

Approximation of heavy-tailed fractional Pearson diffusions in Skorokhod topology

N. N. Leonenko^a, I. Papić^b, A. Sikorskii^c, N. Šuvak^b

^a School of Mathematics, Cardiff University, Senghennydd Road, Cardiff CF244AG, UK

^b Department of Mathematics, J.J. Strossmayer University of Osijek, Trg Ljudevita Gaja 6, HR-31 000 Osijek, Croatia

^c Departments of Psychiatry and Statistics and Probability, Michigan State University, East Lansing, MI 48824, USA

Abstract. Continuous time random walks (CTRWs) have random waiting times between particle jumps. We establish fractional diffusion approximation via correlated CTRWs. Instead of a random walk modeling particle jumps in the classical CTRW model, we use discrete-time Markov chain with correlated steps. The waiting times are selected from the domain of attraction of a stable law.

Key words. fractional diffusion approximation; Skorokhod topology; fractional diffusions; correlated continuous time random walks; Markov chains; Pearson diffusions

Mathematics Subject Classification (2010): 47D07, 60J10, 60J60, 60G22, 60G50.

1 Introduction

Stochastic modeling of many phenomena such as trading on financial markets or pollution in ecology often includes modeling of rest periods between events. This means that there is a need for stochastic models with random waiting times between state changes. Models that recently received attention as suitable include fractional diffusions governed by Kolmogorov forward and backward partial differential equations with the fractional derivative in time-variable and their Skorokhod J_1 topology approximations using continuous-time random walks (CTRWs). For example, the fractional derivative in time is used to reflect delays between trades on financial markets and to derive the Black-Scholes formula in this framework (see [35, 24, 39]). Fractional derivative in time is also used for modeling of sticking and trapping of a pollutant particle in a porous medium or in the river flow (see [36, 6]). Additional applications of CTRWs and related fractional diffusions in engineering and finance could be found in [11, 24, 32, 33, 35, 39].

Independence of waiting times and particle jumps yields the model known as decoupled CTRW [29, 28, 10]. Furthermore, for independent and identically distributed (iid) particle jumps Y_1, \dots, Y_n , the rescaled random walk $S(n) = Y_1 + \dots + Y_n$ converges to either the Brownian motion or a stable Lévy process (see [31, Chapter 4], [38]) in the Skorokhod space $\mathbb{D}([0, 1])$ (the space of right-continuous functions on $[0, 1]$ with left limits) in the J_1 topology introduced by Skorokhod in 1956 [37]. If the waiting times between particle jumps are modeled by iid random variables G_1, \dots, G_n from the domain of attraction of a positively skewed stable law with stability index $0 < \beta < 1$, the CTRW process $S(N(t))$, where $T(n) = G_1 + \dots + G_n$, $N(t) = \max\{n \geq 0: T(n) \leq t\}$, gives the location of a particle at time $t \geq 0$. Then by applying the continuous mapping theorem (see [31, Theorem 4.19]), it follows that $S(N(\lfloor ct \rfloor))$ converges to the process $A(E(t))$ as $c \rightarrow \infty$. The outer process A is either the Brownian motion or a stable Lévy process, and the inner process $E(t)$ is the inverse of a standard β -stable subordinator ($D(t)$, $t \geq 0$). This convergence holds in both M_1 and J_1 Skorokhod topologies (see [30, 40]). The differences among Skorokhod topologies are discussed in [4].

If the particle jumps are correlated, a similar procedure yields the correlated CTRW (see, for example, [27]). In particular, correlated CTRW appears by replacing the outer random walk that represents particle jumps in decoupled case by a suitably chosen discrete-time Markov chain. This Markov chain then yields the fractional diffusion in the weak limit. Therefore, we refer to such correlated CTRW as a fractional diffusion approximation in Skorokhod J_1 topology.

A very recent development of the topic of space-time fractional processes is due to Toniazzi [41], who considered classical stochastic solutions for the fractional evolution equation on a bounded domain. An alternative approach to studying the limiting behavior of CTRWs has been developed in [15, 16] for the spatially non-homogeneous case, in which the jump distribution depends on position of a particle. In order to obtain the limiting process, Kolokoltsov develops the theory of subordination of Markov processes by the hitting-time processes.

Leonenko and colleagues [20] and [21] have constructed the correlated CTRWs converging to non-heavy-tailed fractional Pearson diffusions (Ornstein-Uhlenbeck (OU), Cox-Ingersoll-Ross (CIR) and Jacobi). They used the well known discrete-time Markov chains arising from the Laplace-Bernoulli urn scheme (see e.g. [14]) for the OU case, Wright-Fischer genetic model (see e.g. [14]) for the CIR and Jacobi cases. For the Jacobi case, they have also used a construction based on the Ehrenfest-Brillouin process, often interpreted in economic terms (see [9]). For motivation and more historical facts on these discrete-time Markov chains we refer to [25] and [12].

In this paper, we extend the approach of constructing a suitable discrete-time Markov chains in order to obtain the fractional Pearson diffusions (fPDs) with heavy-tailed invariant distributions in the limit. The paper is organized as follows. Section 2 includes the theoretical background on transition operators of discrete-time Markov chains (Subsection 2.1). Subsection 2.2 presents the mechanism for construction of diffusion approximation via Markov chains, with fractional diffusion approximations considered in Subsection 2.3. The methods presented in Section 2 are then applied in Section 3 for the construction of heavy-tailed fPDs and their approximations in Skorokhod J_1 topology. In particular, Pearson diffusions and fractional Pearson diffusions are described in Subsections 3.1 and 3.2, respectively. Subsection 3.3 contains the specific constructions of all three heavy-tailed Pearson diffusions: the Student diffusion is constructed in 3.3.1; the reciprocal gamma and Fisher-Snedecor diffusions are constructed in 3.3.2. Finally, Section 3.4 contains construction of all three heavy-tailed fPDs and their Skorokhod J_1 topology approximations, i.e., the corresponding correlated CTRWs.

2 General framework for fractional diffusion approximation

In this section we explain the general ideas for the construction of a Markov chain that leads to desired diffusion process as the weak limit in the Skorokhod space endowed with J_1 topology. In Subsection 2.1 we explain the necessary technicalities, while in Subsection 2.2 we give a concrete algorithm on how to construct generally parametrized diffusion via Markov chain in our setting.

2.1 Transition operators of the discrete-time Markov chains

Let μ be an arbitrary probability kernel on a measurable space (S, \mathcal{S}) . The associated transition operator T is defined as

$$Tf(x) = (Tf)(x) = \int \mu(x, dy)f(y), \quad x \in S, \quad (2.1)$$

where $f: S \rightarrow \mathbb{R}$ is assumed to be measurable and either bounded or nonnegative. For details we refer to [13, Chapter 19].

Denote by $\mathbb{D}(S)$ the space of right continuous functions with left limits defined on \mathbb{R}^+ with values in S . Throughout this paper, we consider the J_1 topology in this space. Consider the Banach space of bounded continuous functions on space S with the supremum norm denoted by $\|\cdot\|_\infty$.

For a closed operator \mathcal{A} with domain \mathcal{D} , a core for \mathcal{A} is a linear subspace $D \subset \mathcal{D}$ such that the restriction $\mathcal{A}|_D$ has closure \mathcal{A} . In that case, \mathcal{A} is clearly uniquely determined by its restriction $\mathcal{A}|_D$. Suitable core is important in order to technically establish connection between desired Markov chains and their limiting diffusions. We work with $C_c^3(S)$ as a core of the diffusion infinitesimal generator, but in general not all diffusions have it as its core. Theorems 1.6 and 2.1 from [8, Section 8] give sufficient conditions for $C_c^\infty(S)$ (and therefore $C_c^3(S)$ as well) to be a core of the diffusion infinitesimal generator.

The main technical tool used for obtaining the non-fractional diffusion approximation via suitably chosen Markov chain with known transition operator is Theorem 19.28 from [13]. We state this Theorem below.

Theorem 2.1. *Let $(Y^{(n)}, n \in \mathbb{N})$ be a sequence of discrete-time Markov chains on S with transition operators $(U_n, n \in \mathbb{N})$. Consider a Feller process X on S with semigroup T_t and generator \mathcal{A} . Fix a core D for the generator \mathcal{A} , and assume that $(h_n, n \in \mathbb{N})$ is the sequence of positive reals tending to zero as $n \rightarrow \infty$. Let*

$$A_n = h_n^{-1}(U_n - I), \quad T_{n,t} = U_n^{\lfloor t/h_n \rfloor}, \quad X_t^n = Y^n(\lfloor t/h_n \rfloor).$$

Then the following statements are equivalent:

- a) If $f \in D$, there exist some $f_n \in \text{Dom}(A_n)$ with $f_n \rightarrow f$ and $A_n f_n \rightarrow Af$ as $n \rightarrow \infty$
- b) $T_{n,t} \rightarrow T_t$ strongly for each $t > 0$
- c) $T_{n,t} f_n \rightarrow T_t f$ for each $f \in C_0$, uniformly for bounded $t > 0$
- d) if $X^{(n)}(0) \Rightarrow X(0)$ in S , then $X^n \Rightarrow X$ in the Skorokhod space $\mathbb{D}(S)$ with the Skorokhod J_1 topology.

The proof could be found in [13, Theorem 19.28, page 387].

Remark 2.2. In section 3 we apply technique established in the following sections for heavy-tailed Pearson diffusions. Moreover, $C_c^3(S)$ can be referred to as a core of these diffusions, where $S = \mathbb{R}$ in the Student diffusion case and $S = [0, +\infty)$ in the Fisher-Snedecor and reciprocal gamma cases. In particular all three heavy-tailed Pearson diffusions satisfy conditions of Theorem 2.1. from [8, Section 8].

