs of hyperplanes can intersect in $U_1 \setminus U_0$. Since hyperplanes separate the square complex X, no two pairs of highlighted hyperplanes intersect in $U_1 \setminus U_0$.

Thus, we can repeat the described construction to create the children of S_1^0 and S_4^0, and then proceed inductively to construct a binary tree that embeds into X.

We now address the case of squares in SC not included as nodes in the binary tree T_0. Such squares will appear in the star of the vertex v_{ij}, which is "between" the hyperplanes γ_{ij} and γ_{ik}. Each such square will be a root of a new binary tree, and arguments similar to the above ones show that the newly introduced hyperplanes will not intersect outside U_i.

We apply this construction inductively as follows. Given a square S_i and two hyperplanes rays α_i and β_j that intersect it, the children of S_i are squares S_{i1} and S_{i2} that share the adjacent edges, as determined by the orientation of the hyperplane rays. Thus, balls of size n in alien quilts have at least exponential size.

We now show that the size of balls is also bounded above by an exponential function. We pick a vertex v_0 in X, let $k = 2p$ where p is the size of the link, and construct an injective function $f : X \to T^k$ from X to a k-regular rooted tree.

The function f maps vertices in X to the nodes of the tree in the way that the images of all vertices in the ball of radius n are contained in the union T_n^k of the levels 0 to n of the tree T^k. Since the number of vertices is larger than the number of squares, and each square has area 1, the areas of balls in X are bounded by the the growth function of T^k.

The vertex v_0 is mapped to the root N_0 of the tree T^k. We denote by $U_0 = \{v_0\}$ and by $V_1 = star(v_0)$ the union of all squares containing v_0 as a vertex. There are $2p$ vertices in $U_1 = V_1 \setminus U_0$ and we map them injectively to the children of N_0. We observe that the ball of radius 1 centered at v_0 is contained in $V_1 = U_0 \cup U_1$.

To proceed by induction, we need some new terminology. For a set V in X we let $\tilde{V}$ be the union of all squares in X that share a vertex or an edge with a square in V. We call the set $U = \tilde{V} \setminus V$ the *ring* of V.

Inductively, assume that we have defined a collection of sets U_i in X, for $i < n$, such that U_i is the *ring* of $V_i = \cup_{j<i} U_j$, and a function f that sends the vertices in

U_i injectively to nodes in $i-th$ level of the tree T^k. Also assume that V_i contains the ball of radius i centered at v_0 for $i < n$. Let U_n be the ring of V_{n-1}. We first observe that $V_n = V_{n-1} \cup U_n$ contains the ball of radius n centered at v_0. Next, $U_n \cap V_{n-2} = \emptyset$, since, if a square S intersects V_{n-2}, then it is contained in U_{n-1}. Moreover, for each vertex in U_{n-1} there are at most $2p$ new vertices in U_n, and we map them injectively to children of the nodes in $(n-1)$-th level of the tree T^k. We conclude that the ball of radius n in an alien quilt has area less then equal to the number of nodes in the regular rooted k-tree for $k = 2p$, thus is exponential in n.

Theorem 8. *Let S be a square complex that has uniformly extra-large links. Then S has exponential growth.*

We now turn to investigating the divergence in spaces with extra large links.

Theorem 9. *Let O be a point on in X and assume that O is the center of square and is contained in the hyperplane* α. *Let P and Q be points on* α *at the distance n from O and on the opposite sides of O. Then the detour distance between P and Q is bounded below by* 2^n.

The theorem follows directly from the above discussion. We take the hyperplane β that intersect α in O. By *Technical Lemma 2* there are at least 2^{n+1} hyperplanes that originate in the squares within the ball or radius n. Since hyperplanes separate, every path connecting P and Q has to cross each hyperplane. Thus, the length of the shortest detour path connecting P and Q is at least 2^{n+1} and the divergence in X is at least exponential.

Since the size of the ball is exponential in n, the upper bound for the length of a detour path between any two points on the sphere is also exponential, thus the divergence is exponential.

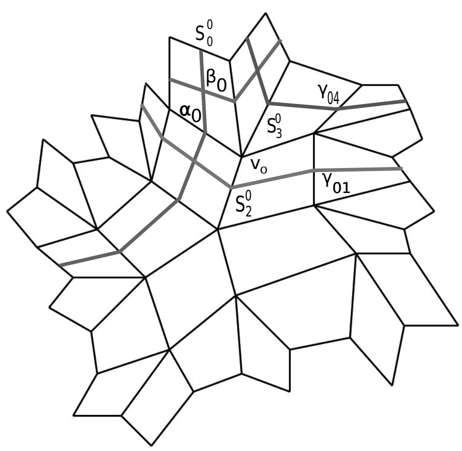

Figure 12. Embeddin binray trees in square complexes with extra large links.

Theorem 10. *Let X be a regular square complex that has extra-large links. Then X has exponential divergence.*

The last geometric property that we discuss is the isoperimetric inequality. We consider a continuous injective function $f : D \to X$, where D is the disk $\{(x,y) \in \mathbf{R} \times \mathbf{R} : x^2 + y^2 \leq 1\}$ in the the Euclidean plane. We assume that f maps the boundary circle $\{(x,y) \in \mathbf{R} \times \mathbf{R} : x^2 + y^2 = 1\}$ to a closed edge path λ in X. We will show that the area of $f(D)$ is linear in the length of λ. We refer to the squares contained in $f(D)$ as squares *inside* the loop λ. We first observe that if every square intersects λ, then the area is linear in the length of λ, and we are done. Thus, we consider squares that do not intersects λ and call such squares *inner squares*.

Let S be an inner square and α and β hyperplanes that intersects it. Since they separate, and λ is a closed curve, each hyperplane intersects λ twice, and $\alpha \cup \beta$ separates X into four components. We denote by C one of the components, note that $f(D) \cap C$ could have more than one component, and consider D_C to be one connected component of $f(D) \cap C$. Its boundary consists of segments of α and β, and an arc of λ, which we denote by λ_C. We will prove that the area of D_C is linear in the length of λ_C using our *Technical Lemmas 1 and 2*.

We use the *Technical Lemma 2* to construct the collection of binary trees, starting with S as the root of the first tree. We observe that a hyperplane γ added to our collection of highlighted hyperplanes will intersect at most one of them (by construction in *Technical Lemma 2*) and thus will also have to intersect λ_C. This follows from the property that γ separates the region bounded by the λ_C and the segments of hyperplanes. Thus, the number of leaves in our tree is equal to the length of λ_C, and our *Technical Lemma 1* implies that the area of D_C is bounded above by twice that length.

Since we can repeat this construction for the remaining components of $f(D) \setminus (\alpha \cup \beta)$, the following theorem holds.

Theorem 11. *Let X be a regular square complex that has extra-large links. Then X has linear isoperimetric inequality.*

References

[1] Bestvina M. and Feighn M. (1992), *A combination theorem for negatively curved groups*, J. Differential Geometry, 35: 85–101.

[2] Bridson M. and Haefliger A. (1999), *Metric spaces of nonpositive curvature*, Grundlehren der Mathematischen Wissenschaften, Vol. 319, Springer-Verlag, Heidelberg.

[3] Bridson M. R. and Gersten S. M. (1996), *The optimal isoperimetric inequality for torus bundles over the circle*, Quart. J. Math. Oxford Ser. (2), 47(185): 1–23.

[4] Gersten S. M. (1993), *Isoperimetric and isodiametric functions of finite presentations*, Geometric group theory, 1 (Sussex, 1991): 79–96.

[5] Gersten S. M. (1994), *Divergence in 3-manifold groups*, Geometric and Functional Analysis, 4(6): 633–647.

[6] GERSTEN S. M. (1994), *Quadratic divergence of geodesics in CAT(0)-spaces*, Geometric and Functional Analysis, 4(1): 37–51.

[7] GERSTEN S. M. (1996), *Banff lectures on hyperbolic and automatic groups*, preprint.

[8] GERSTEN S. M., HOLT D. F. AND RILEY T. (2003), *Isoperimetric inequalities for nilpotent groups*, GAFA, 13(4): 795–814.

[9] MIKHAEL GROMOV (1981), *Groups of polynomial growth and expanding maps*, in Jacques Tits, editor, Publications Mathematiques, N 53, Springer-Verlag.

[10] DAVID W. HENDERSON AND DAINA TAIMINA (2004), *Experiencing Geometry*, Pearson, 3rd edition.

[11] ERICA KLARREICH (2012), *Getting into Shapes: From Hyperbolic Geometry to Cube Complexes*, Quanta Magazine, December 26.

[12] MACURA N. (2013), *CAT(0) spaces with polynomial divergence of geodesics*, Geometria Dedicata, (163): 361–378.

[13] NIBLO G. A. AND REEVES L. D. (1998), *The geometry of cube complexes and the complexity of their fundamental groups*, Topology, 37(3): 621–633.

[14] GERARD VENEMA (2005), *Foundations of Geometry*, Prentice Hall (now Pearson).

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Holes in alien quilts

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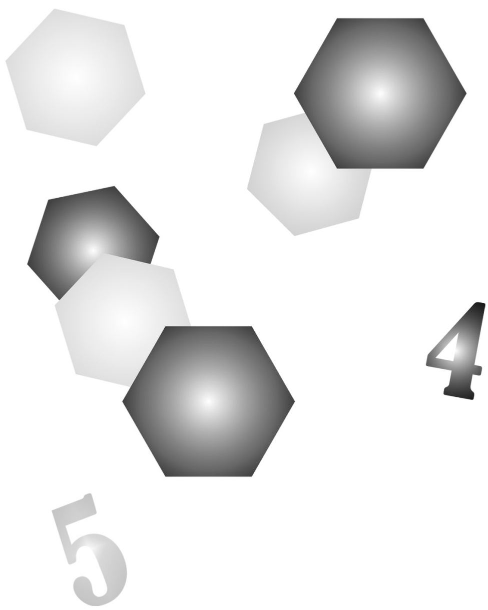
Abstract. Mathematical developments in last two decades brought cube complexes in a spotlight of the research in theoretical mathematics. The concept of CAT(0) cube complexes emerged as a particularly elegant structure suitable for modeling a variety of problems. We will start with the definition of cube complex and then introduce the large link and extra large link conditions, two combinatorial properties of CAT(0) cube complexes. Using examples of two-dimensional CAT(0) cube complexes, we will explain how these properties affect the geometry of the complex, in particular, the divergence and the isoperimetric inequality.

Keywords: spaces, growth, divergence, isoperimetric inequlity

Mathematics Subject Classification 2010: *20F65, 20F67, 57M20.*

3.
The role and importance of mathematics textbooks



Teachers' beliefs on mathematics as a background for their teaching practice

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Abstract. This paper discusses the impact that teachers' beliefs on mathematics and mathematics education have on their teaching practice. The issues concerning teaching practice focus in particular on the utilization of mathematics textbooks in the classroom. It is argued that teachers' beliefs about teaching and learning mathematics are significant in the utilization of a particular resource. We conducted a case study involving lower secondary mathematics teachers (grades five to eight) where we examined teachers' classroom practice and beliefs about mathematics, mathematics education, teaching mathematics and using textbooks as curriculum resources. The aim was to find out whether the teachers' attitudes correspond to their practice in the observed lessons or whether there are inconsistencies. The findings suggest that teachers' decisions regarding the use of textbooks in mathematics are influenced by their perception of the educational value of the textbooks and their personal beliefs about mathematics and mathematics education.

Keywords: mathematics teacher, beliefs, textbook, teacher-curriculum relationship

1. Introduction

Over the years, a variety of curriculum resources have emerged in mathematics education, designed with the practitioner in mind and made accessible to them (Kieran, Tanguay & Solares, 2012). However, the textbook still plays a dominant role in mathematics education (Fan, 2013). In curriculum models, textbooks are understood as potentially implemented curriculum. They present a link between the intentions of the official curriculum and the implementation of that curriculum in the classroom (Valverde et al., 2002). Moreover, the textbook is regarded as a

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book designed to provide an official and authorized pedagogic version of human knowledge (Stray, 1994) and as such it is considered to have authority over students and teachers.

Adler (2012) pointed out that the process of using a curriculum resource, for instance a textbook, is not a straightforward implementation; rather there exists a dynamic interaction between curriculum resource and teacher. The crucial point in understanding textbook utilization depends on the process of understanding what a teacher does with mathematics curriculum materials and why as well as how his/her choices influence the classroom environment (Remillard, 2009). Also, teachers' beliefs and attitudes are found to be a very influential factor in understanding the teacher-textbook interaction (Stipek, Givvin, Salmon & MacGyvers, 2001). Therefore, we conducted a study in which we investigated the impact of teachers' views on mathematics and mathematics education on their teaching practice and the utilization of textbooks in their classrooms.

2. Literature review

2.1. Textbook utilization

Textbooks have a great influence on teaching and learning mathematics in schools (Pepin & Haggarty, 2001). Learning mathematics is traditionally supported by the use of textbooks in many countries, in a manner that each student has a copy of the official textbook in his/her possession during the school year (Shield & Dole, 2013). Mathematics teachers follow the textbook content and structure in planning their instruction (e.g. Thomson & Fleming, 2004; Pepin, Geudet & Trouche, 2013; Fan, Zhu & Miao, 2013). Pepin and Haggarty (2001) found that the teachers they examined relied on the textbook to a great extent, using the proposed pedagogy, the same language as found in the textbook and reproduced similar worked examples during the lesson. The practice exercises and tasks for homework were also assigned from the textbook (Pepin & Haggarty, 2001; Johansson, 2006). Pepin and Haggarty (2001) also pointed out teachers' mediatory role in textbook utilization because the teacher is the one who decides which textbook to use, when and how it is used, which parts to use and in what order.

Another aspect of textbook utilization is connected with the change in teacher practice by using new textbooks (e.g. Sherin & Drake, 2009). Remillard (2000) studied how two elementary teachers used a reform-oriented textbook, their learning about mathematics and the teaching of mathematics. She discovered that the textbook offered teachers a variety of opportunities for learning and that activities which included analysis of students' possible answers and analysis of characteristics of mathematical tasks, influenced changes in the teachers' practice. Further, Remillard and Bryans (2004) detected stability in teachers' patterns of textbook use, but the authors also observed changes in the two teachers' practice who adopted the material most comprehensively. Stein and Kim (2009) investigated how 10 elementary teachers were using a reform-oriented textbook for the first time. Using the curriculum strategy framework *reading-evaluating-implementing* they were able to examine changes in the ways that teachers used the textbook and accompanying material over time. They concluded that "teachers' use of reform-based materials, even in their first year of use, is not haphazard" (p. 410). Moreover, they suggested that through curriculum strategy framework, insight could be gained into what and how teachers learn from using curriculum materials.

2.2. Mathematics textbooks in Croatian schools

Content analysis of Croatian mathematics textbooks for lower secondary education (grades five to eight) showed that operation activities with lower cognitive demands and intra-mathematical content (i.e. symbolic exercises without context) prevail in Croatian mathematics textbooks (Glasnović Gracin, 2011). The results indicate that Croatian textbooks put greater emphasis on algorithms and consider mathematics more as a tool than as a medium of communication. This picture of mathematics as a set of algorithms reveals a more traditional than contemporary view of mathematics. The content analysis also showed that the requirements of the intended curriculum match the ones in the textbooks, thus the Croatian mathematics textbook can be perceived as a 'conveyor of the curriculum' (Fan et al., 2013, p. 635).

Glasnović Gracin and Domović (2009) conducted a survey to determine the attitudes of lower secondary mathematics teachers towards the role of the textbook in mathematics education in Croatia. The results show that teachers use textbooks to a great extent for various activities: lesson preparation, teaching new topics, exercises, and assigning homework and that textbooks were used more than other curriculum resources. Around 52 % of the surveyed teachers said they almost always use the textbook for lesson preparation and an additional 45 % do so often; 97 % confirmed that they use the textbook as a source for mathematics exercises (51 % almost always and 46 % often); 99 % of participants stated that they use textbooks for giving homework (74 % almost always and further 25 % often). The results show a strong reliance on the officially approved textbooks in Croatian mathematics education and indicate that the classroom practice relies considerably on the textbook content and structure. The extension of this study encompassed qualitative methods such as on-site observations and interviews with teachers with the aim of finding out whether teachers' self-reports on textbook utilization differ from the actual situation in the classroom (Glasnović Gracin & Jukić Matić, 2016). The findings confirmed the central role of the textbook in Croatian mathematics classrooms: the textbook is used extensively in teachers' lesson preparation, as well as in the selection of worked examples and practice exercises for the students, and assigning homework.

2.3. Teachers' Beliefs about Mathematics Education and Textbook Use

The teacher's view of mathematics is the basis for the teacher's mental models and approaches to teaching and learning mathematics (Ernest, 1998). In this paper we will discuss traditional and contemporary views and approaches to learning mathematics as they are commonly used to highlight the differences between a person's beliefs and teaching and learning. A traditional view of mathematics sees mathematics as a collection of rules to be mastered, arithmetic computations, algebraic equations, and geometric proofs. In the traditional teaching of mathematics, great attention is paid to procedural knowledge: knowing how or the knowledge of the steps required to attain various goals (Mišurac, 2014). The memorization of definitions, statements and various facts plays an important role and the emphasis is on the content. Here the teacher is guiding students, telling them exactly how to use the materials in a prescribed manner. The focus of the lesson is primarily on getting answers and students rely on the teacher to determine if their answers are correct. A contemporary view of mathematics education encourages more understanding of mathematical ideas. In contemporary teaching, the emphasis is on the gradual building of conceptual knowledge i.e. networks of mathematical concepts and their relationships, as well as the flexible application of different procedures and problem solving. Here the students are encouraged to use varied approaches, strategies, results and interpretations. The emphasis is on communicating about the way of thinking, argumentations and exchanges of experience among students. The role of the teacher is to assist students while they create their own knowledge structures (Mišurac, 2014).

The research literature shows that a teacher's beliefs about mathematics education are also a very influential factor in their decisions concerning the use of curriculum resources, particularly textbooks. For instance, Robert and Rogalski (2005) state that the teachers' experience and professional history in a given activity, as well as knowledge and beliefs about mathematics and teaching, impact on their teaching practice and the ways in which they use curriculum resources. Teachers' beliefs about mathematics teaching and learning, and mathematics education in general, influence both how frequently textbooks are used and the way in which they are used (Manouchehri et al., 2000; Stipek et al., 2001). Findings reported in Stipek et al. (2001) showed substantial coherence among teachers' beliefs and consistent associations between their beliefs and their practices. On one hand, the authors found that the teachers who hold beliefs about mathematics education consistent with a traditional approach to teaching and learning are more likely to rely on and use textbooks. Also, they tend to follow the pedagogy and sequences embedded within the textbook and the accompanying teacher guide. On the other hand, teachers who hold beliefs more in line with a constructivist approach to mathematics are more likely to either teach without a textbook, or adapt the textbook activities (Stipek et al., 2001).

However, some studies found that the connection between teachers' beliefs and their practices may be inconsistent. For instance, Speer (2005) found some discrepancy between the teachers' beliefs about mathematics and mathematics education in relation to his/her use of textbooks in mathematics. Such inconsistency emerged from the complexity of the belief system and the fact that classroom practices are mediated by a number of factors (Rezat, 2013). For example, the decision to use the textbook in the classroom or not is influenced by the teacher's own level of confidence in mathematics and teaching, but also by other external factors such as colleagues in school as well as parents and society in general (Jukić Matić &

Glasnović Gracin, 2016; Pehkonen, 2004; Stipek et al., 2001). Raymond (1997) analysed teaching in terms of traditional or contemporary approach to teaching and learning, and examined teachers' beliefs about mathematics, and mathematics teaching and learning. She detected a mix of teachers' practices and their beliefs. In spite of holding non-traditional beliefs of mathematics teaching and learning, some teachers held traditional beliefs about the nature of mathematics and demonstrated a traditional teaching approach in the classroom. Moreover, Raymond (ibid) pointed out that early and continued reflection on mathematics beliefs and practices could be the key to improving the quality of mathematics instruction and minimizing inconsistencies between beliefs and practice.

3. Model of teacher-curriculum relationship

The findings presented in the text above indicate the need to investigate, understand and explain the relationship between teachers and the use of textbooks as curriculum resources. Brown (2009) developed a theoretical framework called Design Capacity for Enactment framework (DCE), which represents the idea that the curriculum resources as well as the teacher's personal resources influence designing and enacting the instruction (Figure 1).

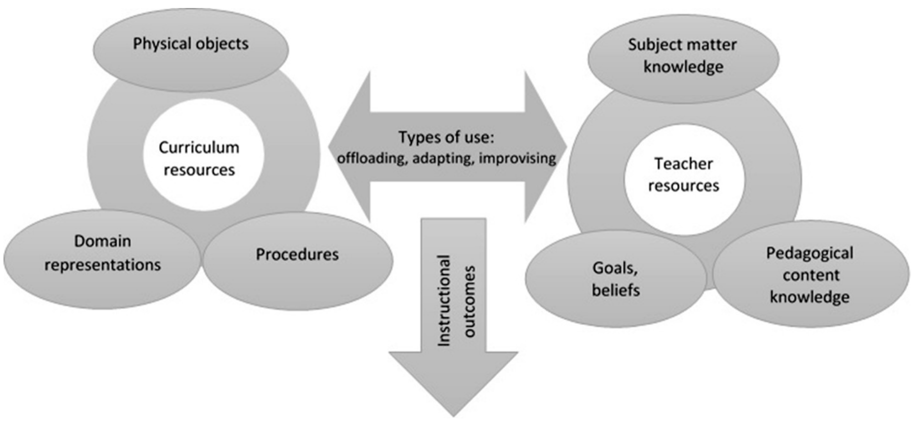

Figure 1. Brown's DCE model.

Curriculum resources encompass physical objects, representations and procedures of the domain, while teachers' personal resources denote subject matter knowledge (SMK), pedagogical content knowledge (PCK), beliefs and goals. The DCE framework "provides a starting point for identifying and situating the factors that can influence how a teacher adapts, offloads, or improvises with curriculum resources" (Brown, 2009, p. 27). *Offloading* denotes relying mostly on the curriculum resources for the delivery of the lesson and giving agency to the materials for guiding instruction. *Adapting* indicates an equally-shared responsibility for the delivery of the lesson between teacher and curriculum resources; it occurs when teachers modify their materials to support instructional goals. *Improvising* occurs when teachers craft instruction spontaneously and without specific guidance from their materials, thus shifting agency to themselves. Here the teacher relies mostly on external and their own resources for delivering the lesson. In our study, we will use the DCE model to investigate how teachers use the textbook in the classroom and how teachers' beliefs about mathematics and mathematics education influence this utilization.

4. Research question

In this study we addressed the following research questions:

1. What is mathematics and mathematics education for the participating teachers?
2. How do teachers organize their teaching practice in terms of textbook utilization?
3. Does the teachers' classroom practice correspond to their attitudes towards mathematics?

5. Methodology

5.1. Design of the study

The research reported in this paper is a follow-up study of a previous study reported in Glasnović Gracin & Jukić Matić (2016). It refers to a case study with a qualitative approach to the attitudes and actions of three mathematics teachers. In order to avoid the disadvantages of the case study, we chose a multi-method approach, combining former quantitative with qualitative methods (e.g. Onwuegbuzie & Collins, 2007). The quantitative study by Glasnović Gracin & Domović (2009) provided background for the case study reported in this paper. Based on the quantitative results, the case study was conducted in order to illustrate the conclusions or to explain some unexpected findings (e.g. Onwuegbuzie & Collins, 2007) of the previous study.

5.2. Participants

The participants of this study are three female mathematics teachers in lower secondary education in Croatia (grades five to eight). They were all identified on the basis of personal acquaintance; therefore, the sample is purposeful (Patton, 2002). To avoid artificial behaviours in the classroom, they were not informed of the specific object of our research. In this study, the teachers we will be called Mrs. R, Mrs. Z and Mrs. S. Mrs. R has 25 years of teaching experience, Mrs. Z has 39 years of teaching experience, and Mrs. S has 15 years of teaching experience. Based on the number of years they have been teaching, we do not consider them to be inexperienced teachers. Mrs. R and Mrs. S obtained their teaching degrees at the mathematics department, while Mrs. Z obtained her degree at the faculty of education.

5.3. Data collection and analysis

The case study encompassed classroom observations and interviews with teachers. We observed four lessons in each teacher's classroom. Participants were observed during October 2013. The choice of grade and content units was left up to the participant teachers, as it comes with the curricular plan and program. For the purposes of observation, we designed observation categories that can be seen in Table 1. One category refers to the impact of the textbook content and structure on instruction, meaning whether the teacher followed the textbook page by page, whether she improvised with the textbook content, or adapted the textbook content (Brown, 2009). Another category refers to the use of the textbook while acquiring new knowledge, while practicing or for homework (Glasnović Gracin & Domović, 2009).

Table 1. Questions in observational table.

Categories	Observational questions
Impact of the textbook structure on instruction (Brown, 2009)	Using textbooks during lesson unit (offloading, adapting and improvising): Does the instruction follow the textbook page by page? What is taken from the textbook? Did the teacher draw attention to any specific figure, frame or picture from the textbook? For what purposes?
Teaching, practising, homework (Glasnović Gracin & Domović, 2009)	How is new content introduced? What is a practice lesson like? Which sources are used for practising and homework?
Other	Does the teacher step outside the textbook? Does the teacher go beyond the national curricula outlines? Is there another source given for learning/practising? What kind?

Table 2. Outlines for semi-structured interview.

Categories	Interview questions
Impact of the textbook structure on instruction (Brown, 2009)	Describe how you usually prepare a mathematics lesson. Does the textbook in your opinion influence the structure of your instruction, e.g. using the same title, definitions, language, symbols, sequence, didactical approach, worked examples, figures? Explain.
Teaching, exercising, homework (Glasnović Gracin & Domović, 2009)	Describe a typical lesson with learning new content. Describe a typical lesson with emphasis on exercising. Describe how you choose homework activities and from which sources. Describe how you prepare your students for examinations.
Mathematics and mathematics education (Raymond, 1997)	What is mathematics for you? What is mathematics education for you? Give your beliefs and ideas about professional development.

In addition to classroom observations, we conducted interviews with all participants. The interview was semi-structured with a set of open-ended questions and outlines which were specified in advance (Table 2). This kind of interview approach enables flexibility and the rich collection of data (Cohen et al., 2007). The interviewers could ask additional questions based on the observed lessons or could expand upon an interesting point arising during the interview. This provides quantitative, as well as qualitative interview potentials, so called prompts and probes (Morrison, 1993, in Cohen et al., 2007).

The gathered data were analysed using the constant comparative method (Cohen et al., 2007). This process allowed us to report on those aspects which were common to the teachers, as well as their differences.

6. Results

6.1. Traditional view, traditional teaching

Mrs. Z conceives mathematics as a set of procedures and as a discipline in which exercises are solved. She was not satisfied with the last reform of mathematics education in 2006 (the Croatian National Educational Standard (HNOS)) because of its promotion of a new way of teaching mathematics. She believes that students did not learn anything in this period and did not work hard; they were "just playing":

> "...when we worked according to HNOS... they had *great* examples [laughter] of how to *entertain* children and it was nothing else but entertaining [laughter]"

She believes that her work in the classroom is better when students practice routine exercises or "when we drill the tasks". She believes that both the textbook and accompanying teacher guide she uses suit and support her teaching, so she relies on them a to a great extent. She also believes that following the textbook is beneficial for students, because students can always learn from the textbook at home:

> "I also teach in a hospital. All kinds of kids are there. I ask them which textbook they are using, so we take it and work according to it."

In all the observed lessons, Mrs. Z adhered to the textbook content and structure. She used the textbook for teaching a new topic as well as for practicing. When she taught new content, she directed students to the textbook saying: "Open the textbook...Turn to page...Look at the title". She also used worked examples from the textbook. In the practice lessons, she assigned exercises from the textbook, which students solved either individually or jointly with the teacher. In the interview, Mrs. Z said she usually organizes practice in such a way that she gives the students tasks from the textbook:

> "There are a lot of exercises in this textbook and workbook... a lot of good exercises"

Most of the time Mrs. Z followed the structure of the textbook blindly:

> "I work according to the textbook. I do not invent any new examples [...] I try to follow it."

In her lessons, Mrs. Z used a traditional approach to teaching mathematics, placing more importance on procedure than on understanding. An example of this is the teaching of division of fractions without any context or model, even though the textbook provided a model of a chocolate bar. In the observed lessons Mrs. Z did not allow students to discover the procedure for dividing fractions; she told them the exact rule:

> "When you divide a fraction by a whole number, you multiply it by the reciprocal of the whole number."

Moreover, she did not encourage students to develop their own understanding of the procedure; she just focused on carrying out the algorithm:

> "...we have to practice the rule [for dividing fractions by a whole number]"

She uses the ready-made lesson plans from the accompanying teacher guide, and accompanying tests provided by the same publisher.

Mrs. Z displayed a traditional approach towards mathematics and mathematics education where mathematics is seen as a set of procedures. This approach was evident in all her observed lessons.

6.2. Contemporary view, contemporary teaching

Mrs. R believes that mathematics is something that one "does not have to *cram* but to understand" and that it is a "main tool for life". Her understanding of mathematics education is that it should enable children to think and prepare them for everyday life. Even though she does not rely on the textbook in her lessons, Mrs. R sees the textbook as a resource for students to study at home, an artefact which helps them when they are stuck, or a resource for parents when they are helping their children:

> "Well, there should be a textbook; children have to have something to go back to, and something their parents can refer to."

Mrs. R thinks that teachers should learn every day and even after 25 years of teaching she believes there is always room for improvement:

> "Even after 25 year of teaching I make sure I'm prepared, I write new lesson plans, I read and consult the literature... I regularly attend professional development courses."

In the observed lessons, Mrs. R used contemporary methods of teaching mathematics. In her instruction, she applied various strategies such as discovery learning or problem based learning. She believes that this way of teaching is the most effective way to learn mathematics.

> “Well, I want them to discover something, to find out something new on their own. If that approach doesn’t work, then I support them [...] But I want them to come to an appropriate conclusion.”

Her lessons were designed to promote students’ conceptual understanding, not just to rely on procedural knowledge. She believes that understanding is crucial in mathematics, what is actually being done and why:

> “I mean it’s silly to have to solve the same exercise five times or three times [...] I want them to make connections...”

Mrs. R only used the textbook directly for assigning homework in two out of four of the observed lessons. She used self-made worksheets intended for individual, pair or group work in her lessons. She also created her own examples for teaching new content which did not resemble examples from the textbook. Her attitude towards the non-utilization of the textbook was made clear in the classroom when she told students to close the textbook they had just opened, saying that what they were going to do in the lesson would be different from what was in the book:

> “Put your textbooks away. They’re only in the way. Everything you need is in your notebook.”

She uses various sources for lesson preparation, taking ideas from many places, but not from the textbook because she is dissatisfied with the textbook’s content and methodological approach. She also noticed that the textbook contains too many routine tasks, only operational requirements and that it does not promote higher cognitive levels:

> “Everything comes down to reproduction, easier or a bit harder, but there are no tasks where the children have to connect [various mathematical topics] and to see the application of what they have learned.”

Mrs. R does not consult the accompanying teacher guide nor does she use the ready-made exams. She approaches mathematics in a more conceptual than procedural way. These beliefs are in line with her teaching practice which places emphasis on and making her own materials discovery learning rather than following the textbook.

6.3. Discrepancies between views on mathematics and teaching practice

The third participant, Mrs. S, sees mathematics as a game and as part of everyday life.

> “Mathematics for me is a game... When I think about it, I see that I have always been into mathematics.”

She believes that one of the goals of mathematics education is to encourage students to become capable and independent users of mathematics literature, especially textbooks.

> “My goal is for students to become independent in using mathematics literature. I noticed that when they reach high school, they are unable to

> differentiate between relevant and irrelevant facts, and they are unable to recognize tasks similar to those they have already learnt [at school in class]."

She shows great appreciation for the higher mathematics she studied at university and believes that the understanding of such mathematics is a good thing for teachers because they can relate various topics to nature and explain a range of tasks:

> "...because we [teachers] think differently...we think how to best explain a task to students or draw attention to a problem around us."

Mrs. S believes that professional development is important, and she attends professional development courses whenever she can.

> "I like to go and listen... I always find something new and I like to implement it."

However, this approach to mathematics and mathematics education was less evident in the classroom observations. In her teaching practice, she used the textbook frequently for many similar tasks, focusing on procedural knowledge with no relation to everyday situations. Asked about this in the interview, she said that she believes that doing many similar examples is a vital part of mathematics education. She likes the structure of the textbook with worked examples followed by plenty of similar exercises.

> "We use the textbook when we practice because there are a lot of exercises there... I assign tasks from it [the textbook], but not always in the given order"

The observations showed her procedural and intra-mathematical approach to operations with fractions. She also stressed that she always liked the rigor in mathematics and following the rules:

> "I like this rigor, following the rules."

Mrs. S organized one lesson as individual work where students were starting a new topic from the textbook by themselves. Two of the observed lessons were practice lessons, where she explicitly told students to open their textbooks and to do the exercises. Mrs. S also uses the lesson plans provided in the teacher guide, as well as the accompanying ready-made tests. She added that she sometimes has to adapt the lesson plans to the students' needs.

We also identified parts of the observed lessons which can be characterized as improvisations or adaptations of textbook content. For example, when she taught division of a whole number with a fraction, it was similar to the model of the chocolate bar given in the textbook (adapting). Also, at the beginning of one of the observed lessons, Mrs. S went over the homework with the students, which was a worksheet she had made herself. This homework was not taken from the textbook; it was an authentic activity connected with a class excursion from the previous week (improvising). She explained that the textbook is the main source of exercises for homework, but sometimes she creates her own worksheets:

"Sometimes I create some worksheets for homework. Sometimes we have projects, so I make connections"

These results point to some discrepancies between Mrs S's attitudes towards mathematics and mathematics education and her teaching practice.

7. Discussion and conclusion

In this case study we wanted to examine how mathematics teachers use textbooks taking into consideration their beliefs about mathematics, mathematics education, teaching mathematics and using textbooks. Moreover, we wanted to establish whether the teacher's beliefs correspond to their practice.

This study showed that the teachers approach textbooks in different ways, depending upon their beliefs about the teaching and learning of mathematics and the role of curriculum materials (Table 3).

Table 3. The relationship between teachers' beliefs about mathematics and their teaching practice.

	Beliefs about mathematics and ME	**Teaching practice**	**Use of textbook**
Mrs. Z	traditional	traditional	offload
Mrs. R	contemporary	contemporary	improvise
Mrs. S	contemporary/traditional	traditional	offload/adapt

Mrs. Z holds traditional beliefs about mathematics where the teacher is in complete control and the students' main goal is to learn operations to get the right answer (Stipek et al., 2001). Here knowing mathematics equals being skilful in performing procedures and manipulating symbols.

This finding corresponds to the results of the survey conducted by Baranović and Štibrić (2009) on mathematics teachers in Croatia. Results of the survey showed that teachers consider solving mathematical tasks, learning basic mathematical facts and applying knowledge as the most vital part of mathematical competence. Moreover, they claimed that they develop students' skills for solving procedures mostly, while the application of mathematics in everyday life and critical thinking about mathematical concepts and procedures seem to be less important. Mrs. Z's traditional approach to mathematics was evident during the observed lessons, where she emphasized the procedural aspect of mathematics and repeatedly stressed the importance of "following the rules". Her understanding of mathematics aligns with the textbook's goals; therefore she closely followed the textbook and teacher guide content in planning her lessons. For her, the textbook represents an authority on what should happen in the classroom (Pepin & Haggarty, 2001). Her teaching practice corresponds to her beliefs.

The teaching practice of Mrs. R relied on improvising and using her own personal resources to create teaching materials. She decentralized classroom teaching, enabled explorations and problem solving. Mrs. R believes that the main outcome of mathematics education should be developing students' thinking, which corresponds to a constructivist approach to learning (Stipek et al., 2001). Her beliefs and the intentions of the textbook used in her school are not aligned, so the textbook does not influence her teaching; she does not use it for lesson planning or enacting. Her teaching practice corresponds to her beliefs.

Even though Mrs. S believes that mathematics can be found all around us, she did not apply this in her teaching. A traditional approach to mathematics education was evident in her teaching, with emphasis on learning rules and applying them efficiently. Mrs. S followed the textbook and shaped her lessons according to the textbook structure. We have purposely chosen to include Mrs. S in this study because of her educational background, which is different than the background of Mrs. Z. She is younger than Mrs. Z, and obtained her degree at the department for mathematics. Nevertheless, she shares many similarities with the teaching of Mrs. Z. There are certain inconsistencies between her beliefs and teaching practice. When she spoke about how she sees mathematics, she emphasised ideas and the implementation of mathematics in everyday life. However, the observations showed traditional teaching with emphasis on symbolic tasks and following rules without providing models or context. When asked about that in the interview, she said that another aspect of mathematics she likes is following rules. This is reflected in her mathematics instruction. However, she did make several attempts to improvise or adapt textbook content (Brown, 2009).

Although the participants did not use the textbook in the same way, all three teachers believe that the textbook has educational value for students; they should use the textbook for learning at home. This is in line with the study of Jamieson-Proctor & Byrne (2008), where even the teachers who used the textbook in their classrooms less frequently, believed the student textbook is a valuable teaching and learning aid in the mathematics classroom, particularly as a resource to provide students with opportunities to practice various mathematical skills.

The results of this study show another interesting result in the domain of teacher's resources and beliefs. All three participants believe that continuous professional education is important for mathematics teachers. But how is it possible that they use the textbooks so differently if they all frequently attend professional development courses? Various researchers (e.g. Ball and Cohen, 1996) highlighted that teachers' personal beliefs can represent an obstacle when they try to implement inquiry-based mathematics instruction. Professional development courses, designed to help teachers to implement such instruction, turned out to be minimally effective, because teachers altered what they learned through their existing beliefs. Therefore, in our study, Mrs. Z and Mrs. S are representative of teachers whose beliefs could possibly constrain the desired outcome of such courses.

Finally, let us turn to Mrs. Z's beliefs about the last educational reform. She believed that the 2006 reform was not good, that children were just playing. This again shows how a teacher's beliefs influence their teaching practice and can create resistance to change. When investigating the effect of educational reform in the USA, various authors found that the reform failed because many teachers held onto their old beliefs of what mathematics is and how mathematics education should look, even though the curriculum resources, especially textbooks, were changed and aligned with a constructivist approach to learning (e.g. Ball & Cohen, 1996; Remillard & Bryans, 2004). Since the new requirements in mathematics education stipulate that more emphasis should be placed on guidance of learning rather than transmission of knowledge, and on constructive teaching rather than demonstrations (van den Berg, 2002), the change of teachers' beliefs systems should be targeted when implementing educational reforms. Drake and Sherin (2009) and Remillard and Bryans (2004) emphasized that teachers have stable beliefs and practices with respect to curriculum use, but that their teaching strategies may change as they develop an understanding of the intent and goals underpinning the curriculum. The cases we presented in this brief study showed that regardless of their experience, teachers can hold traditional beliefs about teaching mathematics and nature of mathematics. We argue that the way of teaching in Croatian mathematics classrooms will not change unless time is devoted to changing teachers' attitudes and beliefs.

Following Raymond (1997), the cases of these three teachers with different educational backgrounds, raise another issue regarding pre-service teacher education: What picture of the nature of mathematics and mathematics education is promoted on university mathematics courses and mathematics education courses? It is very important that mathematics teacher education programs assess their effectiveness in how well they nurture beliefs that are consistent with the philosophy of learning and teaching (Hart, 2002).

References

[1] ADLER, J. (2012), *Knowledge resources in and for school mathematics teaching*, In B. Gueudet, B. Pepin, and L. Trouche (Eds.), From text to 'lived resources', pp. 3–22, Dordrecht: Springer.

[2] BALL, D. L., & COHEN, D. K. (1996), *Reform by the book: What is: or might be: the role of curriculum materials in teacher learning and instructional reform?*, Educational researcher, 25(9), 6–14.

[3] BARANOVIĆ, B. & ŠTIBRIĆ, M. (2009), *Math teachers' perceptions of mathematics education in elementary and secondary schools in Croatia*, Results of an empirical research, in M. Pavleković (Ed.), Second International Scientific Colloquium "Mathematics and Children" – How to Teach and Learn Mathematics, Zagreb: Element.

[4] BROWN, M. (2009), *The Teacher-Tool Relationship: Theorizing the Design and Use of Curriculum Materials*, in J. T. Remillard, B. A. Herbel-Eisenmann & G. M. Lloyd (Eds.), Mathematics Teachers at Work: Connecting Curriculum Materials and Classroom Instruction (pp. 17–36), NY: Routledge.

[5] COHEN, L., MANION, L., MORRISON, K. (2007), *Research methods in education*, Abingdon: Routledge.

[6] DRAKE, C., & SHERIN, M. G. (2009), *Developing curriculum vision and trust: Changes in teachers' curriculum strategies*, in J. T. Remillard, B. Herbel-Eisenmann, & G. Lloyd (Eds.), Mathematics teachers at work: Connecting curriculum materials and classroom instruction, pp. 321–337, New York: Routledge.

[7] ERNEST, P. (1989), *The impact of beliefs on the teaching of mathematics*, in P. Ernest (ed.), Mathematics teaching, The state of art, pp. 249–254, New York: The Falmer Press.

[8] FAN, L. (2013), *Textbook research as scientific research: towards a common ground on issues and methods of research on mathematics textbooks*, ZDM – The International Journal on Mathematics Education, 45(5), 765–777.

[9] FAN, L., ZHU, Y. & MIAO, Z. (2013), *Textbook research in mathematics education: development status and directions*, ZDM Mathematics Education, 45(5), 633–646.

[10] GLASNOVIĆ GRACIN, D. & DOMOVIĆ, V. (2009), *Upotreba matematičkih udžbenika u nastavi viših razreda osnovne škole*, Odgojne znanosti, 11, 297–317, in Croatian.

[11] GLASNOVIĆ GRACIN, D. (2011), *Requirements in mathematics textbooks and PISA assessment*, Doctoral dissertation, University of Klagenfurt, Klagenfurt.

[12] GLASNOVIĆ GRACIN, D., & JUKIĆ MATIĆ, LJ. (2016), *The Role of Mathematics Textbooks in Lower Secondary Education in Croatia: An Empirical Study*, The Mathematics Educator, 16(2), 29–56.

[13] HART, L. C. (2002), *Preservice Teachers' Beliefs and Practice After Participating in an Integrated Content/Methods Course*, School Science and Mathematics, 102, 4–14.

[14] HILL, H. C., BLUNK, M. L., CHARALAMBOUS, C. Y., LEWIS, J. M., PHELPS, G. C., SLEEP, L., & BALL, D. L. (2008), *Mathematical knowledge for teaching and the mathematical quality of instruction: An exploratory study*, Cognition and Instruction, 26(4), 430–511.

[15] JAMIESON-PROCTOR, R., & BYRNE, C. (2008), *Primary teachers' beliefs about the use of mathematics textbooks*, in M. Goos, R. Brown & K. Makar (Eds.), Navigating Currents and Charting Directions (Proceedings of the 31st Annual Conference of the Mathematics Education Research Group of Australasia), Vol. 1, pp. 295–302, Brisbane: MERGA.

[16] JOHANSSON, M. (2006), *Teaching mathematics with textbooks. A classroom and curricular perspective*, Doctoral dissertation, Luleå University of Technology, Luleå.

[17] JUKIĆ MATIĆ, LJ.,& GLASNOVIĆ GRACIN, D. (2016), *The use of the textbook as an artefact in the classroom: A case study in the light of a socio-didactical tetrahedron*, Journal für Mathematik-Didaktik, 37(2), 349–374.

[18] KIERAN, C., TANGUAY, D., & SOLARES, A. (2012), *Researcher-designed resources and their adaptation within classroom teaching practice: Shaping both the implicit and the explicit*, in G. Gueudet, B. Pepin, & L. Trouche (Eds.), From text to 'lived' resources: Mathematics curriculum material and teacher development, pp. 189–213, New York: Springer.

[19] MANOUCHEHRI, A., & GOODMAN, T. (2000), *Implementing Mathematics Reform: The Challenge Within*, Educational Studies In Mathematics, 42(1), 1–34.

[20] MIŠURAC, I. (2014), *Suvremeni standardi matematičkih kompetencija u početnoj nastavi matematike*, Split: Filozofski fakultet.

[21] ONWUEGBUZIE, A. J., & COLLINS, K. M. T. (2007), *A typology of mixed methods sampling designs in social science research*, The Qualitative Report, 12(2), 281–316, Retrieved September 30, 2016, from `http://www.nova.edu/ssss/QR/QR12-2/onwuegbuzie2.pdf`.

[22] PATTON, M. Q. (2002), *Qualitative research and evaluation methods* (3rd ed.), Thousand Oaks, CA: Sage.

[23] PEHKONEN, L. (2004), *The Magic Circle of the Textbook – An Option or an Obstacle for Teacher Change*, in M. J. Hoines & A. B. Fuglestad (Eds.), Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education, Vol. 1, pp. 14–18, Bergen, Norway.

[24] PEPIN, B., & HAGGARTY, L. (2001), *Mathematics textbooks and their use in English, French and German classrooms: a way to understand teaching and learning cultures*, ZDM – Mathematics Education, 33(5), 158–175.

[25] PEPIN, B., GUEUDET, G., & TROUCHE, L. (2013), *Investigating textbooks as crucial interfaces between culture, policy and teacher curricular practice: two contrasted case studies in France and Norway*, ZDM – The International Journal on Mathematics Education, 45(5), 685–698.

[26] RAYMOND, A. M. (1997), *Inconsistency between a beginning elementary school teacher's mathematics beliefs and teaching practice*, Journal for Research in Mathematics Education, 28(5), 550–576.

[27] REMILLARD, J. T. (2000), *Can curriculum materials support teachers' learning? Two fourth-grade teachers' use of a new mathematics text*, The Elementary School Journal, 100, 331–350.

[28] REMILLARD, J. T., & BRYANS, M. B. (2004), *Teachers' orientations toward mathematics curriculum materials: Implications for teacher learning*, Journal for Research in Mathematics Education, 35(5), 352–388.

[29] REMILLARD, J. T. (2009), *Part II commentary: Considering what we know about the relationship between teachers and curriculum materials*, in J. T. Remillard, B. A. Herbel-Eisenmann & G. M. Lloyd (Eds.), Mathematics Teachers at Work: Connecting Curriculum Materials and Classroom Instruction, NY: Routledge.

[30] REZAT, S. (2013), *The textbook-in-use: students' utilization schemes of mathematics textbooks related to self-regulated practicing*, ZDM – The International Journal on Mathematics Education, 45(5), 659–670.

[31] ROBERT, A., & ROGALSKI, J. (2005), *A cross-analysis of the mathematics teacher's activity*, Educational Studies in Mathematics, 59, 269–298.

[32] SHERIN, M. G., & DRAKE, C. (2009), *Curriculum strategy framework: Investigating patterns in teachers' use of a reform-based elementary mathematics curriculum*, Journal of Curriculum Studies, 41(4), 467–500.

[33] SHIELD, M. J., & DOLE, S. (2013), *Assessing the potential of mathematics textbooks to promote deep learning*, Educational Studies in Mathematics, 82(2), 183–199.

[34] SPEER, N. M. (2005), *Issues of Methods and Theory in the Study of Mathematics Teachers' Professed and Attributed Beliefs*, Educational Studies in Mathematics, 58(3), 361–391.

[35] STEIN, M. K., & KIM, G. (2009), *The role of mathematics curriculum materials in large-scale urban reform: An analysis of demands and opportunities for teacher learning*, in J. T. Remillard, B. A. Herbel-Eisenmann & G. M. Lloyd (Eds.), Mathematics Teachers at Work: Connecting Curriculum Materials and Classroom Instruction, pp. 37–55, New York, NY: Routledge.

[36] STIPEK, D. J., GIVVEN, K. B., SALMON, J. M., & MACGYVERS, V. L. (2001), *Teachers' beliefs and practices related to mathematics instruction*, Teaching and Teacher Education, 17, 213–226.

[37] STRAY, C. (1994), *Paradigms regained: Towards a historical sociology of the textbook*, Journal of Curriculum Studies, 26(1), 1–29.

[38] THOMSON, S., & FLEMING, N. (2004), *Summing it up: Mathematics achievement in Australian schools in TIMSS 2002*, Melbourne: Australian Council for Educational Research.

[39] VALVERDE, G. A., BIANCHI, L. J., WOLFE, R. G., SCHMIDT, W. H., & HOUANG, R. T. (2002), *According to the book. Using TIMSS to investigate the translation of policy into practice through the world of textbooks*, Dordrecht: Kluwer Academic Publishers.

[40] VAN DEN BERG, R. (2002), *Teachers' Meanings Regarding Educational Practice*, Review of Educational Research, 72(4), 577–625.

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Učiteljeva uvjerenja o matematici kao pozadina njihove nastavne prakse

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Sažetak. U ovom radu promatra se utjecaj učiteljevih uvjerenja i stavova o matematici i nastavi matematike na njihovu nastavnu praksu. Pitanje nastavne prakse posebno se usredotočuje na upotrebu udžbenika u nastavi matematike.

Učiteljevi stavovi i uvjerenja o učenju i poučavanju matematike važni su kod upotrebe određenog kurikularnog materijala tj. nastavnog sredstva. Proveli smo studiju slučaja koja uključuje učitelje matematike u osnovnoj školi. Istraživanje smo proveli pomoću opservacija i intervjua koji su se fokusirali na učiteljeva uvjerenja o matematici, matematičkom obrazovanju, podučavanju matematike i korištenju udžbenika kao kurikularnog materijala. Cilj je bio utvrditi odgovaraju li stavovi promatranih učitelja njihovoj nastavnoj praksi ili postoje nedosljednosti. Rezultati upućuju na zaključak da na odluku o tome treba li rabiti udžbenik u nastavi ili ne, utječe percipirana obrazovna vrijednost udžbenika kao i osobna uvjerenja učitelja o matematici i matematičkom obrazovanju.

Ključne riječi: učitelj matematike, uvjerenja, kurikularna sredstva, odnos učitelj-kurikulum

Asymptote as a body of knowledge to be taught in textbooks for Croatian secondary education

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Abstract. Asymptote and asymptotic behaviour are bodies of knowledge present in elementary as well as in theoretical and applied mathematics. Some aspects of these concepts make a common part of upper secondary mathematics curricula worldwide. The notion of an asymptote appears as a theoretical concept, a part of a procedure or as a self-sufficient task in different contexts and different educational cycles of upper secondary mathematics education.

In this paper, we present results of a textbook analysis on asymptote and asymptotic behaviour in two most common series of gymnasium mathematics textbooks in Croatia. It is a part of a larger comprehensive study regarding this body of knowledge in general upper secondary education in Croatia. The research is conducted within the theoretical framework of the Anthropological theory of the didactics (ATD), developed by a French mathematician Y. Chevallard. This framework provides *praxeology* $[T, \tau, \theta, \Theta]$ as a tool for describing mathematical knowledge and activities, with its practical component – *a task* T and *a technique* τ, and discursive or theoretical component – *a technology* θ and *a theory* Θ.

The results show that the asymptote is a complex body of knowledge consisted of several praxeologies for which it is a part of practical or theoretical component. Hence, we present the related praxeological organization of considered textbooks along with its direct implications to mathematics teaching and learning.

Keywords: asymptote, asymptotic behaviour, ATD, praxeology, textbook analysis, general secondary mathematics education

1. Introduction

Asymptote and asymptotic behaviour are bodies of knowledge present in various mathematical domains. In the theoretical and applied mathematics asymptotic behaviour is a powerful tool for describing and solving various problems, supported by a well-established abstract theory. In the elementary mathematics asymptote and asymptotic behaviour are a significant property of some functions and curves. That aspect of these concepts is a common part of upper secondary mathematics curricula worldwide. The notion of an asymptote appears as a theoretical concept – when describing or analyzing elementary functions behaviour at infinity and near singularities, a part of a procedure – when graphing some elementary functions or plane curves, or as a self-sufficient task. Also, that notion appears in different contexts – algebra, geometry or calculus, and different educational cycles of upper secondary mathematics education. Hence, the variety of forms in which it is represented, connections to other mathematical bodies of knowledge, mathematical contexts and education levels in which it appears, make asymptote an interesting object of research.

The results presented in this paper are a part of a comprehensive research on the didactic transposition of asymptote as a body of knowledge in general upper secondary education in Croatia. The research is conducted within the theoretical framework of the anthropological theory of the didactic (ATD), developed by a French mathematician Y. Chevallard. The research methodology is compliant with the theoretical framework and it included (1) textbook analysis, (2) three questionnaires with open-ended questions for the final, fifth year mathematics education students at the largest mathematical department in Croatia, and (3) a semi-structured interview with two mathematicians.

In this paper, we present results of a textbook analysis of the two most common series of gymnasium mathematics textbooks in Croatia used in the period 2007. – 2011., containing the teaching content related to asymptote and asymptotic behaviour.

2. Literature review

2.1. Theoretical framework

The ATD provides tools for investigating how a mathematical body of knowledge is represented in various institutions and how it is related to individuals within those institutions. For the conducted research the notion of didactical transposition is relevant (Bosch & Gascón, 2006, 2014; Chevallard & Bosch, 2014). That is a three step process where mathematical (*scholarly*) knowledge is first transformed, transposed, to meet various requirements of an educational system, which results in the constitution of *knowledge to be taught*. In the second step, the body of knowledge, in its changed form, is interpreted by the teacher in the classroom, as *taught knowledge*, and in the end each student forms his version of this body of knowledge as his *available knowledge* (Figure 1). In this paper, the state of *knowledge to be taught* is presented, since ATD posits textbooks as a relevant product of the first step of the didactical transposition (Bosch & Gascón, 2006, 2014; Chevallard & Bosch, 2014).

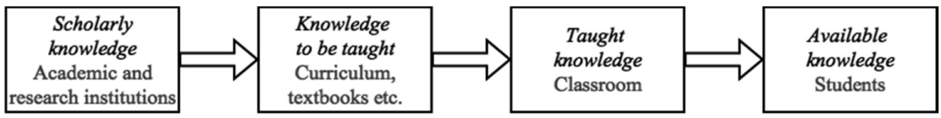

Figure 1. Didactical transposition.

The ATD postulates that any knowledge is or should arise from answering a question or solving a problem (Bosch & Gascón, 2014). Hence, this framework provides means to describe knowledge and activities with *praxeology* $[T, \tau, \theta, \Theta]$ (Barbé, Bosch, Espinoza, & Gascón, 2005; Bosch & Gascón, 2006; Chevallard & Sensevy, 2014). That means one should establish what practical activities – *praxis*, and thinking, reasoning or other discursive activities – *logos* are presupposed and brought about for the considered mathematical knowledge. Both practical and discursive components of a praxeology, regarding some knowledge or activity, ought to be available and valuable; they should be interrelated and inseparable.

A *point-praxeology* is a quadruple $[T, \tau, \theta, \Theta]$ whose terms denote: *task* T, *technique* τ, *technology* θ and *theory* Θ (Chevallard & Sensevy, 2014). Here, a task T and a technique τ make the practical block of a praxeology, also called the *praxis*, and a technology θ with a theory Θ make the discursive block called *logos*. A type of task is set up with some question or a problem, and any action made with the intention to resolve the task is a technique corresponding to that type of task. Technology is a set of concepts, procedures and assertions used to explain, control, verify and construct techniques and theory is a coherent and abstract system of notions, definitions, models, theorems and proofs that upholds technology. The same technology can withhold several different practical blocks, which altogether make a *local praxeology*; those are connected under the same theory into *regional praxeology* and their systematization constructs *global praxeologies* such as algebra, geometry, arithmetic, calculus, analysis and other (Barbé et al., 2005; Bosch & Gascón, 2014). In the ATD a body of knowledge is a praxeology, a part of a praxeology or a set of composite praxeologies that is deemed important by some institution (Chevallard, 2007; Chevallard & Sensevy, 2014).

2.2. Previous research

In the scientific literature, research on the notions of asymptote and asymptotic behaviour are scarce, and no textbook analysis regarding those notions was found. More research was done on limit and tangent, which are related to the notion of an asymptote.

Kidron (2011) investigated how one student's concept image of an asymptote changes when graphing various functions. The initial student's image of a horizontal asymptote was that a graph is approaching and not touching its asymptote when function tends to infinity. Student graphed two rational functions, and a function that intersects a line an infinite number of times. In the latter case student used graphical, numerical and analytical approach to solve the task and concluded that graph does not have to steadily approach its asymptote but function values at infinity approach (and can be equal to) some value that determines the horizontal asymptote (Kidron, 2011). Yerushalmy (1997) investigated how students develop definitions of vertical, horizontal and oblique asymptote when graphing rational functions. For the purpose of this research students used software which displays a function graphically, algebraically and arithmetically, i.e. rational function as a quotient of two polynomials. Due to such representations students differed removable discontinuity versus vertical asymptote at a point for which the function is not defined, established the relation between asymptote equation (horizontal or oblique) and the quotient of the polynomials in the numerator and denominator of the rational function, and used informal infinitesimal argument on the value of the remainder to validate that relation.

Zarhoutti, Mouradi and Maroufi (2014) investigated how students' perceive that two steps of the didactical transposition i.e. textbooks and teacher's interpretation, affect their performance. Students were supposed to manipulate in the graphical register and convert between graphical and algebraic register, with regard to the *praxis* of graphing function and interpreting function properties using derivatives and limits. Authors discussed graphical misrepresentation of changing concavity, tangency and approaching asymptote, is due to the lack of relevant examples in textbooks and classes, or the lack of formal discourse. Further, students were unsuccessful in (1) conversion from the evaluated limit to graphing asymptotic direction or asymptote, (2) conversion from the *praxis* of exploring derivatives to graphing inflection point due to complexity of that *praxis*, and (3) interpreting function domain or limit value from the given function graph due to typical *praxis* of converting from algebraic to graphical register. Zarhoutti et al. (2014) concluded that students' difficulties originate from the results of the didactical transposition i.e. the textbooks and teacher's interpretation, which neglect diversity of examples and representations of functions and emphasize punctual properties at the expense of global function properties.

