Graphs whose Wiener index does not change when a specific vertex is removed

Martin Knor[a], Snježana Majstorović[b,*], Riste Škrekovski[c]

[a] *Slovak University of Technology in Bratislava, Faculty of Civil Engineering, Department of Mathematics, Bratislava, Slovakia*
[b] *Department of Mathematics, Josip Juraj Strossmayer University of Osijek, Osijek, Croatia*
[c] *FMF, University of Ljubljana & Faculty of Information Studies, Novo Mesto & FAMNIT, University of Primorska, Slovenia*

Abstract

The Wiener index $W(G)$ of a connected graph G is defined to be the sum of distances between all pairs of vertices in G. In 1991, Šoltés studied changes of the Wiener index caused by removing a single vertex. He posed the problem of finding all graphs G so that equality $W(G) = W(G-v)$ holds for all their vertices v. The cycle with 11 vertices is still the only known graph with this property. In this paper we study a relaxed version of this problem and find graphs which Wiener index does not change when a particular vertex v is removed. We show that there is a unicyclic graph G on n vertices with $W(G) = W(G-v)$ if and only if $n \geq 9$. Also, there is a unicyclic graph G with a cycle of length c for which $W(G) = W(G-v)$ if and only if $c \geq 5$. Moreover, we show that every graph G is an induced subgraph of H such that $W(H) = W(H-v)$. As our relaxed version is rich with solutions, it gives hope that Šoltes's problem may have also some solutions distinct from C_{11}.

Keywords: Wiener index, transmission, unicyclic graph, pendant vertex, induced subgraph

2010 MSC: 05C12, 05C90

*Corresponding author

Email addresses: knor@math.sk (Martin Knor), smajstor@mathos.hr (Snježana Majstorović), skrekovski@gmail.com (Riste Škrekovski)

1. Introduction

Throughout this paper all graphs will be finite, simple and undirected. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The number of vertices in G is usually denoted by $n(G)$. For $u, v \in V(G)$ the *distance* $d_G(u, v)$ between vertices u and v is defined as the number of edges on a shortest path connecting these vertices in G. The distance, or transmission, $t_G(v)$ of a vertex $v \in V(G)$ is the sum of distances between v and all other vertices of G. By $G - v$ we denote a graph obtained from G when v and all edges incident with v are deleted.

The *Wiener index* $W(G)$ of a connected graph G is a graph invariant, i.e. a property preserved under all possible isomorphisms of a graph. It is defined as the sum of distances between all (unordered) pairs of vertices in G:

$$W(G) = \sum_{\{u,v\} \subseteq V(G)} d_G(u, v) = \frac{1}{2} \sum_{v \in V(G)} t_G(v). \tag{1}$$

Wiener index is named after Wiener, who introduced it in 1947. In his article [15] he gave the approximation formula for the boiling point of paraffin which includes the quantity equivalent to the one given by expression (1), realizing that there are correlations between the boiling points of paraffins and the structure of their molecules. Since then, Winer index has become one of the most frequently used topological indices in chemistry, since molecules are usually modelled by undirected graphs.

The definition of Wiener index in terms of distances between vertices of a graph, such as in (1), was first given by Hosoya [6]. The same quantity has also been studied in pure mathematics under various names. It seems that the first mathematical paper on Wiener index was published in 1976 [4]. Since then, a lot of mathematicians have studied this quantity very extensively. A great deal of knowledge on Wiener index is accumulated in survey papers [3, 7, 16]. Wiener index is also closely related to some centrality measures in complex networks. Nowadays it has been frequently used in sociometry and the theory of social networks [5]. Although many papers have been devoted to Wiener index, there are still a lot of open problems and recent researches concerning this quantity;

see for instance [1, 2, 8, 10, 11]. Therefore, it is still a very popular subject of study in pure and applied mathematics.

