Ehrenfest-Brillouin-type correlated continuous time random walk and fractional Jacobi diffusion

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The paper is dedicated to Professor O. V. Ivanov on the occasion of his 70th birthday

Abstract. Continuous time random walks (CTRWs) have random waiting times between particle jumps. Based on Ehrenfest-Brillouin-type model motivated by economics, we define the correlated CTRW that converge to the fractional Jacobi diffusion $Y(E(t))$, $t \geq 0$, defined as a time change of Jacobi diffusion process $Y(t)$ to the inverse $E(t)$ of the standard stable subordinator. In the CTRW considered in this paper, the jumps are correlated so that in the limit the outer process $Y(t)$ is not a Lévy process but a diffusion process with non-independent increments. The waiting times between jumps are selected from the domain of attraction of a stable law, so that the correlated CTRWs with these waiting times converge to $Y(E(t))$.

Key words. Correlated continuous time random walk; Ehrenfest-Brillouin Markov chain; Fractional diffusion; Jacobi diffusion; Pearson diffusion

Mathematics Subject Classification (2010): 47D07, 60J10, 60J60, 60G22, 60G50.

1 Introduction

Recent development of theory and applications of the anomalous diffusion have often been motivated by models for tracking movements of the state-changing particle, assuming random waiting times between its transitions from one state to another (jumps). Such models are known as continuous time random walks (CTRWs). When jumps and waiting times are independent, the CTRW is called decoupled. For more information and applications of coupled and uncoupled CTRW we refer to [16, 5].

In the simplest case when the particle jumps $Y_1, \dots, Y_n, \dots$ are independent and identically distributed (iid), the random walk $S(n) = Y_1 + \ldots + Y_n$ converges to either the Brownian motion or a stable Lévy process [19, Chapter 4]. More precisely, if the waiting times between particle jumps are modeled by iid random variables $G_1, \dots G_n$ from the domain of attraction of a positively skewed stable law with stability index $0 < \beta < 1$, the CTRW process $S(N(t))$, where $T(n) = G_1 + \ldots + G_n$, $N(t) = \max\{n \geq 0 \colon T(n) \leq t\}$, gives the location of a particle at time $t \geq 0$. Then by applying the continuous mapping argument (see [19, Theorem 4.19]), it follows that with proper scaling, $S(N(\lfloor ct \rfloor))$ converges as $c \to \infty$, to $A(E(t))$, where A is either the Brownian motion or a stable Lévy process, and $E(t)$ is the inverse of a standard β-stable subordinator $(D(t),\, t \geq 0)$. Meerschaert and colleagues have shown the convergence to hold in M_1 and J_1 Skorokhod topology ([17, 25]), and have obtained other Lévy processes as the outer process in the limit by employing triangular arrays [18].

CTRWs were first proposed in 1965 by Montroll and Weiss in [21] and developed further in [9], [20], [22] and [23]. Correlated CTRWs are obtained when the particle jumps or waiting times in CTRW are correlated. Concept of correlated particle jumps given by the stationary linear process were considered in [15], where the outer process in the limit was either a stable Lévy process or a linear fractional stable motion, depending on the strength of the dependence in the particle jump sequence. Situation when the jump distribution depends on the current particle position was treated in [10], where Kolokoltsov develops the theory of subordination of Markov processes by the hitting-time process, showing that this procedure leads to generalized fractional evolutions. Recently, in [11] he develops more general CTRW concepts, the so-called controlled CTRWs. Similarly, the case of correlated waiting times in CTRW model was considered in [26].

Applications of correlated CTRW with time averaged waiting times in statistical mechanics are studied in [14]. Moreover, investigation of limiting correlated CTRW processes was considered in [1] and [24]. In [24], strongly motivated by applications in biology and physics, the terms CTRW and correlated CTRW and their exploration is placed in a slightly different and broader environment than in this paper. Namely, the composition of the outer continuous time Markov process and the inverse of the stable subordinator (as the inner process) is refered to as the CTRW. In our approach this corresponds to a fractional weak limit of what we call CTRW. In construction of correlated CTRW in [24], the authors begin with a discrete time random walk in continuous space with iid jump lengths, called the uncorrelated random walk. This process is then generalized by allowing the correlation between jump lengths, by the means of a non-random function called the correlation kernel. In such process the discrete time is replaced by a continuous time variable and for the correlation kernel the power-law is chosen. The correlation is introduced in the waiting time structure as well. Finally, the correlated CTRW is defined by combining the inverse of correlated waiting time process with a correlated motion process. Such a construction incorporates limiting process of correlated CTRW observed in this paper, which is based on uncorrelated structure of observed waiting times.