Biza, Christou and Zachariades (2008) and Biza and Zachariades (2010) investigated concept images of a tangent line from students exiting secondary and entering university education. Their results showed students could be classified in three groups; those with *Analytical Local*, *Intermediate Local* and *Geometrical Global* perspective of tangent line. In that order students (1) consider local, analytical properties of a tangent line in the context of differential calculus, (2) apply global properties to the neighbourhood of a point, e.g. a curve and a tangent line can have common points but not in the neighbourhood of a tangency point, (3) consider only global properties of a tangent line, e.g. a curve must be in the same half-plane determined by the tangent line (Biza et al., 2008; Biza & Zachariades, 2010). Biza and Zachariades (2010) found that the tasks in the textbooks and exams require mostly knowing and applying formulae and rarely representing concepts in various forms and solving problems to develop concept image.

Within the ATD several textbook analysis, on different educational levels and different body of knowledge, are conducted and in general results show that available praxeologies are disconnected and incomplete, they do not change and evolve to increase in complexity, discourse is given as ‘transparent rule’ and separated from practical activities or unavailable to students (Barbé et al., 2005; Bergé, 2007; García & Ruiz Higueras, 2005; González-Martín, Giraldo, & Souto, 2013; Hardy, 2009). Hardy (2009, 2011) investigated how models of evaluating limit vary depending on the position of an individual in a institution. Students developed different models than the teachers; in particular they chose techniques that were routinely used to solve given task or task that appears similar, and the technology to justify the chosen technique is the consequence of a institutional norm (Hardy, 2009, 2011). Further, when dealing with non-routine tasks students applied various techniques and justified their *praxis* (Hardy, 2011). Barbé et al. (2005) investigated how limit is formulated in knowledge to be taught in the Spanish secondary education and what kind of restrictions hinder the development of this body of knowledge. They examined curriculum and textbooks and found *praxis* of evaluating limit with no discourse and *logos* of limit definition with no related practical activities, and that limit of a function quotient is never used to discuss their asymptotic behaviour (Barbé et al., 2005). Further, Barbé et al. (2005) concluded curriculum and textbook impede teachers to set up complete and connected praxeological organization of taught knowledge.

3. Methodology

We have conducted a praxeological analysis of the two most common series of gymnasium mathematics textbooks in Croatia from different authors and publishers, used in the period 2007. – 2011., containing the teaching content related to asymptote and asymptotic behaviour. That kind of detailed and careful examination results in a list of all available praxeologies relevant to the body of knowledge asymptote. First, discursive elements in textbooks are identified and organised in corresponding technologies. For each technology all available definitions, properties and theorems are named and listed (Bergé, 2007; González-Martín et al., 2013).

Irrelevant to the format of the observed textbook block, every request to perform something in order to gain some kind of a solution, an answer or a result, is recognized as a task (Bergé, 2007; González-Martín et al., 2013; Hardy, 2009). These requests can be of the kind: to determine, calculate, solve, draw, show, explore, recognize or other. Each recognized task is complemented with a distinct technique. Technique is determined as being applicable to such type of task, or as evident from the solved examples, notes and solutions of the exercises given later in the textbook (Bergé, 2007; Hardy, 2009, 2011). A point-praxeology is made of a single type of task, technique and corresponding technology, if it is evoked. A practical block is associated to *logos* if and only if it is conducted or justified with reference to some discursive content; otherwise it is an incomplete, practical praxeology. Theoretical component is determined by structure and organisation of the textbook topics.

The frequency with which each identified point-praxeology appears in the observed textbooks, is determined (Barbé et al., 2005; González-Martín et al., 2013). A task that contains several items of the same type was counted as one instance of such point-praxeology. Further, all types of tasks and techniques recognized in the textbooks are listed (Barbé et al., 2005; González-Martín et al., 2013), the point-praxeologies are systematized and organized into local praxeologies whenever possible and a graphical representation to show relations between praxeologies and towards discourse and the body of knowledge asymptote is created (Barbé et al., 2005; González-Martín et al., 2013).

4. Results

In the context of the larger research study we intended to demonstrate the state of the knowledge to be taught in Croatian upper secondary mathematics education. Hence, we did not aim to compare the chosen textbooks or textbook sets in its own right, rather to explore all items that they include related to the asymptote and asymptotic behaviour. Teaching content related to the body of knowledge asymptote is present in the gymnasium textbooks for the second, third and fourth grade. Relevant content is found in several textbook units and available discourse is organised accordingly into technologies:

- In the second grade textbook within the section *Exponential and logarithmic function*, a just called technology includes function graph and properties.
- In the third grade textbooks within the sections related to trigonometry, four technologies are differentiated: function definition and representation, function properties, trigonometric identities and function graph.
- In the third grade textbooks within the section *Conics*, three technologies are differentiated: hyperbola definition, hyperbola equation and relationship between a line and a hyperbola.
- In the fourth grade textbook within the sections *Sequences,* containing *Limit of a sequence*, *Function*, containing *Limit of a function* and *Derivative*, containing *Function graph*, four technologies are differentiated: sequence properties, function properties, limiting behaviour (of both sequence and function) and differential calculus including function graph technology.

Asymptote and asymptotic behaviour are a part of *logos* for the function, function graph, sequence and curve properties and a part of graphing *praxis*. They can also be a discourse for function and sequence values, especially their limit and limiting behaviour. Each textbook unit in both textbook sets is consisted of practical and theoretical blocks and ends with an exercise block. Total number of identified point-praxeologies within each textbook unit is given in Table 1.

Table 1. Frequencies of point-praxeologies in the textbook units for each textbook set.

Textbook units		Point-prax. in total	Tasks in total	Mean for point-p.	Mode for point-p.	Min. for point-p.	Lower quartile for p-p.	Median for point-p.	Upper quartile for p-p.	Max. for point-p.
First textbook set	Exp.&log. function	45	134	2,98	1	1	1	2	4	11
	Trigon. function	23	75	3,26	1	1	1	3	5	7
	Conics	26	77	2,96	1	1	1	2,5	3,75	11
	Sequence	38	122	3,21	1	1	1	2	4	12
	Function	26	67	2,58	2	1	1,25	2	3	7
	Function graph	24	98	4,08	1	1	1	2	4	35
Second textbook set	Exp.&log. function	52	255	4,9	1	1	1	3	6,25	31
	Trigon. function	18	97	5,39	5	1	3,25	5	7	12
	Conics	20	43	2,15	1	1	1	1,5	2,25	7
	Sequence	44	146	3,32	1	1	1	2	4,25	11
	Function	37	126	3,41	1	1	1	2	5	14
	Function graph	16	56	3,5	3	1	1,75	3	4,25	12

In this section we will present praxeological organisation for the first two items, i.e. textbooks with sections related to exponential and logarithmic function and tangent and cotangent function. This limitation to the display of the results of the performed textbook analysis will not disturb the results of the paper since the praxeological organisations of all textbooks are discussed in the next section. We chose those examples as they provide both typical and specific information about the means of setting up discursive content, the types of available praxeologies, relations among *praxis* and *logos*, interrelation, complexity and coherence of praxeologies included and motivation or application of available praxeologies for solving various types of mathematical or real-life context tasks. Other results are extensively elaborated in Katalenić (2017).

4.1. Praxeological organization in the first textbook set

4.1.1. Exponential and logarithmic function

Definition of an exponential function is given via expanding the set of allowed exponents for powering positive number, different from 1, from the set of rational numbers to the set of real numbers. A graph of an exponential function is closely related to tabular representation of function. Available technique for function graphing is connecting plotted points in coordinate plane by a smooth curve. The graphing *praxis* serves as a support for the *logos* on function domain, graph symmetry, relationship between function values with different basis and argument, etc.

Function behaviour is described as rapid growth for positive and rapid decrease for negative abscissas, where it is nesting toward negative part of the x-axis. That line is named asymptote of the exponential function. For exponential function with base less than 1 it is written that it is decreasing and its asymptote is the positive part of the x-axis. As basic properties of an exponential function there are listed: function domain and range, properties inherited from powering, monotonicity, symmetry for functions with reciprocal bases and y-intercept.

The notion of logarithm is motivated by *praxis* of finding an exponent for which the given power value is achieved by interpreting function graph. Various definitions, description and properties of a logarithm are given. Graph of a logarithmic function motivates discourse of symmetry between function graphs and such functions are then called inverse. From illustrations of a logarithmic function vertical asymptote in y-axis and x-intercept of logarithmic function are recognized; and in comparison to the graph of the exponential function, function domain and monotonicity. A demonstrated technique for solving inequalities is modifying the expression to an algebraically equivalent inequality in the terms of an argument and it is supported by function monotonicity.

Applications of an exponential and a logarithmic function make separate textbook unit. Function $f(t) = f_0 \cdot e^{kt}$ is noted to describe growth processes and logistic function is presented as a model of bounded growth. It is described and illustrated that the values of a logistic function $f(x) = \dfrac{a}{a + b \cdot e^{-kx}}$ increase and approach upper bound when $k > 1$ and vice versa.

The list of available tasks, techniques and discourse as well as graphical representation of praxeologies is given on Figure 2. Amongst most frequent point-praxeologies are those of evaluating function, algebraic expressions, power or logarithm, with no discursive block. There are two frequent complete praxeologies: 'comparing values of an exponential function with reference to the monotonicity' and 'calculating with numbers of large magnitude with reference to the relationship between power and logarithm and scientific number notation'. Other complete praxeologies are: 'exploring logarithm by arguing there is no solution due to the domain of a logarithmic function', 'evaluating exponential function for real argument by approximations with reference to monotonicity' and 'graphing a function via connecting plotted points and describing function behaviour', with emphasis on graph symmetry and monotonicity. The relevant techniques are algebraic manipulation utilizing some property or formulae from *logos*, for instance in the tasks 'to evaluate number with large magnitude' or 'to solve an inequality'.

Other mathematical tasks implement graphing *praxis* in order to solve an equation or to find the number of solutions of an equation, and evaluating *praxis* in order to determine the number of digits in a given number and the number of prime numbers smaller than a given number or to calculate with numbers of large magnitude. The majority of tasks with real-life context require evaluating algebraic expression with power or logarithm which models a given situation. A single task requires graphing the given model and there are five instances of the task 'to determine an expression which models a given situation'.

$\mathbf{T}_{A1}$ Evaluate algebraic expression with power and/or logarithm.

$\mathbf{T}_{A2}$ Evaluate logarithm or argument when the floor function value is given.

$\mathbf{T}_{EA}$ Find an argument, when the exponential function graph or value is given.

$\mathbf{T}_{EG}$ Graph an exponential function.

$\mathbf{T}_{EK}$ Graph a composition of an exponential function with a linear or absolute value function.

$\mathbf{T}_{EM}$ Evaluate power with large magnitude.

$\mathbf{T}_{ER}$ Evaluate exponential function for a real argument.

$\mathbf{T}_{EU}$ Compare exponential function values for different arguments or bases.

$\mathbf{T}_{EV}$Evaluate power or exponential function value for different arguments.

$\mathbf{T}_{LA}$ Find the base or an argument, when logarithm function value is given.

$\mathbf{T}_{LG}$ Graph a logarithmic function.

$\mathbf{T}_{LK}$ Graph a composition of a logarithmic function with absolute value function.

$\mathbf{T}_{LM1}$ Find an argument when logarithm value of large magnitude is given.

$\mathbf{T}_{LM2}$ Evaluate logarithm function when argument with small magnitude is given.

$\mathbf{T}_{LV}$ Evaluate logarithm function for different arguments.

$\mathbf{T}_{NJ}$ Solve an exponential or logarithmic inequality.

$\mathbf{Q}_{M1}$ Solve an equation or find the number of solutions of a given equation.

$\mathbf{Q}_{M2}$ Evaluate an algebraic expression which models a mathematical context situation.

$\mathbf{Q}_{R1}$ Evaluate an algebraic expression with power or logarithm which models a real-life context situation.

$\mathbf{Q}_{R2}$ Determine real-life context value with large or small magnitude.

$\mathbf{Q}_{R3}$ Determine real-life context value.

τ_A Algebraic manipulation

τ_{AP} Approximation

τ_{LG} Recognizing exponent with regard to logarithm definition

τ_M Modelling

τ_O Interpreting from graphical representation

τ_{PA}Determining algebraically equivalent inequality in the terms of the function argument

τ_R Calculating

τ_{TO} Drawing a curve through corresponding plotted points

τ_{TR} Transforming basic function graph

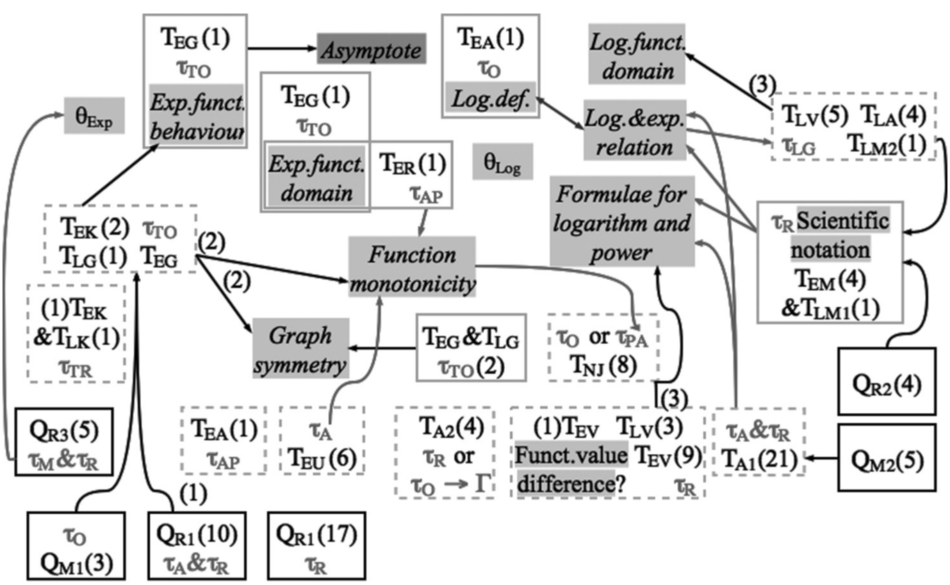

Figure 2. Graphical representation of the praxeological organization of the textbook unit Exponential and logarithmic function in the 1st textbook set.

4.1.2. Trigonometry

Tangent and cotangent functions are defined on a unit circle. Discourse is extensively elaborated: how tangent function is not defined for $t = \dfrac{\pi}{2}$ since the corresponding line is parallel to the tangent line, also that when approaching this value such line gets steeper and tangent values increase and are unbounded. Such discourse is relevant for function graph technology. Function definition technology supports techniques of finding an argument or function values on the unit circle and solving simple trigonometric equation and inequalities. Trigonometric identities technology provides formulae which are used for *praxis* and *logos* throughout trigonometry related textbook units. For instance, cotangent is evaluated as reciprocal to tangent value, or identities are used to justify that tangent is an odd function within the property technology. Function property technology is demonstrated on unit circle.

The function graph technology heavily relies on components from other technologies. Definition technology is applied for a constructive graphing technique of the tangent function (Figure 3). Tangent function graph is described as approaching the line $x = \dfrac{\pi}{2}$ which is then called vertical asymptote of tangent function. In detail, tangent function raises and approaches the asymptote when argument increases toward the value $\dfrac{\pi}{2}$, for which tangent is not defined. Cotangent function is graphed by transforming the tangent function graph with regard to a known trigonometric identity. Function graph technology is noted to include: function domain, x-intercepts, function range, monotonicity, function behaviour in the domain break and asymptotes.

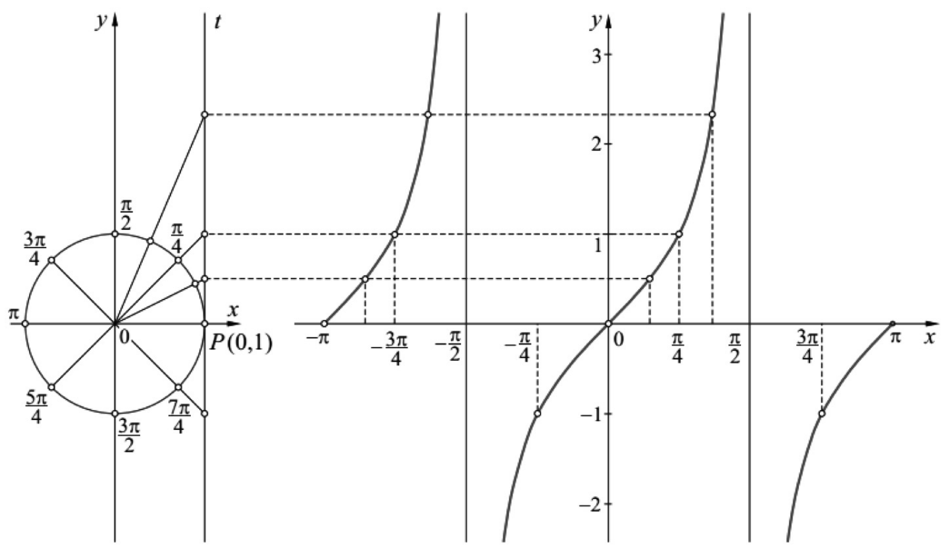

Figure 3. Constructive graphing technique of the tangent function via transferring lengths from unit circle to ordinate in coordinate plane from 1st textbook set.

The list of available tasks, techniques and discourse as well as graphical representation of praxeologies is given on Figure 4. Amongst most frequent point-praxeologies are those of evaluating function and finding argument or function value when other trigonometric function value is given, with no discursive block. There often occur praxeologies: ‘solving trigonometric equation or inequality’ and ‘finding an argument when function value is given’ with regard to function definition technology, and ‘graphing the composition of trigonometric and linear function via transforming basic trigonometric graph’. Complete praxeologies are almost exclusively related to graphing *praxis* and function graph technology.

$\mathbf{T}_{A1}$ Evaluate tangent or cotangent function when other trigonometric function value is given.

$\mathbf{T}_{A2}$ Determine boundary values of cotangent function when boundary values of sine function are given.

$\mathbf{T}_{A3}$ Determine trigonometric function value when tangent or cotangent function value for the same argument is given.

$\mathbf{T}_{KG}$ Graph a cotangent function.

$\mathbf{T}_{TG}$ Graph a tangent function.

$\mathbf{T}_{PG}$ Recognize the graph of a composition of tangent or cotangent function with a linear function.

$\mathbf{T}_{TA}$ Determine an argument when tangent or cotangent function value is given.

$\mathbf{T}_{TJ}$ Solve an equation with tangent and/or cotangent function.

$\mathbf{T}_{TK}$ Graph a composition of tangent or cotangent function with linear or absolute value function.

$\mathbf{T}_{TN}$ Solve an inequality with tangent or cotangent function.

$\mathbf{T}_{TV}$ Evaluate tangent or cotangent function.

τ_A Algebraic manipulation

τ_K Constructing

τ_O Interpreting from graphical representation

τ_R Calculating

τ_{TR} Transforming basic function graph

τ_{SV} Drawing a curve with regard to some properties

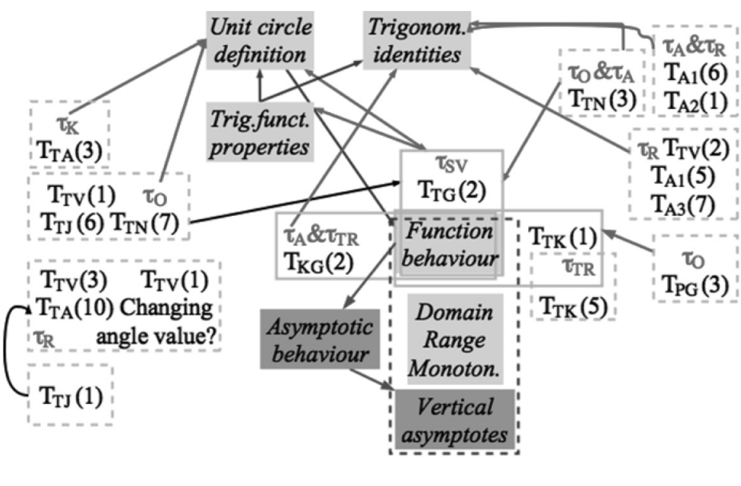

Figure 4. Graphical representation of the praxeological organization of the textbook units related to trigonometric function in the 1st textbook set.

4.2. Praxeological organization in the second textbook set

4.2.1. Exponential and logarithmic function

An introduction to the unit is the *praxis* of exploring population changes through time period. The constant quotient value gives an algebraic expression with power which models a given real-life context situation and this discourse is presented as a prominent property of an exponential function. Using calculator to evaluate function is appropriate technique for some tasks, for example exponential function with real argument. The graphing *praxis* is used for establishing discourse of expanding function domain, choosing values for function basis, recognizing function domain, range, y-intercept and monotonicity. It is described that graph of a function $f(x) = b^x$, regardless of the base value, is arbitrarily approaching x-axis, whence it is an asymptote, which is also described as a tangent at a point in the infinity. Exponential functions with reciprocal basis are noted to have symmetric graphs and equal function values for opposite arguments, and that discourse is validated with power properties.

The context of population change calls on the *praxis* of finding the exponent for a given power value when the result is not an integer. Successive approximation is given as an appropriate technique and that *praxis* is sound because the searched value obviously exists in the given context. Existence of logarithm as a solution of an exponential equation is founded on the properties of an exponential function interpreted from an illustration. Logarithmic function is motivated with *praxis* of finding exponent for given exponential function value which is demonstrated with adjoint points on number lines representing domain and range of an exponential function or, in adverse, its inverse, logarithmic function (Figure 5). Graphing *praxis* gives rise to discourse: defining logarithmic function, recognizing function range, x-intercept, monotonicity and relation with exponential function. The ordinate axis is an asymptote of the logarithmic function graph, which, for any base, approaches but does not intersect the line. Techniques of solving inequalities are demonstrated: (1) in general algebraic manipulation to reach equivalent inequalities in the terms of an argument, (2) for simple inequalities, finding values with regard to monotonicity or interpreting graphical representation and (3) for logarithmic inequalities, interpreting from number line representation (Figure 5). Tasks with real-life context are found within entire textbook unit, but logarithmic scale and application of exponential and logarithmic functions make separate textbook units. It is advised to use an expression of the form $a \cdot b^{kt}$, when the considered values have constant quotient or in other cases to use growth function with general form $N(t) = N_0 e^{kt}$. Interest account formulae are derived.

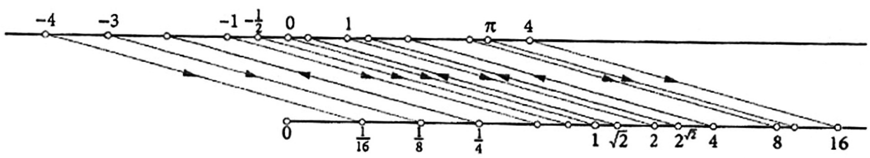

Figure 5. Technique of joining domain and codomain points for exponential/logarithmic function in 2nd textbook set.

The list of available tasks, techniques and discourse as well as graphical representation of praxeologies is given on Figure 6. Amongst most frequent point-praxeologies are those of evaluating function, algebraic expressions, power or logarithm, with no discursive block. Graphing praxeology is often found as a practical block or complete praxeology. The task 'to solve an inequality' is frequently found and solved with three different available techniques. A frequent complete praxeology is 'describing function behaviour with regard to graphing *praxis*', especially symmetry, graph transformation or function domain, range and monotonicity. Other complete praxeologies are: 'recognizing function from given graphical representation with regard to function monotonicity', 'comparing power values for large magnitudes via algebraic manipulations with regard to logarithm properties and function monotonicity' and those that are part of discourse or motivation. There is a single instance of graphing logarithm with discourse of establishing connection between vertical asymptote and function domain. Most of the relevant techniques are algebraic manipulation utilizing some property or formulae from *logos*. Technique of successive approximation has no visible discursive support and technique of interpreting the number line representation is relevant to logarithm definition *logos*.

$\mathbf{T}_{A1}$ Evaluate algebraic expression with power and/or logarithm.

$\mathbf{T}_{A2}$ Determine between which two integers is the value of a given logarithm.

$\mathbf{T}_{A3}$ Compare power values of large magnitude.

$\mathbf{T}_{EA}$ Find an argument, when the exponential function graph or value is given.

$\mathbf{T}_{EG}$ Graph an exponential function.

$\mathbf{T}_{EK}$ Graph a composition of an exponential function with a linear function.

$\mathbf{T}_{EV}$ Evaluate power or exponential function value for a given argument.

$\mathbf{T}_{LG}$ Graph a logarithmic function.

$\mathbf{T}_{LK}$ Graph a composition of a logarithmic function with a linear or absolute value function.

$\mathbf{T}_{LV}$ Evaluate logarithm function for a given argument.

$\mathbf{T}_{NJ}$ Solve an exponential or logarithmic inequality.

$\mathbf{T}_{PG}$ Recognize an exponential or a logarithmic function graph.

$\mathbf{Q}_{M1}$ Find the number of solutions of a given equation.

$\mathbf{Q}_{M2}$ Determine function domain.

$\mathbf{Q}_{R1}$ Evaluate an algebraic expression with power or logarithm which models a real-life context situation.

$\mathbf{Q}_{R2}$ Determine value for which a given value is attained, for an algebraic expression with power which models a real-life context situation.

$\mathbf{Q}_{R3}$ Represent graphically an algebraic expression with power which models a real-life context situation.

$\mathbf{Q}_{R4}$ Determine the value range for an algebraic expression with power or logarithm, which models a real-life context situation.

$\mathbf{Q}_{R5}$ Find the maximum value for an algebraic expression with power which models a real-life context situation.

$\mathbf{Q}_{R6}$ Find the coefficients of an algebraic expression with power or logarithm, which models a variable quantity.

$\mathbf{Q}_{R7}$ Determine an algebraic model for a variable quantity.

τ_A Algebraic manipulation

τ_{AP} Approximation

τ_{LG} Recognizing exponent with regard to logarithm definition

τ_M Modelling

τ_O Interpreting from graphical representation

τ_{PA} Determining algebraically equivalent inequality in the terms of the function argument

τ_R Calculating

τ_{TO} Drawing a curve through corresponding plotted points

τ_{TR} Transforming basic function graph

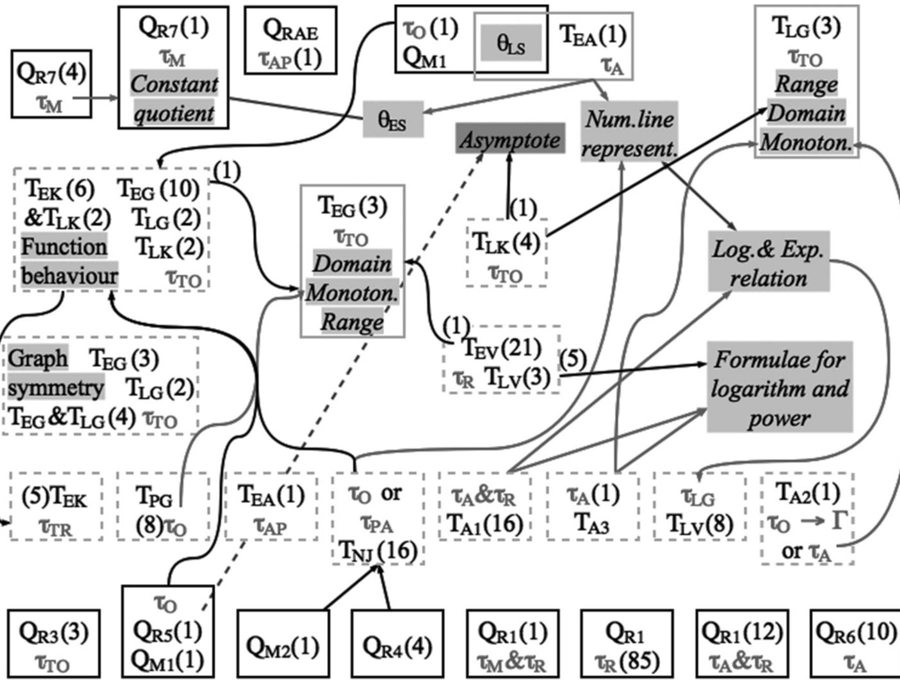

Figure 6. Graphical representation of the praxeological organization of the textbook unit Exponential and logarithmic function in 2nd textbook set.

Other mathematical tasks implement graphing *praxis* or the praxeology 'solving logarithmic inequality' in order to find the number of solutions of a given equation or to determine the domain of a given function. The majority of tasks with real-life context advert to graphing praxeology or 'solving inequalities'. A single task requires the *praxis* 'recognizing function limiting value via interpretation of a graphical representation' where horizontal asymptote could be recognized. The tasks with real-life context are mainly described with an expression containing power or logarithm, and it is required to evaluate or find coefficient of the given model. Other tasks require modelling with regard to constant quotient *logos*, and only one instance of a task should be modelled in other manner.

4.2.2. Trigonometry

Tangent and cotangent functions are defined as a quotient of sine and cosine functions, which are defined on a unit circle. Therefore the values for which those functions are undefined depend on the denominator values. Further, tangent and cotangent function are graphically interpreted on a unit circle, which is justified with regard to the segment of some right triangles. All of that discourse makes the function definition and representation technology. Interpreting values on the unit circle is a technique used for evaluating and solving equations and inequalities. The function properties technology includes function domain, oddness and periodicity, which are supported by definition and representation technology. Trigonometric identities technology supports function evaluation *praxis*, for instance, cotangent function is not available on calculator and it is evaluated as reciprocal to tangent value.

For sine and cosine, function graph makes the *logos* for evaluation *praxis* and function properties. The tangent and cotangent function graphs are represented for first time in the summary section and function graph technology is presented in a separate textbook unit. Function representation technology supports interpretation of tangent values, on unit circle, as unboundedly increasing from zero to infinity when the argument increases from zero to $\frac{\pi}{2}$, for which tangent function is undefined. Constructive graphing technique of the tangent function is applied in the same manner as in Figure 3. The resulting graph is extended with regard to the function properties technology. The lines $x = (2k+1) \cdot \frac{\pi}{2}, k \in \mathbf{Z}$, are recognized as asymptotes, since the function graph approaches, but doesn't intersect them. Cotangent function is graphed in the same manner as tangent function graph and its asymptotes are the lines $x = k \cdot \pi, k \in \mathbf{Z}$.

The list of available tasks, techniques and discourse as well as graphical representation of praxeologies is given on Figure 7. Amongst most frequent point-praxeologies are those of evaluating function or argument using calculator or the common values of trigonometric functions, with no discursive block or addressing function properties: oddness and periodicity. There often occur praxeologies: 'evaluating function when other trigonometric function value is given using trigonometric identities' and 'solving trigonometric equations via interpreting on a unit circle'. There are two complete praxeologies: 'constructing point on the unit circle when function value is given with regard to function representation' and 'graphing function with discussing function graph shape, including asymptotes'. The function definition technology gives rise to the technique of evaluating quotient for tangent and cotangent function values. Mathematical task 'to solve an equation or an inequality' require finding an argument for given function value. Corresponding techniques are evaluating on a calculator and interpreting from the unit circle or from the function graph.

$\mathbf{T}_{A1}$ Evaluate tangent or cotangent function when other trigonometric function value is given.

$\mathbf{T}_{A2}$ Determine trigonometric function value when tangent or cotangent function value is given.

$\mathbf{T}_{PG}$ Recognize the graph of a composition of tangent or cotangent function with a linear function.

$\mathbf{T}_{TA}$ Determine an argument when tangent or cotangent function value is given.

$\mathbf{T}_{TG}$ Graph a tangent or cotangent function.

$\mathbf{T}_{TJ}$ Solve an equation with tangent and/or cotangent function.

$\mathbf{T}_{TK}$ Graph a composition of tangent or cotangent function with linear function.

$\mathbf{T}_{TN}$ Solve an inequality with tangent and/or cotangent function.

$\mathbf{T}_{TV}$ Evaluate tangent or cotangent function.

τ_A Algebraic manipulation

τ_K Constructing

τ_O Interpreting from graphical representation

τ_R Calculating

τ_{TR} Transforming basic function graph

τ_{SV} Drawing a curve with regard to some properties

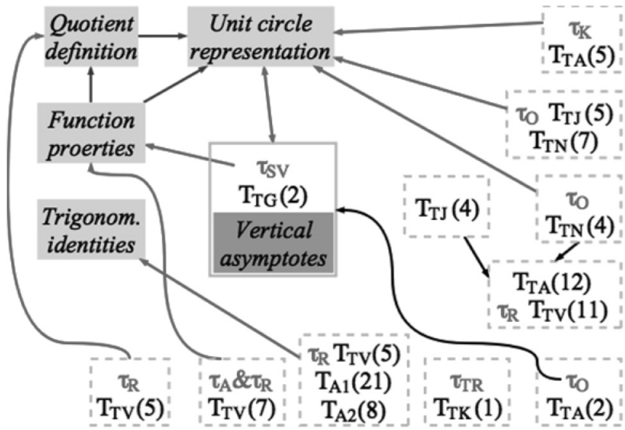

Figure 7. Graphical representation of the praxeological organization of the textbook units related to trigonometric function in 2nd textbook set.

5. Discussion and conclusion

In both textbook sets praxeological organisation is incoherent, available praxeologies are mainly practical and unconnected blocks, even when the corresponding *logos* is evident. Similar results can be found in Barbé et al. (2005), Biza & Zachariades (2010) and Hardy (2011). Available complete praxeologies are mainly those of recognizing function properties from graphing *praxis*, describing sequence and function behaviour at infinity or near a singularity from evaluating *praxis*, etc. Discourse is mainly irrelevant and separated from *praxis*, except when properties and formulae from *logos* are used for manipulating algebraic expressions or equations, with no relevant application or discourse. In the second textbook set the bodies of knowledge are motivated with real-life context problems, but in general bodies are presented without proper motivation or requirement. Claims, formulae, procedures and properties are introduced as given rules instead of as a result of some *praxis*, and discourse to support those is not available. On several occasions it is not evident whether given *logos* is a definition or a property and whether a performed *praxis* is a proof or an example of some claim. Formal mathematical discourse is characteristic to the units related to limits in both textbook sets; however it is more emphasized in the first textbook set, whereas in the second textbook set the emphasis is on examples and counterexamples.

Interpreting graphical representation and exploring first derivative are mainly implemented in order to solve other type of tasks, i.e. to prove a given inequality, to find the number of solutions of a given equation, to determine set of points satisfying some geometrical property, etc. Problems with real-life context are mainly a priori modelled with an algebraic expression and require evaluating or manipulating, rarely graphing or interpreting and just once exploring limiting behaviour of a given model, which could lead to recognizing a horizontal asymptote. Tasks that require examining function behaviour are set on extrema and extrema values, and solved by investigating first derivative.

Asymptote is introduced as a function property when initially graphing elementary function. For tangent and cotangent function it is extensively elaborated with regard to function values and other properties and it makes a prominent part of function behaviour *logos*. For elementary functions asymptote is described informally, using the expressions “curve is nesting toward the line”, “line is a tangent to the curve at infinity” or “line is never touching the curve”.

Asymptote of a hyperbola is accessed by manipulating the line equation. Discourses of distance from a point on a hyperbola to its asymptote, coordinate values of a point on a hyperbola or using asymptote for graphing a hyperbola are mentioned seldom or never. The illustrations and discourse regarding the line and conic relationship *logos* are incomplete. A task is given where no technological and theoretical content is available to resolve it.

Asymptote and asymptotic behaviour are not mentioned in the textbook unit related to function limit in both textbook sets, even when the given example or illustration alludes to that body of knowledge. Actually, those bodies of knowledge are located within textbook unit related to applications of derivatives to function graphing. In the first textbook set the discourse is more informative and formal. The *logos* includes definition, limit formulae and rules to accordingly find asymptotes of a rational function. However, in the limit condition $\lim_{x \to a} f(x) = \infty$ for vertical asymptote, the origin of the argument a is not mentioned. In the second textbook set, only techniques of finding asymptotes are given, with emphasis on the equations for oblique asymptotes. Also, vertical asymptotes are (incorrectly) related solely to the denominator null-value. In the both textbook sets discourse supporting formulae or techniques is unavailable.

Graphing *praxis* is implemented as following: exponential and logarithmic functions graph by plotting corresponding points; tangent and cotangent function graph by drawing a curve with regard to its properties or by transforming basic function graph; hyperbola in the first textbook set by drawing a curve with regard to its properties or in the second textbook set by plotting corresponding points; polynomials and rational functions graph by drawing a curve with regard to its properties explored within calculus (limits, derivatives, etc.). It is advised to determine the asymptotes when graphing a rational function, and for the given rational functions, relations between the graph and its asymptotes are diverse but not systematically or explicitly discussed. Graphing *praxis* is often complemented with function behaviour *logos*. However, discourse mainly concerns function domain, monotonicity or graph symmetry and rarely limiting behaviour or asymptotes. The relationship between asymptotic behaviour and other function properties is almost never found in the discursive content or task realization. Asymptotes and asymptotic behaviour are not related to other relevant bodies of knowledge, such as convergent sequences or logistic function.

The results show that the asymptote is a composite body of knowledge consisted of several praxeologies for which it is a part of practical or theoretical component. However, this body of knowledge is not fully utilized and realized in the considered textbooks. Therefore, our suggestions for the teaching practice are that: (1) asymptotes and all relevant properties of a function or a curve should be emphasized and implemented in various modes of representation, i.e. using asymptotes for graphing *praxis*; (2) relationships among properties of a function or a curve should be established and adverted to, i.e. distinguishing types of singularities, limit value at a point and vertical asymptotes; (3) a technological-theoretical support should be provided for discursive content, i.e. justify asymptotes equation with regard to definition and limits; (4) graphing techniques should be chosen as most appropriate for a given function or curve; (5) evaluating limits should be complemented with appropriate discourse or implemented for problem solving, i.e. describing asymptotic behaviour of given functions or curves.

The analysed textbooks do not take into account the results of epistemological research, scholarly knowledge or pedagogical standards. González-Martín et al. (2013) reported similar findings in their research of the set of real numbers as body of knowledge in the Brasilian secondary school textbooks. The number of simple, isolated and practical praxeologies should be reduced and those should be organized into more coherent and complex praxeologies. Textbooks should be reorganized in a way to shift the focus from calculating, evaluating and manipulation algebraic expressions and equations to representing graphically, engaging properties and interpreting behaviour to solve problems with mathematical, scientific or real-life context.

References

[1] Barbé, J., Bosch, M., Espinoza, L., & Gascón, J. (2005), *Didactic Restrictions on the Teacher's Practice: The Case of Limits of Functions in Spanish High Schools*, Educational Studies in Mathematics, 59(1/3), 235–268.

[2] Bergé, A. (2007), *The completeness property of the set of real numbers in the transition from calculus to analysis*, Educational Studies in Mathematics, 67(3), 217–235.

[3] Biza, I., Christou, C., & Zachariades, T. (2008), *Student perspectives on the relationship between a curve and its tangent in the transition from Euclidean Geometry to Analysis*, Research in Mathematics Education, 10(1), 53–70.

[4] Biza, I., & Zachariades, T. (2010), *First year mathematics undergraduates' settled images of tangent line*, The Journal of Mathematical Behavior, 29(4), 218–229.

[5] Bosch, M., & Gascón, J. (2006), *Twenty-five years of the didactic transposition*, Bulletin of the International Commission on Mathematical Instruction, 58, 51–65.

[6] Bosch, M., & Gascón, J. (2014), *Introduction to the Anthropological Theory of the Didactic (ATD)*, In A. Bikner-Ahsbahs & S. Prediger (Eds.), Networking of Theories as a Research Practice in Mathematics Education (pp. 67–83), Springer International Publishing.

[7] Chevallard, Y. (2007), *Readjusting Didactics to a Changing Epistemology*, European Educational Research Journal, 6(2), 131–134.

[8] CHEVALLARD, Y., & BOSCH, M. (2014), *Didactic Transposition in Mathematics Education*, In S. Lerman (Ed.), Encyclopedia of Mathematics Education (pp. 170–174), Springer Netherlands.

[9] CHEVALLARD, Y., & SENSEVY, G. (2014), *Anthropological Approaches in Mathematics Education, French Perspectives*, In S. Lerman (Ed.), Encyclopedia of Mathematics Education (pp. 38–43), Springer Netherlands.

[10] GARCÍA, F. J., & RUIZ HIGUERAS, L. (2005), *Mathematical praxeologies of increasing complexity*, In Proceedings of the Fourth Congress of the European Society for Research in Mathematics Education (pp. 1645–1654), Sant Feliu de Guíxols, Spain, Retrieved from `http://fractus.uson.mx/Papers/CERME4/Papers%20definitius/13/GarciaRuiz.pdf`.

[11] GONZÁLEZ-MARTÍN, A. S., GIRALDO, V., & SOUTO, A. M. (2013), *The introduction of real numbers in secondary education: an institutional analysis of textbooks*, Research in Mathematics Education, 15(3), 230–248.

[12] HARDY, N. (2009), *Students' perceptions of institutional practices: the case of limits of functions in college level Calculus courses*, Educational Studies in Mathematics, 72(3), 341–358.

[13] HARDY, N. (2011), *Students' praxeologies of routine and non-routine limit finding tasks: Normal vs. mathematical behaviour*, In M. Bosch, J. Gascón, A. Ruiz Olarria, M. Artaud, A. Bronner, Y. Chevallard, . . . M. Larguier (Eds.), Un panorama de la TAD: An overview of ATD (Vol. 10, pp. 349–366), Bellaterra: Centre de Recerca Matemàtica.

[14] KATALENIĆ, A. (2017), *Asymptote and asymptotic behavior – didactic transposition in general secondary education in Croatia* (Doctoral thesis), Faculty of Science, Department of Mathematics, Zagreb.

[15] KIDRON, I. (2011), *Constructing knowledge about the notion of limit in the definition of the horizontal asymptote*, International Journal of Science and Mathematics Education, 9, 1261–1279.

[16] YERUSHALMY, M. (1997), *Reaching the Unreachable: Technology and the Semantics of Asymptotes*, International Journal of Computers for Mathematical Learning, 2(1), 1–25.

[17] ZARHOUTI, M. K., MOURADI, M., & MAROUFI, A. E. (2014), *The teaching of the function at high school: The graphic representation of a function at first year, section experimental sciences*, IOSR Journal of Research & Method in Education, 4(3), 56–65.

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Asimptota kao objekt znanja za poučavanje u udžbenicima za srednju školu u Hrvatskoj

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Sažetak. Asimptota i asimptotsko ponašanje objekti su matematičkog znanja prisutni u elementarnoj, teorijskoj i primijenjenoj matematici. Neki aspekti tih koncepata dijelovi su srednjoškolskog matematičkog kurikuluma diljem svijeta. Pojam asimptote se javlja kao teorijski koncept, dio tehnike ili zaseban zadatak te u više obrazovnih razdoblja i u različitim nastavnim sadržajima predmeta matematike.

U ovom radu prikazujemo rezultate analize udžbenika s obzirom na asimptotu i asimptotsko ponašanje u dva najčešće korištena seta udžbenika za gimnazijsko matematičko obrazovanje. Provedena analiza je dio opsežnog istraživanja vezanog uz taj objekt znanja u srednjoškolskom obrazovanju u Republici Hrvatskoj. Teorijski okvir unutar kojega je istraživanje provedeno jest antropološka teorija didaktike (ATD), koju je razvio francuski matematičar Y. Chevallard. U tom je teorijskom okviru *prakseologija* $[T, \tau, \theta, \Theta]$ model kojim se opisuju matematičko znanje i aktivnosti, koji ima praktičnu komponentu – zadatak T i tehniku τ, te diskurzivnu ili teorijsku komponentu – tehnologiju θ i teoriju Θ.

Rezultati pokazuju kako je asimptota složen objekt znanja koji se sastoji od više prakseologija kojima je ona dijelom praktične ili teorijske komponente. Prikazat ćemo s time povezanu prakseološku organizaciju odabranih udžbenika i odgovarajuće implikacije na učenje i poučavanje matematike.

Ključne riječi: asimptota, asimptotsko ponašanje, ATD, prakseologija, analiza udžbenika, srednjoškolsko matematičko obrazovanje

Language of Croatian mathematical textbooks

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Abstract. Mathematical textbooks, as part of a potentially implemented curriculum have been increasingly analysed, mostly from the point of their structural features, performance expectations and mathematical content. In the previous work, we have performed such kind of analysis of Croatian mathematical textbooks for the 4^{th} grade of secondary schools. During this process, the question of language used in the books came out.

In this paper, we use the framework proposed by Lisa O'Keefe and John O'Donoghue (2014), based on the Halliday's functional grammar analysis (Halliday (1985), Halliday and Martin (1993), Morgan (1998), to perform a preliminary analysis of two Croatian mathematical textbooks.

Keywords: functional grammar analysis, mathematics language analysis, Croatian mathematical textbook

1. Introduction

The language of mathematics, as well as the language of science in general, has certain distinguished features such as: impersonal style, specialist vocabulary, use of symbolism, passive voice and nominalizations (Halliday & Martin, 1993; Martin, 1998). However, this language is not uniform and varies in different mathematical contexts (Morgan, 1998). There are substantial differences between the language used in academic, professional and school mathematical texts.

Different case studies (Pepin & Haggarty, 2001; Haggarty & Pepin, 2002; Fan & Kaeley, 2000) have shown the great impact of textbooks on both teaching and learning mathematics. Studies conducted in Croatia (Glasnović Gracin & Domović, 2009; Domović, Glasnović Gracin & Jurčec, 2012; Glasnović Gracin &

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Jukić Matić, 2014; Glasnović Gracin, 2014) have also shown the prominent role of textbooks in mathematics classroom. On one hand, teachers use textbooks as a guideline for lesson preparation and as a selection of student activities. On the other hand, students use textbooks as a source of exercises for practice.

Hence, it can be said that textbooks give an insight into actual way of teaching and learning mathematics in the classroom. For this reason, we find it valuable to analyse the language of mathematics textbooks.

2. Literature review

Detailed overview of the textbook research in mathematics education can be found in papers of Glasnović-Gracin (2014) and of Fan, Zhu & Miao (2013). Fan, Zhu & Miao's (2013) survey pointed out that there have been very little and scattered research on mathematics textbooks published before the 1980s, but also that since then the number of papers in this field has been rapidly growing. According to Fan (2013), the area of mathematics textbook research is „still at an early stage of development", and theoretical frameworks and methods are still to be developed. Fan, Zhu & Miao's (2013) framework classifies the literature on textbook research in mathematics in four categories: Role of textbooks, Textbook analysis and comparison, Textbook use and Other areas. Their results show that most of the observed literature concerned textbook analysis and comparison (this category occupied approximately 63 % of all the studies identified). Furthermore, most studies in textbook analysis have been concentrated on treatment and presentation of various mathematics content or topics.

A comprehensive TIMSS study (Valverde et al., 2002) involving an analysis of textbooks from 48 countries adopted a concept of a textbook as a mediator between potential and implemented curriculum. Within their analytical framework sections of a textbook are divided into smaller parts, blocks, which are then analysed by their content, performance expectations and perspective. In this way, they describe the structure or "morphology" of a textbook.

O'Keefe and O'Donoghue (2015) propose an extension of this framework by integrating the fourth component – linguistic analysis of a textbook. They present the framework for such analysis, based on Morgan's analysis of students' written mathematical texts (1998), which is, in turn, based on Halliday's systemic functional grammar (Halliday and Matthiessen, 2014). O'Keefe and O'Donoghue illustrate usage of the proposed framework on mathematical textbooks for Irish junior cycle (lower level secondary school). As one of the main conclusions of their analysis, they point out that for a fuller understanding, results of the two analysis should be considered in relation to one another.

3. Theoretical framework: Functional systemic grammar

Halliday's functional systemic grammar (Halliday and Matthiessen, 2014) considers a clause through its usage in communication. Hence it is necessary to take a perspective on clauses not as isolated units, but as part of a wider context – paragraph, section, article, book, speech... In such consideration one can realize that a clause carries more than one uniform function – Halliday recognizes at least three distinct functions: ideational, interpersonal and textual.

As a motivation for the introduction of these three functions, Halliday and Matthiessen (2014) focus on different ways of defining a concept of a subject in a clause:

"(i) that which is the concern of the message
(ii) that of which something is being predicated (i.e. on which rests the truth of the argument)
(iii) the doer of the action" (Halliday & Mathiessen, 2014)

As an illustration, we take two examples from Halliday & Mathiessen (2014).

The duke gave my aunt this teapot.

The subject of this first clause is *the duke*; it fits to all three definitions of a subject. But the same thing could have been said in another way:

This teapot my aunt was given by the duke.

In the second clause, the three functions of a subject split – *this teapot* takes the role of that which is the concern of the message, *my aunt* becomes that of which something is being predicated, while *the duke* remains the doer of the action.

We see that the first notion of a subject is in fact composite. In order to make a clearer picture, it is reasonable to separate the three concepts and consider a language through three different functions: ideational (experiential), interpersonal and textual. Each of these functions brings different structural features on a clause and our previous notion of a subject, now split into three different notions, takes a part in each of these structural concerns. The doer of the action now becomes the Actor; it is a concern of ideational function of language. That of which something is predicated becomes the grammatical subject, or just Subject, for short; it corresponds to interpersonal function of language. Finally, that which is the concern of the message becomes the Theme and it corresponds to the textual function of language. We next briefly describe these three functions of language.

3.1. Ideational (experiential) function

Ideational function of language is concerned of the way in which text reconstructs the real world, both external and internal. The world is realized in a text through processes, participants and circumstances. In every process, there is a distinguished participant, the doer of the action, which is called the Actor. Regarding the world that is being described, there are six different types of processes: material, mental, relational, verbal, existential and behavioural.

Material processes are mostly concerned with the physical, material world, concrete actions that one takes. These are the processes of doing and happening. In the learning environment, it also includes processes of active learning. Examples of material processes include verbs such as *calculate*, *divide*, *derive*, ...

Mental processes are concerned with the internal psychological world, with sensations, feelings, emotions, thoughts. As an example, we list verbs such as *remember*, *realize*, *look*, *concern*, ...

Relational processes describe relations between different participants or objects. We use these processes are used to identify, characterize, classify... As the most prominent examples we point out processes of *being* and *having*.

Verbal processes are concerned with any exchange of meaning between participants, either through writing, speaking or some other medium. As examples, we give processes like *saying*, *pointing out*, *talking*, ...

Existential processes are used to establish existence of some object or participant. In most of clauses of this kind there are an actor and a participant present, without other participants. The most usually these are processes of being and existing.

Behavioural processes describe someone's behaviour, physiological reactions etc. Examples are *He sneezed* or *He laughed*, or in a mathematical context *Values of the function are decreasing.*

3.2. Interpersonal function

In language, social relationships between different participants of verbal exchange are construed. In a written mathematical text, these are usually relationships between the author, a reader and some wider, maybe expert, community. Features that can be identified involve the usage of different forms of verbs (first, second or third person, plural or singular), speech function (statement, command, question, offer), polarity (yes, no) and modality (certainty).

For instance, O'Keefe and O'Donoghue (2015) noted a considerable usage of a pronoun "We". This can be interpreted as putting an author in an authoritative position of a member of an expert community, but can also mean inclusion of a reader in this community or a close relationship between the author and the reader. Also, they noted that authoritative position of an author was established through the usage of imperatives.

3.3. Textual function

When we look at clauses as messages, or parts of a message, we talk about their textual function. It concerns with ways that clauses are bound in a broader whole, like a text, speech, story... Functional systemic grammar considers this function through a system of Theme and Rheme. Here, a Theme is the beginning, introductory, part of a clause through which the clause is connected to the context. It determines that which one wants to talk about. The rest of the clause is called the Rheme.

In our first example from the beginning of the section,

The duke gave my aunt this teapot,

the Theme is *the duke*, while in the second example

This teapot my aunt was given by the duke

the Theme is *this teapot*. It is clear, that a choice of Theme is not determined itself by the process being described, but with the context and relationships that a speaker wants to put in the foreground.

A Theme ends with the first experiential element – either a participant, or a process, or a circumstance. Beside these, a Theme can also have other elements like continuative, conjunctions, conjunctive adjuncts, vocatives, modal adjuncts or finite verbal operators.

O'Keefe and O'Donoghue (2015) reported on thematic use of conjunctions (*by, then*), conjunctive adjuncts (*therefore*), adverbs (*thus*). They interpreted this as a reflection of deductive processes and logical reasoning (*by, then, therefore, thus*) and recalling and recount (*then*).

4. Research question and methodology

In this paper, we consider the following question:

What are the characteristics of language found in mathematical textbooks?

To partially answer this question, we choose two textbooks for the fourth grade of science oriented secondary schools in Croatia:

Đuro Kurepa, Ignacije Smolec, Stjepan Škreblin. (1976). *Matematika: za IV. razred gimnazije*. Zagreb: Školska knjiga.

Svetozar Kurepa, Nikola Sarapa, Aleksandra Kurepa, Juraj Hrnčević. (2001). *Matematika 4: za četvrti razred prirodoslovno-matematičke gimnazije*. Zagreb: Školska knjiga.

Although these two textbooks are not in use anymore, we have chosen them because, during course of our earlier analysis of structure of mathematical textbooks (Trupčević, Valent (2015); Valent, Trupčević (2016)), we felt that, from the linguistic standpoint, the two are most distant from each other and can thus be observed as two poles of a linguistic spectrum of Croatian mathematical textbooks for that educational level. In the following exposition, we denote these two books by KSŠ and KSKH, respectively.

In both books, we have chosen three sections in which the notion of a derivative of a function has been introduced, basic examples of finding a derivative were given, and derivatives of a sum, difference, product and quotient of functions were considered. In this way, we tried to minimize the impact of the differences in content on the differences in language that would be observed. Both selected parts were 11 pages long, hence they were also comparable in volume.

This kind of selection of parts of the book for analysis differs from the one made by O'Keefe and O'Donoghue (2015). They have considered first pages of every section in textbooks that they analysed. We opted for a different approach because we thought that in secondary school textbooks there is a difference in language used in different parts of the text (motivational parts, definitions, theorems, worked examples, remarks, exercises). Hence, we made this choice, hoping that in this way we obtain a more complete picture on the language used.

We have then followed approach of O'Keefe and O'Donoghue (2015), which was inspired by the work of Morgan (1998), and used Hallyday's systemic functional grammar (2014) to analyse language used in the selected portions of text. In our analysis, we have focused on ideational function, and grasped interpersonal function only briefly. We have analysed and classified processes found in texts. Also, we have considered ways in which human action was established in texts. Hence, we have distinguished clauses in which different human beings appear as Actors, clauses in which human action is expressed in a passive form, and also, clauses in which different form of imperatives were being used.

5. Results and discussions

Table 1 lists results for both textbooks. Rows correspond to data for process types, while columns correspond to human role in processes. Codes used are explained in Table 2.

Table 1. Types of processes and human role.

KSŠ

Subject/ Process	AS	AG	AI	II	NS	AP	PP	X	Total
Ma	17	5	7	5	1	6	1		42
Me	11	2		2		3			18
Rel	5					7		89	101
Ver	1		2	1	1	2			7
Egz						2	1	2	5
Beh								7	7
Total	40	7	9	8	2	20	2	98	180

KSKH

Subject/ Process	AS	AI	II	NS	AP	PP	X	Total
Ma	19	6	11		8	1		45
Me	5	2			1			8
Rel	22				12	1	121	156
Ver	2			3	2			7
Egz							1	1
Beh							9	9
Total	48	8	11	3	23	1	132	226

Table 2. Descriptions of codes for human role.

AS	particular human as actor *Različite načine pisanja definicije derivacije napisat ćemo zajedno. (Different ways of writing a definition of a derivative we shall write together.)*
AG	general human as actor *Treba razlikovati 2 slučaja. (One should distinguish 2 cases.)*

AI imperative, 2nd person
Odredite derivacije navedenih funkcija. (Determine derivatives of the following functions.)

II imperative, 1st person plural
Nađimo derivaciju funkcije. (Let us find a derivative of the function.)

NS minor clause, process is implied but not presented explicitly
Riječima: (In words:)

AP passive voice
Sa (1) je dano pravilo za derivaciju sume. (By (1) it was given the rule for derivative of a sum.)

PP human activity is implied, but not foregrounded
Zato je dobio i posebno ime. (For this reason, it got a special name.)

X no human activity
Formula (9) vrijedi za sve cijele brojeve n. (Formula (9) holds for all integers n.)

Tables 3 and 4 list verbs used and their frequencies, in each textbook.

Table 3. Frequencies of verbs used in KSŠ.

Material	#	**Mental**	#	**Relational**	#	**Verbal**	#
dobiti (get)	6	vidjeti (see)	5	biti (be)	68	izreći (say)	3
dokazati (prove)	5	dobiti (get)	1	imati (have)	15	istaknuti (point up)	1
nacrtati (draw)	5	izvoditi (derive)	1	čitati (read)	4	spomenuti(mention)	1
odrediti (determine)	4	interpretirati (interpret)	1	kazati (say)	4	upotrebljavati (use)	1
tražiti (search)	3	naučiti (learn)	1	izlaziti (results)	3	∅	1
postupati (act)	2	pogledati (look)	1	značiti (mean)	3	**Existential**	#
staviti (set)	2	razlikovati (distinguish)	1	vrijediti (hold)	2	govoriti o (biti govora o) (speak of, be no question)	2
izračunati (calculate)	2	spoznati (comprehend)	1	bilježiti (denote)	1	imati (have)	1
dati (give)	1	razmatrati (consider)	1	nazvati (call)	1	javljati se (occur)	1
doći (come)	1	sjetiti se (remember)	1			postojati (exist)	1
ispitati (examine)	1	smetati (bother)	1			**Behavioural**	#
naći (find)	1	pokazati se (appear)	1			dolaziti (come)	2
napisati (write)	1	upamtiti (remember)	1			sijeći (intersect)	2
podijeliti (split)	1	zanimati (interest)	1			težiti (tend)	2
pokazati (show)	1					mijenjati (change)	1
predočiti (visualize)	1						
pridružiti (associate)	1						
supstituirati (substitute)	1						
uzimati (take)	1						
zamijeniti (substitute)	1						
∅	1						

Table 4. Frequencies of verbs used in KSKH.