2.2 General approach to diffusion approximation via Markov chains

Let $(N^{(n)}(r), r \in \mathbb{N})$ be the starting Markov chain with state space $S_n \subseteq \mathbb{N}_0$ and transition probabilities p_{ij} , $i, j \in S_n$. Let $X = (X(t), t \geq 0)$ be the desired diffusion process with state space S . The process X is the solution of the stochastic differential equation (SDE)

$$dX(t) = \mu(x) dt + \sqrt{\sigma^2(x)} dW(t), \quad t \geq 0, \quad x \in S,$$

where $(W(t), t \geq 0)$ is the standard Brownian motion. The infinitesimal generator of the process X is

$$Af(x) = \mu(x)f'(x) + \frac{1}{2}\sigma^2(x)f''(x), \quad f \in C_c^3(S). \quad (2.2)$$

First, one needs to connect starting points $N^{(n)}(0) = i \in S_n$ with $X(0) = x \in S$, i.e., to connect the state space of the starting Markov chain and the state space of the desired diffusion process. Define a strictly monotonic function $g_n : S \rightarrow \mathbb{R}$, such that

$$i = \lfloor g_n(x) \rfloor$$

for n large enough and

$$\lim_{n \rightarrow \infty} \|g_n^{-1}(i+1) - g_n^{-1}(i)\|_\infty = 0.$$

According to the state space S of the desired diffusion process X , one constructs new Markov chain $(M^{(n)}(r), r \in \mathbb{N})$ via the transformation

$$M^{(n)}(r) = g_n^{-1} \left(N^{(n)}(r) \right), \quad (2.3)$$

so that $(M^{(n)}(r), n \in \mathbb{N})$ has state space $g_n^{-1}(S_n)$ and transition operator

$$T_n f(g_n^{-1}(i)) = \sum_{j=0}^n p_{ij} f(g_n^{-1}(j)). \quad (2.4)$$

We now define operator

$$A_n := h_n^{-1}(T_n - I), \quad f_n \in \text{Dom}(A_n), \quad f_n(x) := f(g_n^{-1}(i)), \quad f \in C_c^3(S), \quad (2.5)$$

where $(h_n, n \in \mathbb{N})$ is sequence of positive reals tending to zero as $n \rightarrow \infty$.

Finally, define continuous-time stochastic process $(X^{(n)}(t), t \geq 0)$ via time change in the Markov chain

$$X^{(n)}(t) := M^{(n)}(\lfloor h_n^{-1}t \rfloor). \quad (2.6)$$

The next theorem gives sufficient conditions on when the diffusion process $(X(t), t \geq 0)$ can be obtained as the limiting process of the time-changed stochastic process $(X^{(n)}(t), t \geq 0)$.

Theorem 2.3. For each $n \in \mathbb{N}$, let $(M^{(n)}(r), r \in \mathbb{N}_0)$ be the Markov chain defined by (2.3) with the transition operator (2.4). For each $n \in \mathbb{N}$, let $X^n = (X^{(n)}(t), t \geq 0)$ be its corresponding time-changed process, with the time-change (2.6). Let operators $(A_n, n \in \mathbb{N})$ be defined by (2.5). If

$$\begin{aligned} \mu_n(x) &:= h_n^{-1} \sum_{j=0}^n p_{ij} (g_n^{-1}(j) - g_n^{-1}(i)), & \sigma_n^2(x) &:= h_n^{-1} \sum_{j=0}^n p_{ij} (g_n^{-1}(j) - g_n^{-1}(i))^2, \\ R_n(x) &:= h_n^{-1} \sum_{j=0}^n p_{ij} \frac{(g_n^{-1}(j) - g_n^{-1}(i))^3}{3!} f'''(\zeta), & |\zeta - g_n^{-1}(i)| &< |g_n^{-1}(j) - g_n^{-1}(i)| \end{aligned} \quad (2.7)$$

have uniform limits

$$\lim_{n \rightarrow \infty} \|\mu_n - \mu\|_\infty = \lim_{n \rightarrow \infty} \|\sigma_n^2 - \sigma^2\|_\infty = \lim_{n \rightarrow \infty} \|R_n\|_\infty = 0, \quad (2.8)$$

where μ and σ^2 are infinitesimal parameters given in (2.2), then

$$X^n \Rightarrow X, \quad \text{as } n \rightarrow \infty$$

in the Skorokhod space $\mathbb{D}(S)$ with J_1 topology, where $X = (X(t), t \geq 0)$ is diffusion with the infinitesimal generator \mathcal{A} given by (2.2).

Proof. First, we prove statement *a*) of Theorem 2.1, i.e., we show that infinitesimal generator (2.2) can be approximated by operator A_n defined in (2.5). According to the definition of function $\lfloor \cdot \rfloor$

$$\lfloor g_n(x) \rfloor \leq g_n(x) < \lfloor g_n(x) \rfloor + 1,$$

therefore

$$i \leq g_n(x) < i + 1. \quad (2.9)$$

Let g_n be a monotone increasing function (monotone decreasing case is analogous). Monotonicity with (2.9) give

$$g_n^{-1}(i) \leq g_n^{-1}(g_n(x)) < g_n^{-1}(i + 1)$$

so that

$$|g_n^{-1}(i) - x| < |g_n^{-1}(i + 1) - g_n^{-1}(i)|.$$

The last inequality implies

$$\lim_{n \rightarrow \infty} \|g_n^{-1}(i) - x\|_\infty \leq \lim_{n \rightarrow \infty} \|g_n^{-1}(i + 1) - g_n^{-1}(i)\|_\infty = 0. \quad (2.10)$$

Therefore for $f \in C_c^3(S)$

$$\lim_{n \rightarrow \infty} \|f_n - f\|_\infty = \lim_{n \rightarrow \infty} \sup_{x \in S} |f_n(x) - f(x)| = \lim_{n \rightarrow \infty} \sup_{x \in S} |f(g_n^{-1}(i)) - f(x)| = 0.$$

Since

$$\begin{aligned} A_n f(g_n^{-1}(i)) &= h_n^{-1} \left[\sum_{j=0}^n p_{ij} f(g_n^{-1}(j)) - f(g_n^{-1}(i)) \right] \\ &= h_n^{-1} \sum_{j=0}^n p_{ij} [f(g_n^{-1}(j)) - f(g_n^{-1}(i))], \end{aligned}$$

Taylor formula for function f around $g_n^{-1}(i)$ with mean-value form of the remainder yields

$$\begin{aligned} A_n f(g_n^{-1}(i)) &= h_n^{-1} \sum_{j=0}^n p_{ij} (g_n^{-1}(j) - g_n^{-1}(i)) f'(g_n^{-1}(i)) + h_n^{-1} \sum_{j=0}^n p_{ij} \frac{(g_n^{-1}(j) - g_n^{-1}(i))^2}{2!} f''(g_n^{-1}(i)) \\ &\quad + h_n^{-1} \sum_{j=0}^n p_{ij} \frac{(g_n^{-1}(j) - g_n^{-1}(i))^3}{3!} f'''(\zeta), \end{aligned} \quad (2.11)$$

where ζ is a real number such that $|\zeta - g_n^{-1}(i)| < |g_n^{-1}(j) - g_n^{-1}(i)|$.
Therefore (2.11) reduces to

$$A_n f(g_n^{-1}(i)) = \mu_n(x) f'(g_n^{-1}(i)) + \frac{\sigma_n^2(x)}{2} f''(g_n^{-1}(i)) + R_n(x).$$

The triangle inequality gives

$$\begin{aligned} \|A_n f_n - \mathcal{A}f\|_\infty &= \sup_{x \in S} |A_n f_n(x) - \mathcal{A}f(x)| = \sup_{x \in S} |A_n f_n(x) - \mathcal{A}f(x)| \\ &\leq \sup_{x \in S} |\mu_n(x) f'(g_n^{-1}(i)) - \mu(x) f'(x)| + \sup_{x \in S} \left| \frac{\sigma_n^2(x)}{2} f''(g_n^{-1}(i)) - \frac{\sigma^2(x)}{2} f''(x) \right| \\ &\quad + \sup_{x \in S} |R_n(x)|. \end{aligned} \quad (2.12)$$

For $f \in C_c^3(S)$, (2.10) implies

$$\lim_{n \rightarrow \infty} \|f'_n - f'\|_\infty = \lim_{n \rightarrow \infty} \|f''_n - f''\|_\infty = 0 \quad (2.13)$$

and uniform limits (2.8) and (2.13) together with (2.12) yields

$$\lim_{n \rightarrow \infty} \|A_n f_n - \mathcal{A}f\|_\infty = 0.$$

This completed the proof of statement *a*) of the Theorem 2.1. Since

$$X^{(n)}(0) \Rightarrow X(0) \iff \lim_{n \rightarrow \infty} \|g_n^{-1}(i) - x\|_\infty = 0,$$

the equivalence of statements *a*) and *d*) in Theorem 2.1 yields

$$X^n \Rightarrow X \text{ in } \mathbb{D}(S).$$

□

Remark 2.4. In all cases considered in this paper, functions $g_n : S \rightarrow \mathbb{R}$ are affine functions of the form

$$g_n(x) = a_n x + b_n,$$

where $(a_n, n \in \mathbb{N})$ and $(b_n, n \in \mathbb{N})$ are sequences of real numbers such that

$$\lim_{n \rightarrow \infty} \|g_n^{-1}(i+1) - g_n^{-1}(i)\|_\infty = \lim_{n \rightarrow \infty} \frac{1}{a_n} = 0,$$

and

$$i = \lfloor g_n(x) \rfloor$$

for n large enough.

Moreover, μ_n , σ_n^2 and R_n reduce to

$$\begin{aligned} \mu_n(x) &= \frac{h_n^{-1}}{a_n} \sum_{j=0}^n p_{ij} (j - i), \quad \sigma_n^2(x) = \frac{h_n^{-1}}{a_n^2} \sum_{j=0}^n p_{ij} (j - i)^2, \\ R_n(x) &= \frac{h_n^{-1}}{6a_n^3} \sum_{j=0}^n p_{ij} \frac{(j - i)^3}{3!} f'''(\zeta), \quad |\zeta - g_n^{-1}(i)| < \left| \frac{j - i}{a_n} \right|. \end{aligned} \quad (2.14)$$

Remark 2.5. In this paper we consider starting Markov chains with transition probabilities of the form

$$p_{i,i+1} > 0, \quad p_{i,i-1} > 0, \quad p_{i,i} = 1 - p_{i,i+1} - p_{i,i-1}, \quad \text{and } 0 \text{ otherwise.}$$

For such Markov chains, (2.14) further reduces to

$$\begin{aligned} \mu_n(x) &= \frac{h_n^{-1}}{a_n} (p_{i,i+1} - p_{i,i-1}), \quad \sigma_n^2(x) = \frac{h_n^{-1}}{a_n^2} (p_{i,i+1} + p_{i,i-1}), \\ R_n(x) &= \frac{h_n^{-1}}{6a_n^3} (p_{i,i+1} - p_{i,i-1}) f'''(\zeta), \quad |\zeta - g_n^{-1}(i)| < \left| \frac{j - i}{a_n} \right|. \end{aligned} \quad (2.15)$$

This procedure simplifies manipulations in the state space and time change in order to obtain the desired diffusion.

