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Complex Friedrichs systems and applications

Abstract

Recently, there has been a significant development of the abstract theory of Friedrichs systems in Hilbert spaces (Ern, Guermond & Caplain, 2007; Antonić & Burazin, 2010), and its applications to specific problems in mathematical physics. However, these applications were essentially restricted to real systems. We check that the already developed theory of abstract Friedrichs systems can be adjusted to the complex setting, with some necessary modifications, which allows for applications to partial differential equations with complex coefficients. We also provide examples where the involved Hilbert space is not the space of square integrable functions, as it was the case in previous works, but rather its closed subspace or the space $\mathrm{H}^s(\mathbf{R}^d;\mathbf{C}^r)$, for real s. This setting appears to be suitable for particular systems of partial differential equations, such as the Dirac system, the Dirac-Klein-Gordon system, the Dirac-Maxwell system, and the time-harmonic Maxwell system, which are all addressed in the paper. Moreover, for the time-harmonic Maxwell system we also applied a suitable version of the two-field theory with partial coercivity assumption which is developed in the paper.

Keywords: symmetric positive first-order system, two-field theory, partial differential equations with complex coefficients, coupled systems of partial differential equations

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1. Introduction

Brief overview

The notion of *positive symmetric systems* or *Friedrichs systems* dates from Kurt Otto Friedrichs [29], who showed that this class of problems encompasses a variety of initial and boundary value problems for various linear partial differential equations of different types. The *classical theory* was developed further by a number of authors (see [36, 37] and references therein), but its rebirth begins with the development [27] of the theory of *abstract Friedrichs systems* in Hilbert spaces, which has been extensively studied during the last decade. Among number of its important properties, theretofore shortcomings in having only the existence of weak solutions to Friedrichs systems and the uniqueness of strong ones have been overcome in the abstract theory (see Theorem 1 below).

This renewed interest in Friedrichs systems was initiated by numerical community, resulting in a number of recent numerical results for Friedrichs systems, mostly based on discontinuous Galerkin methods and their variants [24, 25, 26]. From these references we understand that approximating partial differential equations in the form of Friedrichs systems presents certain general advantages: some numerical algorithms based on mixed finite element methods are generally more suitable for first order systems than for higher order equations or systems, which makes the framework of Friedrichs systems more convenient setting for numerical treatment of higher order equations. Secondly, it provides a more accurate reconstruction of the fluxes (the gradient of the primal variable for diffusion-like problems and the stress tensor for linear elasticity problems), as fluxes are naturally integrated within the corresponding Friedrichs system. Finally, it provides a single constructive procedure in devising numerical schemes for different (including elliptic, parabolic and hyperbolic type) partial differential equations. There is also a number of specific benefits of this approach that are related to a chosen particular numerical approach (for example, see [15, 10, 8]).

Alongside the well-posedness result for the stationary theory [27], important recent results include the equivalence of different representations of boundary conditions [2, 11, 27], its relationship with the classical theory [3, 4, 5], the existence and uniqueness results for non-stationary (semi-linear) systems [13], development of the homogenisation theory [14], development of different numerical schemes [9, 10, 16, 24, 25, 26] (see also [20, 23, 31]), as well as applications to various (initial-) boundary value problems of elliptic, hyperbolic and parabolic type [3, 4, 6, 9, 13, 16, 19, 24–27, 33].

As these applications and developments have essentially been restricted to real systems, in the literature one can mostly find the definition of abstract Friedrichs systems on *real* Hilbert spaces. With intention to provide a concise and self-contained description on complex spaces as well, as it was the case with the classical theory [30], in this paper we study the theory of abstract Friedrichs systems in complex Hilbert spaces, and apply the derived results to a number of particular linear and semi-linear equations with complex coefficients of interest in mathematical physics. In contrast to the real case, where in applications to partial differential equations of interest the usual pivot space is $\mathrm{L}^2(\Omega;\mathbf{R}^r)$, in the complex case we also consider certain closed subspaces of $\mathrm{L}^2(\Omega;\mathbf{C}^r)$, as well as $\mathrm{H}^s(\mathbf{R}^d;\mathbf{C}^r)$, for real s. Since a good well-posedness theorem is desirable for the convergence analysis of a numerical scheme, we hope that our paper will open the way for new numerical results in context of complex equations.

The paper is organised as follows: in the second section we introduce abstract Friedrichs operators in the setting of complex Hilbert spaces. The results are mostly analogous to the real case, thus we only pinpoint the differences and omit most of the proofs. After the well-posedness result, as an example of abstract Friedrichs operators we consider the classical Friedrichs operator in two different settings mentioned above, and we prove the well-posedness result for two-field theory with partial coercivity assumption. We proceed by discussing different representations of boundary conditions, and conclude the section with corresponding results in the non-stationary case. In the third section we apply these results to the initial-value problem for linear Dirac system in $\mathrm{H}^s(\mathbf{R}^3;\mathbf{C}^4)$, the semi-linear Dirac system with quadratic nonlinearity in $\mathrm{H}^2(\mathbf{R}^3;\mathbf{C}^4)$, the Dirac-Klein-Gordon system in $\mathrm{H}^2(\mathbf{R}^3;\mathbf{C}^9)$ and the Maxwell-Dirac system in $\mathrm{H}^2(\mathbf{R}^3;\mathbf{C}^{24})$. The last example we consider is a boundary-value problem for the stationary (time-harmonic) Maxwell system.

We finish this introductory part with a brief overview of basic notions known from the classical theory in complex attire.

The classical setting on complex spaces

Let $d, r \in \mathbf{N}$ and let $\Omega \subseteq \mathbf{R}^d$ be an open and bounded set with Lipschitz boundary Γ. The *(classical) Friedrichs operator* is a first-order differential operator $\mathcal{L} : \mathrm{L}^2(\Omega;\mathbf{C}^r) \longrightarrow \mathcal{D}'(\Omega;\mathbf{C}^r)$ defined by $\mathcal{L}\mathsf{u} := \sum_{k=1}^d \partial_k(\mathbf{A}_k\mathsf{u}) + \mathbf{D}\mathsf{u}$, where complex matrix functions $\mathbf{A}_k \in \mathrm{W}^{1,\infty}(\Omega;\mathrm{M}_r(\mathbf{C}))$, $k \in 1..d$, and $\mathbf{D} \in \mathrm{L}^\infty(\Omega;\mathrm{M}_r(\mathbf{C}))$ satisfy:

(F1) $$\mathbf{A}_k \text{ is hermitian:} \quad \mathbf{A}_k = \mathbf{A}_k^*\,,$$

(F2) $$(\exists\,\mu_0 > 0) \quad \mathbf{D} + \mathbf{D}^* + \sum_{k=1}^d \partial_k\mathbf{A}_k \geqslant 2\mu_0\mathbf{I} \qquad (\text{a.e. on } \Omega)\,.$$

Since the left-hand side of (F2) is hermitian we use the usual order on hermitian matrices: $\mathbf{A} \geqslant \mathbf{B}$ if $\mathbf{A}\boldsymbol{\xi}\cdot\boldsymbol{\xi} \geqslant \mathbf{B}\boldsymbol{\xi}\cdot\boldsymbol{\xi}$, $\boldsymbol{\xi} \in \mathbf{C}^d$.

There are three known equivalent ways of assigning boundary (or initial) conditions [29, 30, 36] associated to the *Friedrichs system* $\mathcal{L}\mathsf{u} = \mathsf{f}$. First we need to define $\mathbf{A}_{\boldsymbol{\nu}} := \sum_{k=1}^d \nu_k\mathbf{A}_k$, where $\boldsymbol{\nu} = (\nu_1, \nu_2, \ldots, \nu_d)$ is the outward unit normal on Γ. Let us now present all three possibilities.

A complex matrix field $\mathbf{M} : \Gamma \longrightarrow \mathrm{M}_r(\mathbf{C})$ is called *an admissible boundary condition* if (for a.e. $\mathbf{x} \in \Gamma$) it holds:

(FM1) $$\mathbf{M}(\mathbf{x}) + \mathbf{M}(\mathbf{x})^* \geqslant 0$$

and

(FM2) $$\mathbf{C}^r = \ker\Big(\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x}) - \mathbf{M}(\mathbf{x})\Big) + \ker\Big(\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x}) + \mathbf{M}(\mathbf{x})\Big)\,.$$

The boundary value problem thus reads: for given $\mathsf{f} \in \mathrm{L}^2(\Omega;\mathbf{C}^r)$ find $\mathsf{u} \in \mathrm{L}^2(\Omega;\mathbf{C}^r)$ such that

$$\begin{cases} \mathcal{L}\mathsf{u} = \mathsf{f} \\ (\mathbf{A}_{\boldsymbol{\nu}} - \mathbf{M})\mathsf{u}_{|_\Gamma} = 0 \end{cases}.$$

Some further refinements of the decomposition (FM2) in the real case and for symmetric $\mathbf{M}$ can be found in [33].

A second approach uses a family $N = \{N(\mathbf{x}) : \mathbf{x} \in \Gamma\}$ of subspaces of $\mathbf{C}^r$ which is said to define *the maximal boundary condition* if (for a.e. $\mathbf{x} \in \Gamma$) $N(\mathbf{x})$ is maximal nonnegative with respect to $\mathbf{A}_{\boldsymbol{\nu}}$, i.e. if it holds:

(FX1) $$(\forall\,\boldsymbol{\xi} \in N(\mathbf{x})) \quad \mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})\boldsymbol{\xi}\cdot\boldsymbol{\xi} \geqslant 0$$

and

(FX2) there is no subspace which is larger than $N(\mathbf{x})$ and satisfies (FX1).

Now the boundary value problem takes the form

$$\begin{cases} \mathcal{L}\mathsf{u} = \mathsf{f} \\ \mathsf{u}(\mathbf{x}) \in N(\mathbf{x})\,, \quad \mathbf{x} \in \Gamma \end{cases}.$$

The third approach is just an alternative to (FX1)–(FX2): it is required that $N(\mathbf{x})$ and $\widetilde{N}(\mathbf{x}) := (\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})N(\mathbf{x}))^{\perp}$ (for a.e. $\mathbf{x} \in \Gamma$) satisfy

$$\text{(FV1)} \qquad \begin{aligned} &(\forall \boldsymbol{\xi} \in N(\mathbf{x})) \quad \mathbf{A}_{\boldsymbol{\nu}}\boldsymbol{\xi} \cdot \boldsymbol{\xi} \geqslant 0\,, \\ &(\forall \boldsymbol{\xi} \in \widetilde{N}(\mathbf{x})) \quad \mathbf{A}_{\boldsymbol{\nu}}\boldsymbol{\xi} \cdot \boldsymbol{\xi} \leqslant 0 \end{aligned}$$

and

$$\text{(FV2)} \qquad \widetilde{N}(\mathbf{x}) := (\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})N(\mathbf{x}))^{\perp} \quad \text{and} \quad N(\mathbf{x}) := (\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x})\widetilde{N}(\mathbf{x}))^{\perp}\,.$$

We have three sets of boundary conditions for the Friedrichs system, and we are going to define three more as their counterparts in the abstract setting below. In order to simplify the notation, when referring to e.g. (FM1)–(FM2) we shall simply write only (FM) in the sequel. However, in order to keep a clear distinction from other conditions, like (F1)–(F2), such abbreviations will be reserved only for various forms of boundary conditions. It can be shown [11, 31] that classical conditions (FM), (FV) and (FX) are mutually equivalent, with $N(\mathbf{x}) = \ker\Big(\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x}) - \mathbf{M}(\mathbf{x})\Big)$.

We can see that, in contrast to the real case, all spaces involved are subsets of $\mathbf{C}^r$ (and not of $\mathbf{R}^r$), and in (FM1) we require that only the hermitian part (or equivalently the real part) of $\mathbf{M}(\mathbf{x})$ is nonnegative. We shall see that the same correspondence between real and complex cases is carried over to the abstract theory.

Remark 1. One might bypass the use of complex spaces by separating the system into its real and complex part. Indeed, instead of $\mathcal{L}\mathsf{u} = \sum_{k=1}^{d} \partial_k(\mathbf{A}_k\mathsf{u}) + \mathbf{D}\mathsf{u}$ one can consider

$$\hat{\mathcal{L}}\hat{\mathsf{u}} := \sum_{k=1}^{d} \partial_k(\hat{\mathbf{A}}_k\hat{\mathsf{u}}) + \hat{\mathbf{D}}\hat{\mathsf{u}}\,,$$

where

$$\hat{\mathbf{A}}_k := \begin{bmatrix} \mathsf{Re}\,\mathbf{A}_k & -\mathsf{Im}\,\mathbf{A}_k \\ \mathsf{Im}\,\mathbf{A}_k & \mathsf{Re}\,\mathbf{A}_k \end{bmatrix}\,, \quad \hat{\mathbf{D}} := \begin{bmatrix} \mathsf{Re}\,\mathbf{D} & -\mathsf{Im}\,\mathbf{D} \\ \mathsf{Im}\,\mathbf{D} & \mathsf{Re}\,\mathbf{D} \end{bmatrix}\,.$$

Hence, $\hat{\mathbf{A}}_k \in \mathrm{W}^{1,\infty}(\Omega; \mathrm{M}_{2r}(\mathbf{R}))$, $k \in 1..d$, and $\hat{\mathbf{D}} \in \mathrm{L}^{\infty}(\Omega; \mathrm{M}_{2r}(\mathbf{R}))$.

It is straightforward to verify that u satisfies $\mathcal{L}\mathsf{u} = \mathsf{f}$ if and only if $\hat{\mathsf{u}} := [\mathsf{Re}\,\mathsf{u}\;\; \mathsf{Im}\,\mathsf{u}]^{\top}$ is a solution of $\hat{\mathcal{L}}\hat{\mathsf{u}} = \hat{\mathsf{f}}$, where $\hat{\mathsf{f}} := [\mathsf{Re}\,\mathsf{f}\;\; \mathsf{Im}\,\mathsf{f}]^{\top}$.

This allows for application of the real theory of Friedrichs systems on $\hat{\mathcal{L}}$. However, since

$$\begin{aligned} \mathbf{A}_k = \mathbf{A}_k^* &\iff \hat{\mathbf{A}}_k = \hat{\mathbf{A}}_k^{\top}\,, \\ \mathbf{D} + \mathbf{D}^* + \sum_{k=1}^{d} \partial_k\mathbf{A}_k \geqslant 2\mu_0\mathbf{I}_r &\iff \hat{\mathbf{D}} + \hat{\mathbf{D}}^{\top} + \sum_{k=1}^{d} \partial_k\hat{\mathbf{A}}_k \geqslant 2\mu_0\mathbf{I}_{2r}\,, \end{aligned}$$

we have that $\mathcal{L}$ is a complex Friedrichs operator (i.e. satisfies (F1)–(F2)) if and only if $\hat{\mathcal{L}}$ is a real Friedrichs operator. Moreover, the description of boundary conditions is also equivalent. For example, if $N(\mathbf{x})$ satisfies (FV), then $\hat{N}(\mathbf{x}) := \{[\boldsymbol{\xi}\;\boldsymbol{\eta}]^{\top} : \boldsymbol{\xi} + i\boldsymbol{\eta} \in N(\mathbf{x})\}$ satisfies (FV) for real spaces, and vice versa.

Nevertheless, some insight into the structure of a complex system might be lost by this approach.

