

Discrete series of metaplectic groups having generic theta lifts

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Abstract. We prove that a discrete series representations of metaplectic group over a non-archimedean local field has a generic theta lift on the split odd orthogonal tower if and only if it is generic. Also, we determine the first occurrence indices of such representations and describe the structure of their theta lifts.

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1. Introduction

Generic representations occupy a specially important part of non-unitary duals of both classical and metaplectic groups. This class of representations appears in numerous problems in representation theory and has important global applications as well as applications in the theory of L -functions and L -packets.

Certain basic facts about the generic representations have been discussed in both classical and metaplectic case. For instance, heredity of Whittaker models, proved for classical groups by Rodier ([25]), has been extended to the metaplectic case in [2] and [28].

However, there are still many known properties of generic representations of classical groups that are not verified in the metaplectic case and it is of particular interest to obtain the knowledge how the theta correspondence treats this class of representations.

Theta lifts of generic discrete series for the symplectic-orthogonal dual pair over a non-archimedean local field of characteristic zero have been described by Muić and Savin in [23]. Among other results, it has been proved that every generic discrete series of symplectic group has a generic theta lift and

that the first occurrence index of such representation of rank n symplectic group equals either n or $n - 1$. These results have been obtained using essentially analytical technics, based on the theory of L -functions developed by Shahidi ([26]) and the description of generic square-integrable representations obtained in [19] and [20].

In this paper we study discrete series representations of metaplectic group over a non-archimedean local field of characteristic zero with odd residual characteristic which have a generic theta lift on the split odd orthogonal tower. It is almost a trivial fact that the generic representation of the metaplectic group has a generic theta lift in the equal rank case. However, it remains to determine discrete series of metaplectic groups which have generic theta lifts on not necessarily equal level of the split odd orthogonal tower. To determine such representations, we use mostly algebraic methods, based on the classification of discrete series of classical groups from [16] and [17], together with the classification of strongly positive representations of metaplectic groups from [12]. We note that the classification of discrete series of classical groups was already complete without any assumptions in the case of generic representations.

The crucial facts for our investigation are contained in the correspondence between irreducible genuine representations of rank n metaplectic group and irreducible representations of odd orthogonal groups of quadratic spaces of discriminant 1 and dimension $2n + 1$, proved by Gan and Savin in [4]. Firstly, this correspondence allows us to transfer certain analytic aspects of the Mœglin – Tadić classification of discrete series from the classical case to the metaplectic one. Thus, we start from genuine discrete series representation of the metaplectic group, study its lifts in the odd orthogonal towers and then transfer the obtained information back to the metaplectic one. Similar double pass approach has already been used in [14] to obtain the explicit first occurrence indices of genuine discrete series of metaplectic groups and in [6] to prove irreducibility of unitary principal series of metaplectic groups.

Secondly, the special aspects of mentioned correspondence related to the subset of generic representations (Section 9 in [4]) present a starting point of our study. As a rather direct consequence of mentioned results we obtain some necessary conditions which discrete series representation of metaplectic group that has a generic theta lift on the split odd orthogonal tower has to satisfy. Further results are deduced from the precise investigation of lifts of particular interesting discrete series (that is, in the case of generic reducibilities), combined with certain important properties of generic discrete series of classical groups (which have been derived in [5]). This also allows us to obtain deeper knowledge about the structure of theta lifts of generic discrete series.

Now we describe the contents of the paper in more detail. In the following section we review some of the standard facts on the representation theory of the considered groups, while in the third section we summarize without proofs the relevant material on the theta correspondence and generic representations. Section 4 and 5 present a technical heart of the paper. Section 4 is devoted to the study of theta lifts of discrete series, while in the Section 5 we obtain precise results on the generic theta lifts of discrete series in the case of so-called generic reducibilities. In the same section our main results are stated and proved.

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2. Notations and preliminaries

Let F be a non-archimedean local field of characteristic zero with odd residual characteristic. In the sequel, we fix a non-trivial additive character ψ of F .

First we discuss the groups that we consider.

Let V_0 denote an anisotropic quadratic space over F of odd dimension. Then $\dim V_0 \in \{1, 3\}$. More details about the invariants of this space can be found in [9] and [11]. In each step we add a hyperbolic plane and obtain an enlarged quadratic space, a tower of quadratic spaces and a tower of corresponding orthogonal groups. In the case when r hyperbolic planes are added to the anisotropic space, enlarged quadratic space will be denoted by V_r , while a corresponding orthogonal group will be denoted by $O(V_r)$. Set $m_r = \frac{1}{2} \dim V_r$.

To a fixed quadratic character χ_{V_0} one can attach two odd orthogonal towers, one with $\dim V_0 = 1$ (+-orthogonal tower or split orthogonal tower) and the other with $\dim V_0 = 3$ (--orthogonal tower), as explained in detail in Chapter V of [10]. We denote by $O(V_r^+)$ and $O(V_r^-)$ corresponding orthogonal groups of the spaces obtained by adding r hyperbolic planes.

We denote by $Irr(O(V_r))$ the set of isomorphism classes of irreducible admissible representations of the orthogonal group $O(V_r)$.

Let $\widetilde{Sp}(n)$ be the metaplectic group of rank n over F , the unique non-trivial two-fold central extension of symplectic group $Sp(n, F)$. In other words, we have the following:

$$1 \rightarrow \mu_2 \rightarrow \widetilde{Sp}(n) \rightarrow Sp(n, F) \rightarrow 1,$$

where $\mu_2 = \{1, -1\}$. The multiplication in $\widetilde{Sp}(n)$ (which is as a set given by $Sp(n, F) \times \mu_2$) is given by the Rao's cocycle ([24]). For more detailed description of the structural theory of metaplectic groups we refer the reader to [7], [10] and [24].

We will study only genuine representations of $\widetilde{Sp}(n)$ (that is, those which do not factor through μ_2), so let $Irr(\widetilde{Sp}(n))$ stand for the set of isomorphism classes of irreducible admissible genuine representations of the group $\widetilde{Sp}(n)$. Further, let $\mathcal{S}(\widetilde{Sp}(n))$ denote the Grothendieck group of the category of all admissible genuine representations of finite length of $\widetilde{Sp}(n)$ (that is, a free abelian group over the set of all irreducible genuine representations of $\widetilde{Sp}(n)$) and set $\mathcal{S} = \bigoplus_{n \geq 0} \mathcal{S}(\widetilde{Sp}(n))$.

Let us denote by $GL(n, F)$ a double cover of $GL(n, F)$, where the multiplication is given by $(g_1, \epsilon_1)(g_2, \epsilon_2) = (g_1 g_2, \epsilon_1 \epsilon_2 (\det g_1, \det g_2)_F)$. Here $\epsilon_i \in \mu_2$ for $i = 1, 2$ and $(\cdot, \cdot)_F$ denotes the Hilbert symbol of the field F .

In the sequel we will fix a character $\chi_{V, \psi}$ of $GL(n, F)$ given by $\chi_{V, \psi}(g, \epsilon) = \chi_V(\det g) \epsilon \gamma(\det g, \frac{1}{2} \psi)^{-1}$, where γ denotes the Weil index ([10]) and χ_V is a character related to the orthogonal tower. Set $\alpha = \chi_{V, \psi}^2$ and note that α is a quadratic character on $GL(n, F)$. Also, we remark that every irreducible genuine representation ρ of $GL(n, F)$ is of the form $\chi_{V, \psi} \rho'$ for some irreducible representation ρ' of $GL(n, F)$.

Write $\mathcal{R}^{\text{gen}} = \bigoplus_{n \geq 0} \mathcal{R}(GL(n, F))_{\text{gen}}$, where $\mathcal{R}(GL(n, F))_{\text{gen}}$ denotes the Grothendieck group of the category of all admissible genuine representations of finite length of $GL(n, F)$.