In the recent paper [12], the correlated CTRWs converging to fractional Pearson diffusions have been constructed using the Laplace-Bernoulli urn scheme and Wright-Fischer genetic model [8]. Fractional Pearson diffusions are obtained via a time change to the inverse stable subordinator in classical Pearson diffusions that are unique strong solutions of the stochastic differential equation (SDE) with polynomial coefficients and standard Brownian motion $(W(t),\, t \geq 0)$ as the driving process:

$$dX(t) = -\theta(X(t) - \mu)dt + \sqrt{2\theta(b_2 X^2(t) + b_1 X(t) + b_0)}dW(t), \quad t \geq 0,$$

where $b_0, b_1, b_2 \in \mathbb{R}$ are such that the square root is well defined on the diffusion state space (l, L), $\mu \in \mathbb{R}$ is the mean of the corresponding stationary distribution with the density $\mathfrak{m}(x)$ satisfying the Pearson differential equation

$$\mathfrak{m}'(x)/\mathfrak{m}(x) = [a(x) - b'(x)]/[b(x)] = [(a_1 - 2b_2)x + (a_0 - b_1)]/[b_2x^2 + b_1x + b_0]$$

and $\theta > 0$ is the autocorrelation parameter, i.e. the scaling of time determining the speed of the reversion to the stationary mean. Pearson diffusions are categorized into six subfamilies (see e.g. [3] and [13]). The Jacobi diffusion considered in this paper is one of the six subfamilies, and its Beta stationary distribution arises in the model motivated by economics, particle physics, and genetics. The model based on the Ehrenfest-Brillouin Markov chain is used in this paper to construct CTRW that converge to the fractional Jacobi diffusion. This construction is different from the ones in [12] and is motivated by applications to economics, where the Markov chain is used to model the movement of objects among states, with the moves separated by waiting times with a heavy-tailed distribution.

The paper is organized as follows: Section 2 provides the details on the Jacobi diffusion and fractional Jacobi diffusion and gives an overview of the general approach to the construction of Markov chains that converge to Feller processes. In section 3, Ehrenfest-Brillouin Markov chain is introduced, then specially transformed and re-scaled to converge to the Jacobi diffusion. Finally, in Section 4, the correlated CTRW is constructed, and its convergence to the fractional Jacobi diffusion is proven.

2 Jacobi diffusion and fractional Jacobi diffusion

The Jacobi diffusion $Y = (Y(t),\, t \geq 0)$ is the strong solution of the SDE

$$dY(t) = -\gamma(Y(t) - \mu)dt + \sqrt{2\gamma\delta Y(t)(1 - Y(t))}dW(t), \quad t \geq 0.$$

Its state space is $[0, 1]$ and infinitesimal generator is

$$\mathcal{A}f(y) = -\gamma(y - \mu)f'(y) + \frac{1}{2}(2\gamma\delta)y(1 - y)f''(y), \quad f \in C_c^3([0, 1]), \tag{2.1}$$

where $C_c^3([0, 1])$ is the space of three times continuously differentiable real-valued functions on $[0, 1]$ with compact support.

Its invariant distribution is the Beta distribution with the density function

$$f(x) = \frac{1}{B\left(\frac{\mu}{\delta}, \frac{1-\mu}{\delta}\right)} x^{\frac{\mu}{\delta}-1}(1 - x)^{\frac{1-\mu}{\delta}-1} \cdot \mathrm{I}_{[0,1]}(x), \quad 0 < \mu < 1, \quad \gamma, \delta > 0, \tag{2.2}$$

where $B\left(\frac{\mu}{\delta}, \frac{1-\mu}{\delta}\right)$ is the standard Beta function.

The transition density of the time-homogeneous diffusion $p(x, t; y) = \frac{d}{dx}P(Y(t) \leq x|Y(0) = y)$ solves the Cauchy problem for the backward Kolmogorov equation:

$$\frac{\partial p(x, t; y)}{\partial t} = \mathcal{A}p(x, t; y)$$

with space-varying polynomial coefficients and the point-source initial condition $p(x, 0; y) = \delta(y)$.

For $0 < \beta < 1$, the fractional Jacobi diffusion $(Y_\beta(t),\, t \geq 0)$ is obtained by a non-Markovian time-change $E(t)$ independent of $Y(t)$, i.e.

$$Y_\beta(t) := Y(E(t)), \quad t \geq 0,$$

where $E(t) = \inf\{x > 0 : D(x) > t\}$ is the inverse of the standard β-stable Lévy subordinator $(D(t),\, t \geq 0)$, $0 < \beta < 1$, with the Laplace transform $E[e^{-sD(t)}] = \exp\{-ts^\beta\}$, $s \geq 0$. Since fractional diffusion rests for periods of time with non-exponential distribution, it is clearly non-Markovian. We say that fractional diffusion $Y_\beta(t)$ has a transition density $p_\beta(x, t; y)$ if

$$P(Y_\beta(t) \in B | Y_\beta(0) = y) = \int_B p_\beta(x, t; y) dx$$

for any Borel subset B of its state space.