∎

2. Abstract setting in complex spaces

Abstract complex Hilbert space formalism

The abstract Hilbert space formalism for Friedrichs systems was developed in [27], and the definition for real spaces was addressed. In order to make the application of the abstract Friedrichs systems to partial differential equations with complex coefficients (e.g. the Dirac system) more transparent, here we present a detailed description of the theory for complex spaces. While recalling the known abstract real Hilbert space formalism we shall emphasise the places where it differs form the complex theory, which in many cases will be just the corresponding complex conjugation. Therefore, we shall present the proofs only where there will be some subtle or important difference to the corresponding proof in the real case.

By L we denote a *complex* Hilbert space, identified with its antidual (the space of antilinear functionals on L) L', and by $\langle \cdot \mid \cdot \rangle_L$ the corresponding complex scalar product which we take to be linear in the first and antilinear in the second argument. Let $\mathcal{D} \subseteq L$ be its dense subspace, and $T, \tilde{T} : \mathcal{D} \longrightarrow L$ (unbounded) linear operators satisfying

$$\text{(T1)} \qquad (\forall \varphi, \psi \in \mathcal{D}) \quad \langle T\varphi \mid \psi \rangle_L = \langle \varphi \mid \tilde{T}\psi \rangle_L \,,$$

$$\text{(T2)} \qquad (\exists c > 0)(\forall \varphi \in \mathcal{D}) \quad \|(T + \tilde{T})\varphi\|_L \leqslant c\|\varphi\|_L \,.$$

Both properties (T1) and (T2) are the same as in the real case, although we need to be aware that in (T1) we have a complex scalar product.

By W_0 we denote the completion of the unitary space $(\mathcal{D}, \langle \cdot \mid \cdot \rangle_T)$, with the graph inner product $\langle \cdot \mid \cdot \rangle_T := \langle \cdot \mid \cdot \rangle_L + \langle T\cdot \mid T\cdot \rangle_L$ (the corresponding norm $\| \cdot \|_T$ is usually called the *graph norm*, which is equivalent to $\| \cdot \|_{\tilde{T}}$ due to (T2)), hence we can extend both T and $\tilde{T}$ [27, 2], first by density to W_0, and then via adjoint operators, having in mind the Gel'fand triple $W_0 \hookrightarrow L \equiv L' \hookrightarrow W_0'$, to $T, \tilde{T} \in \mathcal{L}(L; W_0')$. Therefore, for any $u \in L$ and $\varphi \in W_0$ we have

$$_{W_0'}\langle Tu, \varphi \rangle_{W_0} = \langle u \mid \tilde{T}\varphi \rangle_L \qquad \text{and} \qquad _{W_0'}\langle \tilde{T}u, \varphi \rangle_{W_0} = \langle u \mid T\varphi \rangle_L \,,$$

and (T1)–(T2) holds for $\varphi, \psi \in W_0$. In particular, since the proof of [27, Lemma 2.2] fits in the complex setting, we have $T+\tilde{T} \in \mathcal{L}(L; L)$ and $(T+\tilde{T})^* = T+\tilde{T}$, while (T2) holds even for $\varphi \in L$.

By

$$W := \{u \in L : Tu \in L\} = \{u \in L : \tilde{T}u \in L\}$$

we denote the *graph space* which, equipped with the graph norm, is a complex Hilbert space (see the proof of [27, Lemma 2.1]).

As it is well known from the real theory, and it will be elaborated in the complex case below, the crucial problem is to find sufficient conditions on a subspace $V \subseteq W$, such that the operator $T_{|_V} : V \longrightarrow L$ is an isomorphism. In the case of a partial differential operator, the subspace V contains information on boundary conditions.

In order to find such sufficient conditions, we first introduce a *boundary operator* $D \in \mathcal{L}(W; W')$ defined by

$$_{W'}\langle Du, v \rangle_W := \langle Tu \mid v \rangle_L - \langle u \mid \tilde{T}v \rangle_L \,, \qquad u, v \in W \,,$$

and with proper placement of complex conjugation in the proof of [27, Lemma 2.3 and 2.4] we get the following generalisation.

Lemma 1. *Under assumptions* (T1)–(T2), *the operator* D *satisfies*

$$(\forall u, v \in W) \quad {_{W'}\langle Du, v \rangle_W} = \overline{_{W'}\langle Dv, u \rangle_W} \,,$$

$$\ker D = W_0 \qquad \textit{and} \qquad \operatorname{im} D = W_0^0 \,,$$

where ^{0} *stands for the annihilator. In particular,* $\operatorname{im} D = D(W)$ *is closed in* W'. ∎

Remark 2. An immediate consequence of Lemma 1 and the discussion above is that for any $u \in L$ and $v \in W$ we have that both $\langle (T + \tilde{T})u \mid u \rangle_L$ and $_{W'}\langle Dv, v \rangle_W$ are real numbers. In addition, from $\mathsf{Im}\, \langle (T + \tilde{T})v \mid v \rangle_L = 0$ we have $\mathsf{Im}\, \langle Tv \mid v \rangle_L = \mathsf{Im}\, \langle v \mid \tilde{T}v \rangle_L$. ∎

We are now ready to describe the subspace V: let V and $\widetilde{V}$ be two subspaces of W satisfying

$$\text{(V1)} \qquad \begin{aligned} &(\forall\, u \in V) \qquad {}_{W'}\langle\, Du, u\,\rangle_W \geqslant 0\,, \\ &(\forall\, v \in \widetilde{V}) \qquad {}_{W'}\langle\, Dv, v\,\rangle_W \leqslant 0\,, \end{aligned}$$

$$\text{(V2)} \qquad V = D(\widetilde{V})^0\,, \qquad \widetilde{V} = D(V)^0\,,$$

where (as before) ^{0} stands for the annihilator. We shall refer to both (V1) and (V2) as (V). Since the annihilator is a closed subspace of W, we have that V and $\widetilde{V}$ are closed. Moreover, by Lemma 1 and (V2) we have that $\ker D = W_0 \subseteq V \cap \widetilde{V}$.

Remark 3. For a closed subspace V in W such that $W_0 \subseteq V$ we have $V = D(D(V)^0)^0$. Indeed, one can prove this by the same arguments as in the proof of [2, Theorem 2(b)]. Let us just remark that this statement in terms of indefinite inner product used in [2] reads $V = V^{[\perp][\perp]}$.

Therefore if we choose V to be a closed subspace of W containing W_0, then (V2) is satisfied if and only if $\widetilde{V} = D(V)^0$.

∎

In order to get the well-posedness result one additional assumption that ensures the coercivity of T and $\tilde{T}$ is needed:

$$\text{(T3)} \qquad (\exists\, \mu_0 > 0)(\forall\, \varphi \in \mathcal{D}) \quad \langle\, (T + \tilde{T})\varphi \mid \varphi\,\rangle_L \geqslant 2\mu_0 \|\varphi\|_L^2\,.$$

It is easy to see that the above inequality extends to $\varphi \in L$. The operator T, satisfying (T1)–(T3) for some $\tilde{T}$, we shall call *the abstract Friedrichs operator*. The proof of the well-posedness result uses the following lemma.

Lemma 2. *Under assumptions* (T1)–(T3) *and* (V1)*, the operators T and $\tilde{T}$ are L–coercive on V and $\widetilde{V}$, respectively. Moreover, we have:*

$$\begin{aligned} &(\forall\, u \in V) \qquad \mathsf{Re}\,\langle\, Tu \mid u\,\rangle_L \geqslant \mu_0 \|u\|_L^2\,, \\ &(\forall\, v \in \widetilde{V}) \qquad \mathsf{Re}\,\langle\, \tilde{T}v \mid v\,\rangle_L \geqslant \mu_0 \|v\|_L^2\,. \end{aligned}$$

Dem. For $u \in V$ we have

$$\langle\, Tu \mid u\,\rangle_L = \frac{1}{2}\langle\, (T + \tilde{T})u \mid u\,\rangle_L + \frac{1}{2}{}_{W'}\langle\, Du, u\,\rangle_W + i\,\mathsf{Im}\,\langle\, u \mid \tilde{T}u\,\rangle_L\,,$$

so by Remark 2 it follows that

$$\mathsf{Re}\,\langle\, Tu \mid u\,\rangle_L = \frac{1}{2}\langle\, (T + \tilde{T})u \mid u\,\rangle_L + \frac{1}{2}{}_{W'}\langle\, Du, u\,\rangle_W\,.$$

Analogously, for $u \in \widetilde{V}$ we have

$$\mathsf{Re}\,\langle\, \tilde{T}u \mid u\,\rangle_L = \frac{1}{2}\langle\, (T + \tilde{T})u \mid u\,\rangle_L - \frac{1}{2}{}_{W'}\langle\, Du, u\,\rangle_W\,.$$

Therefore, the above estimates follow by taking into account (T3) and (V1).

The L–coercivity property now follows from simple inequalities $|\langle\, Tu \mid u\,\rangle_L| \geqslant \mathsf{Re}\,\langle\, Tu \mid u\,\rangle_L$ and $|\langle\, \tilde{T}u \mid u\,\rangle_L| \geqslant \mathsf{Re}\,\langle\, \tilde{T}u \mid u\,\rangle_L$.

Q.E.D.

Let us take $u \in V$. By Lemma 2 we have

$$\|Tu\|_L \|u\|_L \geqslant |\langle\, Tu \mid u\,\rangle_L| \geqslant \mu_0 \|u\|_L^2\,,$$

which gives us that $\frac{1}{\mu_0}\|Tu\|_L \geqslant \|u\|_L$, and therefore $\|Tu\|_L \geqslant \left(1 + \frac{1}{\mu_0}\right)^{-1} \|u\|_W$. This, together with the complex version of Banach-Nečas-Babuška theorem (see Theorem 2 below), enables us to repeat the steps in the proof of [27, Theorem 3.1] and get the well-posedness result.

Theorem 1. *If* (T1)–(T3) *and* (V) *hold, then the restrictions of operators* $T_{|_V} : V \longrightarrow L$ *and* $\tilde{T}_{|_{\widetilde{V}}} : \widetilde{V} \longrightarrow L$ *are isomorphisms, where* V *and* $\widetilde{V}$ *are equipped with the graph norm.* ∎

In the literature it is more common to find the Banach-Nečas-Babuška theorem on real spaces [23, Theorem 2.6], but the proof in the complex case goes along the same lines, so we skip it.

Theorem 2. (Banach-Nečas-Babuška) *For two (complex) Banach spaces* V *and* L *the following statements are equivalent:*

i) $T \in \mathcal{L}(V; L)$ *is bijective.*

ii) One has

$$(\exists\, \alpha > 0)(\forall\, u \in V) \qquad \sup_{v \in L' \setminus \{0\}} \frac{|{}_{L'}\langle\, v, Tu\,\rangle_L|}{\|v\|_{L'}} \geqslant \alpha \|u\|_V$$

and

$$(\forall\, v \in L') \quad \Big((\forall\, u \in V) \quad {}_{L'}\langle\, v, Tu\,\rangle_L = 0\Big) \quad \Longrightarrow \quad v = 0\,.$$

∎

Theorem 1 gives sufficient conditions on subspaces V and $\widetilde{V}$ that ensure well-posedness of the following problems:

1) for given $f \in L$ find $u \in V$ such that $Tu = f$;
2) for given $f \in L$ find $v \in \widetilde{V}$ such that $\tilde{T}v = f$.

Its importance arises from the relative simplicity of geometric conditions (V) which, in the case when T is a partial differential operator, do not involve the notion of traces (and all the intricacies this brings into the analysis) for functions in the graph space of T. However, for numerical computations it is more appropriate to write the problem above in terms of a suitable bilinear form [24, Theorem 2.8]. The existence (and a classification) of such pairs of subspaces $(V, \widetilde{V})$ satisfying conditions (V) is studied in [7].

Although the abstract theory for Friedrichs systems is developed in arbitrary Hilbert spaces, in all its applications to partial differential equations that we are aware of in the literature, we have $L = \mathrm{L}^2$. Here we shall present some examples with a different situation: the first example deals with the case $L = \mathrm{H}^s$, while for the second one we take L to be a particular closed subspace of L^2. To the best of our knowledge, these examples are not known even in the context of real Friedrichs systems, and both cases shall be used in the next section, where we shall apply the theory presented above to particular examples of interest in mathematical physics.

Example 1. (Classical complex Friedrichs operator on H^s spaces) For $s \in \mathbf{R}$ we consider

$$\mathrm{H}^s(\mathbf{R}^d; \mathbf{C}^r) := \Big\{ u \in \mathcal{S}'(\mathbf{R}^d; \mathbf{C}^r) : \langle \boldsymbol{\xi} \rangle^s \hat{\mathsf{u}} \in \mathrm{L}^2(\mathbf{R}^d; \mathbf{C}^r) \Big\}\,,$$

where $\mathcal{S}'(\mathbf{R}^d; \mathbf{C}^r)$ is the space of ($\mathbf{C}^r$ valued) tempered distributions, $\langle \boldsymbol{\xi} \rangle := \sqrt{1 + |2\pi\boldsymbol{\xi}|^2}$, and $\hat{\mathsf{u}}(\boldsymbol{\xi}) := \int_{\mathbf{R}^d} e^{-2\pi i \boldsymbol{\xi}\cdot\mathbf{x}} \mathsf{u}(\mathbf{x})\, d\mathbf{x}$ denotes the Fourier transform. Equipped with the inner product $\langle\, \mathsf{u} \mid \mathsf{v}\,\rangle_{\mathrm{H}^s(\mathbf{R}^d;\mathbf{C}^r)} := \langle\, \langle \boldsymbol{\xi} \rangle^s \hat{\mathsf{u}} \mid \langle \boldsymbol{\xi} \rangle^s \hat{\mathsf{v}}\,\rangle_{\mathrm{L}^2(\mathbf{R}^d;\mathbf{C}^r)}$, $\mathrm{H}^s(\mathbf{R}^d; \mathbf{C}^r)$ becomes a (complex) Hilbert space with $\mathrm{C}_c^\infty(\mathbf{R}^d; \mathbf{C}^r)$ as a dense subspace [38, Chapter 15].

Let $L = \mathrm{H}^s(\mathbf{R}^d; \mathbf{C}^r)$, for $s \in \mathbf{R}$, and $\mathcal{D} = \mathrm{C}_c^\infty(\mathbf{R}^d; \mathbf{C}^r)$. Furthermore, assume that constant matrices $\mathbf{D}$, $\mathbf{A}_k$, $k \in 1..d$, satisfy (F1) and (F2), i.e. all matrices $\mathbf{A}_k$ are hermitian, while $\mathbf{D} + \mathbf{D}^*$ is a positive definite matrix.

We shall show that operators $\mathcal{L}, \tilde{\mathcal{L}} : L \longrightarrow \mathcal{D}'$ given by $\mathcal{L}\mathsf{u} := \sum_{k=1}^d \partial_k(\mathbf{A}_k \mathsf{u}) + \mathbf{D}\mathsf{u}$ and $\tilde{\mathcal{L}}\mathsf{u} := -\sum_{k=1}^d \partial_k(\mathbf{A}_k \mathsf{u}) + \mathbf{D}^*\mathsf{u}$, where ∂_k represents the distributional derivative, are indeed the (extensions of) abstract Friedrichs operators.

To this end we define operators $T, \tilde{T} : \mathcal{D} \longrightarrow L$ given by the same expressions as $\mathcal{L}$ and $\tilde{\mathcal{L}}$, but with the distributional derivatives replaced by the classical derivatives. The first step is to check that these operators satisfy (T1)–(T3).