2.3 Fractional diffusion approximation

Let $T(r) = G_1 + \dots + G_r$, $r \in \mathbb{N}_0$, $T(0) = 0$ be a random walk with iid waiting times $G_r \geq 0$ between particle jumps. Assume that these waiting times are independent of the Markov chain $(H^{(n)}(r), r \in \mathbb{N}_0)$. Further, assume that G_1 is in the domain of attraction of the β -stable distribution with index $0 < \beta < 1$, and that the waiting time of the Markov chain until its r -th move is described by $T(r)$. Let

$$N(t) = \max\{r \geq 0: T(r) \leq t\} \quad (2.16)$$

be the number of jumps up to time $t \geq 0$. Then the continuous time stochastic process $H^{(n)}(N(t))$ gives the state of the Markov chain at time $t \geq 0$ and is a correlated CTRW process. The next Theorem provides fractional diffusion approximation via correlated CTRWs.

Theorem 2.6. *Let $(A^{(n)}(t), t \geq 0)$ be the weak limit of $(A(t), t \geq 0)$, where all processes are càdlàg and*

$$A^n \Rightarrow A \text{ in } \mathbb{D}(S)$$

with J_1 topology, where S is the state space for the process A .

Let $(N(t), t \geq 0)$ be the renewal process defined in (2.16), and $(E(t), t \geq 0)$ be the inverse of the standard β -stable subordinator $(D(t), t \geq 0)$ with $0 < \beta < 1$. Then

$$A^{(n)} \left(n^{-1} N \left(n^{1/\beta} t \right) \right) \Rightarrow A(E(t)), \quad n \rightarrow \infty$$

in the Skorokhod space $\mathbb{D}(S)$ with J_1 topology.

Proof. The result directly follows from the proof of Theorem 8.1. in [20]. □

Remark 2.7. It is clear that once a non-fractional diffusion approximation is obtained, its fractional counterpart approximation follows immediately from Theorem 2.6. Therefore, in order to obtain specific fractional diffusion approximation, one needs to establish specific Markov chain, which will lead to the non-fractional diffusion in the sense of Theorem 2.3. In the next section, we apply this approach to Pearson family of diffusions.

3 Application to heavy-tailed fractional Pearson diffusions

We start by defining Pearson diffusions and summarizing their properties.

3.1 Pearson diffusions

Pearson diffusion $(X(t), t \geq 0)$ is defined as the unique strong solution of the following SDE

$$dX(t) = -\theta(X(t) - \mu)dt + \sqrt{2\theta(b_2 X(t)^2 + b_1 X(t) + b_0)}dW(t), \quad t \geq 0,$$

where $\mu \in \mathbb{R}$ is the mean of the stationary distribution, $\theta > 0$ is the scaling of time determining the speed of reversion to the stationary mean μ , and b_0 , and b_1 and b_2 are such that the square root in the diffusion coefficient is well defined when $X(t)$ is in its state space (l, L) . Beside this SDE, Pearson diffusions can be defined by the partial differential equations (PDEs) for the transition density $p(x, t; y, s) = \frac{d}{dx} P(X_1(t) \leq x | X_1(s) = y)$, describing the time evolution of diffusion, that is, Kolmogorov forward (Fokker-Planck) and backward PDEs. Since we consider time-homogeneous diffusions for which $p(x, t; y, s) = p(x, t - s; y, 0)$ for $t > s$, we can write $p(x, t; y) = \frac{d}{dx} P(X_1(t) \leq x | X_1(0) = y)$. Kolmogorov forward or Fokker-Planck equation

$$\frac{\partial p(x, t; y)}{\partial t} = -\frac{\partial}{\partial x} (\mu(x)p(x, t; y)) + \frac{1}{2} \frac{\partial^2}{\partial x^2} (\sigma^2(x)p(x, t; y))$$

describes the "forward evolution" of the diffusion, the current state y being a constant. Kolmogorov backward equation

$$\frac{\partial p(x, t; y)}{\partial t} = \mu(y) \frac{\partial p(x, t; y)}{\partial y} + \frac{\sigma^2(y)}{2} \frac{\partial^2 p(x, t; y)}{\partial y^2}.$$

describes the "backward evolution" of the diffusion, the future state x being a constant. The second-order differential operator in this equation is the infinitesimal generator of the diffusion

$$\mathcal{A}g(y) = \left(\mu(y) \frac{\partial}{\partial y} + \frac{\sigma^2(y)}{2} \frac{\partial^2}{\partial y^2} \right) g(y). \quad (3.1)$$

It is a closed, generally unbounded, negative semidefinite, self-adjoint operator densely defined on the space $L^2((l, L), \mathbf{m})$ of square integrable functions with respect to the diffusion invariant density $\mathbf{m}(x)$:

$$\{f \in L^2((l, L), \mathbf{m}) \cap C^2((l, L)) : \mathcal{A}f \in L^2((l, L), \mathbf{m}) \text{ and } f \text{ satisfies boundary conditions at } l \text{ and } L\}. \quad (3.2)$$

For more details on Kolmogorov forward and backward PDEs we refer to [26].

Pearson diffusions are categorized into six subfamilies (see [18]), based on the properties of the stationary distribution. Namely, in 1931 Kolmogorov noticed that the differential equation for the invariant density $\mathbf{m}(x)$, $x \in \mathbb{R}$ of the classical Markovian diffusion with a linear drift $a(x) = a_1x + a_0 = -\theta(x - \mu)$ and the quadratic squared diffusion coefficient $\sigma^2(x) = 2\theta b(x)$ is the famous Pearson differential equation (see [34])

$$\mathbf{m}'(x)/\mathbf{m}(x) = [a(x) - b'(x)]/[b(x)] = [(a_1 - 2b_2)x + (a_0 - b_1)]/[b_2x^2 + b_1x + b_0].$$

According to the degree of the polynomial $\sigma^2(x)$ and further according to the sign of the leading coefficient b_2 and the sign of the discriminant $\Delta = b_1^2 - 4b_0b_2$ in the quadratic case of $\sigma^2(x)$, Pearson diffusions are classified into six subfamilies:

- constant $b(x)$ - the Ornstein-Uhlenbeck (OU) process, characterized by normal stationary distribution,
- linear $b(x)$ - the Cox-Ingersoll-Ross (CIR) process, characterized by gamma stationary distribution,
- quadratic $b(x)$ with $b_2 < 0$ - the Jacobi (JC) diffusion, characterized by beta stationary distribution,
- quadratic $b(x)$ with $b_2 > 0$ and $\Delta(b) > 0$ - the Fisher-Snedecor (FS) diffusion, characterized by the Fisher-Snedecor stationary distribution,
- quadratic $b(x)$ with $b_2 > 0$ and $\Delta(b) = 0$ - the reciprocal gamma (RG) diffusion, characterized by reciprocal gamma stationary distribution,
- quadratic $b(x)$ with $b_2 > 0$ and $\Delta(b) < 0$ - the Student (ST) diffusion, characterized by the Student stationary distribution.

The first three types have non-heavy-tailed stationary distributions. These diffusions are very well studied and widely applied, e.g., in financial practice. The properties on these diffusions can be found in the classical book [14], while more recent developments relying on spectral representation of their transition densities, are covered in [18], [17], [19], [20] and [21]. Heavy-tailed Pearson diffusions have not yet found their wide applications, in part due to complex properties of the spectrum of their infinitesimal generators (3.1). Namely, the transition densities of heavy-tailed PDs are not known in explicit form without knowing the structure of the spectrum of the infinitesimal generator. For these diffusions the spectrum consists of two disjoint parts: the finite discrete part consisting of finitely many simple eigenvalues in $\langle 0, \Lambda \rangle$ and the absolutely continuous part which is exactly the interval $\langle \Lambda, \infty \rangle$. This structure generates the spectral representation in form of a finite sum, including eigenvalues and classical orthogonal polynomials (Bessel, Romanovski and Fisher-Snedecor) as eigenfunctions, and the integral part over the absolutely continuous part of the spectrum, including confluent and generalized hypergeometric functions related to that part of the spectrum. For detailed spectral analysis of these diffusions and the application of the spectral representation of the transition density to statistical analysis of these diffusions we refer to the series of papers [22], [23], [3], [1] and [2].

3.2 Fractional Pearson diffusions (fPDs)

For a Pearson diffusion $(X(t), t \geq 0)$, the corresponding fractional Pearson diffusion (fPD) $(X_\beta(t), t \geq 0)$ is defined via a non-Markovian time-change $E(t)$ independent of $X_1(t)$:

$$X_\beta(t) := X(E(t)), \quad t \geq 0.$$

Here $E(t) = \inf\{x > 0 : D_x > t\}$ is the inverse of the standard β -stable Lévy subordinator $(D(t), t \geq 0)$ of order $0 < \beta < 1$, with the Laplace transform $E[e^{-sD(t)}] = \exp\{-ts^\beta\}$, $s \geq 0$. Since $E(t)$ rests for periods of time with non-exponential distribution, the process $(X_\beta(t), t \geq 0)$ is non-Markovian. Although $X_\beta(t)$ is not Markovian, we will refer to the function $p_\beta(x, t; y)$ as the transition density of fPD $X_\beta(t)$. This transition density satisfies

$$P(X_\beta(t) \in B | X_\beta(0) = y) = \int_B p_\beta(x, t; y) dx.$$

for any Borel subset $B \subset (l, L)$.

