REDUCIBILITY OF REPRESENTATIONS INDUCED FROM THE ZELEVINSKY SEGMENT AND DISCRETE SERIES

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Abstract. Let G_n denote either the group $SO(2n+1, F)$ or $Sp(2n, F)$ over a non-archimedean local field. We determine the reducibility criteria for a parabolically induced representation of the form $\langle\Delta\rangle \rtimes \sigma$, where $\langle\Delta\rangle$ stands for a Zelevinsky segment representation of the general linear group and σ stands for a discrete series representation of G_n, in terms of the Mœglin-Tadić classification.

1. Introduction

In his seminal work [25], Zelevinsky introduced two classes of irreducible representations attached to segments of cuspidal representations, which made a huge impact on latter investigations in the representation theory of general linear and classical p-adic groups. By a segment of cuspidal representations, we mean a set of the form $\Delta = \{\nu^a\rho, \nu^{a+1}\rho, \ldots, \nu^b\rho\}$, where ρ is an irreducible cuspidal representation of a general linear group and $b-a$ is a non-negative integer. Following [25], to the segment Δ we attach irreducible representations $\langle\Delta\rangle$ and $\delta(\Delta)$, the unique irreducible subrepresentation and the unique irreducible quotient of the induced representation

$$\nu^a\rho \times \nu^{a+1}\rho \times \cdots \times \nu^b\rho,$$

respectively. The irreducible representation $\langle\Delta\rangle$ is now called the Zelevinsky segment representation, while irreducible representations of the form $\delta(\Delta)$ exhaust the set of irreducible essentially square-integrable representations of p-adic general linear groups, and play a fundamental role in the Langlands classification.

Naturally, representations of classical p-adic groups induced from ones having representations of the form $\delta(\Delta)$ on the general linear group parts have been extensively studied over the years. In some particular cases are obtained precise results on the structure of induced representations of such form. For instance, reducibility of generalized principal series and standard representations can be seen in [17] and [18], while complete description of the composition factors in some cases when the representation is induced from the strongly positive discrete series on the classical group part can be seen in [11] and [16].

In [25] is also provided a purely algebraic classification of irreducible representations of p-adic general linear groups, which is completely based on the Zelevinsky segments. In terms of that classification, a complete description of the unitary dual of p-adic general linear

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group has been obtained in [21]. We note that there is also a classification of irreducible representations of classical p-adic groups based on the Zelevinsky segments, dual to the Langlands one, given in [5].

Our aim is to study the structure of representations induced from the ones having the Zelevinsky segment on the general linear group part. We provide the reducibility criteria for representations of the form $\langle \Delta \rangle \rtimes \sigma$, induced from the representation $\langle \Delta \rangle \otimes \sigma$ of the maximal Levi subgroup, having the Zelevinsky segment on the general linear group part, and a discrete series representation σ of G_n on the classical group part. Here G_n denotes a symplectic or a special odd orthogonal group of split rank n. Our criteria are expressed in terms of the admissible triples, which are in one-to-one correspondence with discrete series, by the Mœglin-Tadić classification. Our results should lead to a rather precise understanding of the composition series of representation of studied type (in a particular case given in [12]), and could be used to construct some elements appearing in the unitary dual of the group G_n.

Our main strategy is rather straightforward. To prove that some $\langle \Delta \rangle \rtimes \sigma$ is irreducible, we show that all its irreducible subquotients are mutually isomorphic, and then we show that $\langle \Delta \rangle \rtimes \sigma$ can contain at most one such an irreducible representation. On the other hand, to obtain reducibility of $\langle \Delta \rangle \rtimes \sigma$, we just construct two mutually non-isomorphic irreducible subquotients.

We use an adjustment of the methods presented in [16, 17] and in [11] to the case of representations induced from the Zelevinsky segment. The structural formula enables us to provide a calculation of some prominent members appearing in the Jacquet modules of $\langle \Delta \rangle \rtimes \sigma$. Starting from the Langlands classification, we use the provided calculation to determine possible irreducible subquotients of $\langle \Delta \rangle \rtimes \sigma$. In the Langlands classification, on the classical group part of an induced representation appears an irreducible tempered member. To handle this part, we use the combinatorics of the classifications of discrete series and tempered representations, given in [8] and [24]. After obtaining the Langlands parameters of a potential irreducible subquotient, we show that the obtained representation is contained in the composition series of $\langle \Delta \rangle \rtimes \sigma$ using a combination of the Jacquet modules method and methods of the intertwining operators.

For the convenience of the reader, we cite the main reducibility criteria here.

Theorem 1.1. *Let ρ stand for an irreducible self-contragredient cuspidal representation of $GL(n_\rho, F)$, and let $a, b \in \mathbb{R}$ such that $a + b \in \mathbb{Z}_{\geq 0}$. Let $\Delta = [\nu^a \rho, \nu^b \rho]$. Let σ denote a discrete series of G_n, and let $(Jord(\sigma), \sigma_{cusp}, \epsilon_\sigma)$ stand for the corresponding admissible triple. Suppose that $Jord_\rho(\sigma) \neq \emptyset$ and that for $x \in Jord_\rho(\sigma)$ we have $2a + 1 - x \in 2\mathbb{Z}$.*