For $\mathsf{u},\mathsf{v}\in\mathcal{D}$, by the properties of the Fourier transform, we obtain

$$\begin{aligned}\langle T\mathsf{u}\mid\mathsf{v}\rangle_{\mathrm{H}^s(\mathbf{R}^d;\mathbf{C}^r)}-\langle\mathsf{u}\mid\tilde{T}\mathsf{v}\rangle_{\mathrm{H}^s(\mathbf{R}^d;\mathbf{C}^r)} &= \int_{\mathbf{R}^d}\langle\boldsymbol{\xi}\rangle^{2s}\widehat{T\mathsf{u}}\cdot\hat{\mathsf{v}}\,d\boldsymbol{\xi}-\int_{\mathbf{R}^d}\langle\boldsymbol{\xi}\rangle^{2s}\hat{\mathsf{u}}\cdot\widehat{\tilde{T}\mathsf{v}}\,d\boldsymbol{\xi}\\ &=\int_{\mathbf{R}^d}\langle\boldsymbol{\xi}\rangle^{2s}\left(\Big(2\pi i\sum_{k=1}^d\xi_k\mathbf{A}_k\hat{\mathsf{u}}+\mathbf{D}\hat{\mathsf{u}}\Big)\cdot\hat{\mathsf{v}}-\hat{\mathsf{u}}\cdot\Big(-2\pi i\sum_{k=1}^d\xi_k\mathbf{A}_k\hat{\mathsf{v}}+\mathbf{D}^*\hat{\mathsf{v}}\Big)\right)d\boldsymbol{\xi}\\ &=\int_{\mathbf{R}^d}\langle\boldsymbol{\xi}\rangle^{2s}\left(2\pi i\sum_{k=1}^d\xi_k\mathbf{A}_k\hat{\mathsf{u}}\cdot\hat{\mathsf{v}}-2\pi i\sum_{k=1}^d\xi_k\mathbf{A}_k\hat{\mathsf{u}}\cdot\hat{\mathsf{v}}+\mathbf{D}\hat{\mathsf{u}}\cdot\hat{\mathsf{v}}-\mathbf{D}\hat{\mathsf{u}}\cdot\hat{\mathsf{v}}\right)d\boldsymbol{\xi}\\ &=0\,,\end{aligned}$$

which proves (T1). Moreover, for every $\mathsf{u}\in\mathcal{D}$ we have (here $|\mathbf{D}|$ denotes the Frobenius norm of matrix $\mathbf{D}$)

$$\|(T+\tilde{T})\mathsf{u}\|_{\mathrm{H}^s(\mathbf{R}^d;\mathbf{C}^r)}=\|(\mathbf{D}+\mathbf{D}^*)\mathsf{u}\|_{\mathrm{H}^s(\mathbf{R}^d;\mathbf{C}^r)}\leqslant|\mathbf{D}+\mathbf{D}^*|\|\mathsf{u}\|_{\mathrm{H}^s(\mathbf{R}^d;\mathbf{C}^r)}\,,$$

hence (T2) is valid. Finally, for every $\mathsf{u}\in\mathcal{D}$, by (F2) we get

$$\begin{aligned}\langle\,(T+\tilde{T})\mathsf{u}\mid\mathsf{u}\,\rangle_{\mathrm{H}^s(\mathbf{R}^d;\mathbf{C}^r)} &=\int_{\mathbf{R}^d}\langle\boldsymbol{\xi}\rangle^{2s}(\mathbf{D}+\mathbf{D}^*)\hat{\mathsf{u}}\cdot\hat{\mathsf{u}}\,d\boldsymbol{\xi}\\ &\geqslant\int_{\mathbf{R}^d}\langle\boldsymbol{\xi}\rangle^{2s}2\mu_0\hat{\mathsf{u}}\cdot\hat{\mathsf{u}}\,d\boldsymbol{\xi}=2\mu_0\|\mathsf{u}\|^2_{\mathrm{H}^s(\mathbf{R}^d;\mathbf{C}^r)}\,,\end{aligned}$$

which proves (T3). Therefore, we can extend operators T and $\tilde{T}$ to $\mathrm{H}^s(\mathbf{R}^d;\mathbf{C}^r)$, while it still remains to be checked whether these extensions coincide with (classical) operators $\mathcal{L}$ and $\tilde{\mathcal{L}}$. Since $\mathcal{D}$ is dense and continuously embedded in W_0 we have that W_0' is continuously embedded in $\mathcal{D}'=\mathcal{D}'(\mathbf{R}^d;\mathbf{C}^r)$, implying that $T,\tilde{T}\in\mathcal{L}(L;\mathcal{D}')$. Finally, these operators coincide with $\mathcal{L},\tilde{\mathcal{L}}$ since they coincide on a dense subset $\mathcal{D}=\mathrm{C}^\infty_c(\mathbf{R}^d;\mathbf{C}^r)$, as the classical derivative of a smooth function is equal to its distributional derivative.

The graph space in this case is given by

$$W=\Big\{\mathsf{u}\in\mathrm{H}^s(\mathbf{R}^d;\mathbf{C}^r):T\mathsf{u}\in\mathrm{H}^s(\mathbf{R}^d;\mathbf{C}^r)\Big\}\,,$$

and since the above calculation for verifying (T1) on $\mathcal{D}$ is also valid for any $\mathsf{u},\mathsf{v}\in W$, we can conclude that the boundary operator D is trivial, hence $\ker D=W=W_0$. Thus, for $V=\widetilde{V}=W$ the condition (V) is trivially satisfied. Moreover, a trivial boundary operator implies that $\mathcal{D}=\mathrm{C}^\infty_c(\mathbf{R}^d;\mathbf{C}^r)$ is dense in the graph space W, which is a well-known fact in the case $s=0$.

∎

Example 2. (Classical complex Friedrichs operator on a closed subspace of L^2) For $\Omega\subseteq\mathbf{R}^d$ open and bounded, let us denote by $\mathcal{D}$ a closed subspace of $\mathrm{C}^\infty_c(\Omega;\mathbf{C}^r)$, while its closure in the L^2 norm by L. Thus, equipped with the L^2 scalar product, L is a complex Hilbert space. One can think of $\mathcal{D}$ to be $\mathrm{C}^\infty_{c,\mathrm{div}=0}(\Omega;\mathbf{C}^r):=\{\boldsymbol{\varphi}\in\mathrm{C}^\infty_c(\Omega;\mathbf{C}^r):\mathsf{div}\,\boldsymbol{\varphi}=0\}$, while for L we then get $\mathrm{L}^2_{\mathrm{div}=0,0}(\Omega;\mathbf{C}^r)$, which can be characterised as the space of square integrable functions whose divergence and normal trace are equal to zero. Note that the trace is well defined for Ω with a Lipschitz boundary.

Moreover, for $\mathsf{u}\in\mathcal{D}$ we define

$$\begin{aligned}T\mathsf{u}:=&P_L\Big(\sum_{k=1}^d\partial_k(\mathbf{A}_k\mathsf{u})+\mathbf{D}\mathsf{u}\Big)\,,\\ \tilde{T}\mathsf{u}:=&P_L\Big(-\sum_{k=1}^d\partial_k(\mathbf{A}_k^*\mathsf{u})+(\mathbf{D}^*+\sum_{k=1}^d\partial_k\mathbf{A}_k^*)\mathsf{u}\Big)\,,\end{aligned}$$

where $P_L : \mathrm{L}^2(\Omega;\mathbf{C}^r) \longrightarrow L$ is the orthogonal projection on L, while ∂_k stands for *the classical* derivative, $\mathbf{D} \in \mathrm{L}^\infty(\Omega;\mathrm{M}_r(\mathbf{C}))$ and $\mathbf{A}_k \in \mathrm{W}^{1,\infty}(\Omega;\mathrm{M}_r(\mathbf{C}))$, $k \in 1..d$, are hermitian matrix functions (i.e. they satisfy (F1)). Hence $T, \tilde{T} : \mathcal{D} \longrightarrow L$ are (in general) unbounded linear operators, if we use the norm of L on $\mathcal{D}$. Since the scalar product on L coincides with the L^2 scalar product, it is straightforward to verify that the above assumptions imply that T and $\tilde{T}$ satisfy (T1)–(T2). Indeed, for $\mathsf{u},\mathsf{v} \in \mathcal{D} \subseteq L$ we have

$$\begin{aligned}
\langle T\mathsf{u} \mid \mathsf{v} \rangle_L &= \Big\langle P_L\Big(\sum_{k=1}^d \partial_k(\mathbf{A}_k\mathsf{u}) + \mathbf{D}\mathsf{u}\Big) \mid \mathsf{v} \Big\rangle_{\mathrm{L}^2(\Omega;\mathbf{C}^r)} \\
&= \Big\langle \sum_{k=1}^d \partial_k(\mathbf{A}_k\mathsf{u}) + \mathbf{D}\mathsf{u} \mid \mathsf{v} \Big\rangle_{\mathrm{L}^2(\Omega;\mathbf{C}^r)} \\
&= \Big\langle \mathsf{u} \mid -\sum_{k=1}^d \partial_k(\mathbf{A}_k^*\mathsf{v}) + (\mathbf{D}^* + \sum_{k=1}^d \partial_k\mathbf{A}_k^*)\mathsf{v} \Big\rangle_{\mathrm{L}^2(\Omega;\mathbf{C}^r)} = \langle \mathsf{u} \mid \tilde{T}\mathsf{v} \rangle_L \,,
\end{aligned}$$

while

$$\|(T+\tilde{T})\mathsf{u}\|_L \leqslant \Big\|(\mathbf{D} + \mathbf{D}^* + \sum_{k=1}^d \partial_k\mathbf{A}_k)\mathsf{u}\Big\|_{\mathrm{L}^2(\Omega;\mathbf{C}^r)} \leqslant C\|\mathsf{u}\|_{\mathrm{L}^2(\Omega;\mathbf{C}^r)} = C\|\mathsf{u}\|_L \,,$$

where C depends on the L^∞ norms of $\mathbf{D}$ and $\partial_k\mathbf{A}_k$.

Therefore, we can extend T and $\tilde{T}$ to elements in $\mathcal{L}(L;W_0')$ by the procedure presented at the beginning of this section. Next we characterise these extensions.

First, let us remark that the restriction operator $\mathcal{R}$ taking a distribution (i.e. an element in $\mathcal{D}'(\Omega;\mathbf{C}^r)$) and reducing it to a functional on $\mathcal{D}$, is continuous from $\mathcal{D}'(\Omega;\mathbf{C}^r)$ to $\mathcal{D}'$, but it is not injective since $\mathcal{D}$ is not dense in $\mathcal{D}'(\Omega;\mathbf{C}^r)$. For example, on $\mathrm{L}^2(\Omega;\mathbf{C}^r)$, $\mathcal{R}$ coincides with the orthogonal projection P_L to L, since L is embedded in $\mathcal{D}'$. By using this restriction $\mathcal{R}$ we define operators $\mathcal{L}, \tilde{\mathcal{L}} : L \longrightarrow \mathcal{D}'$ by

$$\begin{aligned}
\mathcal{L}\mathsf{u} &:= \mathcal{R}\Big(\sum_{k=1}^d \partial_k(\mathbf{A}_k\mathsf{u}) + \mathbf{D}\mathsf{u}\Big)\,, \\
\tilde{\mathcal{L}}\mathsf{u} &:= \mathcal{R}\Big(-\sum_{k=1}^d \partial_k(\mathbf{A}_k^*\mathsf{u}) + (\mathbf{D}^* + \sum_{k=1}^d \partial_k\mathbf{A}_k^*)\mathsf{u}\Big)\,,
\end{aligned}$$

where ∂_k is now *the distributional* derivative. Since $\mathcal{R}$ is continuous, we have that $\mathcal{L}$ and $\tilde{\mathcal{L}}$ are continuous as well. Let us show that extensions of T and $\tilde{T}$ coincide with $\mathcal{L}$ and $\tilde{\mathcal{L}}$.

As $\mathcal{D}$ is continuously and densely embedded in W_0, we have that W_0' is continuously embedded in $\mathcal{D}'$, implying that $T, \tilde{T} \in \mathcal{L}(L;\mathcal{D}')$. As derivatives of smooth functions are equal to their distributional derivatives and $\mathcal{R}$ coincides with P_L on $\mathrm{L}^2(\Omega;\mathbf{C}^r)$, we have that operators $T, \tilde{T}$ coincide with $\mathcal{L}, \tilde{\mathcal{L}}$ on the dense subspace $\mathcal{D}$, thus by the continuity on the whole L as well.

The corresponding graph space is given by

$$W = \Big\{\mathsf{u} \in L : \mathcal{R}\Big(\sum_{k=1}^d \partial_k(\mathbf{A}_k\mathsf{u}) + \mathbf{D}\mathsf{u}\Big) \in L\Big\}\,.$$

If in addition we have that $\mathbf{D}$ and $\mathbf{A}_k$, $k \in 1..d$, satisfy (F2), then the condition (T3) follows directly. Indeed, for $\mathsf{u} \in \mathcal{D}$ we have

$$\begin{aligned}
\langle (T+\tilde{T})\mathsf{u} \mid \mathsf{u} \rangle_L &= \Big\langle P_L\Big((\mathbf{D} + \mathbf{D}^* + \sum_{k=1}^d \partial_k\mathbf{A}_k)\mathsf{u}\Big) \mid \mathsf{u} \Big\rangle_L \\
&= \Big\langle (\mathbf{D} + \mathbf{D}^* + \sum_{k=1}^d \partial_k\mathbf{A}_k)\mathsf{u} \mid \mathsf{u} \Big\rangle_{\mathrm{L}^2(\Omega;\mathbf{C}^r)} \geqslant 2\mu_0\|\mathsf{u}\|^2_{\mathrm{L}^2(\Omega;\mathbf{C}^r)} = 2\mu_0\|\mathsf{u}\|^2_L \,,
\end{aligned}$$

where in the second equality we have used that $\mathsf{u} \in \mathcal{D} \subseteq L$. Therefore, with a suitable choice of subspaces V and $\widetilde{V}$, by Theorem 1 we get that $T_{|_V} : V \longrightarrow L$ and $\tilde{T}_{|_{\widetilde{V}}} : \widetilde{V} \longrightarrow L$ (or equivalently $\mathcal{L}_{|_V} : V \longrightarrow L$ and $\tilde{\mathcal{L}}_{|_{\widetilde{V}}} : \widetilde{V} \longrightarrow L$) are isomorphisms.