We denote by ν the character of $GL(n, F)$ defined by $|\det|_F$.

If ρ is an irreducible cuspidal representation of $GL(n_\rho, F)$ (this defines n_ρ), or such genuine representation of $GL(n_\rho, F)$, we say that $\Delta = \{\nu^a \rho, \nu^{a+1} \rho, \dots, \nu^{a+k} \rho\}$ is a segment, where $a \in \mathbb{R}$ and $k \in \mathbb{Z}_{\geq 0}$. From now on, we denote the segment $\{\nu^a \rho, \nu^{a+1} \rho, \dots, \nu^{a+k} \rho\}$ briefly by $[\nu^a \rho, \nu^{a+k} \rho]$. The unique irreducible subrepresentation of the induced representation $\nu^{a+k} \rho \times \nu^{a+k-1} \rho \times \dots \times \nu^a \rho$ will be denoted by $\delta(\Delta)$. We note that $\delta(\Delta)$ is an essentially square-integrable representation attached to the segment Δ , and if ρ is a genuine representation then $\delta(\Delta)$ is also genuine.

The trivial representation of group F^\times will be denoted by 1_{F^\times} .

For an ordered partition $s = (n_1, n_2, \dots, n_i)$ of some $m \leq n$, we denote by P_s a standard parabolic subgroup of $Sp(n, F)$ (consisting of block upper-triangular matrices), whose Levi factor equals $GL(n_1, F) \times GL(n_2, F) \times \dots \times GL(n_i, F) \times Sp(n - m, F)$. Then the standard parabolic subgroup \widetilde{P}_s of $\widetilde{Sp}(n)$ is the preimage of P_s in $\widetilde{Sp}(n)$. There is an analogous notation for the Levi factors of the metaplectic groups (described in more detail in [7], Section 2.2) and for the standard parabolic subgroups of $O(V_r)$ (that is, those containing the upper triangular Borel subgroup). We will denote by $\rho_1 \times \rho_2 \times \dots \times \rho_i \rtimes \sigma$ the representation that is parabolically induced from the representation $\rho_1 \otimes \rho_2 \otimes \dots \otimes \rho_i \otimes \sigma$ of the Levi factor of \widetilde{P}_s . The normalized Jacquet module

of a smooth representation σ of $\widetilde{Sp}(n)$ with respect to standard parabolic subgroup \widetilde{P}_s will be denoted by $R_{\widetilde{P}_s}(\sigma)$. Similarly, $R_{P_s}(\tau)$ will stand for the normalized Jacquet module of a smooth representation τ of $O(V_r)$ with respect to standard parabolic subgroup P_s . By abuse of notation, we write $R_{\widetilde{P}_l}(\sigma)$ (resp. $R_{P_l}(\tau)$) instead of $R_{\widetilde{P}_{(l)}}(\sigma)$ (resp. $R_{P_{(l)}}(\tau)$), for $l \in \mathbb{N}$.

For an irreducible cuspidal representation ρ of $GL(n, F)$ (resp., $GL(n, F)$), we denote by $R_{\widetilde{P}_n}(\sigma)(\rho)$ (resp., $R_{P_n}(\sigma)(\rho)$) the maximal ρ -isotypic quotient of $R_{\widetilde{P}_n}(\sigma)$ (resp., of $R_{P_n}(\sigma)$). It is a maximal direct summand of $R_{\widetilde{P}_n}(\sigma)$ (resp., $R_{P_n}(\sigma)$) on which $\widetilde{GL}(n, F)$ (resp., $GL(n, F)$) acts by ρ .

Now we recall some results related to calculations with Jacquet modules, which will be used in the paper. Let σ denote an irreducible genuine representation of $\widetilde{Sp}(n)$. Then $R_{\widetilde{P}_k}(\sigma)$, for $0 \leq k \leq n$, can be interpreted as a genuine representation of $\widetilde{GL}(k, F) \times \widetilde{Sp}(n - k)$, i.e., is an element of $\mathcal{R}^{\text{gen}} \otimes \mathcal{S}$. For such σ we can introduce $\mu^*(\sigma) \in \mathcal{R}^{\text{gen}} \otimes \mathcal{S}$ by

$$\mu^*(\sigma) = \sum_{k=0}^n \text{s.s.}(R_{\widetilde{P}_k}(\sigma))$$

(s.s. denotes the semisimplification) and extend μ^* linearly to the whole of \mathcal{S} . In the same way μ^* can be defined for irreducible representations of classical groups.

The following metaplectic version of Tadić’s structure formula ([29]) has been proved in [7].

Lemma 2.1. *Let $\rho \in \mathcal{R}^{\text{gen}}$ be an irreducible cuspidal representation and $a, b \in \mathbb{R}$ such that $a+b \in \mathbb{Z}_{\geq 0}$. Let σ be an admissible genuine representation of finite length of $\widetilde{Sp}(n)$. Write $\mu^*(\sigma) = \sum_{\pi, \sigma'} \pi \otimes \sigma'$. Then the following holds:*

$$\begin{aligned} \mu^*(\delta([v^{-a}\rho, v^b\rho]) \rtimes \sigma) &= \sum_{i=-a-1}^b \sum_{j=i}^b \sum_{\pi, \sigma'} \delta([v^{-i}\alpha\tilde{\rho}, v^a\alpha\tilde{\rho}]) \\ &\quad \times \delta([v^{j+1}\rho, v^b\rho]) \times \pi \otimes \delta([v^{i+1}\rho, v^j\rho]) \rtimes \sigma'. \end{aligned}$$

We omit $\delta([v^x\rho, v^y\rho])$ if $x > y$.

3. Preliminary results on theta correspondence and generic representations

In this section we review some results about theta correspondence which will be used afterwards in the paper.

The pair $(Sp(n), O(V_r))$ is a reductive dual pair in $Sp(n \cdot \dim V_r)$. The group $Sp(n)$ does not split in $Sp(n \cdot \dim V_r)$, since the space V_r has the odd dimension. Thus, the theta correspondence relates the representations of the metaplectic group $\widetilde{Sp(n)}$ and those of the orthogonal group $O(V_r)$. Set $n_1 = n \cdot \dim V_r$. Let $\omega_{n,r}$ denote the pull-back of the Weil representation $\omega_{n_1,\psi}$ of the group $Sp(n_1)$, restricted to the dual pair $\widetilde{Sp(n)} \times O(V_r)$ (as in Chapter II of [10]).

For $\sigma \in Irr(\widetilde{Sp(n)})$, we let $\Theta(\sigma, r)$ denote a smooth representation of $O(V_r)$ given as the full lift of σ to the r -th level of the orthogonal tower. $\Theta(\sigma, r)$ is the biggest quotient of $\omega_{n,r}$ on which $\widetilde{Sp(n)}$ acts as a multiple of σ . Specially, we write $\Theta^+(\sigma, r)$ (resp., $\Theta^-(\sigma, r)$) for the full lift of σ to the r -th level of the $+$ -orthogonal tower (resp., $-$ -orthogonal tower).

Likewise, for $\tau \in Irr(O(V_r))$ we denote by $\Theta(\tau, n)$ the full lift of representation τ , which is a smooth genuine representation of $\widetilde{Sp(n)}$.

The following theorem summarizes basic facts about the theta correspondence, proved in [10] and [18]. We emphasize the assumption that the residual characteristic of the field F is different than 2.