In 1991, Šoltés [12] posed the following problem:

Problem 1. *Find all such graphs G that the equality $W(G) = W(G - v)$ holds for all their vertices v.*

Till now, only one such graph is known: it is a cycle with 11 vertices. This problem is still unsolved, but there are some computational results on graphs which preserve Wiener index after a particular vertex is removed. More precisely, in [3] there are some examples of graphs G with subtree of a same Winer index. In addition, there are several unicyclic examples that satisfy the requirements of the above problem. Every unicyclic graph contains exactly one cycle and its number of vertices and the number of edges are equal. The are many results concerning Wiener index of unicyclic graphs. We will mention only few of them: Wiener [15] calculated the largest and the smallest Wiener index among all n-vertex unicyclic graphs. Yu and Feng [17] determined the graphs having the largest and smallest Wiener index among all n-vertex unicyclic graphs of given girth. Tan et al. [13, 14] studied Wiener index of unicyclic graphs with given girth, maximum degree, number of pendant vertices and cut-vertices.

Motivated by Šoltés's problem and by some examples presented in [3], in this paper we construct an infinite family of unicyclic graphs which preserve Wiener index after removal of a particular vertex. In fact, we show that there are infinitely many unicyclic graphs with this property even when we fix the length of the cycle. Further, we characterize all n's such that there is a unicyclic graph G with a vertex v for which $W(G) = W(G - v)$. Finally, we show that for every graph G there are infinitely many graphs H such that G is an induced subgraph of H and $W(H) = W(H - v)$ for some vertex $v \in V(H) \setminus V(G)$. Our contribution shows that the class of graphs, which Wiener index does not change when a particular vertex is removed, is rich. This gives hope that Šoltes's problem may have another solution beside C_{11}.

2. Preliminaries

Let G be a connected graph. By $d_G(v)$ we denote the degree of vertex v. A *pendant vertex* is a vertex of degree one and a *pendant edge* is an edge incident with a pendant vertex. One can easily verify the formulae for Wiener index of the path P_n and cycle C_n. Wiener index of path P_n is

$$W(P_n) = \binom{n+1}{3} \tag{2}$$

and Wiener index of a cycle C_n is

$$W(C_n) = \begin{cases} \dfrac{n^3}{8} & \text{if } n \text{ is even} \\ \\ \dfrac{n(n^2-1)}{8} & \text{if } n \text{ is odd.} \end{cases} \tag{3}$$

Proposition 2. *Let G be a connected graph and $v \in V(G)$ be a pendant vertex. Let uv be the corresponding pendant edge in G and $G' = G - v$. Then*

$$W(G) = W(G') + t_{G'}(u) + n(G').$$

The next statement was proved in [9].

Theorem 3. *Let G_u and G_v be two graphs with n_u and n_v vertices, respectively, and let $u \in V(G_u)$, $v \in V(G_v)$.*

(a) If G arises from G_u and G_v by connecting u and v by an edge, then

$$W(G) = W(G_u) + W(G_v) + n_u t_{G_v}(v) + n_v t_{G_u}(u) + n_u n_v.$$

(b) If G arises from G_u and G_v by identifying u and v, then

$$W(G) = W(G_u) + W(G_v) + (n_u - 1)t_{G_v}(v) + (n_v - 1)t_{G_u}(u).$$

Our construction of unicyclic graphs G for which $W(G) = W(G - v)$ goes in the following way. Let C_c be a cycle of length c. We denote its vertices consecutively by $v_0, v_1, \ldots, v_{c-1}$. We add to C_c a pendant vertex, to obtain a new graph, then we add another pendant vertex (which may be connected to previously added vertex) and so on, until we get a unicyclic graph G with $W(G) = W(G - v_0)$. Then we continue with adding pendant vertices to create infinitely many graphs G with the property $W(G) = W(G - v_0)$. Of course, since $G - v_0$ has to be connected, we cannot add pendant vertices to v_0. In fact, most of our graphs will be obtained from C_c by adding a path to v_{c-1} and a tree to v_1, that is, usually the vertices $v_2, v_3, \ldots, v_{c-2}$ will all have degree 2 in G. Let H be a unicyclic graph containing C_c as a subgraph, and let $x \in V(G)$, $x \neq v_0$. We denote

$$\delta_H(x) = t_H(x) - t_{H-v_0}(x) \quad \text{and} \quad \Delta(H) = W(H) - W(H - v_0).$$

We start with several simple lemmas.