Before concluding this discussion, let us mention another equivalent way of defining operators $\mathcal{L}$ and $\tilde{\mathcal{L}}$. On $\mathcal{D}'$ we can define derivatives by transposition, analogously to the definition of usual distributional derivatives on $\mathcal{D}'(\Omega; \mathbf{C}^r)$. Namely, for $S \in \mathcal{D}'$ and $\varphi \in \mathcal{D}$ we define $\tilde{\partial}_j S$ by

$$\langle \tilde{\partial}_j S, \varphi \rangle = -\langle S, \partial_j \varphi \rangle \,,$$

which is well defined since $\partial_j \varphi \in \mathcal{D}$ by the closedness of $\mathcal{D}$ in $\mathrm{C}_c^\infty(\Omega; \mathbf{C}^r)$ (as the derivative is a continuous operator in the topology of $\mathrm{C}_c^\infty(\Omega; \mathbf{C}^r)$). Hence, $\tilde{\partial}_j S$ is a well defined object in $\mathcal{D}'$. The immediate consequence is that for $S \in \mathcal{D}'(\Omega; \mathbf{C}^r)$ we have $\mathcal{R}(\partial_j S) = \tilde{\partial}_j(\mathcal{R} S)$, where ∂_j is the usual distributional derivative. Therefore, we have

$$\mathcal{L}\mathsf{u} = \sum_{k=1}^{d} \tilde{\partial}_k \Big(\mathcal{R}(\mathbf{A}_k \mathsf{u}) \Big) + \mathcal{R}(\mathbf{D}\mathsf{u}) = \sum_{k=1}^{d} \tilde{\partial}_k P_L(\mathbf{A}_k \mathsf{u}) + P_L(\mathbf{D}\mathsf{u}) \,,$$

and similarly for $\tilde{\mathcal{L}}$. ∎

Two-field theory

Unfortunately, for some systems of partial differential equations describing problems in mathematical physics the condition (T3) is violated (e.g. for the time-harmonic Maxwell system presented by the end of the paper). However, if T and $\tilde{T}$ are of the special *two-field* structure [25, 26], we can weaken (T3), while still preserving the result of Theorem 1 [26, Theorem 3.1]. This special structure assumes that r equations partially decouple into the first r_1, where the second set of r_2 unknowns enters with constant coefficients in front of the derivatives, and the remaining r_2 equations, where the first r_1 unknowns enter in the same way, as we shall made precise below. Here we present the description of such systems and give the generalisation of [26, Theorem 3.1] to the complex setting, as well as to the case where L is a (suitable) subspace of L^2.

Let d and $r = r_1 + r_2$ be in $\mathbf{N}$ and let $\Omega \subseteq \mathbf{R}^d$ be an open and bounded set with Lipschitz boundary. Let $\mathcal{D}_1$ be a closed subspace of $\mathrm{C}_c^\infty(\Omega; \mathbf{C}^{r_1})$ (in its standard topology of strict inductive limit) and $\mathcal{D}_2 := \mathrm{C}_c^\infty(\Omega; \mathbf{C}^{r_2})$, let $L_2 = \mathrm{L}^2(\Omega; \mathbf{C}^{r_2})$ and let L_1 be the closure of $\mathcal{D}_1$ in $\mathrm{L}^2(\Omega; \mathbf{C}^{r_1})$. Therefore, $L := L_1 \times L_2$ is a closed subspace of $\mathrm{L}^2(\Omega; \mathbf{C}^{r_1}) \times \mathrm{L}^2(\Omega; \mathbf{C}^{r_2}) = \mathrm{L}^2(\Omega; \mathbf{C}^r)$, and $\mathcal{D} := \mathcal{D}_1 \times \mathcal{D}_2$ is its dense subset.

Let $\mathbf{D} \in \mathrm{L}^\infty(\Omega; \mathrm{M}_r(\mathbf{C}))$ and $\mathbf{A}_k \in \mathrm{W}^{1,\infty}(\Omega; \mathrm{M}_r(\mathbf{C}))$, $k \in 1..d$, be of the form

(F0) $$\mathbf{D} = \begin{bmatrix} \mathbf{D}^{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{D}^{22} \end{bmatrix} \,, \quad \mathbf{A}_k = \begin{bmatrix} \mathbf{A}_k^{11} & \mathbf{A}_k^{12} \\ \mathbf{A}_k^{21} & \mathbf{0} \end{bmatrix} \,, \ k \in 1..d \,,$$

where we have used that for $r \times r$ matrix $\mathbf{M}$, $\mathbf{M}^{ij}$ is its $r_i \times r_j$ submatrix, $i, j \in 1..2$. Moreover, let these matrices in addition satisfy

(F1) $$\mathbf{A}_k \text{ is hermitian: } \mathbf{A}_k^{21} = (\mathbf{A}_k^{12})^* \text{ and } \mathbf{A}_k^{11} = (\mathbf{A}_k^{11})^* \,,$$

(F2A) $$(\exists\, \mu_0 > 0) \quad \mathbf{D} + \mathbf{D}^* + \sum_{k=1}^{d} \partial_k \mathbf{A}_k \geqslant 2\mu_0 \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{r_2} \end{bmatrix} \qquad \text{(a.e. on } \Omega) \,,$$

(F2B) $$\mathbf{A}_k^{12} \text{ (and thus } \mathbf{A}_k^{21} \text{ as well) are constant matrices on } \Omega \,.$$

Let us now consider differential operators $T, \tilde{T} : \mathcal{D} \longrightarrow L$ given by

$$
\begin{aligned}
T\mathsf{u} :=& P_L\Big(\sum_{k=1}^{d} \partial_k(\mathbf{A}_k\mathsf{u}) + \mathbf{D}\mathsf{u}\Big) = \begin{bmatrix} P_{L_1}\Big(\sum_{k=1}^d \partial_k(\mathbf{A}_k^{11}\mathsf{u}^1 + \mathbf{A}_k^{12}\mathsf{u}^2) + \mathbf{D}^{11}\mathsf{u}^1\Big) \\ \sum_{k=1}^d \mathbf{A}_k^{21}\partial_k\mathsf{u}^1 + \mathbf{D}^{22}\mathsf{u}^2 \end{bmatrix}, \\
\tilde{T}\mathsf{u} :=& P_L\Big(-\sum_{k=1}^{d} \partial_k(\mathbf{A}_k^*\mathsf{u}) + (\mathbf{D}^* + \sum_{k=1}^{d}\partial_k\mathbf{A}_k^*)\mathsf{u}\Big) \\
=& \begin{bmatrix} P_{L_1}\Big(-\sum_{k=1}^d \partial_k(\mathbf{A}_k^{11}\mathsf{u}^1 + \mathbf{A}_k^{12}\mathsf{u}^2) + ((\mathbf{D}^{11})^* + \sum_{k=1}^d \partial_k\mathbf{A}_k^{11})\mathsf{u}^1\Big) \\ -\sum_{k=1}^d \mathbf{A}_k^{21}\partial_k\mathsf{u}^1 + (\mathbf{D}^{22})^*\mathsf{u}^2 \end{bmatrix},
\end{aligned}
$$

where $\mathsf{u} = [\mathsf{u}^1\ \mathsf{u}^2]^\top$ and $P_L = [P_{L_1}\ P_{L_2}]$ denotes the orthogonal projection on the space L, with P_{L_1} being the orthogonal projection on the space L_1 in $\mathrm{L}^2(\Omega;\mathbf{C}^{r_1})$, while P_{L_2} is the identity operator on $\mathrm{L}^2(\Omega;\mathbf{C}^{r_2})$. In Example 2 we have shown that these operators extend to operators (denoted by the same symbol) $T, \tilde{T} : L \longrightarrow \mathcal{D}' = \mathcal{D}_1' \times \mathcal{D}_2'$ given by

$$
\begin{aligned}
T\mathsf{u} :=& \mathcal{R}\Big(\sum_{k=1}^{d} \partial_k(\mathbf{A}_k\mathsf{u}) + \mathbf{D}\mathsf{u}\Big) = \begin{bmatrix} \mathcal{R}_1\Big(\sum_{k=1}^d \partial_k(\mathbf{A}_k^{11}\mathsf{u}^1 + \mathbf{A}_k^{12}\mathsf{u}^2) + \mathbf{D}^{11}\mathsf{u}^1\Big) \\ \sum_{k=1}^d \mathbf{A}_k^{21}\partial_k\mathsf{u}^1 + \mathbf{D}^{22}\mathsf{u}^2 \end{bmatrix}, \\
\tilde{T}\mathsf{u} :=& \mathcal{R}\Big(-\sum_{k=1}^{d} \partial_k(\mathbf{A}_k^*\mathsf{u}) + (\mathbf{D}^* + \sum_{k=1}^{d}\partial_k\mathbf{A}_k^*)\mathsf{u}\Big) \\
=& \begin{bmatrix} \mathcal{R}_1\Big(-\sum_{k=1}^d \partial_k(\mathbf{A}_k^{11}\mathsf{u}^1 + \mathbf{A}_k^{12}\mathsf{u}^2) + ((\mathbf{D}^{11})^* + \sum_{k=1}^d \partial_k\mathbf{A}_k^{11})\mathsf{u}^1\Big) \\ -\sum_{k=1}^d \mathbf{A}_k^{21}\partial_k\mathsf{u}^1 + (\mathbf{D}^{22})^*\mathsf{u}^2 \end{bmatrix},
\end{aligned}
$$

where ∂_k denotes the distributional derivative and $\mathcal{R}_1 : \mathcal{D}'(\Omega;\mathbf{C}^{r_1}) \longrightarrow \mathcal{D}_1'$ is the restriction operator, as above.

Similarly as it has already been shown in Example 2, by (F2A) we get that for any $\mathsf{u} \in L$ we have

$$\langle\, (T+\tilde{T})\mathsf{u} \mid \mathsf{u}\,\rangle_L \geqslant 2\mu_0\|\mathsf{u}^2\|_{L_2}^2\,,$$

which is a weaker condition than (T3). However, this is still sufficient to obtain L_2-coercivity of both T and $\tilde{T}$ on suitable subspaces. In particular, let V and $\widetilde{V}$ satisfy (V), then for $\mathsf{u} \in V$ we have

$$\mathsf{Re}\,\langle\, T\mathsf{u} \mid \mathsf{u}\,\rangle_L = \frac{1}{2}\langle\, (T+\tilde{T})\mathsf{u} \mid \mathsf{u}\,\rangle_L + \frac{1}{2}{}_{W'}\langle\, D\mathsf{u}, \mathsf{u}\,\rangle_W \geqslant \mu_0\|\mathsf{u}^2\|_{L_2}^2\,,$$

while for $\mathsf{u} \in \widetilde{V}$ we get

$$\mathsf{Re}\,\langle\, \tilde{T}\mathsf{u} \mid \mathsf{u}\,\rangle_L = \frac{1}{2}\langle\, (T+\tilde{T})\mathsf{u} \mid \mathsf{u}\,\rangle_L - \frac{1}{2}{}_{W'}\langle\, D\mathsf{u}, \mathsf{u}\,\rangle_W \geqslant \mu_0\|\mathsf{u}^2\|_{L_2}^2\,.$$

However, in order to control the norm of the first component we need one more condition. First, let V and $\widetilde{V}$ be two subspaces satisfying (V). Then we assume that there exists a constant $C > 0$ such that

$$
\text{(T3A)}\qquad
\begin{aligned}
&(\forall\,\mathsf{u} \in V) \quad \|\mathsf{u}^1\|_{L_1} \leqslant C\Big(\sqrt{\mathsf{Re}\,\langle\, T\mathsf{u} \mid \mathsf{u}\,\rangle_L} + \|A^{21}\mathsf{u}^1\|_{L_2}\Big)\,, \\
&(\forall\,\mathsf{u} \in \widetilde{V}) \quad \|\mathsf{u}^1\|_{L_1} \leqslant C\Big(\sqrt{\mathsf{Re}\,\langle\, \tilde{T}\mathsf{u} \mid \mathsf{u}\,\rangle_L} + \|A^{21}\mathsf{u}^1\|_{L_2}\Big)\,,
\end{aligned}
$$

where $A^{21} := \sum_{k=1}^d \mathbf{A}_k^{21}\partial_k$. The above inequalities are nontrivial since for $\mathsf{u} \in V$ we have

$$\|A^{21}\mathsf{u}^1\|_{L_2} = \|(T\mathsf{u})^2 - \mathbf{D}^{22}\mathsf{u}^2\|_{L_2} \leqslant \|T\mathsf{u}\|_L + C_1\|\mathsf{u}^2\|_{L_2} < \infty\,,$$

where $C_1 := \|\mathbf{D}^{22}\|_{\mathrm{L}^\infty(\Omega;\mathrm{M}_{r_2}(\mathbf{C}))}$, and similarly for $\mathsf{u} \in \widetilde{V}$.

Applying the above inequality to (T3A), for $\mathsf{u} \in V$ we get

$$\begin{aligned}
\frac{1}{C}\|\mathsf{u}^1\|_{L_1} \leqslant& \sqrt{\|T\mathsf{u}\|_L\|\mathsf{u}\|_L} + \|T\mathsf{u}\|_L + C_1\|\mathsf{u}^2\|_{L_2} \\
\leqslant& \frac{C}{2}\|T\mathsf{u}\|_L + \frac{1}{2C}\|\mathsf{u}\|_L + \|T\mathsf{u}\|_L + C_1\|\mathsf{u}^2\|_{L_2} \\
\leqslant& \Bigl(1+\frac{C}{2}\Bigr)\|T\mathsf{u}\|_L + \Bigl(C_1 + \frac{1}{2C}\Bigr)\|\mathsf{u}^2\|_{L_2} + \frac{1}{2C}\|\mathsf{u}^1\|_{L_1} ,
\end{aligned}$$

where in the second line we have used the Young inequality. Hence,

$$\begin{aligned}
\|\mathsf{u}^1\|_{L_1} \leqslant& 2C\Bigl(1+\frac{C}{2}\Bigr)\|T\mathsf{u}\|_L + 2C\Bigl(C_1+\frac{1}{2C}\Bigr)\|\mathsf{u}^2\|_{L_2} \\
\leqslant& C_3\Bigl(\|T\mathsf{u}\|_L + \|\mathsf{u}^2\|_{L_2}\Bigr) ,
\end{aligned}$$

with $C_3 := 2C\max\{1+\frac{C}{2}, C_1+\frac{1}{2C}\}$. It is easy to see that the above estimate remains valid on $\widetilde{V}$ (with the same constant C_3) if we replace T by $\tilde{T}$.

It remains to be seen that the above assertion implies that T is L-coercive on V, while $\tilde{T}$ is L-coercive on $\widetilde{V}$. Since the inequalities involved are the same for both T and $\tilde{T}$, we shall present this final computation only for T. Hence, for $\mathsf{u} \in V$ we have

$$\begin{aligned}
\mu_0\|\mathsf{u}^2\|_{L_2}^2 \leqslant& \mathsf{Re}\,\langle\, T\mathsf{u} \mid \mathsf{u}\,\rangle_L \\
\leqslant& \|T\mathsf{u}\|_L(\|\mathsf{u}^1\|_{L_1} + \|\mathsf{u}^2\|_{L_2}) \\
\leqslant& (1+C_3)\|T\mathsf{u}\|_L\|\mathsf{u}^2\|_{L_2} + C_3\|T\mathsf{u}\|_L^2 \\
\leqslant& \frac{\mu_0}{2}\|\mathsf{u}^2\|_{L_2}^2 + \Bigl(\frac{(1+C_3)^2}{2\mu_0} + C_3\Bigr)\|T\mathsf{u}\|_L^2 ,
\end{aligned}$$

where in the last line we have again applied the Young inequality. Thus, $\|\mathsf{u}^2\|_{L_2} \leqslant C_4\|T\mathsf{u}\|_L$ with $C_4 := \sqrt{\frac{(1+C_3)^2}{\mu_0^2} + \frac{2C_3}{\mu_0}}$. Therefore,

$$\|\mathsf{u}\|_L \leqslant \|\mathsf{u}^1\|_{L_1} + \|\mathsf{u}^2\|_{L_2} \leqslant (1+C_3)\|\mathsf{u}^2\|_{L_2} + C_3\|T\mathsf{u}\|_L \leqslant ((1+C_3)C_4 + C_3)\|T\mathsf{u}\|_L ,$$

and finally

$$\|T\mathsf{u}\|_L \geqslant \frac{1}{(1+C_3)C_4 + C_3 + 1}\|\mathsf{u}\|_V .$$

After obtaining the above estimate we have got all the ingredients needed to repeat the steps of the proof of [27, Theorem 3.1], i.e. to fulfill both conditions of Theorem 2(ii). Therefore, we just state the conclusion.