Analogously to the non-fractional case, time-evolution of fPDs can be (partially) described by time-fractional forward and backward Kolmogorov equations, with the time-fractional derivative (regularized non-local operator) defined in the Caputo sense (see [31]):

$$\frac{\partial^\beta u}{\partial t^\beta} = \begin{cases} \frac{\partial u}{\partial t}(t, x), & \text{if } \beta = 1 \\ \frac{1}{\Gamma(1-\beta)} \frac{\partial}{\partial t} \int_0^t (t-\tau)^{-\beta} u(\tau, x) d\tau - \frac{u(0, x)}{t^\beta}, & \text{if } \beta \in (0, 1). \end{cases}$$

In [18] the spectral representations for the transition densities of non-heavy-tailed fPDs (OU, CIR, Jacobi) were obtained. Namely, it has been shown that the series

$$p_\beta(x, t; y) = \mathbf{m}(x) \sum_{n=0}^{\infty} \mathcal{E}_\beta(-\lambda_n t^\beta) Q_n(y) Q_n(x) \quad (3.3)$$

converges for fixed $t > 0$, $x, y \in (l, L)$, where $\mathcal{E}_\beta(-z) = \sum_{j=0}^{\infty} (-z)^j / \Gamma(1 + \beta j)$, $z \geq 0$ is the Mittag-Leffler function, and $(Q_n, n \geq 0)$ are classical orthogonal polynomials that are eigenfunctions of the infinitesimal generator of the corresponding non-fractional Pearson diffusion. In the generalized sense, series (3.3) satisfies $p_\beta(x, 0; y) = \delta(x - y)$, where $\delta(\cdot)$ is the Dirac delta function.

Spectral representations of transition densities for fractional reciprocal gamma and Fisher-Snedecor diffusions were obtained in [19] using the asymptotic properties of confluent and Gauss hypergeometric functions (see [7] and [5]) related to the continuous part of the spectrum of the infinitesimal generator of the corresponding non-fractional Pearson diffusion. Here we point out that the case of the spectral representation of Student diffusion, having absolutely continuous part of the spectrum of multiplicity two, is still not completely resolved. For partial results on spectral analysis of Student diffusion we refer to [23].

Spectral representations of the transition densities of fPDs can be used to obtain the explicit strong solutions of the corresponding fractional Cauchy problems for both backward and forward equations. For relevant results we refer to [18] for non-heavy-tailed fPDs and to [19] for the reciprocal gamma and Fisher-Snedecor fractional diffusions.

3.3 Student, Fisher-Snedecor and reciprocal gamma diffusion approximations

In this section, we construct discrete-time Markov chains whose scaling limits are heavy-tailed Pearson diffusions.

3.3.1 Student diffusion approximation

The Student diffusion $X = (X(t), t \geq 0)$ is defined as the solution of the SDE

$$dX(t) = -\theta(X(t) - \mu) dt + \sqrt{\frac{2\theta\delta^2}{\nu-1} \left(1 + \left(\frac{X(t) - \mu}{\delta}\right)^2\right)} dW(t), \quad t \geq 0, \quad \theta > 0, \quad \mu \in \mathbb{R}, \quad \nu > 1, \quad \delta > 0,$$

with the infinitesimal generator

$$\mathcal{A}f(x) = -\theta(x - \mu) f'(x) + \frac{1}{2} \frac{2\theta\delta^2}{\nu-1} \left(1 + \left(\frac{x - \mu}{\delta}\right)^2\right) f''(x), \quad f \in C_c^3(\mathbb{R}). \quad (3.4)$$

The corresponding invariant distribution is symmetric scaled student distribution with probability density function

$$\mathbf{st}(x) = \frac{\Gamma(\frac{\nu+1}{2})}{\delta\sqrt{\pi}\Gamma(\frac{\nu}{2})} \left(1 + \left(\frac{x - \mu}{\delta}\right)^2\right)^{-\frac{\nu+1}{2}}, \quad x \in \mathbb{R}, \quad (3.5)$$

where $\delta > 0$ is scale parameter, $\mu \in \mathbb{R}$ is location parameter, and $\nu > 1$ is degrees of freedom of the invariant distribution. When $\nu > 2$, the mean and variance are finite.

Let $(Z^{(n)}(r), r \in \mathbb{N})$ be the Markov chain with state space $\{0, 1, 2, \dots, n\}$ and transition probabilities

$$p_{0,1} = 1, \quad p_{n,n-1} = 1,$$

$$p_{i,i+1} = \frac{1}{2c} \left(1 - \frac{2i}{n}\right)^2 + \frac{1}{n} \left(1 - \frac{i}{n}\right)^2, \quad p_{i,i-1} = \frac{1}{2c} \left(1 - \frac{2i}{n}\right)^2 + \frac{1}{n} \left(\frac{i}{n}\right)^2, \quad p_{i,i} = 1 - p_{i,i+1} - p_{i,i-1} \quad (3.6)$$

and 0 otherwise, where $i \in \{1, 2, \dots, n-1\}$, $0 < d < 1$, $c > 1$ and n is large enough to ensure $p_{i,i+1} + p_{i,i-1} < 1$.

This Markov chain is clearly irreducible since each state can be reached with positive probability. Finiteness of the state space $\{0, 1, 2, \dots, n\}$ with the irreducibility implies that the Markov chain is also recurrent, which in turn implies that the chain has the unique (up to a constant) invariant measure. Furthermore, finiteness of the state space implies this Markov chain has the unique stationary distribution π :

$$\pi(n) = \pi(0) = \left(2 + \frac{2cn^3}{n(n-2)^2 + 2c} \left(1 + \sum_{x=2}^{n-1} \frac{\prod_{k=1}^{x-1} [n(n-2k)^2 + 2c(n-k)^2]}{\prod_{k=2}^x [n(n-2k)^2 + 2ck^2]} \right) \right)^{-1},$$

$$\pi(x) = \frac{2cn^3}{n(n-2)^2 + 2c} \cdot \frac{\prod_{k=1}^{x-1} [n(n-2k)^2 + 2c(n-k)^2]}{\prod_{k=2}^x [n(n-2k)^2 + 2ck^2]} \cdot \pi(0), \quad x \in \{1, 2, 3, \dots, n-1\}.$$

This Markov chain is also periodic, since states 0 and n have periods of 2.

For $n \in \mathbb{N}$, define the function $g_n : \mathbb{R} \rightarrow \mathbb{R}$,

$$g_n(x) = \frac{1}{2} (n + (ax + b) \sqrt{n}), \quad a > 0, \quad b \in \mathbb{R}.$$

We assume that the initial states of the Markov chain $(Z^{(n)}(r), r \in \mathbb{N}_0)$ and the Student diffusion $X = (X(t), t \geq 0)$ are given by $Z^{(n)}(0) = i$ and $X(0) = x$ respectively, where

$$i(x) = i = \lfloor g_n(x) \rfloor = \left\lfloor \frac{1}{2} (n + (ax + b) \sqrt{n}) \right\rfloor, \quad x \in \mathbb{R}.$$

We also assume that n is always large enough so that $i(x)$ is in the state space of Markov chain $(Z^{(n)}(r), r \in \mathbb{N}_0)$. Furthermore, we assume that the initial Markov chain $(Z^{(n)}(r), r \in \mathbb{N}_0)$ never starts from states 0 or n . Notice that the initial state is a function of x , but we will use notation i for simplicity.

For $n \in \mathbb{N}$, define the new Markov chain

$$H^{(n)}(r) = g_n^{-1}(Z^{(n)}(r)) = \frac{1}{a\sqrt{n}} (2Z^{(n)}(r) - n - b\sqrt{n}), \quad (3.7)$$

with the state space $\left\{ \frac{1}{a\sqrt{n}} (-n - b\sqrt{n}), \frac{1}{a\sqrt{n}} (2 - n - b\sqrt{n}), \dots, \frac{1}{a\sqrt{n}} (n - b\sqrt{n}) \right\}$. The transition operator T_n of the Markov chain $(H^{(n)}(r), n \in \mathbb{N})$ is given by

$$\begin{aligned} T_n f \left(\frac{2i - n - b\sqrt{n}}{a\sqrt{n}} \right) &= \sum_{j=0}^n p_{ij} f \left(\frac{2j - n - b\sqrt{n}}{a\sqrt{n}} \right) \\ &= p_{i,i-1} f \left(\frac{2(i-1) - n - b\sqrt{n}}{a\sqrt{n}} \right) + p_{i,i} f \left(\frac{2i - n - b\sqrt{n}}{a\sqrt{n}} \right) + \\ &\quad + p_{i,i+1} f \left(\frac{2(i+1) - n - b\sqrt{n}}{a\sqrt{n}} \right). \end{aligned} \quad (3.8)$$

Now for $n \in \mathbb{N}$, define operator

$$A_n := \frac{\theta}{2} n^2 (T_n - I), \quad f_n \in \text{Dom}(A_n), \quad f_n(x) := f(g_n^{-1}(i)) = f \left(\frac{2i - n - b\sqrt{n}}{a\sqrt{n}} \right), \quad (3.9)$$

where $\theta > 0$ and $f \in C_c^3(\mathbb{R})$. Apply the scaling of time in $(H^{(n)}(r), r \in \mathbb{N}_0)$ to obtain the corresponding continuous-time process $(X^{(n)}(t), t \geq 0)$:

$$X^{(n)}(t) := H^{(n)} \left(\left\lfloor \frac{\theta}{2} n^2 t \right\rfloor \right). \quad (3.10)$$

The next theorem states that the Student diffusion could be obtained as the limiting process of the time-changed processes $(X^{(n)}(t), t \geq 0)$.