Lemma 4. *Let H be a unicyclic graph containing the cycle C_c as a subgraph, $d_H(v_0) = 2$, and let $u \in V(H)$, $u \neq v_0$. Take a new vertex z, connect it to u by a pendant edge, and denote the resulting graph by H^+. Then*

$$\begin{aligned} \delta_{H^+}(z) &= \delta_H(u) + 1 \\ \Delta(H^+) &= \Delta(H) + \delta_H(u) + 1. \end{aligned}$$

Proof. Observe that $t_{H^+}(z) = t_H(u) + n(H)$ while $t_{H^+-v_0}(z) = t_{H-v_0}(u) + n(H - v_0)$. Hence, $\delta_{H^+}(z) = \delta_H(u) + 1$.

By Proposition 2, we have

$$\begin{aligned} W(H^+) &= W(H) + t_H(u) + n(H) \qquad \text{and} \\ W(H^+ - v_0) &= W(H - v_0) + t_{H-v_0}(u) + n(H - v_0). \end{aligned}$$

So $\Delta(H^+) = \Delta(H) + \delta_H(u) + 1$. □

For some vertices x of the graph H from Lemma 4 it may happen that $\delta_{H^+}(x) \neq \delta_H(x)$, while for other vertices y it holds $\delta_{H^+}(y) = \delta_H(y)$. Since

$$t_{H^+}(x) = t_H(x) + d_{H^+}(x, z) \quad \text{and} \quad t_{H^+-v_0}(x) = t_{H-v_0}(x) + d_{H^+-v_0}(x, z),$$

we have $\delta_{H^+}(x) = \delta_H(x)$ for all vertices $x \in V(H)$ for which there is a shortest x-z path in H^+ avoiding v_0. Hence, by Lemma 4 we get the following observation.

Lemma 5. *Let H be a unicyclic graph containing the cycle C_c as a subgraph, such that $d_H(v_0) = 2$. Further, let H^+ be obtained from H by adding a pendant vertex z to a tree T_k attached to v_k, $1 \leq k \leq c-1$. Let $t = d_{H^+}(v_k, z)$. Then $\delta_{H^+}(z) = \delta_H(v_k) + t$ and for every vertex $x \in V(T_k)$, $x \neq z$, we have $\delta_{H^+}(x) = \delta_H(x)$.*

Observe that although $\delta_{H^+}(v_k) = \delta_H(v_k)$ in the previous lemma, in general $\delta_{H^+}(v_i) \neq \delta_H(v_i)$ if $i \neq k$. The reason is that v_0 is not on a shortest v_k-z path, i.e $d_{H^+}(v_k, z) = d_{H^+-v_0}(v_k, z)$, while for some $i \neq k$ it can happen that the unique shortest path from v_i to z passes trough v_0 so $d_{H^+}(v_i, z) \neq d_{H^+-v_0}(v_i, z)$.
As mentioned above, for some vertices x of the graph H in Lemma 4 it may happen that $\delta_{H^+}(x) \neq \delta_H(x)$. However if c is small, namely if $c \in \{3, 4\}$, then for every vertex x of H, $x \neq v_0$, there is a shortest x-z path in H^+ avoiding v_0. Hence, we have the following observation.

Lemma 6. *Let H be a unicyclic graph containing the cycle C_c as a subgraph, where $c \in \{3, 4\}$, $d_H(v_0) = 2$, and let $u \in V(H)$, $u \neq v_0$. Take a new vertex z, connect it to u by a pendant edge and denote the resulting graph by H^+. Then $\delta_{H^+}(x) = \delta_H(x)$ for every vertex $x \in V(H)$, $x \neq v_0$.*

3. Results for unicyclic graphs

Theorem 7. *Let $c \geq 8$. There exists infinitely many unicyclic graphs G with a cycle of length c and $\Delta(G) = 0$.*

Proof. Let G_1 be isomorphic to the cycle C_c. If c is even, assume that $c = 2a$, otherwise $c = 2a+1$. Our aim is to evaluate $\Delta(G_1)$ and $\delta_{G_1}(v_1)$. We distinguish two cases according to the parity of c.