Theorem 3. *Let $\Omega \subseteq \mathbf{R}^d$ be an open and bounded set with Lipschitz boundary, and let $\mathbf{D} \in \mathrm{L}^\infty(\Omega; \mathrm{M}_r(\mathbf{C}))$ and $\mathbf{A}_k \in \mathrm{W}^{1,\infty}(\Omega; \mathrm{M}_r(\mathbf{C}))$, $k \in 1..d$, satisfy* (F0), (F1), (F2A), (F2B). *For $\mathcal{D} = \mathcal{D}_1 \times \mathcal{D}_2$, where $\mathcal{D}_2 := \mathrm{C}_c^\infty(\Omega; \mathbf{C}^{r_1})$ and $\mathcal{D}_1$ is a closed subspace in $\mathrm{C}_c^\infty(\Omega; \mathbf{C}^{r_2})$ $(r = r_1 + r_2)$, and $L := \mathsf{Cl}_{\,\mathrm{L}^2(\Omega;\mathbf{C}^r)}\mathcal{D}$ we define $T, \tilde{T} : L \longrightarrow \mathcal{D}'$ by*

$$T\mathsf{u} = \mathcal{R}\Bigl(\sum_{k=1}^d \partial_k(\mathbf{A}_k\mathsf{u}) + \mathbf{D}\mathsf{u}\Bigr) \qquad \text{and} \qquad \tilde{T}\mathsf{u} = \mathcal{R}\Bigl(-\sum_{k=1}^d \partial_k(\mathbf{A}_k^*\mathsf{u}) + (\mathbf{D}^* + \sum_{k=1}^d \partial_k\mathbf{A}_k^*)\mathsf{u}\Bigr) .$$

Let V and $\widetilde{V}$ satisfy (V). *If in addition* (T3A) *is fulfilled, then operators $T_{|_V} : V \longrightarrow L$ and $\tilde{T}_{|_{\widetilde{V}}} : \widetilde{V} \longrightarrow L$ are isomorphisms.* ∎

Different representations of boundary conditions

Let us assume that operators T and $\tilde{T}$ satisfy (T1)–(T3), as above. We have already seen how to assign boundary conditions via conditions (V) which are the abstract counterparts of conditions (FV). Next we shall present the other two options, being appropriate analogues of (FX) and (FM).

A subspace V of graph space W is called *maximal nonnegative* if

$$\text{(X1)} \qquad (\forall u \in V) \quad {}_{W'}\langle Du, u \rangle_W \geqslant 0$$

and

$$\text{(X2)} \qquad \text{there is no subspace which is larger than } V \text{ and satisfies (X1)}.$$

These conditions can be expressed using an indefinite scalar product which then can be fitted in the theory of Kreĭn spaces that proved to be the right tool for establishing the equivalence between (V) and (X) in the real case [2]. Since the theory of Kreĭn spaces is also valid in complex spaces, the proof of [2, Theorem 2] completely fits into that framework as well. Therefore, the equivalence of (V) and (X), i.e. the statement of [2, Theorem 2] is valid also in the complex setting.

The conditions corresponding to (FM) read: let $M \in \mathcal{L}(W; W')$ be an operator satisfying

$$\text{(M1)} \qquad (\forall u \in W) \quad {}_{W'}\langle (M + M^*)u, u \rangle_W \geqslant 0$$

and

$$\text{(M2)} \qquad W = \ker(D - M) + \ker(D + M),$$

where for any $u, v \in W$ we have ${}_{W'}\langle M^*u, v \rangle_W = \overline{{}_{W'}\langle Mv, u \rangle_W}$. It is straightforward to see that in (M1) we could have equivalently considered $\mathsf{Re}\, {}_{W'}\langle Mu, u \rangle_W \geqslant 0$ or even $\mathsf{Re}\, {}_{W'}\langle M^*u, u \rangle_W \geqslant 0$.

Similarly, as in the classical theory, the only explicit difference in comparison to the real case is the condition (M1), while the remaining differences are incorporated via corresponding complex spaces and antilinear functionals.

The difference in condition (M1) resulted in some changes in the part of the proof of [27, Lemma 4.1] which, for the sake of completeness, we present here.

Lemma 3. *If M satisfies* (M), *then*

$$\ker D = \ker M = \ker M^* , \quad \textit{and}$$

$$\operatorname{im} D = \operatorname{im} M = \operatorname{im} M^* .$$

Dem. Let us prove only $\ker M = \ker M^*$, while the rest of the proof goes along the same lines as in [27, Lemma 4.1].

Take $u \in \ker M$; then for every $v \in W$ and $\lambda \in \mathbf{R}$ we have

$$0 \leqslant {}_{W'}\langle (M + M^*)(v + \lambda u), v + \lambda u \rangle_W = {}_{W'}\langle (M + M^*)v, v \rangle_W + 2\lambda \mathsf{Re}\, {}_{W'}\langle M^*u, v \rangle_W ,$$

which implies $\mathsf{Re}\, {}_{W'}\langle M^*u, v \rangle_W = 0$. Indeed, if this were not the case, we could choose λ such that the inequality above is disrupted. Similarly, from

$$0 \leqslant {}_{W'}\langle (M + M^*)(v + i\lambda u), v + i\lambda u \rangle_W = {}_{W'}\langle (M + M^*)v, v \rangle_W - 2\lambda \mathsf{Im}\, {}_{W'}\langle M^*u, v \rangle_W$$

we get $\mathsf{Im}\, {}_{W'}\langle M^*u, v \rangle_W = 0$, and therefore ${}_{W'}\langle M^*u, v \rangle_W = 0$. From the arbitrariness of $v \in W$ we finally have $u \in \ker M^*$, hence $\ker M \subseteq \ker M^*$.

The opposite inclusion can be established in the same manner.

Q.E.D.

Since $\ker M = \ker D = W_0$, it is a common practice to call M *the boundary operator* as well. Moreover, by the last lemma and standard arguments of operator theory, we can show that, under (M1), the condition (M2) is equivalent to the one where M is replaced by M^* (see [27, Lemma 4.2]). Therefore, M and M^* play symmetric roles.

If (M) is satisfied, then, as in the real case [27, Theorem 4.2], one can prove the existence of V and $\widetilde{V}$ satisfying (V).

Lemma 4. *Let (T1)–(T3) hold and let $M \in \mathcal{L}(W; W')$ satisfies (M). Then the subspaces*

$$V := \ker(D - M) \quad \textit{and} \quad \widetilde{V} := \ker(D + M^*)$$

satisfy (V).

Dem. For an arbitrary $v \in V$ we have

$$_{W'}\langle\, Dv, v\,\rangle_W = \mathsf{Re}\, _{W'}\langle\, Dv, v\,\rangle_W = \mathsf{Re}\, _{W'}\langle\, Mv, v\,\rangle_W \geqslant 0\,,$$

and analogously for $\widetilde{V}$, which implies (V1).

The proof of (V2) is the same as the proof in [27, Theorem 4.2].

Q.E.D.

In the (standard) real setting, the opposite implication happens to be more tedious, since most of the proofs are based on the explicit construction of an operator M satisfying (M), which appears to be difficult without any additional condition. One approach is presented in [27, Theorem 4.3], and the other one in [2, Theorem 8]. Since the statements of both theorems are quite technical and involve certain projectors, while the proof of both statements naturally extends to the complex setting, we shall omit them here. The final conclusion is that the statements of Theorem 4.3, Lemma 4.4 and Corollary 4.1 in [27] are valid also in the complex setting, as well as [2, Theorem 8], with the remark that all scalar and dual products in the statements should be consider to be linear in the first and antilinear in the second argument.

Nevertheless, the equivalence between (M) and (V) has been shown in [2, Corollary 3], where the conclusion followed by the application of some abstract results from the theory of Kreĭn spaces. As we have already mentioned, the theory of Kreĭn spaces is developed for complex spaces as well, thus the result of [2, Corollary 3] follows in the same manner also in the complex setting, providing that in the complex case the equivalence between (V) and (M) is preserved. However, this is an abstract result, thus the explicit formula for M is known only in the situations mentioned in the previous paragraph.

Non-stationary complex Friedrichs systems

In [13] non-stationary theory for Friedrichs systems was developed via semigroup theory. Since the semigroup theory can be applied on arbitrary complex Hilbert spaces, the results from [13] naturally extend to the complex space. More precisely, let us consider the abstract Cauchy problem:

$$\text{(P)} \qquad \begin{cases} \mathsf{u}'(t) + T\mathsf{u}(t) = \mathsf{f} \\ \qquad\quad \mathsf{u}(0) = \mathsf{u}_0 \end{cases},$$

where $\mathsf{u} : [0,\tau\rangle \longrightarrow L$, for $\tau > 0$, is the unknown function, while the right-hand side $\mathsf{f} : \langle 0,\tau\rangle \longrightarrow L$ (or $\mathsf{f} : \langle 0,\tau\rangle \times L \longrightarrow L$ in the semi-linear case), the initial data $\mathsf{u}_0 \in L$ and the abstract operator T, not depending on the time variable t, are given.

In the context of this non-stationary theory, we can slightly relax the definition of the abstract Friedrichs operator. Namely, if T satisfies (T1)–(T2) and a weaker positivity assumption

$$\text{(T3}'\text{)} \qquad (\forall\, \varphi \in \mathcal{D}) \quad \langle\, (T + \tilde{T})\varphi \mid \varphi\,\rangle_L \geqslant 0\,,$$

we still call T *the abstract Friedrichs operator*.

Remark 4. If the condition (T3′) is disrupted, but $\langle (T+\tilde{T})\varphi \mid \varphi \rangle_L$ is still uniformly bounded from below, by substituting $\mathsf{v} := e^{-\lambda t}\mathsf{u}$, for some suitable $\lambda > 0$, we can get that $T + \lambda\mathbf{I}$ (where $\mathbf{I}$ is the identity operator) satisfies (T3) (and then also (T3′)) as presented in [13]. Therefore, without loss of generality, in all examples we shall assume that a weaker condition (T3′) is satisfied. ∎

Similarly as in [13, Theorem 2], we can show that $-T$ is the infinitesimal generator of a contraction C_0-semigroup on L, where T is an abstract Friedrichs operator in the above sense, which implies the solvability of (P). We summarise the existence and uniqueness results in the following theorem [17, 34].

Theorem 4. *Let T be an operator that satisfies* (T1)–(T2) *and* (T3′)*, and V a subspace of its graph space satisfying* (V)*.*

a) If $\mathsf{f} \in \mathrm{L}^1(\langle 0, \tau\rangle; L)$, then for every $\mathsf{u}_0 \in L$ the problem (P) *has the unique mild solution $\mathsf{u} \in \mathrm{C}([0,\tau]; L)$ given by*

$$\mathsf{u}(t) = S(t)\mathsf{u}_0 + \int_0^t S(t-s)\mathsf{f}(s)ds\,, \qquad t \in [0,\tau]\,,$$

where $(S(t))_{t\geqslant 0}$ is a contraction C_0-semigroup generated by $-T_{|_V}$.

b) If additionally $\mathsf{u}_0 \in V$ and $\mathsf{f} \in \mathrm{W}^{1,1}(\langle 0,\tau\rangle; L) \cup \Big(\mathrm{C}([0,\tau]; L) \cap \mathrm{L}^1(\langle 0,\tau\rangle; V)\Big)$, with V equipped by the graph norm, then the above weak solution is the classical solution of (P) *on $[0, \tau\rangle$.*

c) If $\mathsf{f} : [0,\tau] \times L \longrightarrow L$ is continuous and locally Lipschitz in the last variable, with Lipschitz constant not depending on the first variable, then for every $\mathsf{u}_0 \in L$ there exists τ_{max}, such that the semi-linear problem (P) *has unique mild solution $\mathsf{u} \in \mathrm{C}([0,\tau_{max}]; L)$.* ∎

Remark 5. Under an additional assumption on f (see eg. [34, Ch. 6] for more details) we shall have existence and uniqueness of *strong* or even *classical solution* of the semi-linear problem. For some estimates on *time τ_{max} of existence of solution* see [12]. ∎

3. Applications of complex Friedrichs systems

Dirac system

Let us consider a system of equations

$$\text{(DS)} \qquad a\boldsymbol{\gamma}^0\partial_t\boldsymbol{\psi} + \boldsymbol{\gamma}^1\partial_1\boldsymbol{\psi} + \boldsymbol{\gamma}^2\partial_2\boldsymbol{\psi} + \boldsymbol{\gamma}^3\partial_3\boldsymbol{\psi} + \mathbf{B}\boldsymbol{\psi} = \mathsf{f}\,,$$

where $\boldsymbol{\psi} : [0,\tau\rangle \times \mathbf{R}^3 \longrightarrow \mathbf{C}^4$ is the unknown function, while the right hand side $\mathsf{f} : \langle 0,\tau\rangle \longrightarrow \mathbf{C}^4$ (or $\mathsf{f} : \langle 0,\tau\rangle \times \mathbf{C}^4 \longrightarrow \mathbf{C}^4$ in the semi-linear case), $a > 0$ and $\mathbf{B} = \begin{bmatrix} b_1\mathbf{I} & 0 \\ 0 & b_2\mathbf{I} \end{bmatrix}$, with $b_1, b_2 : \mathbf{R}^3 \longrightarrow \mathbf{C}$ and $\mathbf{I}$ denotes 2×2 unit matrix, are given. Moreover, $\boldsymbol{\gamma}^0$ and $\boldsymbol{\gamma}^k$, $k \in 1..3$, are constant 4×4 matrices of the form:

$$\boldsymbol{\gamma}^0 = \begin{bmatrix} \mathbf{I} & 0 \\ 0 & -\mathbf{I} \end{bmatrix}\,, \quad \boldsymbol{\gamma}^k = \begin{bmatrix} 0 & \boldsymbol{\sigma}^k \\ -\boldsymbol{\sigma}^k & 0 \end{bmatrix}\,,$$

where $\boldsymbol{\sigma}^k$ are Pauli matrices:

$$\boldsymbol{\sigma}^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\,, \quad \boldsymbol{\sigma}^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\,, \quad \boldsymbol{\sigma}^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\,.$$

Remark 6. If $\mathsf{f} \equiv 0$, $b_1 = b_2 = \dfrac{imc}{\hbar}$ and $a = \frac{1}{c\hbar}$, where $\hbar$ represents the Planck constant divided by 2π and c is the speed of light, then $\boldsymbol{\psi} = \boldsymbol{\psi}(t, \mathbf{x})$ is the wave function for electron of the rest mass m, while system (DS) is then the well known (linear) free Dirac equation [28]. ∎

The system above can be written in the form

$$\partial_t \boldsymbol{\psi} + T\boldsymbol{\psi} = \mathsf{F}\,,$$

where $\mathsf{F} = \frac{1}{a}\boldsymbol{\gamma}^0\mathsf{f}$, while $T\boldsymbol{\psi} = \sum_{k=1}^{3} \frac{1}{a}\boldsymbol{\gamma}^0\boldsymbol{\gamma}^k\partial_k\boldsymbol{\psi} + \frac{1}{a}\boldsymbol{\gamma}^0\mathbf{B}\boldsymbol{\psi} = \sum_{k=1}^{3}\mathbf{A}_k\partial_k\boldsymbol{\psi} + \mathbf{D}\boldsymbol{\psi}$ with

$$\mathbf{A}_k := \frac{1}{a}\begin{bmatrix} 0 & \boldsymbol{\sigma}^k \\ \boldsymbol{\sigma}^k & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{D} = \frac{1}{a}\boldsymbol{\gamma}^0\mathbf{B}\,.$$

Operator T fits in our framework of complex non-stationary Friedrichs systems. Indeed, the condition (F1) is trivially satisfied, while the positivity condition (F2), if not valid, can be obtained by the substitution $\mathsf{v} = e^{-\lambda t}\boldsymbol{\psi}$, as remarked above (see Remark 4).