Material	#	**Mental**	#	**Relational**	#	**Verbal**	#
naći (find)	20	zaključiti (conclude)	2	biti (be)	80	∅	3
pisati (write)	4	imati (have)	1	imati (have)	22	iskazati (state)	1
prikazati (show)	2	naslutiti (sense, make a conjecture)	1	vrijediti (hold)	16	isticati (assert)	1
primijeniti (apply)	2	poznavati (know, be familiar)	1	definirati (define)	8	navesti (mention)	1
izvesti (derive)	2	uvjeriti se (convince)	1	zvati (call)	6	reći (say)	1
dobiti (get)	2	vidjeti (see)	1	pisati (write)	4	**Existential**	#
ići (go)	2	uočiti (notice)	1	dati (give)	4	postojati (exist)	1
dati (give)	1			kazati (say)	3	**Behavioural**	#
napisati (write)	1			označavati (denote)	3	težiti (tend)	4
izvući (bring out)	1			povlačiti (imply)	3	događati (happen)	1
izbjeći (avoid)	1			proizlaziti (result)	2	dolaziti (come)	1
olakšati (make easier)	1			odgovarati (correspond)	1	padati (decrease)	1
povući (draw)	1			čitati (read)	1	rasti (increase)	1
uvoditi (introduce)	1			smatrati (consider)	1	približavati (approach)	1
tražiti (search)	1			voditi (lead)	1		
zadati (assign, give)	1			vezati (relate)	1		
raditi (do)	1						
učiniti (do)	1						

Relational processes have the highest frequencies, followed by material, and then mental processes. Verbal, behavioural and existential processes are much less used.

Material processes are used to similar extent in both textbooks, although in relative terms they are somewhat more frequent in KSŠ. They all relate to human activity, in most cases in active voice. KSKH includes higher number of imperatives, which come in solved examples (*Let us find a derivative…).* We can also notice that the total number of different verbs used is much higher in KSŠ than in KSKH. Most material processes refer to performing calculations and procedures; in KSKH this is achieved by use of verb *find*, while KSŠ uses wider spectrum of verbs. In addition to these we also find a smaller number of verbs relating to representing (*draw, visualize, show*) and mathematical reasoning (*prove*, *get*, *give*, *derive*).

Mental processes in KSŠ are more frequent than verbal, behavioural and existential ones, which is not the case in KSKH. In KSKH, there is roughly an equal number of verbal and behavioural processes. These processes are used in KSKH to describe acts of perceiving and deducing, while in KSŠ in addition to perceiving we also have cognitive (*learning, remembering, differentiating*) as well as metacognitive and affective ones (*interest, bother*).

In both texts, relational processes come mostly without human actor. Usually these are the processes of *being, having* and *holding*. To a smaller extent (more in KSKH than in KSŠ) we find relational processes used in establishing logical connections between mathematical statements. Relational processes in KSKH in which a human actor takes part usually appear in definitions (*we say*, *we write*, *we call*) and in certain mathematical statements (*For* x_0 *we have*...) that could have also been expressed without human actor, through the use of verbs *be* or *hold* (*For* x_0 *the following holds*...). Similar is valid for KSŠ, where passive voice is used in definitions. In KSKH passive voice is used in definitions, and in relating different notions (*relate, give*).

Verbal processes are usually used to express a rule or a property. In KSŠ these are sometimes in an imperative form, left as a task for a student, while in KSKH rules are exclusively formulated by authors. We also find several examples of minor clauses in which a process is not explicit but can be inferred from the context (*In words: ...*). Another way in which verbal processes are used is in describing language of mathematics, or language used in these textbooks (*We won't be asserting this specifically*. or *Other notations for a derivative are also used in higher mathematics*.)

Behavioural processes in both texts do not refer to a human activity but to a behaviour of various mathematical objects (variables, functions), like in the statement: *Derivative of a function* f *at* x_0 *is the limit to which a difference quotient approaches.*

6. Conclusion

In the general case, Halliday (2014) states that texts are dominated by material, mental and relational processes. O'Keefe's and O'Donoghue's analysis (2015) of textbooks for the Irish junior cycle has established dominance of material and relational processes, with mental processes less represented, roughly at the same level as verbal processes. Behavioural and existential processes have found to be rare.

In the present case, we notice dominance of relational processes, followed by material ones. Mental processes are generally less present, in one case to the same degree as verbal, behavioural and existential ones, and in other case to a somewhat higher degree. The smaller share of material and verbal processes can be attributed to a departure from school mathematics, where there is high presence of doing and communicating, towards university level mathematics in which exposition of mathematical relationships takes primary part.

This also suggests that for the full explanation of data obtained in this way one also needs to consider other elements of textbook analysis such as mathematical content, performance expectations and structure.

As we have said earlier, we see the observed textbooks as two poles of a linguistic spectrum of Croatian mathematical textbooks. It is our belief that analysis of other, more recent textbooks would show that they “sit” somewhere in between KSKH and KSŠ, i.e. that their languages share some of characteristics of one or another textbook.

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References

[1] DOMOVIĆ, V., GLASNOVIĆ GRACIN, D., & JURČEC, L. (2012), *Use of Mathematical Textbooks with Regard to Sex and Years of Service of Mathematics Teachers*, Napredak: časopis za pedagogijsku teoriju i praksu, 153(2), 187–202.

[2] FAN, L. (2013), *Textbook research as scientific research: towards a common ground on issues and methods of research on mathematics textbooks*, ZDM: The International Journal on Mathematics Education, 45(5), 765–777.

[3] FAN, L., & KAELEY, G. S. (2000), *The influence of textbooks on teaching strategies: An empirical study*, Mid-Western Educational Researcher, 13 (4), 2–9.

[4] FAN, L., ZHU, Y., & MIAO, Z. (2013), *Textbook research in mathematics education: development status and directions*, ZDM: The International Journal on Mathematics Education, 45(5), 633–646.

[5] GLASNOVIĆ GRACIN, D. (2014), *Mathematics Textbook as an Object of Research*, Croatian journal of education, 16(3), 211–237.

[6] GLASNOVIĆ GRACIN, D. & DOMOVIĆ, V. (2009), *Using maths textbooks in lower secondary education*, Odgojne znanosti, 11(2), 297–317.

[7] GLASNOVIĆ GRACIN, D., & JUKIĆ MATIĆ, LJ. (2014), *Schulbuch als Teil des implementierten Curriculums*, 48. Jahrestagung der Gesellschaft für Didaktik der Mathematik, Koblenz, 431–435.

[8] HAGGARTY, L., & PEPIN, B. (2002), *An Investigation of Mathematics Textbooks and Their Use in English, French and German Classrooms: Who Gets an Opportunity to Learn What?*, British Educational Research Journal, 28(4), 567–590.

[9] HALLIDAY, M. A. K., & MARTIN, J. R. (1993), *Writing Science Literacy and Discursive Power*, London: Falmer Press.

[10] HALLIDAY, M. A. K. & MATTHIESSEN., C. M. I. M. (2014), *Halliday's Introduction to Functional Grammar*, 4th edition, Oxon: Routledge.

[11] KUREPA, Đ., SMOLEC, I., ŠKREBLIN, S. (1976), *Matematika: za IV. razred gimnazije*, Zagreb: Školska knjiga.

[12] KUREPA, S., SARAPA, N., KUREPA, A., HRNČEVIĆ, J. (2001), *Matematika 4: za četvrti razred prirodoslovno-matematičke gimnazije*, Zagreb: Školska knjiga.

[13] MARTIN., J. R. (1998), *Reading Science: Critical and Functional Perspectives on Discourses of Science*, London: Routledge.

[14] MORGAN., C. (1998), *Writing Mathematically: The Discourse of Investigation*, London: Falmer Press.

[15] O'HALLORAN, K. (2005), *Mathematical Discourse: Language, Symbolism and Visual Images*, London: Continuum.

[16] O'KEEFFE, L., & O'DONOGHUE, J. (2015), *A role for language analysis in mathematics textbook analysis*, International Journal of Science and Mathematics Education, 13(3), 605–630.

[17] PEPIN, B., & HAGGARTY, L. (2001), *Mathematics textbooks and their use in English, French and German classrooms: A way to understand teaching and learning cultures*, ZDM: The International Journal on Mathematics Education, 33(5), 158–175.

[18] TRUPČEVIĆ, G., & VALENT, A. (2015), *Structures of Croatian Mathematics Textbooks*, In: Z. Kolar-Begović, R. Kolar-Šuper, I. Đurđević Babić, (Eds.), Higher Goals in Mathematics Education (pp. 318–329), Zagreb: Element.

[19] VALENT, A., & TRUPČEVIĆ, G. (2016), *Structures of Croatian mathematics textbooks: before and after*, Unpublished poster presentation at: The 6th Croatian Mathematical Congress, July 14–17 2016, Zagreb, Croatia.

[20] VALVERDE, G. A., BIANCHI, L. J., WOLFE, R. G., SCHMIDT, W. H., & HOUANG, R. T. (2002), *According to the book: Using TIMSS to investigate the translation of policy into practice through the world of textbooks*, Dordrecht: Kluwer.

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Jezik u hrvatskim matematičkim udžbenicima

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Sažetak. Matematički udžbenici, kao dio potencijalno implementiranog kurikuluma se analiziraju u sve većoj mjeri, uglavnom obzirom na njihove strukturne karakteristike, očekivanja izvedbe i matematičke sadržaje. U prethodnom radu, na taj smo način analizirali hrvatske matematičke udžbenike za 4. razred gimnazije. Tijekom analize, nametnulo nam se pitanje jezika korištenog u udžbenicima.

U ovom radu koristimo okvir predložen od Lise O'Keefe i Johna O'Donoghuea (2014.), koji se bazira na Hallidayevoj funkcionalnoj gramatici (Halliday (1985.), Halliday and Martin (1993.), Morgan (1998.), kako bi proveli preliminarnu jezičnu analizu dva hrvatska matematička udžbenika.

Ključne riječi: funkcionalna gramatika, analiza matematičkog jezika, hrvatski matematički udžbenici

4.
Integration of ICT in the teaching mathematics

The impact of using GeoGebra interactive applets on conceptual and procedural knowledge

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Abstract. This work deals with the impact of using digital textbooks consisting of GeoGebra interactive applets on students' conceptual and procedural knowledge in mathematics. The authors propose the model of computer-guided discovery learning by using dynamic geometry software which includes three basic elements: learning objects, tasks for students and learning outcomes. The model is primary based on the theory of constructivism, Pólya's heuristic strategy and Schoenfeld's problem-solving model.

The experimental research plan with experimental and control groups was used and there were 703 students in twelve elementary schools in Croatia participating in the research. The experimental group of students was taught by the model of computer-guided discovery learning and the control group of students was taught by teachers using traditional teaching methods. The covariance analysis determined the impact of the model of computer-guided discovery learning on students' conceptual and procedural knowledge and the results have shown statistically significant differences between the groups in favor of the experimental group.

The qualitative and quantitative analysis of students' and teachers' questionnaires and forum discussions was used to assess their attitudes towards the proposed learning model. The importance of teachers' experience in teaching using this model has also been noted. This learning model indicates the potential of computers and dynamic geometry software GeoGebra for scaffolding support in learning mathematics.

Keywords: conceptual knowledge, digital textbook, discovery learning, dynamic geometry software, GeoGebra, interactive applet, learning model, procedural knowledge

1. Introduction

Modern mathematics teaching is based on the theory of constructivism and focused on the student. Being an active participant in the teaching and learning process, he/she observes, investigates, asks questions, creates hypotheses, solves problems, cooperates with others, analyzes and evaluates his/her own work and the work of others. The student develops new knowledge on his/her own experiences and observations in physical and social environment, and, according to the radical constructivism learning theory, this knowledge is unique to each individual (von Glasersfeld, 1995). The teacher's role in such teaching is twofold: (1) organizing the teaching process and preparing appropriate activities for students and (2) guiding students in the learning process and encouraging them to think. Von Glasersfeld (1995) stands for scaffolding teacher support which means that the teacher initially stimulates and leads students to use their knowledge and skills and helps them learn some new strategies for solving the problem, but simultaneously lets them come to the solution on their own and be responsible for their progress (Dennen, 2004). Except for the teacher, this kind of support can be provided by computers. A computer can be a very useful tool for helping students in their active knowledge construction and especially for the implementation of student experiments and discovery learning.

2. Overview of previous research and the research question

The computer simulation is considered to be exceptionally suitable learning environment for discovery learning (Pinto, Couso and Hernández Rodrigues, 2010). De Jong and van Joolingen (1998) recognize some problems that students may encounter in discovery learning using computers, and emphasize the need for good scaffolding. They suggest three principles for successful discovery process: (1) displaying information and simulation simultaneously, (2) giving the students specific tasks and (3) structuring the learning environment in a few steps. Mayer (2004) shows that structured or guided discovery learning, with specific teaching goals and stimulating students' cognitive activities, best supports constructivist learning approach. Wangpipatwong and Papasratorn (2008) design the constructivist e-learning environment with a focus on research, cooperation and the construction of new knowledge and they conclude it gives better results than traditional learning environment.

In mathematics teaching, Glasnović Gracin (2008) recommends the use of dynamic geometry software because it supports multiple representations of mathematical objects and enables simple transferring from one mathematical view to another. As an example, she proposes GeoGebra since it supports graphic, symbolic and spreadsheet views. Furthermore, GeoGebra provides experimental learning that Schneider (2002) describes as active, independent and revealing learning primarily aimed at the development of intuitive perceptions and ideas, as well as at deeper and more comprehensive understanding of mathematical concepts. Bu, Spector and Haciomeroglu (2011) also emphasize the importance of multiple representations and students' interactive dealing with learning objects. They also suggest

GeoGebra as an extremely useful mathematical tool which provides an intellectual bridge between certain domains of mathematics theory, school mathematics and instructional design frameworks. According to Iranzo and Fortuny (2011) the use of GeoGebra has a positive impact on the mathematical problem solving strategies since it helps students visualize the problem and avoid algebraic obstacles so students can focus on understanding geometry more. Shadaan and Leong (2013) find that GeoGebra is extremely useful as students' scaffolding support in understanding mathematical concepts and also as an excellent motivational tool for increasing self-confidence and fostering higher-level student thinking.

The previous researches show the great potential of GeoGebra as a tool for discovering and acquiring new mathematical knowledge which is especially related to conceptual knowledge. So the subject of this study is to examine the impact of GeoGebra and computer-guided discovery learning on conceptual and procedural knowledge of students. For this purpose GeoGebra is used to create various interactive applets and GeoGebraTube service is used to organize the applets in a digital textbook. In addition to these learning objects, it is necessary to plan the activities for students according to the desired learning outcomes. All three elements together form a learning model which is described in the next chapter.

3. The model of computer-guided discovery learning

Elliott, Sweeney and Irving (2009) develop the conceptual model of e-learning, which is based on the theory of constructivism, the discovery learning strategy and the problem-based learning. They explore its effectiveness in the natural sciences (Figure 1). The tasks for students are defined as a simple research cycle which is similar to the stages of scientific research. The idea of solving problem in four steps is taken from Pólya (1945): (1) problem analysis, (2) inquiry planning, (3) research loop and (4) problem closure.

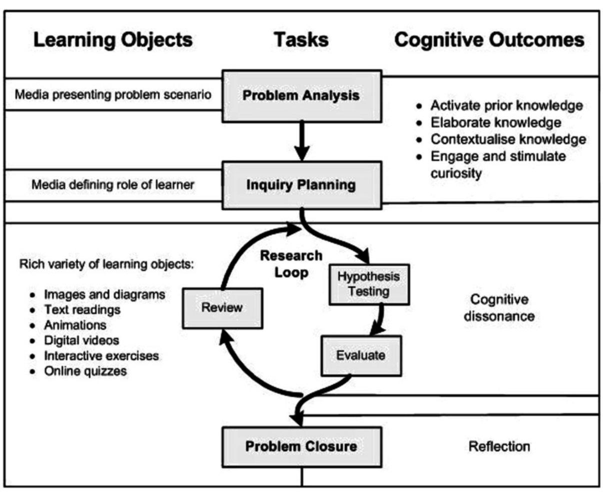


Figure 1. The conceptual model of e-learning (Elliott, Sweeney and Irving, 2009, p. 660).

On the basis of previous model, the model of computer guided discovery learning by using dynamic geometry software in mathematics teaching[1] is developed. Pólya's heuristic problem solving strategy is embedded in GeoGebra's dynamic learning environment, and according to Karadag and McDougall (2009) it can be used for explaining, exploring and modeling. If we want a student to successfully solve a given problem, with or without a computer, except possessing the basic mathematical knowledge (resources), the problem solving techniques (heuristics), proper selection of previous knowledge and techniques (control), it is necessary for the student to believe in his/her own abilities and have a positive attitude regarding mathematics (Schoenfeld, 1985). This Schoenfeld's frame of mathematical problem solving model is complemented with socio-cultural factors related to the interaction of students with teachers, other students and the learning environment (Lester, Garofalo and Kroll, 1989), or, in this case, the dynamic discovery learning environment of GeoGebra.

The computer-guided discovery learning model consists of three elements: (1) learning objects, (2) activities for students and (3) learning outcomes (Figure 2).

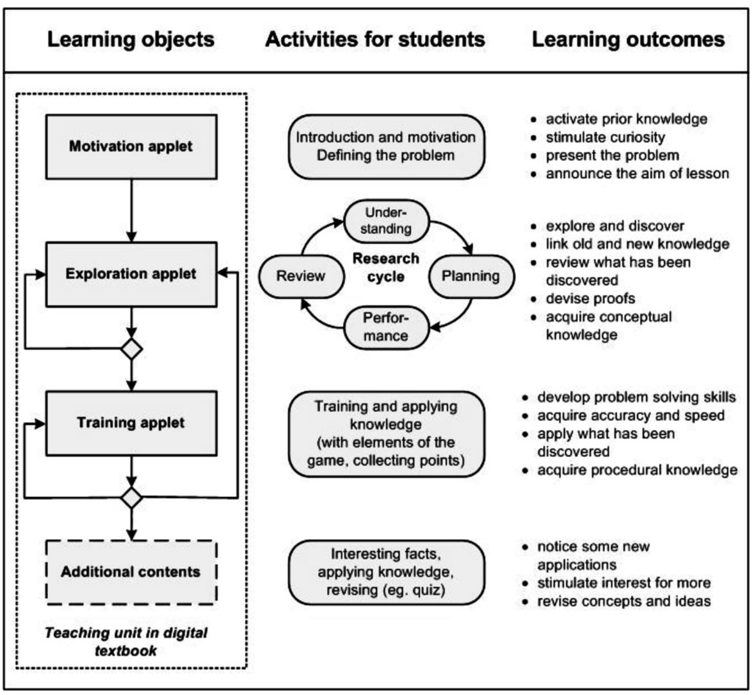


Figure 2. Scheme of the computer-guided discovery learning model (Dijanić, 2017, p. 152).

[1] This learning model is developed as first author's doctoral thesis "Development of the model of computer guided discovery learning by using dynamic geometry software in teaching mathematics" under the supervision of second author and Mario Dumančić, PhD, and presented at The Faculty of Teacher Education, University of Zagreb on 16th February 2017.

Learning objects are created like interactive applets or mathlets. According to Hohenwarter and Preiner (2007) mathlet is a small learning object dealing with specific mathematical topic or problem. It can be used for teacher demonstration to the whole class or for individual student's learning. The mathlet is made of a static and a dynamic part. Bjelanović Dijanić (2009) explains that the dynamic section includes mathematical objects (such as points, lines, geometric shapes, etc.) that can be moved so students could observe their interactivity and perform experimental work. The static section of mathlet contains instructions, questions, tasks, suggestions or help, which allow guiding student along his/her way to conclusions with minimal risk of failure. According to Dijanić (2017) these interactive applets are organized in the digital textbook with chapters for each teaching unit and every applet has one of the following functions:

1. Motivation applet, at the beginning of each teaching unit. Its main role is to present a problem or topic to the student and to motivate him/her for further work.
2. Exploration applet; its role is to enable performing mathematical experiments to the student and help him/her discover new mathematical knowledge. There are one or more research cycles in each applet which stages are taken from Pólya's problem solving strategy: understanding the problem, devising a plan, carrying out the plan and looking back. At this learning step the basic outcomes are to acquire conceptual knowledge and understand the basic ideas in mathematics.
3. Training applet; its role is to enable students solve some math problems related to the topic. For additional students' motivation these applets include the elements of game and students' progress is encouraged by scoring points. The learning outcomes are to apply discovered skills in solving problems and acquire procedural knowledge.
4. Additional contents; like interesting facts from the history of mathematics, some specific knowledge application in everyday life, extended mathematical contents for gifted pupils, additional teaching activities for students with less developed mathematical abilities, quizzes and the like.

For the purpose of the research which included the development of computer guided discovery learning model and the verification of this learning model's efficiency (Dijanić, 2017), four teachers and the first author of this paper collaboratively created three digital textbooks:

6th grade: *Angle and triangle*
(`http://tube.geogebra.org/student/b272303`)
(Dijanić, Belavić, Grgić, Pein i Vuković, 2014),

7th grade: *The similarity of triangles and polygons*
(`http://tube.geogebra.org/student/b364307`)
(Dijanić, Grgić, Pein i Vuković, 2014),

8th grade: *Pythagorean theorem*
(`http://tube.geogebra.org/student/b297339`)
(Dijanić, Belavić, Grgić i Pein, 2014).

In these digital textbooks the emphasis is on the applets that enable discovering new mathematical knowledge with the scaffolding support of GeoGebra. For example, Figure 3 shows the exploration mathlet from digital textbook *The similarity of triangles and polygons*, teaching unit *The perimeter and area of similar triangles*. The idea is to enable students observe several cases of similar triangles, write down their observations in the table and discover the relationship between the areas of similar triangles by inductive reasoning. The mathematical proof of discovered assertion is interactively demonstrated to students in the next applet of this textbook.

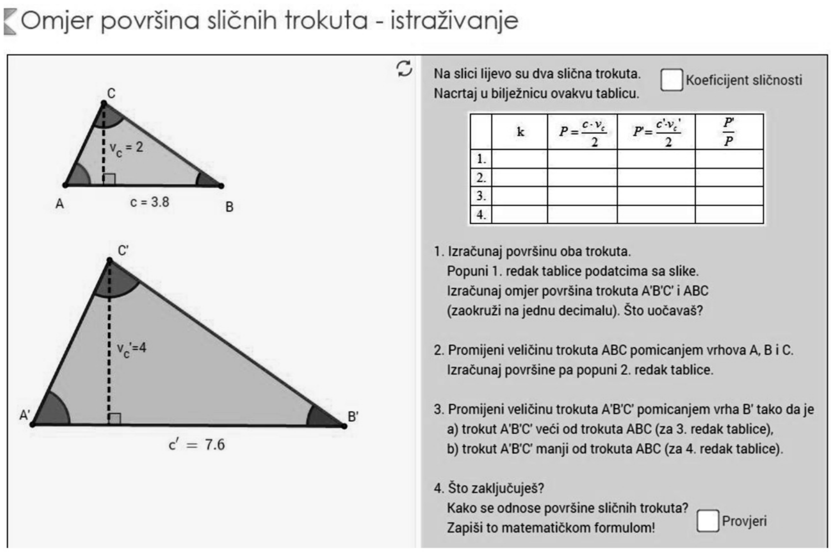


Figure 3. Exploration applet: The ratio of the areas of two similar triangles.

Some other types of interactive applets that are used in the proposed learning model with a detailed description of activities and tasks for the students can be found in Dijanić (2015; 2017).

4. Research methodology

The aim of the research is to investigate the impact of using proposed computer-guided discovery learning model on students' conceptual and procedural knowledge in mathematics.

There were 15 mathematics teachers and 703 of their students from the sixth, seventh and eighth grades from twelve Croatian school participating in the research. The study was conducted during mathematics classes during school year 2014/2015. Before the experiment started, the students had taken the exams testing their prior knowledge in mathematics. The experimental research plan with experimental and control groups was used. Each teacher participated with two parallel classes, one being experimental and the other control group. The experimental group of students was taught by the model of computer-guided discovery learning (as the intervention) working individually or in pairs in IT classroom and the control group of students was taught by teachers using traditional teaching methods. The research process included five lessons of learning new geometry content in the sixth, seventh and eighth grades for both groups of students. All the teachers used the same lesson plans for teaching students in the control group and used digital textbooks mentioned in the previous section for teaching students in the experimental group. The final examinations of acquired knowledge were conducted in the same way as the initial testing.

Besides the quantitative analysis of students' results at testing conceptual and procedural knowledge, the qualitative analysis of the teachers' discussions in the forum of learning management system Moodle was conducted, as well as the analysis of the semi-structured interview of the teachers. The results indicate multiple advantages of the proposed learning model.

5. Results

The covariance analysis has been used to determine the impact of the model of computer-guided discovery learning on students' conceptual and procedural knowledge. The students' pretest results were used as a covariate in the analysis to control the potential impact of the difference in the initial testing results of students on the differences between the groups in the final testing results.

The following tables show the differences between groups (control and experimental) and statistical significance of these differences for two dependent variables: conceptual knowledge (Table 1) and procedural knowledge (Table 2).

Table 1. The results: conceptual knowledge.

	Control group	**Experimental group**		
Grade	*M* (*SD*)	*M* (*SD*)	*F* (*df*)	*p*
6^{th}	7.283 (2.940)	7.996 (3.068)	5.608 (1/238)	$p<\mathbf{0.05}$
7^{th}	8.308 (2.694)	8.516 (3.310)	0.346 (1/177)	$p>0.05$
8^{th}	8.780 (2.595)	9.459 (2.774)	6.524 (1/279)	$p<\mathbf{0.05}$
all together ($6^{th}-8^{th}$)	8.166 (2.802)	8.699 (3.098)	8.296 (1/700)	$p<\mathbf{0.01}$

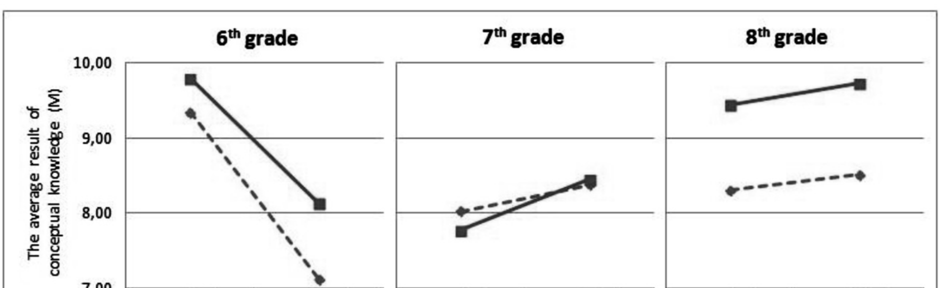
Table 2. The results: procedural knowledge.

	Control group	**Experimental group**		
Grade	*M* (*SD*)	*M* (*SD*)	*F* (*df*)	*p*
6^{th}	3.734 (2.627)	4.060 (2.900)	1.443 (1/238)	$p>0.05$
7^{th}	3.485 (2.837)	3.370 (2.806)	0.173 (1/177)	$p>0.05$
8^{th}	4.254 (2.831)	5.300 (3.167)	16.413 (1/279)	$p<\mathbf{0.01}$
all together ($6^{th}-8^{th}$)	3.888 (2.776)	4.369 (3.095)	9.173 (1/700)	$p<\mathbf{0.01}$

If we look at the whole sample of students together, the covariance analysis gives the following results. The students' average scores at the conceptual knowledge variable are statistically significantly different between the intervention and the comparable group ($F = 8.296$, $df = 1/700$; $p < 0.01$). Concurrently, the students taught using the computer-guided discovery learning model show higher conceptual knowledge ($M = 8.699$) than the students taught traditionally ($M = 8.166$). The students' average scores at the procedural knowledge variable are statistically significantly different between the intervention and the comparable group ($F = 9.173$, $df = 1/700$; $p < 0.01$). Concurrently, the students taught using the computer-guided discovery learning model show higher procedural knowledge ($M = 4.369$) than the students taught traditionally ($M = 3.888$).

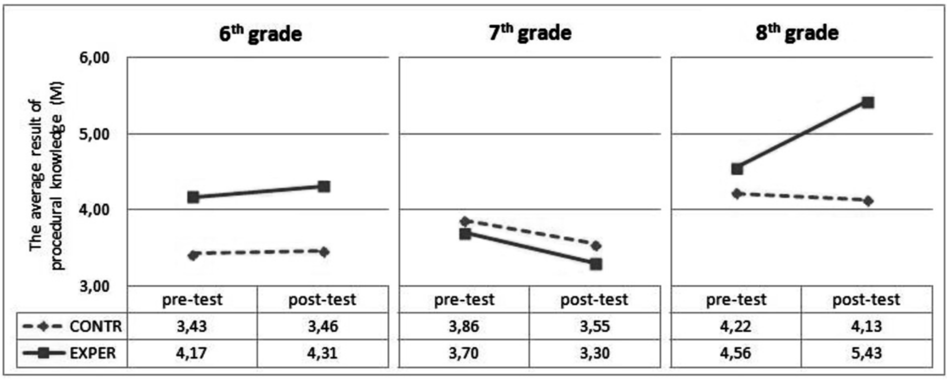
If we look at the grades individually, the students' average scores at the conceptual knowledge variable are statistically significantly different between the intervention and the comparable group only at the sixth-grade ($F = 5.608$; $df = 1/238$; $p < 0.05$) and eighth-grade students ($F = 6.524$; $df = 1/279$; $p < 0.05$). Concurrently, the students taught using the computer-guided discovery learning model show higher conceptual knowledge ($M_6 = 7.996$; $M_8 = 9.459$) than the students taught traditionally ($M_6 = 7.283$; $M_8 = 8.780$). The students' average scores at the procedural knowledge variable are statistically significantly different between the intervention and the comparable group only at the eighth-grade students ($F = 16.413$; $df = 1/279$; $p < 0.01$). Concurrently, the students taught using the computer-guided discovery learning model show higher conceptual knowledge ($M_8 = 5.300$) than the students taught traditionally ($M_8 = 4.254$). There is no statistically significant difference between the groups of seventh-grade students at both conceptual and procedural knowledge variable.

The average results comparison of initial (pre-test) and final (post-test) assessment of students' knowledge is given in the Figure 4 and Figure 5 separately for the sixth, seventh and eighth-grade students.



	6th grade pre-test	6th grade post-test	7th grade pre-test	7th grade post-test	8th grade pre-test	8th grade post-test
CONTR	9,35	7,13	8,02	8,37	8,32	8,51
EXPER	9,79	8,13	7,78	8,45	9,44	9,73

Figure 4. The average pre-test and post-test results of conceptual knowledge.



	6th grade pre-test	6th grade post-test	7th grade pre-test	7th grade post-test	8th grade pre-test	8th grade post-test
CONTR	3,43	3,46	3,86	3,55	4,22	4,13
EXPER	4,17	4,31	3,70	3,30	4,56	5,43

Figure 5. The average pre-test and post-test results of procedural knowledge.

The difference in achievement at sixth, seventh and eighth-grade students can be observed. Eighth-grade students achieved the best results using computer-guided discovery learning model, while such intervention in mathematics teaching and learning had no impact on knowledge of seventh-grade students. We presume there are some other factors that influence achievement in mathematics, like the complexity of studied mathematical content, the puberty stages of students, teacher's experience in teaching using this learning model, students' experience in discovery learning and in learning individually using ICT.

The qualitative analysis of teachers' questionnaires and forum discussions shows their attitudes towards the proposed learning model. The results indicate multiple advantages: an individualized approach to each student, active work of all the students at their own pace, the possibility of repeating the lesson several times or returning to the previous lesson, the exploration and discovery of new knowledge for all the students, the visualization of mathematical contents and reasoning based on the visualization, the interactivity and dynamism of digital learning materials that enables experimental work, immediate feedback, easier and more interesting learning and solving problems through playing a game, additional contents for gifted pupils and teachers also noted that students with learning difficulties can get more attention and help from them (whereas others work independently).

Some of the perceived deficiencies of the computer-guided discovery learning are: disorientation of some students in the new learning environment, the lack of understanding the operating instructions and reading without understanding, some students rush through the learning material without proper reasoning, some students have the feeling they are left to themselves, the lack of feedback to the teacher about the level of acquired mathematics knowledge for each student as well as the inadequacy of this learning model for lessons in which the drawing and the geometric constructions are the key features. The importance of the teachers' experience in teaching by this model has also been noted.

6. Conclusion

The results show that computer-guided discovery learning model by using dynamic geometry software and GeoGebra interactive applets in mathematics teaching has certain potential and can provide better results in acquiring both, conceptual and procedural knowledge than traditional teaching methods can. Similar results were obtained in some other experimental research dealing with conceptual knowledge (Svedružić, 2006; Shadaan and Leong, 2013), procedural knowledge (Nguyen and Kulm, 2005) or both (Güzeller and Akın, 2012; Jupri, Drijvers and van den Heuvel-Panhuizen, 2015). According to Cheung and Slavin (2013) the use of ICT in mathematics teaching shows positive but small effects on mathematical performance. Therefore, they highlight the need for new and better digital tools to take advantage of the possibilities of ICT to improve mathematics achievement of all students. We believe that proposed learning model, which uses digital textbooks consisting of GeoGebra interactive applets with carefully planned activities for students, can make some scientific contribution in that direction. This learning model also indicates the potential of computers and dynamic geometry software GeoGebra for scaffolding support in learning mathematics.

However, there were differences in acquiring conceptual and procedural knowledge of students considering their age and also related to the specific learning content (different grades students used different digital textbooks). Therefore, the influence of students' age (but also gender which plays an important role during teenage years) requires additional examination, as well as the complexity of mathematical content to be studied.

Likewise, there have been noted some limitations of this research. Firstly, using the proposed learning model, students have acquired new knowledge in geometry since the visualization has been highlighted as one of the most significant potential for using GeoGebra. So, there is the question to what extent would this learning model influence the discovery of knowledge from other mathematical areas (such as arithmetic, algebra, analysis, statistics). Secondly, the research has been conducted in Croatian schools during regular mathematics classes which are mostly traditional (and the results have confirmed some deficiencies of this traditional classroom environment). There is the question how this learning model would influence the conceptual and procedural knowledge of students in some other educational systems (e.g. where problem solving, communication and reasoning are emphasized). Furthermore, besides the problem with technology equipment in Croatian schools, there is also the problem related to in-service teacher training on using technology, as well as their readiness for applying modern teaching and learning models like the one described in this issue.

References

[1] BJELANOVIĆ DIJANIĆ, Ž. (2009), *Mathlet – interaktivni digitalni materijal namijenjen samostalnom učenju*, Pogled kroz prozor 6, http://pogledkrozprozor.wordpress.com/2009/03/31/mathlet-interaktivni-digitalni-materijal-namijenjen-samostalnom-ucenju/ [Accessed 10/01/2017].

[2] BU, L., SPECTOR, J. M., & HACIOMEROGLU, E. S. (2011), *Toward model-centered mathematics learning and instruction using GeoGebra: A theoretical framework for learning mathematics with understanding*, In: L. Bu, & R. Schoen (Eds.), Model-Centered Learning: Pathways to Mathematical Understanding Using GeoGebra (pp. 13–40), Rotterdam: Sense Publishers.

[3] CHEUNG, A. C., & SLAVIN, R. E. (2013), *The effectiveness of educational technology applications for enhancing mathematics achievement in K-12 classrooms: A meta-analysis*, Educational Research Review, 9, 88–113.

[4] DE JONG, T., & VAN JOOLINGEN, W. R. (1998), *Scientific discovery learning with computer simulations of conceptual domains*, Review of educational research, 68(2), 179–201.

[5] DENNEN, V. P. (2004), *Cognitive apprenticeship in educational practice: Research on scaffolding, modeling, mentoring, and coaching as instructional strategies*, In D. H. Jonassen (Ed.), Handbook of Research on Educational Communications and Technology (2nd ed., pp. 813–828), Mahwah, NJ: Lawrence Erlbaum Associates.

[6] DIJANIĆ, Ž., BELAVIĆ, D., GRGIĆ, N., PEIN, B., & VUKOVIĆ, A. M. (2014), *E-udžbenik: Kut i trokut* [Elektronički udžbenik], http://tube.geogebra.org/student/b272303 [Accessed 10/01/2017].

[7] DIJANIĆ, Ž., GRGIĆ, N., PEIN, B., & VUKOVIĆ, A. M. (2014), *E-udžbenik: Sličnost i mnogokuti* [Elektronički udžbenik], http://tube.geogebra.org/student/b364307 [Accessed 10/01/2017].

[8] DIJANIĆ, Ž., BELAVIĆ, D., GRGIĆ, N., & PEIN, B. (2014), *E-udžbenik: Pitagorin poučak* [Elektronički udžbenik], http://tube.geogebra.org/student/b297339 [Accessed 10/01/2017].

[9] DIJANIĆ, Ž. (2015), *Zadatci u kontekstu e-udžbenika*, U: R. Gortan (Ur.), Zbornik radova Devetog stručno-metodičkog skupa Metodika nastave matematike u osnovnoj i srednjoj školi: Zadaci u nastavi matematike (str. 56–71), Pula: Matematičko društvo Istra.

[10] DIJANIĆ, Ž. (2017), *Development of the model of computer guided discovery learning by using dynamic geometry software in teaching mathematics* (Unpublished doctoral dissertation), Faculty of Teacher Education, University of Zagreb.

[11] ELLIOTT K., SWEENEY K., & IRVING H. (2009), *A learning design to teach scientific inquiry*, In: L. Lockyer, S. Bennett, S. Agostinho, & B. Harper (Eds.), Handbook of Research on Learning Design and Learning Objects: Issues, Applications, and Technologies (pp. 652–675), Hershey & London: IGI Global.

[12] GLASNOVIĆ GRACIN, D. (2008), *Računalo u nastavi matematike: Teorijska podloga i metodičke smjernice*, Matematika i škola, 46, 10–15.

[13] GÜZELLER, C. O., & AKIN, A. (2012), *The effect of web-based mathematics instruction on mathematics achievement, attitudes, anxiety and self-efficacy of 6th grade students*, International Journal of Academic Research in Progressive Education and Development, 1(2), 42–54.

[14] HOHENWARTER, M., & PREINER, J. (2007), *Creating mathlets with open source tools*, Journal of Online Mathematics and its Applications, 7, ID 1574, `http://www.maa.org/external_archive/joma/Volume7/Hohenwarter2/index.html` [Accessed 15/04/2015].

[15] IRANZO, N., & FORTUNY, J. M. (2011), *Influence of GeoGebra on problem solving strategies*, In: L. Bu, & R. Schoen (Eds.), Model-Centered Learning: Pathways to Mathematical Understanding Using GeoGebra (pp. 91–103), Rotterdam: Sense Publishers.

[16] JUPRI, A., DRIJVERS, P., & VAN DEN HEUVEL-PANHUIZEN, M. (2015), *Improving grade 7 students' achievement in initial algebra through a technology-based intervention*, Digital Experiences in Mathematics Education, 1(1), 28–58.

[17] KARADAG, Z., & MCDOUGALL, D. (2009), *Dynamic worksheets: visual learning with guidance of Pólya*, MSOR Connections, 9(2), 13–16.

[18] LESTER, F. K., GAROFALO, J., & KROLL, D. (1989), *The role of metacognition in mathematical problem solving: A study of two grade seven classes* (Final report NSF MDR 85-50346), Bloomington: Indiana university, Mathematics Education Development Center, `http://files.eric.ed.gov/fulltext/ED314255.pdf%20ERIC%20ED314255` [Accessed 18/06/2015].

[19] MAYER, R. E. (2004), *Should there be a three-strikes rule against pure discovery learning? The case for guided methods of instruction*, American Psychologist, 59(1), 14–19.

[20] NGUYEN, D. M., & KULM, G. (2005), *Using web-based practice to enhance mathematics learning and achievement*, Journal of interactive online learning, 3(3), 1–16.

[21] PINTÓ, R., COUSO, D., & HERNÁNDEZ RODRÍGUEZ, M. I. (2010), *An inquiry-oriented approach for making the best use of ICT in the classroom*, eLearning Papers, 20 1–14. `http://www.uh.cu/static/documents/RDA/An%20inquiry-oriented%20approach.pdf` [Accessed 08/08/2016].

[22] PÓLYA, G. (1945), *How to solve*. Princeton, New Jersey: Princeton University Press.

[23] SCHNEIDER, E. (2002), *Computeralgebrasysteme in einem allgemeinbildenden Mathematikunterricht: didaktische Orientierungen – praktische Erfahrungen*, München-Wien: Profil Verlag.

[24] SCHOENFELD, A. (1985), *Mathematical problem solving*, Orlando, Florida: Academic Press.

[25] SHADAAN, P., & LEONG, K. E. (2013), *Effectiveness of Using GeoGebra on Students' Understanding in Learning Circles*, Malaysian Online Journal of Educational Technology, 1(4), 1–11.

[26] SVEDRUŽIĆ, A. (2006), *Vrednovanje učinkovitosti učenja fizike putem računala za učenike osnovne škole*, Odgojne znanosti, 8(2), 535–549.

[27] VON GLASERSFELD, E. (1995), *Radical Constructivism: A Way of Knowing and Learning*, London: Falmer Press.

[28] WANGPIPATWONG, T., & PAPASRATORN, B. (2008), *The influence of constructivist e-learning system on student learning outcomes*, In: C. van Slyke (Ed.), Information Communication Technologies: Concepts, Methodologies, Tools, and Applications (Vol. 1, pp. 1109–1121), Hershey & London: IGI Global.

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Utjecaj uporabe interaktivnih apleta kreiranih u GeoGebri na konceptualno i proceduralno znanje učenika

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Sažetak. Ovaj se rad bavi utjecajem uporabe digitalnih udžbenika koji sadrže interaktivne aplete kreirane u GeoGebri na konceptualno i proceduralno znanje učenika. Autori predlažu model računalno vođenoga učenja otkrivanjem korištenjem programa dinamične geometrije koji uključuje tri osnovna elementa: objekte učenja, aktivnosti za učenike i ishode učenja. Model se temelji na teoriji konstruktivizma, Pólyinoj heurističkoj strategiji i Schoenfeldovom modelu rješavanja problema.

U istraživanju je sudjelovalo 703 učenika iz dvanaest osnovnih škola u Hrvatskoj te je korišten eksperimentalni nacrt s eksperimentalnom i kontrolnom skupinom. Eksperimentalna skupina učenika učila je po modelu računalno vođenoga učenja otkrivanjem, a kontrolna skupina učenika u tradicionalno organiziranoj nastavi. Analizom kovarijance utvrđen je utjecaj modela računalno vođenoga učenja otkrivanjem na konceptualno i proceduralno znanje učenika, a rezultati pokazuju statistički značajne razlike između skupina u korist učenika eksperimentalne skupine.

Provedena je kvalitativna i kvantitativna analiza upitnika za učenike i učitelje te analiza rasprave učitelja na forumu kako bi se ustanovili stavovi učenika i učitelja prema predloženome modelu učenja. Uočena je važnost iskustva učitelja u primjeni ovoga modela učenja. Predloženi model učenja ukazuje na potencijal računala i programa dinamične geometrije *GeoGebra* kao *scaffolding* podrške u učenju matematike.

Ključne riječi: digitalni udžbenik, GeoGebra, interaktivni aplet, konceptualno znanje, model učenja, proceduralno znanje, program dinamične geometrije, učenje otkrivanjem

The use of the computer program *Graph* in teaching application of differential calculus

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Abstract. The paper presents the authors' own teaching experience in the application of the computer program *Graph* in teaching unit *Problem of determining the tangent line and the normal line to the graph of real functions of one variable* within the mathematical courses in professional study programs of Zagreb University of Applied Sciences. The typical problems encountered during teaching the aforementioned unit and concrete examples how these problems can be successfully solved methodically are presented.

Keywords: computer program *Graph*, differential calculus, tangent line, normal line, graphs of functions of one variable

1. Introduction

One of the teaching units within the mathematical courses on the first year of professional studies of Zagreb University of Applied Sciences is *The problem of determining the tangent line and the normal line to the graph of real functions*. This teaching unit shows the typical applications of differential calculus of real functions of one variable. Within that unit, various tasks related to the tangent line and the normal line of the planar curve whose equation is given in explicit, implicit, or parametric form are solved.

The official curriculum of the mathematical courses bounds the teacher to show the students how to solve the mentioned tasks analytically, i.e. without the use of suitable computer programs. Possible sketches that are used in solving the tasks are related only to drawing tangent lines and normal lines as separated lines without any connection with the corresponding planar curve. However, experience in teaching, and especially the experience gained in the oral exams, shows that

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the analytical solving of these tasks must be accompanied with the corresponding graphical display of the planar curve, the tangent line and / or the normal line because most students have no idea what actually "happens" in that task or what is the relative position of the given curve to obtained tangent or normal line. Since in most tasks accurate sketching of the planar curve is technically relatively slow (requires determining domain, zeroes, monotony intervals etc.), methodically suitable computer program *Graph* is chosen to help.

2. The problem of (in)correct understanding of the concept of the tangent line to the planar curve

Based on the experience with teaching and oral examinations, we claim that the majority of students asked the question "What is a tangent line to a planar curve?" will answer "This is a line that intersects the given curve in exactly one point." Such a "definition" students met most likely during the secondary school education. By giving that answer, the students have in mind a situation like the one shown in Figure 1. In this figure the straight line intersects parabola in exactly one point (compare to [7]).

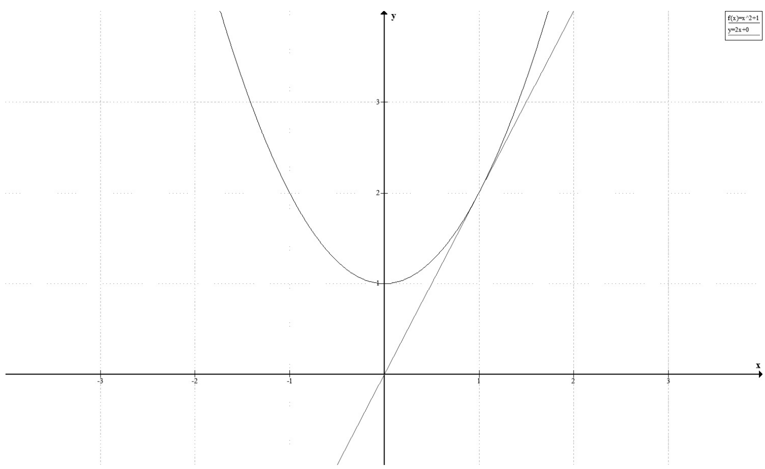

Figure 1. Parabola $y = x^2 + 1$ and a tangent line.

But when the students are asked whether the straight lines shown in Figures 2 and 3 are the tangent lined to the corresponding planar curves, their typical response is: "They are not, because they intersect the curves in more than one point." Even more, in the case shown in Figure 3. they consider the question practically meaningless because, based on the figure, they conclude that the drawn line intersects the sinusoid in the infinite number of points.

Clearly, the above response is obviously wrong. In this article, we have no intention to discuss which definition of the concept of a tangent line is methodically the most appropriate, but we are using the following definition ([2, 5]):

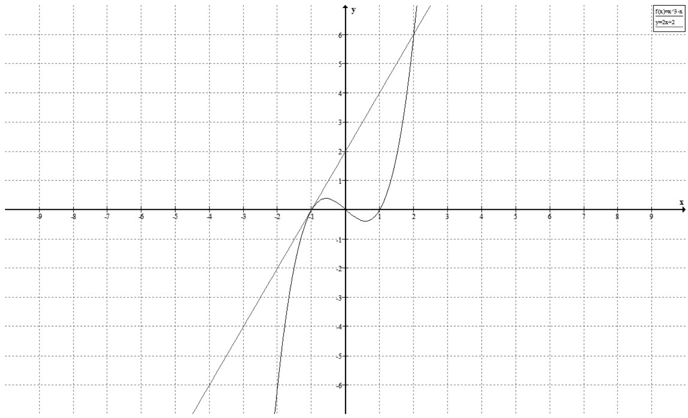

Figure 2. Curve $y = x^3 - x$ and its tangent line.

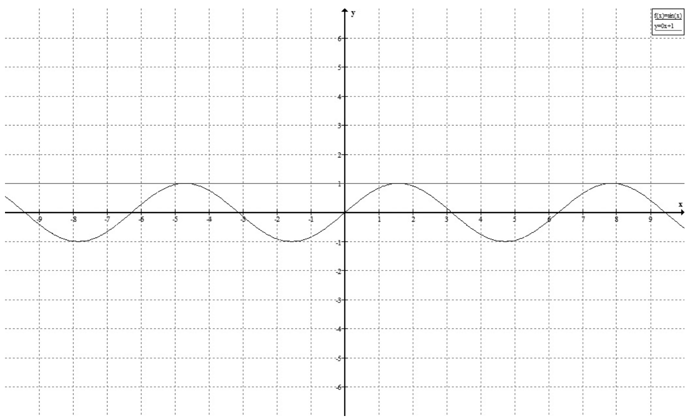

Figure 3. Sinusoid $y = \sin x$ and its tangent line.

Definition 1. Let f be a real function of a real variable differentiable at a point x_0 and Γ_f the graph of the function f. Let us denote $y_0 := f(x_0)$. We say that the straight line t is the *tangent line* to Γ_f *at the point* $T = (x_0, y_0) \in \Gamma_f$ if the line t passes through the point T and the slope of the line equals $f'(x_0)$.

Note that the above definition says nothing about whether the line t intersects the graph of the function f in any other point (except point T), and that the definition does not apply to the planar curves which are not the graphs of a real function of one real variable (such as circle, ellipse, etc.). However, taking into account the objective of teaching unit, and that is the application of differential calculus to solve the problem of determining the tangent and normal lines to the graph of real functions, we consider the above definition to be methodologically and pedagogically justified.

For completeness, we give the definition of a normal line given in accordance with the aim of the lesson stated above ([1, 3]):

Definition 2. Let f be a real function of a real variable differentiable at a point x_0 and Γ_f the graph of the function f. Let us denote $y_0 := f(x_0)$. We say that the straight line n is the *normal line* to Γ_f *at the point* $T = (x_0, y_0) \in \Gamma_f$ if the line n passes through the point T and is perpendicular to the tangent line to Γ_f at the point T.

Remark 1. In the above definition we deliberately do not mention that the slope of the normal n equals to $k_n = -\dfrac{1}{f'(x_0)}$. This would require that the inequality $f'(x_0) \neq 0$ must be valid necessarily, which would complicate the definition of the normal line in any stationary point of f. Definitions above actually assume that the equation of *any* tangent line may be written in the form $y = kx + l$, where $k, l \in \mathbf{R}$, but that the equation of the normal line may be a form of $x = a$, for $a \in \mathbf{R}$.

The computer program *Graph* also applies the aforementioned assumptions. Before presenting its application, we give basic information about the program.

3. Basic information about the computer program *Graph*

A computer program *Graph* belongs to the open-source[2] computer programs intended primarily for drawing graphs of mathematical functions in a rectangular coordinate system in the plane. The functions may be given in the standard (explicit), parametric or polar form. Also, the program allows drawing of point series, tangent lines and normal lines to the curve at some point, cross-hatching to indicate the plane figures, etc. The interested reader may refer to [2]. The latest version of the program is available on the website of `http://www.padowan.dk/download/` for free.

For the successful use of the program or setting function rules, it is enough to know the syntax that is used in MS Excel. Students learn working in MS Excel during their secondary education, so at the time of processing the lesson of the problem of determining tangents and normals they should have the necessary IT knowledge. We consider it appropriate to point out that the most of the students at the first year of professional studies at Zagreb University of Applied Sciences generally does not have (enough) experience with various programming languages, and by the time of processing the lesson of the problem of determining tangents and normals they still do not complete the course *Programming*. Therefore, the computer program in which this experience is not required is selected as methodically appropriate.

Working window of the program *Graph* is shown in Figure 4. How to use certain menus and options, we will show using concrete examples in the following section.

[2] Syntagma *open source* means the whole set of methods that are used in the computer applications development, with the entire development process and its results publicly available without special restrictions. This phrase is often (mistakenly!) translated to Croatian language as *application with open source*. This is only partly true, because the source code is actually the end result of the process of application development.

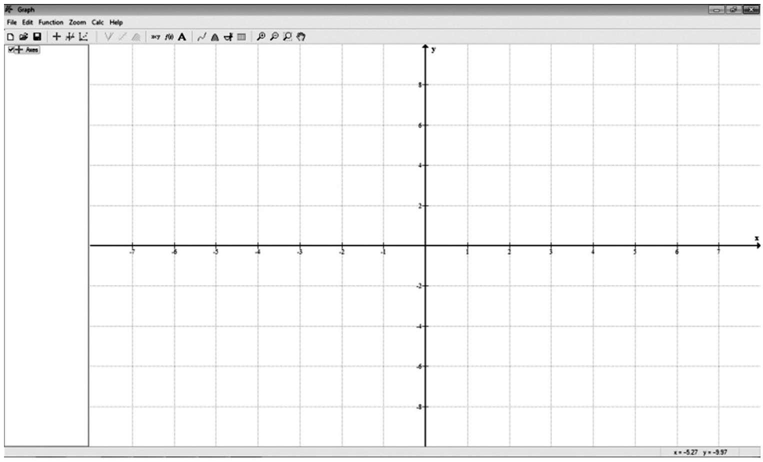

Figure 4. Working window of *Graph*.

4. Examples of the use of computer program *Graph*

To avoid increasing the volume of the text, we are giving only the end result of analytical solutions for each task. Details of task solving are left to the reader.

Example 1. Determine the equations of the tangent line and the normal line of the curve $y = x^2$ in the point $T = (1, y_T)$ of that curve ([1, 3, 6]).

Analytical solution:

The equation of the tangent line: $t ... \; y = 2x - 1$.

The equation of the normal line: $n ... \; y = -\frac{1}{2}x + \frac{3}{2}$.

Solution using the program Graph:

Open the program *Graph*. Press the *Ins* key on the keyboard or choose the option *Insert function* from the drop-down menu *Function*. We get the dialog box shown in Figure 5.

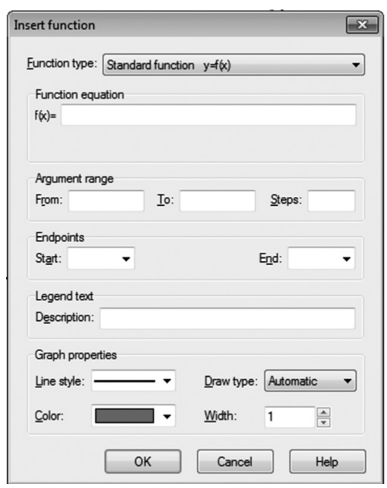

Figure 5. Dialog box for inserting a new function in *Graph*.

Click the mouse in the empty rectangle right next to the label `f(x) =`. We type: `x ^ 2`. All other rectangles are left unchanged. Then click OK. We get Figure 6.

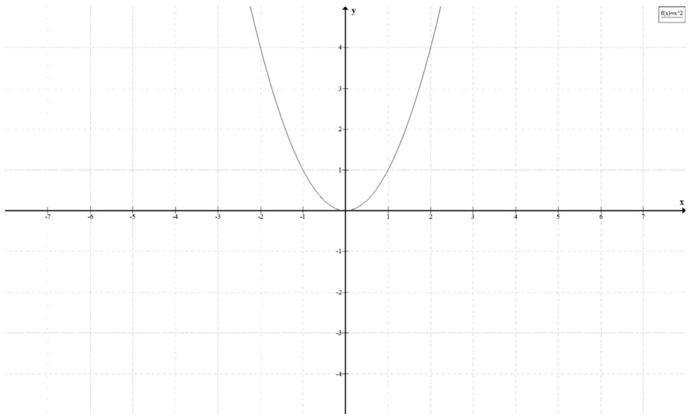

Figure 6. Curve $y = x^2$.

Then press the F2 key or choose the option *Insert tangent/normal* from the drop-down menu *Function*. We get the dialog box shown in Figure 7.

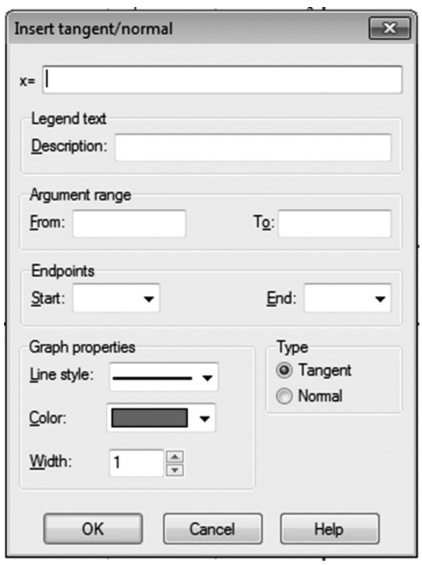

Figure 7. Dialog box for inserting tangent line or normal line in *Graph*.

Click the mouse in the rectangle right next to the label `x =`. The first coordinate of the point T equals 1, so we type 1 in this rectangle. First, we draw a tangent line and determine its equation. This option is already offered (as initial one) in the part of the box called *Type*, choose it and then click on OK. We get Figure 8.