Case 1: $c = 2a$. By (3) we have $W(G_1) = a^3$, and it is easy to see that

$$t_{G_1}(v_i) = 2\binom{a+1}{2} - a = a^2, \quad i = 1, 2, \ldots, c-1.$$

Since $G_1 - v_0$ is a path on $2a - 1$ vertices, by (2) we have

$$W(G_1 - v_0) = \binom{2a}{3} = \frac{1}{3}\Big(4a^3 - 6a^2 + 2a\Big),$$

and so

$$\Delta(G_1) = \frac{1}{3}\Big(- a^3 + 6a^2 - 2a \Big).$$

Next,

$$t_{G_1 - v_0}(v_1) = \binom{2a-1}{2} = 2a^2 - 3a + 1,$$

hence

$$\delta_{G_1}(v_1) = -a^2 + 3a - 1.$$

Observe that $\delta_{G_1}(v_1) < 0$ if $a \geq 3$.

Case 2: $c = 2a + 1$. By (3)

$$W(G_1) = \frac{1}{2}(2a+1)(a^2+a) = \frac{1}{2}\Big(2a^3 + 3a^2 + a\Big),$$

and we have

$$t_{G_1}(v_i) = 2\binom{a+1}{2} = a^2 + a.$$

Since $G_1 - v_0$ is a path on $2a$ vertices, by (2) we have

$$W(G_1 - v_0) = \binom{2a+1}{3} = \frac{1}{3}\Big(4a^3 - a\Big),$$

and

$$\Delta(G_1) = \frac{1}{6}\Big(- 2a^3 + 9a^2 + 5a \Big).$$

Further,

$$t_{G_1 - v_0}(v_1) = \binom{2a}{2} = 2a^2 - a,$$

and hence

$$\delta_{G_1}(v_1) = -a^2 + 2a.$$

Again, $\delta_{G_1}(v_1) < 0$ if $a \geq 3$.

Now attach to v_1 a path P^d of length $d = -\delta_{G_1}(v_1)$, namely $P^d = v_1^d v_1^{d-1} \cdots v_1^0$, where $v_1 = v_1^d$, and denote the resulting graph by G_2. By Lemma 5, we have $\delta_{G_2}(v_1^i) = -i$, where $0 \leq i \leq d$, and by Lemmas 4 and 5 we conclude

$$\Delta(G_2) = \Delta(G_1) - (d-1) - (d-2) - \cdots - (d-d) = \Delta(G_1) - \binom{d}{2}.$$

If c is even, then

$$\Delta(G_2) = \frac{1}{3}\Big(- a^3 + 6a^2 - 2a \Big) - \binom{a^2 - 3a + 1}{2} = \frac{1}{6}\Big(- 3a^4 + 16a^3 - 18a^2 + 5a \Big).$$

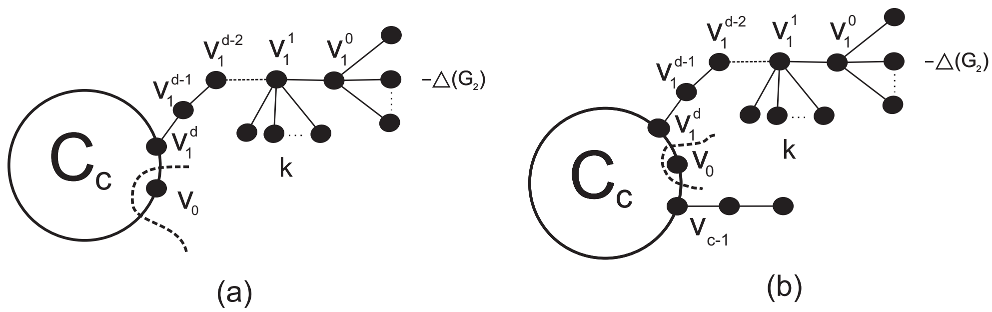

Figure 1: Configurations from the proofs of Theorems 7 and 8.