By Example 1, we can take L to be $\mathrm{H}^s(\mathbf{R}^3;\mathbf{C}^4)$ for some $s \in \mathbf{R}$, and $\mathcal{D} = \mathrm{C}_c^\infty(\mathbf{R}^3;\mathbf{C}^4)$, while the graph space is given by

$$W = \Big\{\mathsf{u} \in \mathrm{H}^s(\mathbf{R}^3;\mathbf{C}^4) : \sum_{k=1}^{3}\mathbf{A}_k\partial_k\mathsf{u} \in \mathrm{H}^s(\mathbf{R}^3;\mathbf{C}^4)\Big\}\,.$$

Furthermore, again relying on the mentioned example, the boundary operator D is trivial, hence for $V = \widetilde{V} = W$ we have that (V) is satisfied, fulfilling all assumptions of Theorem 4. Therefore, we get the result of existence and uniqueness of the solution to (DS). For example, in the case of a linear equation and classical solutions, we have the following result.

Theorem 5. *For $s \in \mathbf{R}$, let $\mathsf{f} \in \mathrm{W}^{1,1}(\langle 0,\tau\rangle;\mathrm{H}^s(\mathbf{R}^3;\mathbf{C}^4)) \cup \Big(\mathrm{C}([0,\tau];\mathrm{H}^s(\mathbf{R}^3;\mathbf{C}^4)) \cap \mathrm{L}^1(\langle 0,\tau\rangle;W)\Big)$ and $\boldsymbol{\psi}_0 \in W$, where W is given as above. Then the abstract initial-value problem*

$$\begin{cases} a\boldsymbol{\gamma}^0\boldsymbol{\psi}' + \boldsymbol{\gamma}^1\partial_1\boldsymbol{\psi} + \boldsymbol{\gamma}^2\partial_2\boldsymbol{\psi} + \boldsymbol{\gamma}^3\partial_3\boldsymbol{\psi} + \mathbf{B}\boldsymbol{\psi} = \mathsf{f} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \boldsymbol{\psi}(0) = \boldsymbol{\psi}_0 \end{cases}$$

has the unique classical solution $\boldsymbol{\psi} \in \mathrm{C}^1(\langle 0,\tau\rangle;\mathrm{H}^s(\mathbf{R}^3;\mathbf{C}^4)) \cap \mathrm{C}([0,\tau];\mathrm{H}^s(\mathbf{R}^3;\mathbf{C}^4))$ such that $\boldsymbol{\psi}(t) \in W$ for $0 < t < \tau$. ∎

Remark 7. In the case of the free Dirac equation we have $\mathsf{f} \equiv \mathsf{0}$, and thus one only needs to ensure $\boldsymbol{\psi}_0 \in W$ for the existence and uniqueness of the classical solution. ∎

If we consider a semilinear equation, with a quadratic nonlinear term $\mathsf{f}(t,\boldsymbol{\psi}) = \mathbf{Q}\boldsymbol{\psi}\cdot\boldsymbol{\psi}$ in (DS), for some constant complex 4×4 matrix $\mathbf{Q}$, by the following lemma and Theorem 4(c), for $s = 2$ we can get the existence and uniqueness of the mild solution of the semilinear abstract Cauchy problem (DS).

Lemma 5. *For $d \leqslant 3$, quadratic form $q(\mathsf{u}) = \mathbf{Q}\mathsf{u}\cdot\mathsf{u}$, where $\mathbf{Q}$ is any complex $r\times r$ matrix, is a locally Lipschitz function from $\mathrm{H}^2(\mathbf{R}^d;\mathbf{C}^r)$ to $\mathrm{H}^2(\mathbf{R}^d;\mathbf{C})$.*

Dem. Since $d \leqslant 3$ the following two inequalities are valid:

$$\begin{aligned} &(\exists C > 0)(\forall \mathsf{u} \in \mathrm{H}^2(\mathbf{R}^d;\mathbf{C}^r)) \quad \|\mathsf{u}\|_{\mathrm{L}^\infty} \leqslant C\|\mathsf{u}\|_{\mathrm{H}^2}\,, \\ &(\exists D > 0)(\forall \mathsf{u} \in \mathrm{H}^2(\mathbf{R}^d;\mathbf{C}^r)) \quad \|\mathsf{u}\|_{\mathrm{W}^{1,4}} \leqslant D\|\mathsf{u}\|_{\mathrm{L}^\infty}^{\frac{1}{2}}\|\mathsf{u}\|_{\mathrm{H}^2}^{\frac{1}{2}}\,, \end{aligned}$$

where, as well as in the rest of the proof, for simplicity we omit to write down the domain and codomain of corresponding spaces in the notation for norms. The first one is due to the Sobolev embedding theorem and the second one arises from interpolation inequalities. All constants in the proof below will be marked with the same letter C.

For $\mathsf{u},\mathsf{v}\in \mathrm{H}^2(\mathbf{R}^d;\mathbf{C}^r)$ let us estimate the L^2 norm of $\mathbf{Q}\mathsf{u}\cdot\mathsf{v}$, its first and second derivatives. With

$$\|\mathbf{Q}\mathsf{u}\cdot\mathsf{v}\|_{\mathrm{L}^2}=\Big\|\sum_{i,j=1}^{r} q_{ij}u^i\overline{v}^j\Big\|_{\mathrm{L}^2}\leqslant\sum_{i,j=1}^{r}|q_{ij}|\|u^i\|_{\mathrm{L}^\infty}\|v^j\|_{\mathrm{L}^2}\leqslant C|\mathbf{Q}|\|\mathsf{u}\|_{\mathrm{L}^\infty}\|\mathsf{v}\|_{\mathrm{H}^2}\leqslant C\|\mathsf{u}\|_{\mathrm{H}^2}\|\mathsf{v}\|_{\mathrm{H}^2}\,,$$

we have provided the estimate for the function, while by

$$\begin{aligned}\|\partial_k(\mathbf{Q}\mathsf{u}\cdot\mathsf{v})\|_{\mathrm{L}^2}=&\Big\|\sum_{i,j=1}^{r} q_{ij}\left((\partial_k u^i)\overline{v}^j+u^i(\partial_k\overline{v}^j)\right)\Big\|_{\mathrm{L}^2}\\ \leqslant&|\mathbf{Q}|\sum_{i,j=1}^{r}\Big(\|\partial_k u^i\|_{\mathrm{L}^2}\|v^j\|_{\mathrm{L}^\infty}+\|u^i\|_{\mathrm{L}^\infty}\|\partial_k v^j\|_{\mathrm{L}^2}\Big)\\ \leqslant&|\mathbf{Q}|\sum_{i,j=1}^{r}\Big(\|\mathsf{u}\|_{\mathrm{H}^1}\|\mathsf{v}\|_{\mathrm{L}^\infty}+\|\mathsf{u}\|_{\mathrm{L}^\infty}\|\mathsf{v}\|_{\mathrm{H}^1}\Big)\\ \leqslant&C\|\mathsf{u}\|_{\mathrm{H}^2}\|\mathsf{v}\|_{\mathrm{H}^2}\,,\end{aligned}$$

where $k\in 1..d$, we have bounded its first derivatives. Similarly, for the second derivatives we have

$$\begin{aligned}\|\partial_l\partial_k(\mathbf{Q}\mathsf{u}\cdot\mathsf{v})\|_{\mathrm{L}^2}=&\Big\|\sum_{i,j=1}^{r} q_{ij}\left((\partial_l\partial_k u^i)\overline{v}^j+(\partial_k u^i)(\partial_l\overline{v}^j)+(\partial_l u^i)(\partial_k\overline{v}^j)+u^i(\partial_l\partial_k\overline{v}^j)\right)\Big\|_{\mathrm{L}^2}\\ \leqslant&|\mathbf{Q}|\sum_{i,j=1}^{r}\Big(\|\mathsf{u}\|_{\mathrm{H}^2}\|\mathsf{v}\|_{\mathrm{L}^\infty}+\|\partial_k u^i\|_{\mathrm{L}^4}\|\partial_l\overline{v}^j\|_{\mathrm{L}^4}+\|\partial_l u^i\|_{\mathrm{L}^4}\|\partial_k\overline{v}^j\|_{\mathrm{L}^4}+\|\mathsf{u}\|_{\mathrm{L}^\infty}\|\mathsf{v}\|_{\mathrm{H}^2}\Big)\\ \leqslant&C\Big(\|\mathsf{u}\|_{\mathrm{H}^2}\|\mathsf{v}\|_{\mathrm{L}^\infty}+2\|\mathsf{u}\|_{\mathrm{W}^{1,4}}\|\mathsf{v}\|_{\mathrm{W}^{1,4}}+\|\mathsf{u}\|_{\mathrm{L}^\infty}\|\mathsf{v}\|_{\mathrm{H}^2}\Big)\\ \leqslant&C\Big(\|\mathsf{u}\|_{\mathrm{H}^2}\|\mathsf{v}\|_{\mathrm{L}^\infty}+2\tilde{C}\|\mathsf{u}\|_{\mathrm{L}^\infty}^{\frac12}\|\mathsf{u}\|_{\mathrm{H}^2}^{\frac12}\|\mathsf{v}\|_{\mathrm{L}^\infty}^{\frac12}\|\mathsf{v}\|_{\mathrm{H}^2}^{\frac12}+\|\mathsf{u}\|_{\mathrm{L}^\infty}\|\mathsf{v}\|_{\mathrm{H}^2}\Big)\\ \leqslant&C\|\mathsf{u}\|_{\mathrm{H}^2}\|\mathsf{v}\|_{\mathrm{H}^2}\,,\end{aligned}$$

where $k,l\in 1..d$. Finally, for $\mathsf{u},\mathsf{v}\in \mathrm{H}^2(\mathbf{R}^d;\mathbf{C}^r)$ we get

$$\|\mathbf{Q}\mathsf{u}\cdot\mathsf{v}\|_{\mathrm{H}^2}\leqslant C\|\mathsf{u}\|_{\mathrm{H}^2}\|\mathsf{v}\|_{\mathrm{H}^2}\,.$$

By the previous estimate the boundedness of q is immediate, so it remains to prove the local Lipschitz property. Using the above estimate again, we get

$$\begin{aligned}\|\mathbf{Q}\mathsf{u}\cdot\mathsf{u}-\mathbf{Q}\mathsf{v}\cdot\mathsf{v}\|_{\mathrm{H}^2}\leqslant&\|\mathbf{Q}\mathsf{u}\cdot(\mathsf{u}-\mathsf{v})\|_{\mathrm{H}^2}+\|\mathbf{Q}(\mathsf{u}-\mathsf{v})\cdot\mathsf{v}\|_{\mathrm{H}^2}\\ \leqslant&C\Big(\|\mathsf{u}\|_{\mathrm{H}^2}\|\mathsf{u}-\mathsf{v}\|_{\mathrm{H}^2}+\|\mathsf{u}-\mathsf{v}\|_{\mathrm{H}^2}\|\mathsf{v}\|_{\mathrm{H}^2}\Big)\\ \leqslant&C\left(\|\mathsf{u}\|_{\mathrm{H}^2}+\|\mathsf{v}\|_{\mathrm{H}^2}\right)\|\mathsf{u}-\mathsf{v}\|_{\mathrm{H}^2}\,,\end{aligned}$$

which concludes the proof.

Q.E.D.

Dirac-Klein-Gordon system

In this example we consider the coupled Dirac-Klein-Gordon system of equations [21]

$$\begin{cases}-i\boldsymbol{\gamma}^0\partial_t\boldsymbol{\psi}-i\boldsymbol{\gamma}^1\partial_1\boldsymbol{\psi}-i\boldsymbol{\gamma}^2\partial_2\boldsymbol{\psi}-i\boldsymbol{\gamma}^3\partial_3\boldsymbol{\psi}+M\boldsymbol{\psi}=\phi\boldsymbol{\psi}\\ \qquad\qquad\qquad\qquad\qquad\partial_t^2\phi-\Delta\phi+m^2\phi=\boldsymbol{\psi}^*\boldsymbol{\gamma}^0\boldsymbol{\psi}\end{cases},$$

where unknown functions are $\boldsymbol{\psi}=\boldsymbol{\psi}(t,x):\mathbf{R}^{1+3}\longrightarrow\mathbf{C}^4$ and $\phi=\phi(t,x):\mathbf{R}^{1+3}\longrightarrow\mathbf{R}$, $M,m\in\mathbf{R}_0^+$ are given, while matrices $\boldsymbol{\gamma}^k, k\in 0..3$, are the same as in the previous example. As one can see, the Dirac-Klein-Gordon system is a semilinear system of equations.