(1) Suppose that $a \geq 1$. The induced representation $\langle \Delta \rangle \rtimes \sigma$ is irreducible if and only if one of the following holds:
- *$2a - 1 \notin Jord_\rho(\sigma)$,*
- *$x \in Jord_\rho(\sigma)$ for all $x \in \{2a-1, 2a+1, \ldots, 2b+1\}$, and $\epsilon_\sigma((x_-, \rho), (x, \rho)) = -1$ for all $x \in \{2a+1, 2a+3, \ldots, 2b+1\}$.*

(2) Suppose that $a \leq \frac{1}{2}$ and let $c = 2b + 3 - 2\lceil b + \frac{1}{2} \rceil$, where $\lceil x \rceil$ denotes the least integer greater than or equal to x. If $a \in \mathbb{Z}$, suppose additionally that $charF = 0$. The

induced representation $\langle\Delta\rangle \rtimes \sigma$ *is irreducible if and only if* $x \in Jord_\rho(\sigma)$ *for all* $x \in \{c, c+2, \ldots, 2b+1\}$, $\epsilon_\sigma((x_-, \rho), (x, \rho)) = -1$ *for all* $x \in \{c+2, c+4, \ldots, 2b+1\}$, *and* $\epsilon_\sigma(2, \rho) = -1$ *if* $2a+1$ *is even.*

The following corollary parallels [23, Theorem 13.2].

Corollary 1.2. *Suppose that* $charF = 0$. *Let* ρ *stand for an irreducible self-contragredient cuspidal representation of* $GL(n_\rho, F)$, *and let* $a, b \in \mathbb{R}$ *such that* $a+b \in \mathbb{Z}_{\geq 0}$. *Let* $\Delta = [\nu^a\rho, \nu^b\rho]$, *and let* σ *denote a discrete series of* G_n. *The induced representation* $\langle\Delta\rangle \rtimes \sigma$ *is irreducible if and only if* $\nu^c \rtimes \sigma$ *is irreducible for all* $\nu^c\rho \in \Delta$.

In the following section we introduce the notation and obtain several technical results which are frequently used in the paper. Reducibility criteria in basic cases are provided in Section 3. The main results are obtained in Sections 4 and 5, using a case-by-case consideration.

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2. Preliminaries

Let F denote a non-archimedean local field of the characteristic different than two. Let us describe the groups that we consider.

Let $J_n = (\delta_{i,n+1-j})_{1\leq i,j\leq n}$ denote the $n \times n$ matrix, where $\delta_{i,n+1-j}$ stands for the Kronecker symbol. For a square matrix g, we denote by g^t its transposed matrix, and by g^τ its transposed matrix with respect to the second diagonal. In what follows, we shall fix one of the series of classical groups

$$Sp(n, F) = \left\{ g \in GL(2n, F) : \begin{pmatrix} 0 & -J_n \\ J_n & 0 \end{pmatrix} g^t \begin{pmatrix} 0 & -J_n \\ J_n & 0 \end{pmatrix} = g^{-1} \right\},$$

or

$$SO(2n+1, F) = \left\{ g \in GL(2n+1, F) : g^\tau = g^{-1} \right\}$$

and denote by G_n the rank n group belonging to the series which we fixed.

The set of standard parabolic subgroups will be fixed in a usual way, i.e., we fix a minimal F-parabolic subgroup in the classical group G_n consisting of upper-triangular matrices in the usual matrix realization of the classical group. Then the Levi factors of standard parabolic subgroups have the form $M \cong GL(n_1, F) \times \cdots \times GL(n_k, F) \times G_{n'}$, where $GL(m, F)$ denotes the general linear group of rank m over F. If δ_i is a representation of $GL(n_i, F)$ and τ a representation of $G_{n'}$, the normalized parabolically induced representation $\mathrm{Ind}_M^{G_n}(\delta_1 \otimes \cdots \otimes \delta_k \otimes \tau)$ will be denoted by $\delta_1 \times \cdots \times \delta_k \rtimes \tau$. We use similar notation to denote a parabolically induced representation of $GL(m, F)$.

By $\mathrm{Irr}(G_n)$ we denote the set of all irreducible admissible representations of G_n. Let $R(G_n)$ denote the Grothendieck group of admissible representations of finite length of G_n

and define $R(G) = \oplus_{n\geq 0} R(G_n)$. In a similar way we define $\mathrm{Irr}(GL(n,F))$ and $R(GL) = \oplus_{n\geq 0} R(GL(n,F))$. In $R(G)$ we have $\pi \rtimes \sigma = \widetilde{\pi} \rtimes \sigma$ and $\pi_1 \times \pi_2 \rtimes \sigma = \pi_2 \times \pi_1 \rtimes \sigma$.

For $\sigma \in \mathrm{Irr}(G_n)$ and $1 \leq k \leq n$ we denote by $r_{(k)}(\sigma)$ the normalized Jacquet module of σ with respect to parabolic subgroup $P_{(k)}$ having the Levi subgroup equal to $GL(k,F) \times G_{n-k}$. We identify $r_{(k)}(\sigma)$ with its semisimplification in $R(GL(k,F)) \otimes R(G_{n-k})$ and consider

$$\mu^*(\sigma) = 1 \otimes \sigma + \sum_{k=1}^{n} r_{(k)}(\sigma) \in R(GL) \otimes R(G).$$

We denote by ν a composition of the determinant mapping with the normalized absolute value on F. Let $\rho \in R(GL)$ denote an irreducible supercuspidal representation. By a segment Δ we mean a set of the form $[\rho, \nu^m \rho] := \{\rho, \nu\rho, \ldots, \nu^m \rho\}$, for a non-negative integer m. The induced representation $\rho \times \nu\rho \times \cdots \times \nu^m \rho$ has a unique irreducible subrepresentation ([25]), denoted by $\langle \Delta \rangle$. Representation $\langle \Delta \rangle$ is called the Zelevinsky segment representation and plays a fundamental role in the description of the unitary dual of the group $GL(n,F)$.