Observe that $\Delta(G_2) < 0$ if $a \geq 4$.

On the other hand, if c is odd, then

$$\Delta(G_2) = \frac{1}{6}\Big(-2a^3 + 9a^2 + 5a\Big) - \binom{a^2 - 2a}{2} = \frac{1}{6}\Big(-3a^4 + 10a^3 - a\Big).$$

Again, $\Delta(G_2) < 0$ if $a \geq 4$.

Now add to G_2 exactly $-\Delta(G_2)$ pendant vertices, connect them to v_1^0 and denote the resulting graph by G_3. Since $\delta_{G_2}(v_1^0) = 0$, by Lemma 4 we conclude that for every $x \in V(G_3) - V(G_2)$ we have $\delta_{G_3}(x) = 1$. Hence by Lemmas 4 and 5 it holds $\Delta(G_3) = \Delta(G_2) - \Delta(G_2) = 0$.

Finally, for arbitrary $k \geq 0$ add to G_3 exactly k pendant vertices, connect them to v_1^1 and denote the resulting graph by G_4, see Figure 1(a). Since $\delta_{G_3}(v_1^1) = -1$, for every $x \in V(G_4) - V(G_3)$ we have $\delta_{G_4}(x) = 0$, by Lemma 4. Hence by Lemmas 4 and 5 it holds $\Delta(G_4) = \Delta(G_3) = 0$. □

Theorem 8. *Let $c \in \{5, 6, 7\}$. Then there are infinitely many unicyclic graphs G with a cycle of length c and $\Delta(G) = 0$.*

Proof. We start with C_c. Attach to v_{c-1} a path of length 2 and denote the resulting graph by G_1. Now $\delta_{G_1}(v_1) = -2$ if $c = 5$; $\delta_{G_1}(v_1) = -5$ if $c = 6$; and $\delta_{G_1}(v_1) = -9$ if $c = 7$.

Analogously as in the proof of Theorem 7, attach to v_1 a path P^d of length $d = -\delta_{G_1}(v_1)$, namely $P^d = v_1^d v_1^{d-1} \cdots v_1^0$, $v_1 = v_1^d$, and denote the resulting graph by G_2. Notice that $G_2 - v_0$ is a path with $c + d + 1$ vertices.

Now if $\Delta(G_2) < 0$ ($c = 6, 7$), add to G_2 exactly $-\Delta(G_2)$ pendant vertices and connect them to v_1^0. On the other hand, if $\Delta(G_2) > 0$ ($c = 5$), add to G_2 exactly $\Delta(G_2)$ pendant vertices and connect them to v_1^2. Denote the resulting graph by G_3. Since $\delta_{G_2}(v_1) \leq -2$, both v_1^2 and v_1^0 exist in G_2. Moreover, $\delta_{G_2}(v_1^2) = -2$ and $\delta_{G_2}(v_1^0) = 0$, by Lemma 5. Hence, by Lemmas 4 and 5 it holds $\Delta(G_3) = 0$.

Finally, for arbitrary $k \geq 0$ add to G_3 exactly k pendant vertices, connect them to v_1^1 and denote the resulting graph by G_4, see Figure 1(b). Since $\delta_{G_3}(v_1^1) = -1$, we get $\Delta(G_4) = \Delta(G_3) = 0$ by Lemmas 4 and 5. □

Theorem 9. *Let $c \in \{3, 4\}$. Then there is no unicyclic graph G with a cycle of length c satisfying $\Delta(G) = 0$.*

Proof. Let $c \in \{3, 4\}$. By way of contradiction, suppose that there is a graph G, containing C_c as a subgraph, and such that $\Delta(G) = 0$. Observe that $\Delta(C_c) = 2$ if $c = 3$ and $\Delta(C_c) = 4$ if $c = 4$. In both cases, $\Delta(C_c) > 0$.