Theorem 3.1. For $n \in \mathbb{N}$, let $(H^{(n)}(r), r \in \mathbb{N}_0)$ be the Markov chain defined by (3.7) with the transition operator (3.8). Let $X^n = (X^{(n)}(t), t \geq 0)$ be its corresponding time-changed process, with the time-change (3.10). Let the operators $(A_n, n \in \mathbb{N})$ be defined by (3.9). Then as $n \rightarrow \infty$

$$X^n \Rightarrow X \text{ in } \mathbb{D}(\mathbb{R}),$$

where $X = (X(t), t \geq 0)$ is the Student diffusion with the infinitesimal generator \mathcal{A} given by (3.4), and

$$\mu = -\frac{b}{a}, \quad \nu = c + 1, \quad \delta = \frac{1}{a} \sqrt{\frac{c}{2}}.$$

Proof. First, notice that function g_n satisfies conditions given in Section 2, i.e., function g_n is strictly monotonic and

$$\lim_{n \rightarrow \infty} \|g_n^{-1}(i+1) - g_n^{-1}(i)\|_\infty = \lim_{n \rightarrow \infty} \left| \frac{2}{a\sqrt{n}} \right| = 0.$$

Taking into account Remark 2.5, state space transformation (3.7) with the time scale $h_n^{-1} = \theta n^2/2$ yield

$$\mu_n(x) = \frac{\theta}{a} n\sqrt{n} (p_{i,i+1} - p_{i,i-1}), \quad \sigma_n^2(x) = \frac{2\theta}{a^2} n (p_{i,i+1} + p_{i,i-1}),$$

$$R_n(x) = \frac{2\theta}{3a^3} \sqrt{n} (p_{i,i+1} - p_{i,i-1}) f'''(\zeta), \quad \left| \zeta - \frac{2i - n - b\sqrt{n}}{a\sqrt{n}} \right| < \left| \frac{2}{a\sqrt{n}} (j - i) \right|.$$

Next, the transition probabilities (3.6) further simplify

$$\begin{aligned} \mu_n(x) &= \frac{\theta}{a} n\sqrt{n} \left(\frac{1}{2c} \left(1 - \frac{2i}{n}\right)^2 + \frac{1}{n} \left(1 - \frac{i}{n}\right)^2 - \frac{1}{2c} \left(1 - \frac{2i}{n}\right)^2 - \frac{1}{n} \left(\frac{i}{n}\right)^2 \right) \\ &= \frac{\theta\sqrt{n}}{a} \left(1 - \frac{2i}{n}\right) = \theta \left(\frac{n-2i}{a\sqrt{n}}\right), \end{aligned} \quad (3.11)$$

$$\begin{aligned} \sigma_n^2(x) &= \frac{2\theta}{a^2} n \left(\frac{1}{2c} \left(1 - \frac{2i}{n}\right)^2 + \frac{1}{n} \left(1 - \frac{i}{n}\right)^2 + \frac{1}{2c} \left(1 - \frac{2i}{n}\right)^2 + \frac{1}{n} \left(\frac{i}{n}\right)^2 \right) \\ &= 2\theta \left(\frac{1}{c} \left(\frac{n-2i}{a\sqrt{n}}\right)^2 + \frac{1}{a^2} \left(1 - \frac{i}{n}\right)^2 + \frac{1}{a^2} \left(\frac{i}{n}\right)^2 \right), \end{aligned} \quad (3.12)$$

$$\begin{aligned} |R_n(x)| &\leq \left| \frac{2\theta}{3a^3} \sqrt{n} \left(\frac{1}{2c} \left(1 - \frac{2i}{n}\right)^2 + \frac{1}{n} \left(1 - \frac{i}{n}\right)^2 - \frac{1}{2c} \left(1 - \frac{2i}{n}\right)^2 - \frac{1}{n} \left(\frac{i}{n}\right)^2 \right) \right| K \\ &= \left| \frac{2\theta}{3a^3} \left(\frac{1}{\sqrt{n}} \left(1 - \frac{i}{n}\right)^2 - \frac{1}{\sqrt{n}} \left(\frac{i}{n}\right)^2 \right) \right| K \\ &= \left| \frac{2\theta}{3a^3} \left(\frac{n-2i}{n\sqrt{n}} \right) \right| K, \end{aligned} \quad (3.13)$$

where K is a constant such that $|f'''(\zeta)| \leq K$. Since $i = \lfloor g_n(x) \rfloor$, it follows

$$\lim_{n \rightarrow \infty} \sup_{x \in \mathbb{R}} \left| \frac{n-2i}{a\sqrt{n}} + \left(x + \frac{b}{a}\right) \right| = 0, \quad \lim_{n \rightarrow \infty} \sup_{x \in \mathbb{R}} \left| \frac{i}{n} - \frac{1}{2} \right| = 0. \quad (3.14)$$

Now, using (3.11), (3.12), (3.13) together with (3.14) and the fact that $f \in C_c^3(\mathbb{R})$, we have

$$\lim_{n \rightarrow \infty} \|\mu_n - \mu\|_\infty = 0, \quad \lim_{n \rightarrow \infty} \|\sigma_n^2 - \sigma^2\|_\infty = 0, \quad \lim_{n \rightarrow \infty} \|R_n\|_\infty = 0, \quad (3.15)$$

where

$$\mu(x) = -\theta \left(x + \frac{b}{a}\right), \quad \sigma^2(x) = 2\theta \left(\frac{1}{c} \left(x + \frac{b}{a}\right)^2 + \frac{1}{2a^2}\right).$$

By re-parametrizing

$$\mu = -\frac{b}{a}, \quad \nu = c + 1, \quad \delta = \frac{1}{a}\sqrt{\frac{c}{2}} \quad (3.16)$$

we obtain

$$\mu(x) = -\theta(x - \mu), \quad \sigma^2(x) = \frac{2\theta\delta^2}{\nu - 1} \left(1 + \left(\frac{x - \mu}{\delta} \right)^2 \right). \quad (3.17)$$

Comparing (3.17) with (3.4) we see that the limits coincide with the infinitesimal parameters of the Student diffusion. Since (3.15) holds, as a direct consequence of Theorem 2.3 we obtain $X^n \Rightarrow X$ in $\mathbb{D}(\mathbb{R})$ as $n \rightarrow \infty$, where X is the generally parametrized Student diffusion. \square

Remark 3.2. Note that re-parametrization (3.16) ensures parameters of the Student diffusion satisfy

$$\theta > 0, \quad \mu \in \mathbb{R}, \quad \nu > 2, \quad \delta > 0,$$

since $a > 0$, $b \in \mathbb{R}$, $c > 1$. In general, parameter ν can be any real number larger than 1, but $\nu > 2$ we obtained ensures that the invariant Student distribution has finite second moment.

Remark 3.3. It is well known that for high degrees of freedom ν , the Student distribution (3.5) can be approximated by the normal distribution. If we let $c \rightarrow \infty$ in the transition probabilities (3.6), they resemble the structure of transition probabilities of the famous Bernoulli-Laplace urn-scheme model (see [20, Section 6]), which leads to the OU diffusion. On the other hand, by taking into account (3.16), the infinitesimal parameters (3.17) of the Student diffusion reduce to the infinitesimal parameters of the OU process as $c \rightarrow \infty$, and $\nu \rightarrow \infty$, $\delta \rightarrow \infty$. Therefore it is not surprising that when $c \rightarrow \infty$, the scaled Markov chain which leads to the Student diffusion resembles the structure of the scaled Markov chain which leads to the OU process.

3.3.2 Fisher-Snedecor and reciprocal gamma diffusion approximations

First, we define starting Markov chain which will lead to the Fisher-Snedecor and reciprocal gamma diffusions with appropriately chosen parameters. Let $(G^{(n)}(r), r \in \mathbb{N})$ be the Markov chain with the state space $\{0, 1, 2, \dots, n\}$ and transition probabilities

$$p_{0,1} = 1, \quad p_{n,n-1} = 1, \\ p_{i,i+1} = \binom{i}{n}^2 \frac{a^*}{n^d} + \frac{b^*}{n^2} + \frac{c^*i}{n^2}, \quad p_{i,i-1} = \frac{a^*i + d^*}{n^2} \frac{i}{n^d} + \frac{c^*i}{n^2}, \quad p_{i,i} = 1 - p_{i,i+1} - p_{i,i-1}, \quad 0 \text{ otherwise}, \quad (3.18)$$

where $i \in \{1, 2, \dots, n-1\}$, $0 < d < 1$, $a^* \geq 0$, $b^* \geq 0$, $c^* \geq 0$, $d^* \geq 0$.

This Markov chain is clearly irreducible since each state can be reached with positive probability. Finiteness of the state space $\{0, 1, 2, \dots, n\}$ and the irreducibility imply that the Markov chain is also recurrent, which in turn implies that the chain has the unique (up to a constant) invariant measure. Furthermore, finiteness of the state space implies this Markov chain has the unique stationary distribution π :

$$\pi(0) = \left(1 + \frac{n^{d+2}}{a^* + b^* + c^*n^d} \left(1 + \frac{(n-1)^2 a^* + n^d(b^* + c^*(n-1))}{n^{d+2}} \frac{\prod_{k=1}^{x-1} [a^*k^2 + c^*n^d k + b^*n^d]}{\prod_{k=2}^x [a^*k^2 + (c^*n^d + d^*)k]} \right. \right. \\ \left. \left. + \sum_{x=2}^{n-1} \frac{\prod_{k=1}^{x-1} [a^*k^2 + c^*n^d k + b^*n^d]}{\prod_{k=2}^x [a^*k^2 + (c^*n^d + d^*)k]} \right) \right)^{-1}, \\ \pi(x) = \frac{n^{d+2}}{a^* + b^* + c^*n^d} \cdot \frac{\prod_{k=1}^{x-1} [a^*k^2 + c^*n^d k + b^*n^d]}{\prod_{k=2}^x [a^*k^2 + (c^*n^d + d^*)k]} \cdot \pi(0), \quad x \in \{1, 2, 3, \dots, n-1\}$$

$$\pi(n) = \pi(n-1) \cdot \left[\left(\frac{n-1}{n} \right)^2 \frac{a^*}{n^d} + \frac{b^*}{n^2} + \frac{c^*(n-1)}{n^2} \right].$$

This Markov chain is also periodic, since states 0 and n have periods of 2.