Theorem 3.1. *For $\sigma \in Irr(\widetilde{Sp(n)})$ exists a non-negative integer r such that $\Theta(\sigma, r) \neq 0$. The smallest such r is called the first occurrence index of σ in the orthogonal tower and will be denoted by $r(\sigma)$. Also, $\Theta(\sigma, r') \neq 0$ for $r' \geq r$. We will write $r^+(\sigma)$ (resp., $r^-(\sigma)$) for the first occurrence index of σ in the $+$ -orthogonal tower (resp., $-$ -orthogonal tower).*

The representation $\Theta(\sigma, r)$ is either zero or it has a unique irreducible quotient. We denote this unique irreducible quotient by $\sigma(r)$. Also, we write $\sigma^+(r)$ (resp., $\sigma^-(r)$) for this irreducible quotient in the $+$ -orthogonal tower (resp., $-$ -orthogonal tower).

The analogous statements hold for $\Theta(\tau, n)$ if τ is an irreducible smooth representation of $O(V_r)$.

If σ is an irreducible cuspidal representation of $\widetilde{Sp(n)}$ then $\sigma(r(\sigma))$ is an irreducible cuspidal representation of $O(V_{r(\sigma)})$.

We take a moment to recall the basic results on the rank-one reducibilities which are of particular importance in the case of generic representations.

Let ρ denote an irreducible self-contragredient cuspidal representation of the group $GL(m, F)$ and let σ denote an irreducible cuspidal representation of $Sp(n)$. It is proved in [27] that there exists a unique non-negative real number s_1 such that $v^{s_1}\rho \rtimes \sigma(r(\sigma))$ reduces. Furthermore, it is a result of Shahidi ([26]) that if $\sigma(r(\sigma))$ is a generic representation of $O(V_{r(\sigma)}^+)$, then $s_1 \in \{0, \frac{1}{2}, 1\}$.

Using the methods of theta correspondence, Hanzer and Muić have proved in [8] that there is a unique non-negative real number s_2 such that $\nu^{s_2} \chi_{V, \psi} \rho \rtimes \sigma$ reduces (this has also been proved independently in [4]). If ρ is not a trivial representation of F^\times , then $s_1 = s_2$. Otherwise, $s_1 = |n - m_{r(\sigma)}|$ and $s_2 = |m_{r(\sigma)} - n - 1|$.

Let us now recall the definition of generic representations.

Let U_r stand for the unipotent radical of a Borel subgroup $B_r = T_r \cdot U_r$ of $O(V_r^+)$ and let λ denote some generic character of U_r . Choice of λ is not important, since any two generic characters of U_r are in the same orbit under the adjoint action of the maximal torus T_r . Representation $\tau \in Irr(O(V_r^+))$ is said to be generic if $\text{Hom}_{U_r}(\tau, \lambda) \neq 0$.

In the following lemma we summarize some important properties of generic representations of split odd orthogonal groups, which can be proved in the same way as in [5], Section 6.

Lemma 3.2.

- (i) *Suppose that the standard generic representation τ of $O(V_r^+)$ contains some square integrable subquotient. Then every irreducible generic subquotient of τ is square integrable.*
- (ii) *The Whittaker model is hereditary, that is, if $\rho_1, \rho_2, \dots, \rho_k$ are irreducible generic representation of $GL(m_1, F), GL(m_2, F), \dots, GL(m_k, F)$, respectively, and τ an irreducible representation of $O(V_r^+)$, then the induced representation $\rho_1 \times \rho_2 \times \dots \times \rho_k \rtimes \tau$ is generic if and only if τ is generic.*

Similarly as in the orthogonal case, let U'_n denote the unipotent radical of a Borel subgroup $\widetilde{B}'_n = \widetilde{T}'_n \cdot U'_n$ of $\widetilde{Sp}(n)$. Observe that $\widetilde{Sp}(n)$ splits over U'_n , and \widetilde{T}'_n denotes the preimage of maximal diagonal torus in $Sp(n)$. By results of Gan, Gross and Prasad ([3]), the \widetilde{T}'_n -orbits of generic characters of U'_n are indexed by non-trivial characters of F modulo the action of $(F^\times)^2$. So, the additive character ψ of F yields a \widetilde{T}'_n -orbit of generic characters λ'_ψ of U'_n , which is picked up by the Weil representation. More explicitly, the λ -twisted Jacquet U_n -module of ω_ψ equals $c - \text{ind}_{U'_n}^{\widetilde{Sp}(n)}(\lambda'_\psi)$, where $c - \text{ind}$ denotes the compact induction. We call a representation $\sigma \in Irr(\widetilde{Sp}(n))$ ψ -generic if $\text{Hom}_{U'_n}(\sigma, \lambda'_\psi) \neq 0$. This immediately shows that any ψ -generic representation of $\widetilde{Sp}(n)$ lifts to the split odd orthogonal tower.

The following results of Gan and Savin (Theorems 1.3 and 8.1 of [4]) are crucial for our investigation.

Theorem 3.3. *Let F be a non-archimedean local field of characteristic 0 with odd residual characteristic. For each non-trivial additive character ψ of*

F , there is an injection

$$\Theta_\psi : Irr(\widetilde{Sp}(n)) \rightarrow Irr(O(V_n^+)) \sqcup Irr(O(V_{n-1}^-))$$

given by the theta correspondence (with respect to ψ). Suppose that representations $\sigma \in Irr(\widetilde{Sp}(n))$ and $\tau \in Irr(O(V))$ correspond under Θ_ψ . Then σ is a discrete series representation if and only if τ is a discrete series representation. Further, if τ is a generic representation of $O(V_n^+)$, then σ is ψ -generic. Finally, if σ is a ψ -generic tempered representation, then τ is generic.

Now we prove a corollary which significantly reduces the procedure of finding the generic theta lifts.

Corollary 3.4. *Let $\sigma \in Irr(\widetilde{Sp}(n))$ such that $r^+(\sigma) \geq n + 1$. Then there is no generic theta lift of σ in the $+$ -orthogonal tower.*

Proof. Suppose that $\tau = \sigma^+(k)$ is generic for some $k \geq r^+(\sigma)$. Then Theorem 3.1 implies that $\tau(m)$ is non-zero for some $m < k$, contradicting Corollary 9.5 of [4]. □

An important consequence of this corollary is that if $\sigma \in Irr(\widetilde{Sp}(n))$ has a generic theta lift in the $+$ -orthogonal tower, then $\sigma^+(k)$ is generic for some $k \leq n$.

The following two lemmas are proved in Section 5 of [13] and they present an elementary but useful criterion for pushing down the lifts of irreducible representations.

Lemma 3.5. *Suppose that σ is an irreducible genuine representation of $\widetilde{Sp}(n)$. Then $\Theta(\sigma, r) \neq 0$ implies $R_{P_1}(\Theta(\sigma, r + 1))(v^{-(m_{r+1}-n-1)} 1_{F^\times}) \neq 0$.*

Further, if $R_{\widetilde{P}_1}(\sigma)(v^{-(m_{r+1}-n-1)} \chi_{V,\psi} 1_{F^\times}) = 0$, then $\Theta(\sigma, r) \neq 0$ if and only if $R_{P_1}(\Theta(\sigma, r + 1))(v^{-(m_{r+1}-n-1)} 1_{F^\times}) \neq 0$. In that case,

$$\sigma(r + 1) \hookrightarrow v^{-(m_{r+1}-n-1)} 1_{F^\times} \rtimes \sigma(r).$$

Lemma 3.6. *Let τ denote an irreducible representation of $O(V_r)$. If $\Theta(\tau, n) \neq 0$, then $R_{\widetilde{P}_1}(\Theta(\tau, n + 1))(v^{m_r-(n+1)} \chi_{V,\psi} 1_{F^\times}) \neq 0$.*

Suppose that $R_{P_1}(\tau)(v^{m_r-(n+1)} 1_{F^\times}) = 0$. Then $\Theta(\tau, n) \neq 0$ if and only if $R_{\widetilde{P}_1}(\Theta(\tau, n + 1))(v^{m_r-(n+1)} \chi_{V,\psi} 1_{F^\times}) \neq 0$. In that case, $\tau(n + 1)$ is a subrepresentation of $v^{m_r-(n+1)} \chi_{V,\psi} 1_{F^\times} \rtimes \tau(n)$.