The induced representation $\nu^m \rho \times \nu^{m-1} \rho \times \cdots \times \rho$ also has a unique irreducible subrepresentation ([25]), denoted by $\delta(\Delta)$. Representation $\delta(\Delta)$ is essentially square-integrable, and every irreducible essentially square-integrable representation of $GL(n,F)$ is of the form $\delta(\Delta)$, by [25].

We study reducibility of the induced representation of the form $\langle \Delta \rangle \rtimes \sigma$, where $\Delta = [\nu^a \rho, \nu^b \rho]$ and $\sigma \in R(G_n)$ is a discrete series. Since in $R(G)$ we have $\langle [\nu^a \rho, \nu^b \rho] \rangle \rtimes \sigma = \langle [\nu^{-b} \widetilde{\rho}, \nu^{-a} \widetilde{\rho}] \rangle \rtimes \sigma$, we may assume $a + b \geq 0$.

We frequently use the following structural formulas, obtained in [5, Theorem 1.4] and in [22]:

Theorem 2.1. *Let $\rho \in Irr(GL(m,F))$ be a supercuspidal representation and $k, l \in \mathbb{R}$ such that $k + l \in \mathbb{Z}_{\geq 0}$. Let σ denote an admissible representation of finite length of G_n. Write $\mu^*(\sigma) = \sum_{\pi, \sigma'} \pi \otimes \sigma'$. Then we have:*

$$\begin{aligned} \mu^*(\langle [\nu^{-k}\rho, \nu^l \rho] \rangle \rtimes \sigma) = & \sum_{i=0}^{k+l+1} \sum_{j=0}^{i} \sum_{\pi, \sigma'} \langle [\nu^{-l} \widetilde{\rho}, \nu^{-i+k} \widetilde{\rho}] \rangle \times \langle [\nu^{-k} \rho, \nu^{j-k-1} \rho] \rangle \times \pi \\ & \otimes \langle [\nu^{j-k} \rho, \nu^{i-k-1} \rho] \rangle \rtimes \sigma', \end{aligned}$$

$$\begin{aligned} \mu^*(\delta([\nu^{-k}\rho, \nu^l \rho]) \rtimes \sigma) = & \sum_{i=-k-1}^{l} \sum_{j=i}^{l} \sum_{\pi, \sigma'} \delta([\nu^{-i} \widetilde{\rho}, \nu^{k} \widetilde{\rho}]) \times \delta([\nu^{j+1} \rho, \nu^{l} \rho]) \times \pi \\ & \otimes \delta([\nu^{i+1} \rho, \nu^{j} \rho]) \rtimes \sigma'. \end{aligned}$$

We briefly recall the Langlands classification for general linear groups. We favor the subrepresentation version of this classification over the quotient one.

For every irreducible essentially square-integrable representation $\delta \in R(GL)$, there is a unique $e(\delta) \in \mathbb{R}$ such that $\nu^{-e(\delta)} \delta$ is unitarizable. Note that $e(\delta([\nu^a \rho, \nu^b \rho])) = (a+b)/2$. Suppose that $\delta_1, \delta_2, \ldots, \delta_k$ are irreducible essentially square-integrable representations such that $e(\delta_1) \leq e(\delta_2) \leq \ldots \leq e(\delta_k)$. Then the induced representation $\delta_1 \times \delta_2 \times \cdots \times \delta_k$

has a unique irreducible subrepresentation, which we denote by $L(\delta_1, \delta_2, \dots, \delta_k)$. This irreducible subrepresentation is called the Langlands subrepresentation, and it appears with multiplicity one in the composition series of $\delta_1 \times \delta_2 \times \cdots \times \delta_k$. Every irreducible representation $\pi \in R(GL)$ is isomorphic to some $L(\delta_1, \delta_2, \dots, \delta_k)$ and, for a given π, the representations $\delta_1, \delta_2, \dots, \delta_k$ are unique up to a permutation.

If we let $\delta_i = \delta([\nu^{a_i}\rho_i, \nu^{b_i}\rho_i])$, note that for $a_i = b_i$, $a_i + 1 = a_{i+1}$ for all i, and $\rho_i \cong \rho_j$ for all i and j, we have $L(\delta_1, \delta_2, \dots, \delta_k) \cong \langle [\nu^{a_1}\rho_1, \nu^{a_1+k-1}\rho_1] \rangle$.

Similarly, throughout the paper we use the subrepresentation version of the Langlands classification for classical groups, since it is more appropriate for our Jacquet module considerations. So, we realize a non-tempered irreducible representation π of G_n as the unique irreducible (Langlands) subrepresentation of an induced representation of the form $\delta_1 \times \delta_2 \times \cdots \times \delta_k \rtimes \tau$, where τ is an irreducible tempered representation of some G_t, and $\delta_1, \delta_2, \dots, \delta_k \in R(GL)$ are irreducible essentially square-integrable representations such that $e(\delta_1) \leq e(\delta_2) \leq \cdots \leq e(\delta_k) < 0$. In this case, we write $\pi = L(\delta_1, \delta_2, \dots, \delta_k, \tau)$.

The next lemma is [6, Lemma 5.5].