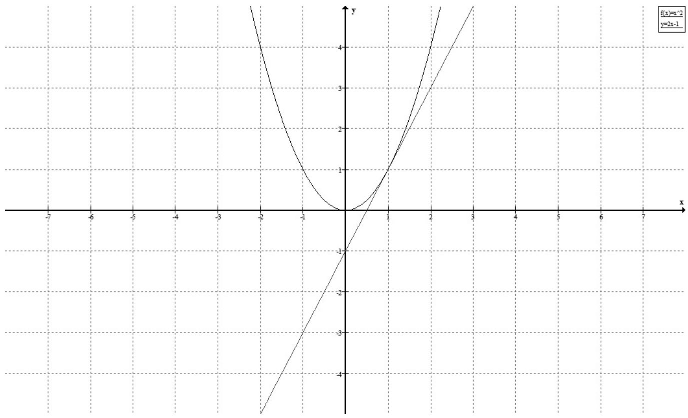

Figure 8. Curve $y = x^2$ and its tangent line at the point $T = (1, 1)$.

We can see the equation of the obtained tangent line in the legend in the upper right corner of the figure:

$$y = 2x - 1,$$

as obtained in the analytical solution. It remains to determine the equation of the normal line. For this purpose, we first click with the mouse on the equality `f (x) = x ^ 2` shown on the left side of the working window (see Figure 9). By clicking on that equality we enable the determination of the equation of the normal line.

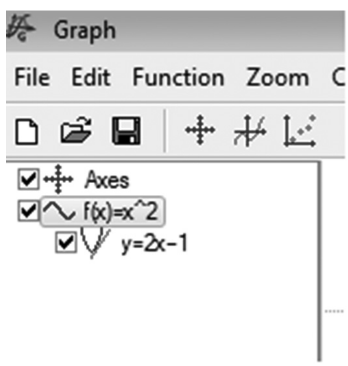

Figure 9. Choosing the curve equation before inserting its normal line.

Then we press the F2 key again. In the rectangle right next to the label `x =` we type 1, and in the part of the box below the label *Type* we click on a circle right next to the label *Normal* (See Figure 10). In order to distinguish the tangent line from the normal line, in part *Graph properties* of the graph we click on the dropdown menu next to the label *Color*, and choose eg. dark green color. Then we click OK.

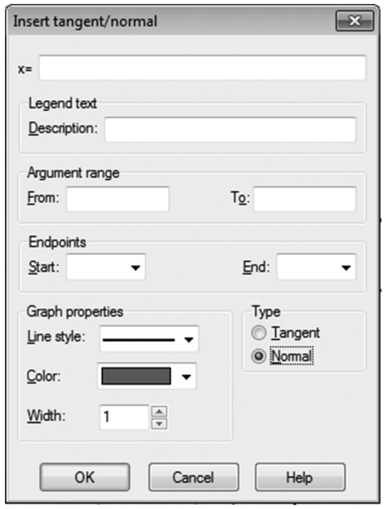

Figure 10. Entering the data needed for inserting the normal line and determining its equation.

We get Figure 11.

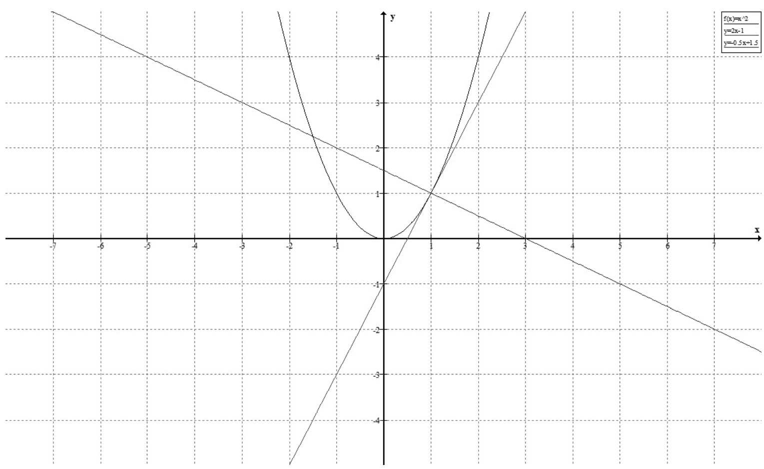

Figure 11. Solution of Example 1.

In the legend in the upper right corner we read the equation of the normal line:

$$y = -0.5x + 1.5.$$

It is easy to see that this equation is equivalent to the equation of the normal line obtained analytically. Note that the program *Graph* can not print fractions, so it replaces them with the appropriate decimal numbers.

Example 2. Determine the equations of the tangent line and the normal line of the curve $y = -x^2 - 4x - 3$ in the point $T = (-2, y)$ of that curve ([1, 3]).

Analytical solution:

The equation of the tangent line: $t \dots y = 1$.

The equation of the normal line: $n \dots x = -2$.

Solution using the program Graph:

We repeat the procedure analogous to that of the previous task. In the rectangle right next to the label `f (x)` we type: `-x ^ 2-4x-3`, and in the rectangle right next to the label `x =` in both cases we type `-2`. Finally we get Figure 12.

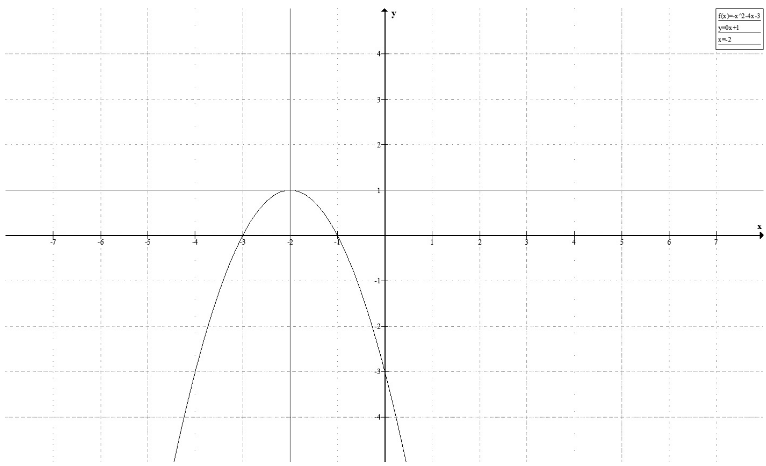

Figure 12. Solution of Example 2.

In the legend in the upper right corner we read the equations of the tangent line and the normal line:

$$
\begin{aligned}
y &= 0x + 1, \\
x &= -2.
\end{aligned}
$$

We note that the equation of the tangent line is not completely simplified, i.e. it is not written in a simpler form $y = 1$. The reason for this is the previously mentioned assumption that the equation of the tangent line must necessarily be written in the form $y = kx + l$, where $k, l \in \mathbf{R}$.

Remark 2. This task is important also because of the empirical experience of teachers, according to which many students enter the equation curve incorrect. The most common mistake is input `(-x)^ 2-4x-3` because students think that the equality $-x^2 = (-x)^2$ holds. Therefore, one of the purposes of this task methodically is emphasizing the validity of inequality $-x^2 \neq (-x)^2$, $\forall x \in \mathbf{R} \setminus \{0\}$.

Remark 3. Unlike eg. MS Excel, *Graph* allows omission of character * for the multiplication of real numbers. That is why we can type the equation of the curve in Example 2. not using that character. Of course, the entry `-x^2-4*x-3` would also be correct.

Example 3. Determine the equations of the tangent line and the normal line of the curve $y = x^3 - x$ in its intersection with the egative part of the axis abscissa ([1, 3]).

Analytical solution:

The equation of the tangent line: $t \ldots\ y = 2x + 2$.

The equation of the normal line: $n \ldots\ y = -\dfrac{1}{2}x - \dfrac{1}{2}$.

Solution using Graph:

We draw the curve using the procedure described in Example 1. We get Figure 13.

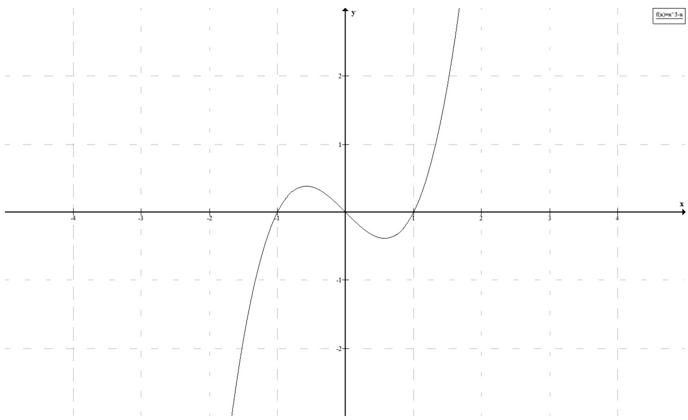

Figure 13. Curve $y = x^3 - x$.

It is clear that the intersection of the curve with a negative part of the axis abscissa is the point $T = (-1, 0)$. Therefore, we construct the tangent line and the normal line to the given curve at the point whose first coordinate is $x_0 = -1$. Continuing the process of Example 1 we get Figure 14.

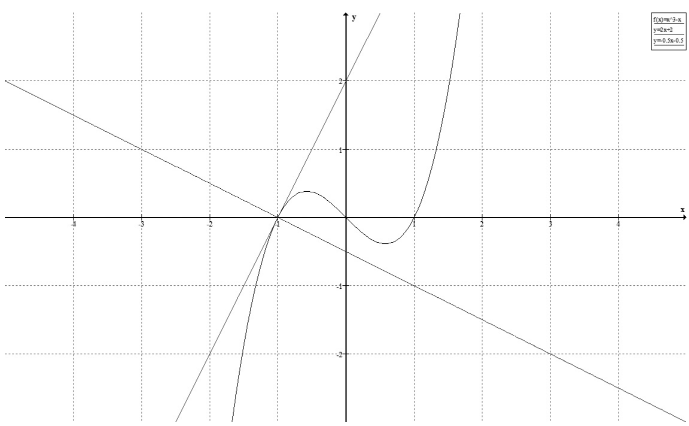

Figure 14. Solution of Example 3.

In the legend in the upper right corner we can see that the equations of the tangent line and the normal line are:

$$y = 2x + 2,$$
$$y = -0.5x - 0.5.$$

respectively.

At first glance, this example does not differ from Example 2. However, we use the possibility of the program *Graph* so that we get the graph of the curve over the segment $[-2, 2]$. For this purpose we use the icon located on the toolbar. We click with the left mouse button a few times on that icon until the entire part of the curve over the segment $[-2, 2]$ becomes visible. We get Figure 15.

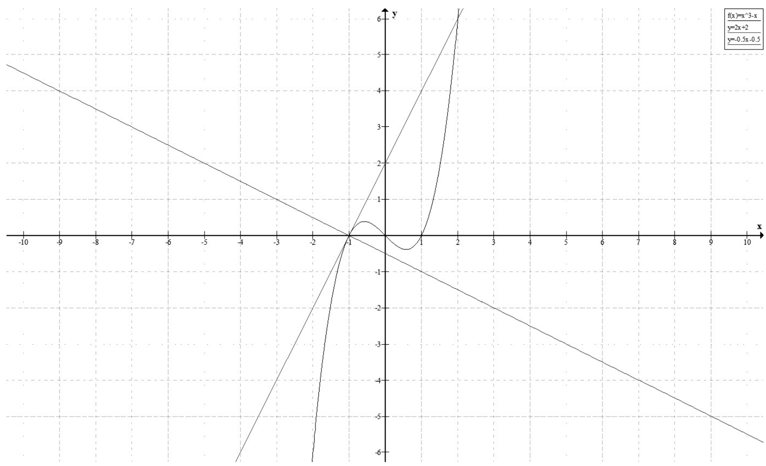

Figure 15. Curve $y = x^3 - x$ over the segment $[-2, 2]$, tangent line and normal line to the curve at the point $T = (-1, 0)$.

The figure shows that the drawn tangent line intersects the curve also in the point $T_1 = (2, 6)$.

According to the previous experience of students, obtained solution is incorrect because the line t cuts the given curve in exactly two points, and not in exactly one point. Therefore, the purpose of this example methodically is the illustration that the tangent line to the planar curve *can not be defined* as "the line which intersects the curve in exactly one point." Note that for the same purpose the example of a sinusoid and its tangent line to the point of local maximum is explained to students. That tangent line intersects sinusoid in infinitely many different points (see Figure 3).

In the last example, we show the determination of the equations of the tangent line and the normal line to planar curve given by parametric equations.

Example 4. The planar curve is given by parametric equations $\begin{cases} x = \sin t, \\ y = \cos t, \end{cases}$

for $t \in [0, 2\pi]$. Determine the equations of the tangent line and the normal line to that curve at the point determined with the parameter $t = \dfrac{\pi}{2}$ ([1, 3]).

Analytical solution:

The equation of the tangent line: $t \ldots x = 1$.

The equation of the normal line: $n \ldots y = 0$.

Solution using Graph:

First, draw the given curve. Press the *Ins* key. In the drop-down menu next to the label *Function type*, select the option *Parametric function x (t), y (t)* (see Figure 16).

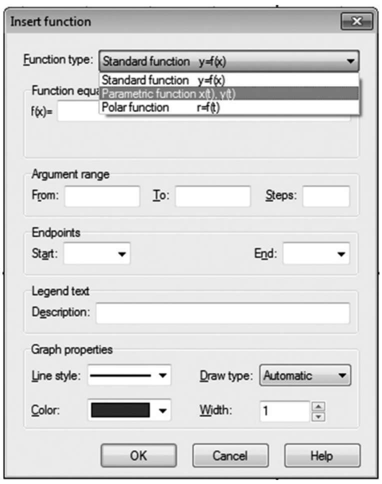

Figure 16. Choosing parametric function. as function type

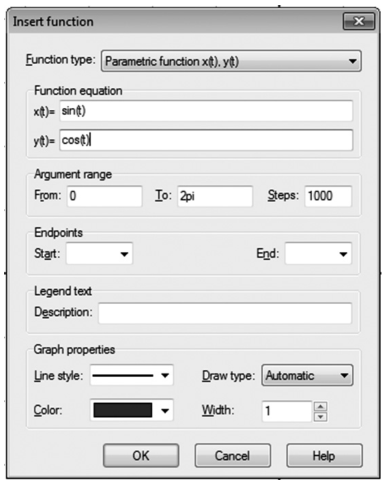

Figure 17. Entering the data for inserting the function from Example 4.

We type `sin(t)` in the rectangle next to the label `x(t)=`, and `cos(t)` in the rectangle next to the label `y(t)=`. Then under the label *Argument range* we type 0 in the rectangle right next to the label *From:* and we type `2pi` in the rectangle right next to the label *To:* (see Figure 17). Then we click OK.

We determine the equations of the tangent line and the normal line using the procedure analogous to that described in Example 1, noting that in the menu *Insert tangent/normal* instead of `x =` it stands `t =`, so in the empty rectangle right next to that label should be written `pi/2`. All other steps of the mentioned procedures remain unchanged. So we get Figure 18.

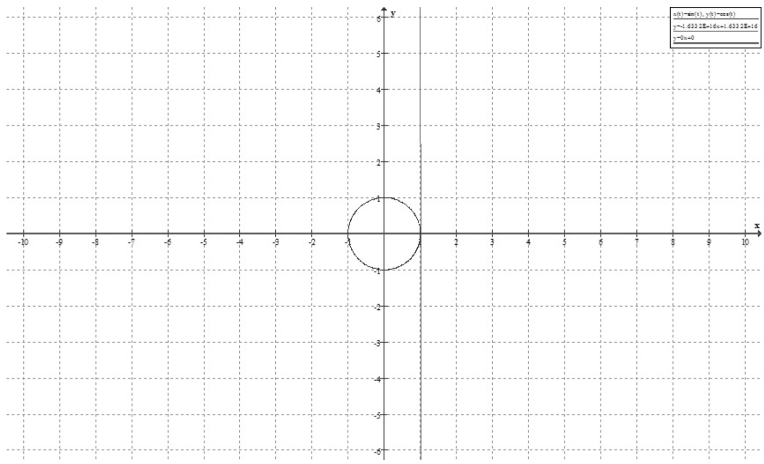

Figure 18. Solution of Example 4.

The solutions for the equations of the tangent line and the normal line are

$$y = -1.6332E + 16x + 1.6332E + 16,$$
$$y = 0x + 0,$$

respectively.

Note that the equation of the tangent line obtained using *Graph* does not coincide with the equation of the tangent line obtained analytically. This is because the slope of the tangent line “equals” $\dfrac{\sin\frac{\pi}{2}}{\cos\frac{\pi}{2}}$, and that numerical term is not defined. Even after the apparent cancellation of “values” $-1.6332E$ and 1.6332, the equation $y = 16x + 16$ is obtained which is obviously incorrect equation of the tangent line. Therefore, we can conclude that using *Graph* we can not determine the exact equation of the tangent line (to the curve given by parametric equations) which is parallel to the axis ordinate.

Remark 4. Example 4 can be solved correctly by selecting the polar form of the equation of the curve. The curve in the Example 4. is obviously central unit circle. Its equation in polar form is: $r = 1$. In the *Insert function* menu we choose the option *Polar function r = f (t)*, and we type 1 in the rectangle next to the label `r(t)=`. All other steps shown in Example 4. remain unchanged (including the entry of the values 0 and 2π). For parameter t we enter 0. So we get Figure 19.

We get the equations:

$$x = 1,$$
$$y = 0x + 0,$$

as obtained also in the analytical solution.

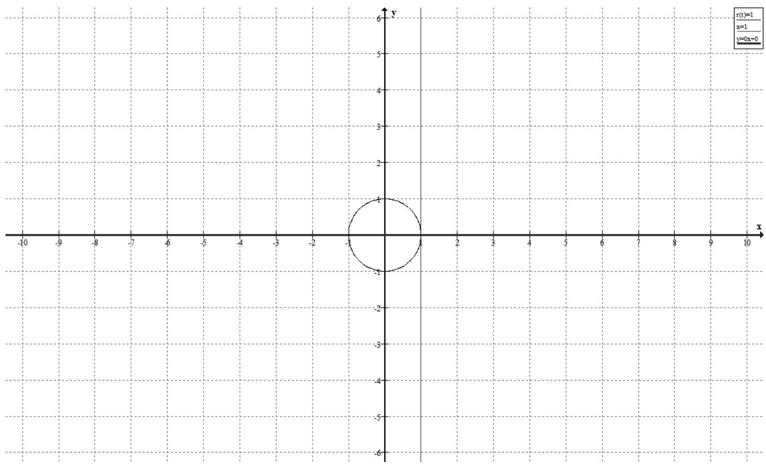

Figure 19. Solution of Example 4. in the case of polar form as function type.

5. Conclusion

At the colleges and independent colleges in Croatia the use of different computer programs as aids in teaching mathematical courses is increasing. Due to very different levels of background IT-knowledge of students, those programs should be methodically suitable and enable the most of the users to learn how to use them quickly and easily. One of such programs is *Graph*. This paper describes the application of this program in teaching unit *Problem of determining the tangent line and the normal line to the graph of real function of one variable*. However, there are other possible applications of the same program on which this paper did not discuss (for example, calculate the length of a planar curve over a segment, calculate the surface area of some plane figures, etc.). We are confident that the application of such computer programs in teaching mathematical courses would contribute positively not only to improve the quality of teaching, but also to increase the interest of students to learn mathematics, and a better understanding of mathematical problems and ways of solving them. Solving certain kinds of problems using different methods, the comparison of solving methods and quality analysis of the obtained results are useful not only in mathematics, but also in many other areas of life.

Furthermore, apart from *Graph*, there are more free programs that teachers could use: *MathGV* (available at `http://www.mathgv.com`), *Desmos* (online program available at `https://www.desmos.com/`), and *Geogebra* (available at `https://www.geogebra.org`).

The program *MathGV* has a function syntax equal to the function syntax in *Graph*, allows to draw curves whose equations are given explicitly or parametrically, but does not allow the determination of a tangent line or a normal line equation in a point of a curve. These equations must first be determined analytically, and then draw a tangent line or a normal line as a separate function graph (without any link to the starting curve).

The program *Desmos*, as an online program, can be used exclusively on the Internet (it is not possible to download and install it on a PC, and use it without access to the Internet). Its syntax is somewhat simpler than the syntax of *Graph* (eg. the function argument does not need to be entered within brackets). It allows drawing tangent lines, but only giving the expression in the form $y = f'(x_T) \cdot (x - x_T) + y_T$ where it is sufficient to define the function f and the point $T = (x_T, y_T)$, i.e. it is not necessary to enter the value for $f'(x_T)$. Analogous statement is valid for drawing normal lines.

The program *Geogebra* allows the determination of the tangent line equation using the `Tangent` function. Unlike *Graph*, *Geogebra* also allows determining tangent lines parallel to the ordinate axis, and the determination of the tangent line equation of the curve given in the implicit form. However, the analogous statement is not valid for determining the normal line. The normal line may be determined using the function `PerpendicularLine` having the point T and the tangent line t as its arguments.

To conclude, we will cite [4]:

"When paper-pencil was used to learn about graphing, the teachers' role was a task setter and an explainer. On the other hand, when students used the software in their learning, teachers' role shifted to a consultant, a facilitator and a fellow investigator."

References

[1] AGLIĆ ALJINOVIĆ, A., et al. (2014), *Matematika 1*, Element, Zagreb.

[2] JOHANSEN, I. (2017), *Graph*, version 4.4., `http://www.padowan.dk/doc/english/` [Accessed 10/02/2017].

[3] SULJAGIĆ, S. (2003), *Matematika 1*, internal script, Zagreb University of Applied Sciences.

[4] SYED, S. M. (2008), *Graphing Software in Understanding of Polynomial Functions*, California State University, Northridge, College of Education, 06/04/2008.

[5] WEISSTEIN, E. W. (2017), *Tangent Line*, `http://mathworld.wolfram.com/TangentLine.html` [Accessed 10/02/2017].

[6] WHITEHEAD, J. (2004), *Finding the Equation of Tangent Line Using First Derivative*, Texas State University, `http://gato-docs.its.txstate.edu/jcr:48ee831e-5969-4419-b9f8-820925a1b46a/Finding%20the%20Equation%20of%20a%20Tangent%20Line.pdf` [Accessed 10/02/2017].

[7] ZIAUDDIN, S. (1974), *Drawing a Tangent to a Curve*, The Physics Teacher, Vol. 12.

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Primjena računalnoga programa *Graph* u poučavanju primjene diferencijalnoga računa

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Sažetak. U radu se izlažu vlastita nastavna iskustva autora u primjeni računalnoga programa *Graph* u obradi nastavne cjeline *Problem određivanja tangente i normale na graf realne funkcije jedne realne varijable* u sastavu matematičkih predmeta na stručnim studijima Tehničkoga veleučilišta u Zagrebu. Navode se tipični problemi uočeni u obradi navedene nastavne cjeline, te na konkretnim primjerima objašnjava kako se ti problemi metodički mogu uspješno riješiti.

Ključne riječi: računalni program *Graph*, diferencijalni račun, tangenta, normala, grafovi funkcija jedne varijable

Applications of free computational software in math courses at Zagreb University of Applied Sciences

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Abstract. In this paper, authors describe their own teaching experience using Maxima and GNU Octave software packages as computational tools in math courses at studies for Electrical and Civil Engineering of Zagreb University of Applied Sciences and discuss possibilities of achieving the same learning outcomes using software package Giac/Xcas. Finally, a way of modernizing math courses by introduction and usage of suitable computer software is proposed, emphasizing the importance of using free software in education.

Keywords: teaching mathematics, free software, computer algebra system, Maxima, Octave, Giac/Xcas

1. Introduction

With aim to modernize teaching of mathematics courses, hand-on lessons have been quite often introduced in traditional teaching process, where problems are solved by using adequate software packages. About the importance of introducing CAS (Computer Algebra System) tools into the teaching process and the advantages and disadvantages of it, numerous scientific-pedagogical researches have been conducted. As an advantage of use CAS tools in math teaching, Glasnović Gracin emphasize that the computer can take on a large part of the operating part of the teaching, and the free space students can use to better understand the basic ideas in mathematics, as well as discussion and creative mathematical thinking (Glasnović Gracin, 2008). According to Schneider, the advantage of use CAS tools in the teaching of mathematics can be divided into four categories (Schneider, 2002; Glasnović Gracin, 2008): *multiple display options* (eq. transition from symbolic to graphic design); *experimental work* (the possibility of students using experimentation in order to gain new knowledge, ideas and problem solving approaches);

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elementarisation of mathematical methods (computers allow the use of basic methods that have been abandoned due to complex calculations); *modularity* (the ability to directly invoke commands and not have to bother with the steps of the algorithms or calculation methods). CAS tools offer great opportunities and maximal exploitation of their potential is need to encourage mathematical creativity of students. The opportunities of CAS tools enable them the ability of individual experimentation. Their use should not be reduced to a high performance calculator (García, 2005).

At Electrical and Civil Engineering studies of Zagreb University of Applied Sciences, teaching in existing mathematics courses has been supplemented with hands-on lessons in laboratories and new courses have been introduced where students acquire basic knowledge and skills on implementation of software for solving mathematical problems. Although Polytechnic has certain number of licenses for the MATLAB package, the idea is to introduce free software for different reasons, such as license price.

This paper presents the introduction of two such free software packages in teaching mathematics courses at Zagreb University of Applied Sciences, namely Maxima and GNU Octave. Finally, another free software package, called Giac/Xcas, is presented as a valuable tool in teaching mathematics. With the aim of making maths more accessible to students, motivating them to work and improving their understanding and adoption of mathematical concepts, phenomena and legality we introduce these software to the teaching process.

2. Maxima

Maxima is a freely available symbolic computation platform that is open source, runs on all major platforms and covers a wide range of mathematical functions. Is used to solve problems in different areas such as Algebra, Trigonometry, Linear Algebra, Differential Calculus and Integral Calculus, including 2D and 3D plotting and animation. wxMaxima is a worksheet based interface for the computer algebra system Maxima. It provides menus and dialogs for many common maxima commands, and features autocompletion, inline plots and simple animations. wxMaxima allows one to export any cellòs contents as text, as LaTeX or as MathML. An entire worksheet can be exported, either as an HTML file or as a LaTeX file (De Souza, 2004).

2.1. wxMaxima software in math courses at study of Civil Engineering

There are four mathematics courses at the undergraduate and specialist graduate professional study of Civil Engineering at Zagreb University of Applied Sciences:

- at undergraduate study – *Mathematics 1* in the first semester including $45+45$ lessons and *Mathematics 2* in the second semester including $45+30$ lessons;
- at the graduate study – *Mathematics* in the first semester including $30+30$ lessons and *Probability and Statistics* in the first semester including $15+15$ lessons.

Of all four courses, only *Probability and Statistics* includes hands-on laboratory classes, performed in the computer room. Other courses are taught in classis manner: lectures + exercises. However, since it is our goal to modernize the traditional teaching of mathematics by implementing software for solving mathematical problems, within *Computer Use in Civil Engineering* course, a portion of hand-on lessons is dedicated to the mathematical topics. Namely, we were not able to incorporate hands-on lessons into the existing hourly quota for these mathematical courses and the only solution was to introduce a new course. Course *Computer Use in Civil Engineering* is taught in the first semester, including 15+15 lectures (lectures + hands-on lessons). Hand-on lessons are performed in the computer lab, equipped with 23 computers: one student – one computer. The course was introduced in academic year 2015/2016 at undergraduate professional study of Civil Engineering, which was in detailed addressed in paper *The introduction of IT course at professional study of civil engineering* (Orlić Bachler, 2016). The course includes two software packages: AutoCAD and wxMaxima. Each package is taught in seven lessons, which, for the wxMaxima package, is sufficient to cover only basic instructions, including the following topics:

- wxMaxima installation. wxMaxima as a calculator,
- Constants and common functions. Defining functions and variables,
- Symbolic Calculations – **commands**: *Expand, Simplify, Factor and trigsimp*,
- Solving Equations and Systems of linear equations,
- Limits. Derivation. Integration (of a univariate real function),
- Matrices calculations,
- 2D plotting.

In order to master these instructions, students should be familiar with basic mathematical knowledge on matrices, systems of linear equations, univariate real function, differentiation and integrals of a univariate real function. Most of these topics were acquired during the high school education. However, these are repeated and additionally taught within *Mathematics 1* course. It is important to mention that hand-on lessons first address the AutoCAD package and then the wxMaxima package. In that way, *Mathematics 1* topics are the base for solving problems using the wxMaxima package.

Along with already mentioned capabilities, the wxMaxima package is suitable for calculations including repeated sequence of instructions, which assumes programming. As the modest number of lessons for that section is a limiting factor, as well as it is for the sections referring 3D graphics, differentiation and animation, we do not make to address these within the course. Programming in wxMaxima would be exceptionally useful for solving problems in numerical mathematics and differential calculus that are taught within the *Mathematics 2* course in the second semester. Presently, the only option for somewhat broader coverage of wxMaxima, which would include programming, differentiation and 3D graphics, is to create an optional course, taught during the second or third semester, in parallel or immediately after completing the *Mathematics 2* course. Programming would enable students to address problems referring to solving equations numerically (Newton's method), Lagrange's interpolation and numerical solution of differential equations (Euler's method). Lessons including the 3D would be fundamental for the topics covered in the *Mathematics* course referring to the graphical representation of multivariate real functions, calculation of volume and gravity center of objects and similar problems where graphical representation of given functions is required.

As already mentioned, the *Probability and Statistics* course includes two hands-on lessons in laboratory. Up to now, Microsoft Excel was used for the purpose, but considering the fact that Maxima provides statistical data processing, it is planned to use both programs for solving such problems starting from the academic year 2017/2018.

2.2. Examples problems solving from math courses by using software wxMaxima

This part of the papers presents how to implement the wxMaxime package for solving some mathematical problems addressed during exercise lessons of mathematical courses at Civil Engineering study.

(%i1) `f(x):=(x^2+3)/(x^2-1);`

(%o1) $f(x) := \frac{x^2+3}{x^2-1}$

(%i3) `eqn1:solve([diff(f(x),x,1)=0], [x]);f(eqn1);`

(eqn1) $[x=0]$

(%o3) $[\frac{x^2+3}{x^2-1}=-3]$

Stationary point is S(0,-3).

(%i4) `diff(diff(f(x),x,1),x,1);`

(%o4) $\frac{2}{x^2-1}-\frac{8x^2}{(x^2-1)^2}-\frac{2(x^2+3)}{(x^2-1)^2}+\frac{8x^2(x^2+3)}{(x^2-1)^3}$

(%i6) `f2(x):=2/(x^2-1)-(8*x^2)/(x^2-1)^2-(2*(x^2+3))/(x^2-1)^2+(8*x^2*(x^2+3))/`
`f2(0);`

(%o5) $f2(x) := \frac{2}{x^2-1}-\frac{8x^2}{(x^2-1)^2}+\frac{-2(x^2+3)}{(x^2-1)^2}+\frac{8x^2(x^2+3)}{(x^2-1)^3}$

(%o6) -8

S(0,-3) is local maximum.

(%i7) `eqn2:solve([diff(diff(f(x),x,1),x,1)=0], [x]);`

(eqn2) $[x=-\frac{\%i}{\sqrt{3}}, x=\frac{\%i}{\sqrt{3}}]$

Inflection point does not exist.

(%i9) `limit(f(x), x, 1);limit(f(x), x, -1);`

(%o8) *infinity*

(%o9) *infinity*

Vertical asymptote are x=1, x=-1.

(%i11) `limit(f(x)/x, x, inf);limit(f(x)/x, x, -inf);`

(%o10) 0

(%o11) 0

Oblique asymptote does not exist.

(%i13) `limit(f(x), x, inf);limit(f(x), x, -inf);`

(%o12) 1

(%o13) 1

Horinzontal asymptote is y=1.

Figure 1. Finding local extrema, inflection points and asymptotes of an univariate real function.

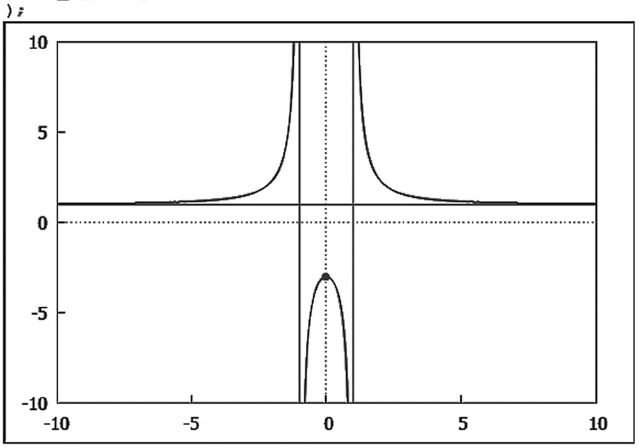
Figure 1 and Figure 2 shows an example of which is processed in *Computer Use in Civil Engineering* and *Mathematics 1* courses.

```
(%i1)   f(x):=(x^2+3)/(x^2-1);
```

$$\text{(\%o1)} \quad f(x) := \frac{x^2+3}{x^2-1}$$

```
(%i3)   wxdraw2d(
        xaxis=true,yaxis=true,
        xrange=[-10,10],yrange=[-10,10],
        color=black,
        explicit((f(x)),x,-10,10),
        color=red,
        explicit((1),x,-10,10),
        parametric(1,t,t,-10,10),parametric(-1,t,t,-10,10),
        point_type=7,points([[0,-3]])
        );
```

(%t3)



(%o3)

Figure 2. Plot of an univariate real function along with all asymptotes and local extrema.

Solution of the problem dealt with in the *Mathematics 2* course using the wxMaxima package is shown in Figure 3.

```
With Newton's method solve equation 2*log(x)+x-3=0
with accuracy of 0.00001. The initial approximation is 1.
(%i1)   nm(t):=subst(t,x,x-f(x)/diff(f(x),x));
```

$$\text{(\%o1)} \quad nm(t) := subst\left(t, x, x - \frac{f(x)}{diff(f(x), x)}\right)$$

```
(%i2)   f(x):=2*log(x)+x-3;
(%o2)   f(x):=2 log(x)+x-3
(%i3)   x0:1;
(x0)    1
(%i4)   x1:nm(x0),numer;
(x1)    1.666666666666667
(%i5)   x2:nm(x1);
(x2)    1.808340342030918
(%i6)   x3:nm(x2);
(x3)    1.811588551325027
(%i7)   x4:nm(x3);
(x4)    1.811590082982117
(%i8)   x5:nm(x4);
(x5)    1.811590082982456
```

Figure 3. Newton's method.

Figure 4 shows solution of the problem dealt with in the *Mathematics* course using the wxMaxima package.

```
(%i1)   z(x,y):=16*y+8/x+x^2/y;
```

$$\text{(\%o1)}\quad z(x,y) := 16\,y + \frac{8}{x} + \frac{x^2}{y}$$

```
(%i2)   zx(x,y):=''diff(z(x,y),x,1);
(%o2)   zx (x,y) :=diff (z (x,y) ,x,1)
(%i3)   zy(x,y):=''diff(z(x,y),y,1);
(%o3)   zy (x,y) :=diff (z (x,y) ,y,1)
(%i4)   algsys([zx(x,y)=0, zy(x,y)=0], [x,y]);
```

$$\text{(\%o4)}\quad [[x=-\%i, y=\frac{\%i}{4}], [x=\%i, y=-\frac{\%i}{4}], [x=0, y=0], [x=-1, y=-\frac{1}{4}], [x=1, y=\frac{1}{4}]]$$

```
(%i5)   H:hessian(z(x,y),[x,y]);
0 errors, 0 warnings
```

$$\text{(H)}\quad \begin{bmatrix} \frac{2}{y}+\frac{16}{x^3} & -\frac{2\,x}{y^2} \\ -\frac{2\,x}{y^2} & \frac{2\,x^2}{y^3} \end{bmatrix}$$

```
(%i7)   H11S1:ev(H[1,1],x=-1,y=-1/4);D1:ev(determinant(H),x=-1,y=-1/4);
(H11S1) -24
 (D1)     2048
 Stationary point S1(-1,-1/4) which is H11(S1)<0
 and det(H11(S1))>0 is a local maximum.
(%i9)   H11S2:ev(H[1,1],x=1,y=1/4);D2:ev(determinant(H),x=1,y=1/4);
(H11S2) 24
 (D2)     2048
 Stationary point S2(1,1/4) which is H11(S1)>0
 and det(H11(S1))>0 is a local minimum.
```

Figure 4. Finding local extrema of a multivariate real function.

3. GNU Octave

GNU Octave (in further text: Octave) is a high-level scientific programming language primarily intended for numerical computations. Development of Octave started back in 1988, almost thirty years ago, making it a rock-solid and stable system which is still actively developed today. Similarly to Maxima, it has a command line interface but also features a native graphical user interface as of version 4.0. Octave is open source freely available for instalation on all major platforms.

Octave is capable of solving numerical linear algebra problems, manipulating with matrices and polynomials, finding the roots of nonlinear equations and systems of equations, integrating ordinary differential equations and basic statistical analysis out of the box. It also features a powerful plotting system capable of creating 2D and 3D plots. Moreover, Octave is designed to be extensible, which means that user can define custom functions and load third-party packages, called *toolboxes*, for example Signal Processing Toolbox or Symbolic Toolbox, among many others. Toolboxes add new functionality to Octave and are also freely available.

Octave language was conceived to be compatible with MATLAB, a proprietary software which provides the *de facto* standard language of science and engineering. From that perspective, Octave is a free replacement for MATLAB and developers pay great attention that any code written in MATLAB can be – without any modification – processed in Octave with the same result. It means that any knowledge of MATLAB core functions is completely usable in Octave, making transition very easy. The two languages are mostly compatible the other way too, meaning that existing Octave code may be used in MATLAB. However, some modifications may be needed because Octave's parser allows some (useful) syntax that MATLAB does not support (see [3]).

3.1. GNU Octave in math courses at study of Electrical Engineering

Octave is used as the main software package in course Math Software in Electrical Engineering, which is mandatory for all students in second semester of undergraduate study of Electrical engineering at Zagreb University of Applied Sciences, consisting of 30 hands-on lessons in a computer lab. The goal of the course is to teach students the basics of Octave, a knowledge usable in subsequent, more specialized courses, where MATLAB is often used. An attention is paid to Octave syntax used in lessons so that the same code can run in MATLAB without errors. In the first half of the course, students learn how to:

- use Octave as a calculator,
- enter and manipulate matrices,
- define custom functions,
- plot graphs of real functions, including implicitly defined and parametrized functions,
- do simple programming with *m*-files (see below).

In the other half of the course, Symbolic Toolbox is utilized to teach students how to:

- enter algebraic expressions,
- differentiate and integrate real functions,
- find limits,
- sum numeric series,
- approximate real functions with Taylor and Fourier polynomials,
- solve ordinary differential equations,
- find Laplace transforms and inverse Laplace transforms.

One of the most useful features of Octave is the ability to interpret *m*-files at runtime. *m*-files are simply text files containing Octave commands, which will be executed in order they're written when the name of *m*-file is typed to the command line. Since Octave language supports "if-else" clauses and "while" and "for" loops, i.e. the basic program flow modifiers, it allows user to write a program in Octave language producing result of the given problem, save it to text file and run it when s/he wishes. Also, *m*-files may be designed to accept any number of parameters at calling time.

It was observed that students find program flow organization when writing *m*-files particularly difficult, especially when it comes to the usage of loops, which indicates significant lack of time high school teachers spent teaching them programming basics in hands-on manner, probably due to the outdated curricula and insufficient resources. It should be noted that vast majority of students graduated at vocational schools.

Two examples of using Octave are shown in Figure 5 and Figure 6.

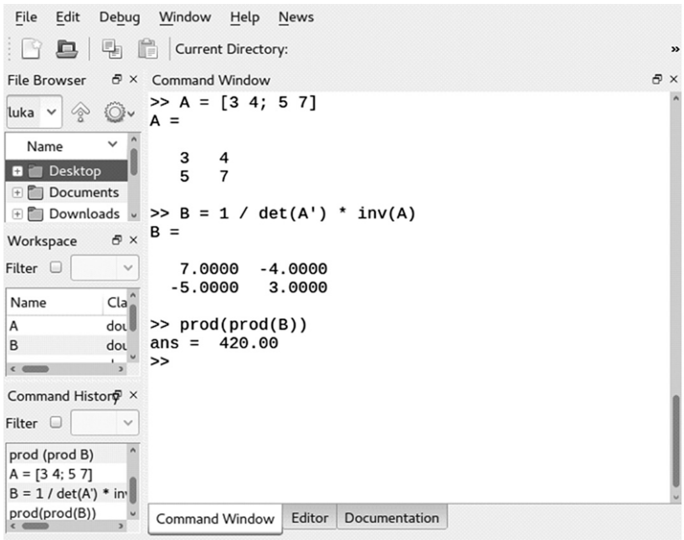


Figure 5. Finding product of all elements of matrix B in Octave.

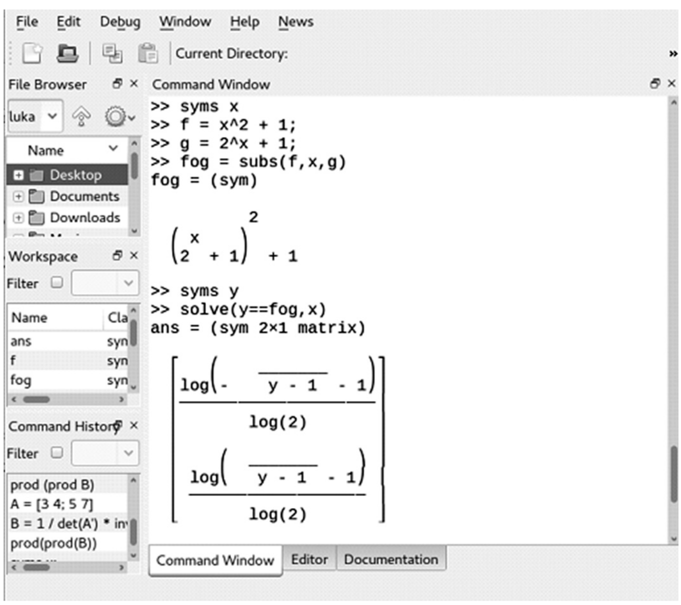


Figure 6. Using Octave's Symbolic Toolbox to determine the inverse function of composition of functions f and g.

4. Giac/Xcas

Giac is a general purpose computer algebra system (CAS) with worksheet-based graphical user interface called Xcas. It is open source package, easily installable in all major platforms. Giac development was initiated in 2000 by Bernard Parisse from Institut Fourier (University of Grenoble I, France), who is still the project's main developer. Version 1.1 of Giac kernel is used in Hewlett-Packard Prime calculators, providing CAS and graphing support (R. de Graeve et al., 2007).

Giac/Xcas, dubbed "the swiss knife for mathematics", can handle:

- arbitrary precision integers and floats,
- integer and polynomial arithmetic,
- simplification,
- equation solving,
- calculus (derivation, integration, finding limits, series expansion),
- differential equations,
- linear algebra,
- graph plotting,
- interactive 2D and 3D geometry,
- spreadsheets,
- interactive programming,
- statistics,
- computation with physical units and constants.

Giac/Xcas language syntax and user interface were conceived to be appropriate for high school and undergraduate students. It is compatible with Maple and MuPad and provides a range of convenient functions. It has many useful and often unique features: for example, it is very easy to solve inequalities, plot solution of an inequality, enter piecewise defined functions, find intersection of graphs, curves and geometrical objects, etc. The welcome feature not found in other CAS applications is the basic but practical support for spreadsheets including importing/exporting of CSV (comma-separated value) files. Also, Xcas features command autocompletion and a simple but very effective interactive help system, which provides a brief description, usage syntax and a few examples for every built-in command. The online documentation is available in French and English.

Worksheets in Xcas are sequences of cells, which can be of different types. A single cell can be a standard command line entry, a 2D or 3D geometry figure, a program, a spreadsheet or a comment. Hence it is easy to create versatile and detailed educational sheets containing detailed step-by-step solutions. Worksheets can be saved and subsequently loaded, and there is a possibility of export to PDF or HTML as well.

Example of using Xcas when drawing tangent to the graph of a function is shown in Figure 7.

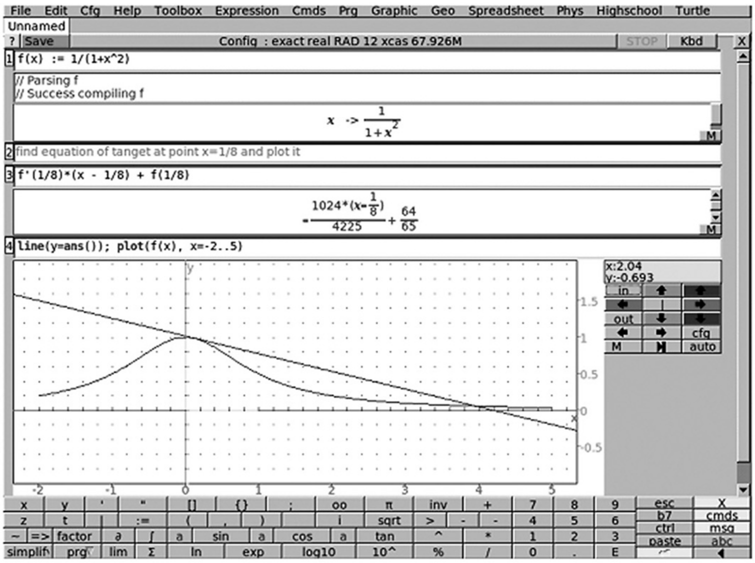


Figure 7. A worksheet in Xcas.

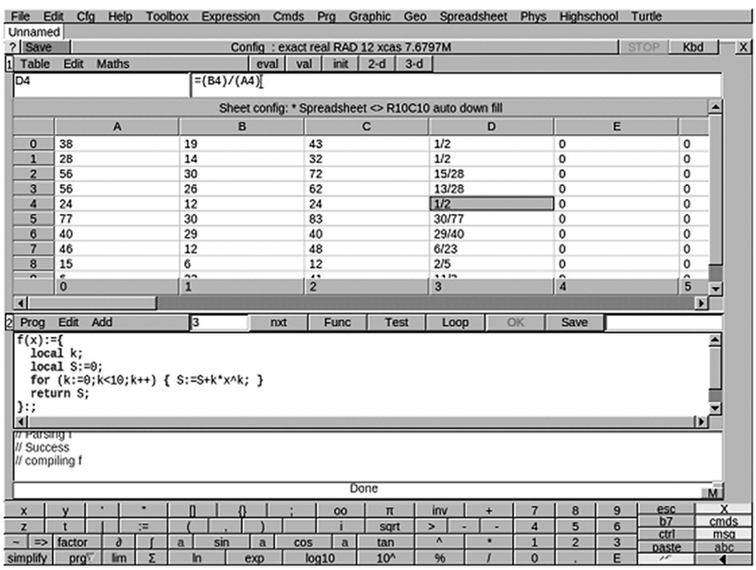


Figure 8. Spreadsheet and programming cells in the Xcas worksheet.

There is an intention to introduce Giac/Xcas instead Octave's Symbolic Toolbox in second half of previously mentioned course Math Software in Electrical Engineering where students learn basics of symbolic computation.

5. Advantages and disadvantages of using CAS software in teaching

Using CAS software in education has certain benefits. Some advantages are listed below.

1. Freeing the student from tedious and technical work saves his/her time and concentration to focus more on the translation of realistic problems into mathematical models and to the interpretation of the results with respect to the context (Drijvers, 2000).

2. It is easy to switch between different types of mathematical representations, such as graphs, tables and formulas, which is best illustrated with a worksheet-based software such as Xcas. In particular, it is easy for the student to visualize two- and three-dimensional concepts.

3. As the CAS software feedback is practically instant for ordinary tasks, the student is motivated to explore and subsequently, through generalization and reflection, reinvent the already known principles and theorems (Drijvers, 2000).

4. The CAS input requires strict syntax. That may be considered as a downside at first, but it actually may help the students to understand the structure of an algebraic expression better. The student can copy a formula by hand into a notebook without understanding how exactly its elements are related; but, when s/he has to copy (enter) the same formula into a computer interface, the expression needs to be syntactically correct.

On the other hand, there are some disadvantages and pitfalls:

1. A CAS software is generally a "black box", giving no insight in the work done and methods used to produce result. These features are difficult to implement as the methods will often be far more sophisticated than those the student learns in the course. The other reason is that algorithms that produce the result usually aren't executable by humans, so there is no reason to show the details of the process. For example, the "work" which CAS does to simplify an algebraic expression is expected to be unreadable as the respective algorithms generally work by "inflating" and then "deflating" the expression, making it possibly very large somewhere around the middle of the process.

2. An algebraic expression being the result of a command may often not be what the student considers "simple" and convenient. One simply cannot expect the beauty of formulas for which mathematics is known to be reflected in CAS results. Also, the results may be presented in unfamiliar ways.

3. The 'CAS-language' is different from mathematical and natural language, and the system does not allow informal language (Drijvers, 2000).

6. Conclusion

The CAS software should not be neglected in teaching mathematics as nowadays it represents a reliable tool for doing technically complex symbolic computation. For example, a theoretically simple operation can easily turn to an exceedingly tedious calculation when done by hand, although the result may be simple. The same operation, however, may present no problem for an ordinary CAS application; such operations include partial fraction decomposition, factoring polynomials and finding higher-order derivations. These topics present great opportunities to demonstrate the power and capabilities of CAS software. The other major benefit of using CAS in teaching is easy visualization of function graphs and geometrical concepts. However, pencil and paper work cannot be eliminated: when presented with an exercise, the student should first formulate the problem and decide how to solve it. The basic steps, including introducing the notation, should be put on paper, enabling the effective usage of a computer as a working horse to solve each step and to combine their results.

Every software package described in this paper is licensed with GNU General Public License. Hence its source code can be freely examined, modified, compiled and forked as long as the license conditions are retained. Although the most obvious benefit of free and open source software (FOSS) is that it is available free of charge, this particular “freedom” is crucial for math software. Mathematical knowledge can’t be patented, and so shouldn’t be any math software, since it implements algorithms (which are part of that same knowledge) and cannot be just “trusted” with no possibility of examining the implementation and improving it. In particular, proprietary software (usually closed-source) should not be endorsed in public education, as “the code is not generally available for examination, [which] raises reliability concerns. In a higher educational context, trusting the software is not so much the issue as it is teaching mathematics upon the same foundations as mathematics is built upon: openness” (Botana et al. 2014). It is therefore important for teachers to explore the possibilities of meeting computational requirements for their lessons in the scope of FOSS. That way students would be able to install the software used in class on their personal computer units and practice at home, and schools could use financial resources to buy more computers instead of paying for expensive and restrictive proprietary software licenses. Also, teachers could contribute by writing textbooks in their own language, thereby improving the package documentation.

References

[1] BACHLER, M. O. (2016, January), *The introduction of IT course at professional study of Civil Engineering*, in 3^{rd} International Multidisciplinary Scientific Conference on Social Sciences and Arts SGEM 2016, pp. 1139–1146.

[2] BOTANA, F., et al. (2014), *Using a free open source software to teach mathematics*, Computer Applications in Engineering Education, vol. 22(4), pp. 728–735.

[3] DE SOUZA, P. N., et al. (2004), *The Maxima Book*, http://maxima.sourceforge.net/docs/maximabook/maximabook-19-Sept-2004.pdf [Accessed 22/1/2017].

[4] *Differences between Octave and MATLAB*, https://en.wikibooks.org/wiki/MATLAB_Programming/Differences_between_Octave_and_MATLAB [Accessed 22/1/2017].

[5] DRIJVERS, P. (2000), *Students encountering obstacles using a CAS*, International Journal of Computers for Mathematical Learning, vol. 5, pp. 189–209.

[6] GARCÍA, J. G., et al. (2005), *Computer Algebra Systems: A basic tool for teaching Mathematics in Engineering*, Recent Research Developments in Learning Technologies, pp. 1060–1064.

[7] GLASNOVIĆ GRACIN, D. (2008), *Računalo u nastavi matematike-teorijska podloga i metodičke smjernice*, Matematika i škola, vol. 10(46), pp. 10–15.

[8] R. DE GRAEVE ET AL. (2007), *Symbolic algebra and Mathematics with Xcas*, http://www-fourier.ujf-grenoble.fr/~parisse/giac/cascmd_en.pdf [Accessed 22/1/2017].

[9] SCHNEIDER, E. (2002), *Computer algebra systeme in einem allgemeinbildenden Mathematikunterricht: didaktische Orientierungen-praktische Erfahrungen*, Profil-Verlag.

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Primjena slobodnih softvera u nastavi matematike na Tehničkom veleučilištu u Zagrebu

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Sažetak. U radu autori opisuju svoja iskustva u primjeni programskih paketa Maxima i GNU Octave u nastvavi matematike na stručnim studijima elektrotehnike i graditeljstva na Tehničkom veleučilištu u Zagrebu, te ukazuju na mogućnosti postizanja istih ishoda učenja primjenom softverskog paketa Giac/Xcas. Na kraju rada, predlažu način modernizacije nastave matematike, uvođenjem i korištenjem odgovarajućeg računalnog softvera, naglašavajući važnost korištenja besplatnog softvera u obrazovanju.

Ključne riječi: podučavanje matematike, slobodni softver, računalni algebarski sustav, Maxima, Octave, Giac/Xcas

5.

Attitudes toward and beliefs about mathematics teaching

Mathematics attitudes among students of Civil Engineering

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Abstract. The aim of this paper is to identify attitudes of undergraduate students of Civil Engineering towards mathematics by studying the responses of 193 students. This study examines whether there is a connection between attitudes towards mathematics and performance on exams in mathematics. Furthermore, differences in attitudes about math regarding gender and prior education (vocational or grammar school) were also analyzed. The Attitude Toward Mathematics Inventory (ATMI) questionnaire was used for this research. Data were analyzed with analytics software package Statistica 13, t-test, Mann-Whitney and Kruskal-Wallis tests were also used.

The survey results suggest a conclusion that there is no difference in attitudes towards mathematics regarding gender. Grammar school students scored significantly higher than vocational students in overall attitudes, but also in all four sub-scales Enjoyment, Motivation, Value and Self-confidence. Students with the lowest scores on the exam have more negative attitudes and vice versa. This also goes for the subscales Enjoyment, Value and Self-confidence, but not for Motivation.

Keywords: math attitudes, math performance, ATMI questionnaire, statistical analysis, undergraduate students

1. Introduction

In recent years, passing rates of all teaching courses at the Faculty of Civil Engineering Osijek have been intensively monitored. The most recent results were for the academic year 2014/2015. In total at the undergraduate and graduate university studies and undergraduate professional studies, six mathematics courses were held. The passing rates on the colloquium ranges from 23 % for Mathematics 2 (2^{nd} semester of undergraduate university study) to 41 % for Mathematics 3 (3^{rd} semester of undergraduate university study). The passing rates on the exams range from 16.04 % for Mathematics 3 to 56.67 % for Probability and Statistics (1^{st}

semester of graduate university study). The average grade for all math courses, with the value of 2.46, is very low.

To improve the students' results, various special measures are being performed (such as the implementation of students' survey, more transparent information of the expectations of students in individual courses, etc.). At this technical faculty, mathematics courses are basic courses and the knowledge of mathematics is the basis for understanding many other professional courses. Students come to the faculty with pre-built attitudes towards mathematics. Those attitudes have been previously built during their schooling. The literature often relates the attitudes towards mathematics with the results achieved in mathematical tasks i.e. exams (Tapia and Marsh, 2004, Goy et al., 2015). We found the study which demonstrated that mathematics attitudes contributed to students' mathematics performance over and above personality and cognitive ability (Lipnevich, et al., 2016). It can be said that positive attitude toward mathematics contribute significantly to the prediction of mathematics achievement (Nicolaidou and Philippou, 2003; Rabab'h and Veloo, 2015). Kibrislioglu (2015) investigated the difference between students' performance in mathematics and attitude scores. He showed that this difference is significant for high and low achiever, but not for low and medium achiever (Kibrislioglu, 2015).

The purpose of this article is to identify attitudes of undergraduate students of Civil Engineering towards mathematics. This study examines whether there is a connection between attitudes towards mathematics and achievements in mathematics. Furthermore, differences in attitudes about math regarding gender and prior education (vocational or grammar school) were also analyzed. We hope that this research will bring useful information to all participants (both students and teachers) in the process of learning and teaching mathematics at higher level institutions and that results will contribute to improve the mathematics performance.

What is, in fact, the meaning of the word *attitude*? Many dictionaries define it as "the way you think and feel about someone or something". The most detailed definition describes *attitude* as follows: "A predisposition or a tendency to respond positively or negatively towards a certain idea, object, person, or situation. Attitude influences an individual's choice of action, and responses to challenges, incentives, and rewards (together called stimuli). Four major components of attitude are affective (emotions or feelings), cognitive (belief or opinions held consciously), conative (inclination for action), evaluative (positive or negative response to stimuli)" (Business Dictionary, 2017). Such a definition leads to conclusion, made by many other scholars as well, that the study of attitudes towards learning in general and in this case mathematics specifically, matters.

There are many scientific paper regarding this topic, that have been either published in journals, or presented at conferences or written as doctoral thesis. Review papers analyzing different methodologies and scientific approaches to this topic are also interesting. There is a further research and development suggestion coming from one of such review papers, and, although it dates from 1970, it is very much up to date: "Suggestions for further research included adequate familiarization with previous studies concerned with the topic, more extensive multivariate experiments extending over longer periods of time, and more attention to techniques for developing positive attitudes and modifying negative attitudes" (Aiken, 1970).