Results obtained in the paper [14], using Lemma 3.5 and the classification of discrete series representations given in [17], provide useful embeddings of the non-zero lifts of discrete series representations of metaplectic groups which we recall in the following lemma. Special case of these results is contained in Corollary 6.4 of [14].

Lemma 3.7. *Let σ denote a discrete series representation of the metaplectic group $\widetilde{Sp}(n)$. If $\Theta(\sigma, r) \neq 0$, then $\sigma(r + 1) \hookrightarrow v^{-(m_{r+1}-n-1)} 1_{F^\times} \rtimes \sigma(r)$.*

At the end of this section we state two propositions, analogous to Remark 5.2 of [22], which enable the use of an inductive procedure for determining the theta lifts of discrete series representations.

Proposition 3.8. *Suppose that an irreducible representation $\sigma \in Irr(\widetilde{Sp}(n))$ may be written as an irreducible subrepresentation of the induced representation of the form $\delta([v^a \chi_{V, \psi} \rho, v^b \chi_{V, \psi} \rho]) \rtimes \sigma'$, where ρ is an irreducible cuspidal representation of $GL(n_\rho, F)$, $\sigma' \in Irr(\widetilde{Sp}(n'))$ and $b - a \geq 0$. Let $\Theta(\sigma, r) \neq 0$ and suppose that $(a, \rho) \neq (m_r - n, 1_{F^\times})$. Then there is an irreducible representation τ of some $O(V_{r'})$ such that $\sigma(r)$ is a subrepresentation of $\delta([v^a \rho, v^b \rho]) \rtimes \tau$. Further, suppose that if $\mu^*(\sigma)$ contains the representation $\delta([v^a \chi_{V, \psi} \rho, v^b \chi_{V, \psi} \rho]) \otimes \sigma''$ for some irreducible genuine representation σ'' of $Sp(n')$, then $\sigma'' \cong \sigma'$. Then $\sigma(r)$ is a subrepresentation of*

$$\delta([v^a \rho, v^b \rho]) \rtimes \sigma'(r - n + n').$$

Proposition 3.9. *Suppose that an irreducible representation $\tau \in Irr(O(V_r))$ may be written as an irreducible subrepresentation of the induced representation of the form $\delta([v^a \rho, v^b \rho]) \rtimes \tau'$, where ρ is an irreducible cuspidal representation of $GL(n_\rho, F)$, $\tau' \in Irr(O(V_{r'}))$ and $b - a \geq 0$. Let $\Theta(\tau, n) \neq 0$ and suppose that $(a, \rho) \neq (n - m_r + 1, 1_{F^\times})$. Then there is an irreducible representation σ of some $\widetilde{Sp}(n')$ such that $\tau(n)$ is a subrepresentation of $\delta([v^a \chi_{V, \psi} \rho, v^b \chi_{V, \psi} \rho]) \rtimes \sigma$. Further, suppose that if $\mu^*(\tau) \supseteq \delta([v^a \rho, v^b \rho]) \otimes \tau''$, for some irreducible genuine representation τ'' of $O(V_{r'})$, then $\tau'' \cong \tau'$. Then $\tau(n)$ is a subrepresentation of*

$$\delta([v^a \chi_{V, \psi} \rho, v^b \chi_{V, \psi} \rho]) \rtimes \tau'(n - r + r').$$

4. Theta lifts of discrete series

This section is devoted to the proof of some general results on the theta lifts of genuine discrete series of metaplectic groups. Also, we introduce some notation and recall the results which will be used afterwards in the paper.

Our description of the theta lifts of discrete series relies on the basic assumption, which now follows from [1]. This assumption is explained in detail in [17], while its metaplectic version has been discussed in the beginning of the sixth section of [14]. We note that the basic assumption for generic representations of classical groups follows already from [26].

An irreducible representation $\sigma \in \mathcal{S}$ is called strongly positive if for every embedding

$$\sigma \hookrightarrow v^{s_1} \rho_1 \times v^{s_2} \rho_2 \times \cdots \times v^{s_k} \rho_k \rtimes \sigma_{\text{cusp}},$$

where $\rho_i \in \mathcal{R}^{\text{gen}}$, $i = 1, 2, \dots, k$, are irreducible cuspidal unitary representations and $\sigma_{\text{cusp}} \in \mathcal{S}$ is an irreducible cuspidal representation, we have $s_i > 0$ for each i .

Obviously, every strongly positive representation is square-integrable. Irreducible strongly positive representations are called strongly positive discrete series. Strongly positive discrete series of classical groups are defined in a completely analogous way.

We take a moment to briefly recall an inductive description of non-supercuspidal strongly positive discrete series, which has been obtained in [12].

Proposition 4.1. *Suppose that $\sigma \in \mathcal{S}(\widetilde{Sp}(n))$ is an irreducible strongly positive representation and let $\rho \in \mathcal{R}(GL(m, F))_{\text{gen}}$ denote an irreducible cuspidal representation such that some twist of ρ appears in the cuspidal support of σ . We denote by σ_{cusp} the partial cuspidal support of σ . Then there exist unique $a, b \in \mathbb{R}$ such that $a > 0$, $b > 0$, $b - a \in \mathbb{Z}_{\geq 0}$, and a unique irreducible strongly positive representation σ' with the property that σ is a unique irreducible subrepresentation of $\delta([v^a \rho, v^b \rho]) \rtimes \sigma'$. Furthermore, there is a non-negative integer l such that $a + l = s$, where $s > 0$ is such that $v^s \rho \rtimes \sigma_{\text{cusp}}$ reduces (note that the basic assumption implies $2s \in \mathbb{Z}$). If $l = 0$ there are no twists of ρ appearing in the cuspidal support of σ' and if $l > 0$ there exist a unique $b' > b$ and a unique strongly positive discrete series σ'' , which contains neither $v^a \rho$ nor $v^{a+1} \rho$ in its cuspidal support, such that σ' can be written as a unique irreducible subrepresentation of $\delta([v^{a+1} \rho, v^{b'} \rho]) \rtimes \sigma''$.*

Obviously, if $v^x \rho$ appears in the cuspidal support of σ then we have $2x \in \mathbb{Z}$.

In the rest of this section σ denotes an arbitrary but fixed discrete series representation of $\widetilde{Sp}(n)$ (that is, an irreducible square-integrable representation of $\widetilde{Sp}(n)$).

The following result has been proved in [14], using Mœglin-Tadić classification of discrete series.