Lemma 2.2. *Suppose that $\pi \in R(G_n)$ is an irreducible representation, λ an irreducible representation of the Levi subgroup M, and π is a subrepresentation of $Ind_M^{G_n}(\lambda)$. If $L > M$, then there is an irreducible subquotient ρ of $Ind_M^L(\lambda)$ such that π is a subrepresentation of $Ind_L^{G_n}(\rho)$.*

Let us now recall the Mœglin-Tadić classification of discrete series for groups that we consider. We note that it now holds unconditionally, due to results of [1], [14, Théorème 3.1.1] and [3, Theorem 7.8]. A shorter and more algebraic form of this classification, which covers both classical and odd general spin groups, can be found in [10]. Every discrete series in G_n is uniquely described by three invariants: the partial cuspidal support, the Jordan block and the ϵ-function.

The partial cuspidal support of a discrete series $\sigma \in \mathrm{Irr}(G_n)$ is an irreducible cuspidal representation σ_{cusp} of some G_m such that there is an irreducible admissible representation π of $GL(n_\pi, F)$ such that σ is a subrepresentation of $\pi \rtimes \sigma_{cusp}$.

The Jordan block of σ, denoted by $\mathrm{Jord}(\sigma)$, is a set of all pairs (x, ρ) where $\rho \cong \widetilde{\rho}$ is an irreducible cuspidal representation of some $GL(n_\rho, F)$ and $x > 0$ is an integer such that the following two conditions are satisfied:

(1) x is even if and only if $L(s, \rho, r)$ has a pole at $s = 0$. The local L-function $L(s, \rho, r)$ is the one defined by Shahidi (see for instance [19], [20]), where $r = \bigwedge^2 \mathbb{C}^{n_\rho}$ is the exterior square representation of the standard representation on $\mathbb{C}^{n_\rho}$ of $GL(n_\rho, \mathbb{C})$ if G_n is a symplectic or even-orthogonal group and $r = \mathrm{Sym}^2\mathbb{C}^{n_\rho}$ is the symmetric-square representation of the standard representation on $\mathbb{C}^{n_\rho}$ of $GL(n_\rho, \mathbb{C})$ if G_n is an odd-orthogonal group.

(2) The induced representation

$$\delta([\nu^{-(x-1)/2}\rho, \nu^{(x-1)/2}\rho]) \rtimes \sigma$$

is irreducible.

To explain the notion of the ϵ-function, we first define the Jordan triples. This are the triples of the form $(\mathrm{Jord}, \sigma', \epsilon)$ where

- σ' is an irreducible cuspidal representation of some G_n.
- Jord is a finite set (possibly empty) of pairs (x, ρ), where ρ is an irreducible selfcontragredient cuspidal representation of $GL(n_\rho, F)$ and x is a positive integer which is even if and only if $L(s, \rho, r)$ has a pole at $s = 0$ (for the local L-function as above). For an irreducible selfcontragredient cuspidal representation ρ of $GL(n_\rho, F)$ we write $\mathrm{Jord}_\rho = \{x : (x, \rho) \in \mathrm{Jord}\}$. If $\mathrm{Jord}_\rho \neq \emptyset$ and $x \in \mathrm{Jord}_\rho$, denote $x_- = \max\{y \in \mathrm{Jord}_\rho : y < x\}$, if it exists.
- ϵ is a function defined on a subset of $\mathrm{Jord} \cup (\mathrm{Jord} \times \mathrm{Jord})$ and attains the values 1 and -1. If $(x, \rho) \in \mathrm{Jord}$, then $\epsilon(x, \rho)$ is not defined if and only if x is odd and $(y, \rho) \in \mathrm{Jord}(\sigma')$ for some positive integer y. Next, ϵ is defined on a pair $(x, \rho), (y, \rho') \in \mathrm{Jord}$ if and only if $\rho \cong \rho'$ and $x \neq y$.

It follows from the compatibility conditions, which can be found in [10], [15] or [17], that it is enough to know the value of ϵ on the consecutive pairs $(x_-, \rho), (x, \rho)$ and on the minimal element of Jord_ρ (if it is defined on elements, not only on pairs).

Suppose that, for the Jordan triple $(\mathrm{Jord}, \sigma', \epsilon)$, there is $(x, \rho) \in \mathrm{Jord}$ such that we have $\epsilon((x_-, \rho), (x, \rho)) = 1$. If we put $\mathrm{Jord}' = \mathrm{Jord} \setminus \{(x_-, \rho), (x, \rho)\}$ and consider the restriction ϵ' of ϵ to $\mathrm{Jord}' \cup (\mathrm{Jord}' \times \mathrm{Jord}')$, we obtain a new Jordan triple $(\mathrm{Jord}', \sigma', \epsilon')$, and we say that such Jordan triple is subordinated to $(\mathrm{Jord}, \sigma', \epsilon)$.

We say that the Jordan triple $(\mathrm{Jord}, \sigma', \epsilon)$ is a triple of alternated type if $\epsilon((x_-, \rho), (x, \rho)) = -1$ holds whenever x_- is defined and there is an increasing bijection $\phi_\rho : \mathrm{Jord}_\rho \to \mathrm{Jord}'_\rho(\sigma')$, where $\mathrm{Jord}'_\rho(\sigma')$ equals $\mathrm{Jord}_\rho(\sigma') \cup \{0\}$ if a is even and $\epsilon(\min(\mathrm{Jord}_\rho), \rho) = 1$ and $\mathrm{Jord}'_\rho(\sigma')$ equals $\mathrm{Jord}_\rho(\sigma')$ otherwise.