If $c = 3$, then $\delta_{C_c}(v_1) = \delta_{C_c}(v_2) = 1$ and if $c = 4$, then $\delta_{C_c}(v_1) = \delta_{C_c}(v_3) = 1$, while $\delta_{C_c}(v_2) = 2$. In both cases, $\delta_{C_c}(v_i) > 0$ for every i, $1 \leq i \leq c - 1$.

Let $t = V(G) - V(C_c)$. Then G was obtained from C_c by adding of t vertices, say $z_1, z_2, \ldots, z_t$. Let G_i be the graph obtained after adding of z_i. Then z_i is a pendant vertex in G_i, $V(G_i) - V(C_c) = \{z_1, z_2, \ldots, z_i\}$, $G_0 = C_c$ and $G_t = G$.

Denote by u_i the unique neighbour of z_i in G_i. By Lemma 4, we have $\delta_{G_i}(z_i) = \delta_{G_{i-1}}(u_i) + 1$, and $\delta_{G_i}(z_i) = \delta_{G_{i+1}}(z_i) = \cdots = \delta_{G_t}(z_i)$, by Lemma 6. It means that if a vertex x appears in G_{j_0}, then for all j, where $j_0 \leq j \leq t$, we have $\delta_{G_j}(x) = \delta_{G_{j_0}}(x)$. Consequently, since all vertices v_k of G_0 have $\delta_{G_0}(v_k) > 0$, it holds $\delta_{G_i}(z_i) > 0$ as well.

By Lemma 4, $\Delta(G) = \Delta(G_0) + \delta_{G_1}(z_1) + \cdots + \delta_{G_t}(z_t)$. Since all terms on the right-hand side of the equation are positive, we have $\Delta(G) > 0$, a contradiction.

□

In the next result we describe for which n there is a unicyclic graph G on

n vertices for which $\Delta(G) = 0$. We remark that for small n, namely if $n \leq 14$, the result was obtained already in [3].

Theorem 10. *A unicyclic graph G on n vertices for which $\Delta(G) = 0$ exists if and only if $n \geq 9$.*

Proof. First suppose that $n \geq 11$. We construct a required graph on n vertices. Let $c = 11$. Then $\Delta(C_c) = 0$. Take $n - 11$ vertices $z_1, z_2, \ldots, z_{n-11}$, attach them to v_3 and denote the resulting graph by G_1. Since $\delta_{C_c}(v_3) = -1$, we have $\delta_{G_1}(z_i) = 0$ for every i, $1 \leq i \leq n - 11$, by Lemma 4. Moreover, by Lemmas 4 and 5 we have

$$\Delta(G_1) = \Delta(C_c) + \sum_{i=1}^{n-11} \delta_{G_1}(z_i) = 0.$$

For $n = 9$, take C_7. Now attach one vertex to v_1, one to v_6 and denote the resulting graph by G_1. Then $\Delta(G_1) = 0$, see G_1 on Figure 6 in [3].

For $n = 10$, take C_8. Attach two pendant vertices to v_1 and denote the resulting graph by G_1. Then $\Delta(G_1) = 0$.

Since $W(C_5) - W(P_4) = 5 \neq 0$, by Theorem 9 there is no required unicyclic graph on n vertices if $n \leq 5$. The cases $n \in \{6, 7, 8\}$ were checked by a computer. □

We remark that for $n = 9$ the graph described in the proof of Theorem 10 is unique, while for $n = 10$ there are two such graphs, the other one has a cycle of length 6, see also Table II in [3].

4. Induced subgraphs

In this section we show that every graph G is an induced subgraph of a larger graph H, such that for a vertex $v_0 \in V(H) \setminus V(G)$ it holds $W(H) = W(H - v_0)$. The main tool is Theorem 3, where one graph contains G as an induced subgraph and the other graph is a cycle. Since the construction in Theorem 3(a) is just a special case of the construction in Theorem 3(b) when $d_{G_u}(u) = 1$, we analyze Theorem 3(b).