The generator of the Jacobi diffusion has a purely discrete spectrum, consisting of infinitely many simple eigenvalues $(\lambda_n,\, n \in \mathbb{N})$ (see [13]), and the corresponding orthonormal eigenfunctions $(Q_n(x), n \in \mathbb{N})$, are the Jacobi orthogonal polynomials. The spectral representations of the transition density of the Jacobi diffusion in terms of the infinite sum including the corresponding eigenvalues and orthogonal polynomials is well known (see e.g. [8]) and is used in [13] to obtain the spectral representation of transition density of fractional diffusion and explicit strong solutions of the corresponding fractional Cauchy problems for both backward and forward Kolmogorov equations.

We finish this section with a brief overview of methodology used to prove the convergence of specially constructed Markov chain to Jacobi diffusion. We consider a suitable homogeneous, irreducible and aperiodic birth-and-death Markov chain with the finite state space $S = \{0, 1, 2, \ldots, n\}$ and only three non-null one-step transition probabilities: the probability of transition from state i to state $(i + 1)$, to state $(i - 1)$ or staying in the same state i. Besides its starting distribution and the one-step transition matrix, the dynamics of such chain can be described by its transition operator T:

$$Tf(i) = (Tf)(i) = \int \mu(i, dy) f(y), \quad i \in S, \tag{2.3}$$

where μ is the corresponding probability kernel on a measurable space $(S, \mathcal{S})$ and $f \colon S \to \mathbb{R}$ is assumed to be measurable and either bounded or nonnegative (for more details we refer to [6, Chapter 19]).

In order to obtain diffusion as the scaling limit of the suitably chosen Markov chain with known transition operator, we state the basic result on which our construction relies. Let $\mathbb{D}(S)$ be the space of right continuous functions with left limits defined on $\mathbb{R}^+$ with values in S endowed with Skorokhod J_1 topology. Since the state space of the Jacobi diffusion is the interval $[0, 1]$, we will only consider Banach space of bounded continuous functions on $S = [0, 1]$ with the supremum norm.

For a closed operator $\mathcal{A}$ with the domain $\mathcal{D}$, a core of operator $\mathcal{A}$ is a linear subspace $D \subset \mathcal{D}$ such that the restriction $\mathcal{A}|_D$ has closure $\mathcal{A}$. In that case, $\mathcal{A}$ is clearly uniquely determined by its restriction $\mathcal{A}|_D$. Note that the Jacobi diffusion satisfies conditions of Theorem 1.6 from [2, Section 8] which gives sufficient conditions for $C_c^\infty(S)$, the space of infinitely differentiable continuous functions with compact support, to be a core of this diffusion infinitesimal generator. Therefore $C_c^3(S)$, as a broader space, can be referred to as the core of the Jacobi diffusion as well.

The next theorem is a key tool for proving the convergence of Markov chains to a Feller process (for proof we refer to [6, Theorem 19.28, page 387]).

Theorem 2.1. *Let $(Y^{(n)}, n \in \mathbb{N})$ be a sequence of discrete-time Markov chains on S with transition operators $(U_n, n \in \mathbb{N})$. Consider a Feller process X on S with semigroup T_t and generator $\mathcal{A}$. Fix a core D for the generator $\mathcal{A}$, and assume that $(h_n, n \in \mathbb{N})$ is the sequence of positive reals tending to zero as $n \to \infty$. Let*

$$A_n = h_n^{-1}(U_n - I), \quad T_{n,t} = U_n^{\lfloor t/h_n \rfloor}, \quad X^{(n)}(t) = Y^{(n)}(\lfloor t/h_n \rfloor).$$

Then the following statements are equivalent:

a) If $f \in D$, there exist some $f_n \in Dom(A_n)$ with $f_n \to f$ and $A_n f_n \to \mathcal{A} f$ as $n \to \infty$

b) $T_{n,t} \to T_t$ strongly for each $t > 0$

c) $T_{n,t} f_n \to T_t f$ for each $f \in C_0$, uniformly for bounded $t > 0$

d) if $X^{(n)}(0) \Rightarrow X(0)$ in S, then $X^{(n)} \Rightarrow X$ in the Skorokhod space $\mathbb{D}(S)$ with the J_1 topology.

3 Ehrenfest-Brillouin Markov chain

In this section, motivated by applications to economics, particle physics and genetics, we present the discrete-time birth-and-death Markov chain which have the Jacobi diffusion as the scaling limit.