The Jordan triple $(\mathrm{Jord}, \sigma', \epsilon)$ dominates the Jordan triple $(\mathrm{Jord}', \sigma', \epsilon')$ if there is a sequence of Jordan triples $(\mathrm{Jord}_i, \sigma', \epsilon_i)$, $0 \leq i \leq k$, such that $(\mathrm{Jord}_0, \sigma', \epsilon_0) = (\mathrm{Jord}, \sigma', \epsilon)$, $(\mathrm{Jord}_k, \sigma', \epsilon_k) = (\mathrm{Jord}', \sigma', \epsilon')$ and $(\mathrm{Jord}_i, \sigma', \epsilon_i)$ is subordinated to $(\mathrm{Jord}_{i-1}, \sigma', \epsilon_{i-1})$ for $i \in \{1, 2, \ldots, k\}$. The Jordan triple $(\mathrm{Jord}, \sigma', \epsilon)$ is called an admissible triple if it dominates a triple of alternated type.

Classification given in [13] and [15] states that there is a one-to-one correspondence between the set of all discrete series in $\mathrm{Irr}(G)$ and the set of all admissible triples $(\mathrm{Jord}, \sigma', \epsilon)$ given by $\sigma = \sigma_{(\mathrm{Jord}, \sigma', \epsilon)}$, such that $\sigma_{cusp} = \sigma'$ and $\mathrm{Jord}(\sigma) = \mathrm{Jord}$.

Throughout the paper, the admissible triple corresponding to discrete series $\sigma \in R(G)$ will be denoted by $(\mathrm{Jord}(\sigma), \sigma_{cusp}, \epsilon_\sigma)$.

We close this section with several useful technical results.

Lemma 2.3. *Let π denote the irreducible representation $L(\nu^{-c}\widetilde{\rho}, \delta_1, \delta_2, \ldots, \delta_k, \tau)$ of G_n such that $-c < e(\delta_1)$. Suppose that $L(\delta_1, \delta_2, \ldots, \delta_k, \tau)$ is an irreducible subquotient of $\langle [\nu^d\rho, \nu^{c-1}\rho] \rangle \rtimes \sigma$, for some discrete series σ and $-c + 1 \leq d$. Then π is an irreducible subquotient of $\langle [\nu^d\rho, \nu^c\rho] \rangle \rtimes \sigma$.*

Proof. Since both $L(\nu^{-c}\widetilde{\rho}, \delta_1, \delta_2, \ldots, \delta_k, \tau)$ and $\nu^{-c}\widetilde{\rho} \rtimes L(\delta_1, \delta_2, \ldots, \delta_k, \tau)$ are subrepresentations of the induced representation $\nu^{-c}\widetilde{\rho} \times \delta_1 \times \cdots \times \delta_k \rtimes \tau$, which contains a unique irreducible subrepresentation, it follows that π is a subrepresentation of $\nu^{-c}\widetilde{\rho} \rtimes L(\delta_1, \delta_2, \dots, \delta_k, \tau)$. Thus, π is an irreducible subquotient of $\nu^{-c}\widetilde{\rho} \times \langle[\nu^d\rho, \nu^{c-1}\rho]\rangle \rtimes \sigma$. In an appropriate Grothendieck group we have

$$\nu^{-c}\widetilde{\rho} \times \langle[\nu^d\rho, \nu^{c-1}\rho]\rangle \rtimes \sigma = \langle[\nu^d\rho, \nu^{c-1}\rho]\rangle \times \nu^c\rho \rtimes \sigma,$$

so there is some irreducible subquotient π' of $\langle[\nu^d\rho, \nu^{c-1}\rho]\rangle \times \nu^c\rho$ such that $\pi \leq \pi' \rtimes \sigma$. It can be easily seen that the only possible irreducible subquotients of $\langle[\nu^d\rho, \nu^{c-1}\rho]\rangle \times \nu^c\rho$ are $\langle[\nu^d\rho, \nu^c\rho]\rangle$ and $L(\nu^d\rho, \dots, \nu^{c-2}\rho, \delta([\nu^{c-1}\rho, \nu^c\rho]))$.

Since π is a subrepresentation of $\nu^{-c}\widetilde{\rho} \rtimes L(\delta_1, \delta_2, \dots, \delta_k, \tau)$, Frobenius reciprocity implies that $\mu^*(\pi) \geq \nu^{-c}\widetilde{\rho} \otimes L(\delta_1, \delta_2, \dots, \delta_k, \tau)$. Using the structural formula, together with the square-integrability of σ, we deduce that $\mu^*(L(\nu^d\rho, \dots, \nu^{c-2}\rho, \delta([\nu^{c-1}\rho, \nu^c\rho])) \rtimes \sigma)$ does not contain $\nu^{-c}\widetilde{\rho} \otimes L(\delta_1, \delta_2, \dots, \delta_k, \tau)$. Thus, $\pi \leq \langle[\nu^d\rho, \nu^c\rho]\rangle \rtimes \sigma$ and the lemma is proved. $\square$