2. Literature review

2.1. Instruments for measuring attitudes toward mathematics

In their research, the scientists have used a variety of methods to collect information for assessment of students' attitudes toward mathematics. Corcoran and Gibb in 1961 systematized methods and divided them into three groups: interviews, observation methods and the methods in which students themselves assess their attitudes – such as questionnaires, attitude scaling and even writing essays (Aiken, 1970).

The focus of this paper is the method of attitude scaling. There are three different techniques of this scaling method: Thurstone, Likert and Guttman scale. The Guttman scaling procedure is rarely used, probably due to its questions that enable only binary responses. As such, it is much more suitable for cognitive tests than for attitude assessment tests (Aiken, 1970).

The Thurstone scale is the first scale used for attitudes measurement. It is the method of equal-appearing intervals, consisted of a series of statements in which respondents are asked to grade in respect of their (dis)agreement. One of the oldest and often used examples of Thurstone scale is a Dutton scale. Dutton used 15 – item scale to assess attitudes toward arithmetic, originally of teachers, and later also to investigate the attitudes of high school students (Dutton, 1962).

Still, most researchers use Likert scale, probably because it is easiest to form. Likert method consists of a series of statements to which respondents allocate a numeric value of 1 to 5, depending on the level of disagreement or agreement with the statement. After that, the data is being summarized and statistically processed. A higher value indicates a more positive attitude towards the subject matter. A one-dimensional scale was developed by Aiken and Dreger (Aiken and Dreger,1961). Later, a two-dimensional scale was introduced, measuring enjoyment of mathematics (with 10 items) as well as the appreciation and use of mathematics (with 11 items) (Aiken, 1974). Finally, different multi-dimensional scales have been introduced. The most popular such scale is Fennema – Sherman Mathematics Attitude Scales, which has been used since 1976. This scale includes in fact 9 sub-scales: (1) Attitude Toward Success in Mathematics Scale, (2) Mathematics as a Male Domain Scale, (3) and (4) Mother/Father Scale, (5) Teacher Scale, (6) Confidence in Learning Mathematics Scale, (7) Mathematics Anxiety Scale, (8) Effective Motivation Scale in Mathematics, and (9) Mathematics Usefulness Scale (Fennema, Sherman, 1976).

One of the subject that is being often highlighted and investigated is the math anxiety. The three most popular scales developed to measure mathematics anxiety are MARS (Mathematics Anxiety Rating Scale) created 1972. by Richardson and Suinn, ten years later Plake and Parker improved that scale and thus developed

MARS Revised and finally MAQ (Mathematics Anxiety Questionnaire), whose authors are Wigfield and Meece (Tapia and Marsh, 2004).

Fennema – Sherman Mathematics Attitude Scales consists of as many as 108 items, and it is often necessary to have a scale that does not require so much time to be applied (Fennema, Sherman, 1976). A shorter procedure has been created because of that and it is called Attitudes Toward Mathematics Inventory (ATMI). ATMI consists of 49 items that have been modeled in a way to explore 6 factors: (1) Confidence, (2) Anxiety, (3) Value, (4) Enjoyment, (5) Motivation and (6) Parent/teacher expectations (Tapia, 1996). Even ATMI was reduced to 40 items and only 4 factors: (1) Self – Confidence, (2), Value of Mathematics, (3) Enjoyment of Mathematics and (4) Motivation (Tapia and Marsh, 2000).

Author of this paper has chosen the very last described method, the shorter version of ATMI, for the method of her own research.

2.2. Some results of measuring attitudes toward mathematics

Many studies investigated relationship among demographic data, such as gender, ages, social and ethnic background as well as success on the math test and attitudes toward mathematics (Aiken, 1970, Mata et al., 2012).

Research suggests that more negative attitudes are there among older students (Terwilliger & Titus, 1995).

The correlation among success in mathematics and attitudes toward mathematics is probably the most researched question that is seeking to be answered. Many studies have proven that there is a strong positive correlation among these two (Aiken, 1970, Kadijevich, 2008, Ma and Kishor, 1997). Individual factors have been often compared as well. It has been concluded that self – confidence positively correlates with achievement, although this may not necessarily imply a cause – and effect relationship (Siew, 2010, Kadijevich, 2008).

There is a significant difference in students' attitudes regarding the type of their previous secondary schooling (Tekere et al., 2011).

It turns out that, in some studies, there are no significant differences of opinion regarding gender (Nicolaidou and Philippou, 2003; Kibrislioglu, 2015), however some studies suggest otherwise (Siew, 2010). Sometimes boys showed significantly higher degree of motivation, self-confidence and interest than girls (Terwilliger & Titus, 1995). In contrast, it has been shown that girls have significantly more positive attitudes towards mathematics in general than boys (Tekere et al., 2011). These inconsistencies in the results may be attributed to the variety in cultures in which the research was conducted as well as the age difference of the respondents.

3. Methodology

A questionnaire has been conducted in this study. It contains two parts: Attitude toward Mathematics Inventory (ATMI) and basic demographic questions (Tapia, 1996; Tapia and Marsh, 2004). ATMI was translated to Croatian language. The data was collected in January 2017. The survey was voluntary and anonymous. The questionnaire was offered to the Faculty of Civil Engineering students of first and second year on all programs of undergraduate studies. They could choose a questionnaire in paper or online form. Most of them preferred paper form and only the one group (first year of professional studies) filled out the online questionnaire, which was made in Moodle.

3.1. Questionnaire

Shorter version of ATMI was selected because this is the instrument that is relatively short (it contains only 40 items) and it doesn't require a lot of time to complete. It was shown that reliability coefficient Cronbach alpha for total scale was 0.97 (Tapia and Marsh, 2004). Tapia and Marsh retested reliability. The new result was 0.89. These data indicate that the scores on the inventory are stable over time (Tapia and Marsh, 2004).

Khine and Afari found out that reliability and stability of ATMI was not violated by translating it into Arabic language (Khine and Afari, 2014).

ATMI items were created using Likert – scale. It means that students would choose one value from five-point scale which the best describe their feelings for each statement. Scale was ranked from (1) strongly disagree, (2) disagree, (3) neutral, (4) agree to (5) strongly agree. Some of the statements, 11 of them, have been written in reverse (negative) form. Their values have been corrected before processing of the statistical data.

Statements from ATMI were grouped into four subscales known like factors, named Enjoyment (10 items), Value (10 items), Motivation (5 items) and Self-confidence (15 items).

3.2. Results and discussion

There were altogether N=193 collected surveys in total. Table 1 shows students' demographic properties such as gender, type of graduated high school and current year and type of study. Table 1 also indicates that there were 10 % more male participants than female participants. Almost 52 % (N=102) students finished some type of grammar school. On Faculty of Civil Engineering Osijek there are three types of first year studying (vocational, civil undergraduate and architecture undergraduate). We don't have second year architecture students yet because study of architecture started this year.

Table 1. Demographic properties of students.

	Property	*f*	%		Property	*f*	%
Gender	Female	85	44.04	**Year of study**	First year vocational	55	28.5
	Male	106	54.92		Second year vocational	28	14.51
Type of graduated High school	Math grammar school	21	10.88		First year civil undergraduate	61	31.61
	Other type of grammar school	81	41.97		Second year civil undergraduate	35	18.13
	Vocational high school	84	43.52		First year architecture undergraduate	14	7.25
	Other high schools	7	3.63				

All statistical analysis was calculated in software package Statistica 13. In Table 2, data is presented from descriptive analysis for each item individually from ATMI. We can see that some questions remain unanswered by some students and because of that, in our analysis we didn't use the sum of variables than theirs means. Also, we wanted to avoid different scales for each factor.

The minimum for each variable was 1, and maximum for each variable was 5 and it was represented in all ratings for each question.

The statement with the lowest mean 2.28 was: "I am willing to take more than the required amount of mathematics". It tells us about the lack of motivation among students. Unfortunately, we can also see that in our daily work. Most of students give a minimum in their work, and often they are not willing answering for higher grade. As opposed to that, students respect the value of mathematics. They are aware of its importance, as the statement with highest mean 4.04 confirms. It was: "Mathematics helps develop the mind and teaches a person to think".

We want to point out another statement: "I would like to avoid using mathematics in college". ATMI was primary constructed for high school students, and for them this question makes more sense. Mean for this question was 3.27 (before reversing answer it was 2.73). Some respondents strongly agreed with this statement, but technical faculty needs strong support of mathematic which is contradictory by itself. It suggests us that maybe some of the students are studying on the wrong faculty.

In Table 3 it is shown a descriptive analysis of four subscales of grouping variables. The data confirms what has been suggested in previous analysis. Factor Motivation has the lowest mean 2.69 and factor Value has the highest mean 3.5. The minimum value 1 of Enjoyment and Motivation means that at least one respondent was assessed the all statement in these groups with mark 1. Analogy, the maximum value 5 of variable Value means that at least one respondent was assessed the all statement in this group with mark 5.

Table 2. Descriptive analysis for all variables of ATMI.

Inventory question	N	M	Median	SD
1. Mathematics is a very worthwhile and necessary subject.	193	3.80	4	0.92
2. I want to develop my mathematical skills.	193	3.92	4	0.89
3. I get a great deal of satisfaction out of solving a mathematics problem.	179	3.01	3	1.07
4. Mathematics helps develop the mind and teaches a person to think.	193	4.04	4	0.86
5. Mathematics is important in everyday life.	193	3.32	3	1.07
6. Mathematics is one of the most important subjects for people to study.	192	3.17	3	1.11
7. High school math courses would be very helpful no matter what I decide to study.	193	3.84	4	0.96
8. I can think of many ways that I use math outside of school.	193	2.95	3	1.04
9. Mathematics is one of my most dreaded subjects.	193	3.16	3	1.21
10. My mind goes blank and I am unable to think clearly when working with mathematics.	193	3.60	4	1.00
11. Studying mathematics makes me feel nervous.	193	3.39	4	1.08
12. Mathematics makes me feel uncomfortable.	193	3.68	4	1.05
13. I am always under a terrible strain in a math class.	182	2.77	3	1.09
14. When I hear the word mathematics, I have a feeling of dislike.	193	3.54	4	1.16
15. It makes me nervous to even think about having to do a mathematics problem.	182	3.57	4	1.12
16. Mathematics does not scare me at all.	193	3.21	3	1.15
17. I have a lot of self-confidence when it comes to mathematics.	193	2.95	3	0.98
18. I am able to solve mathematics problems without too much difficulty.	193	2.95	3	0.95
19. I expect to do fairly well in any math class I take.	193	2.99	3	0.91
20. I am always confused in my mathematics class.	193	3.47	4	0.99
21. I feel a sense of insecurity when attempting mathematics.	182	3.38	4	1.02
22. I learn mathematics easily.	193	3.16	3	0.98
23. I am confident that I could learn advanced mathematics.	193	2.85	3	1.09
24. I have usually enjoyed studying mathematics in school.	193	2.60	3	0.99
25. Mathematics is dull and boring.	193	3.69	4	1.00
26. I like to solve new problems in mathematics.	193	2.85	3	1.03
27. I would prefer to do an assignment in math than to write an essay.	193	3.64	4	1.30
28. I would like to avoid using mathematics in college.	193	3.27	3	1.22
29. I really like mathematics.	193	3.03	3	1.12
30. I am happier in a math class than in any other class.	193	2.44	2	0.93

31. Mathematics is a very interesting subject.	193	3.10	3	0.96
32. I am willing to take more than the required amount of mathematics.	193	2.28	2	0.97
33. I plan to take as much mathematics as I can during my education.	193	2.37	2	0.98
34. The challenge of math appeals to me.	185	2.64	3	1.05
35. I think studying advanced mathematics is useful.	185	2.97	3	1.04
36. I believe studying math helps me with problem solving in other areas.	184	3.37	3	1.03
37. I am comfortable expressing my own ideas on how to look for solutions to a difficult problem in math.	193	2.88	3	1.10
38. I am comfortable answering questions in math class.	193	2.91	3	1.13
39. A strong math background could help me in my professional life.	185	3.47	4	1.02
40. I believe I am good at solving math problems.	193	3.12	3	0.92

Table 3. Descriptive analysis of four factors of attitude.

Variable	Descriptive Statistics					
	N	M	Median	Min	Max	SD
Enjoyment	193	3.02	3.1	1.0	4.9	0.77
Motivation	193	2.69	2.8	1.0	4.8	0.79
Value	193	3.50	3.6	1.7	5.0	0.68
Self-confidence	193	3.27	3.3	1.4	4.8	0.66
total	193	3.12	3.12	1.5	4.7	0.64

A simple descriptive statistic of all four factors (Enjoyment, Motivation, Value, Self-confidence), plus variable total attitude with requirements due to gender was conducted. Table 4 suggests that the differences in means of each of the five variables between male and female students are very small. This also suggests that there is no significant difference in attitudes toward math due to gender.

Table 4. Comparison of means by gender.

gender	Enjoyment	Motivation	Value	Self-confidence	total
Male (N=106)	3.00	**2.74**	3.5	3.26	**3.13**
Female (N=85)	**3.04**	2.63	3.5	**3.28**	3.12

This assumption was tested with Mann-Whitney nonparametric test. P-value was larger than 0.05 for each variable. That means that we don't reject null hypothesis about equality of attitudes toward mathematic due to gender.

Table 5. Mann-Whitney test for means by gender.

	Enjoyment	**Motivation**	**Value**	**Self-confidence**	**total**
p-value	0.65	0.63	0.94	0.79	1.00
U value	4332	4320.5	4476	4402.5	4505
r	0.0024	0.0025	3.927	0.0014	0

Furthermore, the differences in attitudes towards mathematic and in regarding the type of graduated high school was checked. From the Table 6, we can immediately see that the math grammar school students had the highest results in each category, each value was higher than 3. The lowest results in also each category were made by vocational high school students. The outcomes and our expectations were matched and they were based on our work experience.

Table 6. Comparison of means by type of graduated high school.

Type of high school	**Enjoyment**	**Motivation**	**Value**	**Self-confidence**	**total**
Math grammar school (N=21)	3.54	3.21	3.89	3.79	3.61
Other type of grammar school (N=81)	3.11	2.78	3.61	3.36	3.21
Vocational high school (N=84)	2.78	2.47	3.27	3.07	2.9
Other high schools (N=7)	3.15	2.66	3.74	3.07	3.16

This time Kruskal-Wallis test was used to check our claim. As seen in the Table 7, p-values for each factor are smaller than 0.05, even than 0.01. Then we reject the null hypothesis of equality, and accept alternative hypothesis, we can conclude that there exists a statistical significant difference, with the level of significance 0.01, between attitudes toward mathematics regarding type of graduated high schools.

Table 7. Kruskal-Wallis test for each four-factor regarding on type of graduated high school.

	Enjoyment	**Motivation**	**Value**	**Self-confidence**	**total**
p-value	0.0005	0.0024	0.0005	0.0001	0.0001
χ^2	17.78	10.32	9.8	16.47	14.11

Finally, we were interested in whether there is a connection between the attitude towards mathematics and exam scores. Data for analysis was collected from the first out of three exams during the semester. We excluded the second-year students from further analysis because they mostly passed first year mathematics.

We can see, from the Table 8, that students who have achieved the score of $0-30$ % have the lowest means in each of the four factors of attitudes and in total attitudes. Furthermore, we note that each succeeding group has a higher mean in all factors. The exception is the last group, with $86-100$ %, but N is very small there, only four. This group has approximately the same results as the group $51-65$ %.

Table 8. Comparison of means by achievement on exam.

Achievement on the first exam	Enjoyment	Motivation	Value	Self-confidence	total
0-30 % (N=30)	2.69	2.37	3.48	2.89	2.85
31-50 % (N=33)	2.92	2.64	3.35	3.16	3.02
51-65 % (N=28)	3.15	2.73	3.51	3.38	3.19
66-85 % (N=32)	3.33	2.98	3.66	3.6	3.39
86-100 % (N=4)	3.2	2.65	3.48	3.47	3.2

In further analysis, parametric t-test was conducted. Table 9 shows us that p-values for almost each factor are much smaller than 0.05, even than 0.01. Only factor where is $p = 0.345442 > 0.05$ is Motivation. Then we reject the null hypothesis of equality, and accept alternative hypothesis for factors Enjoyment, Value and Self-confidence and for total attitudes. And conclude that statistical significant difference exists there, with the level of significance of 0.01, between attitudes toward mathematics, especially in Enjoyment, Value and Self-confidence regarding the achievement on first exam.

Table 9. t- test for each four-factor regarding on achievement on exam.

	Enjoyment	Motivation	Value	Self-confidence	total
p-value	0.000099	0.35	0.00	0.00	0.00
t statistic	3.94	0.94	8.73	6.57	5.19
Cohen's d	0.43	0.10	0.94	0.71	0.57

4. Conclusion

Nowadays, there is an increasingly present popularization of science, technology, engineering and mathematics (STEM) subjects. It is considered that these are the key subjects and that the knowledge of them will be needed for further development in the future of humanity. Goal of this popularization is, among other things, the development of new competences and the improvement of competitiveness. Considering that mathematics is a fundamental subject at technical faculties, failure in mathematics would also be reflected in other subjects, or it could even cause the student to drops from the faculty. Issues of students' performance in mathematics subjects have become very popular because of that. Many studies have confirmed a positive relationship between attitudes toward mathematics and performance in mathematics. In this research, Attitudes Mathematics Inventory (ATMI) questionnaire (Tapia and Marsh, 2004) was conducted among students of first and second year of Civil Engineering Faculty in Osijek in order to identify their attitudes.

This study found no difference in attitudes towards mathematics regarding gender. Furthermore, it was shown that there is a statistically significant difference in attitudes toward mathematics regarding the type of finished high school. The students who graduated some type of grammar high schools have a more positive attitude than the students who have graduated from some type of vocational high schools. Math grammar school students have especially had more positive attitudes, in regards to the other grammar school students. This applies not only to the overall attitude, but also on each of four subscales, which are Enjoyment, Motivation, Value and Self-confidence. Also, we showed that there is a statistically significant difference in attitudes toward mathematics regarding the achievement on mathematics exam. Students with the lowest scores on the exam have more negative attitudes and vice versa. And this again goes for the subscales Enjoyment, Value and Self-confidence, but not for Motivation. Another interesting conclusion was made by comparing results for the sub-scales in all analysis. It says that our students are aware of the value of mathematics, but they are not motivated enough to make themselves study it more.

We believe that the achievements in mathematics courses can be improved if the attitudes toward mathematics, especially motivation, can be increased. Attitudes are built throughout all of the education process. This means that the problems of motivation or overall attitudes should be identified and dealt with already at lower levels of education, in primary or secondary school. Therefore, in further research it should be expanded to a target group on secondary and primary school students.

Further, we recommend that extra attention should be given not only to teaching math but also to increasing students' motivation and relaxation as well as providing a sensation of pleasure in math classes, which will lead to improved attitudes towards mathematics. We hope that the results of this research will help to improve the students' mathematics performance at higher education institutions.

References

[1] AIKEN, L. R. (1970), *Attitudes Toward Mathematics*, Review of Educational Research, 40(4), 551–596.

[2] AIKEN, L. R. (1974), *Two Scales of Attitude toward Mathematics*, Journal for Research in Mathematics Education, 5(2), 67–71.

[3] AIKEN, L. R., & DREGER, R. M. (1961), *The effect of attitudes on performance in mathematics*, Journal of Educational Psychology, 52, 19–24.

[4] BUSINESS DICTIONARY (2017), Retrieved from `http://www.businessdictionary.com/definition/attitude.html`.

[5] CHILDS, K. (2013), *A Comparison Of Students' And Parents' Mathematics Attitudes And Achievement At A Private Historically Black* (Doctoral dissertation), Orlando: University of Central Florida.

[6] DUTTON, W. H. (1962), *Attitude change of prospective elementary school teachers toward arithmetic*, The Arithmetic Teacher, 9(8), 418–424.

[7] FENNEMA E., SHERMAN, J. A. (1976), *Fennema-Sherman Mathematics Attitudes Scales: Instruments Designed to Measure Attitudes toward the Learning of Mathematics by Females and Males*, Journal for Research in Mathematics Education, 7(5), 324–326.

[8] GUY, G. M., CORNIK, J., BECKFORD, I. (2015), *More than Math: On the Affective Domain in Developmental Mathematics*, International Journal for the Scholarship of Teaching and Learning, 9(2), 1–5, doi: 10.20429/ijsotl.2015.090207.

[9] KADIJEVICH, DJ. (2008), *TIMSS 2003: Relating Dimensions of Mathematics Attitude to Mathematics Achievement*, *Зборник Института за педагошка истраживања*, 40(2), 327–346.

[10] KHINE, M. S., AFARI, E. (2014), *Psychometric properties of an inventory to determine the factors that affect students' attitudes toward mathematics*, Psychology, Society & Education, 6(1), 1–15.

[11] KIBRISLIOGLU, N. (2015.), *An Investigation About 6th Grade Students' Attitudes Towards Mathematics*, Procedia – Social and Behavioural Sciences, 186, 64–69, `http://www.sciencedirect.com/science/journal/18770428`.

[12] LIPNEVICH, A. A., PRECKEL, F., KRUMM, S. (2016), *Mathematics attitudes and their unique contribution to achievement: Going over and above cognitive ability and personality*, Learning and Individual Differences, 47, 70–79.

[13] MA, X., & KISHOR, N. (1997), *Assessing the Relationship between Attitude toward Mathematics and Achievement in Mathematics: A Meta-Analysis*, Journal for Research in Mathematics Education, 28(1), 26–47.

[14] MATA, M. L., MONTEIERO, V., & PEIXOTO, F. (2012), *Attitudes towards Mathematics: Effects of Individual, Motivational, and Social Support Factors*, Child Development Research, 2012, 1–10, doi: 10.1155/2012/876028.

[15] NICOLAIDOU, M., & PHILIPPOU, G. (2002), *Attitudes towards Mathematics, Self-Efficacy and Achievement in Problem Solving*, Paper presented at 3rd European Research in Mathematics Education, Bellaria, Italy.

[16] RABAB'H, B. S., & VELOO, A. (2015), *Prediction of Mathematics Learning Strategies on Mathematics Achievement among 8th Grade Students in Jordan*, Asian Social Science, 11(2), 276–283.

[17] SIEW, Y. L. (2010), *Mathematics Attitudes and Achievement of Junior College Students in Singapore*, In L. Sparrow, B. Kissane & C. Hurst (Eds.), Proceedings of the 33rd annual conference of the Mathematics Education Research Group of Australasia (pp. 681–689), Freemantle: MERGA

[18] TAPIA, M. (1996), *The Attitudes Toward Mathematics Instrument*, Paper presented at the annual meeting of the Mid-south Educational Research Association, Tuscaloosa, AL (ERIC Reproduction Service No. ED 404165).

[19] TAPIA, M., & MARSH, G. E. II. (2004), *An Instrument to Measure Mathematics Attitudes*, Academic Exchange Quarterly, 8(2), 16–21.

[20] TEKEREK, M., YENITERZI, B., ERCAN, O. (2011), *Math Attitudes of Computer Education and Instructional Technology Students*, The Turkish Online Journal of Educational Technology, 10(3), 168–174.

[21] TERWILLIGER, J. S., TITUS, J. (1995), *Gender Differences in Attitudes and Attitude Changes Among Mathematically Talented Youth*, Gifted Child Quarterly, 39(1), 29–35.

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Stavovi prema matematici studenata Građevinskog fakulteta

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Sažetak. Cilj ovoga rada je identificirati stavove prema matematici studenata Građevinskog fakulteta analizirajući odgovore 193 studenta. U ovom članku se ispituje postoji li veza između stavova prema matematici i uspješnosti u polaganju ispita iz matematike. Također, analizirane su razlike u stavovima prema matematici s obzirom na spol i prijašnje obrazovanje (stručna srednja škola ili gimnazija). Za prikupljanje podataka je korišten upitnik ATMI. Podaci su analizirani uz pomoć programskog paketa Statistica 13, u tu svrhu su korišteni sljedeći testovi: t-test, Mann-Whitney i Kruskal-Wallis test.

Iz rezultat istraživanja zaključujemo da nema razlike u stavovima prema matematici u odnosu na spol. Studenti koji su završili neku od gimnazija su postigli značajno više rezultate od studenata koji su završili neku od stručnih srednjih škola u ukupnim stavovima, ali i u sve četiri kategorije pitanja: zadovoljstvo, motivacija, vrijednost i samopouzdanje. Pokazano je da studenti s najnižim ocjenama na ispitu imaju najlošije (najnegativnije) ukupne stavove prema matematici, ali i obrnuto. Isto vrijedi i za kategorije pitanja zadovoljstvo, vrijednost i samopouzdanje, ali ne i za motivaciju.

Ključne riječi: stavovi prema matematici, uspješnost u matematici, upitnik ATMI, statistička analiza, preddiplomski studenti

Targeting additional effort for students' success improvement: The highest effect group selection method

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Abstract. Goal of our research is to present a method for selection of a group of students. The selected group of students is supposed to receive additional teaching attention in order to improve their performance in the course. Guiding line used, for the group selection, is: value of an action is proportional to the benefit that produces to a customer. In our case customers are students. The selection method is based on multinomial logistic regression, Poisson-type discrete variables modelling the number of points achieved in a term exam, and transition probability matrix. We use data of a study progress during monitoring program in an undergraduate mathematical course. Demographic data and other attributes about previous performance were not included in the analyses. In the first part of the paper we present methodology, while in the second part we introduce data used for demonstration of the proposed method. At the end of the paper, an individual approach for the final selection is proposed. Criteria for selection is clear: increased probabilities of obtaining desired final grade for a student. Weighing of criteria is subjective depending on the goals of the decision makers.

Keywords: learning analytics, predictive analytics, students' success, multinomial logistic regression, Poisson distribution

1. Introduction

Development of information and communication technologies provides collection of large amounts of data about the students learning process and their study progress. Learning analytics, as a discipline whose purpose is to analyze and improve learning process, accordingly gains additional development possibilities.

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Learning analytics does not involve only the measurement and data collection by some institution, but also the understanding of these data, analysing and creating reports based on the data in order to understand and optimize the learning and the environment in which learning occurs (Society for Learning Analytics Research, 2011), (U.S. Department of Education, Office of Educational Technology, 2012). Significant challenges (Ferguson, 2012) that Learning Analytics must deal with are systematized in the following: to consolidate experience from the learning sciences, to work with a wider range of datasets, to engage with learner perspectives and to develop a set of ethical guidelines directions. To overcome the gap in knowledge exchange and conversation between researchers, vendors and experts, Siemens (Siemens, 2012) pointed out some improvements inside Learning Analytics, which involves new tools, techniques and people development, addressing concerns connected with the data openness and ethics, expanding and transitioning the goal of analytics activity and enhancing connections to related areas. In order to determine how the Learning Analytics tools are successful and what is their impact on learning, Scheffel et al. (Scheffel et al., 2014) proposed a five-dimensional framework of quality indicators to standardize the evaluation of the Learning Analytics tools. The proposed framework consists of five criteria and quality indicators as follows: Objectives, Learning Support, Learning Measures and Output, Data Aspects and Organisational Aspects.

Reyes (Reyes, 2015) concluded that Learning Analytics will provide students to take the initiative in learning process advance by using the data about the factors of student's success, about allocation of resources and the effectiveness of teaching which will finally contribute to the learning effectiveness. Authors Mor et al. (Mor et al., 2015) showed that the synergistic effects of Learning Analytics and two approaches, Learning design and Teacher inquiry, can improve the quality of teaching and learning. In this context, learning design contributes with its semantic structure for analytics and teacher inquiry with defining profound questions to analyse. As concluded in (Gašević, 2015), Learning Analytics should be developing, building and connecting with existing knowledge about learning and teaching. It is important to inform a wider audience with the possibilities that Learning Analytics provide. Various activities, such as projects (e.g. Learning Analytics Community Exchange, 2017), conferences (e.g. Learning Analytics and Knowledge, 2017) and journals (e.g. Journal of Learning Analytics, 2017) are held to accomplish the goal of providing the information. It can be concluded that Learning Analytics nowadays receives importance in academic as well as in professional circles.

This paper presents the results of estimating probability of obtaining the final grade on mathematical course based on points achieved at the two term exams as well as the impact of the additional points obtained in the first term exam on the final grade. Two research questions are: "Which student should be chosen to effect on to increase probability of passing the course?" and "Which student should be chosen to increase final grade on the course?" Student at risk is student who needs a little assistance from professor for passing the course or for increasing the final grade. In the paper identification of students at risk is presented. The developed methodology for the detection of these students is based on multinomial logistic regression. Computation is done in R Studio, a user interface for R programming language.

2. Literature review

This section provides an overview of relevant research related to the use of Learning Analytics in order to improve students' success as well as predict students at risk, i.e. students who are at greater risk of failing the course or dropping out of the course.

By using visualization, decision trees, class association rules and clustering approaches authors Kotsiantis et al. (Kotsiantis et al., 2013) analyzed students' perceptions of Moodle and students' interaction with Moodle. As the collected data combined with Learning Analytics approaches can provide efficient information about the educational process, in this study the analysis showed that failure in the course was connected with negative students' attitudes and perceptions according to Moodle, while the increased use of Moodle led to excellent grades. By using the logistic regression, the authors Barber and Sharkey (Barber, 2012) created a model which identifies students who are at risk that will not succeed in enrolled course. The probability that a student will fail the course is calculated based on data obtained from learning management system and student system. Knowing such indicators can help students succeed in the course by beforehand intervention and, if necessary, by providing additional services. Munđar and Erjavec (Munđar, 2015), by using logistic model, assessed the probability of passing the exam in mathematics based on the students' results on Matura state exam and high school grades. Identification of such data is important not only for successfully passing the mathematical courses, but also for the study success at whole. Similarly, a model for predicting mathematics course passing rates was developed by applying classification trees and neural networks (Keček, 2012). Regarding the accuracy of classification, which students have passed the course and which have not, classification trees have proven to be a better method in predicting mathematics course passing rates than neural networks. Prediction of the study success was done in (Munđar et al., 2015) by using multiple regression. Study success was measured by number of ECTS credits which student achieved one year after enrolment at the faculty, while enrolment points achieved based on the results of State Matura exams and points achieved based on high school grades were considered as predictors for that study success. The results of that research showed that the high school grades in general are better predictor in achieving more ECTS credits than the State Matura exam. In order to improve retention, completion and graduation rates authors Essa and Ayad (Essa, 2012) developed a Student Success System that enables, through the machine intelligence and statistical techniques, monitoring student with the aim of improving its study success. The System can identify a student who is at risk, understand why the student is endangered, plan interventions to alleviate this risk and can provide feedbacks about effectiveness of undertaken interventions.

3. Methodology

Based on data, multinomial logistic regression and modelling with Poisson distribution type of random variables as the basic methods, we are proposing the following method for detection of targeting group. Strategy for detection students at risk is based on transition rates from 1st term exam to 2nd term exam and from achieved points at 2nd term to the final grade. The emphasis is placed on points increase at the 2nd term exam whose shift results in the greatest change in success probability. Our method has three main parts. The first part is presentation of multinomial logistic regression for final grade prediction. The second part is presentation of a model for transition probabilities which are used to predict success of a student depending of success in the first part of the course. The third, presentation of a method for valuation of additional effort by estimation of the effect it produces.

3.1. Grade prediction

Multinomial logistic regression is a method developed to estimate the probabilities of the categorical response variable, which has more than two possible outcomes. Generally, for l possible outcomes $\{1,\ 2,\ \ldots,\ l\}$ of the response variable Y, where outcome 1 is chosen as a standard category, multinomial probabilities $P[Y=k]=p_k$, $k=1,\ldots,l$ are parameterized as

$$p_1 = P[Y=1] = \frac{1}{1+\sum_{i=2}^{l} e^{\alpha_i + x\beta_i}} \tag{1}$$

and

$$p_k = P[Y=k] = \frac{e^{\alpha_k + x\beta_k}}{1+\sum_{i=2}^{l} e^{\alpha_i + x\beta_i}}, \tag{2}$$

for $k=2,\ldots,l$. Those probabilities sum to 1 and any other outcome, instead of outcome 1, could be chosen as a standard category. (Ledolter, 2013).

In our paper, we use multinomial logistic regression in two occasions. Firstly, we predict grade based on success on the 1st term exam. In the second part of the analysis, we predict grade by using points achieved on the first two term exams.

3.2. Transition rates modelling

Prediction of number of points achieved on the 2nd term exam is estimated by using Poisson distribution variables. Expected value of those variables depends on the number of points achieved on the 1st term exam. Transition matrix, $T=[t_{ij}]$, of type $(M+1)\times(M+1)$, contains transition probabilities from the 1st term exam score, i.e. number of points achieved, to score achieved on the 2nd term exam. M is maximal number of points that a student can achieve in a term exam (same for all term exams in our case).

$$t_{x_1+1,\, x_2+1} = P(X_2 = x_2 \mid X_1 = x_1), \text{ where } x_1, x_2 \in \{0,\ldots,M\} \tag{3}$$

As mentioned before, we assume that the number of points achieved for a student at a term exam follows Poisson distribution. Probability of scoring points are given by

$$P(X_2 = x_2) = \frac{(\lambda_{x_1})^{x_2} \exp(-\lambda_{x_1})}{x_2!} \tag{4}$$

Expected value of number of points achieved, λ_{x1} , depends on number of points x_1 achieved at the first term exam.

$$\lambda_{x_1} = \lambda\ (\beta_0 + \beta_1 x_1 + \beta_2 x_1^2) \tag{5}$$

The parameters $\lambda,\ \beta_0$, β_1, and β_2 are estimated using maximum likelihood estimation method, i.e. by determining value of parameters that minimize the value cost function C

$$C(\beta_0, \beta_1, \beta_2) = -\sum_{(x_1,x_2)\in S} \ln\left(p(X_2 = x_2 \mid X_1 = x_1)\right) \tag{6}$$

Summation goes for each observation, combination of points achieved $(x_1\ , x_2)$ at the 1st and the 2nd term exam for each student, in an academic year which is denoted by S.

3.3. Effect measurement

In order to compute estimated effect of addition teaching effort we need to estimate parameters of another multivariate model, i.e. we estimate probability of a student receiving the grade g, when we know his/her score at the first two term exams:

$$P(G = g \mid X_2 = x_2,\ X_1 = x_1). \tag{7}$$

In our case, we do that by modelling probabilities of each grade with multinomial logistic regression

$$P[Y = 4] = \frac{1}{1 + \sum_{i=1}^{3} e^{\alpha_i + x\beta_i}};\ \ P[Y = k] = \frac{e^{\alpha_k + x\beta_k}}{1 + \sum_{i=1}^{3} e^{\alpha_i + x\beta_i}},\quad k = 1, 2, 3 \tag{8}$$

where $x = (x_1, x_2)$ are results of the 1st and the 2nd term exam.

Initial probability that student achieve grade g, knowing his/her result at the 1st term exam, i.e. knowing number of points x_1 achieved at the 1st term exam, is given by:

$$P(G = g \mid X_1 = x_1) = \sum_{x_2=0}^{M} P(G = g \mid X_2 = x_2, X_1 = x_1) P(X_2 = x_2 \mid X_1 = x_1). \tag{9}$$

Probability after treatment, i.e. probability that student who achieved x_1 points at the 1st term exam will achieve grade g, after receiving treatment of additional teaching action equivalent to raise of m points, is:

$$P_{+m}(G{=}g|X_1{=}x_1){=}\sum_{x_2=0}^{M} P(G{=}g|X_2{=}\min(x_2{+}m,M),X_1{=}x_1)\; P(X_2{=}x_2|X_1{=}x_1) \tag{10}$$

Difference in probabilities in out focused measure is then:

$$\Delta_{+m}(G=g\mid X_1=x_1)=\; P_{+m}-P_0 \tag{11}$$

If decision maker do not value increase in each grade equally, it is possible to take the opinion in valuation by weighing increased probabilities, for example by formula:

$$V(x_1)=\sum_{g=2}^{4} w(G=g)\Delta_{+m}(G\geq g\mid X_1=x_1). \tag{12}$$

4. Data

In this paper, we conducted the analysis for the course Financial Mathematics (FM) performed at the study program Information and Business Systems (IBS) at the Faculty of Organization and Informatics, University of Zagreb. In this analysis, 158 full-time students of academic year 2015/2016 and 144 full-time students of academic year 2016/2017 were involved, while part-time students were not included in the analysis.

Elements of students' work monitoring in this course included: Homework assignments (HW), Short examinations (SE), Project (PRO), Term exam 1 (TE1), Term exam 2 (TE2) and Term exam 3 (TE3). Students can achieve maximum of 20 points at each Term exam as well as at the Project, maximum of 10 points at Homework assignment and maximum of 10 points at Short examinations. We based this analysis on the term exams scores. Term exams scores are the most significant variables for students' success prediction (when other attributes about students, except their performance on course, are not available).

Table 1 presents descriptive statistics of variables Homework assignment (HW), Short examinations (SE), Project (PRO), 1st term exam (TE1), 2nd term exam (TE2) and 3rd term exam (TE3) for academic year 2015/2016 and for academic year 2016/2017. Total number of points (TOTAL) is equal to the sum of points achieved through HE, SE, PRO, TE1, TE2 and TE3. It can be noted that all of the observed measures of location, mean, median, first and third quartile, for most variables in academic year 2016/2017 had a bit higher values than in academic year 2015/2016. Values of variables HW and SE in two observed academic years were almost equal. The highest total number of points achieved in academic year 2015/2016, out of the maximum 100, was 90 and in academic year 2016/2017 was

95 points. It can also be noted that all measures of location for variable TOTAL were very similar in academic year 2016/2017 and academic year 2015/2016.

Table 1. Descriptive statistics of students' points achieved during the course.

Financial mathematics (FM)	**Academic Year**							
	2015/2016				**2016/2017**			
	1st Q.	**Median**	**Mean**	**3rd Q.**	**1st Q.**	**Median**	**Mean**	**3rd Q.**
Home work assignment (HW)	5.00	6.00	6.03	7.00	3.50	5.50	5.35	7.50
Short examination (SE)	3.87	4.78	4.88	6.01	3.08	5.00	4.59	6.00
Project (PRO)	12.00	14.00	13.75	16.00	13.00	15.00	14.29	16.00
Term exam 1 (TE1)	5.63	9.00	8.83	11.38	6.00	9.00	9.09	12.00
Term exam 2 (TE2)	7.50	11.00	10.82	14.00	10.00	12.00	11.63	14.00
Term exam 3 (TE3)	7.00	10.00	9.53	12.00	8.00	11.00	10.42	14.00
Total number of points (TOTAL)	41.00	54.00	50.97	62.00	47.75	56.00	52.65	62.00

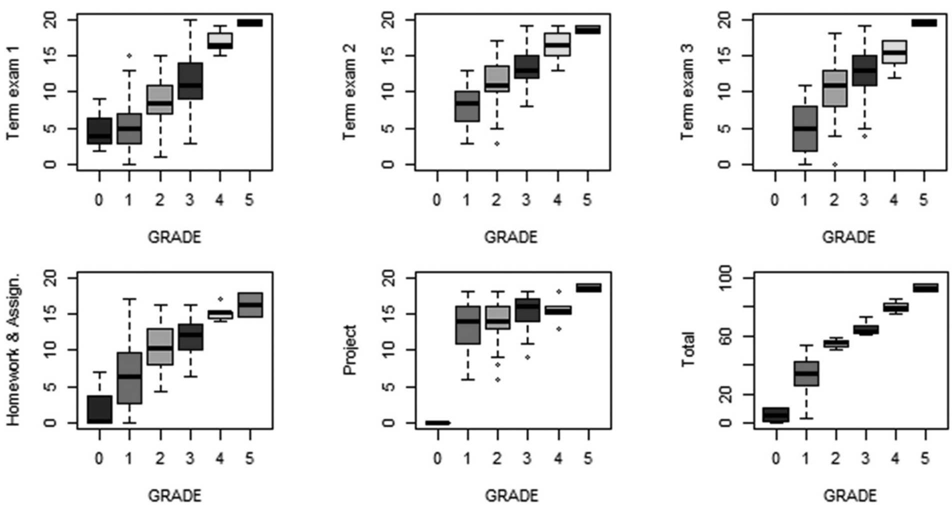

Figure 1. Graphical representation of points and grades dependency (ac. year 2016/2017).

Outcomes of the categorical response variable named GRADE are: 0 – student did not accomplish their minimum commitments related to the course, 1 – student accomplished their minimum commitments related to the course, but didn't pass the course, while categories 2, 3, 4 and 5 represents grades according to the Croatian national grading scale. Grade 2 is equivalent to sufficient, 3 is equivalent to good, 4 to very good and 5 to excellent. Grades are assigned according to the previously agreed scoring scale rating and according to the achieved total number of points. According this scale, in academic year 2015/2016 around 68 percent of students passed the course and two students got excellent grade. In academic year

2016/2017 the results are even a bit better. Around 71 percent of students passed the course, and two of those were given excellent grade. In both academic years, the largest number of students got grade 2.

For better understanding of above results, Figure 1. shows a graphical representation of the results through box plots, where each of the box plot shows how the variable GRADE depends on the points achieved on the individual element of students' work monitoring.

5. Results

5.1. Grade prediction: Case of Financial Mathematics course in academic year 2016/2017

For further data analysis, outcomes 0 and 1 of the variable GRADE were merged, as well as outcomes 4 and 5, since grades categories 0 and 5 included only few students. Thus, the analysis continued with four outcomes of variable GRADE: 1, 2, 3 and 4. For those outcomes of the variable GRADE, where outcome 4 represents a standard category, multinomial probabilities are as follows:

$$P[Y=4] = \frac{1}{1+\sum_{i=1}^{3} e^{\alpha_i + x\beta_i}} = \frac{1}{1+e^{\alpha_1+x\beta_1}+e^{\alpha_2+x\beta_2}+e^{\alpha_3+x\beta_3}} \tag{13}$$

$$P[Y=k] = \frac{e^{\alpha_k+x\beta_k}}{1+\sum_{i=1}^{3} e^{\alpha_i + x\beta_i}} = \frac{e^{\alpha_k+x\beta_k}}{1+e^{\alpha_1+x\beta_1}+e^{\alpha_2+x\beta_2}+e^{\alpha_3+x\beta_3}},\ k=1,\ 2,\ 3. \tag{14}$$

Prediction of students' success in this analysis is based on the accomplishments at course assignments. Other variables, such as social variables e.g. gender, success at previous courses were not taken in consideration since the goal of the paper is introduction of the methodology for success assessment. Multinomial logistic regression, shortly explained in the Methodology part of this paper, can be used for estimation of probability for obtaining final grade.

Table 2. Coefficients of the multinomial logistic model: Case financial mathematics A.Y. 2016/2017.

Coefficients	Estimate	Std. Error	Z value	Pr(<|z|)
α_1	16.162	3.970	4.078	4.54e−05
α_2	14.412	3.947	3.651	0.000261
α_3	12.080	3.892	3.104	0.001909
β_1	−1.209	0.253	−4.786	1.7e−06
β_2	−0.912	0.244	−3.741	0.000183
β_3	−0.699	0.246	−2.967	0.003010

Figure 2 presents estimated probabilities calculated using only results of the 1st term exam, for academic year 2016/2017. Grade 0 and grade 1 are joint in one class called Grade 1. Similar approach is used for Grade 4 and Grade 5 since we have small number of students receiving Grade 4 (3 students) or Grade 5 (2 students).

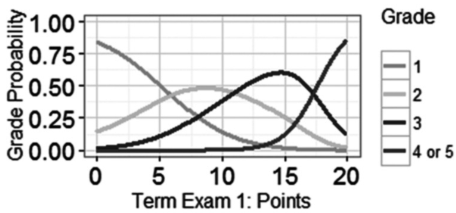

Figure 2. Estimated probabilities of final grade based on the 1st term exam: Case Financial Mathematics (ac. year. 2016/2017).

5.2. Transition probabilities: Case of Financial Mathematics course in academic year 2016/2017

Transition matrix, $T = [t_{ij}]$, represents matrix of transition probabilities from 1st term exam score of points achieved to scoring point achieved at the 2nd term exam. Using methodology for assessment of parameters β_0, β_1, and β_2, we determined values of the transition probabilities. Table 3 presents submatrix of matrix T. Rows and columns names are number of points achieved at the 1st term exam, x_1, and the 2nd term exam, x_2, of values of 0, 5, 10, 15 and 20.

Table 3. Estimated transition probabilities for points at the 1st and the 2nd term exam.

	0	**...**	**5**	**...**	**10**	**...**	**15**	**...**	**20**
0	0.03 %	...	8.67 %	...	10.27 %	...	1.02 %	...	0.02 %
...	...	...		...	...	...	...	...	...
5	0.01 %	...	5.87 %	...	11.95 %	...	2.04 %	...	0.07 %
...	...	...		...	...	...	...	...	...
10	0.00 %	...	2.78 %	...	12.30 %	...	4.56 %	...	0.33 %
...	...	...		...	...	...	...	...	...
15	0.00 %	...	0.84 %	...	9.16 %	...	8.43 %	...	1.50 %
...	...	...	...	...	...	...	...	...	...
20	0.00 %	...	0.15 %	...	4.22 %		10.19 %	...	4.76 %

5.3. Effect measurement: Case of financial mathematics in academic year 2016/2017

In order to compute estimated effect of addition teaching effort we need to estimate parameters of another multivariate model, i.e. we estimate probability of a student receiving the grade g, when we know his/her score at the first two term exams:

$$P(G = g \mid X_2 = x_2,\ X_1 = x_1) \tag{15}$$

So in our case of modelling by multinomial logistic regression:

$$P[Y{=}4]{=}\frac{1}{1+\sum_{i=1}^{3} e^{\alpha_i+x\beta_i}};\ \ P[Y{=}k]{=}\frac{e^{\alpha_k+x\beta_k}}{1+\sum_{i=1}^{3} e^{\alpha_i+x\beta_i}},\ \ k = 1,2,3, \tag{16}$$

where are$x = (x_1, x_2)$ results of the 1st and the 2nd term exam. Estimated values of coefficients with their standard error, and accompanied z score and associated p-value are given in the Table 4.

Table 4. Parameters for multinomial logistic regression of grade probability based on results of the first two term exams.

Coefficients	**Estimate**	**Std. Error**	**Z value**	**Pr(<|z|)**
α_1	25.249	4.506	5.603	2.10e−08
α_2	19.570	4.290	4.561	5.08e−06
α_3	12.870	3.941	3.266	0.001091
β_{11}	−1.036	0.281	−3.685	0.000228
β_{12}	−0.734	0.265	−2.769	0.005614
β_{13}	−0.503	0.256	−1.965	0.049466
β_{21}	−1.010	0.290	−3.477	0.000508
β_{22}	−0.597	0.269	−2.221	0.026367
β_{23}	−0.256	0.251	−1.019	0.308090

For example, our model for Grade 4 (combines Grade 4 and 5) is:

$$P[Y{=}4|X_2{=}x_2, X_1{=}x_1]{=}\frac{1}{1+e^{25.2-1.04x_1-0.50x_2}+e^{19.6-0.73x_1-0.60x_2}+e^{12.9-0.50x_1-0.26x_2}} \tag{17}$$

In special case, for students that achieved 15 points at the 1st and 15 points at the 2nd term exam, model gives probability of 16.33 % to receive Grade 4 or 5:

$$P[Y = 4 \mid X_2 = 15, X_1 = 15] = 0.1633 \tag{18}$$

Additional effort (with estimated effect of m additional points at second term exam) given to a student, which received poix_1 nts at the first term, and would without treatment received atx_2 the second term exam, improves probability of the grade 4 or 5.

$$\Delta P_{+m|x_1}{=}P[Y{=}4 \mid X_2{=}\min(x_2{+}m, M), X_1{=}x_1]{-}P[Y{=}4 \mid X_2{=}x_2, X_1{=}x_1] \tag{19}$$

$$\Delta P_{+1|15}{=}P[Y{=}4 \mid X_2{=}16, X_1{=}15]{-}P[Y{=}4 \mid X_2{=}15, X_1{=}15]{=}0.2077 \tag{20}$$

In special case, for students that achieved 15 points at the 1st and would receive 15 points at the 2nd term exam effect of effort of 1 (m=1) increases probability of receiving Grade 4 or 5 for 4.44 %, or to value of 20.77 %.

If the effect of the effort is weighted with estimated probability of receiving a special number of point in the case of absence of additional effort.

$$\Delta P_{+m} = \sum_{x_2=0}^{M} \Delta P_{+m|x_1} P(X_2 = x_2 \mid X_1 = x_1) \tag{21}$$

Table 5 presents increased probability of becoming equal of higher grade than initial estimated by multinomial logistic regression for each case of points at the 1st term exam. The table can be used to estimate effect of additional effort. For example, if we choose to give additional attention, equivalent to knowledge needed to earn one additional point, to a student who have 5 points at the 1st term exam, we increase probability of passing the course (receiving Grade 2 or higher) from 0.437 to value of 0.819.

Table 5. Table of probabilities and improved probabilities of final grade based of the 1st term exam.

	Probability of the grade					**Increased probability of the grade**		
Points at the 1st term exam	**2**	**2 or higher**	**3**	**3 or higher**	**4 or 5**	**2 or higher**	**3 or higher**	**4 or 5**
0	0.137	0.148	0.011	0.011	0.000	0.487	0.093	0.000
1	0.172	0.189	0.016	0.016	0.000	0.557	0.122	0.000
2	0.213	0.237	0.024	0.024	0.000	0.628	0.159	0.000
3	0.256	0.296	0.036	0.036	0.000	0.697	0.203	0.000
4	0.311	0.363	0.052	0.052	0.000	0.761	0.255	0.000
5	0.361	0.437	0.074	0.074	0.000	0.819	0.314	0.001
6	0.413	0.517	0.104	0.104	0.000	0.867	0.380	0.001
7	0.6	0.598	0.142	0.142	0.000	0.907	0.450	0.003
8	0.488	0.678	0.189	0.190	0.001	0.937	0.523	0.006
9	0.504	0.751	0.245	0.247	0.002	0.960	0.595	0.011
10	0.503	0.816	0.309	0.313	0.004	0.975	0.665	0.020
11	0.482	0.869	0.379	0.387	0.009	0.985	0.731	0.037
12	0.445	0.911	0.449	0.466	0.017	0.992	0.790	0.065
13	0.394	0.943	0.515	0.548	0.034	0.995	0.842	0.113
14	0.335	0.965	0.566	0.629	0.063	0.998	0.885	0.184
15	0.272	0.979	0.595	0.707	0.112	0.999	0.921	0.284
16	0.210	0.988	0.591	0.778	0.187	0.999	0.949	0.408
17	0.153	0.994	0.550	0.841	0.291	1.000	0.969	0.544
18	0.105	0.997	0.475	0.892	0.418	1.000	0.983	0.672
19	0.066	0.999	0.378	0.932	0.554	1.000	0.991	0.780
20	0.039	0.999	0.278	0.961	0.683	1.000	0.996	0.861

6. Conclusions

Learning analytics is a discipline whose purpose is to analyze and improve learning process. Analysis of available data about the learning process gives additional development possibilities of the discipline. In our paper, we have introduced a simple probabilistic model for assessment of effect that additional teaching effort can produce for different subgroups of students. Subgroups of students are, in our example, divided considering only results in term exams. Analysts could take more variables or other variables in consideration when analyzing student progress data.

Presented method applied on data about progress on the Financial Mathematics course, an undergraduate mathematics, gave us assessment of the effect that teaching effort needed for a student to gain additional five points at the 2$^{\text{nd}}$ term exam. Example, presented in the Results part of this paper, for a student who has 5 points at 1$^{\text{st}}$ term exam assessed effect in increased probability of passing the course (receiving Grade 2 or higher) from 0.437 to value of 0.819. Similar conclusions can be deducted from the results for any student depending on the number of points achieved at the 1$^{\text{st}}$ term exam.

Teaching effect presented in term exam points increase (m) can be transformed to final grade probabilities, using presented method, for any teaching effect m. In special case, for students that achieved 15 points at the 1$^{\text{st}}$ term exam and would receive 15 points at the 2$^{\text{nd}}$ term exam effect of effort of 1 ($m = 1$) increases probability of receiving Grade 4 or 5 for 4.44 %, or to value of 20.77 %.

References

[1] Barber, R. and Sharkey M. (2012), *Course Correction: Using Analytics to Predict Course Success*, in LAK12: 2nd International Conference on Learning Analytics & Knowledge, Vancouver, BC, Canada, 259–262.

[2] Essa, A. and Ayad H. (2012), *Improving student success using predictive models and data visualisations*, Research in Learning Technology, Supplement: ALT-C 2012 Conference Proceedings, 20, 58–70.

[3] Ferguson, R. (2012), *Learning analytics: drivers, developments and challenges*, International Journal of Technology Enhanced Learning, 4(5/6), 304–317.

[4] Gašević, D. (2015), *Let's not forget: Learning analytics are about learning*, TechTrends, 59(1), 64–71.

[5] Journal of Learning Analytics (2017), `http://learning-analytics.info/` [Accessed 02/03/2017].

[6] Keček, D., Milković, M. and Hajdek, K. (2012), *Data mining methods for predicting course passing rates*, in Proceedings of the 16th International Conference on Printing, Design and Graphic Communications, Senj, Croatia, 392–401.

[7] Kotsiantis, S., Tselios, N., Filippidi, A. and Komis, V. (2013), *Using learning analytics to identify successful learners in a blended learning course*, International Journal of Technology Enhanced Learning, 5(2), 133–150.

[8] Learning Analytics and Knowledge (LAK) Conference (2017), `http://lak17.solaresearch.org/` [Accessed 02/03/2017].

[9] Learning Analytics Community Exchange (LACE) (2017), `http://www.laceproject.eu/` [Accessed 03/01/2017].

[10] LEDOLTER, J. (2013), *Data mining and business analytics with R*, John Wiley & Sons, Inc., Hoboken, New Jersey.

[11] MOR, Y., FERGUSON, R. AND WASSON, B. (2015), *Editorial: Learning design, teacher inquiry into student learning and learning analytics: A call for action*, British Journal of Educational Technology, 46(2), 221–229.

[12] MUNĐAR, D. AND ERJAVEC, Z. (2015), *Pass rates in mathematical courses: relationship with the state matura exams scores and high school grades*, in Proceedings of The fifth international scientific colloquium mathematics and children, Osijek, Croatia, 153–162.

[13] MUNĐAR, D., KEČEK, D. AND MATOTEK, D. (2015), *Relationship between enrolment criteria and first-year students' study-success*, in Proceedings HASSACC (HASSACC – Human And Social Sciences at the Common Conference), 111–115, DOI: 10.18638/hassacc.2015.3.1.179.

[14] REYES, J. A. (2015), *The skinny on big data in education: Learning analytics simplified*, TechTrends, 59(2), 75–80.

[15] SCHEFFEL, M., DRACHSLER, H., STOYANOV, S. AND SPECHT, M. (2014), *Quality Indicators for Learning Analytics*, Educational Technology & Society, 17(4), 117–132.

[16] SIEMENS, G. (2012), *Learning Analytics: Envisioning a Research Discipline and a Domain of Practice*, in LAK12: 2nd International Conference on Learning Analytics & Knowledge, Vancouver, BC, Canada, 4–8.

[17] Society for Learning Analytics Research (SoLAR), (2011), *1st International Conference on Learning Analytics and Knowledge 2011*, Banff, Alberta, `https://tekri.athabascau.ca/analytics/about`.

[18] U.S. Department of Education, Office of Educational Technology (2012), *Enhancing Teaching and Learning Through Educational Data Mining and Learning Analytics: An Issue Brief*, Washington, D.C.

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Usmjeravanje dodatnog angažmana za poboljšanje uspjeha studenata: odabir grupe s najvećim učinkom

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Sažetak. Cilj našeg rada je predstaviti jednu metodu za odabir grupe studenata. Odabranoj grupi studenata trebao bi biti usmjeren dodatni nastavnički angažman kako bi studenti te grupe popravili uspjeh na predmetu. Princip vrednovanja uspješnosti odabira grupe je: vrijednost poduhvata proporcionalna je koristi koju poduhvat stvara kod korisnika. U našem slučaju korisnici su studenti, a korist je povećana vjerojatnost uspjeha studenata. Metoda odabira bazira se na multinomijalnoj logističkoj regresiji, modeliranju broja bodova ostvarenih na kolokviju diskretnim slučajnim varijablama Poissonovog tipa i vjerojatnosnim matricama prijelaza. U radu su korišteni podaci s kontinuiranog praćenja napretka studenata na matematičkom predmetu preddiplomskog studija. Demografski podaci o studentima, kao ni podaci o uspjehu studenata na prethodnim predmetima, nisu korišteni u analizi. U prvom dijelu predstavljena je metodologija, a u drugom dijelu predstavljaju se podatci koji se koriste za demonstraciju metode. Na kraju rada predstavljamo mogućnost individualnog pristupa konačnom odabiru grupe. Kriteriji za odabir su jednoznačni: povećane vjerojatnosti ostvarenja željene završne ocjene studenta. Težine kriterija su subjektivne i ovise o ciljevima donositelja odluke.

Ključne riječi: analitika učenja, prediktivna analitika, uspjeh studenata, multinomijalna logistička regresija, Poissonova distribucija

Discovering student profiles with regard to the use of mathematics tutoring services at university level

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Abstract. One could argue that there is a growing trend in Croatian society of using tutoring services from different subjects at all levels of education. This research, using classification tree approach, deals with the use of mathematics tutoring services at university level and focuses on building student profiles related with its use. Along with questions that provided some general information about students, scores obtained by the *Willingness to Listen Measure* (Richmond & Hickson, 2001) and *Teacher Apprehension Test* (Richmond, Wrench & Gorham, 2001) as well as statements describing students' attitudes towards mathematics tutoring were used for classification trees modelling. Several classification tree models were built and from the most successful one student profiles were extracted.