3.1 Ehrenfest-Brillouin Markov chain

The dynamics of this model, in which n objects move within N categories according to prescribed transition probabilities, could be viewed as the generalization of the famous Ehrenfest's model (see, for example, [7]). In the Ehrenfest-Brillouin model, the destruction mechanism is the same as in Ehrenfest's mode, but the creation mechanism is more general and more complex than in the Ehrenfest's case. Here we give a brief overview of the facts on model dynamics, according to [4], inheriting the notation.

To explain the destruction-creation mechanism of this Markov chain, consider a population of n objects that could be interpreted as particles in a physical system, genes in applications in genetics or agents in economics models. The state of the system is given by the occupation number vector

$$\mathbf{n} = (n_1, \ldots, n_i, \ldots, n_N), \quad n_k \geq 0, \quad \forall k \in \{1, \ldots N\}, \quad \sum_{k=1}^{N} n_k = n.$$

Obviously, the state space is the set of N-tuples with non-negative components summing up to n, denoted here as S_N^n. The dynamics of the system observed here is simple: the state of the system in one step changes from initial state $\mathbf{n} = (n_1, \ldots, n_i, \ldots, n_k, \ldots, n_N)$ to the final state $\mathbf{n}_i^k = (n_1, \ldots, n_i - 1, \ldots, n_k + 1, \ldots, n_N)$. This change of state could be viewed as the two-component transition:

- the destruction of the object on the ith coordinate (category) in the initial state $\mathbf{n}$ (the "Ehrenfest's term"), resulting in the state vector

  $$\mathbf{n}_i = (n_1, \ldots, n_i - 1, \ldots, n_k, \ldots, n_N),$$

  which happens with probability

  $$P(\mathbf{n}_i | \mathbf{n}) = \frac{n_i}{n}$$

- the creation of the object in the kth coordinate (category) given the state vector $\mathbf{n}_i$, resulting in the final state vector $\mathbf{n}_i^k$, with probability

$$P(\mathbf{n}_i^k|\mathbf{n}_i) = \frac{\alpha_k + n_k - \delta_{k,i}}{\alpha + n - 1},$$

  where $\alpha = (\alpha_1, \dots, \alpha_N)$ is the vector of parameters such that $\sum_{k=1}^{N} \alpha_k = \alpha$ and $\delta_{k,i}$ is the usual Kronecker's delta symbol, taking value 1 when $k = i$ and zero otherwise.

Interpretation of parameter α_i is related to the probability of accommodation on the coordinate (category) i if it is empty. In [4] two interesting cases are discussed. In the first case all α_i are negative. Then the population size is limited by $|\alpha|$ and categories by $|\alpha_i|$. In this case the transition probability is

$$P(\mathbf{n}_i^k|\mathbf{n}) = P(\mathbf{n}_i|\mathbf{n}) \cdot P(\mathbf{n}_i^k|\mathbf{n}_i) = \frac{n_i}{n} \cdot \frac{\alpha_k + n_k - \delta_{k,i}}{\alpha + n - 1}. \tag{3.1}$$

In the second case, all $\alpha_i > 0$. Then, starting from initial state $\mathbf{n}$ by repeated application of the previous transition probabilities, each state from the state space S_N^n can be reached with positive probability, meaning that the Ehrenfest-Brillouin Markov chain is irreducible. Finiteness of the state space together with the irreducibility implies that Ehrenfest-Brillouin Markov chain is recurrent, and therefore it has a unique invariant measure $\pi(\mathbf{n})$. Furthermore, the transition probability doesn't exclude the case $k = j$, so this Markov chain is aperiodic. It implies that the invariant measure $\pi(\mathbf{n})$ is the equilibrium distribution as well. The standard procedure recovers the N-dimensional Pólya distribution

$$\pi(\mathbf{n}) = Pólya(\mathbf{n}; \alpha) = \frac{n!}{\alpha^{[n]}} \prod_{i=1}^{N} \frac{\alpha_i^{[n_i]}}{n_i!}, \quad \sum_{i=1}^{N} n_i = n, \quad \sum_{i=1}^{N} \alpha_i = \alpha, \tag{3.2}$$

$$\alpha^{[n]} = \alpha \cdot (\alpha + 1) \cdot \ldots \cdot (\alpha + n - 1),$$

as the equilibrium distribution (see [4, P.175]). This distribution comprises some famous multivariate distributions of quantum physics:

- If all $\alpha_i > 0$, the special case of equilibrium distribution (3.2) for $\alpha_i = 1$ and $\alpha = N$ is the Bose-Einstein distribution.
- If all $\alpha_i < 0$, (3.2) is the N-dimensional hypergeometric distribution whose special case, for $\alpha_i = -1$ and $\alpha = -N$ is the Fermi-Dirac distribution.
- As $|\alpha| \to \infty$, the limit of (3.2) is the multinomial distribution whose symmetric case is known as the Maxwell-Boltzmann distribution.