Remark 8. In writing the Dirac-Klein-Gordon system as a Friedrichs system we are going to write the Dirac part and the wave part of the system as Friedrichs systems separately, and then use the block diagonal structure in order to get the required form. More precisely, it is easy to see that if

$$\partial_t \mathsf{u}_1 + T_1 \mathsf{u}_1 = \mathsf{f}_1$$

and

$$\partial_t \mathsf{u}_2 + T_2 \mathsf{u}_2 = \mathsf{f}_2$$

are two Friedrichs systems, where T_1 and T_2 are classical Friedrichs operators, then

$$\partial_t \mathsf{u} + T\mathsf{u} = \mathsf{f} \; ,$$

for $T = \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix}$, $\mathsf{u} = \begin{bmatrix} \mathsf{u}_1 \\ \mathsf{u}_2 \end{bmatrix}$ and $\mathsf{f} = \begin{bmatrix} \mathsf{f}_1 \\ \mathsf{f}_2 \end{bmatrix}$ is also a Friedrichs system and T is a classical Friedrichs operator. ∎

The first set of equations in the above Dirac-Klein-Gordon system is the Dirac system and, like in the preceding example, we can write it as a Friedrichs system

$$\partial_t \boldsymbol{\psi} + T_1 \boldsymbol{\psi} = \mathsf{f}_1 \; ,$$

where $T_1\boldsymbol{\psi} = \sum_{k=1}^{3} \partial_k \tilde{\mathbf{A}}_k \boldsymbol{\psi} + \mathbf{D}_1 \boldsymbol{\psi}$, with $\tilde{\mathbf{A}}_k = \begin{bmatrix} 0 & \boldsymbol{\sigma}^k \\ \boldsymbol{\sigma}^k & 0 \end{bmatrix}$ as before and $\mathbf{D}_1 = \begin{bmatrix} iM\mathbf{I} & 0 \\ 0 & -iM\mathbf{I} \end{bmatrix}$, while $\mathsf{f}_1 = i\phi\gamma^0\boldsymbol{\psi}$. The last equation in the Dirac-Klein-Gordon system is the wave equation, for which we introduce the following substitution

$$\mathsf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \mathsf{v}_3 \end{bmatrix} = \begin{bmatrix} \phi \\ \partial_t \phi \\ -\nabla \phi \end{bmatrix} \; ,$$

in order to get a non-stationary Friedrichs system

$$\partial_t \mathsf{v} + T_2 \mathsf{v} = \mathsf{f}_2 \; ,$$

where $T_2\mathsf{v} = \sum_{k=1}^{3} \partial_k \bar{\mathbf{A}}_k \mathsf{v} + \mathbf{D}_2 \mathsf{v}$ with

$$\bar{\mathbf{A}}_1 = \begin{bmatrix} 0&0&0&0&0 \\ 0&0&1&0&0 \\ 0&1&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{bmatrix} , \quad \bar{\mathbf{A}}_2 = \begin{bmatrix} 0&0&0&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&0 \\ 0&1&0&0&0 \\ 0&0&0&0&0 \end{bmatrix} , \quad \bar{\mathbf{A}}_3 = \begin{bmatrix} 0&0&0&0&0 \\ 0&0&0&0&1 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&1&0&0&0 \end{bmatrix} ,$$

and

$$\mathbf{D}_2 = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 \\ m^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} ,$$

while $\mathsf{f}_2 = \begin{bmatrix} 0 & \boldsymbol{\psi}^*\gamma^0\boldsymbol{\psi} & 0 & 0 & 0 \end{bmatrix}^\top$. Now, by using the block diagonal structure, we can write the Dirac-Klein-Gordon system as an evolution Friedrichs system

$$\partial_t \mathsf{u} + T\mathsf{u} = \mathsf{f} \, ,$$

where $\mathsf{u} = [\,\boldsymbol{\psi} \quad \mathsf{v}\,]^\top$ and the operator T is a classical Friedrichs operator with $\mathbf{A}_k = \begin{bmatrix} \tilde{\mathbf{A}}_k & 0 \\ 0 & \bar{\mathbf{A}}_k \end{bmatrix}$, $k \in 1..3$, $\mathbf{D} = \begin{bmatrix} \mathbf{D}_1 & 0 \\ 0 & \mathbf{D}_2 \end{bmatrix}$ and $\mathsf{f} = [\,\mathsf{f}_1 \quad \mathsf{f}_2\,]^\top$. All matrices in the system are hermitian and constant so condition (F1) is trivially satisfied. Condition (F2) can be obtained by substituting $\mathsf{v} = e^{-\lambda t}\mathsf{u}$, for λ large enough, if needed. For $L = \mathrm{H}^2(\mathbf{R}^3; \mathbf{C}^9)$, we have

$$W = \left\{\boldsymbol{\psi} \in \mathrm{H}^2(\mathbf{R}^3; \mathbf{C}^4) : T_1\boldsymbol{\psi} \in \mathrm{H}^2(\mathbf{R}^3; \mathbf{C}^4)\right\} \times \mathrm{H}^2(\mathbf{R}^3) \times \mathrm{H}^3(\mathbf{R}^3) \times \mathrm{H}^2_{\mathrm{div}}(\mathbf{R}^3; \mathbf{C}^3)\;,$$

where $\mathrm{H}^2_{\mathrm{div}}(\mathbf{R}^3; \mathbf{C}^3) = \{\mathsf{u} \in \mathrm{H}^2(\mathbf{R}^3; \mathbf{C}^3) : \operatorname{div}\mathsf{u} \in \mathrm{H}^2(\mathbf{R}^3; \mathbf{C})\}$. Function f is given by

$$\begin{bmatrix} iu_5u_1 \\ iu_5u_2 \\ -iu_5u_3 \\ -iu_5u_4 \\ 0 \\ |u_1|^2 + |u_2|^2 - |u_3|^2 - |u_4|^2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

and it is a locally Lipshitz function on $\mathrm{H}^2(\mathbf{R}^3; \mathbf{C}^9)$, by Lemma 5. Therefore, by Theorem 4(c) we have the existence and uniqueness result.

Maxwell-Dirac system

Let us now consider a coupled Maxwell-Dirac system of equations [22]

$$\begin{cases} -\dfrac{i}{2\pi}(\boldsymbol{\gamma}^0\partial_t + \boldsymbol{\gamma}^1\partial_1 + \boldsymbol{\gamma}^2\partial_2 + \boldsymbol{\gamma}^3\partial_3)\boldsymbol{\psi} + m\boldsymbol{\beta}\boldsymbol{\psi} = \displaystyle\sum_{k=0}^{3} \mathcal{A}_k\boldsymbol{\gamma}^k\boldsymbol{\psi}\;, \\ \qquad\qquad (-\partial_t^2 + \Delta)\mathcal{A}_k = -\,\boldsymbol{\gamma}^k\boldsymbol{\psi}\cdot\boldsymbol{\psi}\;, \quad k \in 0..3\;, \end{cases}$$

where $\boldsymbol{\gamma}^0 = \mathbf{I}$ and $\boldsymbol{\gamma}^k$, $k \in 1..3$ as before. The unknown functions are $\boldsymbol{\psi} : \mathbf{R}^{1+3} \longrightarrow \mathbf{C}^4$ and $\mathcal{A} = [\,\mathcal{A}_0 \quad \mathcal{A}_1 \quad \mathcal{A}_2 \quad \mathcal{A}_3\,]^\top$, while $m \geqslant 0$ and $\boldsymbol{\beta} = \begin{bmatrix} \mathbf{I} & 0 \\ 0 & -\mathbf{I} \end{bmatrix}$. We can use an analogous procedure to the one in the previous example to get the evolution Friedrichs system

$$\partial_t\mathsf{u} + T\mathsf{u} = \mathsf{F}\,,$$

where we introduce vector functions

$$\mathsf{u} = \begin{bmatrix} \boldsymbol{\psi} \\ \mathsf{v}_0 \\ \mathsf{v}_1 \\ \mathsf{v}_2 \\ \mathsf{v}_3 \end{bmatrix}\;, \quad \mathsf{v}_k = \begin{bmatrix} \mathcal{A}_k \\ \partial_t\mathcal{A}_k \\ -\nabla\mathcal{A}_k \end{bmatrix}\;, \quad \mathsf{f}_k = \begin{bmatrix} 0 \\ \boldsymbol{\gamma}^k\boldsymbol{\psi}\cdot\boldsymbol{\psi} \\ 0 \\ 0 \\ 0 \end{bmatrix}\;, \quad k \in 0..3\;,$$

and

$$\mathsf{F} = \begin{bmatrix} \displaystyle\sum_{k=0}^{3} \mathcal{A}_k\boldsymbol{\gamma}^k\boldsymbol{\psi} & \mathsf{f}_0 & \mathsf{f}_1 & \mathsf{f}_2 & \mathsf{f}_3 \end{bmatrix}^\top\;.$$

Operator T is a classical Friedrichs operator with $\mathbf{D} = \begin{bmatrix} \tilde{\mathbf{D}} & 0 \\ 0 & 0 \end{bmatrix} \in \mathrm{M}_{24}(\mathbf{C})$,$\tilde{\mathbf{D}} = \begin{bmatrix} 2\pi im & 0 \\ 0 & -2\pi im \end{bmatrix}$ and

$$\mathbf{A}_k = \begin{bmatrix} \tilde{\mathbf{A}}_k & 0 & 0 & 0 & 0 \\ 0 & \bar{\mathbf{A}}_k & 0 & 0 & 0 \\ 0 & 0 & \bar{\mathbf{A}}_k & 0 & 0 \\ 0 & 0 & 0 & \bar{\mathbf{A}}_k & 0 \\ 0 & 0 & 0 & 0 & \bar{\mathbf{A}}_k \end{bmatrix}\;, \quad k \in 1..3\;,$$

with $\tilde{\mathbf{A}}_k$ and $\bar{\mathbf{A}}_k$ as in the previous example. All matrices in the system are hermitian and constant so (F1) is trivially satisfied, and (F2) can be obtained by substitution $\mathsf{w} = e^{-\lambda t}\mathsf{u}$, for λ large enough. The spaces involved are

$$L = \mathrm{H}^2(\mathbf{R}^3; \mathbf{C}^{24})$$

and

$$W = \Big\{\boldsymbol{\psi} \in \mathrm{H}^2(\mathbf{R}^3) : \tilde{\mathbf{A}}_1\partial_1\boldsymbol{\psi} + \tilde{\mathbf{A}}_2\partial_2\boldsymbol{\psi} + \tilde{\mathbf{A}}_3\partial_3\boldsymbol{\psi} \in \mathrm{H}^2(\mathbf{R}^3)\Big\} \times [\mathrm{H}^2(\mathbf{R}^3) \times \mathrm{H}^3(\mathbf{R}^3) \times \mathrm{H}^2_{\mathrm{div}}(\mathbf{R}^3)]^4 \,.$$

Moreover, boundary operator D is trivial, F is a Lipschitz function due to Lemma 5 and as a consequence of Theorem 4(c) we have the existence and uniqueness result.

Time-harmonic Maxwell system

Let $\Omega \subseteq \mathbf{R}^3$ be an open and convex bounded set. In [32, Section 1.3.3] (see also [18], and [18, Section 5.2.5] for a simplified version) the stationary or the time-harmonic Maxwell equations on Ω are given as the following system

$$\text{(MS)} \qquad \begin{cases} -i\omega\mu\mathsf{H} + \mathsf{rot}\,\mathsf{E} = \mathsf{f}_1 \\ (-i\omega\varepsilon + \sigma)\mathsf{E} - \mathsf{rot}\,\mathsf{H} = \mathsf{f}_2 \end{cases},$$

where $\mu, \varepsilon, \omega \in \mathbf{R}$ and $\sigma \in \mathbf{R}^+$ are constants. Functions $\mathsf{f}_1, \mathsf{f}_2 : \Omega \longrightarrow \mathbf{C}^3$ are given and $\mathsf{E}, \mathsf{H} : \Omega \longrightarrow \mathbf{C}^3$ are the unknown functions.

Remark 9. The above system can be derived from the (standard) Maxwell equations by assuming that all fields are periodic in time with the same frequency ω. Then $\mathsf{f}_1 \equiv \mathsf{0}$ and μ, ε, σ represent the magnetic susceptibility, the electric permeability and the conductivity of the medium, while the applied current density, the magnetic and electric fields are given by the inverse Fourier transform of f_2, H and E.

Here we shall consider a slightly more general situation where the first equation is not necessarily homogeneous. ∎

We want to write the above system as a (stationary) complex Friedrichs system.

Let us introduce $\mathsf{u} = [\mathsf{u}_1\ \mathsf{u}_2]^\top := [\mathsf{H}\ \mathsf{E}]^\top$, and define

$$\mathbf{D} := \begin{bmatrix} -i\omega\mu\mathbf{I} & \mathbf{0} \\ \mathbf{0} & (-i\omega\varepsilon + \sigma)\mathbf{I} \end{bmatrix}, \quad \mathbf{A}_k := \begin{bmatrix} \mathbf{0} & \mathbf{B}_k \\ \mathbf{B}_k^\top & \mathbf{0} \end{bmatrix}, \quad k \in 1..3\,,$$

where $\mathbf{B}_k$ are 3×3 constant matrices associated to the differential operator rot [27, 13]. If we denote by $\mathsf{e}_1, \mathsf{e}_2, \mathsf{e}_3$ the standard basis in $\mathbf{C}^3$, and define twice the antisymmetric part of tensor product of two vectors by $\wedge$ (i.e. $\mathsf{u} \wedge \mathsf{v} := \mathsf{u} \otimes \mathsf{v} - \mathsf{v} \otimes \mathsf{u}$), we have that $\mathbf{B}_1 = \mathsf{e}_3 \wedge \mathsf{e}_2$, $\mathbf{B}_2 = \mathsf{e}_3 \wedge \mathsf{e}_1$ and $\mathbf{B}_3 = \mathsf{e}_2 \wedge \mathsf{e}_1$.

For $\mathsf{u} \in \mathrm{C}^\infty_c(\Omega; \mathbf{C}^6)$ we define

$$T\mathsf{u} := \sum_{k=1}^{3} \partial_k(\mathbf{A}_k\mathsf{u}) + \mathbf{D}\mathsf{u}\,,$$

so the system above reads $T\mathsf{u} = \mathsf{f}$, with $\mathsf{f} = [\mathsf{f}_1\ \mathsf{f}_2]^\top$.

Since constant matrices $\mathbf{A}_k$ are hermitian (in fact, real and symmetric), the condition (F1) is clearly satisfied, but the positivity assumption (F2) is not, as

$$\mathbf{D} + \mathbf{D}^* = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 2\sigma\mathbf{I} \end{bmatrix}.$$

Therefore, we shall use *the two-field* theory introduced in the second section.

Let us check the assumptions of Theorem 3. The conditions (F0), (F1), (F2A) and (F2B) are trivially satisfied. In order to analyse the remaining condition (T3A) we first need to fix the function spaces, and then to identify the graph space W and the boundary operator D.

As it will be clear later, the space $\mathrm{L}^2(\Omega;\mathbf{C}^6)$ is, unfortunately, not suitable, since for condition (T3A) to be satisfied we need a higher regularity in the first component. Therefore, for the space of test functions we take $\mathcal{D}=\mathcal{D}_1\times\mathcal{D}_2 := \mathrm{C}^\infty_{c,\mathrm{div}=0}(\Omega;\mathbf{C}^3)\times \mathrm{C}^\infty_c(\Omega;\mathbf{C}^3)$, where the space of smooth compactly supported divergence free functions we denote by

$$\mathrm{C}^\infty_{c,\mathrm{div}=0}(\Omega;\mathbf{C}^3) := \Big\{\mathsf{u}\in \mathrm{C}^\infty_c(\Omega;\mathbf{C}^3) : \operatorname{div}\mathsf{u}=0\Big\}\,,$$

and it is clearly a closed subspace of $\mathrm{C}^\infty_c(\Omega;\mathbf{C}^3)$. If we denote by $\mathrm{L}^2_{\mathrm{div}=0,0}(\Omega;\mathbf{C}^3)$ the closure of $\mathrm{C}^\infty_{c,\mathrm{div}=0}(\Omega;\mathbf{C}^3)$ in the space $\mathrm{L}^2(\Omega;\mathbf{C}^3)$, we have that $L=L_1\times L_2 := \mathrm{L}^2_{\mathrm{div}=0,0}(\Omega;\mathbf{C}^3)\times \mathrm{L}^2(\Omega;\mathbf{C}^3)$ is a closed subspace of $\mathrm{L}^2(\Omega;\mathbf{C}^6)$, thus a complex Hilbert space when equipped with the standard L^2 scalar product, with $\mathcal{D}$ as a dense subset.

In this setting, by Example 2, we have that $T,\tilde T : L\longrightarrow \mathcal{D}'$ are given by

$$T\mathsf{u} := \begin{bmatrix} \mathcal{R}_1(-i\omega\mu\mathsf{u}_1+\operatorname{rot}\mathsf{u}_2)\\ (-i\omega\varepsilon+\sigma)\mathsf{u}_2-\operatorname{rot}\mathsf{u}_1\end{bmatrix} \qquad\text{and}\qquad \tilde T\mathsf{u} := \begin{bmatrix} \mathcal{R}_1(i\omega\mu\mathsf{u}_1-\operatorname{rot}\mathsf{u}_2)\\ (i\omega\varepsilon+\sigma)\mathsf{u}_2+\operatorname{rot}\mathsf{u}_1\end{bmatrix}\,,$$

where $\mathcal{D}'=\mathcal{D}_1'\times\mathcal{D}_2'$ is the antidual of $\mathcal{D}$. Here $\mathcal{R}_1$ denotes the restriction operator from (standard) distributions to $\mathcal{D}_1'$, as it is explained in Example 2.