Lemma 2.4. *Suppose that $\rho \cong \widetilde{\rho}$. Let π denote the irreducible representation $L(\nu^{-c}\rho, \nu^{-c}\rho, \delta_1, \delta_2, \dots, \delta_k, \tau)$ of G_n such that $-c < e(\delta_1)$. Suppose that $L(\delta_1, \delta_2, \dots, \delta_k, \tau)$ is an irreducible subquotient of $\langle[\nu^{-c+1}\rho, \nu^{c-1}\rho]\rangle \rtimes \sigma$, for some discrete series σ. Then π is an irreducible subquotient of $\langle[\nu^{-c}\rho, \nu^c\rho]\rangle \rtimes \sigma$.*

Proof. In the same way as in the proof of the previous lemma we deduce that there is an irreducible subquotient π' of $\nu^{-c}\rho \times \langle[\nu^{-c+1}\rho, \nu^{c-1}\rho]\rangle \times \nu^c\rho$ such that $\pi \leq \pi' \rtimes \sigma$. Since $\mu^*(\pi)$ contains an irreducible constituent of the form $\nu^{-c}\rho \times \nu^{-c}\rho \otimes \pi''$, it follows directly from the structural formula and square-integrability of σ that $\pi' \cong \langle[\nu^{-c}\rho, \nu^c\rho]\rangle$. This ends the proof. $\square$

Lemma 2.5. *Suppose that $\rho \cong \widetilde{\rho}$. Let π denote the irreducible representation $L(\nu^{-\frac{1}{2}}\rho, \tau)$ of G_n such that τ is an irreducible subquotient of $\nu^{\frac{1}{2}}\rho \rtimes \sigma$, for some discrete series σ. Then π is an irreducible subquotient of $\langle[\nu^{-\frac{1}{2}}\rho, \nu^{\frac{1}{2}}\rho]\rangle \rtimes \sigma$.*

Proof. Obviously, $\pi \leq \nu^{-\frac{1}{2}}\rho \times \nu^{\frac{1}{2}}\rho \rtimes \sigma$. In the appropriate Grothendieck group we have

$$\nu^{-\frac{1}{2}}\rho \times \nu^{\frac{1}{2}}\rho \rtimes \sigma = \langle[\nu^{-\frac{1}{2}}\rho, \nu^{\frac{1}{2}}\rho]\rangle \rtimes \sigma + \delta([\nu^{-\frac{1}{2}}\rho, \nu^{\frac{1}{2}}\rho]) \rtimes \sigma.$$

Frobenius reciprocity shows that $\mu^*(\pi) \geq \nu^{-\frac{1}{2}}\rho \otimes \tau$. From Theorem 2.1 and square-integrability of σ we deduce that $\mu^*(\delta([\nu^{-\frac{1}{2}}\rho, \nu^{\frac{1}{2}}\rho]) \rtimes \sigma)$ does not contain $\nu^{-\frac{1}{2}}\rho \otimes \tau$. Consequently, π has to be an irreducible subquotient of $\langle[\nu^{-\frac{1}{2}}\rho, \nu^{\frac{1}{2}}\rho]\rangle \rtimes \sigma$. $\square$

The following technical lemma follows directly from Theorem 2.1 and Casselman square-integrability criterion.

Lemma 2.6. *Let $\rho \in R(GL)$ stand for an irreducible cuspidal representation and let $a, b \in \mathbb{R}$ such that $a+b \in \mathbb{Z}_{\geq 0}$, and let $\Delta = [\nu^a\rho, \nu^b\rho]$. Let $\sigma \in R(G)$ denote a discrete series representation. Suppose that $L(\delta_1, \delta_2, \dots, \delta_k, \tau)$ is an irreducible non-tempered subquotient of $\langle\Delta\rangle \rtimes \sigma$. We have:*

(1) If $\delta_1 \cong \nu^x\rho'$, for some cuspidal $\rho' \in R(GL)$, then $\rho' \in \{\rho, \widetilde{\rho}\}$, $x \in \{a, -b\}$ and $L(\delta_2, \dots, \delta_k, \tau)$ is an irreducible subquotient of $\langle\Delta'\rangle \rtimes \sigma$, for $\Delta' \in \{[\nu^a\rho, \nu^{b-1}\rho], [\nu^{a+1}\rho, \nu^b\rho]\}$. Also, if $a \geq 0$, then $\rho' \cong \widetilde{\rho}$, $x = -b$ and $\Delta' = [\nu^a\rho, \nu^{b-1}\rho]$.

(2) If $\delta_1 \cong \delta([\nu^x\rho', \nu^{x+1}\rho'])$, for some cuspidal $\rho' \in R(GL)$, then $\rho' \cong \rho \cong \widetilde{\rho}$, $2a \in \mathbb{Z}$, $x = -b = a-1$ and $L(\delta_2, \dots, \delta_k, \tau)$ is an irreducible subquotient of $\langle[\nu^{a+1}\rho, \nu^{b-1}\rho]\rangle \rtimes \sigma$.

3. Reducibility criteria in some basic cases

Let $\rho \in R(GL)$ stand for an irreducible cuspidal representation, and let $a, b \in \mathbb{R}$ such that $a+b \in \mathbb{Z}_{\geq 0}$. Let $\Delta = [\nu^a\rho, \nu^b\rho]$. Let $\sigma \in R(G)$ denote a discrete series representation.

We start with a description of tempered irreducible subquotients of some induced representations of studied type.

Lemma 3.1. *If $2a \notin \mathbb{Z}$, then $\langle\Delta\rangle \rtimes \sigma$ does not contain an irreducible tempered subquotient. Suppose that $2a \in \mathbb{Z}$ and that one of the following holds:*
(1) $\rho \not\cong \widetilde{\rho}$,
(2) $Jord_\rho(\sigma) \neq \emptyset$, $2a \in \mathbb{Z}$, and $2a+1-x \notin 2\mathbb{Z}$, for $x \in Jord_\rho(\sigma)$.
Then $\langle\Delta\rangle \rtimes \sigma$ contains an irreducible tempered subquotient if and only if $a = b = 0$.