Theorem 4.2. *There exists an ordered pair (k, S_k) , consisting of an integer $k > 0$ and an ordered k -tuple $S_k = (\sigma_1, \sigma_2, \dots, \sigma_k)$ of discrete series representations with the following properties:*

- (i) $\sigma_i \in \text{Irr}(\widetilde{Sp}(n_i))$, $n_i < n_j$ for $i < j$;
- (ii) σ_1 is a strongly positive discrete series and $\sigma \cong \sigma_k$;

- (iii) For every $i \in \{2, 3, \dots, k\}$ there is a self-contragredient cuspidal representation $\rho_i \in \text{Irr}(GL(m_i, F))$ (this defines m_i) and non-negative real numbers a_i, b_i such that $b_i - a_i$ is a positive integer, satisfying

$$\sigma_i \hookrightarrow \delta([v^{-a_i} \chi_{V, \psi} \rho_i, v^{b_i} \chi_{V, \psi} \rho_i]) \rtimes \sigma_{i-1}$$

and $R_{\widetilde{P}_{m_i}}(\sigma_{i-1})(v^x \chi_{V, \psi} \rho_i) = 0$ for $a_i \leq x \leq b_i$;

- (iv) If $\rho_i \cong \rho_j$ for $i < j$, then $\rho_i \cong \rho_l$ for $l \in \{i + 1, i + 2, \dots, j\}$;
- (v) If $\rho_i \cong \rho_{i+1}$ then $a_i < a_{i+1}$;
- (vi) If there is some $i \in \{2, 3, \dots, k\}$ such that $\rho_i \cong 1_{F^\times}$, then $\rho_2 \cong 1_{F^\times}$.

The ordered k -tuple S_k , attached to discrete series σ as in the previous theorem, may not be unique. We denote by $U(\sigma)$ the set of all such ordered k -tuples of discrete series representations. To obtain an ordered k -tuple which is the most appropriate for determining the first occurrence indices of σ , to each $S_k \in U(\sigma)$ we attach a non-negative real number $\min(S_k)$ in the following way:

Definition 4.3. For $S_k \in U(\sigma)$ we denote by $\min(S_k)$ the minimal $x \in \mathbb{R}$ such that σ_1 can be written as a unique irreducible subrepresentation of the induced representation of the form $\delta([v^x \chi_{V, \psi} 1_{F^\times}, v^b \chi_{V, \psi} 1_{F^\times}]) \rtimes \sigma_{sp}$, $b \geq x$, where σ_{sp} is a strongly positive discrete series. If there is no such x , let $\min(S_k) = 0$.

An ordered k -tuple $S_k \in U(\sigma)$ will be called minimal if $\min(S_k) \leq \min(S'_k)$ for every $S'_k \in U(\sigma)$.

In what follows we fix one minimal ordered k -tuple $S_k = (\sigma_1, \sigma_2, \dots, \sigma_k)$ and write $\sigma_i \hookrightarrow \delta([v^{-a_i} \chi_{V, \psi} \rho_i, v^{b_i} \chi_{V, \psi} \rho_i]) \rtimes \sigma_{i-1}$ again.

By Theorem 3.3, there exists exactly one $\epsilon \in \{+, -\}$ such that $\Theta^\epsilon(\sigma, n - t_\epsilon) \neq 0$, where $t_+ = 0$ and $t_- = 1$.

Proposition 6.2 of [14] shows $\Theta^\epsilon(\sigma_1, n_1 - t_\epsilon) \neq 0$ (note that σ_1 is an irreducible genuine representation of $\widetilde{Sp}(n_1)$). Let us denote a half-integer $n_1 - t_\epsilon - r^\epsilon(\sigma_1)$ by m . The following proposition is the main result of the paper [14]:

Proposition 4.4. Let us denote by r the largest integer l , $2 \leq l \leq k$, such that $(a_i, \rho_i) = (m + i - \frac{3}{2}, 1_{F^\times})$ and $R_{\widetilde{P}_1}(\sigma)(v^{a_i} \chi_{V, \psi} 1_{F^\times}) = 0$ hold for $i = 2, 3, \dots, l$. If there is no such l , set $r = 1$. Then $r^\epsilon(\sigma) = n - t_\epsilon - m - r + 1$ and $r^{\epsilon'}(\sigma) = 2n - r^{\epsilon'}(\sigma)$ for $\epsilon' \in \{+, -\}$, $\epsilon' \neq \epsilon$. Furthermore, $\sigma_i^\epsilon(r^\epsilon(\sigma_i))$ is a discrete series subrepresentation of $\delta([v^{-a_i+1} 1_{F^\times}, v^{b_i} 1_{F^\times}]) \rtimes \sigma_{i-1}^\epsilon(r^\epsilon(\sigma_{i-1}))$ for $i = 2, 3, \dots, r$.

It has been proved in [4], and recalled in Theorem 3.3, that $\sigma^\epsilon(n - t_\epsilon)$ is a discrete series representation. An analogous result for the lower lifts is contained in the following proposition.

Proposition 4.5. *The representation $\sigma^\epsilon(i)$ is in the discrete series for $r^\epsilon(\sigma) \leq i \leq n - t_\epsilon$.*

Proof. We prove this proposition using induction over k . Let us first assume $k = 1$. Consequently, $\sigma \cong \sigma_1$ is strongly positive discrete series and it has been proved in [14], Section 4, that $\sigma^\epsilon(i)$ is also strongly positive for $r^\epsilon(\sigma) \leq i \leq n - t_\epsilon$. Furthermore, if there is some irreducible cuspidal representation ρ of $GL(l, F)$ such that $R_{\tilde{P}_1}(\sigma)(v^x \chi_{V, \psi} \rho) = 0$ and $R_{P_1}(\sigma^\epsilon(i))(v^x \rho) \neq 0$ for some $x \in \mathbb{R}$ and $r^\epsilon(\sigma) \leq i \leq n - t_\epsilon$, then $\rho \cong 1_{F^\times}$; if also $i \neq r^\epsilon(\sigma)$ holds, then $x = n - i + \frac{1}{2} - t_\epsilon$. Moreover, $\sigma^\epsilon(n - t_\epsilon)$ is a subrepresentation of the induced representation of the form

$$v^{\frac{1}{2}} 1_{F^\times} \times v^{\frac{3}{2}} 1_{F^\times} \times \dots \times v^{n-r^\epsilon(\sigma)-\frac{1}{2}-t_\epsilon} 1_{F^\times} \rtimes \sigma^\epsilon(r^\epsilon(\sigma)) \tag{1}$$

and $R_{\tilde{P}_1}(\sigma)(v^x \chi_{V, \psi} 1_{F^\times}) = 0$ if and only if $R_{P_1}(\sigma^\epsilon(r^\epsilon(\sigma)))(v^x 1_{F^\times}) = 0$, for $x \in \mathbb{R}$.

To simplify notation, let us denote $n - r^\epsilon(\sigma) - \frac{1}{2} - t_\epsilon$ by s .

Now we consider the case $k > 1$.

Since the representation σ_i is square-integrable and $R_{\tilde{P}_{m_i}}(\sigma_{i-1})(v^x \chi_{V, \psi} \rho_i) = 0$ for $a_i \leq x \leq b_i$, using Lemma 2.1 and following the same lines as in the proof of Theorem 2.3 of [22], we obtain that if $\mu^*(\sigma_i)$ contains an irreducible representation $\delta([v^{-a_i} \chi_{V, \psi} \rho_i, v^{b_i} \chi_{V, \psi} \rho_i]) \otimes \sigma'$ then $\sigma' \cong \sigma_{i-1}$. Thus, Proposition 3.8 gives $\sigma_i^\epsilon(n_i - t_\epsilon) \hookrightarrow \delta([v^{-a_i} \rho_i, v^{b_i} \rho_i]) \rtimes \sigma_{i-1}^\epsilon(n_{i-1} - t_\epsilon)$.

Applying Proposition 3.8 $k - 1$ times we obtain

$$\sigma^\epsilon(n - t_\epsilon) \hookrightarrow \delta([v^{-a_k} \rho_k, v^{b_k} \rho_k]) \times \dots \times \delta([v^{-a_2} \rho_2, v^{b_2} \rho_2]) \rtimes \sigma_1^\epsilon(n_1 - t_\epsilon).$$

If $\rho_2 \cong 1_{F^\times}$, as a direct consequence of square-integrability of $\sigma^\epsilon(n - t_\epsilon)$, Proposition 2.1 of [17] and the fact that $\sigma_1^\epsilon(n_1 - t_\epsilon)$ can be embedded in the induced representation of the form (1), we obtain $a_2 > s$.