Keywords: mathematics tutoring, classification tree, university level, mathematics performance, students

1. Introduction

Overall impression is that demand for tutoring services for elementary, high school or university students, in any given type (private, group, online, etc.), is constantly increasing. One could consider that the reason for this lies in the increasing number of students who fail to achieve intended learning outcomes. However, there are different motives that drive students to use tutoring services and different profiles of students use them.

Focusing on revealing the profiles of university students who use mathematics tutoring services, derived from classification tree model, this research makes meaningful contribution to this field and to the understanding of this current problem at university sphere.

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Overview of findings concerning the observed problem is given in the next section followed by the representation of used methodology. Obtained classification tree models and extracted rules are given in Section 4 and the conclusion in Section 5.

2. Literature review

Respective number of studies concentrated on variables that describe socio-economic background and their impact on the use of tutoring services. For instance, Bregvadze (2012) revealed positive influence of tutoring service use on university admission with logistic regression analysis. Guimarães and Sampaio (2013) came to similar conclusion in their research. They used least square and quantile regression to explore the influence of parental schooling and income as well as private and public tutoring use on students' success. Their results confirmed significant correlation between students' success and parental schooling and family's income. Furthermore, the connection between students' success and tutoring use was found. In addition, Berberoğlu and Tansel (2014) using linear regression discovered that tutoring services have a small positive affect on students' mathematics success as well as success in Turkish language. Their results were partially confirmed by Zhang and Xie (2015) who used ordinary least square regression for predicting verbal and math success by tutoring use. However, they did not found statistically significant connection between tutoring use and mathematics success when considering students' family characteristics. These results are also consistent with results gained by Navarra-Madsen and Ingram (2010) who used descriptive statistics and correlation analysis to explore the effect of tutoring service use on university students' grades from mathematics subjects and did not found the connection between them. Kersaint et al. (2011) investigated students' willingness to use an online tutoring service from algebra and concluded that students who were less prepared for this course were more likely to use this service.

In view of previous research results, some socio-economic variables were applied in this research while the emphasis was on discovering the profiles of students who use mathematics tutoring services.

3. Methodology

The research was carried out at winter semester of the academic year 2016/2017 at the Faculty of Education, University of Osijek. 130 students involved in this research were requested to fulfil Willingness to Listen (Richmond & Hickson, 2001) measure and Teacher Apprehension Test (Richmond, Wrench & Gorham, 2001) as well as to answer on 14 general questions. Besides general information (gender, age, study year, number of family members), those questions were related to students' financial situation and accommodation while studying (3 variables), parents' education level (2 variables), current employment status of parents (2 variables) and frequencies of mathematics tutoring services use at different levels of education (3 variables). Students were also asked to assess their attitude toward tutoring services in general (10 variables). Brief description of these variables and their basic statistics is given in Table 1.

Table 1. Descriptive statistics of variables.

Variable	**Brief description**	**Descriptive statistics**
V1	gender	1=6.92 %; 2=93.08 %
V2	age	1=37.69 %; 2=23.85 %; 3=33.08 %; 4=5.38 %
V3	study year	1=40.00 %; 2=16.15 %; 3=0.00 %; 4=21.54 %; 5=22.31 %
V4	number of family members	1=0.77 %; 2=3.08 %; 3=6.92 %; 4=40.77 %; 5=33.85 %; 6=9.23 %; 7=4.62 %; 8=0.00 %; 9=0.00 %; 10=0.77 %
V5	accommodation while studying	1=22.31 %; 2=2.31 %; 3=13.85 %; 4=61.54 %; 5=0.00 %
V6	financing during studying	1=8.46 %; 2=81.54 %; 3=9.23 %; 4=0.00 %
V7	Student Service use	1=53.85 %; 2=19.23 %; 3=15.38 %; 4=4.62 %; 5=6.92 %
V8	mothers' education level	1=2.31 %; 2=10.00 %; 3=60.77 %; 4=6.92 %; 5=19.23 %; 6=0.77 %
V9	fathers' education level	1=1.54 %; 2=3.08 %; 3=66.15 %; 4=16.15 %; 5=13.08 %; 6=0.00 %
V10	current employment status of mother	1=15.38 %; 2=55.38 %; 3=26.92 %; 4=2.31 %
V11	current employment status of father	1=11.54 %; 2=54.62 %; 3=11.54 %; 4=22.31 %
V12	math tutoring services use at elementary school	1=86.15 %; 2=8.46 %; 3=2.31 %; 4=3.08 %; 5=0.00 %
V13	math tutoring services use at high school	1=48.46 %; 2=31.54 %; 3=11.54 %; 4=4.62 %; 5=3.85 %
V14	preparation for exam	1=1.54 %; 2=10.77 %; 3=21.54 %; 4=38.46 %; 5=27.69 %
V15	lower grade correction	1=3.85 %; 2=8.46 %; 3=28.46 %; 4=35.38 %; 5=23.85 %
V16	extension of knowledge	1=19.23 %; 2=32.31 %; 3=26.92 %; 4=16.15 %; 5=5.38 %
V17	tutoring service efficiency	1=0.00 %; 2=1.54 %; 3=22.31 %; 4=44.62 %; 5=31.54 %
V18	tutoring services as one's own desire	1=0.77 %; 2=3.85 %; 3=14.62 %; 4=34.62 %; 5=46.15 %
V19	inability to pass and master without tutoring services	1=36.15 %; 2=16.92 %; 3=25.38 %; 4=9.23 %; 5=12.31 %
V20	parents and siblings help during schooling	1=14.62 %; 2=23.85 %; 3=18.46 %; 4=23.85 %; 5=19.23 %
V21	learning in study group	1=11.54 %; 2=13.08 %; 3=30.77 %; 4=30.00 %; 5=14.62 %
V22	regularly going to consultations	1=22.31 %; 2=42.31 %; 3=29.23 %; 4=6.15 %; 5=0.00 %
V23	learning regularly	1=9.23 %; 2=22.31 %; 3=42.31 %; 4=20.77 %; 5=5.38 %
WL	final score on Willingness to Listen Measure	M=78.98; SD=9.23
TAT	final score on Teacher Apprehension Test	M=55.52; SD=5.69
MTSU	mathematics tutoring services use at university	0= 57.69 %; 1=42.31 %

Depending on the answer given to the question about the use of mathematics tutoring services at university level, in order to develop classification tree models, students were categorised into two categories: those that did not use mathematics tutoring services at university level (category labelled with 0) and those that did (category labelled with 1). This variable was set as the dependent variable for the classification tree modelling.

In the interest of classification tree modelling, overall sample was randomized and divided into train (80 % of the data) and test (20 %) sample. Overall sample was randomized (random sample was computer-generated) in order to insure that each sample case had the same chance of getting into train or test sample.

3.1. Measures used in research

Willingness to Listen Measure (WL) (Richmond & Hickson, 2001) was used to assess students' readiness to listen. It consists of 24 statements where students indicated the level of relevance for each statement using a five-point Likert scale (1- strongly disagree, 5- strongly agree). Teacher Apprehension Test (TAT) of Richmond et al. (2001) was used as a measure to determine the level of apprehension students feel in communication with their teacher. TAT is built of 20 statements where students also expressed their agreement on a five-point Likert scale. Authors of measures used in this research provided scoring scheme for both measures and for this research students' final scores were used as input variables for the classification tree modelling.

3.2. Classification tree method

It is powerful and popular method for data mining used in many scientific fields (Izydorczyk and Wojciechowski, 2016). Main advantage of this method lies in its simple interpretation of results and relations between variables, which are often quite complex (Song and Lu, 2015). Classification tree is constructed by repeatedly splitting the data in two mutually exclusive classes with regard to some simple conditions (De'ath and Fabricius, 2000). Tree starts with root node, which includes entire learning sample, and it splits into two nodes (subsets) (Wilkinson, 2004). Each internal node is connected with edges to its parent node and its child nodes (Song and Lu, 2015). It is obvious that each node consists of the union of data in the child nodes (Wilkinson, 2004). Finally, tree ends with child nodes called leaves or terminal nodes that represent final categories of data (Song and Lu, 2015).

4. Results

As recommended in Zekić-Sušac and Đurđević Babić (2015), at the beginning of modelling procedure variables independence was tested. Based on the type of the variables, Pearson Chi-square test (also known in literature as the chi-square test for independence or the chi-square test of association) was used to see whether variables are independent. Observed significant ($p<0.05$) relationships are reported in Table 2.

Table 2. Association between variables.

Variable	Association (p<0.05)
V1	V20, V23
V2	V3, V6, V7, V13
V3	V6, V7, V13
V4	V6, V8, V11
V5	V6, V7, V10, V11, V16
V6	V2, V3, V4, V5, V7
V7	V2, V3, V5, V6
V8	V9, V11
V9	V8, V11
V10	V5, V11
V11	V4, V5, V8, V9, V10
V12	V13
V13	V2, V3, V12
V14	V5, V8, V11, V15, V18, V20
V15	V6, V14
V16	V5
V17	V12, V13, V18, V21
V18	V5, V7, V13, V14, V17
V20	V1, V14
V21	V10, V17, V22
V22	V2, V21, V23
V23	V1, V22

Since the results given in Table 2 only inform us that some associations between these variables exist, the relationships between dependent variables and categorical variables (in other words, the independence between each categorical variable and dependent variable) were also analysed by Pearson Chi-square test. The results showed, at the level of significance 0.05, that there is dependency between the use of mathematics tutoring services at university level and gender, age, current study year, the use of mathematics tutoring services at elementary and high school as well as the variable describing tutoring service efficiency.

For determining the difference between the group of students that did not use mathematics tutoring services at university level and the group of students that did use it regarding the final scores obtained by the TAT and WL scale, the t-test for independent samples was used. The results showed that at 0.05 significance level there is no statistically significant difference in means between the group of students that did not use mathematics tutoring services at university level and the group of students that did use it regarding TAT scores ($t(128)=1.07$, $p=0.29$) or WL score ($t(128)=-1.14$, $p=0.26$).

Due to the obtained results, variables V1 (gender), V2 (age), V3 (current study year), V12 (mathematics tutoring services use at elementary school), V13 (mathematics tutoring services use at high school) and V17 (tutoring service efficiency) were excluded from the modelling.

Two split selection methods (discriminant-based univariate splits for categorical and ordered predictors and C&RT-style exhaustive search for univariate splits) available in Statistica 12 software were used. For more information about these methods, see Hill and Lewicki (2007). Discriminant-based univariate splits method produced the highest overall classification accuracy (46.15 %) when using FACT-style direct stopping rule and fraction of objects set to 0.03. As explained in Hill and Lewicki (2007) fraction of objects is a way to control splitting process by stopping it when there is less cases than defined minimum proportion of categories in tree leaves (terminal nodes). C&RT split selection method produced higher overall classification accuracy. All three used measures of goodness of fit had the same level of overall classification success (see Table 3).

Table 3. Results of classification tree models.

Split selection method	Goodness of fit	Stopping rule	Stopping parameters	Number of splits	Accuracy		
					0	1	Overall
C&RT	Gini	FACT-style	fraction of objects=0.07	9	80.00	45.45	65.39
C&RT	**Chi-square**	**FACT-style**	**fraction of objects=0.03**	**19**	**73.33**	**54.54**	**65.38**
C&RT	G-square	FACT-style	fraction of objects=0.07	16	86.67	36.36	65.38
C&RT	G-square	FACT-style	fraction of objects=0.03	22	73.33	54.54	65.38

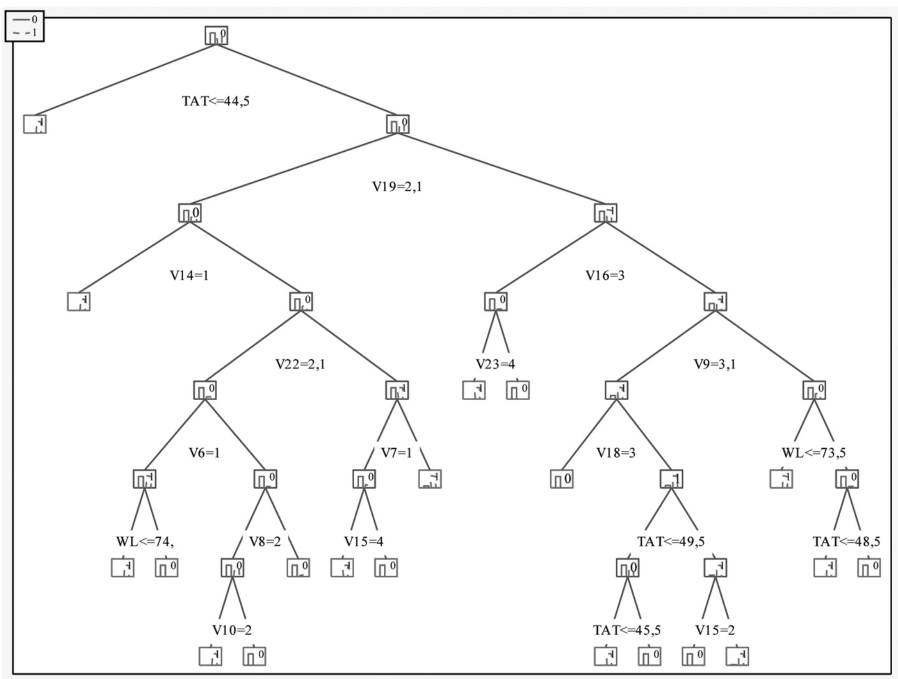

Figure 1. Graphical representation of chosen classification tree model.

Among models presented in Table 3, model with the most acceptable (over 50 %) classification accuracy per each category and the smaller number of splits was considered for rule extraction and profile creation. Sensitivity analysis was also conducted in order to see which variables influence the most on classification achievement of this model. From the results presented in Table 4, it can be concluded that variable V16 has the greatest, while the variable V5 has minor influence on model classification achievement.

Table 4. Results obtained by sensitivity analysis.

Variable	**Rank**
V16	100
V14	92
TAT	85
V7	84
V10	84
WL	81
V23	79
V9	75
V8	73
V6	64
V18	64
V19	62
V4	55
V21	55
V22	52
V15	42
V20	35
V11	34
V5	19

Extracted rules from the chosen model are:

1) IF TAT <= 44.5 THEN add to category 1
2) IF TAT > 44.5 AND (V19=2 or V19=1) AND V14=1 THEN add to category 1
3) IF TAT > 44.5 AND (V19=2 or V19=1) AND V14< >1 AND (V22=2 or V22=1) AND V6=1 AND WL <= 74 THEN add to category 1
4) IF TAT > 44.5 AND (V19=2 or V19=1) AND V14< >1 AND (V22=2 or V22=1) AND V6=1 AND WL > 74 THEN add to category 0
5) IF TAT > 44.5 AND (V19=2 or V19=1) AND V14< >1 AND (V22=2 or V22=1) AND V6< >1 AND V8< >2 THEN add to category 0
6) IF TAT > 44.5 AND (V19=2 or V19=1) AND V14< >1 AND (V22=2 or V22=1) AND V6< >1 AND V8=2 AND V10=2 THEN add to category 1

7) IF TAT > 44.5 AND (V19=2 or V19=1) AND V14< >1 AND (V22=2 or V22=1) AND V6< >1 AND V8=2 AND V10< >2 THEN add to category 0
8) IF TAT > 44.5 AND (V19=2 or V19=1) AND V14< >1 AND (V22< >2 AND V22< >1) AND V7=1 AND V15=4 THEN add to category 1
9) IF TAT > 44.5 AND (V19=2 or V19=1) AND V14< >1 AND (V22< >2 AND V22< >1) AND V7=1 AND V15< >4 THEN add to category 0
10) IF TAT > 44.5 AND (V19=2 or V19=1) AND V14< >1 AND (V22< >2 AND V22< >1) AND V7< >1 THEN add to category 1
11) IF TAT > 44.5 AND (V19< >2 AND V19< >1) AND V16=3 AND V23=4 THEN add to category 1
12) IF TAT > 44.5 AND (V19< >2 AND V19< >1) AND V16=3 AND V23< >4 THEN add to category 0
13) IF TAT > 44.5 AND (V19< >2 AND V19< >1) AND V16< >3 AND (V9=3 or V9=1) AND V18=3 THEN add to category 0
14) IF TAT > 44.5 AND (V19< >2 AND V19< >1) AND V16< >3 AND (V9=3 or V9=1) AND V18< >3 AND TAT <= 49.5 AND TAT <= 45.5 THEN add to category 1
15) IF TAT > 44.5 AND (V19< >2 AND V19< >1) AND V16< >3 AND (V9=3 or V9=1) AND V18< >3 AND TAT <= 49.5 AND TAT > 45.5 THEN add to category 0
16) IF TAT > 44.5 AND (V19< >2 AND V19< >1) AND V16< >3 AND (V9=3 or V9=1) AND V18< >3 AND TAT > 49.5 AND V15=2 THEN add to category 0
17) IF TAT > 44.5 AND (V19< >2 AND V19< >1) AND V16< >3 AND (V9=3 or V9=1) AND V18< >3 AND TAT > 49.5 AND V15< >2 then add to category 1
18) IF TAT > 44.5 AND (V19< >2 AND V19< >1) AND V16< >3 AND (V9< >3 AND V9< >1) AND WL <= 73.5 THEN add to category 1
19) IF TAT > 44.5 AND (V19< >2 AND V19< >1) AND V16< >3 AND (V9< >3 AND V9< >1) AND WL >73.5 AND TAT <= 48.5 THEN add to category 1
20) IF TAT > 44.5 AND (V19< >2 AND V19< >1) AND V16< >3 AND (V9< >3 AND V9< >1) AND WL >73.5 AND TAT >48.5 THEN add to category 0

The established profiles of students who use mathematics tutoring services at university level are:

- Profile 1: students whose final score on TAT was 44.5 or less
- Profile 2: students whose final score on TAT was higher than 44.5, strongly disagreed or disagreed with the statement that some subjects can't be mastered

without the use of tutoring service and strongly disagreed with the statement that tutoring services serve as preparation for exams

- Profile 3: students whose final score on TAT was higher than 44.5, strongly disagreed or disagreed with the statement that some subjects can't be mastered without the use of tutoring service, don't strongly disagree with the statement that tutoring services serve as preparation for exams, strongly disagreed or disagreed with the statement that they are frequently going on consultation with teacher, are mostly self-financed during study and whose final score on WL was 74 or less
- Profile 4: students whose final score on TAT was higher than 44.5, strongly disagreed or disagreed with the statement that some subjects can't be mastered without the use of tutoring service, don't strongly disagree with the statement that tutoring services serve as preparation for exams, strongly disagreed or disagreed with the statement that they are frequently going on consultation with teacher, are not mostly self-financed during study, whose mothers' level of education is 2 and whose mothers have permanent work
- Profile 5: students whose final score on TAT was higher than 44.5, strongly disagreed or disagreed with the statement that some subjects can't be mastered without the use of tutoring service, don't strongly disagree with the statement that tutoring services serve as preparation for exams, don't strongly disagree or disagree with the statement that they are frequently going on consultation with teacher, who never used Student Service for employment and did agree with the statement *Tutoring services are mostly used for lower grade correction*
- Profile 6: students whose final score on TAT was higher than 44.5, strongly disagreed or disagreed with the statement that some subjects can't be mastered without the use of tutoring service, don't strongly disagree with the statement that tutoring services serve as preparation for exams, don't strongly disagree or disagree with the statement that they are frequently going on consultation with teacher, who used Student Service for employment
- Profile 7: students whose final score on TAT was higher than 44.5, who didn't strongly disagreed or disagreed with the statement that some subjects can't be mastered without the use of tutoring service, who were neutral concerning agreement with the statement *Tutoring services are mostly used for extension of knowledge* and agreed with the statement *I learn regularly*
- Profile 8: students whose final score on TAT was higher than 44.5, who didn't strongly disagreed or disagreed with the statement that some subjects can't be mastered without the use of tutoring service, who were not neutral concerning agreement with the statement *Tutoring services are mostly used for extension of knowledge*, whose fathers gained level 1 or level 3 of education, were not neutral concerning agreement on the statement *One goes to tutoring services depending on one's own desire* and scored on TAT 45.5 or less
- Profile 9: students whose final score on TAT was higher than 44.5, who didn't strongly disagreed or disagreed with the statement that some subjects can't be mastered without the use of tutoring service, who were not neutral concerning agreement with the statement *Tutoring services are mostly used for extension*

of knowledge, whose fathers gained level 1 or level 3 of education, were not neutral concerning agreement on the statement *One goes to tutoring services depending on one's own desire*, scored on TAT higher than 49.5 and did not disagree with the statement *Tutoring services are mostly used for lower grade correction*

- Profile 10: students whose final score on TAT was higher than 44.5, who didn't strongly disagreed or disagreed with the statement that some subjects can't be mastered without the use of tutoring service, who were not neutral concerning agreement with the statement *Tutoring services are mostly used for extension of knowledge*, whose fathers didn't gained level 1 or level 3 of education and scored 73.5 points or less on WL
- Profile 11: students whose final score on TAT was higher than 44.5, who didn't strongly disagreed or disagreed with the statement that some subjects can't be mastered without the use of tutoring service, who were not neutral concerning agreement with the statement *Tutoring services are mostly used for extension of knowledge*, whose fathers didn't gained level 1 or level 3 of education and scored higher than 73.5 points on WL and 48.5 or less on TAT.

The most straightforward profile among them is Profile 1. It distinguishes students who do not express high level of teacher apprehension as those that use mathematics tutoring services at university. Nevertheless, students who have moderate or high level of teacher apprehension in addition with some other features are also recognized as the ones that use mathematics tutoring services (profiles from 2 to 11).

5. Conclusion

The purpose of this paper was to extract profiles of students who use mathematics tutoring services at university. Classification tree method was selected as approach and several classification tree models, with different architectures, were created. Models using C&RT split selection method showed the same high overall classification ability (65.38 %) and the less complex model with efficient classification accuracy per class among them was chosen for rule extraction and profile interpretation. Eleven profiles of students that use mathematics tutoring services were drown out. Despite this relatively large number of profiles, that clearly shows diverseness of student population, it was revealed that students who possess lower level of anxiety toward mathematics teacher are the ones that use mathematics tutoring services at university. For assessing and improving course quality, it is useful for educators to know profiles of students that use mathematics tutoring services and to be aware that students who don't have negative attitude toward mathematics teachers use mathematics tutoring services at university. In order to enhance students' understanding of course, besides putting additional materials and activities, educators could reflect on course aims and design, strengthen the communication with students within course and empower them to master the course on they own.

References

[1] BERBEROĞLU, G., & TANSEL, A. (2014), *Does private tutoring increase students' academic performance?*, Evidence from Turkey, International Review of Education, 60(5), 683–701.

[2] BREGVADZE, T. (2012), *Analysing the shadows: Private tutoring as a descriptor of the education system in Georgia*, International Education Studies, 5(6), 80–89.

[3] DE'ATH, G., FABRICIUS, K. E. (2000), *Classification and regression trees: a powerful yet simple technique for ecological data analysis*, Ecology, 81(11), 3178–3192.

[4] GUIMARÃES, J., & SAMPAIO, B. (2013), *Family background and students' achievement on a university entrance exam in Brazil*, Education Economics, 21(1), 38–59.

[5] HILL, T., & LEWICKI, P. (2007), *STATISTICS: Methods and Applications*, StatSoft, Tulsa, OK.

[6] IZYDORCZYK, B., & WOJCIECHOWSKI, B. (2016), *Differential diagnosis of eating disorders with the use of classification trees (decision algorithm)*, Archives of Psychiatry and Psychotherapy, 4, 53–62.

[7] KERSAINT, G., DOGBEY, J., BARBER, J., & KEPHART, D. (2011), *The effect of access to an online tutorial service on college algebra student outcomes*, Mentoring & Tutoring: Partnership in Learning, 19(1), 25–44.

[8] NAVARRA-MADSEN, J., & INGRAM, P. (2010), *Mathematics tutoring and student success*, Procedia-Social and Behavioral Sciences, 8, 207–212.

[9] RICHMOND, V. P., & HICKSON, M. III. (2001), *Going public: A practical guide to public talk*, Boston: Allyn & Bacon.

[10] RICHMOND, V. P., WRENCH, J. S., & GORHAM, J. (2001), *Communication, affect, and learning in the classroom*, Acton, MA: Tapestry Press.

[11] SONG, Y. Y., & LU, Y. (2015), *Decision tree methods: applications for classification and prediction*, Shanghai archives of psychiatry, 27(2), 130–135.

[12] WILKINSON, L. (2004), *Classification and regression trees*, Systat, 11, 35–56.

[13] ZEKIĆ-SUŠAC, M., & ĐURĐEVIĆ BABIĆ, I. (2015), *Discovering patterns of student behaviour in e-learning environment*, In Z. Kolar-Begović, R. Kolar-Šuper & I. Đurđević Babić (Eds.) Higher goals in mathematics education (94), Zagreb: Element.

[14] ZHANG, Y., & XIE, Y. (2016), *Family Background, Private Tutoring, and Children's Educational Performance in Contemporary China*, Chinese Sociological Review, 48(1), 64–82.

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Otkrivanje profila studenata s obzirom na korištenje instrukcija iz matematike kod sveučilišnog obrazovanja

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Sažetak. Moglo bi se reći da je rastući trend u hrvatskom društvu korištenje instrukcija iz raznih predmeta na svim razinama obrazovanja. Koristeći pristup klasifikacijskih stabala, ovaj rad bavi se upotrebom instrukcija iz matematike na razini sveučilišnog obrazovanja i fokusiran je na izgradnju profila studenata koji koriste takve instrukcije. Zajedno s pitanjima kojima su dobiveni neki opći podatci o studentima, rezultati skala *Willingness to Listen* (Richmond & Hickson, 2001.) i *Teacher Apprehension* (Richmond, Wrench & Gorham, 2001.) kao i izjave koje opisuju stav studenata prema instrukcijama iz matematike, korišteni su kao varijable za modeliranje klasifikacijskih stabala. Izrađeno je nekoliko modela klasifikacijskih stabala te su iz najuspješnijeg izvučeni traženi profili studenata.

Ključne riječi: instrukcije iz matematike, klasifikacijsko stablo, sveučilišno obrazovanje, uspješnost iz matematike, studenti

Identifying mathematical anxiety with MLP and RBF neural networks

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Abstract. This research addresses the problem of mathematical anxiety which is usually associated with inadequate mathematical performance and achievement. It aims to develop neural network model for classification of students according to the degree of mathematical anxiety in order to examine and better understand the relationship and effects of physical activity along with some other factors on mathematical anxiety. For this purpose, Multilayer Perceptron (MLP) and Radial Basis Function (RBF) neural networks were used. The results of this research showed that neural network models were efficient in identifying students' mathematical anxiety. With the purpose of exploring the relationships between the mathematical anxiety and input variables, sensitivity analysis was conducted and reported for the model with the highest overall classification accuracy.

Keywords: mathematical anxiety, physical activity, neural network, MLP, RBF

1. Introduction

As one of major school subjects, mathematics is in focus of parents and educators around the world. Numerous research deal with the issue of mathematics performance and investigate why some students have weak results in mathematics. In many cases, the reasons for these unsuited results are found in mathematical anxiety. Desa and others (2016) highlight that math anxiety can manifest itself as failing to remember how to solve mathematical problem or even project on ones' self-confidence in mathematics. Therefore, researchers such as Luo and others (2009), look at math anxiety as on illness with certain symptoms.

Instead of trying to find the cause of math anxiety, this research, using nonlinear models, attempts to identify students with above average tendency to mathematical

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anxiety. Because of their characteristics, neural networks were chosen as a method for achieving this. Additionally, this research is also concentrated with determining the influence of physical activity on detection of mathematical anxiety in the context of nonlinear modeling.

2. Literature review

Mathematical anxiety is well covered topic in previous research, some of which put attention on preservice teachers. For instance, Isiksal and others (2009) investigated the influence that mathematical anxiety has on mathematical self-concept. They focused on early childhood and elementary school preservice teachers enrolled in educational programs at two countries (USA and Turkey). MANOVA was applied to test what influence educational programs and grades have on mathematical anxiety and mathematical self-concept. They discovered significant influence from both of them. Using ANOVA on relevant variables, they confirmed the negative effect of mathematical anxiety on mathematical self-concept. Zakaria and Nordin (2008) explored the connection between mathematical anxiety and performance of college students and, after performing Pearson's correlation, reported significant negative association between mathematical anxiety and students' performance in mathematics as well as mathematical anxiety and motivation. With these results, they additionally confirmed some of the results already gained by Betz (1978) or Baloglu and Kocak (2006). Jackson (2008) also investigated the impact of mathematical anxiety on preservice teachers and concluded that students' attitude toward mathematics has powerful impact on their mathematical performance. Peker (2009) examined math-teaching anxiety of preservice teachers considering their learning style and with ANOVA showed that there is a statistically significant difference among different learning styles and math-teaching anxiety. Research conducted by Bursal and Paznokas (2006) showed that preservice teachers with noticeable mathematical anxiety feel that they will not be effective mathematics teachers. From the used method point of view relevant to this research, Herawan and others (2011) research is significant. They used data mining method (association rules) for extraction of basic principles concerning mathematical anxiety among engineering students.

3. Methodology

135 students from Faculty of Education, University of Osijek, took part in this research. Besides answering on the Abbreviated Math Anxiety Scale (AMAS; Hopko et al., 2003) and International Physical Activity Questionnaire (IPAQ, n.d.), they answered on additional 20 questions. Some of these additional questions were demographic information questions (age, gender) and others were related to success, especially math success in different levels of education (elementary school, high school and university).

The Abbreviated Math Anxiety Scale (AMAS) developed by Hopko and others (2003) was used for estimating the intensity of mathematical anxiety in this research. This scale is composed of 9 statements where participants express their attitudes toward these statements on a typical 5-point scale varying from strongly disagree to strongly agree. Hopko and others (2003) assessed and presented acceptable measures of internal consistency ($\alpha = 0.90$) and reliability (r= 0.85) for AMAS (for more information see Hopko et al., 2003).

A short version of self-administered International Physical Activity Questionnaire (IPAQ) was used for estimating physical activity of participants. Croatian version of IPAQ as well as scoring protocol is available on the IPAQ homepage (IPAQ, n.d.) where large number of research that used IPAQ are also listed.

Neural networks ability to work with sizable dataset and to generalize well (Santos et al., 2014) was main reason why this method was chosen for data mining in this research. Two types of feedforward neural networks were used: Multilayer Perceptron (MLP) and Radial Basis Function (RBF). Both of these types of networks are composed of the input, hidden and the output layer, but while MLP neural networks can have more than one hidden layer, RBF neural networks have only one hidden layer (Behrang et al., 2010). Morajda (2003) presents the process of obtaining output of the MLP neural networks with equation:

$$y = f\left(\sum_i x_i w_i\right) \tag{1}$$

where f is non-linear activation function, x_i the i-th input signal and w_i the weight parameter. As stated in Wesolowski and Suchacz (2003) the correct weights are unknown at the beginning of the learning process and they are updated during this process with some training algorithm such as backpropagation algorithm. Other training algorithms like conjugate gradient descent can be also used (Hill and Lewicki, 2007). RBF neural network produces output in a following manner: inputs are not multiplied by the weights, merely forwarded to the hidden layer, which measures the distance between the input vector and the centre of its radial function (Yilmaz et al., 2012). Gaussian function is frequently used as radial basis function, although many others can be used (Yilmaz et al., 2012). Detailed information about the neural network architecture and learning algorithms can be found at Haykin (2005).

4. Results

At the beginning of modeling procedure, 21 variables were considered for input variables. In accordance with procedure presented in Zekić-Sušac and Đurđević Babić (2015), the association between the considered input variables and the target variable was determined in order to shrink the large dataset. Pearson Chi-square test was used. The variables that showed significant association ($p < 0.05$) with the target variable were rejected from modeling. Pearson Chi-square test was also used for detecting relationships between the input variables. However, this test does not provide the information about the intensity of the observed correlations, so Cramer's V test was used to detect the level of association. Those variables that showed medium and large effect size (Cramer's V>0.3) were also removed from the modeling (see Table 1).

Table 1. Variables that showed medium and strong strength of the relationship.

Variable	Relationship (Cramer's V>0.3)
study year	gender, age
gender	study year
age	study year
elementary school success	math elementary school success, remedial teaching (elementary school), additional teaching (elementary school), high school success
math elementary school success	elementary school success, remedial teaching (elementary school), additional teaching (elementary school), high school success
remedial teaching (elementary school)	elementary school success, math elementary school success, remedial teaching (high school)
additional teaching (elementary school)	elementary school success, math elementary school success, math competition (elementary school)
math competition (elementary school)	additional teaching (elementary school)
high school success	elementary school success, math elementary school success, math success (high school)
math success (high school)	high school success
remedial teaching (high school)	remedial teaching (elementary school)
additional teaching (high school)	math competition (high school)
math competition (high school)	additional teaching (high school)
completed mathematics courses (university)	average grade from completed mathematics courses (university), number of attempts to pass exam (university), student assistant at math classes (university)
average grade from completed mathematics courses (university)	completed mathematics courses (university), number of attempts to pass exam (university), student assistant at math classes (university)
number of attempts to pass exam (university)	completed mathematics courses (university), average grade from completed mathematics courses (university), student assistant at math classes (university)
student assistant at math classes (university)	completed mathematics courses (university), average grade from completed mathematics courses (university), number of attempts to pass exam (university)

This resulted in removal of 12 variables (study year, overall elementary school success, math elementary school success, remedial teaching (elementary school), additional teaching (elementary school), overall high school success, additional teaching (high school), completed mathematics courses (university), average grade from completed mathematics courses (university), number of attempts to pass exam (university), math competition (elementary school) and remedial teaching (high school)).

Remaining 9 variables were taken as the input variables while the indicator of mathematical anxiety was the dependent variable for neural network models. These variables were: gender (MV1), age (MV2), math success in high school (MV3), participating in math competition in high school (MV4), student assistant (MV5), work during studying (MV6), accommodation during studying (MV7), studying at place of residence (MV8), physical activity (MV9).

Achieved average value on the mathematical anxiety questionnaire was 26. For the purpose of this research and with the goal of constructing neural network models, students who achieved above average value (in the case of participants in this research higher than 26) of mathematical anxiety were classified into category labeled as 1. The rest of the students (those who achieved average and below average value of mathematical anxiety) were classified into category labeled as 0.

Overall sample was randomized and 70 % of its data was used to build training sample, 20 % test sample and 10 % validation sample. By changing different parameters needed for creation of neural network models (number of hidden neurons, error function, training algorithm and activation functions for MPL networks, stopping conditions), four hundred MLP and RBF neural networks were trained, tested and validated. In order to gain the best classification model with the strong generalization ability and avoid overfitting, models performance on the validation sample were compared and assessed. The MLP model with 4 neurons in a hidden layer, entropy as error function and tangent hyperbolic as activation function, achieved the highest overall classification accuracy of 71.43 %. This model correctly classified 77.78 % students who obtained average and below average value of mathematical anxiety (category labeled as 0) and 60.00 % of those students who obtained above average value of mathematical anxiety (category labeled as 1). Sensitivity analysis (see Table 2) showed that the highest importance on this specific model performance had the variable *studying at place of residence*, while the variable *physical activity* exhibited the lowest impact on this model.

Table 2. Sensitivity analysis of variables.

MV7	MV4	MV1	MV6	MV2	MV9	MV5	MV3	MV8
1.63	1.81	1.76	1.63	1.50	0.89	1.09	1.19	2.09

5. Conclusion

The main objective of this research was to create neural network model capable of recognizing students with above average values of mathematical anxiety. MLP and RBF neural network type were used to accomplish this aspiration. Although, relatively large number of variables was considered for models input variables, after conducted Pearson's Chi-square test and Cramer's V test the number of variables was decreased and only 9 relevant variables were used. For generalization purpose, models were compared on validation sample. The most powerful model was obtained by MLP neural network. This model had the highest overall classification rate (71.43 %) and was able to identify 60.00 % of students with above average value of mathematical anxiety. It was more successful (77.78 %) in recognizing students without tendency to mathematical anxiety (those who had average and below average value of mathematical anxiety). Additionally, the effects of physical activity and other essential variables for development of this model were estimated with sensitivity measure. It revealed that physical activity had insignificant influence on this models performance and is shown to be redundant in this context.

Results derived from this research could serve to mathematics teachers as help in detecting students that have tendency to mathematical anxiety when observing some variables that are not evidently connected with it. Teachers could use these insights to raise the level of quality of their teaching by applying more individualized approach in their class and take into consideration different teaching and learning styles, which may help in creating more enjoyable and stimulating classroom environment.

References

[1] Baloglu, M., & Kocak, R. (2006), *A multivariate investigation of the differences in mathematics anxiety*, Personality and Individual Differences, 40(7), 1325–1335.

[2] Behrang, M. A., Assareh, E., Ghanbarzadeh, A., & Noghrehabadi, A. R. (2010), *The potential of different artificial neural network (ANN) techniques in daily global solar radiation modeling based on meteorological data*, Solar Energy, 84(8), 1468–1480.

[3] Betz, N. E. (1978), *Prevalence, distribution, and correlates of math anxiety in college students*, Journal of counseling psychology, 25 (5), 441.

[4] Bursal, M., & Paznokas, L. (2006), *Mathematics anxiety and preservice elementary teachers' confidence to teach mathematics and science*, School Science and Mathematics, 106(4), 173–180.

[5] Desa, A. M., Saad, S. A., Zakaria, S. A., & Zakaria, M. H. (2016, October), *Exploring mathematics anxiety among first year business students: UniMAP experience*, In N. Rusli, W. M. K. A. W. Zaimi, K. A. M. Khazali, M. J. Masnan, W. S. W. Daud, N. Abdullah, & Z. Yahya (Eds.), AIP Conference Proceedings (Vol. 1775, No. 1, p. 030069), AIP Publishing.

[6] Haykin, S. (2005), *Neural networks: A comprehensive foundation*, Second edition, Pearson Education, Singapure.

[7] Herawan, T., Vitasari, P., & Abdullah, Z. (2011, June), *Mining interesting association rules of student suffering mathematics anxiety*, In International Conference on Software Engineering and Computer Systems (pp. 495–508), Springer Berlin Heidelberg.

[8] HILL, T., & LEWICKI, P. (2007), *STATISTICS: Methods and Applications*, StatSoft, Tulsa, OK.

[9] HOPKO, D. R., MAHADEVAN, R., BARE, R. L., & HUNT, M. K. (2003), *The abbreviated math anxiety scale (AMAS) construction, validity, and reliability*, Assessment, 10(2), 178–182.

[10] IPAQ, (N.D.), Retrieved from `https://sites.google.com/site/theipaq/`.

[11] ISIKSAL, M., CURRAN, J. M., KOC, Y., & ASKUN, C. S. (2009), *Mathematics anxiety and mathematical self-concept: Considerations in preparing elementary-school teachers*, Social Behavior and Personality: an international journal, 37(5), 631–643.

[12] JACKSON, E. (2008), *Mathematics anxiety in student teachers*, Practitioner Research in Higher Education, 2(1), 36–42.

[13] LUO, X., WANG, F., & LUO, Z. (2009), *Investigation and analysis of mathematics anxiety in middle school students*, Journal of Mathematics Education, 2(2), 12–19.

[14] MORAJDA, J. (2003), *Neural networks and their economic applications*, In Artificial intelligence and security in computing systems (pp. 53–62), Springer US.

[15] PEKER, M. (2009), *Pre-service teachers' teaching anxiety about mathematics and their learning styles*, Eurasia Journal of Mathematics, Science & Technology Education, 5(4), 335–345.

[16] SANTOS, R. B., RUPPB, M., BONZI, S. J., & FILETIA, A. M. F. (2013), *Comparison between multilayer feedforward neural networks and a radial basis function network to detect and locate leaks in pipelines transporting gas*, Chem. Eng. Trans, 32(1375), e1380.

[17] WESOLOWSKI, M., & SUCHACZ, B. (2012), *Artificial neural networks: theoretical background and pharmaceutical applications: a review*, Journal of AOAC International, 95(3), 652–668.

[18] YILMAZ, I., MARSCHALKO, M., BEDNARIK, M., KAYNAR, O., & FOJTOVA, L. (2012), *Neural computing models for prediction of permeability coefficient of coarse-grained soils*, Neural Computing and Applications, 21(5), 957–968.

[19] ZAKARIA, E., & NORDIN, N. M. (2008), *The effects of mathematics anxiety on matriculation students as related to motivation and achievement*, Eurasia Journal of Mathematics, Science & Technology Education, 4(1), 27–30.

[20] ZEKIĆ-SUŠAC, M., & ĐURĐEVIĆ BABIĆ, I. (2015), *Discovering patterns of student behaviour in e-learning environment*, In Z. Kolar-Begović, R. Kolar-Šuper & I. Đurđević Babić (Eds.), Higher goals in mathematics education (94), Zagreb: Element.

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Identificiranje matematičke anksioznosti uz pomoć MLP i RBF neuronskih mreža

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Sažetak. U ovom istraživanju razmatra se problem matematičke anksioznosti koja je obično povezana s nedostatnim postignućima i neodgovarajućim uspjehom iz matematike. Istraživanje je usmjereno na razvijanje modela neuronske mreže u svrhu klasifikacije studenata prema stupnju matematičke anksioznosti s namjerom da se istraži i bolje razumije povezanost i utjecaj tjelesne aktivnosti i nekih drugih čimbenika na matematičku anksioznost. U tu svrhu korištene su mreže višeslojni perceptron (engl. multilayer perceptron, MLP) i mreža s radijalno zasnovanom funkcijom (engl. radial basis function, RBF). Rezultati ovog istraživanja pokazali su da su modeli neuronskih mreža bili učinkoviti u prepoznavanju matematičke anksioznosti kod studenata. Za model s kojim se postigla najviša ukupna točnost klasifikacije provedena je analiza osjetljivosti kako bi se istražila povezanost između matematičke anksioznosti i ulaznih varijabli.

Ključne riječi: matematička anksioznost, tjelesna aktivnost, neuronske mreže, MLP, RBF

Standardization of learning outcomes in teaching mathematics

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Abstract. This work presents critical approach in analysis of recent curriculum documents and literature relevant for evaluation of learning outcomes. Author focuses on advantages and disadvantages of standardization using didactical and pedagogical approach to mathematics education. In recent literature related for teaching mathematics there is growing focus on social and pedagogical dimensions of teaching and learning as key assumptions of students' success in mathematics. Therefore, learning outcomes should evaluate all parts of mathematics curriculum, but in national curriculum documents only educational outcomes of mathematics content are being relevant. Author indicates negative implications of neglecting pedagogical dimensions of teaching and learning mathematics, and emphasizes contradiction between standardization and pedagogical evaluation of learning outcomes. Considering the totality of factors influencing mathematics education, validation of learning outcomes should include pedagogical and psychological dimensions of teaching process and emotional and affective components of learning outcomes. In the conclusion, author suggests guidelines toward upgrading learning outcomes by integrating students' abilities, needs, work habits, attitudes and beliefs with clear objectives, goal setting methods and assessments specified in the curriculum.

Keywords: learning outcomes, social and affective dimensions of teaching and learning mathematics

1. Introduction

The fundamental characteristic of the *National Curriculum Framework* [NCF] (MZOS, 2010) is a transition to a system based on competence and student achievements, i.e. educational standards. The term "educational standards" in this case does not stand for learning and educational goals, but implies a new way of setting goals in teaching and educating processes. (Bašić, 2007, p. 117). The essential part of educational standards are measurable competences for particular areas, i.e.

teaching subjects. They represent a set of knowledge, skills, abillities and values which comprise learning and educational standard, together with indicators of acquisition (Vican, Bognar, Previšić, 2007, p. 171).

Standardization of student achievements in this case refers to attaining learning goals, which in mathematics reflects in acquired skills and competences related to logical thinking, concluding and providing mathematical arguments in solving various tasks modelled on real life situations and problems. However, according to the curriculum proposed by the expert group, it is obvious that the educational goals in learning and educational standards are set quite high in theoretical domain, but remain in the background in practical (operative) one when referring to the area of mathematics. According to the NCF (MZOS, 2010) and Mathematics Curriculum (proposed by the working group in 2016, educational goals in teaching mathematics are specifically named in only one of the goals of teaching and learning mathematics and are not listed at all among learning and educational outcomes in cycles and domains regarding mathematical area. Furthermore, educational outcomes determined within the mathematics curriculum display absence of values and other educational aspects of competences otherwise so strongly advocated in the recent considerations on mathematics teachings. The fact is, the curriculum determines educational standards according to the levels of acquisition at the end of each academic year, while those are not in compliance with motivational aspects of teaching, individualization, numerous student challenges and, ultimately, the overall concept of modern teaching, which takes students and their individual characteristics as its starting point. There also remains a question of operational plans and programs used in education programs, which have not yet been critically approached, nor has there been a change in their content, application, or individual approach to students' needs.

2. Educational standards in mathematics

As both a scientific discipline and teaching subject, mathematics is guided by the general principle of science and thought which is essential to science as well as everyday life. It is, therefore, necessary in class to develop a set of comprehensive skills and abilities as well as to understand the practical aspects of mathematics whose scope of knowledge and skills extends its study domain. Within the National Curriculum Framework (2010) expected student achievements have been defined for educational areas per cycles and subject structure of each area has been outlined. The NCF (2010) forms a starting point for developing school and subject curricula based on elaborate achievements for educational areas. Learning and teaching mathematics incorporates acquisition of knowledge, skills and computing competences as well as assessment and logical and spatial thinking (MZOS, 2010). In line with this, general educational goals in the area of mathematics have been developed and expected student achievements per cycles elaborated. Curriculum goals (MZOS, 2010, p. 80) in mathematics area which provide basis for educational outcomes per classes and domains are:

- acquire principal mathematical knowledge, skills and processes, establish and comprehend mathematical relations and connections; develop a positive attitude towards mathematics, a sense of personal responsibility for success and progress as well as self-awareness in relation to mathematical achievements;
- capability for abstract and spatial thinking and logical deduction;
- capability for solving mathematical problems and applying mathematics in various contexts, including professional life;
- recognize and understand the historical and societal importance of mathematics in science, culture, art and technology and its potential in the future of society;
- exchange efficiently mathematical knowledge, ideas and results using different presenting formats;
- apply technology efficiently;
- acquire sound basis for lifelong learning and continuing education.

Educational standards as the curriculum instrument for achieving the set goals serve at the same time as criteria as well as objectives, which demands a clear distinction and formulation of learning outcomes so they can be used as criteria for the quality of the class, that is the efficiency of the learning and teaching process (Bašić, 2007). Based on those, the teacher derives learning goals which are the starting point for planning, organizing and managing educational process (Jurčić, 2012). The author mentioned classifies these goals as general and specific, oriented towards the change in student's knowledge, skills, values, attitudes and habits, resulting from organized teaching and learning in each class, and accomplished by student activity and learning. Standards of teaching contents present the basis for learning and enable students to extend knowledge and abilities, as well as acquire skills.

Taking into account the diversity of challenges in teaching mathematics, it is key to precisely determine the content and achievement level standards in the mathematics area. Considering the cumulative development of mathematical knowledge, one of the primary problems in teaching mathematics is establishing the level of mathematical knowledge and skills acquisition, so as to enable the students to apply them in new areas, but also to ensure as many students as possible develop skills necessary for understanding further lesson materials. It is also vital in teaching mathematics to elaborate student achievements not only according to school criteria, educational orientation or class, but also to take into consideration the alignment of achievement standards with the applicability of mathematical competence expected of students to acquire.

Previously mentioned general goals in mathematical area within the national curriculum have been subsequently somewhat adjusted in line with the comprehensive curriculum reform, but standards and levels of student achievements in mathematical areas and educational cycles have been elaborated precisely and in great detail. The intention of achievement level standards was to define clearly the minimum criteria necessary for a passing grade according to the educational guidelines. Along with this, possibly the most relevant part of the achievement standards within the comprehensive curriculum reform is the so called intermediate standard expected in most students due to the cumulative aspect of mathematical contents, which is required as sufficient for understanding further lesson materials determined by the lesson plan and program. The mentioned standard should be determined by the overall mathematical competence according to key competences of the European Union. However, it is recognized that many students accumulate problems related to mathematics as early as the beginning of their education, and do not acquire mathematical contents at the desired intensity or range, due to the cumulative aspect of knowledge. According to Poljak (1991) we distinguish the following levels based on knowledge acquisition quality: remembering, recognition, reproduction, operative and creative knowledge. For example, the level of knowledge reproduction indicates knowledge acquisition independent of functional applicability, which in the context of mathematical competence based on applicability as primary goal implies the need for a higher level knowledge. Therefore, the minimum level required is operative knowledge, which presumes complete comprehension and application of the acquired knowledge in later contents.

The ongoing question is whether to reduce and/or change teaching plans and programs in order to achieve a more profound quality of basic forms of knowledge. On the other hand, it is worthwhile considering the amount to which teacher training and modernization of teaching methods can help lessen the restrictions in teaching process and goals set within the teaching plans and programs. Taking into account the level of complaints coming from the teaching profession regarding the overly extensive aspect of the teaching plan and program and the lack of any substantial evidence gathered from the external assessment tests (e.g. State Matura Examination, PISA) to support the desired knowledge applicability and quality, the question which presents itself is how to reach the knowledge quality without reducing the quantity, that is, the scope of the teaching plans and programs?

Within the critical opinions regarding the comprehensive curriculum reform in mathematical area, Mišurac-Zorica (2016) points out the fact that the focus of the entire document remains traditionally on teaching contents and not as much on processes or skills. The author also brings to attention the approach within the reform to "appoint all secondary schools having 3 math classes, regardless of being vocational or grammar schools, the same math teaching program. This includes, e.g. language schools and numerous vocational schools (mechatronics technician, clothing technician, maritime analyst or vocations in the field of forestry, wood processing etc.)" This inevitably implies lack of a comprehensive and methodologically based approach regarding the appropriateness, modernity and pedagogical legitimacy in the selection of teaching contents for each educational orientation. The fact is, if the set goals cannot be achieved through current teaching plans and programs, then the compatibility of educational standards and teaching plans and programs, along with appropriate teacher training, must all be equally analyzed. Contemporary didactic approaches in teaching should be examined as well and incorporated on a practical level into teaching mathematics in order to determine skills and knowledge required from teachers.

3. Between normative standards and pedagogical aspects of school and teaching

On a daily basis students come across different stressful situations in class, which may trigger various emotions towards all the subjects in educational process. Students competence is assessed by examining the acquired skills and knowledge according to the defined educational outcomes, often neglecting the affective aspects of both learning and teaching. Students who encounter problems with learning mathematics most often have insufficient prior knowledge, negative attitudes and beliefs regarding mathematics, lack motivation in learning and consequently underestimate their own capabilities. The interaction of the mentioned circumstances and unfavorable factors in school and family surrounding results in animosity towards mathematics and lack of motivation as early as elementary school. These circumstances and situations are pervaded by students' emotional reactions towards all the subjects in educational process. Pekrun(2006) introduced the term "academic emotions" alongside emotional reactions in class, which refers to emotional reactions closely related to the teaching process, i.e. learning, teaching and student achievements. The same are reflected in emotions students associate with mathematics, preconceived beliefs as well as attitudes towards mathematics. It is a generally acknowledged fact that the most dominant emotion in teaching mathematics is fear of mathematics.

On the other hand, according to the curriculum draft introduced by the expert group in 2016, in the segment referring to *assessment* it is clearly stated that its purpose is to foster learning and student development, that it is based on a comprehensive approach of evaluating and supporting individual student development, with aim to encourage students to recognize individual success and promote positive patterns to motivation and learning. According to the assessment criteria within the same document, three approaches to student achievement assessment are highlighted: assessment for learning, assessment as learning and assessment of the learned content.

However, the same document states: "The final grade (evaluation) is the most efficient and valid measure when it reflects acquisition of educational and learning outcomes, and not some otherwise relevant elements of student behavior: cooperation, effort, attendance, participation, neatness, etc. The criteria mentioned need to be reported on specifically, unless their inclusion is defined within educational outcomes for the particular subject. Because of the fact that final grades cannot encompass all the mentioned aspects of learning, they are combined in final report with a qualitative feedback on student achievements and progress (1st and 2nd cycle) or assessment scales which include those relevant aspects: student responsibility, self-reliance, self-initiative, communication and collaboration (assessment of generic competences in 3rd, 4th and 5th cycle" (p. 27).

Although various documents within the curriculum draft (Group of experts, 2016) emphasize the importance of pedagogical approach in evaluation and assessment, many contradictions regarding educational standards still exist. In particular, their reduction to a more narrow term "student achievement standards" does not imply pedagogical intention and is contradictory to the fundamental requirement of contemporary education, the need for individualization, not standardization (Herman, 2005 in Bašić, 2007, p. 150). A contemporary teacher should take into account a child's individuality, his needs and social being, regarding school as a social being too (Previšić, 1999). The above mentioned quotation from the curriculum draft on assessment points out that: *"The final grade (evaluation) is the most efficient and valid measure when it reflects acquisition of educational outcomes, and not otherwise relevant elements of student behavior...*" and raises the issue who is this measure most efficient and valid for – educational system of students? The question of educational aspects of teaching and confrontation of two approaches follows: standardization and prescription by the educational policy and student oriented approach based on pedagogical and psychological findings on school, class and teaching. Bašić (2007) points out the fact that: "we can infer from international experiences that school does not require standards as means of (economic) school competitiveness and external control. It requires high quality school and teaching standards which are pedagogically and psychologically legitimized, based on contemporary scientific grounds and culturally individualized" (Bašić, 2007, p. 149). The author further claims that "the trend of considering education as product in a technocratic society diminishes individual learning responsibility, due to the fact that the focal point has become school achievements and competitiveness and not students as subjects of their own education" (Bašić, 2007, p. 123).

The only highlighted component regarding educational goals mentioned in the curriculum draft by the expert group in 2016 along with the one in the 2010 curriculum is: "Students will build self-esteem and awareness of their personal mathematical abilities as well as persistence, entrepreneurship, responsibility, respect and positive attitude towards mathematics and work in general" (Group of experts, 2016). There remains a fact that this component is left out of the learning outcomes with the intent to introduce differentiated curriculum following the guidelines of the *Framework for Improving Learning Experience and Assessing Achievements of Students with Difficulties* and with the aim of meeting educational needs of students with difficulties. On the other hand, the mentioned framework is at risk of being too general-based and not specified for mathematical area, not to mention particular problem areas concerning learning and teaching mathematics. Recent literature in teaching mathematics brings to attention a valid example of "anti-anxiety curriculum" aimed at helping students to overcome hindrances faced when learning mathematics (Geist, 2010). Senior students are facing increasing negative attitudes towards mathematics, (Vizek-Vidović et al., 1997), which suggests the necessity of such a curriculum as early as primary school and extending to the completion of high school. Aimed primarily at educational aspects in teaching mathematics, it is designed to become an integral part of learning outcomes as well. The lack of confidence among students as well as the feeling of incompetence in learning mathematics are well recognized and result in insufficient motivation often combined with memorising the mathematical tasks instead of applying logical thinking when solving them.

The main goal of the anti-anxiety curriculum is for students to reach the *success cycle* in developing mathematical skills and knowledge (Koshy et al., 2009). The cycle is commonly described in following terms:

- confidence in personal abilities, positive beliefs towards mathematics;
- effort, persistence and interest in challenging tasks;
- achievement and success in mathematics.

In the context of the anti-anxiety curriculum becomes apparent the need for a therapeutic dimension of math teachers in order to successfully direct negative student attitudes and beliefs about learning mathematics towards success, i.e. activities not restricted to negative aspects from the affective domain (Horvat, 2016). In doing so, two problems occur: the ones of success and comparison. The fact is that weaker students have the ability to solve only simpler tasks, being aware of the effort invested in the process and the success and the amount of effort made by better students in class. Taking this into consideration, the success cycle does not have a total effect, due to the fact that the achievement does not meet objectivity criteria as with the students who know (and believe) they are able to solve the more advanced tasks.

For that reason precisely should the program be individualized and realized through additional classes to help students overcome the difficulties in learning mathematics. Although this requires specific teacher competence, the crucial factor in its realization is time limitation within teaching plans and programs and additional teacher engagement combined with finances within the state budget invested currently in education. Taking into account the fact that the idea of reducing teaching plans and programs was absent from the considerations within the recent curriculum reform, the obvious question is *when* and *how* are teachers supposed to invest additional time for students with learning challenges?

The entire issue is based on dealing with educational aspects of mathematical competence, which requires teacher effort and amount of individualization currently impossible to put in practice for every single student. In practice this results in students attending private lessons. The educational aspect of teaching which needs to be achieved in the process is fostering values and attitudes (moral, aesthetic, physical and work), i.e. developing personality traits which belong to the affective and conative domain (Poljak, 1991). Education is directed towards developing positive and noble personality qualities. The fact is that unless material and functional tasks include appropriate and suitable educational values, they can never be fully realized.

According to all relevant curriculum documents and research on teaching mathematics, the crucial factor is to develop and nurture a positive attitude towards mathematics as well as personal progress responsibility. The most efficient means within the scope of teaching to foster the above mentioned is by systematic evaluation and assessment. Questioning, student evaluation, assessment of knowledge are integral part of educational process. Assessing student knowledge provides insight into the quality of mathematical content acquisition, monitoring of their progress, feedback on their work, guidelines for further improvement of learning quality. In assessing and monitoring student achievement teachers mainly rely on normative and criteria-based evaluation. Normative evaluation is based on comparison with other students, while the criteria-based one is established on the curriculum level through educational and learning outcomes. It is important here to refer to student self-evaluation, including the current level of attained skills and knowledge. Is it possible to achieve the set educational (as well as learning) goals and at the same time rely on standardized norms and external criteria which usually have little relevance to a specific class, class environment, individual aspects of students, motivational factors in assessment as well as numerous difficulties students encounter when learning mathematics?