An important observation, directly connecting one particular case of this model to Jacobi diffusion, is that in case of two categories, from the invariant Pólya distribution $Pólya(k, n-k; 1/2, 1/2)$ the distribution of the ratio k/n is the Beta distribution (2.2) with $\mu = 1/2$ and $\delta = 1$. For more details we refer to [4, Section 7.3].

One example of the Ehrenfest-Brillouin Markov chain is the taxation-redistribution economics model, see [4, p.212], where n coins are redistributed among N agents. A taxation is a step in which coin is randomly taken out of the set of n coins (destruction) and a redistribution is a step in which the coin is given to one of N agents (creation). The (destruction) probability of selecting one coin belonging to the ith agent is n_i/n, while in the redistribution step there are several possible schemes, e.g. favoring the agents already having many coins or those having few coins. For example, if it is assumed that the probability of giving the coin taken from agent i to agent j is proportional to $(w_j + n_j)$, where n_j is the wealth of jth agent and w_j is the corresponding weight, then depending on the choice of the weight different equilibrium distributions could be obtained. In this general framework one could assume that the transition probability is of the following form:

$$P(\mathbf{n}_i^j|\mathbf{n}) = \frac{n_i}{n} \cdot \frac{w_j + n_j - \delta_{i,j}}{w + n - 1}, \quad w = \sum_{i=1}^{N} w_i.$$

If no agent is favored in this scheme, then $w_j = \alpha$ for all $j \in \{1, \dots, N\}$, and therefore

$$P(\mathbf{n}_i^j|\mathbf{n}) = \frac{n_i}{n} \cdot \frac{\alpha + n_j - \delta_{i,j}}{N\alpha + n - 1}, \tag{3.3}$$

which is exactly the Ehrenfest-Brillouin model with unary moves. For more details on the taxation-redistribution model see [4, Section 8.2], while [4, Section 8.3] contains more applications of the Ehrenfest-Brillouin model to economics. One particularly interesting example is the relationship of this model to the Aoki-Yoshikawa model for sectoral productivity, which is detailed in [4, Section 8.3].

3.2 Jacobi diffusion as a scaling limit of Ehrenfest-Brillouin Markov chain

In this subsection we use the margin of the two-dimensional Ehrenfest-Brillouin Markov chain from Subsection 3.1 to construct a transformed and rescaled chain converging to the Jacobi diffusion, using Theorem 2.1. First we introduce the needed notation and technical details.

For each $n \in \mathbb{N}$, denote by $(G^{(n)}(r),\, r \in \mathbb{N}_0)$ the marginal Ehrenfest-Brillouin Markov chain with the state space $\{0, 1, 2, \dots, n\}$. The transition probabilities for this Markov chain are as follows:

$$p_{i,i+1} = \frac{n-i}{n} \cdot \frac{\alpha_1 + i}{\alpha_1 + \alpha_2 + n - 1}, \quad p_{i,i-1} = \frac{i}{n} \cdot \frac{\alpha_2 + n - i}{\alpha_1 + \alpha_2 + n - 1}, \quad p_{i,i} = 1 - p_{i,i+1} - p_{i,i-1}, \tag{3.4}$$

0 otherwise, where $\alpha_1 > 0$, $\alpha_2 > 0$. In light of the taxation-redistribution model with uniformly weighted agents (with weight α), these transition probabilities could be interpreted in terms of the number of coins belonging to agent 1 in time t. If we start with i coins, $p_{i,i+1}$ is the probability that a randomly chosen coin, out from the set of $(n - i)$ coins belonging to other agents, is redistributed to agent 1; $p_{i,i-1}$ is the probability that a randomly chosen coin, out of i coins belonging to agent 1, is redistributed to one of the other agents; $p_{i,i}$ is the probability that reflects agent 1 invariance to the coin "destruction-creation".

As previously stated, we assume that the initial state of the chain is given by

$$i(y) = i = G^n(0) = \lfloor ny \rfloor,$$

$y \in [0, 1]$ being the initial state of the corresponding limiting diffusion. Even though the initial state is a function of y, we will use the notation i for simplicity. For each $n \in \mathbb{N}$ we define the new Markov chain $(H^{(n)}(r),\, r \in \mathbb{N}_0)$, where

$$H^{(n)}(r) := \frac{G^{(n)}(r)}{n} \tag{3.5}$$

with state space $\{0, \frac{1}{n}, \frac{2}{n}, \ldots, 1\}$.
The transition operator T_n of the new Markov chain $(H^{(n)}(r),\, r \in \mathbb{N}_0)$ is defined as follows:

$$T_n f\left(\frac{i}{n}\right) = \sum_{j=0}^{n} p_{ij} f\left(\frac{j}{n}\right) = p_{i,i-1}\, f\left(\frac{i-1}{n}\right) + p_{i,i}\, f\left(\frac{i}{n}\right) + p_{i,i+1}\, f\left(\frac{i+1}{n}\right). \tag{3.6}$$

Define the operator

$$A_n := \theta n^2 (T_n - I), \quad f_n \in Dom(A_n), \quad f_n(y) := f\left(\frac{\lfloor ny \rfloor}{n}\right) = f\left(\frac{i}{n}\right), \tag{3.7}$$

where $f \in C_c^3\left([0,1]\right)$.