For $c \geq 3$, let C_c be a cycle with c vertices $v_0, v_1, \ldots, v_{c-1}$. Moreover, let G_m be a graph on m vertices. For $u \in V(G_m)$ let H be the graph with $m+c-1$ vertices obtained from G_m and C_c by identifying vertices u and v_i. By symmetry, we may assume that $1 \leq i \leq \frac{c}{2}$. We distinguish two cases according to the parity of c.

Case 1: $c = 2a$, $a \geq 2$. By Theorem 3(b) we have

$$\begin{aligned} W(H) &= W(G_m) + W(C_{2a}) + (m-1)t_{C_{2a}}(v_i) + (2a-1)t_{G_m}(u) \\ W(H - v_0) &= W(G_m) + W(P_{2a-1}) + (m-1)t_{P_{2a-1}}(v_i) + (2a-2)t_{G_m}(u), \end{aligned}$$

so

$$\begin{aligned} \Delta(H) &= W(C_{2a}) - W(P_{2a-1}) + (m-1)[t_{C_{2a}}(v_i) - t_{P_{2a-1}}(v_i)] + t_{G_m}(u) \\ &= a^3 - \binom{2a}{3} + (m-1)[a^2 - 2a^2 + a + (2a-i)i] + t_{G_m}(u) \\ &= \frac{1}{3}\Big[-a^3 + 6a^2 - 2a\Big] + (m-1)[a - a^2 + (2a-i)i] + t_{G_m}(u). \end{aligned}$$

Hence, $\Delta(H) = 0$ if and only if

$$t_{G_m}(u) = \frac{a}{3}\Big[a^2 - 6a + 2\Big] + (m-1)[i^2 - 2ai + a^2 - a]. \tag{4}$$

Define $f(i) = i^2 - 2ai + a^2 - a$. Observe that if $f(i) = 0$ then $t_{G_m}(u)$ does not depend on m in (4). Therefore, we set i so that $f(i) = 0$. Since $1 \leq i \leq a$, we obtain

$$i = a - \sqrt{a}.$$

Of course, i should be integer. We get the following lemma.

Lemma 11. *Let C_c be a cycle of even length, $c = 2a$, such that a is a square. Moreover, let G_m be a graph with a vertex u for which $t_{G_m}(u) = \frac{a}{3}[a^2 - 6a + 2]$. Let H be obtained from G_m and C_c by identifying u with v_i, where $i = a - \sqrt{a}$. Then $\Delta(H) = 0$.*

Case 2: $c = 2a + 1$, $a \geq 1$. Analogously as above we get

$$\Delta(H) = \frac{1}{6}\Big[-2a^3 + 9a^2 + 5a\Big] + (m-1)[-a^2 - i^2 + i(2a+1)] + t_{G_m}(u).$$

Define $f(i) = i^2 - (2a{+}1)i + a^2$. Then $f(i) = 0$ if

$$i = \frac{2a + 1 - \sqrt{4a + 1}}{2}.$$

We obtain the following.

Lemma 12. *Let C_c be a cycle of odd length, $c = 2a + 1$, such that $4a + 1$ is a square. Moreover, let G_m be a graph with a vertex u for which $t_{G_m}(u) = \frac{a}{6}[2a^2 - 9a - 5]$. Let H be obtained from G_m and C_c by identifying u with v_i, where $i = \frac{1}{2}(2a + 1 - \sqrt{4a + 1})$. Then $\Delta(H) = 0$.*

Now using Lemmas 11 and 12 we obtain the following result in which G does not need to be connected.