Let us comment the case $k = 2$.

If $r^\epsilon(\sigma) = n - t_\epsilon - m$, Proposition 3.8 shows

$$\sigma^\epsilon(i) \hookrightarrow \delta([v^{-a_2} \rho_2, v^{b_2} \rho_2]) \rtimes \sigma_1^\epsilon(n_1 - n + i),$$

for all i such that $r^\epsilon(\sigma) \leq i \leq n - t_\epsilon$. Previous discussion shows $R_{P_{m_2}}(\sigma_1^\epsilon(n_1 - n + i))(v^x \rho_2) = 0$ for $a_2 \leq x \leq b_2$. In the same way as in Theorem 2.1 of [21] we obtain that the induced representation $\delta([v^{-a_2} \rho_2, v^{b_2} \rho_2]) \rtimes \sigma_1^\epsilon(n_1 - n + i)$ has exactly two irreducible subrepresentations which are both in the discrete series. Consequently, $\sigma^\epsilon(i)$ is a discrete series representation.

If $r^\epsilon(\sigma) = n - t_\epsilon - m - 1$, for $r^\epsilon(\sigma) < i \leq n - t_\epsilon$ we again have

$$\sigma^\epsilon(i) \hookrightarrow \delta([v^{-a_2} 1_{F^\times}, v^{b_2} 1_{F^\times}]) \rtimes \sigma_1^\epsilon(n_1 - n + i)$$

and square-integrability of $\sigma^\epsilon(i)$ follows in the same way as in previously considered case, while the square-integrability of $\sigma^\epsilon(r^\epsilon(\sigma))$ is noted in Proposition 4.4. Note that we have also proved that if there is some $x \in \mathbb{R}$ such that $R_{P_{m_2}}(\sigma_2^\epsilon(i))(v^x \rho_2) \neq 0$ for $r^\epsilon(\sigma_2) \leq i \leq n - t_\epsilon$, then either $x \leq a_2$ or $x \geq b_2$.

We suppose that the claim holds for all numbers less than k , $k \geq 3$, and prove it for k . Also, we inductively assume that if there is some $x \in \mathbb{R}$ such that $R_{P_{m_{k-1}}}(\sigma_{k-1}^\epsilon(i))(v^x \rho_{k-1}) \neq 0$ for $r^\epsilon(\sigma_{k-1}) \leq i \leq n - t_\epsilon$, then either $x \leq a_{k-1}$ or $x \geq b_{k-1}$.

Let i denote an element of the set $\{r^\epsilon(\sigma), r^\epsilon(\sigma) + 1, \dots, n - t_\epsilon\}$.

For $r^\epsilon(\sigma) = n - t_\epsilon - m - k + 1$ and $i = r^\epsilon(\sigma)$, $\sigma^\epsilon(i)$ is in the discrete series by Proposition 4.4, while otherwise using Proposition 3.8 we get

$$\sigma^\epsilon(i) \hookrightarrow \delta([v^{-a_k} \rho_k, v^{b_k} \rho_k]) \rtimes \sigma_{k-1}^\epsilon(n_{k-1} - n + i).$$

By the inductive assumption, $\sigma_{k-1}^\epsilon(n_{k-1} - n + i)$ is a discrete series representation. If ρ_k is not isomorphic to ρ_{k-1} , then the fact that there is no $x \in \mathbb{R}$, $a_k \leq x \leq b_k$, such that $R_{P_{m_k}}(\sigma_{k-1}^\epsilon(n_{k-1} - n + i))(v^x \rho_k) \neq 0$ follows from Theorem 4.2 and the already observed case $k = 1$. On the other hand, if $\rho_k \cong \rho_{k-1}$ holds, non-existence of such x follows directly from the inductive assumption. Repeating the arguments of Theorem 2.1 of [21] we deduce that every irreducible subrepresentation of $\delta([v^{-a_k} \rho_k, v^{b_k} \rho_k]) \rtimes \sigma_{k-1}^\epsilon(n_{k-1} - n + i)$ is in discrete series. Consequently, $\sigma^\epsilon(i)$ is a discrete series representation.

Also, using the classical group version of Lemma 2.1 and the inductive assumption, we see at once that if there is some x such that $R_{P_{m_k}}(\sigma_k^\epsilon(i))(v^x \rho_k) \neq 0$, for $r^\epsilon(\sigma_k) \leq i \leq n - t_\epsilon$, then either $x \leq a_k$ or $x \geq b_k$. This completes the proof. \square

5. Generic theta lifts of discrete series

The first purpose of this section is to determine the first occurrence indices of discrete series of metaplectic groups which have a generic theta lift on the $+$ -orthogonal tower. Further, we give a description of the first non-zero lifts of such representations.

Let $\sigma \in \widetilde{Irr}(Sp(n))$ denote a discrete series representation and let $\sigma_{\text{cusp}} \in \widetilde{Irr}(Sp(n_{\text{cusp}}))$ denote its partial cuspidal support. To the representation σ we attach an ordered k -tuple $S_k = (\sigma_1, \sigma_2, \dots, \sigma_k)$ as in Theorem 4.2, minimal in the sense of Definition 4.3. Again we write $\sigma_i \hookrightarrow \delta([v^{-a_i} \chi_{V, \psi} \rho_i, v^{b_i} \chi_{V, \psi} \rho_i]) \rtimes \sigma_{i-1}$.

Suppose that there is some l such that $\sigma^+(l)$ is generic representation of $O(V_l^+)$.

Corollary 3.4 implies $l \leq n$, thus $r^+(\sigma) \leq n$. Also, heredity of Whittaker models and description of cuspidal supports of theta lifts imply that the representation $\sigma_{\text{cusp}}^+(r^+(\sigma_{\text{cusp}}))$ is generic. In the sequel we denote this cuspidal representation by τ_{cusp}^+ . By the results of Shahidi, this implies that for each cuspidal selfcontragredient representation ρ whose twist appears in the cuspidal support of $\sigma^+(n)$, the induced representation $v^s \rho \rtimes \tau_{\text{cusp}}^+$ reduces for $s \in \{0, \frac{1}{2}, 1\}$. Also, the induced representation $v^s 1_{F^\times} \rtimes \tau_{\text{cusp}}^+$ reduces for $s = \frac{1}{2}$. By the rank-one reducibilities described in the Section 3, it follows $r^+(\sigma_{\text{cusp}}) \in \{n - 1, n\}$. This also shows $r^+(\sigma_{\text{cusp}}) \leq r^-(\sigma_{\text{cusp}})$.

If $r^+(\sigma_{\text{cusp}}) = n - 1$, then the induced representation $v^s \chi_{V,\psi} 1_{F^\times} \rtimes \sigma_{\text{cusp}}$ reduces for $s = \frac{3}{2}$. If $r^+(\sigma_{\text{cusp}}) = n$, then this representation reduces for $s = \frac{1}{2}$.