Let us now compute the graph space, i.e. find sufficient and necessary conditions on $\mathsf{u}\in L$ to have $T\mathsf{u}\in L$. For the second component we have that $(-i\omega\varepsilon+\sigma)\mathsf{u}_2-\operatorname{rot}\mathsf{u}_1\in L_2$, which implies that $\operatorname{rot}\mathsf{u}_1\in L_2=\mathrm{L}^2(\Omega;\mathbf{C}^3)$, therefore $\mathsf{u}_1\in \mathrm{L}^2_{\mathrm{rot}}(\Omega;\mathbf{C}^3)\cap \mathrm{L}^2_{\mathrm{div}=0,0}(\Omega;\mathbf{C}^3)$, where $\mathrm{L}^2_{\mathrm{rot}}(\Omega;\mathbf{C}^3)=\{\mathsf{v}\in \mathrm{L}^2(\Omega;\mathbf{C}^3) : \operatorname{rot}\mathsf{v}\in \mathrm{L}^2(\Omega;\mathbf{C}^3)\}$. The first component gives $\mathcal{R}_1(-i\omega\mu\mathsf{u}_1+\operatorname{rot}\mathsf{u}_2)\in L_1$ or in other words $-i\omega\mu\mathsf{u}_1+\mathcal{R}_1(\operatorname{rot}\mathsf{u}_2)\in L_1$, which is equivalent to $\mathcal{R}_1(\operatorname{rot}\mathsf{u}_2)\in L_1$. Thus the graph space is given by

$$\begin{aligned} W &= \Big(\mathrm{L}^2_{\mathrm{rot}}(\Omega;\mathbf{C}^3)\cap \mathrm{L}^2_{\mathrm{div}=0,0}(\Omega;\mathbf{C}^3)\Big)\times\Big\{\mathsf{v}\in \mathrm{L}^2(\Omega;\mathbf{C}^3) : \mathcal{R}_1(\operatorname{rot}\mathsf{v})\in \mathrm{L}^2_{\mathrm{div}=0,0}(\Omega;\mathbf{C}^3)\Big\}\\ &=: W_1\times W_2\,. \end{aligned}$$

By [35, Lemma 3] we have $W_1\subseteq \mathrm{H}^1(\Omega;\mathbf{C}^3)$, while for $\operatorname{rot}\mathsf{v}\in \mathrm{L}^2(\Omega;\mathbf{C}^3)$ we have $\mathcal{R}_1(\operatorname{rot}\mathsf{v}) = P_{L_1}(\operatorname{rot}\mathsf{v})\in L_1$ (see Example 2), thus $\mathrm{L}^2_{\mathrm{rot}}(\Omega;\mathbf{C}^3)\subseteq W_2$. The last inclusion appears to be strict, but we do not study that question since it is irrelevant for the final conclusion in this example.

Moreover, the graph norm $\|\cdot\|_T$ is equivalent to

$$\|\mathsf{u}\|_*^2 = \|\mathsf{u}\|^2_{\mathrm{L}^2(\Omega;\mathbf{C}^6)} + \|\operatorname{rot}\mathsf{u}_1\|^2_{\mathrm{L}^2(\Omega;\mathbf{C}^3)} + \|\mathcal{R}_1(\operatorname{rot}\mathsf{u}_2)\|^2_{\mathrm{L}^2(\Omega;\mathbf{C}^3)}\,,$$

which is weaker than the standard norm [1] on $\mathrm{L}^2_{\mathrm{rot}}(\Omega;\mathbf{C}^6)$. However, one should be aware that in the case where $\omega\mu=0$ (and hence $\|\mathsf{u}_1\|_{\mathrm{L}^2(\Omega;\mathbf{C}^3)}$ is not present in the graph norm), in order to reach the same conclusion, one has to use the fact that on W_1, by [35, Theorem 5], we have $\|\mathsf{u}_1\|_{\mathrm{L}^2(\Omega;\mathbf{C}^3)}\leqslant C\|\operatorname{rot}\mathsf{u}_1\|_{\mathrm{L}^2(\Omega;\mathbf{C}^3)}$ for some constant $C>0$.

Thus, the closure of $\mathcal{D}$ in the graph norm is equal to

$$W_0 = \mathrm{H}^2_{\mathrm{div}=0,0}(\Omega;\mathbf{C}^3)\times \mathrm{L}^2_{\mathrm{rot},0}(\Omega;\mathbf{C}^3)\,,$$

where $\mathrm{H}^2_{\mathrm{div}=0,0}(\Omega;\mathbf{C}^3)=\{\mathsf{v}\in \mathrm{H}^1_0(\Omega;\mathbf{C}^3) : \operatorname{div}\mathsf{v}=0\}$ and $\mathrm{L}^2_{\mathrm{rot},0}(\Omega;\mathbf{C}^3)$ is the space of functions from $\mathrm{L}^2_{\mathrm{rot}}(\Omega;\mathbf{C}^3)$ whose tangential trace vanishes. Here we have used that on $\mathcal{D}$ the norm $\|\cdot\|_*$ is equal to the norm of the space $\mathrm{L}^2_{\mathrm{rot}}(\Omega;\mathbf{C}^6)$, i.e. we can remove the restriction operator $\mathcal{R}_1$. This is in accordance with the fact that $\operatorname{rot}\mathrm{L}^2_{\mathrm{rot},0}(\Omega;\mathbf{C}^3)=\mathrm{L}^2_{\mathrm{div}=0,0}(\Omega;\mathbf{C}^3)$ (see [35]).

Although the second component of the graph space is a slightly unusual space, for checking the condition (T3A) the first component is sufficient. Indeed, let $V,\widetilde V\subseteq W$ satisfy (V). Then

$\mathsf{u} \in V \cup \widetilde{V}$ implies $\mathsf{u}_1 \in \mathrm{L}^2_{\mathrm{rot}}(\Omega;\mathbf{C}^3) \cap \mathrm{L}^2_{\mathrm{div}=0,0}(\Omega;\mathbf{C}^3)$, thus by the convexity of Ω and [35, Theorem 5] there exists $C > 0$ (independent of u_1) such that

$$\|\mathsf{u}_1\|_{L_1} = \|\mathsf{u}_1\|_{\mathrm{L}^2(\Omega;\mathbf{C}^3)} \leqslant C\|\mathsf{rot}\,\mathsf{u}_1\|_{\mathrm{L}^2(\Omega;\mathbf{C}^3)}\,,$$

which implies (T3A).

Therefore, for suitable V and $\widetilde{V}$ we can apply Theorem 3 to these operators T and $\tilde{T}$.

To this end, let us present one possible choice of V and $\widetilde{V}$. Namely, let

$$V = \widetilde{V} = W_1 \times \mathrm{L}^2_{\mathrm{rot},0}(\Omega;\mathbf{C}^3)\,,$$

which is clearly contained in the graph space. Let us show that V (and then also $\widetilde{V}$) is closed in W. The crucial fact is that on V the norm $\|\cdot\|_*$ is equal to the norm of the space $\mathrm{L}^2_{\mathrm{rot}}(\Omega;\mathbf{C}^6)$, as it has already been commented. The statement follows since $\mathrm{L}^2_{\mathrm{rot},0}(\Omega;\mathbf{C}^3)$ is closed in $\mathrm{L}^2_{\mathrm{rot}}(\Omega;\mathbf{C}^3)$.

For $\mathsf{u},\mathsf{v} \in W_1 \times \mathrm{C}^\infty_c(\mathbf{R}^3;\mathbf{C}^3)$ we have

$$\begin{aligned} {}_{W'}\langle D\mathsf{u},\mathsf{v}\rangle_W &= \langle T\mathsf{u} \mid \mathsf{v}\rangle_L - \langle \mathsf{u} \mid \tilde{T}\mathsf{v}\rangle_L \\ &= {}_{\mathrm{H}^{-\frac12}}\langle \mathcal{T}_{\mathrm{rot}}\mathsf{u}_2, \mathcal{T}_{\mathrm{H}^1}\mathsf{v}_1\rangle_{\mathrm{H}^{\frac12}} + {}_{\mathrm{H}^{-\frac12}}\langle \mathcal{T}_{\mathrm{rot}}\mathsf{v}_2, \mathcal{T}_{\mathrm{H}^1}\mathsf{u}_1\rangle_{\mathrm{H}^{\frac12}}\,, \end{aligned}$$

where $\mathcal{T}_{\mathrm{H}^1} : \mathrm{H}^1(\Omega;\mathbf{C}^3) \longrightarrow \mathrm{H}^{\frac12}(\Gamma;\mathbf{C}^3)$ is the trace operator and $\mathcal{T}_{\mathrm{rot}} : \mathrm{L}^2_{\mathrm{rot}}(\Omega;\mathbf{C}^3) \longrightarrow \mathrm{H}^{-\frac12}(\Gamma;\mathbf{C}^3)$ is the tangential trace operator (see [1], [11]). Since $\mathrm{C}^\infty_c(\mathbf{R}^3;\mathbf{C}^3)$ is dense in $\mathrm{L}^2_{\mathrm{rot}}(\Omega;\mathbf{C}^3)$ and the topology of $W_1 \times \mathrm{L}^2_{\mathrm{rot}}(\Omega;\mathbf{C}^3)$ is stronger than the topology of the graph space W, we can extend the above identity for D to $W_1 \times \mathrm{L}^2_{\mathrm{rot}}(\Omega;\mathbf{C}^3)$. It is straightforward then to see that for any $\mathsf{u},\mathsf{v} \in V$ we have ${}_{W'}\langle D\mathsf{u},\mathsf{v}\rangle_W = 0$, implying that (V1) is satisfied, and that $V \subseteq D(V)^0$.

Let us prove that $D(V)^0 \subseteq D(D(V)^0)^0$. Since $D(V)^0 \subseteq W$ and V is closed in W, there exists an orthogonal projection P_V in W on V. Moreover, it is obvious that for any $\mathsf{v} \in W$ the first component of $\mathsf{v} - P_V\mathsf{v}$ is equal to zero, implying that for any $\mathsf{u},\mathsf{v} \in W$ we have

$$\langle T(\mathsf{u} - P_V\mathsf{u}) \mid \mathsf{v} - P_V\mathsf{v}\rangle_L = \langle \mathsf{u} - P_V\mathsf{u} \mid \tilde{T}(\mathsf{v} - P_V\mathsf{v})\rangle_L = 0\,.$$

Hence, for $\mathsf{u},\mathsf{v} \in D(V)^0$, by symmetry of D and by using $P_V\mathsf{u}, P_V\mathsf{v} \in V \subseteq D(V)^0$, we have

$$\begin{aligned} 0 &= \langle T(\mathsf{u} - P_V\mathsf{u}) \mid \mathsf{v} - P_V\mathsf{v}\rangle_L - \langle \mathsf{u} - P_V\mathsf{u} \mid \tilde{T}(\mathsf{v} - P_V\mathsf{v})\rangle_L \\ &= {}_{W'}\langle D(\mathsf{u} - P_V\mathsf{u}), \mathsf{v} - P_V\mathsf{v}\rangle_W \\ &= {}_{W'}\langle D\mathsf{u},\mathsf{v}\rangle_W - {}_{W'}\langle D\mathsf{u}, P_V\mathsf{v}\rangle_W - {}_{W'}\langle DP_V\mathsf{u},\mathsf{v}\rangle_W + {}_{W'}\langle DP_V\mathsf{u}, P_V\mathsf{v}\rangle_W \\ &= {}_{W'}\langle D\mathsf{u},\mathsf{v}\rangle_W\,, \end{aligned}$$

resulting in $D(V)^0 \subseteq D(D(V)^0)^0$. Since V is closed in W and $W_0 \subseteq V$ by Remark 3 we have that $V = D(D(V)^0)^0$, obtaining $V = D(V)^0$, hence (V2) is satisfied.

Therefore, by Theorem 3, for any $\mathsf{f}_1 \in \mathrm{L}^2_{\mathrm{div}=0,0}(\Omega;\mathbf{C}^3)$ and $\mathsf{f}_2 \in \mathrm{L}^2(\Omega;\mathbf{C}^3)$ there exists a unique pair of fields $\mathsf{H} \in \mathrm{L}^2_{\mathrm{rot}}(\Omega;\mathbf{C}^3) \cap \mathrm{L}^2_{\mathrm{div}=0,0}(\Omega;\mathbf{C}^3)$ and $\mathsf{E} \in \mathrm{L}^2_{\mathrm{rot},0}(\Omega;\mathbf{C}^3)$ such that

$$\begin{cases} \mathcal{R}_1(-i\omega\mu\mathsf{H} + \mathsf{rot}\,\mathsf{E}) = \mathsf{f}_1 \\ (-i\omega\varepsilon + \sigma)\mathsf{E} - \mathsf{rot}\,\mathsf{H} = \mathsf{f}_2 \end{cases}.$$

Moreover, as $-i\omega\mu\mathsf{H} + \mathsf{rot}\,\mathsf{E} \in \mathrm{L}^2_{\mathrm{div}=0,0}(\Omega;\mathbf{C}^3)$, we can remove $\mathcal{R}_1$, obtaining that H and E are indeed the unique solution of the starting system (MS).

Remark 10. The choice of space $\mathrm{L}^2_{\mathrm{div}=0,0}(\Omega;\mathbf{C}^3)$ for the first component of our system may look strange at first glance. However, if we have in mind the physical motivation for this example in which $\mathsf{f}_1 = 0$, than from the first equation we can easily conclude that H belongs to $\mathrm{L}^2_{\mathrm{div}=0,0}(\Omega;\mathbf{C}^3)$ if E belongs to $\mathrm{L}^2_{\mathrm{rot},0}(\Omega;\mathbf{C}^3)$, which justifies this choice. ∎

Remark 11. The above result can easily be generalised (with the same existence and uniqueness result) for $\varepsilon, \sigma \in \mathrm{L}^\infty(\Omega; \mathbf{C})$ such that $\omega \mathsf{Im}\, \varepsilon + \mathsf{Re}\, \sigma > \sigma_0$ for some $\sigma_0 > 0$. These assumptions encompass some cases which, to the best of our knowledge, are not treated in the classical literature (see [32, pp. 188–192] for a typical setting for the time-harmonic Maxwell system).

However, relaxing assumptions on μ is more challenging since it is desirable to have $\mu \mathsf{u}_1 \in L_1$, which is violated for non-constant μ. ∎

4. Concluding remarks

We have shown that the already developed theory of abstract Friedrichs systems can be adjusted to the complex Hilbert space setting, by proving a number of results, including a well-posedness result for the stationary theory, for the non-stationary (semi-linear) theory and for the two-field theory with partial coercivity in which the pivot space is some closed subspace of the space of square integrable functions. We also proved equivalence of different representations of boundary conditions and applied the derived results to a number of particular linear and semi-linear equations with complex coefficients. We particularly emphasise situations in which the pivot space is the space $\mathrm{H}^s(\mathbf{R}^d; \mathbf{C}^r)$, for real s, or some specific closed subspace of the space of square integrable functions.

We hope that these extensions will broaden the applicability of the theory of Friedrichs systems to various complex equations of mathematical physics, e.g. their numerical analysis by some well developed numerical schemes for Friedrichs systems.

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