In line with recent literature in pedagogy it is vital to perform a systematic evaluation of student progress based on pedagogical, diagnostic, psychological, and therapeutic approach to each student. From the pedagogical aspect student assessment needs to be an integral part of teacher work and at the same time not fixed on and binned by national standards, but guided by pedagogically and psychologically supported standards, set and adjusted by the teacher who is familiar with the student and his individual characteristics. The assessment should aim to encompass student achievements and progress in order to form a final grade, reflecting student's individual progress and not serving as means of measuring and comparison with other students, classes and schools.

4. Final considerations

Contemporary school has an educational responsibility to ensure child development taking into consideration all his intellectual, physical, emotional, societal and moral aspects. Such a school needs to have purpose along with self-criticism towards its methods, contents and teaching plans and programs in order to continue to modernize its curriculum in compliance with the imminent social changes. The class has an obligation to provide students with both educational and learning values, not only as a reflection of school values, or what the school represents, but also because learning goals can never be fully realized if they do not entail educational values. The greatest challenge in teaching mathematics is the growing number of students with difficulties in learning mathematics, which demonstrates insufficient knowledge in comparison with their peers who participated in a number of PISA international education assessment tests.

The necessary requirement for a shift toward a more didactic approach to teaching mathematics is to prescribe educational standards in assessment and evaluation, along with learning. This would also result in mathematics pedagogy which binds teachers to include a pedagogical framework into their assessment and evaluation of students and their overall work. Learning and educational standards should be defined based on a scientific approach deriving from theory and practice of learning and teaching mathematics, so as to didactically transform key elements of mathematics into a teaching subject, primarily taking into consideration psychological and physical aspects of students at a specific age. Taking into account the fact that most difficulties in teaching mathematics lie directly or indirectly in emotional or affective domain, the obvious conclusion is that those can be dealt with only by interdisciplinary approach to teaching mathematics, including pedagogy, psychology and other relevant sciences in the area of education. Similar approach could be applied to the documents from the comprehensive curriculum reform, which prescribe educational standards for learning, teaching and assessment, but lack a clear link with the curriculum of a specific area in order to validate current learning and educational outcomes as the ones which do not serve merely as means of student comparison, but also as means of their personal progress in particular areas.

References

[1] BAŠIĆ, S. (2007), *Obrazovni standardi – didaktički pristup metodologiji izgradnje kurikuluma*, In V. Previšić (Ed.), Kurikulum – Teorije, metodologija, sadržaj, struktura (p. 117–155), Zagreb: Zavod za pedagogiju Filozofskog fakulteta u Zagrebu i Školska knjiga.

[2] GEIST, E. (2010), *The Anti-Anxiety Curriculum: Combating Math Anxiety in the Classroom*, Journal of Instructional Psychology, 37(1), 24–31

[3] GROUP OF EXPERTS (2016), *Mathematics Curriculum – Proposition*, Retrieved December, 2016, from `http://www.kurikulum.hr/`.

[4] HORVAT, Z. (2016), *Motivacija u nastavi matematike*, Sedmi kongres nastavnika matematike Republike Hrvatske, Zagreb.

[5] JURČIĆ, M. (2012), *Pedagoške kompetencije suvremenog učitelja*, Zagreb: Recedo.

[6] KOSHY, V., ERNEST, P., & CASEY, R. (2009), *Mathematically gifted and talented learners: theory and practice*, International Journal of Mathematical Education in Science and Technology, 40(2), 213–228.

[7] MZOS [MINISTARSTVO ZNANOSTI, OBRAZOVANJA I ŠPORTA REPUBLIKE HRVATSKE] (2010), *Nacionalni okvirni kurikulum*, Ministarstvo znanosti, obrazovanja i športa Republike Hrvatske, Zagreb.

[8] POLJAK, V. (1991), *Didaktika*, Zagreb: Školska knjiga.

[9] PEKRUN, R. (2006), *The Control Value Theory of Achievement Emotions: Assumptions, Collollaries and Implications for Educational Research and Practice*, Educational Psychology Review, 18(1), 315–341.

[10] PREVIŠIĆ, V. (1999), *Škola budućnosti: humana, stvaralačka i socijalna zajednica*, Napredak, 140(1), 7–16.

[11] VICAN, D., BOGNAR, L., & PREVIŠIĆ, V. (2007), *Hrvatski nacionalni kurikulum*, In V. Previšić (Ed.), Kurikulum – Teorije, metodologija, sadržaj, struktura (pp. 157–204), Zagreb: Zavod za pedagogiju Filozofskog fakulteta u Zagrebu i Školska knjiga.

[12] VIZEK-VIDOVIĆ, V., VLAHOVIĆ-ŠTETIĆ, V., & ARAMBAŠIĆ, L. (1997), *Konativne i emocionalne karakteristike matematički nadarene i prosječno sposobne djece*, Društvena istraživanja, 6(4-5), 619–634.

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Standardizacija obrazovnih ishoda u nastavi matematike

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Sažetak. Autor u ovom radu kritički analizira recentne kurikularne dokumente i literaturu relevantnu za evaluaciju (odgojno)–obrazovnih ishoda. Naglasak je pritom na prednostima i nedostacima standardizacije sa didaktičkog i pedagoškog aspekta matematičkog obrazovanja. U recentnoj literaturi učenja i poučavanja matematike sve je veći naglasak na socijalnim i pedagoškim dimenzijama nastave matematike kao ključnim pretpostavkama uspjeha učenika. U skladu s time obrazovni ishodi trebaju evaluirati sve komponentne kurikuluma matematike, no u aktualnim kurikularnim dokumentima relevantni su jedino standardi obrazovnih postignuća. Autor ističe negativne posljedice zapostavljanja odgojnih dimenzija nastave matematike te ističe proturječja između standardizacije i pedagoške evaluacije učeničkih postignuća. Uvažavajući sveukupnost čimbenika koji utječu na matematičko obrazovanje, konceptualizacija obrazovnih ishoda mora sadržavati pedagoško–psihološku dimenziju nastavnog procesa, te emocionalne i afektivne komponente učeničkih postignuća. Zaključno, autor predlaže smjernice za nadogradnju obrazovnih ishoda integracijom stavova, sposobnosti, radnih navika i uvjerenja učenika sa ishodima, metodama i evaluacijom zadanim u kurikulumu.

Ključne riječi: obrazovni ishodi, socijalne i afektivne dimenzije nastave matematike

Teaching and learning mathematics in inclusive settings: Analysis of curriculum of compulsory education in five European countries

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Abstract. Global perspectives on inclusive education advocate one's effective social functioning. In that perspective, math education is a priority due to its presence in everyday life. Independent living presupposes presence of basic math skills. To identify the presence of math skills that supports independence (i.e. functional mathematical skills), content analysis of five European countries national curriculums was conducted: Great Britain, Finland, Germany, France and Croatia. Results showed that mathematical skills promoted in these national agendas are inherent part of independent living. According to these agendas, functional math can be interpreted as math fluency, numeracy skills, and connection between science and everyday life. Math curriculum is focused on personal experiences and possibilities to achieve basic knowledge and understandings of mathematics, applicable in everyday life. To achieve this goal, some curriculum accommodations should be present, such as applying special teaching strategies, adequate equipment, and facilitate experiential learning.

Keywords: inclusive math curriculum, inclusive settings, pupils with disability, functional mathematics

1. Introduction

Over last two decades, mathematics has become one of main field of interests of stakeholders at international level. There are several reasons. One of them is low achievement of pupils across Europe in PISA. According to Froese-Germain (2010)

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achievement in PISA reflects true state of one's state education, which should be reconsidered during educational reforms. Analysis showed that achievement in math is overall lower than in other academic fields such as language, which suggests that educational policies should reconsider the role of mathematic in a pupils' life (OECD, 2015). For that reason, Niss (2015) analyze term "mathematical literacy" suggesting that math is more than sequential learning, but an activity founded in everyday life. It is a continuous process which has strong practical uphold. Similarly, Fox and Surtees (2010) notice the fact that children, even at the earliest stages of life, come in touch with math, especially during play. For instance, children before their 2^{nd} birthday, through songs, pictures and story books which contain mathematical vocabulary, experience math. It is a key feature for future abstract knowledge and functional demonstration in natural settings. Obviously, math is widely present in a child's everyday life: it can be taught by intuitive interaction schemes, which leads to conclusion that functional math needs no or very little didactic accommodation, and thus should be easy to teach. Yet, some authors (Fox and Surtees, 2010; Gordon, 2016) notice teachers have difficulties during curriculum accommodation for pupils with disability suggesting that teachers overlook pupil's overall knowledge and get focused onto just one particular math activity, and not a learning process itself.

Another reason for social interest in math is growing presence of ICT in classrooms which presuppose teachers' knowledge on math (UNESCO, 2015). Globally, over the last decade, ICT became a strategic field for education, and thus teachers should strength their competencies in math, as a pillar of ICT. Nowadays, math and ICT (as inherent elements of STEM) are seen as possibility for overcoming current economic crisis across nations. In that context, math has been identified as one of the key educational subject for developing competence. Math competence should result in one's active citizenship, social inclusion and economic autonomy in adulthood, as one of strategic task of knowledge society (OECD, 2015). This functional approach to math is practically oriented, but is it achievable in reality? Since math competence encompasses variety of knowledge, skills, abilities, and values, they are achievable if a pupil's development is typical. The question is, what about pupils with disabilities, especially intellectual ones? Do they need math, and to which extent? These and similar questions slowly become key topics in inclusive education agendas. The reason is that inclusive education presupposes paradigm shift from valuing outcomes to valuing processes. In pedagogy, this approach is closely connected with quality of education, i.e. quality in process of learning and teaching has become a key indicator of overall quality of education. This idea is supported by Moss and Dahlberg (2008) who claim that quality in education can been seen in process dimension, i.e. in the quality of interaction and level of teachers' support, rather than in grades. Despite the fact that quality of inclusive education has been researched in the last 20 years, there is still present strong need for changing approaches in education, including math, because inclusion is about designing appropriate curriculums, strategies and methods (Dixon and Verenikina, 2007), and not about "matching" a pupil with a educational program. For Norwich (2008) main problem in curriculum design is teacher's perception of pupil's abilities. Teachers are more concerned about things a pupil don't know, and should learn, and less about his/hers learning potentials and experiences (ibid).

This reductionist approach to teaching is derived from teachers' orientation towards measurable indicators (Romstein, Irović and Vego, 2015), as only reliable strategy for measuring own professional success. As Cole (2013) states, teachers have own perception of lessons, which results with different approaches to teaching, depending on teachers' capacity to develop own, unique approach in teaching. So, quality in education is closely connected with teachers' orientation towards process, which presuppose teachers' autonomy in teaching strategies, and what's more important, their reflection upon own competencies, i.e. nourishing meta-approach to own teaching practice. Therefore, quality in inclusive math education is supported by teachers' knowledge in curriculum design, including developmentally appropriate teaching strategies which are the true indicators of presence of inclusive values in education.

Summarizing findings about math in the context of inclusive education, it all comes to one major goal: to make math a sort of help in everyday life of pupils with disabilities. Functional approach is foundation stone in inclusive education, so math should serve a purpose of supporting one's everyday life. Basically, this approach is appropriate for all key competences needed for full social fulfillment, yet it looks like math needs more advocacy than other educational fields. To overarch this problem, some European countries have developed inclusive math education embedded in their national curriculums. In the last 10 years, inclusive math education has emerged as a priority in many disability agendas. Math is seen as a tool for successful and autonomous living (OECD, 2015). It is seen as a living skill, and thus should be present in everyday activities, starting from kindergarten.

2. Methods

The purpose of this survey was to explore national curriculums in five European countries: Great Britain, Finland, Germany, France and Croatia, and to find out to which extent these agendas have considered pupils with disabilities in context of math education? Main goal was to identify key competences in inclusive math education. For that purpose, a content analysis of five national curriculums of primary education was conducted: Great Britain, Finland, Germany, France and Croatia. Content analysis was qualitative, i.e. keywords in context technique were applied. Key words were inclusion, disability, and competencies.

3. Results and interpretation

Analyzed national curriculums were presented to general public during period of 10 years, i.e. from 2004 (Germany) to 2016 (Finland). Interesting fact is that some countries are evaluating their national curriculums continuously (Germany, Great Britain and Finland), which contribute to its implementation, while Croatia has published curriculum in 2010., but never conducted educational reform, and subsequently never conducted curriculum evaluation. At this point, in Croatia, education is driven by agenda named Croatian National Educational Standard, introduced to the public in 2005. It represents a strategic document, which later derived national curriculum, analyzed in this survey. This indicates that educational practices vary across European Union despite common agendas and guidelines. It is important to say that Germany has several curriculums due to its geo-political organization of education. Berlin-Brandenburg county curriculum was analyzed, because here is situated German's capital city of Berlin. Also, in French language the term program is widely accepted, not the curriculum.

3.1. Presence of inclusive approach in math education

Math education in all national curriculums was identified as a priority for all pupils, but when it comes to pupils with disability there is still a lack of guidelines. Mostly, math for pupils with disability is interpreted as prerequisite of pupils' autonomy in later life, as part of social inclusion. Emphasis on math as inherent part of social life is clear only in curriculums of Great Britain, Finland, and Germany. In Croatian and French national curriculum math is primarily considered as numeracy skill needed for later employment, not a social skill *per se*.

Table 1. Mathematics within the national curriculums.

Country	**Main task in math education**	**Accommodation for pupils with disability**
Great Britain	To develop pupils' math fluency.	Access to special equipment and appropriate approach during teaching session.
Germany	To develop personal autonomy.	Continuous follow-up of pupil's performances and individual approach.
Finland	To develop and maintain knowledge and skills in math that will remain strong in the future.	Enhancing pupil's participation, and facilitating experiential learning.
France	To develop pupils' knowledge about interconnection between math and other sciences.	Concrete models and schemes, and appropriate teaching strategies.
Croatia	To develop numeracy skills needed for successful employment.	Adequate teaching strategies.

Although national curriculums are oriented towards inclusive education, there is no clear taxonomy on key competences which should derive from leaning. This problem is even more visible when it comes to math education. All five European Countries have embraced key competences prescribed by European Commission, with one important difference, and that's Finland interests in intercultural competence, and individuality as key issues for successful lifelong learning. However, only two European Countries (Great Britain and Finland) have considered competencies for pupils with disability, as part of inclusive education. Taxonomy on key competencies allows follow ups, developing standards and raising quality of education.

Table 2. Key mathematical competencies in national curriculums.

Country	Key competencies for all	Key math competencies for pupils with disability
Great Britain	English/literacy/reading; mathematics/numeracy; science; modern foreign languages; digital competence/ICT; social and civic competence (including sex and relationships education and citizenship); initiative, entrepreneurship and enterprise education	Basic numeracy (counting, measuring)
Germany	Command of the German language; proficiency in a modern foreign language; key elements of mathematics, scientific culture, and technology; mastery of ordinary information and communication skills; humanist culture; social and civic skills; autonomy and initiative.	None
Finland	Command of the Finnish language; proficiency in a modern foreign language; key elements of mathematics, scientific culture, and technology; mastery of ordinary information and communication skills; humanist culture; social and civic skills; autonomy and initiative; individuality and intercultural competence.	Basic numeracy and transfer of basic mathematical skills into daily living.
France	Command of the French language; proficiency in a modern foreign language; key elements of mathematics, scientific culture, and technology; mastery of ordinary information and communication skills; humanist culture; social and civic skills; autonomy and initiative.	None
Croatia	Communication in mother language; communication in foreign language; mathematical competencies, and competencies in sciences and technology; digital competencies; learn how to learn; social and civic competence.	None

4. Discussion

Math in inclusive education should have one important key feature and that's transferability from classroom to real life. To achieve this, teachers must continuously reflect own practice, and scrutinize the way they teach children. As Cole (2013) noticed, reflections and dialogue on particular math task leads to its understanding on higher cognitive level, thus teachers and pupils should reflect together. This claim is supported by curriculums of Great Britain and Finland, which are focused on a pupils' participative learning. Although participative education, as pedagogical idea, is present almost 20 years, there are still some barriers for its implementation. Major problem is teacher's understanding of participative learning, as a key process of quality in education (Moss and Dahlberg, 2008; Tagucchi, 2010). This could be changed by joint reflection of teacher and pupils, as suggested in Finnish curriculum. This curriculum is the only one which put an emphasis on a pupil's personal experience, as a didactic recommendation for teaching math. Other countries do recognize pupils' actions as a key to a learning math, but not to that extension, i.e. in Finland teachers are supported to reflect with children upon math tasks, which, subsequently leads to change in math class, and development of flexible curriculum.

Inclusive approach in math curriculum can be revealed through presence of curriculum accommodations for pupils with disabilities. There are several accommodation strategies, and each country has its own vision on this issue: in Great Britain the key is access to equipment, in Germany the emphasis is on follow-ups of pupils' performance, in Finland the most important thing is to enhance pupils' participative learning, and in Croatia the most important strategy is to apply adequate ones. The main purpose of accommodation strategies is to raise competence of pupils with disabilities, and that's the major problem in analyzed curriculums because only Great Britain and Finland clearly articulates desirable competence. Its basic numeracy (Great Britain), and basic numeracy and its' transfer to daily routines (Finland). Other countries have failed to describe competence for pupils with disabilities, despite presence of strategies for accommodations. As Gordon (2016) claims, enabling pupils with disabilities in math presuppose supportive classrooms, which could help them gain the control over their actions, relationships and emotions. This represents a positivistic approach to inclusive math education, which is important if teachers wish their subject have a positive impact on a pupil's life (Fox and Surtees, 2010). Since inclusive education is more about reorganization of current educational practice (Dixon and Verenikina, 2007), accommodation of math curriculum within supportive classrooms is a key to success. However, at this point, it looks like all the responsibility for pupils with disabilities achievement in math is placed solely on teachers: they have to make accommodations according to own drive. Absence of articulated strategies on accommodations, and absence of description of competences supports that claim. For that reason, it is important to continuously scrutinize educational reality, in which mathematics have (or should have) a key role. Dividing responsibility between stakeholders and professionals is prerequisite of inclusive math education.

5. Conclusion

Inclusive math education is an interdisciplinary scientific area. It's between social sciences and STEM, with strong focus on particular pupil's potential to become independent adult. Inclusion in context of math education is about curriculum accommodations, and as such should be present in everyday teaching (Gordon, 2016). Including pupils' own goals (Gordon, 2016), and reflection on performance (Tagguchi, 2010), are key features of inclusive math education. The emphasis in this approach is on process of learning and thus should be focused on a particular pupils' life experience (Fox and Surtees, 2010), and math teachers should reconsider their own role in inclusive education. Math teachers should be able to confront new challenges in educational process, and adopt their practice to this relatively new idea on purpose of mathematics. Inclusive math curriculum has two major benefits for pupils with disabilities: one is to prevent additional disabilities such as dyscalculia, and the other is to facilitate new mathematical knowledge and skills applicable in everyday life. This can be achieved when teachers shift their focus from grades to process and, what's more important experiences. It means that math in inclusive education should be contextually situated whenever possible. Within this approach demonstration, practical methods, pupil's participation and task-reflections are main teaching strategies. In that way mathematics becomes fully functional, and a key feature of inclusive education.

References

[1] DIXON, R. M., VERENIKINA, I. (2007), *Towards inclusive schools: an examination of socio-cultural theory and inclusive practices and policies in New South Wales DET schools*, `http://www.ro.uow.edu.au/llrg` [Accessed 01/05/2008].

[2] *Finnish National Agency for Education* (2014), Core curriculum for 2016, `http://www.oph.fi` [Accessed 05/02/2017].

[3] FOX, S., & SURTEES, L. (2010), *Mathematics Across the Curriculum*, New York: Continuum.

[4] FROESE-GERMAIN, B. (2010), *The OECD, PISA and the Impacts on Educational Policy*, Toronto: Canadian Teachers' Foundation.

[5] GORDON, M. (2016), *Enabling Students in Mathematics*, Bern: Springer.

[6] *Ministere de l'Education nationale* (2014), Programme d'etudeds de mathematique, Paris: MEN.

[7] *Ministerium für Bildung, Wissenschaft und Kultur* (2004), Grundschule Mathematik Curriculum, `http://www.bildungsserver.berlin-brandenburg.de` [Accessed 10/03/2008].

[8] MOSS, P., DAHLBERG, G. (2008), *Beyond Quality in Early Childhood Education and Care – Languages of Evaluation*, New Zealand Journal of Teachers' Work, 5 (1), 3–12.

[9] MZOS (2011), *Nacionalni okvirni kurikulum za predškolski odgoj i obrazovanje te opće obvezno i srednjoškolsko obrazovanje*, Zagreb: MZOS, [in Croatian].

[10] NISS, N. (2015), *Mathematical Competencies and PISA*, In K. Stacey & R. Turner (Eds.), Assessing Mathematical Literacy (pp. 1–22), Bern: Springer Verlag.

[11] NORWICH, B. (2008), *Perspectives and Purposes of Disability Classification Systems: Implication for Teachers and Curriculum and Pedagogy*, In L. Florian, M. J. McLaughlin, (Eds.), Disability Classification in Education (pp. 131–153), Thousand Oaks: Corvin Press.

[12] OECD (2015), *PISA results of 2015*, `http://www.oecd.org` [Accessed 02/02/2017].

[13] ROMSTEIN, K., IROVIĆ, S., VEGO, M. (2015), *Teaching Mathematics in early education: current issues in classrooms*, In Z. Kolar-Begović, R. Kolar-Šuper, & I. Đurđević Babić, (Eds.), Teaching and Learning Mathematics (pp. 187–202), Osijek: FOOZOS.

[14] TAGUCHI, H. L. (2010), *Going Beyond the Theory/ Practice Divide in Early Childhood Education – Introducing an intraactive pedagogy*, London and New York: Routledge.

[15] UNESCO (2015), *ICT Competency Framework for Teachers 2015*, `http://www.unesdoc.unesco.org` [Accessed 02/02/2017].

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Učenje i poučavanje matematike u inkluzivnom okruženju: analiza kurikuluma obveznog odgoja i obrazovanja u pet europskih zemalja

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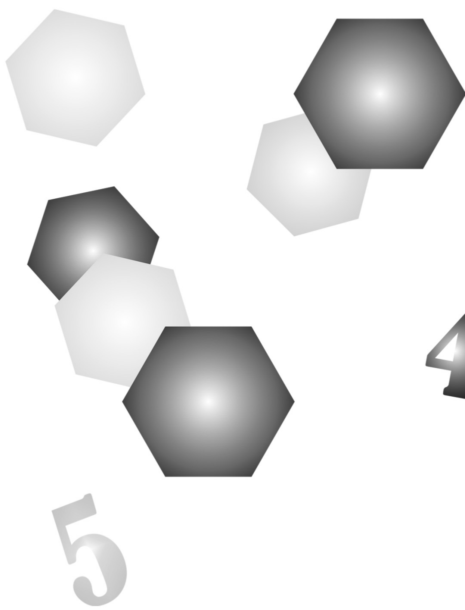
Sažetak. Globalne perspektive o inkluzivnom obrazovanju stavljaju naglasak na djetetovo učinkovito socijalno funkcioniranje. U toj perspektivi, matematika je prioritet, ponajprije zbog njezine prisutnosti u svakodnevnom životu. Samostalno življenje zahtjeva posjedovanje temeljnih matematičkih vještina. Kako bi se identificirale matematičke vještine koje mogu podržavati pojedinčevu samostalnost (tj. funkcionalnih matematičkih vještina), načinjena je analiza sadržaja pet europskih nacionalnih kurikuluma: Velike Britanije, Finske, Njemačke, Francuske i Hrvatske. Rezultati pokazuju kako su matematičke vještine u navedenim dokumentima sastavnice samostalnog življenja. Prema njima, funkcionalna matematika se može interpretirati kao matematička fluentnost, vještine mjerenja i brojenja, te povezivanje znanosti i svakodnevnog života. Kurikulumi matematike su usmjereni na osobna iskustva i osiguravanje mogućnosti za jačanje temeljnog matematičkog znanja i razumijevanja matematike, s primjenom u svakodnevnom životu. Kako bi se to postiglo potrebno je načiniti određene kurikulumske prilagodbe, kao što su primjena specifičnih strategija poučavanja, primjerena opremljenost učionica, te podržavanje iskustvenog učenja.

Ključne riječi: inkluzivni kurikulum matematike, inkluzivno okruženje, učenici s teškoćama u razvoju, funkcionalna matematika

6.
Some approaches to teaching mathematics



Enhancing positive attitude towards mathematics through introducing Escape Room games

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Abstract. By researching the attitudes towards mathematics, research agreed that these are important factors that appear in the analysis of variation in students' accomplishments in mathematics. Various authors have interpreted the attitudes towards mathematics as an emotional disposition. Moreover, Ma (1997) has shown the proportional correlation between positive attitudes and achievements in mathematics. It is very important for teachers to bare this piece of information in mind and act accordingly. It is considered that positive relation towards mathematics is transmitted by choosing appropriate and innovative methods of teaching and the increase of intrinsic motivation of students. The teachers should be up to date with the contemporary and realistic world, the development of technology, they should follow the students' affinities and according to that create the school lessons. In the past two years, real-life escape room games have achieved a great level of popularity in Croatia. Escape Room games require from participants (in groups of 2 to 5) team work, tolerance towards others, acceptance of diversities for solving logical tasks and problematical situations, while applying wide spectre of knowledge and common culture all under the time pressure. In this paper we investigate MathEscape as relevant method to revise or systematize mathematical (Quadratic equation) content as a method that could affect pupils' attitudes towards mathematics. MathEscape combines technology and teachers' involvement and creativity in crating interesting and real on mathematical content. The research contributes to better understanding of using technology and innovative methods to enhance positive attitude towards mathematics. We also want to encourage teachers to integrate technology and hands on, interesting activities into their teaching since it would help their pupils increase intrinsic motivation. We demonstrate practical example of using Escape Room games as basis for such type of lessons and activities in order to give teachers some starting point for some more ideas on how to make mathematical content more appealing in order to create positive attitudes.

Keywords: attitude, teaching methods, escape room games, math lesson, applying knowledge

1. Introduction

The first research of attitudes towards mathematics date back to the 70s of 20th century, and numerous results have shown great, if not even the greatest influence of affective factors (attitudes, beliefs, emotions) on school achievements in mathematics. Tapia (1996) considers motivation as the key factor of attitude, while Klostermann (1996) states that attitudes are related to the motivational process, i.e. with the desire to learn mathematics. Moreover, Ma (1997) has shown the proportional correlation between positive attitudes and achievements in mathematics. That means, that students with equal abilities which have more positive attitudes towards mathematics also have better achievements (meaning of school success) in relation to class colleagues which have negative attitudes. The fact that numerous research show that students in general have neutral or positive attitude towards Math in the beginning of school which transforms into negative attitude as the years go by, is very important and interesting. Mohamed and Waheed (2011) assume that this happens during the transition from the Primary school into High school education, when mathematical subject becomes abstract. However, it is still not determined when that change occurs and what causes it. The complex factors, which influence mathematical success, are defined by Singh, Granville and Dika (2002), explaining it as a function of several variables related to students, school and parents emphasizing the important role of a teacher, as a motivator. Among numerous factors by which teachers influence on the creation of attitudes and beliefs about mathematics, and finally achievements, for Singh, Granville and Dika (2002) the greatest influence has teacher's attitudes and beliefs about mathematics, teacher's knowledge, teaching methods and social skills, managing the class, forming the class atmosphere, cognitive modelling, teacher's anticipation, teacher's style and his personal characteristics. It is considered that positive relation towards mathematics is transmitted by choosing appropriate and innovative methods of teaching and the increase of intrinsic motivation of students. The teachers should be up to date with the contemporary and realistic world, the development of technology; they should follow the students' affinities and according to that create school lessons. In the past two years, real-life escape room games have achieved a great level of popularity in Croatia. Escape Room games require from participants (in groups of 2 to 5) team work, tolerance towards others, acceptance of diversities for solving logical tasks and problematical situations, while applying a wide spectre of knowledge and common culture all under time pressure. Exactly this type of game is very effective as a method in math class with the aim of popularising mathematics and forming positive attitudes towards mathematics. The adjustment of mathematical subject tasks is very simple, almost for every lesson, and it is best prepared for the revision of the matter. Except for that, it can be used as the lesson of integration of various math contents, correlation with other subjects and as an inter-subjective class. In this paper, the process of planning, realisation and the implementation of one activity to the school lesson on the example of Quadratic equation, intended for 2nd year of high school, according to the program of grammar school, will be explained in detail. Students who attend the 4^{th} year of Department of mathematics took part in research as sample group. Also, the results of the research and success of such lesson with math students will be stated as well as any impact on attitudes towards mathematics.

1.1. Attitudes

By researching the attitudes towards mathematics, research agreed that these are important factors that appear in the analysis of variation in students' accomplishments in mathematics. Various authors have interpreted the attitudes towards mathematics as an emotional disposition. Neale (1969), however, defines attitude towards mathematics as an aggregated measure of "a liking or disliking of mathematics, a tendency to engage in or avoid mathematical activities, a belief that one is good or bad at mathematics, and a belief that mathematics is useful or useless" (p. 632). Zan and Martino (2007) gave a more concise definition. They believed that the attitude towards mathematics is rather a positive or negative emotional state of mind.

Tapia (1996) considers motivation as the key factor of attitude, while Klostermann (1996) states that attitudes are related to the motivational process, i.e. with the desire to learn mathematics. They are connected and therefore have an impact on the overall achievement in mathematics. Besides that, Ma (1997) shows a reciprocal connection of positive and negative attitudes towards the achievement in mathematics.

That means that between students with equal capabilities better results in school show students that have a more positive attitude towards mathematics.

They also show better results in comparison to themselves. An individual that is more motivated, and has a more positive attitude, will try and work more, and achieve better results. Also a student with lower achievement in mathematics will have a more negative attitude toward mathematics and vice versa.

Among the most noticeable instruments for measuring and defining student's attitudes towards mathematics, was the Fennema-Sherman scale (1976), which was too comprehensive. Today, simpler versions are used, and the following aspects are measured: real-life usefulness, pleasure/well-being and relevance/importance.

Some examples of the statements that students had to agree upon on a scale-level are: "Mathematics is an important subject" and "Things learned in mathematics are rarely useable in everyday life" (Vlahović-Štetić, Rovan & Arambašić, 2005).

Numerous research showed an interesting fact. Students in the beginning of school have neutral or positive attitude towards mathematics, which becomes more negative later on.

Mohamed and Waheed (2011) assume that this happens between elementary and high school, when mathematics becomes more abstract. But, it is still not proved when this phenomenon starts and what causes it. This is a matter that has yet to be researched, and the findings of it could help shape more positive attitudes towards mathematics.

Factors that influence the development and process of shaping attitudes considering motivation can be divided into two groups: internal and external factors.

Internal factors (intrinsic motivation) are factors that involve personal goals, while external factors (extrinsic motivation) occur through the influence of someone else like parents, school and friends.

1.2. Motivation

Besides Tapia and Kloosterman, Becker (2005) concluded that a certain number of students cannot fully accomplish their potential because of the lack of motivation caused by negative beliefs and loss of desire for learning. Although all people have inborn qualities, their achievement also depends on other (motivational) factors.

Motivation incites, guides and defines the intensity and duration of an activity. According to the theory of self-determination, motivation can be put into the following categories: demotivation, extrinsic and intrinsic motivation (Deci & Ryan, 1985).

Extrinsic motivation involves external factors that lead into certain activity involvement. Factors that influence extrinsic motivation are as already said: school, parents, society. Demotivation is connected to a bad attitude towards yourself and with the feeling of incompetence for solving mathematical problems, which is reflected by negative attitudes and beliefs (Ryan & Deci, 2000). It is also reflected in self-underestimated statements about mathematical capabilities.

Intrinsic motivation consists of attitudes and beliefs about usefulness and well-being.

The analysis of Lambić and Lipovski's (2011) research "Measurement of the influence of student's attitudes on the process of acquiring mathematical knowledge", concludes that attitudes towards well-being and pleasure have more influence on motivation than the importance of mathematical usefulness.

Concerning these thoughts about getting negative attitudes towards mathematics in High school by introducing abstractness, Mohamed and Waheed (2011) said that the lack of intrinsic motivation can be related with low appliance of what has been learned, and the absence of interesting topics. Therefore, it is important to introduce teachers with the end results and see what can be done to improve the interest in mathematics. That is important because students retain information longer if they find something more interesting and useful.

1.3. Teachers' role

The complexity of factors affecting success in mathematics was defined by Singh, Granville and Dika (2002), explaining it as functions of several variables in relation to pupils, school and parents stressing the important role of teacher. Among numerous manners through which teachers influence moulding of pupils' attitudes and beliefs on mathematics and finally their success what stands out are teachers' attitudes, knowledge, teaching methods and organization methods, class management, shaping class atmosphere, cognitive moulding, teachers' expectation and teachers' personality.

Various authors claim that teachers who have positive attitudes towards mathematics transmits those attitudes to their pupils who then develop same positive attitudes. (Sherman & Christian, 1999).

It has been considered that positive attitude is transmitted throughout selection of apt teaching methods and pupils' motivation (Brophy, 1999). Highlights cognitive moulding of tasks in other words providing pupils with an idea of solution. Research have confirmed that method as a manner of transferring different approaches, techniques and strategies used for task solving, which influence attitudes and beliefs. (Lester, Garofalo & Kroll, 1989).

Furthermore, Borphy finds cognitive moulding is a way through emotions and attitudes are transferred in relation to their solving (enthusiasm, usefulness, and conviction that success depends on implemented effort).

Likewise, many authors state teacher – pupil relation to have a great impact in process of emerging negative attitudes, as well as teaching style and the manner in which pupils perceive teachers. Pupils who perceive their teachers in positive manner, they are not afraid of them (authoritarian style) because teacher managed to create positive atmosphere, cooperative and active studying (democratic style), are judged as supportive, approachable, involved, dependable and with a positive attitude towards mathematics. Creating enjoyable classroom surroundings, it is possible to increase intrinsic motivation and pupils' perception on their own competences of task solving. (Deci, & Ryan, 1985).

2. Empirical part

2.1. Problem definition

In the empirical part of the study conducted with future teachers of mathematics, the aim was to explore their attitudes towards teachers' role in motivating pupils for mathematics and influence their position on the subject, implementing technology in mathematics classroom and Escape Room games as innovative method of systematization of knowledge. For this purpose, researchers developed "MathEscape" Escape Room game, specialized for revising Quadratic equations, content taught in second year of High school in Croatia, and often considered as not very interesting content to learn or to tech.

Hypothesis and research questions are set as follows:

RQ1: What is the opinion of future mathematics teachers on a difference making to the pupils' motivation and attitudes towards mathematics?

RQ2: What is the opinion of future mathematics teachers on a technology enhancing pupils interest for mathematics? Does MathEscape introduce technology accordingly?

RQ3: Are Escape Room games adjustable for systematization of mathematical content of quadratic equations?

RQ4: Are Escape Room games providing amusement?

RQ5: Do Escape Room games prepare pupils for problems and situations they are going to deal with in future?

H1: There is a correlation between systematization mathematical content through Escape Room games and positive attitude towards mathematics.

2.2. Sample description

This study was conducted at University of J. J. Strossmayer in Osijek, Department of Mathematics. The sample consisted of 24 students of 4^{th} year, majoring in Education of mathematics and Computer science, who went through MathEscape, Escape Room game designed especially for revising quadratic equation content.

Table 1. Sample description.

Feature	Category	Frequency	Percent
Gender	Male	6	25,0
	Female	18	75,0
Age group	$20-25$	24	100,0
Teaching preference	Mathematics	22	91,7
	Computer science	1	4,2
	Either	1	4,2
Escape Room experience	None	17	70,8
	Virtual	5	20,8
	Once	1	4,2
	More than once	1	4,2

3. Data gathering and methodology

3.1. MathEscape

Real life escape rooms games were created as a simulation of computer games based on solving problems within a certain theme (e.g. escaping an elevator, deactivating a bomb, searching for Tesla's invention etc.) by searching for clues, solving and connecting riddles, solving a series of logic-based tasks and deciphering codes within a given time frame.

MathEscape is an Escape room game that, aside for standard riddles and tasks, also contains tasks connected to mathematical content of a certain teaching unit.

A version of a lesson meant for systematization of knowledge on quadratic equation intended for second grade students of Croatian Grammar school program has been conducted in this research.

In the beginning of each escape room game, the game master (a teacher) explains the rules, gives detailed instructions of the game and tells the students a theme and a motivation story of the game. For the implemented MathEscape games the storyline is the following:

"You are a team of mathematicians working on a curriculum reform for the Ministry of education. Rumor has it that the minister came up with a solution for a 150-year-old mathematical problem, a so called Riemann's hypothesis that has a million-dollar reward for the person who solves it. You, as a mathematician, are very intrigued and have decided to find this solution in the minister's office while he is gone. You approach the building of the Ministry and want to use the elevator as the quickest way to the office, but unfortunately, the elevator gets stuck on one of the floors. Your first step is to unjam the elevator, reach the minister's office and find the solution to the hypothesis. However, this won't be an easy test because the minister did a great job locking the solution in his office and protecting the locks with different mathematical tasks. Also, since the minister is coming back in 60 minutes, you will have to be very fast and get out of the office by then."

In the MathEscape game with a topic of quadratic equation there are 19 mathematical and 9 non-mathematical tasks. Correlation with different teaching subjects has been established through non-mathematical tasks e.g. Chemistry (recognition of chemical element and the use of the periodic table of elements), Music (listening to Morse code), Croatian language (discourse comprehension), Arts (mixing primary into secondary colors, jigsaw puzzles), Physics (optics), English language (reading and comprehension), Biology (animate and inanimate world), Physical education (space orientation). Also, some other affective, psychomotor and cognitive skills such as team work, tolerance for others and different opinions, creativity, self-criticism, self-initiative, competitive spirit, communication, trust, managing a new situation and many others are developed through playing a MathEscape game.

Mathematical tasks were related around following outcomes:

apply formulas for solving Quadratic equations, recognize and categorize solutions of equation, evaluate whether solutions are real or complex using discriminant, solving equations which are reduced to quadratic (irrational, equations of third degree), solving system of linear and quadratic equations, solving linear equation, solving special types of quadratic equations, word problems, demonstrate Viete formulas, apply absolute value, rearrange of biquadratic equations etc.

Tasks used to question outcomes mentioned above:

1. Determine coefficient b, c if:

$$\begin{aligned} a &= 5 \\ x_1 + x_2 &= 0 \\ x_1 \cdot x_2 &= 7 \end{aligned} \tag{1}$$

2. Determine $x_1 \geq 0$ if:

$$\begin{aligned} 2 \cdot a &= 2 \\ \frac{-4}{105} b &= 8 \\ \frac{-c}{144} &= 9 \end{aligned} \tag{2}$$

3. Solve: $x^2 - 25x = 0$ (3)
4. Many little lively monkeys,
   have finished eating and play.
   Of them, squared part eighth,
   On the meadow had fun.

   And the rest twelve monkeys on the end of the meadow had played,
   started to jump really high. . .
   How many little monkeys were in that crowd,
   You tell me. . .
   Bhaskara (4)
5. $\sqrt{x+2} = x$ (5)
6. $\sqrt{x-2} = 4 - x$ (6)
7. $1 - x = \sqrt{x+1}$ (7)
8. $\sqrt[3]{x} - 3\sqrt[6]{x} + 2 = 0$ (8)
9. Determine $x_1 + y_1$, $|y_1 - y_2|$ if it is given that:

$$\begin{cases} x + y = 10 \\ x^2 = 4xy \end{cases} \tag{9}$$

10. Determine truthfulness of the claim:
   a) Equation $x^4 - 2x^2 + 1 = 0$ is biquadratic. (10)
   b) $x = -2$is solution of equation

$$\left(\frac{x-1}{x+3}\right)^2 + 3 = 4\left(\frac{1-x}{x+3}\right). \tag{11}$$

   c) Equation $5x^2 - x - 2 = 0$ has no real solutions. (12)
   d) If $a = 1, x_1 = 1, x_2 = 4$, then equating type is

$$x^2 - 5x + 4 = 0. \tag{13}$$

   e) Sum of solutions of equation $ax^2 + x + 2 = 0$
   has two different real solutions for

$$a < \frac{1}{8}. \tag{14}$$

f) Quadratic equation cannot have complex solutions. (15)
g) It is valid: $x^2 - 2x + 1 = (x - 1)(x + 1)$. (16)
h) Equation $\sqrt{x + 10} = x - 2$ has one solution. (17)
i) Equation $\sqrt{\frac{2}{x}} = 0$ has solution in set of real numbers. (18)

3.2. Data gathering

After the "playing" MathEscape game the students are given some time to discuss and reflect on their recent experience followed by limitless time to complete the survey. A survey was used to collect data to answer research questions and verify hypothesis. There are 24, 4^{th} year students of Education of mathematics and Computer science at the Department of Mathematics and all of them got the invitation to participate, 24 of them decide to participate in MathEscape and 24 of them complete the survey. The survey instrument was developed by the researchers and contains 32 questions. At the front part of the survey there are some practical information and definitions of terms, Escape Room and MathEscape. First part of the survey covers demographic questions regarding gender, age group, question about teaching preferences and previous Escape Room experience. Following, there is part of survey shaped in 26 claims of Likert scale. Students were labelling a grade given to certain statement, where 1 stands for "I strongly disagree" and 5 stands for "I strongly agree". Those statements comprise students' perspective on quadratic equation as interesting content as itself, teachers' influence on pupils' attitudes towards mathematics, technology as factor to enhance positive attitude towards mathematics, MathEscape as relevant method to revise or systematize mathematical content and to influence pupils' interest for mathematics. In the third part of the survey, student gave their opinions on value of MathEscape as a method in teaching mathematics. Data gathered, was analysed by SPSS statistics program.

4. Results and interpretation

In this study we were curious to know whether future mathematics' teachers believe that teachers have influence on pupils' motivation and attitudes towards mathematics. 4 (16.7 %) students neither agrees or disagrees with the statement proposed, 9 (37.5 %) students agree and 11 (45.8 %) completely agree with the statement proposed.

It is found, that over 70 % of participants finds that using technology in mathematical lessons would increase pupils interest for the subject.

According to results presented, almost 80 % of participants find MathEscape appropriate to revise exercise and content related to the content tested.

It has been mentioned earlier that introducing technology could be of great importance for positive attitude and developing interest for mathematics in pupils. Also results of the study, shows that future mathematics' teachers agree upon that statement. It is clear from the results, that those participants more than 90 % of them agree or completely agree that technology introduced in MathEscape is suitable as well.

Table 2. Influence of teachers, technology and MathEscape on pupils' attitudes towards mathematics.

Statements	Strongly agree	Agree	Neither	Disagree	Strongly disagree
Mathematics teachers can influence pupils' motivation and attitudes towards mathematics.	11 (46 %)	9 (37 %)	4 (17 %)	0 (0 %)	0 (0 %)
Introducing technology in mathematics lessons would increase pupils' interest for the subject.	8 (33 %)	9 (38 %)	5 (21 %)	2 (8 %)	0 (0 %)
MathEscape game is appropriate for revising (systematization) of Quadratic equation content.	9 (38 %)	10 (42 %)	4 (18 %)	0(0 %)	0(0 %)
Technology in MathEscape is introduced suitably.	16 (67 %)	6 (25 %)	1 (4 %)	1 (4 %)	0 (0 %)
"Playing" MathEscape was interesting.	19 (79 %)	3 (13 %)	1 (4 %)	1 (4 %)	0 (0 %)
I consider method presented as method that prepares pupils for problems and situations with which they are going to encounter in future.	7 (29 %)	10 (42 %)	6 (25 %)	0 (0 %)	0 (0 %)
Revising and systematizing mathematical content through Room Escape method would influence pupils' attitudes towards mathematics.	14 (59 %)	6 (25 %)	2 (8 %)	2 (8 %)	0 (0 %)

According to literature reviewed and previous research it is important for pupils to feel comfortable and judge the content as interesting in order to have positive perspective of a subject. More than 90 % of participants consider MathEscape to be interesting.

Table 3 presents cross tabulation of two questions in survey one regarding Quadratic equation as an interesting mathematical content and the other concerning MathEscape as interesting method of revising that content. 17 of examined 24 participants would not agree with the statement that Quadratic equation is an interesting mathematical content as it is, but 15 of those 17 finds that revising Quadratic equation through MathEscape game would make Quadratic equation lessons more interesting. Moreover, all of the 7 participants who find Quadratic equation to be an interesting mathematical content as it is also find that revising Quadratic equation through MathEscape game would make Quadratic equation lessons more interesting.

It is very important that lessons provided prepare our pupils for the future challenges and problems they are to encounter. Also it is important for pupils to see and understand the purpose of mathematics' lessons in order to develop intrinsic motivation and genuine interest for the matter. As we can read from the results presented, over 80 % of participants agree or completely agree that method presented fulfils function mentioned above.

Table 3. Cross tabel – students' attitudes on quadratic equation and revising quadratic equation through MathEscape.

		Quadratic equation is an interesting mathematical content as it is.					**Total**
		I do not agree	I agree to some extant	I do not agree nor disagree	I agree	I agree completely	
Revising Quadratic equation through MathEscape game would make Quadratic equation lessons more interesting.	I do not agree nor disagree	0	0	2	0	0	2
	I agree	0	1	5	4	0	10
	I agree completely	3	2	4	1	2	12
Total		3	3	11	5	2	24

Finally, results also shows that future mathematics' teachers in percentage over than 80 % find that revising and systematizing mathematical content through Room Escape method would influence pupils' attitudes towards mathematics.

5. Conclusion

The matter of popularizing mathematics as school subject has been discussed since 70s of 20th century. Some research has shown that pupils have neutral emotions towards mathematics transitioning form elementary school level of education to high school level. However, during the high school those emotions changes into negative ones. It is assumed that emersion of more abstract content influence on pupils' attitudes and emotions towards mathematics. Factors that influence the development and process of shaping attitudes considering motivation can be divided into two groups: internal and external factors.

Lack of intrinsic motivation can be related with low appliance of what has been learned, and the absence of interesting topics. Therefore, it is important to introduce teachers with the end results and see what can be done to improve the interest in mathematics. That is important because students retain information longer if they find something more interesting and useful.

(Lambić & Lipkovski, 2011) concludes that attitudes towards well-being and pleasure have more influence on motivation than the importance of mathematical usefulness. It has been considered that positive attitude is transmitted throughout selection of apt teaching methods and pupils' motivation. Creating enjoyable classroom surroundings, it is possible to increase intrinsic motivation and pupils' perception on their own competences of task solving. (Deci, & Ryan, 1985).

MathEscape is an Escape room game that, aside for standard riddles and tasks, also contains tasks connected to mathematical content of a certain teaching unit.

A version of a lesson meant for systematization of knowledge on quadratic equation intended for second grade students of Croatian Grammar school program has been conducted in this research. 4^{th} year students of Department of Mathematics majoring in Education of mathematics and Computer science participated in the study. All 24 students "played" MathEscape in teams of four or five. According to survey conducted after the game, participants find suitable to systematize content of Quadratic equations. Data collected shows that MathEscape is amusing, fun and enjoyable to be involved in. Students share opinion that Quadratic equation is not among the most interesting mathematical content, however they find MathEscape could be the method to make Quadratic equation revision more than interesting lesson, and any other mathematical content revision lesson for that matter. According to survey results there is a correlation between systematization mathematical content through Escape Room games and positive attitude towards mathematics.

References

[1] ARAMBAŠIĆ, L., VLAHOVIĆ-ŠTETIĆ, V., & SEVERINAC, A. (2005), *Je li matematika bauk? Stavovi, uvjerenja i strah od matematike kod gimnazijalaca*, Društvena istraživanja, 14(6), 1081–1102.

[2] BECKER, M. (2005), *Motivacija za učenje*, Pedagoško društvo Srbije, Beograd, BEL.

[3] BROPHY, J. (1999), *Toward a model of the value aspects of motivation in education: Developing appreciation for particular learning domains and activities*, Educational Psychologist, 34(2), 75–85.

[4] DECI, E. L., RYAN, R. M. (1985), *Intrinsic motivation and self-determination in human behaviour*, New York, NY: Plenum Press.

[5] KLOOSTERMAN, P. (1996), *Students' Beliefs About Knowing and Learning Mathematics: Implications for Motivation*, In M. Carr (Ed.), Motivation in Mathematics (pp. 131–156), Cresskill: Hampton Press, Inc.

[6] LAMBIĆ, D., & LIPKOVSKI, A. (2011), *Measuring the influence of student's attitudes on the process of acquiring knowledge in mathematics*, Croatian Journal of Education, 14(1), 187–205.

[7] LESTER, F., GAROFALO, J., & KROLL, D. (1989), *The role of metacognition in mathematical problem solving: A study of two grade seven classes*, Final report to the National Science Foundation of NSF Project, MDR 85–50346

[8] LINDBERG, S. M., HYDE, J. S., PETERSEN, J. L., & LINN, M. C. (2010), *New trends in gender and mathematics performance: a meta-analysis*, Psychological Bulletin, 136(16), 1123–1135.

[9] MA, X. (1997), *Reciprocal relationships between attitude toward mathematics and achievement in mathematics*, The Journal of Educational Research, 90(4), 221–229.

[10] MOHAMED, L., & WAHEED, H. (2011), *Secondary students' attitude towards mathematics in a selected school of Maldives*, International Journal of Humanities and Social Science, 1(15), 277–281.

[11] NEALE, D. C. (1969), *The role of attitudes in learning mathematics*, Arithmetic Teacher, 16(8), 631–640.

[12] RYAN, R. M., & DECI, E. L. (2000), *Self-determination theory and the facilitation of intrinsic motivation, social development, and well-being*, American Psychologist, 55(1), 68–78.

[13] SHERMAN, H. J. & CHRISTIAN, M. (1999), *Mathematics attitudes and global self-concept: An investigation of the relationship*, College Student Journal, 33(1), 95–102.

[14] SINGH, K., GRANVILLE, M., & DIKA, S. (2002), *Mathematics and science achievement: effects of motivation, interest, and academic engagement*, Journal of Educational Research, 95(6), 323–332.

[15] TAPIA, M. (1996), *The Attitudes Toward Mathematics Instrument*, Paper presented at the annual meeting of the Mid-south Educational Research Association, Tuscaloosa, AL.

[16] VLAHOVIĆ-ŠTETIĆ, V., ROVAN, D. & ARAMBAŠIĆ, L. (2005.), *Ljestvica za ispitivanje stavova i uvjerenja o matematici*, Unpublished instrument. Odsjek za psihologiju Filozofskog fakulteta u Zagrebu. Zagreb

[17] ZAN, R., & MARTINO, P. D. (2007), *Attitudes towards mathematics: Overcoming positive/negative dichotomy*, The Montana Mathematics Enthusiasts Monograph, 3, 157–168.

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Poticanje razvoja pozitivnih stavova prema matematici kroz primjenu Escape Room igara

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Sažetak. Istražujući stavove prema matematici, istraživači su se složili kako su stavovi relevantan faktor koji utječe na ostvarenje matematičkih postignuća. Različiti autori, stavove prema matematici smatraju emocionalnom dispozicijom. Ma (1997) ukazuje na proporcionalnu korelaciju pozitivnih stavova i matematičkih postignuća. Vrlo je važno da učitelji te informacije osvijeste i nastavu prilagode tim saznanjima. Smatra se, kako se pozitivan odnos prema matematici prenosi odabirom primjerenih i inovativnih metoda poučavanja kao i podupiranjem razvoja intrinzične motivacije učenika. Strogo se savjetuje da učitelji pri stvaranju plana i programa svoje nastave budu upoznati i prate razvoj suvremene tehnologije, i svakodnevnice te imaju na umu afinitete učenika. U posljednje dvije godine, real-life escape room igre dosegle su veliku razinu popularnosti u Hrvatskoj. Escape Room igre podrazumijevaju timski rad dionika (u grupama od 2 do 5), međusobnu toleranciju i prihvaćanje različitosti kako bi riješili logičke zadatke s kojima su suočeni u različitim problemskim situacijama. Od dionika se također traži primjena širokog spektra znanja opće kulture i sve to pod pritiskom vremenskog roka. U ovom istraživanju, ispitali smo MathEscape kao relevantnu metodu vježbanja i sistematizacije matematičkog (Kvadratne jednadžbe) sadržaja koja bi mogla utjecati na razvoj pozitivnih stavova prema matematici. MathEscape kombinira tehnologiju, učiteljev doprinos i kreativnost pri stvaranju zanimljivog i realističnog matematičkog sadržaja. Istraživanje doprinosi dubljem razumijevanju utjecaja, uporabe tehnologije i inovativnih metoda, na poticanje pozitivnih stavova prema matematici. Također, cilj nam je potaknuti učitelje na uključivanje tehnologije te zanimljivih fizičkih aktivnosti u poučavanje matematike kako bi pomogli svojim učenicima razviti intrinzičnu motivaciju za učenje matematike. Predstavili smo i praktičan primjer Escape Room igre kao podlogu razvoja takvog tipa nastave i nastavnih aktivnosti. Ponuđeni primjer sam je početna točka učiteljima za razvoj vlastitih ideja kojima će matematički sadržaj učiniti privlačnijim kako bi stvorili pozitivne stavove kod svojih učenika.

Ključne riječi: stavovi, metode poučavanja, escape room igre, nastava matematike, primjena znanja

The presence of mathematical games in primary school

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Abstract. One of the most important goals of teaching mathematics that needs to be taken into account is to stimulate interest in and love for mathematics. A number of studies have confirmed that the introduction of different types of educational games in mathematics education increases student interest in mathematics and has a positive impact on student achievement in different fields of mathematics. Through well-designed games that are implemented correctly in mathematics classes students acquire knowledge in a new and more interesting way and develop a positive attitude towards mathematics, which represents the greatest value and one of the basic goals of teaching mathematics. The aim of this paper is to investigate the frequency of using mathematical games in mathematics revision classes in the lower grades of primary school (the first cycle) and to learn about the types of games teachers use when teaching mathematics, including computer games in mathematics classes.

Keywords: mathematical game, computer game, motivation, mathematics teaching, mathematics education

1. Introduction

Mathematics is a game played according to certain simple rules with meaningless marks on paper.

David Hilbert

One of the important factors that any teacher must take into account is student interest in the subject. Increasing interest in learning mathematics can be achieved in various ways. One of the ways is to integrate educational games into teaching mathematics. Contemporary educational theory emphasizes the need for engaging students in learning activities and considers the integration of games into the

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mathematics classroom to be an inevitable activity at the present day. A large body of research deals with multiple benefits this form of activity employed in teaching mathematics results in. Teaching mathematics, especially in the early years of schooling, can be successful only if students find it interesting, inspirational, stimulating, and adapted to their mental age. Most students in their lower grades of primary school fail to keep their attention for a long time and sometimes they can't even sit still. They need action, and this action implies frequent change of activities and the introduction of new methods, forms of teaching, as well as different teaching resources and aids. Therefore the introduction of educational games in mathematics classes represents an interesting activity to students.

Playing games has long been a natural human activity in leisure time. The presence of games dates back thousands of years. Many ancient objects have been found that were used while playing games. The oldest games were probably racing games and strength games. The rules of these games have not been preserved. Much evidence supports the existence of dice games. To the extent that playing games is the natural instinct of all people, game analysis is the natural instinct of mathematicians. Who should win? What is the best move? What are the odds for a particular random event? How long does a game last? When we solve a riddle, are there any standard techniques that help us find a solution? These are the natural issues of mathematical interest (Beasley, 1990).

2. Literature review

As an inevitable part of class, games play an important role, especially in the lower grades of primary school.

Student motivation increases, they easily accept activities and thus change their attitudes toward mathematics. When participating in mathematical games, students talk about moves, discuss correctness of answers and use different strategies. In this way, mathematical games get students to debate both with each other and with teachers (Ernest, 1986). Pasek, Golinkoff and Eyer (2003) point out that much of what we teach has little impact on our students' real lives. The role of a teacher is to put learning into context. It makes learning fun and releases natural curiosity and creativity among students. The best and the only way to do this is through play. The aforementioned authors believe that learning in context is the best way of learning and that the game is the best teacher. They emphasize the equality PLAYING = LEARNING.

Oldfield (1991) gives the definition of a mathematical game as an activity which:

— involves a challenge, usually against one or more opponents,
— is governed by a set of rules and has a clear underlying structure,
— normally has a distinct finishing point,
— has specific mathematical cognitive objectives.

Through the application of various games in teaching mathematics, students learn about themselves, their capabilities, their abilities, they develop tolerance, self-confidence, imagination, creativity, increase attention, gain positive experience, become more responsible, develop their thinking, cooperate with each other, get acquainted, relax, and reduce test anxiety. Games are a natural way to learn. Games help children learn to concentrate, exercise visualization, try out ideas, practice adult behavior, and develop control over their own world Marzollo and Lloyd (as cited in Sharma, 2001).

In his paper, Vankúš (2008) analyzes the following four main parts of educational games:

— milieu of the game,

— goals of the game,

— teacher and student activities, which are determined by the rules of the game,

— final evaluation.

He says that the interaction between pupils and milieu of an educational game should motivate students to work. This work actually leads to the realization of educational goals of the game. These goals determine the form of the game. Educational games have a value only if they enable us to reach these educational goals. The activities of teachers and students in the game are determined by the rules, and they should be attractive and motivating for students. Activities should be tailored to students' ages, abilities and interests. The rules of the game determine the form and organization of student work in the game. Final evaluation verifies whether the goals of the game have been achieved and students should be rewarded and motivated for other activities.