In the same way as in the proof of Proposition 4.5 we obtain

$$\sigma^+(n) \hookrightarrow \delta([v^{-a_k} \rho_k, v^{b_k} \rho_k]) \times \cdots \times \delta([v^{-a_2} \rho_2, v^{b_2} \rho_2]) \rtimes \sigma_1^+(n_1).$$

This shows $r^+(\sigma_1) \leq n_1$ and the last case considered in Section 4 of [14] yields that the representation $v^{\frac{1}{2}} \chi_{V,\psi} 1_{F^\times}$ does not appear in the cuspidal support of σ_1 . It is easily seen that the representation $v^{\frac{1}{2}} \chi_{V,\psi} 1_{F^\times}$ either does not appear in the cuspidal support of $\delta([v^{-a_i} \chi_{V,\psi} \rho_i, v^{b_i} \chi_{V,\psi} \rho_i])$ or appears two times there, for $i \in \{2, 3, \dots, k\}$. Thus, we have proved the following:

Lemma 5.1. *Suppose that genuine discrete series representation σ of $\widetilde{Sp}(n)$ has generic lift on the $+$ -orthogonal tower. Then the representation $v^{\frac{1}{2}} \chi_{V,\psi} 1_{F^\times}$ appears even number of times in the cuspidal support of σ .*

Previous lemma, together with Proposition 4.1, shows that either there are no twists of $\chi_{V,\psi} 1_{F^\times}$ appearing in the cuspidal support of σ_1 or there is a unique half integer $b, b \geq \frac{3}{2}$, such that σ_1 is a unique irreducible subrepresentation of $\delta([v^{\frac{3}{2}} \chi_{V,\psi} 1_{F^\times}, v^b \chi_{V,\psi} 1_{F^\times}]) \rtimes \sigma'_1$, where σ'_1 is a strongly positive discrete series without any twists of $\chi_{V,\psi} 1_{F^\times}$ in the cuspidal support. Note that the latter case can occur only if the induced representation $v^s \chi_{V,\psi} 1_{F^\times} \rtimes \sigma_{\text{cusp}}$ reduces for $s = \frac{3}{2}$.

In the following theorem we obtain important results regarding the first occurrence in the case of generic reducibilities.

Theorem 5.2. *Let $\sigma \in \text{Irr}(\widetilde{Sp}(n))$ denote a discrete series representation that has some generic lift in the $+$ -orthogonal tower. Then $r^+(\sigma)$ is an element of the set $\{n - 1, n\}$.*

Proof. We have already shown $r^+(\sigma) \leq n$. If σ is a supercuspidal representation, theorem follows from the discussion made in the beginning of this section.

If σ is a non-supercuspidal strongly positive representation, then the representation $v^{\frac{1}{2}}\chi_{V,\psi}1_{F^\times}$ does not appear in the cuspidal support of σ , since otherwise it would appear exactly once, by the classification of strongly positive discrete series (this has also been discussed in the proof of Lemma 3.6 in [15]). Thus, if the induced representation $v^s\chi_{V,\psi}1_{F^\times} \rtimes \sigma_{\text{cusp}}$ reduces for $s = \frac{1}{2}$, there are no twists of the representation $\chi_{V,\psi}1_{F^\times}$ appearing in the cuspidal support of σ . Further, if $v^s\chi_{V,\psi}1_{F^\times} \rtimes \sigma_{\text{cusp}}$ reduces for $s = \frac{3}{2}$ and there are some twists of $\chi_{V,\psi}1_{F^\times}$ appearing in the cuspidal support of σ , it follows from Proposition 4.1 that the minimal non-negative real number x such that $v^x\chi_{V,\psi}1_{F^\times}$ appears in the cuspidal support of σ equals $\frac{3}{2}$.

Consequently, the first two cases considered in Section 4 of [14] yield $r^+(\sigma) \in \{n, n-1\}$. Moreover, $r^+(\sigma) = n-1$ if and only if $v^s\chi_{V,\psi}1_{F^\times} \rtimes \sigma_{\text{cusp}}$ reduces for $s = \frac{3}{2}$ and there are no twists of the representation $\chi_{V,\psi}1_{F^\times}$ appearing in the cuspidal support of σ .

It remains to prove Theorem 5.2 for non-strongly positive discrete series σ , that is, for $k \geq 2$.

Assumption of the theorem implies that there is some $l \in \{r^+(\sigma), r^+(\sigma) + 1, \dots, n\}$ such that $\sigma^+(l)$ is generic. Corollary 6.4 of [14] shows that $\sigma^+(l)$ is a subrepresentation of

$$v^{n-l+\frac{1}{2}}1_{F^\times} \times v^{n-l+\frac{3}{2}}1_{F^\times} \times \dots \times v^{n-r^+(\sigma)-\frac{1}{2}}1_{F^\times} \rtimes \sigma^+(r^+(\sigma))$$

and from Lemma 3.2 (ii) follows that $\sigma^+(r^+(\sigma))$ is generic.

We will prove that if $r^+(\sigma) \leq n-2$ then there is no generic lift of the representation σ in the $+$ -orthogonal tower. Two possibilities will be considered separately.

- Suppose $r^+(\sigma) < n-2$.

Since $r^+(\sigma) \leq n-3$ and $n_1-3 < r^+(\sigma_1)$, Proposition 4.4 shows that there is an $i \in \{3, 4\}$ such that $(a_i, \rho_i) = (\frac{5}{2}, 1_{F^\times})$. Also, $R_{\tilde{\rho}_1}(\sigma)(v^{\frac{5}{2}}\chi_{V,\psi}1_{F^\times}) = R_{\tilde{\rho}_1}(\sigma_i)(v^{\frac{5}{2}}\chi_{V,\psi}1_{F^\times}) = 0$ and $R_{\tilde{\rho}_1}(\sigma)(v^{\frac{3}{2}}\chi_{V,\psi}1_{F^\times}) = R_{\tilde{\rho}_1}(\sigma_{i-1})(v^{\frac{3}{2}}\chi_{V,\psi}1_{F^\times}) = 0$. For simplicity of notation, we denote the representation $\sigma_{i-1}^+(r^+(\sigma_{i-1}))$ briefly by τ_{i-1} .

Combining an inductive application of Proposition 3.8 with Proposition 4.4, we deduce that $\sigma^+(r^+(\sigma))$ is a subrepresentation of

$$\begin{aligned} &\delta([v^{x_k}\rho_k, v^{b_k}\rho_k]) \times \delta([v^{x_{k-1}}\rho_{k-1}, v^{b_{k-1}}\rho_{k-1}]) \times \dots \\ &\times \delta([v^{-\frac{3}{2}}1_{F^\times}, v^{b_i}1_{F^\times}]) \rtimes \tau_{i-1} \end{aligned}$$

where x_j equals either $-a_j$ or $-a_j + 1$ for $j = i+1, i+2, \dots, k$.

Proposition 4.5 shows that both representations $\sigma^+(r^+(\sigma))$ and τ_{i-1} are in the discrete series. Note that both these representations are generic.

Combining [17], Proposition 2.1 and Lemma 5.1, with Proposition 3.1 of [5], we obtain $R_{P_1}(\sigma^+(r^+(\sigma)))(v^{\frac{3}{2}}1_{F^\times}) \neq 0$. Therefore, there is some irreducible representation τ of the appropriate odd orthogonal group such that $\sigma^+(r^+(\sigma))$ is an irreducible subrepresentation of $v^{\frac{3}{2}}1_{F^\times} \rtimes \tau$. Since $r^+(\sigma) \leq n - 3$, Proposition 3.9 implies $R_{\tilde{P}_1}(\sigma)(v^{\frac{3}{2}}\chi_{V,\psi}1_{F^\times}) \neq 0$, a contradiction.

- Suppose $r^+(\sigma) = n - 2$.

In this case, Proposition 4.4 shows that there is an $i \in \{2, 3\}$ such that $(a_i, \rho_i) = (\frac{3}{2}, 1_{F^\times})$. Further, $R_{\tilde{P}_1}(\sigma)(v^{\frac{3}{2}}\chi_{V,\psi}1_{F^\times}) = R_{\tilde{P}_1}(\sigma_i)(v^{\frac{3}{2}}\chi_{V,\psi}1_{F^\times}) = 0$. For abbreviation, we denote the representation $\sigma_{i-1}^+(n_{i-1} - 1)$ by τ_{i-1} .