By the scaling of time in $(H^{(n)}(r),\, r \in \mathbb{N}_0)$, for each $n \in \mathbb{N}$ we obtain the corresponding continuous-time process $(Y^{(n)}(t),\, t \geq 0)$:

$$Y^{(n)}(t) := H^{(n)}(\lfloor \theta n^2 t \rfloor), \ \theta > 0. \tag{3.8}$$

The next theorem states that the Jacobi diffusion could be obtained as the limiting process of the previously defined time-changed processes $(Y^{(n)}(t),\, t \geq 0)$.

Theorem 3.1. *For each $n \in \mathbb{N}$, let $(H^{(n)}(r),\, r \in \mathbb{N}_0)$ be the Markov chain defined by* (3.5) *with the transition operator* (3.6)*. For each $n \in \mathbb{N}$, let $Y^n = (Y^{(n)}(t),\, t \geq 0)$, be its corresponding time-changed process with the time-change* (3.8)*. Let the operators $(A_n,\, n \in \mathbb{N})$ be defined by* (3.7)*. Then*

$$Y^n \Rightarrow Y \ \textit{in} \ \mathbb{D}([0,1]),$$

where $Y = (Y(t),\, t \geq 0)$ is the Jacobi diffusion with the infinitesimal generator $\mathcal{A}$ given by (2.1)*, and*

$$\delta = \frac{1}{\alpha_1 + \alpha_2}, \quad \gamma = \theta(\alpha_1 + \alpha_2), \quad \mu = \frac{\alpha_1}{\alpha_1 + \alpha_2}.$$

Proof. We are now in the setting of the Theorem 2.1. We first prove that statement *a*) of Theorem 2.1 is valid for A_n defined by (3.7) and $\mathcal{A}$ defined by (2.1). Then we use the equivalence of statements *a*) and *d*) from Theorem 2.1 to obtain convergence $Y^n \Rightarrow Y$ in $\mathbb{D}([0,1])$ under assumption $Y^{(n)}(0) \Rightarrow Y(0)$, $n \to \infty$.

First, we prove the statement *a*) from Theorem 2.1 in our setting:

$$||f_n - f||_\infty = \sup_{y \in [0,1]} |f_n\left(y\right) - f\left(y\right)| = \sup_{y \in [0,1]} |f\left(i/n\right) - f\left(y\right)|.$$

According to the well-known property $\lfloor ny \rfloor \leq ny < \lfloor ny \rfloor + 1$ of the function $\lfloor \cdot \rfloor$, it follows that

$$\frac{i}{n} \leq y < \frac{i}{n} + \frac{1}{n}$$

and therefore

$$\lim_{n \to \infty} \frac{i}{n} = y.$$

From this we obtain

$$||f_n - f||_\infty \to 0, \ n \to \infty$$

and we have

$$||A_n f_n - \mathcal{A} f||_\infty = \sup_{y \in [0,1]} |A_n f_n(y) - \mathcal{A} f(y)| = \sup_{y \in [0,1]} |A_n f(i/n) - \mathcal{A} f(y)|.$$

According to (3.7) it follows that

$$A_n f\left(\frac{i}{n}\right) = \theta n^2 \left(\sum_{j=0}^{n} p_{ij} f\left(\frac{j}{n}\right) - f\left(\frac{i}{n}\right)\right) = \theta n^2 \sum_{j=0}^{n} p_{ij} \left(f\left(\frac{j}{n}\right) - f\left(\frac{i}{n}\right)\right).$$

By the Taylor formula for function f around $\frac{i}{n}$ with the mean-value form of the remainder we obtain

$$A_n f\left(\frac{i}{n}\right) = \theta n \sum_{j=0}^{n} p_{ij}\left(j-i\right) f'\left(\frac{i}{n}\right) + \theta \sum_{j=0}^{n} p_{ij} \frac{\left(j-i\right)^2}{2} f''\left(\frac{i}{n}\right) + \theta \sum_{j=0}^{n} p_{ij} \frac{\left(j-i\right)^3}{6n} f'''\left(\zeta\right), \tag{3.9}$$

where ζ is a real number such that $|\zeta - \frac{i}{n}| < |\frac{j}{n} - \frac{i}{n}|$.
Next, denote

$$\mu(y) := \lim_{n\to\infty} n \sum_{j=0}^{n} p_{ij}\left(j-i\right), \quad \sigma^2(y) := \lim_{n\to\infty} \sum_{j=0}^{n} p_{ij}\left(j-i\right)^2, \quad R_n(y) := \theta \sum_{j=0}^{n} p_{ij} \frac{\left(j-i\right)^3}{6n} f'''\left(\zeta\right).$$