Proof. By the classification of discrete series, $\langle\Delta\rangle \rtimes \sigma$ does not contain a discrete series subquotient.

Let us now suppose that $\langle\Delta\rangle \rtimes \sigma$ contains an irreducible tempered subquotient which is not square-integrable. Then there are irreducible cuspidal representation $\rho' \in R(GL)$, $x \geq 0$, and irreducible tempered representation $\tau \in R(G)$ such that $\mu^*(\langle\Delta\rangle \rtimes \sigma) \geq \delta([\nu^{-x}\rho', \nu^x\rho']) \otimes \tau$. By the structural formula, there are $0 \leq j \leq i \leq b-a+1$ and an irreducible constituent $\pi \otimes \sigma'$ of $\mu^*(\sigma)$ such that

$$\delta([\nu^{-x}\rho', \nu^x\rho']) \leq \langle[\nu^{-b}\widetilde{\rho}, \nu^{-i-a}\widetilde{\rho}]\rangle \times \langle[\nu^a\rho, \nu^{j+a-1}\rho]\rangle \times \pi.$$

Square-integrability of σ implies that $(x, \rho') \in \{(-a, \rho), (b, \widetilde{\rho})\}$. Since $\nu^y\rho'$ does not appear in the cuspidal support of σ for y such that $x+y \in \mathbb{Z}$, it follows that $\delta([\nu^{-x}\rho', \nu^x\rho']) \leq \langle[\nu^{-b}\widetilde{\rho}, \nu^{-i-a}\widetilde{\rho}]\rangle \times \langle[\nu^a\rho, \nu^{j+a-1}\rho]\rangle$. Since $\delta([\nu^{-x}\rho', \nu^x\rho'])$ is irreducible essentially square-integrable, either $x = 0$ or $x = \frac{1}{2}$. This implies $2a \in \mathbb{Z}$. But, $x = \frac{1}{2}$ implies $a = b = \frac{1}{2}$, $i = 0$ and $j = 1$, which is impossible. Consequently, $x = a = 0$, $\rho' \cong \rho$, and $\tau \leq \langle[\nu\rho, \nu^b\rho]\rangle \rtimes \sigma$. Obviously, $\mu^*(\tau)$ does not contain an irreducible constituent of the form $\rho'' \otimes \tau'$, for some irreducible cuspidal $\rho'' \in R(GL)$, so τ has to be square-integrable, but this is possible only for $b = 0$.

On the other hand, if $a = b = 0$, we have $\langle\Delta\rangle \rtimes \sigma \cong \rho \rtimes \sigma$, which contains an irreducible tempered representation. □

We frequently use the following multiplicity one result. For a real number x, by $\lceil x \rceil$ we denote the least integer greater than or equal to x.

Lemma 3.2. *If a is an integer and $a \leq 0$, let τ denote an irreducible subrepresentation of $\rho \rtimes \sigma$, otherwise let $\tau = \sigma$. If $a = 0$, let $c = -1$ and if $a > 0$ let $c = -a$.*
(1) If $a \geq 0$, let π denote $\nu^{-b}\widetilde{\rho} \otimes \nu^{-b+1}\widetilde{\rho} \otimes \cdots \otimes \nu^c\widetilde{\rho} \otimes \tau$.
(2) If $a < 0$ and either $\rho \not\cong \widetilde{\rho}$ or $2a \notin \mathbb{Z}$, let π denote

$$\nu^{-b}\widetilde{\rho} \otimes \nu^{-b+1}\widetilde{\rho} \otimes \cdots \otimes \nu^{-b+\lceil b \rceil -1}\widetilde{\rho} \otimes \nu^a\rho \otimes \nu^{a+1}\rho \otimes \cdots \otimes \nu^{a-\lceil a \rceil}\rho \otimes \tau,$$

(3) If $a<0$, $\rho \cong \widetilde{\rho}$ and $2a \in \mathbb{Z}$, let π denote
$\nu^{-b}\rho \otimes \nu^{-b+1}\rho \otimes \cdots \otimes \nu^{a-1}\rho \otimes \nu^{a}\rho \times \nu^{a}\rho \otimes \nu^{a+1}\rho \times \nu^{a+1}\rho \otimes \cdots \otimes \nu^{\lceil a \rceil -a-1}\rho \times \nu^{\lceil a \rceil -a-1}\rho \otimes \tau$.
Then the multiplicity of π in the Jacquet module of $\langle \Delta \rangle \rtimes \sigma$ with respect to the appropriate parabolic subgroup equals one.