Fisher-Snedecor diffusion

The Fisher-Snedecor diffusion $Y = (Y(t), t \geq 0)$ is defined as the solution of the SDE

$$dY(t) = -\theta \left(Y(t) - \frac{\beta}{\beta-2} \right) dt + \sqrt{\frac{4\theta}{\gamma(\beta-2)}} Y(t)(\gamma Y(t) + \beta) dW(t), \quad t \geq 0, \quad \theta > 0, \quad \beta > 2, \quad \gamma > 0,$$

with the infinitesimal generator

$$\mathcal{A}f(y) = -\theta \left(y - \frac{\beta}{\beta-2} \right) f'(y) + \frac{1}{2} \frac{4\theta}{\gamma(\beta-2)} y(\gamma y + \beta) f''(y), \quad f \in C_c^3([0, +\infty)). \quad (3.19)$$

Let $(G^{(n)}(r), r \in \mathbb{N})$ be the Markov chain with the state space $\{0, 1, 2, \dots, n\}$ and transition probabilities (3.18) with parameters

$$a^* = a, \quad b^* = a + b, \quad c^* = c, \quad d^* = b, \quad a > 0, \quad b > 0, \quad c > 0,$$

i.e.,

$$p_{0,1} = 1, \quad p_{n,n-1} = 1, \\ p_{i,i+1} = \left(\frac{i}{n} \right)^2 \frac{a}{n^d} + \frac{a+b}{n^2} + \frac{ci}{n^2}, \quad p_{i,i-1} = \frac{ai+b}{n^2} \frac{i}{n^d} + \frac{ci}{n^2}, \quad p_{i,i} = 1 - p_{i,i+1} - p_{i,i-1}, \quad 0 \text{ otherwise}, \quad (3.20)$$

where $i \in \{1, 2, \dots, n-1\}$ and n is large enough, ensuring $p_{i,i+1} + p_{i,i-1} < 1$. Define the function $g_n : [0, +\infty) \rightarrow \mathbb{R}$,

$$g_n(y) = n^d y.$$

We assume that the initial states of the Markov chain $(G^{(n)}(r), r \in \mathbb{N}_0)$ and Fisher-Snedecor diffusion $Y = (Y(t), t \geq 0)$ are given by $G^{(n)}(0) = i$ and $Y(0) = y$ respectively, where

$$i(y) = i = \lfloor g_n(y) \rfloor = \lfloor n^d y \rfloor, \quad y \in [0, +\infty).$$

We also assume that n is always large enough so that $i(y)$ is in the state space of Markov chain $(G^{(n)}(r), r \in \mathbb{N}_0)$. Furthermore, we assume that the initial Markov chain $(G^{(n)}(r), r \in \mathbb{N}_0)$ never starts from states 0 or n . Notice that the initial state is a function of y , but we will use notation i for simplicity. For $n \in \mathbb{N}$, we define the new Markov chain $(H^{(n)}(r), r \in \mathbb{N})$ with the state space $\{0, 1/n^d, \dots, 1/n^{d-1}\}$

$$H^{(n)}(r) = g_n^{-1}(G^{(n)}(r)) = \frac{G^{(n)}(r)}{n^d}. \quad (3.21)$$

The transition operator T_n of the Markov chain $(H^{(n)}(r), n \in \mathbb{N})$ is given by

$$T_n f \left(\frac{i}{n^d} \right) = \sum_{j=0}^n p_{ij} f \left(\frac{j}{n^d} \right) = p_{i,i-1} f \left(\frac{i-1}{n^d} \right) + p_{i,i} f \left(\frac{i}{n^d} \right) + p_{i,i+1} f \left(\frac{i+1}{n^d} \right). \quad (3.22)$$

For $n \in \mathbb{N}$, define operator

$$A_n := n^{2+d}(T_n - I), \quad f_n \in \text{Dom}(A_n), \quad f_n(y) := f \left(g_n^{-1}(y) \right) = f \left(\frac{i}{n^d} \right), \quad (3.23)$$

where $f \in C_c^3([0, +\infty))$ and by the following scaling of time in $(H^{(n)}(r), r \in \mathbb{N}_0)$, for $n \in \mathbb{N}$ we obtain the corresponding continuous-time process $(Y^{(n)}(t), t \geq 0)$:

$$Y^{(n)}(t) := H^{(n)}(\lfloor n^{2+d}t \rfloor). \quad (3.24)$$

The next theorem states that the Fisher-Snedecor diffusion could be obtained as the limiting process of the time-changed processes $(Y^{(n)}(t), t \geq 0)$.

Theorem 3.4. For $n \in \mathbb{N}$, let $(H^{(n)}(r), r \in \mathbb{N}_0)$ be the Markov chain defined by (3.21) with the transition operator (3.22). Let $Y^n = (Y^{(n)}(t), t \geq 0)$, for each $n \in \mathbb{N}$, be its corresponding time-changed process, with the time-change (3.24). Let the operators $(A_n, n \in \mathbb{N})$ be defined by (3.23). Then as $n \rightarrow \infty$

$$Y^n \Rightarrow Y \text{ in } \mathbb{D}([0, +\infty)),$$

where $Y = (Y(t), t \geq 0)$ is the Fisher-Snedecor diffusion with the infinitesimal generator \mathcal{A} given by (3.19), and

$$\theta = b, \quad \beta = 2 \left(\frac{b}{a} + 1 \right), \quad \gamma = \frac{2(a+b)}{c}.$$

Proof. First, notice that function g_n satisfies conditions given in Section 2, i.e., function g_n is strictly monotone and

$$\lim_{n \rightarrow \infty} \|g_n^{-1}(i+1) - g_n^{-1}(i)\|_\infty = \lim_{n \rightarrow \infty} \left| \frac{1}{n^d} \right| = 0.$$

Taking into account Remark 2.5, state space transformation (3.7) together with the time scale $h_n^{-1} = n^{2+d}$ yield

$$\mu_n(x) = n^2 (p_{i,i+1} - p_{i,i-1}), \quad \sigma_n^2(x) = n^{2-d} (p_{i,i+1} + p_{i,i-1}),$$

$$R_n(x) = \frac{1}{6} n^{2(1-d)} (p_{i,i+1} - p_{i,i-1}) f'''(\zeta), \quad \left| \zeta - \frac{i}{n^d} \right| < \left| \frac{j-i}{n^d} \right|.$$

Next, transition probabilities (3.6) further simplify

$$\begin{aligned} \mu_n(y) &= n^2 \left(\left(\frac{i}{n} \right)^2 \frac{a}{n^d} + \frac{a+b}{n^2} + \frac{ci}{n^2} - \frac{ai+b}{n^2} \frac{i}{n^d} - \frac{ci}{n^2} \right) \\ &= a + b - b \frac{i}{n^d}, \end{aligned} \tag{3.25}$$

$$\begin{aligned} \sigma_n^2(y) &= n^{2-d} \left(\left(\frac{i}{n} \right)^2 \frac{a}{n^d} + \frac{a+b}{n^2} + \frac{ci}{n^2} + \frac{ai+b}{n^2} \frac{i}{n^d} + \frac{ci}{n^2} \right) \\ &= 2a \left(\frac{i}{n^d} \right)^2 + 2c \frac{i}{n^d} + \frac{a+b}{n^d} + b \frac{i}{n^{2d}}, \end{aligned} \tag{3.26}$$

$$\begin{aligned} |R_n(y)| &\leq \left| \frac{n^{2-2d}}{6} \left(\left(\frac{i}{n} \right)^2 \frac{a}{n^d} + \frac{a+b}{n^2} + \frac{ci}{n^2} - \frac{ai+b}{n^2} \frac{i}{n^d} - \frac{ci}{n^2} \right) \right| K \\ &= \left| \frac{1}{6} \left(\frac{a+b}{n^{2d}} - b \frac{i}{n^{3d}} \right) \right| K, \end{aligned} \tag{3.27}$$

where K is a constant such that $|f'''(\zeta)| \leq K$. Since $i = \lfloor g_n(y) \rfloor$, it follows

$$\lim_{n \rightarrow \infty} \sup_{y \in [0, +\infty)} \left| \frac{i}{n^d} - y \right| = 0. \tag{3.28}$$

Now, using (3.25), (3.26), (3.27) together with (3.28) and the fact that $f \in C_c^3([0, +\infty))$ we have

$$\lim_{n \rightarrow \infty} \|\mu_n - \mu\|_\infty = 0, \quad \lim_{n \rightarrow \infty} \|\sigma_n^2 - \sigma^2\|_\infty = 0, \quad \lim_{n \rightarrow \infty} \|R_n\|_\infty = 0, \tag{3.29}$$

where

$$\mu(y) = a + b - by, \quad \sigma^2(y) = 2ay^2 + 2cy.$$

By re-parametrizing

$$\theta = b, \quad \beta = 2 \left(\frac{b}{a} + 1 \right), \quad \gamma = \frac{2(a+b)}{c} \tag{3.30}$$

it follows

$$\mu(y) = -\theta \left(y - \frac{\beta}{\beta-2} \right), \quad \sigma^2(y) = \frac{4\theta}{\gamma(\beta-2)} y(\gamma y + \beta). \tag{3.31}$$

Note that re-parametrization (3.30) ensures the generality of parameters of the Fisher-Snedecor diffusion, i.e.

$$\theta > 0, \quad \beta > 2, \quad \gamma > 0$$

since $a > 0$, $b > 0$, $c > 0$. Comparing the obtained limits (3.31) with (3.19) we see that the limits coincide with the infinitesimal parameters of the Fisher-Snedecor diffusion. Since (3.29) holds, as a direct consequence of Theorem 2.3 we obtain $Y^n \Rightarrow Y$ in $\mathbb{D}([0, +\infty))$ as $n \rightarrow \infty$, where Y is the generally parametrized Fisher-Snedecor diffusion. \square

Reciprocal gamma diffusion

The reciprocal gamma diffusion $Z = (Z(t), t \geq 0)$ is defined as the solution of the SDE

$$dZ(t) = -\theta \left(Z(t) - \frac{\gamma}{\beta - 1} \right) dt + \sqrt{\frac{2\theta}{\beta - 1}} Z^2(t) dW(t), \quad t \geq 0, \quad \theta > 0, \quad \beta > 1, \quad \gamma > 0,$$

with the infinitesimal generator

$$\mathcal{A}f(z) = -\theta \left(z - \frac{\gamma}{\beta - 1} \right) f'(z) + \frac{1}{2} \frac{2\theta}{\beta - 1} z^2 f''(z), \quad f \in C_c^3([0, \infty)). \quad (3.32)$$