Taking into account (3.9), we obtain

$$\lim_{n\to\infty} A_n f\left(\frac{i}{n}\right) = \theta\mu(y) f'\left(y\right) + \frac{\theta}{2}\sigma^2(y) f''\left(y\right) + \lim_{n\to\infty} R_n(y). \tag{3.10}$$

It follows that

$$\begin{aligned}
\mu(y) &= \lim_{n\to\infty} n \sum_{j=0}^{n} p_{ij}\left(j-i\right) = \lim_{n\to\infty} n\left(p_{i,i+1} - p_{i,i-1}\right) \\
&= \lim_{n\to\infty}\left((n-i)\cdot\frac{\alpha_1 + i}{\alpha_1+\alpha_2+n-1} - i\cdot\frac{\alpha_2+n-i}{\alpha_1+\alpha_2+n-1}\right) \\
&= \lim_{n\to\infty}\left(\frac{n\alpha_1 - i(\alpha_1+\alpha_2)}{\alpha_1+\alpha_2+n-1}\right) \\
&= \lim_{n\to\infty}\left(\frac{\alpha_1 - \frac{i}{n}(\alpha_1+\alpha_2)}{\frac{\alpha_1}{n}+\frac{\alpha_2}{n}+1-\frac{1}{n}}\right) \\
&= -(\alpha_1+\alpha_2)y + \alpha_1,
\end{aligned} \tag{3.11}$$

$$\begin{aligned}
\sigma^2(y) &= \lim_{n\to\infty} \sum_{j=0}^{n} p_{ij}\left(j-i\right)^2 = \lim_{n\to\infty}\left(p_{i,i+1} + p_{i,i-1}\right) \\
&= \lim_{n\to\infty}\left(\frac{n-i}{n}\cdot\frac{\alpha_1+i}{\alpha_1+\alpha_2+n-1} + \frac{i}{n}\cdot\frac{\alpha_2+n-i}{\alpha_1+\alpha_2+n-1}\right) \\
&= \lim_{n\to\infty}\left((1-\frac{i}{n})\cdot\frac{\frac{\alpha_1}{n}+\frac{i}{n}}{\frac{\alpha_1}{n}+\frac{\alpha_2}{n}+1-\frac{1}{n}} + \frac{i}{n}\cdot\frac{\frac{\alpha_2}{n}+1-\frac{i}{n}}{\frac{\alpha_1}{n}+\frac{\alpha_2}{n}+1-\frac{1}{n}}\right) \\
&= (1-y)y + y(1-y) = 2y(1-y),
\end{aligned} \tag{3.12}$$

$$\begin{aligned}
\lim_{n\to\infty}|R_n(y)| &\le K\left|\theta \lim_{n\to\infty}\sum_{j=0}^{n} p_{ij}\frac{\left(j-i\right)^3}{6n}\right| = K\left|\theta \lim_{n\to\infty}\frac{1}{6n}\left(p_{i,i+1} - p_{i,i-1}\right)\right| \\
&= K\left|\theta \lim_{n\to\infty}\frac{1}{6n}\left(\frac{n-i}{n}\cdot\frac{\alpha_1+i}{\alpha_1+\alpha_2+n-1} - \frac{i}{n}\cdot\frac{\alpha_2+n-i}{\alpha_1+\alpha_2+n-1}\right)\right| \\
&= K\left|\theta \lim_{n\to\infty}\frac{1}{6}\left(\left(\frac{1}{n}-\frac{i}{n^2}\right)\cdot\frac{\frac{\alpha_1}{n}+\frac{i}{n}}{\frac{\alpha_1}{n}+\frac{\alpha_2}{n}+1-\frac{1}{n}} - \frac{i}{n^2}\cdot\frac{\frac{\alpha_2}{n}+1-\frac{i}{n}}{\frac{\alpha_1}{n}+\frac{\alpha_2}{n}+1-\frac{1}{n}}\right)\right| \\
&= 0,
\end{aligned} \tag{3.13}$$

where K is a constant such that $|f'''(\zeta)| \leq K$.