Since $\sigma^+(n - 2)$ is a generic discrete series, we may apply results of Hanzer ([5]) to obtain that every generic irreducible subquotient of the standard representation $v^{\frac{3}{2}}1_{F^\times} \rtimes \sigma^+(n - 2)$ is a subrepresentation. By Lemma 3.7, $\sigma^+(n - 1)$ is a subrepresentation of $v^{\frac{3}{2}}1_{F^\times} \rtimes \sigma^+(n - 2)$.

Combining Propositions 3.8 and 4.4, we deduce that $\sigma^+(n - 2)$ is a subrepresentation of

$$\delta([v^{-a_k}\rho_k, v^{b_k}\rho_k]) \times \cdots \times \delta([v^{-a_{i+1}}\rho_{i+1}, v^{b_{i+1}}\rho_{i+1}]) \\ \times \delta([v^{-\frac{1}{2}}1_{F^\times}, v^{b_i}1_{F^\times}]) \rtimes \tau_{i-1}.$$

If $i = 2$, τ_{i-1} is a strongly positive representation containing no twists of 1_{F^\times} in its cuspidal support. On the other hand, if $i = 3$ then $b_2 \geq b_3 > \frac{3}{2}$ and τ_{i-1} is a discrete series representation such that if $R_{P_1}(\tau_{i-1})(v^x1_{F^\times}) \neq 0$ then $x \geq b_2$.

Since $(a_j, \rho_j) \neq (\frac{3}{2}, 1_{F^\times})$ for $j \in \{2, 3, \dots, k\}, j \neq i$, and $(b_j, \rho_j) \neq (\frac{3}{2}, 1_{F^\times})$ for $j \in \{2, 3, \dots, k\}$, it follows immediately that $R_{P_1}(\sigma^+(n - 2))(v^{\frac{3}{2}}1_{F^\times}) = 0$.

We directly obtain that irreducible representation $v^{\frac{3}{2}}1_{F^\times} \otimes \sigma^+(n - 2)$ appears exactly once in $\mu^*(v^{\frac{3}{2}}1_{F^\times} \rtimes \sigma^+(n - 2))$. Hence $\sigma^+(n - 1)$ is the unique irreducible subrepresentation of $v^{\frac{3}{2}}1_{F^\times} \rtimes \sigma^+(n - 2)$, and it follows that $\sigma^+(n - 1)$ is generic.

Further, Lemma 3.7 provides an embedding $\sigma^+(n) \hookrightarrow v^{\frac{1}{2}}1_{F^\times} \rtimes \sigma^+(n - 1)$ and in the same way as in the previously considered case it can be seen that $\sigma^+(n - 1)$ is a subrepresentation of

$$\delta([v^{-a_k}\rho_k, v^{b_k}\rho_k]) \times \cdots \times \delta([v^{-a_i}1_{F^\times}, v^{b_i}1_{F^\times}]) \rtimes \tau_{i-1}.$$

Repeating the same arguments as before, we conclude that $\sigma^+(n)$ is also a generic discrete series, given as the unique irreducible subrepresentation of the standard representation $v^{\frac{1}{2}}1_{F^\times} \rtimes \sigma^+(n - 1)$.

We have the following embeddings and intertwining operators:

$$\begin{aligned} \sigma^+(n) &\hookrightarrow v^{\frac{1}{2}}1_{F^\times} \rtimes \sigma^+(n-1) \\ &\hookrightarrow v^{\frac{1}{2}}1_{F^\times} \times v^{\frac{3}{2}}1_{F^\times} \rtimes \sigma^+(n-2) \\ &\rightarrow v^{\frac{3}{2}}1_{F^\times} \times v^{\frac{1}{2}}1_{F^\times} \rtimes \sigma^+(n-2). \end{aligned}$$

Note that the kernel of the last intertwining operator equals

$$L(v^{\frac{1}{2}}1_{F^\times}, v^{\frac{3}{2}}1_{F^\times}) \rtimes \sigma^+(n-2), \tag{2}$$

where $L(v^{\frac{1}{2}}1_{F^\times}, v^{\frac{3}{2}}1_{F^\times})$ stands for the unique irreducible subrepresentation of the induced representation $v^{\frac{1}{2}}1_{F^\times} \times v^{\frac{3}{2}}1_{F^\times}$. Since the representation (2) is degenerate, it follows that $\sigma^+(n)$ is a subrepresentation of $v^{\frac{3}{2}}1_{F^\times} \times v^{\frac{1}{2}}1_{F^\times} \rtimes \sigma^+(n-2)$.

Consequently, there is some irreducible representation $\tau \in Irr(O(V_{n-1}^+))$ such that $\sigma^+(n) \hookrightarrow v^{\frac{3}{2}}1_{F^\times} \rtimes \tau$. Proposition 3.9 shows $R_{\widetilde{P}_1}(\sigma)(v^{\frac{3}{2}}\chi_{V,\psi}1_{F^\times}) \neq 0$, which is impossible.

Therefore $r^+(\sigma)$ equals either $n-1$ or n and the proof is complete. □

As a consequence of the previous theorem, we obtain the main result of this paper:

Corollary 5.3. *Suppose that discrete series representation σ of $\widetilde{Sp}(n)$ has some generic lift in the $+$ -orthogonal tower. The σ is ψ -generic.*

Proof. By the results of Gan and Savin, summarized in Theorem 3.3, it is enough to prove that $\sigma^+(n)$ is generic.

By Theorem 5.2 and consequence of Corollary 3.4, either $\sigma^+(n)$ or $\sigma^+(n-1)$ is generic. If $\sigma^+(n)$ is generic, there is nothing to prove. Suppose that $\sigma^+(n-1)$ is generic. By Lemma 3.7, $\sigma^+(n)$ is subrepresentation of $v^{\frac{1}{2}}1_{F^\times} \rtimes \sigma^+(n-1)$. Further, by the first part of Lemma 3.2, every irreducible generic subquotient of $v^{\frac{1}{2}}1_{F^\times} \rtimes \sigma^+(n-1)$ is square-integrable and by the results of Hanzer ([5]) it is a subrepresentation.

By Theorem 4.5, both representations $\sigma^+(n)$ and $\sigma^+(n-1)$ are in the discrete series. It follows directly from Proposition 2.1 and Lemma 3.6 of [17] that $R_{P_1}(\sigma^+(n-1))(v^{\frac{1}{2}}1_{F^\times}) = 0$. Consequently, $v^{\frac{1}{2}}1_{F^\times} \otimes \sigma^+(n-1)$ appears exactly once in $R_{P_1}(v^{\frac{1}{2}}1_{F^\times} \rtimes \sigma^+(n-1))$.

Thus, the representation $\sigma^+(n)$ is the unique irreducible subrepresentation of $v^{\frac{1}{2}}1_{F^\times} \rtimes \sigma^+(n-1)$ and, in consequence, it is generic. □

We also note the following corollary.

Corollary 5.4. *Let $\sigma \in \widetilde{\text{Irr}}(\text{Sp}(n))$ denote a ψ -generic discrete series representation. Then $\sigma^+(r^+(\sigma))$ is generic.*

Proof. Theorem 5.2 shows $r^+(\sigma) \in \{n-1, n\}$. If $r^+(\sigma) = n$, the corollary follows from Theorem 3.3. On the other hand, if $r^+(\sigma) = n-1$, we deduce that the generic discrete series $\sigma^+(n)$ is a subrepresentation of $\nu^{\frac{1}{2}}1_{F^\times} \rtimes \sigma^+(n-1)$. Thus, genericity of $\sigma^+(n-1)$ is a consequence of Lemma 3.2 (ii). This proves the corollary. \square

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