Let $(G^{(n)}(r), r \in \mathbb{N})$ be the Markov chain with the state space $\{0, 1, 2, \dots, n\}$ and transition probabilities (3.18) with parameters

$$a^* = a, \quad b^* = c, \quad c^* = 0, \quad d^* = b, \quad a > 0, \quad b > 0, \quad c > 0,$$

i.e.

$$p_{0,1} = 1, \quad p_{n,n-1} = 1, \\ p_{i,i+1} = \left(\frac{i}{n} \right)^2 \frac{a}{n^d} + \frac{c}{n^2}, \quad p_{i,i-1} = \frac{ai + b}{n^2} \frac{i}{n^d}, \quad p_{i,i} = 1 - p_{i,i+1} - p_{i,i-1}, \quad 0 \text{ otherwise}, \quad (3.33)$$

where $i \in \{1, 2, \dots, n-1\}$ and n is large enough, ensuring $p_{i,i+1} + p_{i,i-1} < 1$. Define the function $g_n : [0, +\infty) \rightarrow \mathbb{R}$,

$$g_n(z) = n^d z.$$

We assume that the initial states of the Markov chain $(G^{(n)}(r), r \in \mathbb{N}_0)$ and reciprocal gamma diffusion $Z = (Z(t), t \geq 0)$ are given by $G^{(n)}(0) = i$ and $Z(0) = z$ respectively, where

$$i(z) = i = \lfloor g_n(z) \rfloor = \lfloor n^d z \rfloor, \quad z \in [0, +\infty).$$

and n is always large enough so that $i(z)$ is in the state space of Markov chain $(G^{(n)}(r), r \in \mathbb{N}_0)$. Furthermore, we assume that the initial Markov chain $(G^{(n)}(r), r \in \mathbb{N}_0)$ never starts from states 0 or n . Notice that the initial state is a function of z , but we will use notation i for simplicity. For $n \in \mathbb{N}$, we define the new Markov chain $(H^{(n)}(r), r \in \mathbb{N})$ with the state space $\{0, 1/n^d, \dots, 1/n^{d-1}\}$

$$H^{(n)}(r) = g_n^{-1}(G^{(n)}(r)) = \frac{G^{(n)}(r)}{n^d}. \quad (3.34)$$

The transition operator T_n of the Markov chain $(H^{(n)}(r), n \in \mathbb{N})$ is given by

$$T_n f \left(\frac{i}{n^d} \right) = \sum_{j=0}^n p_{ij} f \left(\frac{j}{n^d} \right) = p_{i,i-1} f \left(\frac{i-1}{n^d} \right) + p_{i,i} f \left(\frac{i}{n^d} \right) + p_{i,i+1} f \left(\frac{i+1}{n^d} \right). \quad (3.35)$$

For $n \in \mathbb{N}$, define operator

$$A_n := n^{2+d}(T_n - I), \quad f_n \in \text{Dom}(A_n), \quad f_n(z) := f(g_n^{-1}(i)) = f \left(\frac{i}{n^d} \right) \quad (3.36)$$

where $f \in C_c^3([0, +\infty))$ and by the following scaling of time in $(H^{(n)}(r), r \in \mathbb{N}_0)$, for each $n \in \mathbb{N}$ we obtain the corresponding continuous-time process $(Z^{(n)}(t), t \geq 0)$:

$$Z^{(n)}(t) := H^{(n)}(\lfloor n^{2+d}t \rfloor). \quad (3.37)$$

The next theorem states that the reciprocal gamma diffusion could be obtained as the limiting process of the time-changed processes $(Z^{(n)}(t), t \geq 0)$.

Theorem 3.5. Forh $n \in \mathbb{N}$, let $(H^{(n)}(r), r \in \mathbb{N}_0)$ be the Markov chain defined by (3.34) with the transition operator (3.35). Let $Z^n = (Z^{(n)}(t), t \geq 0)$, for each $n \in \mathbb{N}$, be its corresponding time-changed process, with the time-change (3.37). Let the operators $(A_n, n \in \mathbb{N})$ be defined by (3.36). Then as $n \rightarrow \infty$

$$Z^n \Rightarrow Z \text{ in } \mathbb{D}([0, +\infty)),$$

where $Z = (Z(t), t \geq 0)$ is the RG diffusion with the infinitesimal generator \mathcal{A} given by (3.32), and

$$\theta = b, \quad \beta = \frac{b}{a} + 1, \quad \gamma = \frac{c}{a}.$$

Proof. First, notice that function g_n satisfies conditions given in Section 2, i.e. function g_n is strictly monotone and

$$\lim_{n \rightarrow \infty} \|g_n^{-1}(i+1) - g_n^{-1}(i)\|_\infty = \lim_{n \rightarrow \infty} \left| \frac{1}{n^d} \right| = 0.$$

Taking into account Remark 2.5, state space transformation (3.34) together with the time scale $h_n^{-1} = n^{2+d}$ yield

$$\mu_n(z) = n^2 (p_{i,i+1} - p_{i,i-1}), \quad \sigma_n^2(z) = n^{2-d} (p_{i,i+1} + p_{i,i-1}),$$

$$R_n(z) = \frac{1}{6} n^{2(1-d)} (p_{i,i+1} - p_{i,i-1}) f'''(\zeta), \quad \left| \zeta - \frac{i}{n^d} \right| < \left| \frac{j-i}{n^d} \right|.$$

Next, transition probabilities (3.33) further simplify

$$\begin{aligned} \mu_n(z) &= n^2 \left(\left(\frac{i}{n} \right)^2 \frac{a}{n^d} + \frac{c}{n^2} - \frac{ai+b}{n^2} \frac{i}{n^d} \right) \\ &= c - b \frac{i}{n^d}, \end{aligned} \tag{3.38}$$

$$\begin{aligned} \sigma_n^2(z) &= n^{2-d} \left(\left(\frac{i}{n} \right)^2 \frac{a}{n^d} + \frac{c}{n^2} + \frac{ai+b}{n^2} \frac{i}{n^d} \right) \\ &= 2a \left(\frac{i}{n^d} \right)^2 + \frac{c}{n^d} + b \frac{i}{n^{2d}}, \end{aligned} \tag{3.39}$$

$$\begin{aligned} |R_n(z)| &\leq \frac{K}{6} \left| n^{2-2d} \left(\left(\frac{i}{n} \right)^2 \frac{a}{n^d} + \frac{c}{n^2} - \frac{ai+b}{n^2} \frac{i}{n^d} \right) \right| \\ &= \frac{K}{6} \left| \left(\frac{c}{n^{2d}} - b \frac{i}{n^{3d}} \right) \right|, \end{aligned} \tag{3.40}$$

where K is a constant such that $|f'''(\zeta)| \leq K$. Since $i = \lfloor g_n(z) \rfloor$, it follows

$$\lim_{n \rightarrow \infty} \sup_{z \in [0, +\infty)} \left| \frac{i}{n^d} - z \right| = 0. \tag{3.41}$$

Now, using (3.38), (3.39), (3.40) together with (3.41) and the fact that $f \in C_c^3([0, +\infty))$ we have

$$\lim_{n \rightarrow \infty} \|\mu_n - \mu\|_\infty = 0, \quad \lim_{n \rightarrow \infty} \|\sigma_n^2 - \sigma^2\|_\infty = 0, \quad \lim_{n \rightarrow \infty} \|R_n\|_\infty = 0, \tag{3.42}$$

where

$$\mu(z) = c - bz, \quad \sigma^2(z) = 2az^2.$$

By re-parametrizing

$$\theta = b, \quad \beta = \frac{b}{a} + 1, \quad \gamma = \frac{c}{a} \tag{3.43}$$

it follows

$$\mu(z) = -\theta \left(z - \frac{\gamma}{\beta - 1} \right), \quad \sigma^2(z) = \frac{2\theta}{\beta - 1} z^2. \tag{3.44}$$

Notice that re-parametrization (3.43) ensures the generality of parameters of the reciprocal gamma diffusion, i.e.,

$$\theta > 0, \quad \beta > 1, \quad \gamma > 0$$

since $a > 0$, $b > 0$, $c > 0$. Now, comparing the obtained limits (3.44) with (3.32) we see that the limits coincide with the infinitesimal parameters of the RG diffusion. Since (3.42) holds, as a direct consequence of Theorem 2.3 we obtain $Z^n \Rightarrow Z$ in $\mathbb{D}([0, +\infty))$ as $n \rightarrow \infty$, where Z is the generally parametrized reciprocal gamma diffusion. □

3.4 Fractional Student, Fisher-Snedecor and reciprocal gamma diffusion approximations in Skorokhod topology

In this section, we apply Theorem 2.6 to obtain fractional Student, Fisher-Snedecor and reciprocal gamma diffusion approximations in Skorokhod topology.

Corollary 3.6. *Let $(H^{(n)}(r), r \in \mathbb{N}_0)$ be the Markov chain defined by (3.7). Let $(X^{(n)}(t), t \geq 0)$ be the corresponding rescaled Markov chain given by (3.10). Let $(N(t), t \geq 0)$ be the renewal process defined in (2.16), and $(E(t), t \geq 0)$ be the inverse of the standard β -stable subordinator $(D(t), t \geq 0)$ with $0 < \beta < 1$. Then*

$$X^{(n)} \left(n^{-1} N \left(n^{1/\beta} t \right) \right) \Rightarrow X(E(t)), \quad n \rightarrow \infty$$

in the Skorokhod space $\mathbb{D}(\mathbb{R})$ with J_1 topology, where $(X(t), t \geq 0)$ is Student diffusion with generator

$$\mathcal{A}f(x) = -\theta(x - \mu) f'(x) + \frac{1}{2} \frac{2\theta\delta^2}{\nu - 1} \left(1 + \left(\frac{x - \mu}{\delta} \right)^2 \right) f''(x), \quad f \in C_c^3(\mathbb{R}).$$

Proof. Stochastic processes $(X^{(n)}(t), t \geq 0)$ and $(X(t), t \geq 0)$ are both càdlàg, and Theorem 3.1 implies

$$X^n \Rightarrow X \text{ in } \mathbb{D}(\mathbb{R})$$

as $n \rightarrow \infty$. Now, simply apply Theorem 2.6 to obtain the desired result. □