Finally, by substituting (3.11), (3.12) and (3.13) in (3.10), we obtain

$$\lim_{n\to\infty} A_n f\left(\frac{i}{n}\right) = \theta(-(\alpha_1+\alpha_2)y+\alpha_1)f'(y) + \frac{1}{2}(2\theta)y(1-y)f''(y).$$

Now by re-parametrizing

$$\delta = \frac{1}{\alpha_1+\alpha_2}, \quad \gamma = \theta(\alpha_1+\alpha_2), \quad \mu = \frac{\alpha_1}{\alpha_1+\alpha_2},$$

the last limit becomes

$$\lim_{n\to\infty} A_n f\left(\frac{i}{n}\right) = -\gamma(y-\mu)f'(y) + \frac{1}{2}(2\gamma\delta)y(1-y)f''(y),$$

which is precisely the infinitesimal generator of the Jacobi diffusion. In the space $C_c^3([0,1])$ all above limits hold uniformly, therefore we obtain

$$||A_n f_n - \mathcal{A}f||_\infty \to 0, \quad n \to \infty,$$

and since

$$Y^{(n)}(0) \Rightarrow Y(0), \quad n\to\infty \quad \Longleftrightarrow \quad i/n \to y, \quad n \to \infty,$$

by Theorem 2.1 we obtain $Y^n \Rightarrow Y$ in $\mathbb{D}([0,1])$, where Y is the generally parametrized Jacobi diffusion. □

4 Fractional Jacobi diffusion as the correlated CTRW limit

Suppose that $(T(r),\, r \in \mathbb{N}_0)$, where $T(0) = 0$, $T(r) = G_1 + \ldots + G_r$, is the random walk where $G_r \geq 0$ are iid waiting times between particle jumps that are independent of the Markov chain $(H^{(n)}(r),\, r \in \mathbb{N}_0)$. We assume G_1 is in the domain of attraction of the β-stable distribution with index $0 < \beta < 1$, and that the waiting time of the Markov chain until its r-th move is described by G_r. Let

$$N(t) = \max\{r \geq 0 \colon T(r) \leq t\} \tag{4.1}$$

be the number of jumps up to time $t \geq 0$. Then the continuous time stochastic process

$$\left(H^{(n)}(N(t)),\, t \geq 0\right),$$

where $H^{(n)}(N(t))$ is the state of the Markov chain at time $t \geq 0$, is the correlated CTRW process. The following theorem is the main ingredient to connect our correlated CTRW with corresponding limit, i.e. fractional Jacobi diffusion.

Theorem 4.1. *Let $(A^{(n)}(t),\, t \geq 0)$ be the càdlàg process with càdlàg process $(A(t),\, t \geq 0)$ as its corresponding weak limit, i.e. let*

$$A^n \Rightarrow A, \quad n \to \infty$$

in the Skorokhod space $\mathbb{D}(S)$ with J_1 topology, where S is the state space for the process A. Let $(N(t),\, t \geq 0)$ be the renewal process defined in (4.1), and $(E(t),\, t \geq 0)$ be the inverse of the standard β-stable subordinator $(D(t),\, t \geq 0)$ with $0 < \beta < 1$. Then

$$A^{(n)}(n^{-1}N(n^{\frac{1}{\beta}}t)) \Rightarrow A(E(t)), \quad n \to \infty$$

in the Skorokhod space $\mathbb{D}(S)$ with J_1 topology.

Proof. The result directly follows from the proof of Theorem 8.1. in [12]. □

Next, we apply this theorem in our setting to obtain the fractional Jacobi diffusion as the correlated CTRW limit from the model motivated by economics.

Corollary 4.2. *Let* $(H^{(n)}(r),\, r \in \mathbb{N}_0)$ *be the Markov chain defined by* (3.5) *with transition probabilities* (3.4)*. Let* $(Y^{(n)}(t),\, t \geq 0)$ *be the corresponding rescaled Markov chain given by* (3.8)*. Let* $(N(t),\, t \geq 0)$ *be the renewal process defined in* (4.1)*, and* $(E(t),\, t \geq 0)$ *be the inverse of the standard* β*-stable subordinator* $(D(t),\, t \geq 0)$ *with* $0 < \beta < 1$*. Then*

$$Y^{(n)}(n^{-1}N(n^{\frac{1}{\beta}}t)) \Rightarrow Y(E(t)), \quad n \to \infty$$

in the Skorokhod space $\mathbb{D}([0,1])$ *with* J_1 *topology, where* $(Y(t),\, t \geq 0)$ *is Jacobi diffusion with generator*

$$\mathcal{A}f(y) = -\gamma(y-\mu)f'(y) + \frac{1}{2}(2\gamma\delta)y(1-y)f''(y), \quad f \in C_c^3([0,1]).$$

Proof. Stochastic processes $(Y^{(n)}(t),\, t \geq 0)$ and $(Y(t),\, t \geq 0)$ are both càdlàg, and Theorem 3.1 implies

$$Y^n \Rightarrow Y \text{ in } \mathbb{D}([0,1]).$$

Now, simply apply Theorem 4.1 to obtain the desired result. □

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