Theorem 13. *Let G be an arbitrary graph. Then there are infinitely many connected graphs H, containing G as an induced subgraph, and such that $W(H) = W(H - v_0)$ for some vertex $v_0 \in V(H) - V(G)$.*

Proof. We use Lemma 11, the proof using Lemma 12 is analogous. Choose a such that a is a square and $\frac{a}{3}[a^2{-}6a{+}2] \geq |V(G)|$. Obviously, there are infinitely many a's satisfying these two assumptions. Now we construct G_m. Take a new vertex u and connect it to all vertices of G. Further, take $\frac{a}{3}[a^2 - 6a + 2] - |V(G)|$ new isolated vertices, connect them to u and denote the resulting graph by G_m. Then u is adjacent to all vertices of G_m, except itself, and so

$$t_{G_u}(u) = |V(G)| + \tfrac{a}{3}[a^2 - 6a + 2] - |V(G)| = \tfrac{a}{3}[a^2 - 6a + 2].$$

Hence, take the cycle C_{2a}, identify u with v_i, where $i = a - \sqrt{a}$, and denote the resulting graph by H. By Lemma 11, $\Delta(H) = W(H) - W(H - v_0) = 0$. □

Acknowledgements. The authors acknowledge partial support by Slovak research grants VEGA 1/0026/16, VEGA 1/0142/17, APVV 0136–12 and APVV–15–0220, National Scholarschip Programme of the Slovak Republic SAIA and Slovenian research agency ARRS, program no. P1–0383.

[1] A. Abiad, B. Brimkov, A. Erey, L. Leshock, X. Martínez–Rivera, O. Suil, S. Y. Song and J. Williford, On the Wiener index, distance cospectrality and transmission-regular graphs, *Discrete Appl. Math.* **230** (2017) 1–10.

[2] Z. Cinkir, Contraction formulas for the Kirchhoff and Wiener indices, *MATCH Commun. Math. Comput. Chem.* **75** (2016) 169–198.

[3] A. A. Dobrynin, R. Entringer and I. Gutman, Wiener index of trees: theory and application, *Acta Appl. Math.* **66** (2001) 211–249.

[4] R. C. Entringer, D. E. Jackson and D. A. Snyder, Distance in graphs, *Czechoslovak Math. J.* **26** (1976) 283–296.

[5] E. Estrada, The Structure of Complex Networks: Theory and Applications, *Oxford University Press* (2011).

[6] H. Hosoya, Topological index, a newly proposed quantity characterizing the topological nature of structural isomers of saturated hydrocarbons, *Bull. Chem. Soc. Jpn.* **44** (1971) 2332–2339.

[7] M. Knor, R. Škrekovski and A. Tepeh, Mathematical aspects on Wiener index, *Ars Math. Contemp.* **11** (2016) 327–352.

[8] M. Krnc, R. Škrekovski, On Wiener inverse interval problem, *MATCH Commun. Math. Comput. Chem.* **75** (2016) 71–80.

[9] O. E. Polansky, D. Bonchev, The Wiener number of graphs. I. General theory and changes due to some graph operations, *MATCH Commun. Math. Comput. Chem.* **21** (1986) 133–186.

[10] H. S. Ramane, V. V. Manjalapur, Note on the bounds on Wiener number of a graph, *MATCH Commun. Math. Comput. Chem.* **76** (2016) 19–22.

[11] H. Shabani, A. R. Ashrafi, Symmetry–moderated Wiener index, *MATCH Commun. Math. Comput. Chem.* **76** (2016) 3–18.

[12] L'. Šoltés, Transmission in graphs: A bound and vertex removing, *Math. Slovaca* **41** (1991) 11–16.

[13] S. Tan, Y. Lin, The largest Wiener index of unicyclic graphs given girth or maximum degree, *J. Appl. Math. Comput.* **53**(1) (2017) 343–363.

[14] S. Tan, Q. Wang, Y. Lin, The Wiener index of unicyclic graphs given number of pendant vertices or cut vertices, *J. Appl. Math. Comput.* (2016) doi:10.1007/s12190-016-1022-y.

[15] H. Wiener, Structural determination of paraffin boiling points, *J. Am. Chem. Soc.* **69** (1947) 17–20.

[16] K. Xu, M. Liu, K. C. Das, I. Gutman and B. Furtula, A survey on graphs extremal with respect to distance-based topological indices, *MATCH Commun. Math. Comput. Chem.* **71** (2014) 461–508.

[17] G. H. Yu, L. H. Feng, On the Wiener index of unicyclic graphs with given girth, *Ars Comb.* **94** (2010) 361–369.