Using educational games in the education of students has a long tradition. Pulos and Sneider (1994, as cited in Vankúš, 2008) found out that when educational games are properly selected, they help students learn new mathematical concepts and adopt skills. In their paper Krejčová and Volfová (1994, as cited in Vankuš, 2008) highlighted the great value of the game as a vital part of education. The authors concluded that inclusion of educational games in teaching increases the overall student interest in mathematics and thus promotes their active participation. Olson (2007) points out that good games are interesting and create opportunities for students to explore mathematical ideas. The introduction of games encourages students to explore important mathematical concepts. In his opinion, a framework that consists of the three Ps (i.e., plan, play, and please be patient) provides a guideline on how we should consider a game's potential for exploring mathematical ideas with students and introducing rich classroom discussion.

The increasing use of information and communication technologies in everyday life and school implies the introduction of computer games. In general students learning mathematics with the use of computer technology, compared to those without computer technology, had higher mathematics achievement (Li and Ma, 2011). Shih and Hsu (2016) argue that the use of computer games in learning can motivate students more than just traditional learning. Van den Heuvel-Panhuizen, Kolovou and Robitzsch (2013) conducted research on the impact of a dynamic online game on developing the ability of students in primary grades to solve algebraic problems and found out that the online game contributed to the student's early algebraic performance.

An important feature of introducing mathematical games in the process of teaching mathematics was given by Holton, Ahmed, Williams and Hill (2010): "One of the other advantages of mathematical play situations is that students can take part at their own level and build on their individual knowledge and understanding. It also enables students to make errors in a supportive environment. It would seem that to achieve a high level of understanding it is as valuable to know that certain things will not work and why they will not work, as it is to know positive results. This certainly appears to be the way we construct our own internal map of a new city. By taking wrong turns and seeing where we end up we achieve a better concept of the layout of the city. In this way, through play and exploration over a larger area than is actually required to solve a particular problem, we provide the foundation for further learning." Juričić Devčić (2011) describes the use of educational games in the teaching of mathematics and states that games can be used to revise mathematical contents, train and drill mathematical actions. Creativity and innovativeness of the teacher stand out when designing and implementing games, as emphasized by Basta and Mesić (2011), who give an overview of educational games that are most commonly used in teaching mathematics and point to class teachers' teaching attitudes towards the use of games in the process of teaching mathematics.

So, even though the introduction of games is not a new idea in the educational process, it still causes great attention. Given all the benefits brought by the introduction of games in mathematics classes, which are the topics of many studies and research, the goal of this paper is to determine the extent to which games are represented in mathematics classes in the lower grades of primary school.

3. Research questions and methodology

In this research, we focused on the representation of educational games in the teaching of mathematics, and the types of games that teachers use, including computer games. Specifically, we tried to answer the following questions:

1) How often do primary school class teachers apply educational games in the teaching of mathematics, especially for the purpose of revision?
2) Which games are most frequently used in mathematics classes?
3) Are there any differences in the use of games with regard to the teacher's professional qualifications/degree?
4) Do teachers use computer games in mathematics classes and which computer games do they use?

Data was collected by an anonymous questionnaire. The questionnaire contains 21 questions, 7 of which are open-ended, and the remaining 14 are closed-ended questions. The research was conducted in primary schools in the towns of

Vinkovci and Osijek and in suburban schools in 2017. 100 primary school teachers participated in the research. All teachers were female. The sample was random. Figure 1 shows respondents' work experience (in years).

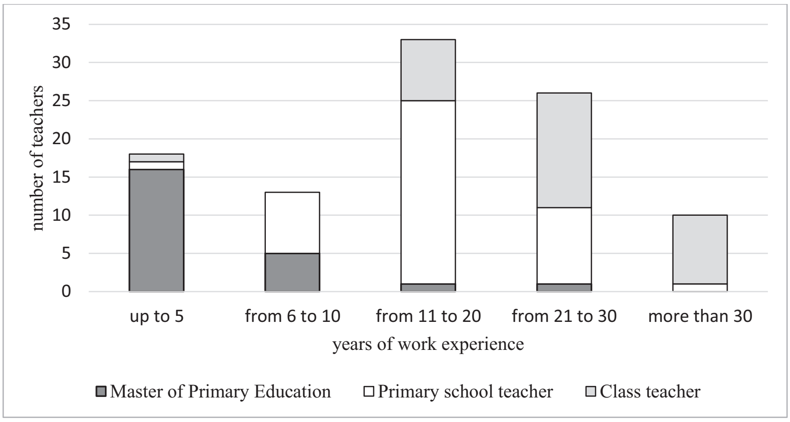

Figure 1. Structure of the sample with regard to respondents' work experience and qualifications.

4. Results

Through the collected data, we came up with the following results that were grouped according to the nature of the questions asked.

a) Professional development

As to their participation in professional meetings, 80 % and 20 % of teachers said that they often and sometimes participate in professional meetings, respectively.

b) The most popular teaching subject

30 % of the total number of the teachers surveyed opted for *Mathematics* as their favorite subject for teaching. 38 % of the teachers surveyed were in favor of the *Croatian Language*, while 27 % selected *Science and Social Studies*. Finally, 5 % of the teachers surveyed opted for other subjects. Additional mathematics classes are held by 74 % of the teachers surveyed.

c) Literature for preparing classes

89 % of the teachers surveyed use additional mathematics literature to prepare their classes. Of all teachers who use additional literature to prepare their Mathematics classes, 79 % use various Internet sources. 28 % and 7 % of them use *Matka* journal and additional class handouts and worksheets, respectively. Alternative textbooks, workbooks and task collections are used by 8 % of teachers. The journal *Mathematics and School* is used by 5 % of respondents.

d) Frequency of using games in mathematics classes

In their mathematics classes, 59 % of teachers use games 1–3 times per one teaching unit, and 34 % of respondents use games 4–6 times.

e) Using mathematical games with respect to part of the lesson

40 % of the teachers surveyed said that most frequently they use games in the introductory part of their mathematics classes. 45 % of teachers use games at the end of their classes. Only 5 % of teachers said that they mostly use games during their classes. 11 % of the teachers surveyed most frequently use games at the beginning and at the end of their mathematics classes. Note that 96 % of respondents stated that mathematical games are more frequently used in arithmetic.

f) Frequency of using games in mathematics revision classes with respect to respondents' degrees and work experience

The data referring to the frequency of using games in mathematics revision classes are shown in Figures 2 and 3. According to Figure 2, 81 % of teachers use games always or often, with the majority represented by Master degree holders in Primary Education and primary school teachers.

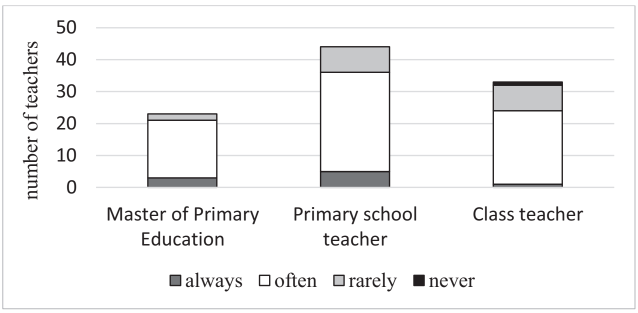

Figure 2. The frequency of using games in mathematics classes with respect to qualifications.

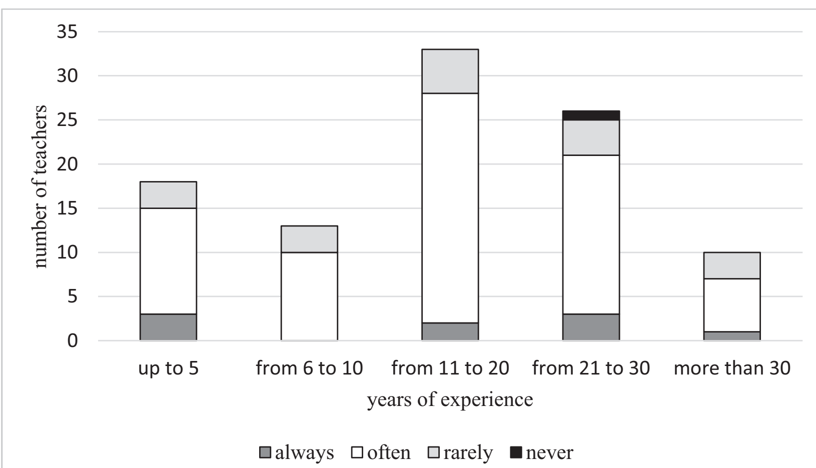

Figure 3. The frequency of using educational games with respect to work experience.

g) Types of games used in mathematics revision classes

Games that are most frequently used by mathematics teachers in their revision classes are *Memory* (37 %), *Napping* (33 %), *Bingo* (32 %), *Dice games* (29 %), and *Quiz* (17 %). Figure 4 gives the distribution of games with respect to respondents' qualifications. The teachers listed a total of 39 different games they use in mathematics revision classes.

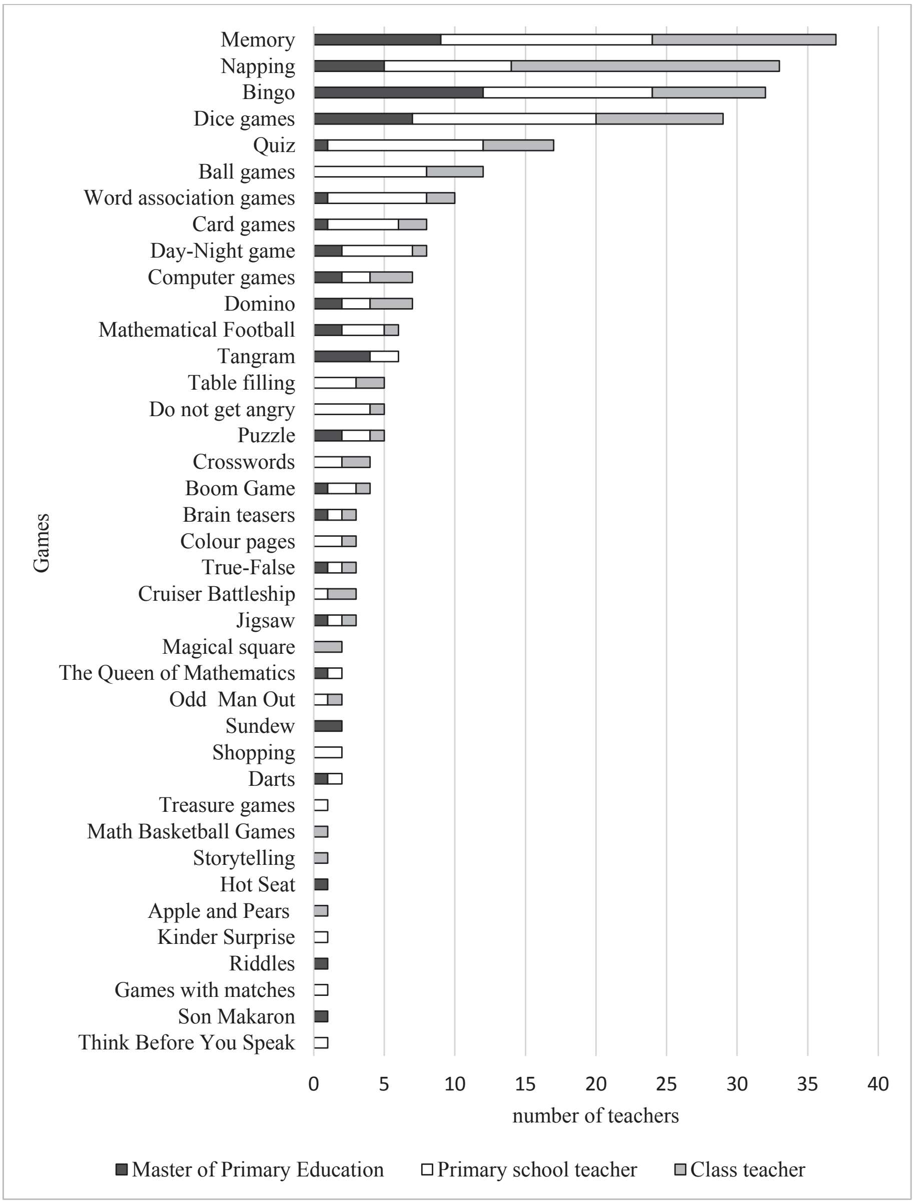

Figure 4. The frequency of using games in mathematics revision classes with respect to respondents' qualifications.

In terms of their efficiency in the educational process, the teachers believe that *Memory* and *Bingo* (14 %) and *Quiz* and *Dice games* (10 %) are especially effective.

Although there is an abundant amount of games that influence the development of geometric thinking, geometry games are not mentioned by a large number of teachers. "Rich and stimulating instruction in geometry can be provided through playful activities with mosaics, such as pattern blocks or design tiles, with puzzles like tangrams." Van-Hiele (1999).

h) Favorite games for students

Figure 5 shows games that students, according to their teachers' opinion, prefer to play in mathematics revision classes. Other games were less frequently chosen. The popularity of games among students is certainly an important factor in choosing the games that teachers use in mathematics classes as these types of games that students are fond of are also the top games used by teachers.

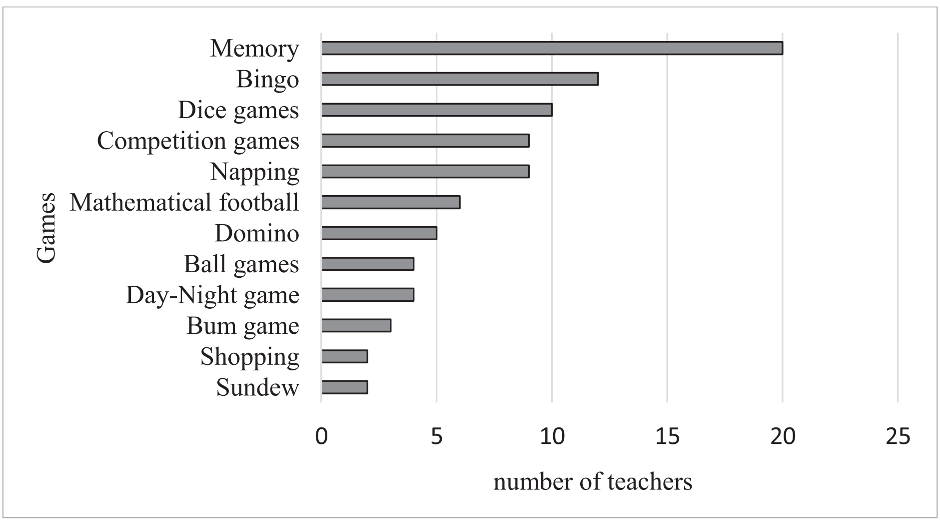

Figure 5. Favorite games for students to play in mathematics revision classes.

i) Representation and types of computer games in mathematics classes

The following results on the use of computer games in the teaching of mathematics were obtained with respect to teachers' qualifications.

1) 78 % of Master degree holders in Primary Education have computer equipment available for teaching, and 61 % of them use computer games in their mathematics classes.

2) 82 % of primary school teachers have computer equipment available for teaching, and 56 % of them use computer games in their mathematics classes.

3) 82 % of class teachers have computer equipment available for teaching, and only 26 % of them use computer games in their mathematics classes.

Figure 6 shows the distribution of teachers using computer games by years of age.

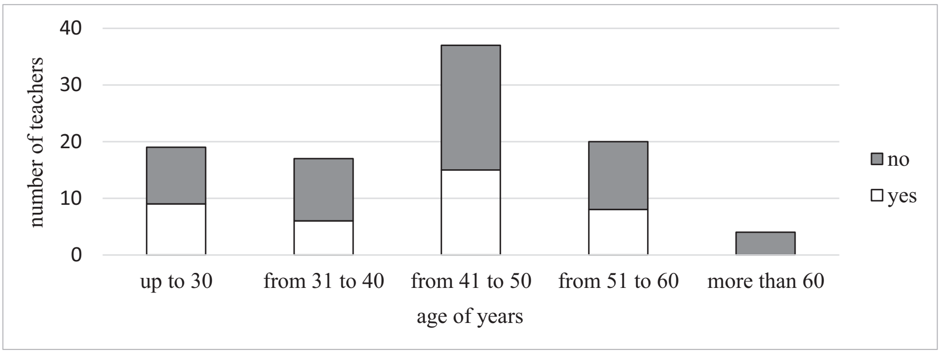

Figure 6. Use of computer games by years of age.

Respondents mentioned that they use the following computer games: *Knowledge quiz, Memory, Word association games, Plickers, The Odd Man Out Game, Zondle, Wheel of Luck, games on "Petica", Crossword Puzzles, Sudoku, Evaluation Games, Gizmo Simulations, games from*: `https://www.matific.com/us/en-us`, `https://wordwall.co.uk/`, `https://learningapps.org`. One Master degree holder in Primary Education mentioned the use of a "Web game that can be independently formatted for any content dealt with in mathematics classes or in any other subject".

j) The effect of games on increasing students' interest in learning mathematics

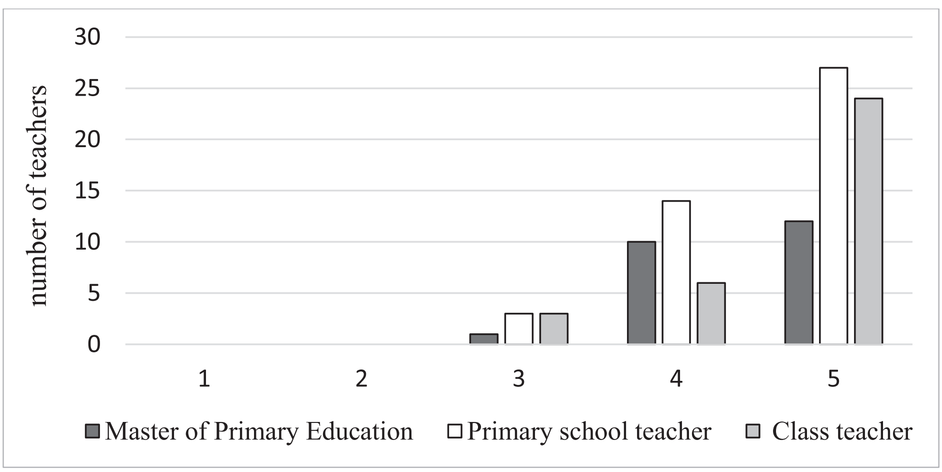

Figure 7. Agreement with the statement: "Mathematics revision by means of playing games increases students' interest in learning mathematics." with respect to respondents' qualifications.

A 5-point Likert scale is used for assessing participants' agreement with the statement: "Mathematics revision by means of playing games increases students' interest in learning mathematics." The respondents marked their level of agreement/disagreement on a five-point Likert type scale 1 (from 1: strongly disagree, through 3: neither agree nor disagree, to 5: strongly agree).

5. Discussion and conclusion

The survey we conducted by means of a questionnaire administered to 100 primary school teachers showed that 59 % of teachers use games one to three times per one unit, whereas 34 % of them use games four or more times. Primary school teachers usually use games at the beginning and at the end of their classes. Mathematics revision games are used by 81 % of teachers, who increase the quality of the teaching of mathematics by integrating educational games into their teaching so frequently. Given the nature of mathematical content, games are much more represented in arithmetic than in geometry. As there is a large number of mathematical games influencing the development of geometric thinking and the transition to a higher Van-Hiel's level of geometric thinking, it would be convenient to familiarize the teachers with the types of games and the good sides of using appropriate games for the development of geometric thinking through various forms of professional training. Primary school teachers and Master degree holders in Primary Education list many games they use and a great number of games that have proved to be particularly effective in the educational process. This result is likely to be a consequence of teacher education, because in recent years special emphasis in modern mathematics teaching is placed on creativity and the use of various educational games in the teaching process.

Computer equipment in classrooms is satisfactory, but teachers rarely use computer games. The results indicate that primary school teachers and Master degree holders in Primary Education who have access to computer equipment use computer games in mathematics classes more frequently than class teachers. The reason for this is probably more advanced computer education of primary school teachers and Master degree holders in Primary Education, which allows for a more contemporary approach to the teaching process and can provide students with more interesting classes. However, the fact that nearly 40 % of Master degree holders in Primary Education do not use computer games in their mathematics classes although IT equipment is available to them, should be important for informatics and mathematics teachers who should make students, i.e. future teachers, better acquainted with the possibilities of using computer games in the classroom. Primary school teachers and class teachers, of whom 44 % and 74 % do not use computer games in their mathematics classes, respectively, although computer equipment is available, should be offered additional training and education aimed at using ICT in teaching. We have highlighted that numerous studies have shown that the use of computer games contributes to increasing the level of adoption of mathematical content and the development of task solving strategies. As today's students spend a large part of their free time playing computer games, including playing various online games, a major contribution would be to introduce students to interesting computer games that affect deeper understanding and drilling of mathematical content. Such an impact would also have a positive effect on increasing students' interest in adopting mathematical contents as well as on better utilization of time spent on computers. As the role of a well-educated and motivated teacher is certainly fundamental in this context, it might be of great benefit to introduce a course within the study program, one of whose goals would be to introduce students to the types of computer games useful for adopting mathematical content and practicing mathematics.

Most teachers argued that mathematics revision classes conducted by playing games increase students' interest in learning mathematics, and this should be the biggest incentive for greater representation of educational games in the teaching of mathematics as well as for the design of new ones.

References

[1] BASTA, S., MESIĆ, D. (2011), *Play-a way towards mathematical competency*, The MathTeacher (editor M. Pavleković), 188–195.

[2] BEASLEY, JOHN D. (1990), *The Mathematics of Games*, Oxford University Press.

[3] ERNEST, P. (1986), *Games. A Rationale for Their Use in the Teaching of Mathematics in School*, Mathematics in School, 15(1) 2–5.

[4] VAN DEN HEUVEL-PANHUIZEN, M. H. A. M., KOLOVOU, A., & ROBITZSCH, A. (2013), *Primary school students' strategies in early algebra problem solving supported by an online game*, Educational Studies in Mathematics, 84(3), 281–307.

[5] HOLTON, D. D., AHMED, A., WILLIAMS, H., & HILL, C. (2010), *On the importance of mathematical play*, International Journal of Mathematical Education in Science and Technology, 32(3), 401–415.

[6] JURIČIĆ DEVČIĆ, M. (2011), *Didactic games in mathematics teaching*, The MathTeacher (editor M. Pavleković), 478–490.

[7] LI, Q., & MA, X. (2010), *A meta-analysis of the effects of computer technology on school students' mathematics learning*, Educational Psychology Review, 22, 215–243.

[8] OLDFIELD, B. (1991), *Games in the learning of mathematics*, Mathematics in Schools, 20(1) 41–43.

[9] OLSON, J. C. (2007), *Developing Students' Mathematical Reasoning through Games*, Teaching Children Mathematics, 13(9), 464–471.

[10] PAPERT, S. (1980), *Mindstorms: Children, computers and powerful ideas*, New York, NY: Basic Books.

[11] PASEK, K. H., GOLINKOFF, R. M., & EYER, D. (2003), *Einstein never used flashcards. How our children really learn – and why they need to play more and memorize less*, Rodale Press.

[12] SHIH, J. L., & HSU, Y. J. (2016), *Advancing Adventure Education Using Digital Motion-Sensing Games*, Journal of Educational Technology & Society, 19(4), 178–189.

[13] C. SHARMA, M. (2001), *Matematika bez suza*, Lekenik, Ostvarenje.

[14] VANKÚŠ, P. (2008), *Games based learning in teaching of mathematics at lower secondary school*, Acta Didactica Universitatis Comenianae Mathematics, 8, 103–120.

[15] VAN HIELE, PIERRE M. (1999), *Developing geometric thinking through activities that begin with play*, Teaching Children Mathematics, 310–316.

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Prisutnost matematičkih igara u osnovnoj školi

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Sažetak. Jedan od najvažnijih ciljeva nastave matematike o kojem je nužno voditi računa je izazivanje interesa i pobuđivanje ljubavi prema matematičkim sadržajima. Nizom istraživanja potvrđeno je da uvođenja različtih vrsta didaktičkih igara na nastavi matematike povećava kod učenika interes prema matematici te ima pozitivan utjecaj na postignuća učenika iz različitih područja matematike. Kroz dobro osmišljene i provedene igru u nastavi matematike učenici uče na nov i zanimljiviji način te formiraju pozitivan stav prema matematici, što predstavlja najveću vrijednost i jedan od osnovnih ciljeva nastave matematike. Cilj ovog rada je istražiti učestalost primjene matematičkih igara pri ponavljanju matematičkih sadržaja u nižim razredima osnovne škole (prvom odgojno obrazovnom ciklusu) te dobiti saznanja o vrstama igara koje učitelji koriste u nastavi matematike uključujući i računalne igre na satu matematike.

Ključne riječi: matematička igra, računalna igra, motivacija, nastava matematike, matematičko obrazovanje

Problem solving in elementary mathematics education

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Abstract. General problem-solving skills are of central importance in mathematics achievement. Problem-solving strategies and cognitive operations should be taught in the classroom consciously. In most cases, the focus is on the problem-solving process, rather than the result. How do we solve problems? There are different models for this. Pólya's model develops reasoning and divides problem solving into four steps (Pólya, 2000).

Strategy keys can be an alternative way to develop students' problem solving capacity by heuristics (Herold-Blasius, Jazby, 2016).

The present paper presents an experiment involving third and fourth grade (9–10 years old) students who have been asked to solve certain mathematical problems. Two experimental groups were involved. The aim was to investigate whether problem solving is more efficient when using the traditional method (working on paper with pencil) or when using keys. We hypothesized that using keys might prove helpful in solving problems.

Keywords: problem-solving skills, unusual problems, heuristics, strategy keys, mathematical model

1. Introduction

The general development of problem solving skills is an important goal in mathematics teaching. The problem solving process involves the development and use of several aspects of the students' cognitive skills; reasoning being one of the most important ones.

With the help of modern imaging techniques (fMRI) O'Boyle from Texas Tech University found that during problem solving the 13 year old S. M., a student gifted in mathematics, used approximately five-six times more areas of the brain than students with average skills (Randall, 2012).

One might raise the question whether problem solving is teachable at all, and if it is how to teach it effectively. We pedagogues believe in teaching problem solving skills, and that we can help students of different ages and different skills through familiarizing them with different strategies.

Problem solving skills can be developed by solving several different problems of the same type. The particular methods for solving word problems are: representation, contrasting, hypotheses, backwards working, rule of three, and the method of balancing.

When students have to find the algorithm, the steps leading to the solution, this is an efficient way to develop their problem solving skills.

Teaching word problems is one of the most difficult methodological issues. Problem-solving strategies and cognitive operations should be taught consciously in the classroom. In most cases, the focus is on the problem-solving process, rather than the result. A repeated emphasis must be laid on the structure of this process (Ambrus, 2002).

Modelling realistic mathematical situations is also important, as well as, familiarizing students with unsolvable problems. In addition, there is a need for problems containing redundant information as considers the question. Teaching these problems in a colourful and varied way is crucial. Bruner already discussed different representations some 60 years ago.Methods should be adapted to age level and consideration given to Bruner's stages of representation (Bruner, 1974).

In the *enactive stage* learning is characterized by concrete, hands-on actions and manipulations. The younger the students, the more concrete actions they need.

The *iconic stage* involves the use of images and other visuals, as well as, imaginary situations.

The effectiveness of the learning process can be increased by consciously varying the modes of representation. The visualization of a problem can facilitate a better understanding. The NCTM 2000 highlighted "Representation" as one of the key activities in mathematical problem solving.

Students have to be made aware of using visual representations.

Goldin & Kaput (1996) argue that the ability to visualize data and their relations in a mathematical problem may contribute to mathematics problem solving. They hold that the interaction between external and internal representations is important, some of the existing relations are interpreted actively and consciously, while others are automatically and passively added to already existing knowledge.

Drawings used for solving word problems can be categorized based on Kozhevnikov et al. (2002) as schematic and pictorial. Seidel (2001) proved the importance of different modalities of representation connected with mathematical giftedness by using modern methods of brain research.

Schematic drawings, modelling, mathematic quantities, and relations play an important role in the cognitive processes involved in mathematical problem solving. Mental representations generated in the process of word problem solving can be introduced as early as elementary education. Students who are made aware of mental representations built on visual imagery perform better and change their convictions related to mathematics (Csíkos et *al*., 2012).

The *symbolic stage* involves using words and mathematical symbols. The right timing for introducing the symbolic stage is very important, since students in the classroom are not on the same level.

Minimal teacher guidance, as a means of exploratory learning is not enough for efficient learning. The other extreme, maximum teacher guidance or frontal teaching is also not too efficient (Kirschner et *al*, 2006). Students need to gain as much experience as possible through individual work. However, if they do not receive any help at all or too little help they might not advance. Teachers should help but not in excess. They should help in such a measure that students are left with a reasonable proportion of work (Pólya, 2000).

How do we solve word problems? There are different models for this. We propose Pólya's model for solving problems in class. Pólya puts forth four principles for solving problems and developing cognitive skills (Pólya, 2000).

I. *Understand the problem!*

II. *Look for connections between data and the unknown! If you cannot find direct connections look for helping problems! Make a plan for solving the problem!*

III. *Carry out the plan!*

IV. *Check the solution!*

We might use the following models for solving problems (Török, 2013):

1. Number problems might be used as a model for introducing basic operations. Representation and acting out might facilitate understanding.

2. In elementary school number lines are primarily used as a model for natural numbers. In word problems they can illustrate the position and relations between elements, as well as, their relation to the unknown.

3. Representation as a model: After analysing the problem, i.e. we discard from the text what is not relevant from a mathematical point of view and establish the relations between the known and unknown data, we translate the text to mathematical language, the outcome is a mathematical model. The mathematical model cannot only be a set of operations or an open sentence, but it can also be the representation of the problem with different tools, such as segments, areas, volume or symbols representing relations. Representation should be schematic. At first there is a concrete relation to the text, later on this lessens. Representation depends on the type of the problem, on the abstracting ability of the person solving the problem, and his/her experience in solving this type of problems. Representation can be a great helping tool in problem solving (Olosz, Olosz, 2000).

4. Charts can be used as a model in the case of word problems involving functions for presenting all the possibilities.

5. Open sentences can be used as a model in the case of substitution word problems.

6. Graphs can be used as a model to illustrate pairs of data that belong together. They are an important tool in shaping reasoning about functions. Graphs can be used in word problems involving movement, shopping, or measuring.

7. The Venn-diagram as a model is used for positioning elements of sets, as well as introducing the number of the elements of subsets in problems dealing with the number of elements in sets.

The eight keys problem-solving method might be used for applying more models at once. This method guides the students through the problem-solving steps and points them to the strategies that can be used to solve the problem. The keys are designed as an alternative way to develop students' ability to use problem solving heuristic. Students have a bunch of eight keys. Each key contains a simple instruction. They may choose from the following (Herold-Blasius, Jazby, 2016):

1. Read the task again!
2. Start with a small number!
3. Find an example!
4. Look for a pattern or a rule!
5. Draw a picture. Use different colours!
6. Make a table!
7. Work backwards!
8. Check the solution!

The keys contain icons which represent the instructions, thus helping students visually in choosing a certain strategy.

The advantages of the model:

1. The keys provide suggestions as how to approach the problem, what the possibilities are and where to start. They also contain the steps of Pólya's model: key 1. facilitates the understanding of the problem, keys 2., 3., and 4. encourage students to look for concrete examples and correlations. Key 5. prompts to checking and verifying the solution.

2. It encourages students to consider the mathematical relations between the given data and the unknown (key 4.).

3. Problem-solving strategies are named, making it easier for students to become aware of them. Key 6. reminds student to make a drawing, key 7. to make a table, key 8. to the possibility of working backwards.

4. Students can become self-motivated.

5. It gives students time to think.

6. Students can track the problem-solving strategies used and they will memorize the name of these as a result of multiple use. (Which keys did we use?)

Table 1. The icons used on the keys.

Work backwards		Find an example	
Make a table		Use different colours	
Read the task again		Draw a picture	
Check the solution		Start with a small number	

2. The research. Aims and methods

The most important characteristic of our longitudinal intervention program is the use of strategy keys, i.e. to understand the problem, to look for connections between data and the unknown, to look for helping problems, to make a plan for solving the problem, to carry out the plan, the use of possible visual representations (sketches, plane figures, segments, symbols and conventional signs or letters, schematic drawings), to make tables, to work backward, and check the solution.

Our aim was shaping students' belief about the importance of the four steps of Pólya's model. The basic idea of the intervention was to develop students' knowledge about word problem solving strategies with an emphasis on the role of problem solving in four steps and visual representations in mathematical modelling (Csíkos et al., 2012), since visualization has a great impact.

Our intervention program had two phases. The 1st phase was carried out between February and March 2016, while the 2nd phase took place between February and March 2017, at the time when teaching word problems was assigned by the curriculum for grades 3 and 4. We used the same experimental group in both phases of the intervention.

Both phases were divided into 10 units (pre-test, 8 lessons, post-test), lasting for about 8 weeks, (at a rate of one lesson/week) and were dedicated to word problems in grade 3, and 4 respectively. We selected word problems with familiar and realistic content.

The research question posed: Will students' problem-solving skills improve if they are taught different methods and models for solving word problems, and are familiarized with different problem-solving strategies?

Our hypothesis was the following: the experimental group will achieve better results in word problem solving using strategy keys.

Method: The students involved in our classic pedagogical experiment were recruited from two different schools: one class was the experimental group (III. B., from Oradea in 2016, respectively IV. B., from Oradea in 2017) and the other was the control group (III. B., from Sacuieni in 2016, respectively IV. A., from Salonta in 2017). In the 1st phase of the program all participants were third-grade students whose mean age was 9 in 2016, while in the 2nd phase of all participants were four-grade students whose mean age was 10 in 2017.

In 2016 the experimental group comprised 20 students and the control group consisted of 12 students. In 2017 the experimental class comprised 23 students (3 newcomers have joined the class) and the control group consisted of 12 students.

Table 2. Distribution of sample according to sex and group.

Group	**Experimental group**		**Control group**	
	Girls	**Boys**	**Girls**	**Boys**
1st phase 2016	6 (30 %)	14 (70 %)	5 (41.67 %)	7 (58.33 %)
2nd phase 2017	6 (26.08 %)	17 (73.91 %)	12 (60 %)	8 (40 %)

In both experiments a pre-test and post-test control was adopted. In the intervention program we used one test as both pre- and post-test. The experiment was carried out in both classes at the time when students were studying word problems.

Both experiments encompassed the three specific stages of pedagogical research. In the assessment stage the knowledge of 20 (respectively 23) subjects from the experimental group and 12 (respectively 20) subjects from the control group was assessed with the help of a pre-test.

The research stage coincides with the experiment stage. Once a week, for a period of 8 weeks, students from the experimental stage solved different types of word problems. The program contained 51 word problems altogether in the 1st phase, and 72 word problems in the 2nd phase. On an average 6 to 9 problems were solved at one occasion. The duration of an intervention lesson was 50 minutes.

We used the observation method for the control group; we did not interfere in the lesson. Students were observed during mathematics class.

Evaluation was the last stage of the research. We employed the post-test for both groups. The conditions and requirements of the test were the same for the two groups.

In the pre- and post-test the problems were of a gradual difficulty, starting from the easily understandable to the more difficult ones, as well as, from using easy operations to more difficult ones.

We used the test below for assessment in the 1st phase

1. In a library there are story books on a total of 6 shelves. There are 4 books on each shelf. How many story books are there in the library?
2. A box of candy costs 5 RON, a box of chocolates costs 4 times more. How much do we have to pay for the box of candy and the box of chocolates together? (Use representation for solving the problem!)
3. A farmer took 35 kg of potatoes to the market. He sold the fifth of the amount on the first day. On the second day he sold the remaining amount. How many kilograms of potatoes did he sell the second day? *(Prepare a problem-solving plan!)*
4. Discover the numbers below!

first number	second number
two-digit number	two-digit number
odd	even
when divided by 9, the quotient is 9	when divided by 9, the quotient is 4

5. Andrea has 20 toys, Anikó has half as many toys as Andrea. How many toys do the two girls have together? *(Prepare a problem-solving plan!)*

Each student's score on the test was calculated based on the number of correctly solved items. The test contained 5 dichotomous items. The Cronbach α-reliability coefficient of the pre-test was 0.69 ($N = 32$). The test contained problems which could be solved with the help of calculation or schematic drawing, or both. (Students in the control group used more drawings, however, the problem solving methods used by the students in the experimental group were more varied).

We proceeded the same way in 2017, in the 2nd phase as in the 1st phase. The Cronbach α-reliability coefficient of the pre-test was 0.72 ($N = 43$). Table 3 shows that the test contained an exercise of each type. We wanted to investigate which type of problem posed the most difficulties and what strategies students used. The problems required simple operations.

Table 3. Problem types (2^{nd} phase).

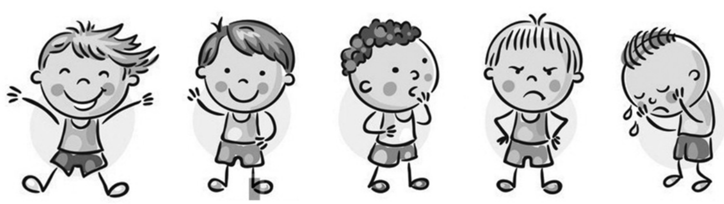
Problem	Type
1.	Simple word problem
2.	Representations
3.	Compare and contrast
4.	Working backwards
5.	Logical

For each type we compared the results of the two groups and analysed the differences between them. The F-test was carried out on the pre- and post-test used in the research to test deviation. The t-test was carried out on the whole population.

The word problems used in the experiment were mostly made, or reformulated by teacher training students. They used different topics and types of problems. The reason for making our own mathematical problems was that the few word problems to be found in course books and workbooks are outdated and do not raise students' interest. When writing the problems we focused on the type of problems we wanted to teach students. We used topics that are more related to children's interest, and are more life-like. In order for the students to relate even more to the problems these were personalized, using students' names, thus making them more enjoyable.

Throughout the program students felt motivated and eagerly participated in problem solving. Next to the solutions they indicated which key(s) they used in solving the problem.

We have also investigated students' attitude. We wanted to find out how students felt while solving word problems, assuming a connection between this and their results.



Last question: How did you feel while solving word problems? Circle you answer!

Here in an example for a problem made by a teacher training student: The *IPhone is a smartphone made by Apple. Its first edition was presented by Steve Jobs in 2007. Its newest generation is the IPhone 7, which appeared on the market in 2016. How many years have passed from the first edition of the IPhone to its seventh edition?* (Letti Rajos, third-year teacher training student)

Here is an example for *development activity*: In the 2nd phase of the program, the fifth unit focused on representations and compare and contrast type word problems. The topic of the lesson was “Encountering sea creatures”. Students were divided into five groups. The problems were presented on posters. Each group was given a poster. They were given the strategy keys. Once they solved the problems on one poster they could move to another one. Each poster presented a problem and illustration related to a sea creature. Each team also received a table in which they had to insert the solutions. If the solution was correct they received a “Well done” stamp.

The problems were made by teacher-training students.

1. Location: *The coast of Indonesia. The creature awaiting the children: the seahorse.*

Problem: *The tourists have encountered two seahorses on the first day. On the second day, when they adventured deeper into the sea they encountered 4 times more seahorses than on the first day. On the third day they have encountered as many seahorses as on the previous two days together. How many seahorses did they see in the three days?*

2. Location: *Red Sea. Sea creature: clownfish*

Problem: *The clownfish are training for the national swimming competition. On the second day of training one of them swam three times the distance it swam on the first day. What distance did it swim on the first and second day, respectively, if the distance on the second day was 50 km more than on the first day?*

3. Location: *Pacific Ocean. Sea creature: firefish*

Problem: *Peter is collecting firefish in the city aquarium. In the first hour he manages to catch 5 fish. In the second hour he catches twice as many as in the first hour. In the third hour he catches two fish more than in the first hour. How many firefish did he catch in three hours?*

4. Location: *Indian Ocean. Sea creature: surgeonfish*

Problem: *A surgeonfish costs one lej more than a firefish. The Indian merchant has bought 2 surgeonfish and 3 firefish for 22 lej. How many lejs does the merchant need if he wants to buy one surgeonfish and two firefish?*

5. Location: *Atlantic Ocean. Sea creature: sea star*

Problem: *Explorers have found 100 more sea stars on the north coast of the Atlantic Ocean than on the south coast. Altogether they have recorded 352 sea stars. How many sea stars did they find on the north coast, and how many did they find on the south coast?*

Teacher training students’ observations about the lesson: “Each student was strongly motivated by this problem-solving method, thus they worked hard on solving the problems. They were excellent team players; they helped and encouraged each other. If one of them encountered difficulties in solving the problem, the other students explained the problem to him/her. By the end of the development activity everybody managed to visit every sea creature. They could solve all the problems without my intervention, which made me really happy, since this is proof for their development. I will use teamwork in the other development activities, too, since it helps a lot in developing problem solving skills" (Letti Rajos, third-year teacher training student).

"In order to ensure that students have understood the solutions to the problems, I have asked them to make a similar word problem and give it to one of the groups to solve it. Students made very interesting problems. The topics ranged from cakes, fruits, clothes, shoes to balls. Students loved solving the problems. Everybody understood them" (Kinga Kis, third-year teacher training student).

"At the ninth and tenth development unit I no longer handed the strategy keys to the students, however I asked them to rely on them when solving the problems. The reason for this was that I wanted them to be able to solve the problems even when they do not have the keys in their hands. It worked out well. The students excelled in solving the problems" (Letti Rajos, third-grade teacher training student).

"The worksheet contained problems of different type. This lesson assured me that students' problem solving skills have improved enormously. They managed to solve the problems on their own. There were students who remembered the helping instructions from the keys, other remembered the icons (Erika Víg, third-grade teacher training student).

3. Results

Before the experiment both the experimental and the control group completed the pre-test. After the experiment both groups completed the post-test. Students sat for the tests in their own classrooms.

3.1. Descriptive statistics and comparison

Throughout the analysis we used different statistical indicators, such as mean, and standard deviation. The table below contains the indicators for the pre- and post-test for the two groups:

Table 4. Means and standard deviations of the experimental and control groups' test on the pre- and post-tests.

Group	Experimental group		Control group	
	Pre-test	**Post-test**	**Pre-test**	**Post-test**
1^{st} phase 2016	$M = 51.4$ $SD = 23.37$	$M = 64.15$ $SD = 26.29$	$M = 50$ $SD = 22.55$	$M = 38.92$ $SD = 29.34$
2^{nd} phase 2017	$M = 57.26$ $SD = 28.42$	$M = 67.52$ $SD = 24.07$	$M = 50.25$ $SD = 25.92$	$M = 39$ $SD = 23.3$

Source: own calculations

In the 1st phase intervention program 2016, the table above shows that in the pre-test there is only a 1.40 % difference in the average of the two groups, thus at the initial stage the level of knowledge in the two groups can be considered equal. However, in the post-test the average of the experimental group has increased 12.75 percentage points, while that of the control group has dropped 11.08 percentage points. In the post-test the deviation for the results in the experimental group is smaller than in the control group.

In the 2nd phase intervention program 2017, table 4 shows that in the pre-test there is only a 7.26 % difference in the average of the two groups, thus at the initial stage the level of knowledge in the two groups can be considered almost the same. However, in the post-test the average of the experimental group has increased 10.26 percentage points, while that of the control group has dropped 11.25 percentage points. In the post-test the deviation for the results in the experimental group is a little bigger than in the control group.

3.2. Comparing the results of the pre- and post-test in the experimental and control group

Table 5 shows the comparison between students' pre- and post-test achievement in the experimental and control group in the 1st and 2nd phase of the intervention program.

Table 5. The F-test and t-test comparisons between the achievement of the experimental and control group in the pre- and post-test.

	1st phase 2016 ($N = 20$, $M = 12$)	2nd phase 2017 ($N = 23$, $M = 20$)
F-test	$F = 0.65$ ($< F$ critical $= 2.65$)	$F = 0.01$ ($< F$ critical $= 2.16$)
t-test	$t = 2.48$ ($> t$ critical $= 1.7$)	$t = 4.59$ ($> t$ critical $= 1.68$)
Level of significance	0.05	

The F-test did not show significant differences between standard deviation (see table 5), thus the two-sample t-test was employed.

The two-sample t-test provides information about the development of students' achievement during the 8-week period in both phases of the intervention program. Table 5 also shows the t-test comparisons between experimental and control group in pre-test and post-test.

The t-test indicates changes during the intervention period (0.05 level of significance); there are significant differences between the achievement of the experimental and control group.

The value 4.59 in table 5 shows that according to the t-test students in the experimental group developed significantly more compared to the pre-test results (0.05 level of significance) than students in the control group.

In order to investigate the impact of the development activities on students with lower or average skills these were listed in a separate group (henceforth called subgroup) when analysing results. This group contained students who reached less than 75 points in the pre-test.

Table 6 shows the comparison between students' pre- and post-test achievement in the experimental and control group in the 1st and 2nd phase of the intervention programs for the subgroup.

Table 6. The F-test and t-test comparisons between the achievement of the experimental and control group in the pre- and post-test for the subgroup.

	1st phase 2016 ($N = 18, M = 9$)	2nd phase 2017 ($N = 15, M = 15$)
F-test	$F = 0.68$ ($<$ F critical $= 3.22$)	$F = 0.04$ ($<$ F critical $= 2.46$)
t-test	$t = 2.21$ ($>$ t critical $= 1.71$)	$t = 4.87$ ($>$ t critical $= 1.7$)
Level of significance	0.05	

The F-test did not show significant differences between standard deviation (see table 6), thus the two-sample t-test was employed. Table 6 also shows the t-test comparisons between the experimental and control group in the pre-test and post-test for the subgroup.

The t-test indicates changes during the intervention period (0.05 level of significance); there are significant differences between the achievement of the experimental and control group.

The value 4.87 in table 6 shows that according to the t-test students with poor of average skills in the experimental group developed significantly more compared to the pre-test results (0.05 level of significance) than students in the control group.

It can be concluded that students with poor or average skills show greater development if they are exposed to more life-like word problems rather than a minimal amount of common problems. This provides further support to our hypothesis.

The table below shows the distribution of students from the experimental group according to their choice of emoticons.

Table 7. Distribution of emoticons in the pre- and post-test in the experimental group.

Measurement	**Emoticons**				
	1 – very well	**2 – well**	**3 – surprised**	**4 – angry**	**5 – sad**
Pre-test	10 (43.47 %)	5 (21.73 %)	8 (34.78 %)	0 (0 %)	0 (0 %)
Post-test	9 (39.13 %)	11 (47.82 %)	3 (13.04 %)	0 (0 %)	0 (0 %)

There was an increase in the number of students who felt well while solving word problems, and less students experienced surprise.

3.3. Effect sizes

Cohen's d index is the standardized mean change for the experimental and control groups, and is estimated by the following equation (Morris, 2005):

$$\Delta = \frac{\left(M_{\text{post, exp}} - M_{\text{pre, exp}}\right) - \left(M_{\text{post, control}} - M_{\text{pre, control}}\right)}{SD_{\text{pre, pooled}}}$$

$$SD_{\text{pre, pooled}} = \sqrt{\frac{(n_{\text{exp}} - 1)SD^2_{\text{pre, exp}} + (n_{\text{control}} - 1)SD^2_{\text{pre, control}}}{n_{\text{exp}} + n_{\text{control}} - 2}}$$

where M is the mean, SD – standard deviation, *exp* – the experimental group, *control* – the control group.

The final d index will contain a c constant for unbiased estimation: $d = c \cdot \Delta$, where

$$c = 1 - \frac{3}{4\left(n_{\text{exp}} + n_{\text{control}} - 2\right) - 1}.$$

The unbiased effect size d may indicate the development increase rate in the experimental group. According to Cohen (1969) $d = 0.8$ can be considered large effect size, $d = 0.5$ medium, and $d = 0.2$ small effect size.

In our case, in the 1st phase of the intervention program $SD_{\text{pre, pooled}} = 23.81$, $c = 0.9747$ and $\Delta = 1.000$. $d = 0.9747$, which means that the first phase had a large effect size.

In the 2nd phase of the intervention program: $SD_{\text{pre, pooled}} = 27.28$, $c = 0.98$ and $\Delta = 0.7884$.

Thus: $d = c \cdot \Delta = 0.98 \cdot 0.7884 = 0.7738$, which means that the second phase also had a great impact (or large effect size).

Table 8. Effect size.

	$SD_{\text{pre, pooled}}$	c	Δ	d
1st phase	23.81	0.9747	1.000	0.9747
2nd phase	27.28	0.98	0.7884	0.7738

4. Conclusion

Our hypothesis was that the experimental group will reach better results on the word problems post-test. Both groups in both experiments had similar results at the beginning of the intervention period. The intervention program proved to be successful in terms of significant development in the experimental group compared to the control group. The experimental effect sizes were estimated by means of Cohen's d statistics. The results suggest that the intervention program had a large effect size on students' word problem solving (students with poor or average skills also showed a significant development compared to the pre-test results).

The research would be worth continuing, thus achieving better results in the case of more complex word problems as well.

The methods and context chosen by teachers largely influences students' motivation and attitude towards word problems. Problem type word problems need to be brought closer to students, i.e. a child-centred approach is needed. Teachers should vary the methods focusing on child-centeredness, real life contexts, and on developing students' intrinsic motivation.

Formulating problems in such a way that it includes children, and addresses them, has a major impact on shaping a positive attitude towards mathematics. It is important to raise students' interest, and provide opportunities for them to use their own experiences during lessons.

The problems designed by the teacher training students raised students' interest. Some of them even gave voice to their positive attitude "These were very interesting problems!", "I would like to solve more such problems!". We were very happy regarding this positive feedback.

The most important part of the research was that teacher training students could experience a classical developmental research; they could take part in the research alongside their tutors and learn from it. They are still in the process of learning how to become a teacher; however they are also expected to do research in the field.

References

[1] AMBRUS, A. (2002), *A problémamegoldás tanításának elméleti alapjai*, Új Pedagógiai Szemle, 2002 október, 157–169.

[2] BRUNER, J. S. (1974), *Új utak az oktatás elméletéhez*, Gondolat Kiadó, 13–40.

[3] COHEN, J. (1969), *Statistical power analysis for the behavioral sciences*, NY Academic.

[4] CSÍKOS, CS., SZITÁNYI, J. AND KELEMEN, R. (2012), *The effects of using drawings in developing young children's mathematical word problem solving: A design experiment with third- grade Hungarian students*, Educational Studies in Mathematics, 81(1), 47–65.

[5] GOLDIN, G. AND KAPUT, J. (1996), *A joint perspective on the idea of representation in learning and doing mathematics*, in L. Steffe, P. Nesher, P. Cobb, G. Goldin, B. Greer (Eds.), Theories of mathematical learning (pp. 397–430), Hillsdale Erlbaum.

[6] HEROLD-BLASIUS AND R., JAZBY, D. (2016), *Can Strategy Keys Help 3rd and 4th Graders Engage in Mathematical Problem Solving?*, in Csíkos, C., Rausch, A., Szitányi, J. (Eds.), Proceedings of the 40th Conference of the International Group for the Psychology of Mathematics Education (pp. 171), Szeged, Hungary: PME.

[7] KIRSCHNER, P. A., SWELLER, J. AND CLARK, R. E. (2006), *Why Minimal Guidance During Instruction Does Not Work: An Analysis of the Failure of Constructivist, Discovery, Problem-based, Experiential and Inquiry-based Teaching*, Educational Psychologist, 41(2), 75–86.

[8] KOZHEVNIKOV, M., HEGARTY, M. AND MAYER, R. (2002), *Revising the visualizer-verbalizer dimension: Evidence for two types of visualizers*, Cognition and Instruction, 20, 47–77.

[9] MORRIS, S. B. (2005), *Effect size estimation from pretest-posttest-control designs with heterogeneous variances*, Paper presented at the 20th Annual Conference of the Society for Industrial and Organizational Psychology, Los Angeles, CA.

[10] NCTM (2000), *Curriculum Standards for School Mathematics*, Reston.

[11] OLOSZ, F. AND OLOSZ, E. (2000), *Matematika és módszertan*, Erdélyi Tankönyvtanács.

[12] PÓLYA, G. (2000), *A gondolkodás iskolája*, Akkord.

[13] RANDALL, T. (Writer, Director, Producer) (2012), *Human Brain: How smart can we get?* [Documentary] United States: Nova science Now for WGBH Boston, `https://www.youtube.com/watch?v=bxbwwEk8A9c` [Accessed 09/11/2016].

[14] SEIDEL, G. (2001), *Ordnung und Multimodalitat im Denken mathematisch Hochbegabter: sequentielle und topologische Eigenschaften kognitiver Mikrozustande*, Dissertation, Friedrich-Schiller Universitat Jena.

[15] TÖRÖK, T. (2013), *Szöveges feladatok és tanításuk*, in Herendiné K. E. (Ed.), A matematika tanítása az alsó osztályokban (pp. 164–191), Budapest: Nemzedékek Tudása tankönyvkiadó.

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Problémamegoldás az elemi matematikatanításban

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Kivonat. A problémamegoldó képesség általános fejlesztése fontos célkitűzés a matematikatanításban. A probléma-megoldási stratégiákat, gondolkodási műveleteket tudatosan kell tanítani a tanórákon. A legtöbb esetben a probléma-megoldási folyamat áll a középpontban, nem pedig a kapott eredmény. A feladatok tanórai feldolgozásának legfontosabb lépései Pólya modellje szerint javasolt, ez a modell a feladatok megoldását négy nagy lépésre bontja. (Pólya, 2000).

A kulcsos problémamegoldási modell egy lehetőség lehet a tanulók feladatmegoldási képességeinek fejlesztésére. (Herold- Blasius, Jazby, 2016).

Ebben a kétlépéses klasszikus pedagógiai kutatásban azt vizsgáltuk, hogy ha feladatmegoldási stratégiákkal ismertetjük meg a tanulókat, ha szöveges feladatok megoldására különböző módszereket, eljárásokat, modelleket tanítunk meg a harmadik, illetve negyedik osztályban (9-10 éves gyerekeknek), akkor javul-e a tanulók feladatmegoldási és problémamegoldási képessége? Feltételeztük, hogy a kulcsok használata elősegíti a probléma-megoldási képességet.

Kulcsszavak: probléma megoldási képesség, szöveges feladatok, heurisztika, kulcsos stratégiák, matematikai modell

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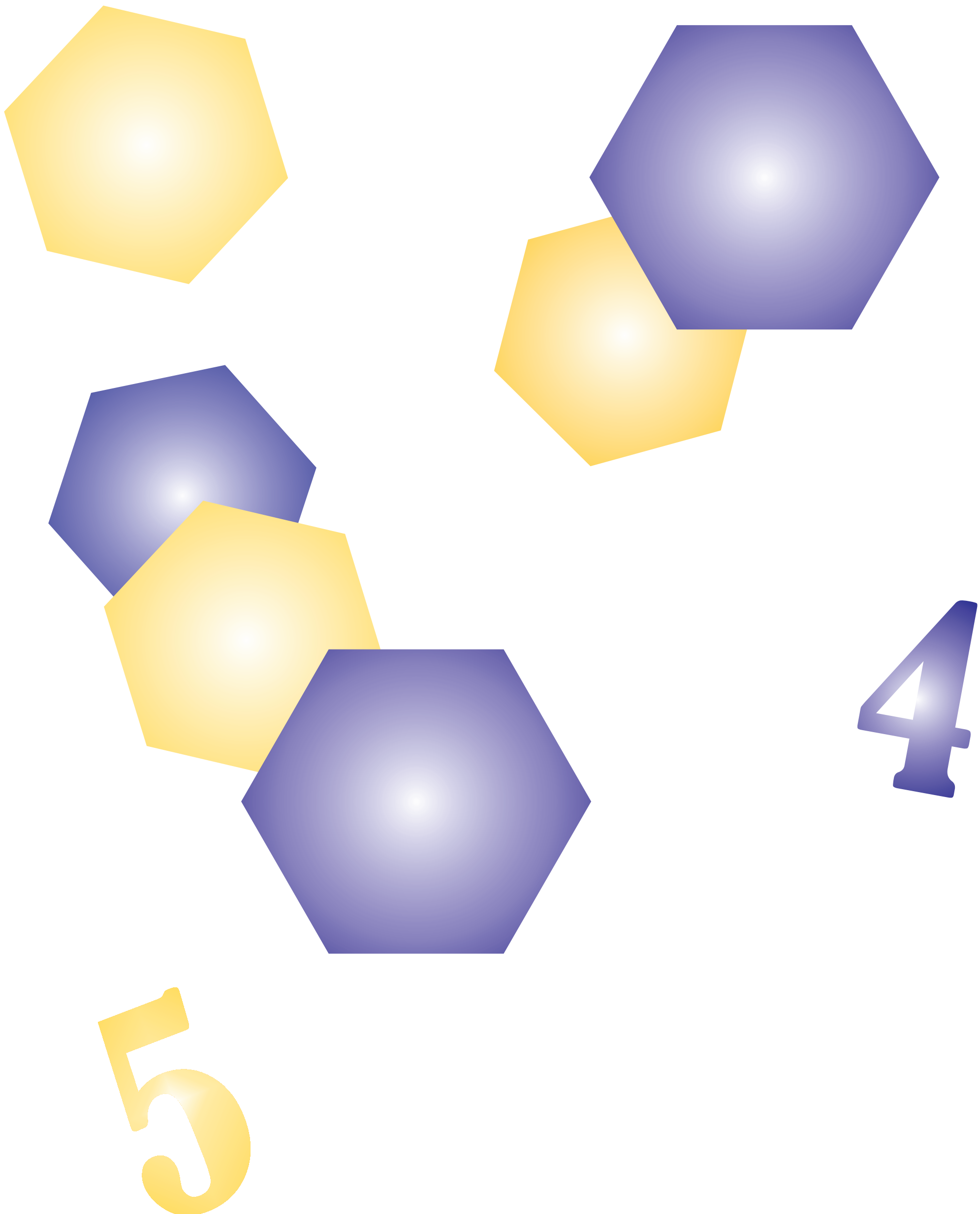

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