Corollary 3.7. *Let $(H^{(n)}(r), r \in \mathbb{N}_0)$ be the Markov chain defined by (3.21). Let $(Y^{(n)}(t), t \geq 0)$ be the corresponding rescaled Markov chain given by (3.24). Let $(N(t), t \geq 0)$ be the renewal process defined in (2.16), and $(E(t), t \geq 0)$ be the inverse of the standard β -stable subordinator $(D(t), t \geq 0)$ with $0 < \beta < 1$. Then*

$$Y^{(n)} \left(n^{-1} N \left(n^{1/\beta} t \right) \right) \Rightarrow Y(E(t)), \quad n \rightarrow \infty$$

in the Skorokhod space $\mathbb{D}([0, +\infty))$ with J_1 topology, where $(Y(t), t \geq 0)$ is the Fisher-Snedecor diffusion with generator

$$\mathcal{A}f(y) = -\theta \left(y - \frac{\beta}{\beta - 2} \right) f'(y) + \frac{1}{2} \frac{4\theta}{\gamma(\beta - 2)} y(\gamma y + \beta) f''(y), \quad f \in C_c^3([0, +\infty)).$$

Proof. Stochastic processes $(Y^{(n)}(t), t \geq 0)$ and $(Y(t), t \geq 0)$ are both càdlàg, and Theorem 3.4 implies

$$Y^n \Rightarrow Y \text{ in } \mathbb{D}([0, +\infty)).$$

as $n \rightarrow \infty$. Now, simply apply Theorem 2.6 to obtain the desired result. □

Corollary 3.8. *Let $(H^{(n)}(r), r \in \mathbb{N}_0)$ be the Markov chain defined by (3.34). Let $(Z^{(n)}(t), t \geq 0)$ be the corresponding rescaled Markov chain given by (3.37). Let $(N(t), t \geq 0)$ be the renewal process defined in (2.16), and $(E(t), t \geq 0)$ be the inverse of the standard β -stable subordinator $(D(t), t \geq 0)$ with $0 < \beta < 1$. Then*

$$Z^{(n)} \left(n^{-1} N \left(n^{1/\beta} t \right) \right) \Rightarrow Z(E(t)), \quad n \rightarrow \infty$$

in the Skorokhod space $\mathbb{D}([0, +\infty))$ with J_1 topology, where $(Z(t), t \geq 0)$ is the reciprocal gamma diffusion with generator

$$\mathcal{A}f(z) = -\theta \left(z - \frac{\gamma}{\beta - 1} \right) f'(z) + \frac{1}{2} \frac{2\theta}{\beta - 1} z^2 f''(z), \quad f \in C_c^3([0, +\infty)).$$

Proof. Stochastic processes $(Z^{(n)}(t), t \geq 0)$ and $(Z(t), t \geq 0)$ are both càdlàg, and Theorem 3.5 implies

$$Z^n \Rightarrow Z \text{ in } \mathbb{D}([0, +\infty))$$

as $n \rightarrow \infty$. Now, simply apply Theorem 2.6 to obtain the desired result. \square

References

- [1] F. Avram, N. N. Leonenko, and N. Šuvak. Parameter estimation for Fisher Snedecor diffusion. *Statistics*, 45(1):27–42, 2011.
- [2] F. Avram, N. N. Leonenko, and N. Šuvak. Hypothesis testing for Fisher Snedecor diffusion. *Journal of Statistical Planning and Inference*, 142(8):2308 – 2321, 2012.
- [3] F. Avram, N. N. Leonenko, and N. Šuvak. Spectral representation of transition density of Fisher Snedecor diffusion. *Stochastics*, 85(2):346–369, 2013.
- [4] F. Avram and M. Taqqu. Weak convergence of sum of moving averages in the α -stable domain of attraction. *Annals of Probability*, 20:483–503, 1992.
- [5] H. Buchholz. *The Confluent Hypergeometric Function: with Special Emphasis on its Applications*. Springer Tracts in Natural Philosophy 15. Springer-Verlag Berlin Heidelberg, 1 edition, 1969.
- [6] P. Chakraborty, M. M. Meerschaert, and C. Y. Lim. Parameter estimation for fractional transport: A particle-tracking approach. *Water Resources Research*, 45(10):n/a–n/a, 2009. W10415.
- [7] A. Erdelyi. *Higher Transcendental Functions, Volume II*. Krieger Pub Co, 1981.
- [8] S. N. Ethier and T. G. Kurtz. *Markov Processes: Characterization and Convergence*. John Wiley & Sons, 2009.
- [9] U. Garibaldi and E. Scalas. *Finitary Probabilistic Methods in Econophysics*. Cambridge University Press, 2010.
- [10] G. Germano, M. Politi, E. Scalas, and R. L. Schilling. Stochastic calculus for uncoupled continuous-time random walks. *Phys. Rev. E*, 79:066102, 2009.
- [11] R. Gorenflo and F. Mainardi. *Processes with Long-Range Correlations: Theory and Applications*, chapter Fractional diffusion Processes: Probability Distributions and Continuous Time Random Walk, pages 148–166. Springer Berlin Heidelberg, Berlin, Heidelberg, 2003.
- [12] M. Jacobsen. Laplace and the origin of the Ornstein-Uhlenbeck process. *Bernoulli*, 2(3):271–286, 1996.
- [13] O. Kallenberg. *Foundations of Modern Probability*. Springer Series in Statistics. Probability and Its Applications. Springer, 2nd edition, 2002.
- [14] S. Karlin and H. Taylor. *A Second Course in Stochastic Processes*. Academic Press, 1981.
- [15] V. N. Kolokoltsov. Generalized continuous-time random walks, subordination by hitting times, and fractional dynamics. *Theory of Probability & Its Applications*, 53(4):594–609, 2009.
- [16] V. N. Kolokoltsov. *Markov Processes, Semigroups and Generators*. De Gruyter, Berlin, Boston, 2011.
- [17] N. N. Leonenko, M. M. Meerschaert, and A. Sikorskii. Correlation structure of fractional Pearson diffusions. *Computers and Mathematics with Applications*, 66(5):737 – 745, 2013.
- [18] N. N. Leonenko, M. M. Meerschaert, and A. Sikorskii. Fractional Pearson diffusions. *Journal of Mathematical Analysis and Applications*, 403(2):532 – 546, 2013.
- [19] N. N. Leonenko, I. Papić, A. Sikorskii, and N. Šuvak. Heavy-tailed fractional Pearson diffusions. *Stochastic Processes and their Applications*, 127(11):3512 – 3535, 2017.
- [20] N. N. Leonenko, I. Papić, A. Sikorskii, and N. Šuvak. Correlated continuous time random walks and fractional Pearson diffusions. *Bernoulli*, 24(4B):3603–3627, 11 2018.
- [21] N. N. Leonenko, I. Papić, A. Sikorskii, and N. Šuvak. Ehrenfest-Brillouin-type correlated continuous time random walk and fractional Jacobi diffusion. *Theory of probability and Mathematical Statistics*, 2019.
- [22] N. N. Leonenko and N. Šuvak. Statistical inference for reciprocal gamma diffusion process. *Journal of Statistical Planning and Inference*, 140(1):30 – 51, 2010.

- [23] N. N. Leonenko and N. Šuvak. Statistical inference for Student diffusion process. *Stoch. Anal. Appl.*, 28(6):972–1002, 2010.
- [24] M. Magdziarz. Black-Scholes formula in subdiffusive regime. *Journal of Statistical Physics*, 136(3):553–564, 2009.
- [25] A. A. Markov. On a problem of Laplace. *Bulletin de l’Académie Impériale des Sciences, Petrograd, VI Série*, 9:87–104, 1915.
- [26] H. McKean. Elementary solutions for certain parabolic partial differential equations. *Transactions of the American Mathematical Society*, 82(2):519–548, 1956.
- [27] M. M. Meerschaert, E. Nane, and Y. Xiao. Correlated continuous time random walks. *Statistics & Probability Letters*, 79(9):1194–1202, 2009.
- [28] M. M. Meerschaert and E. Scalas. Coupled continuous time random walks in finance. *Physica A: Statistical Mechanics and its Applications*, 370(1):114 – 118, 2006.
- [29] M. M. Meerschaert and H.-P. Scheffler. Limit theorems for continuous-time random walks with infinite mean waiting times. *J. Appl. Probab.*, 41(3):623–638, 09 2004.
- [30] M. M. Meerschaert and H.-P. Scheffler. Limit theorems for continuous-time random walks with infinite mean waiting times. *Journal of Applied Probability*, pages 623–638, 2004.
- [31] M. M. Meerschaert and A. Sikorskii. *Stochastic Models for Fractional Calculus*. De Gruyter, 2011.
- [32] R. Metzler and J. Klafter. The random walk’s guide to anomalous diffusion: a fractional dynamics approach. *Physics Reports*, 339(1):1 – 77, 2000.
- [33] R. Metzler and J. Klafter. The restaurant at the end of the random walk: recent developments in the description of anomalous transport by fractional dynamics. *J. Phys. A*, 37(1):161–208, 2004.
- [34] K. Pearson. *Tables for Statisticians and Biometricians, Part I*. Biometrics Laboratory, University College London, 1914.
- [35] E. Scalas. Five years of continuous-time random walks in econophysics. *Complex Netw. Econ. Interactions*, 567(1):3–16, 2006.
- [36] R. Schumer, D. Benson, M. Meerschaert, and B. Baeumer. Fractal mobile/immobile solute transport. *Water Resour. Res.*, 39(1):12–69, 2003.
- [37] A. Skorokhod. Limit theorems for stochastic processes. *Theory Prob. Appl.*, 1:261–290, 1956.
- [38] A. Skorokhod. Limit theorems for stochastic processes with independent increments. *Theory Prob. Appl.*, 2:138–171, 1957.
- [39] A. Stanislavsky. Black-Scholes model under subordination. *Physica A*, 18(1):469–474, 2009.
- [40] P. Straka and B. Henry. Lagging and leading coupled continuous time random walks, renewal times and their joint limits. *Stochastic Processes and their Applications*, 121(2):324 – 336, 2011.
- [41] L. Toniazzi. Stochastic classical solutions for space-time fractional evolution equations on a bounded domain. *Journal of Mathematical Analysis and Applications*, 469(2):594–622, 2019.