Proof. We prove only the third part of the lemma, other two parts can be proved in the same way but more easily. We assume that $-a<b$, since the case analogous to $-a=b$ then appears as a subcase. Transitivity of Jacquet modules implies that there is an irreducible constituent $\nu^{-b}\rho \otimes \pi_1$ of $\mu^*(\langle \Delta \rangle \rtimes \sigma)$ such that $\nu^{-b+1}\rho \otimes \cdots \otimes \nu^{a-1}\rho \otimes \nu^{a}\rho \times \nu^{a}\rho \otimes \nu^{a+1}\rho \times \nu^{a+1}\rho \otimes \cdots \otimes \nu^{\lceil a \rceil -a-1}\rho \times \nu^{\lceil a \rceil -a-1}\rho \otimes \tau$ is contained in the Jacquet module of π_1 with respect to the appropriate parabolic subgroup. Thus, there are $0 \leq j \leq i \leq b-a+1$ and an irreducible constituent $\pi' \otimes \sigma'$ of $\mu^*(\sigma)$ such that

$$\nu^{-b}\rho \leq \langle [\nu^{-b}\rho, \nu^{-i-a}\rho] \rangle \times \langle [\nu^{a}\rho, \nu^{j+a-1}\rho] \rangle \times \pi'.$$

It follows at once that $i=b-a+1$, $j=0$, and $\pi_1 \leq \langle [\nu^{a}\rho, \nu^{b-1}\rho] \rangle \rtimes \sigma$. Also, $\pi_1 \leq \langle [\nu^{a}\rho, \nu^{b-1}\rho] \rangle \rtimes \sigma$ and the multiplicity of π in the Jacquet module of $\langle \Delta \rangle \rtimes \sigma$ with respect to the appropriate parabolic subgroup equals the multiplicity of $\nu^{-b+1}\rho \otimes \cdots \otimes \nu^{a-1}\rho \otimes \nu^{a}\rho \times \nu^{a}\rho \otimes \nu^{a+1}\rho \times \nu^{a+1}\rho \otimes \cdots \otimes \nu^{\lceil a \rceil -a-1}\rho \times \nu^{\lceil a \rceil -a-1}\rho \otimes \tau$ in the Jacquet module of $\langle [\nu^{a}\rho, \nu^{b-1}\rho] \rangle \rtimes \sigma$ with respect to the appropriate parabolic subgroup.

Repeating this procedure, we get that the multiplicity of π in the Jacquet module of $\langle \Delta \rangle \rtimes \sigma$ with respect to the appropriate parabolic subgroup equals the multiplicity of $\nu^{a}\rho \times \nu^{a}\rho \otimes \nu^{a+1}\rho \times \nu^{a+1}\rho \otimes \cdots \otimes \nu^{\lceil a \rceil -a-1}\rho \times \nu^{\lceil a \rceil -a-1}\rho \otimes \tau$ in the Jacquet module of $\langle [\nu^{a}\rho, \nu^{-a}\rho] \rangle \rtimes \sigma$ with respect to the appropriate parabolic subgroup.

Again, transitivity of Jacquet modules implies that there is an irreducible constituent $\nu^{a}\rho \times \nu^{a}\rho \otimes \pi_2$ of $\mu^*(\langle [\nu^{a}\rho, \nu^{-a}\rho] \rangle \rtimes \sigma)$ such that $\nu^{a+1}\rho \times \nu^{a+1}\rho \otimes \cdots \otimes \nu^{\lceil a \rceil -a-1}\rho \times \nu^{\lceil a \rceil -a-1}\rho \otimes \tau$ is contained in the Jacquet module of π_2 with respect to the appropriate parabolic subgroup. In the same way as before we deduce that $\pi_2 \leq \langle [\nu^{a+1}\rho, \nu^{-a-1}\rho] \rangle \rtimes \sigma$ and the multiplicity of π in the Jacquet module of $\langle \Delta \rangle \rtimes \sigma$ with respect to the appropriate parabolic subgroup equals the multiplicity of $\nu^{a+1}\rho \times \nu^{a+1}\rho \otimes \cdots \otimes \nu^{\lceil a \rceil -a-1}\rho \times \nu^{\lceil a \rceil -a-1}\rho \otimes \tau$ in the Jacquet module of $\langle [\nu^{a+1}\rho, \nu^{-a-1}\rho] \rangle \rtimes \sigma$ with respect to the appropriate parabolic subgroup.

Repeating this procedure, we obtain that the multiplicity of π in the Jacquet module of $\langle \Delta \rangle \rtimes \sigma$ with respect to the appropriate parabolic subgroup equals the multiplicity of τ in $\rho \rtimes \sigma$ if $a \in \mathbb{Z}$, and the multiplicity of τ in σ otherwise. Thus, π appears in the Jacquet module of $\langle \Delta \rangle \rtimes \sigma$ with respect to the appropriate parabolic subgroup with multiplicity one. □

We provide several basic reducibility criteria.

Proposition 3.3. *If $\rho \not\cong \widetilde{\rho}$ or $2a$ is not an integer, then $\langle \Delta \rangle \rtimes \sigma$ is irreducible.*

Proof. We determine all irreducible subquotients of $\langle \Delta \rangle \rtimes \sigma$. Thus, suppose that we have $L(\delta_1, \delta_2, \ldots, \delta_k, \tau) \leq \langle \Delta \rangle \rtimes \sigma$, and let $\delta_i \cong \delta([\nu^{a_i}\rho_i, \nu^{b_i}\rho_i])$, for $i=1,2,\ldots,k$. Let us first prove that $a_i = b_i$ for all i. Suppose, on the contrary, that there is some j such that $a_j \leq b_j - 1$ and let $m = \min\{j : a_j \leq b_j - 1\}$. Then $a_i = b_i$ for $i < m$ and a repeated application of Lemma 2.6 implies that there are non-negative integers s, t such that $L(\delta_m, \delta_{m+1}, \ldots, \delta_k, \tau)$ is an irreducible subquotient of $\langle[\nu^{a+s}\rho, \nu^{b-t}\rho]\rangle \rtimes \sigma$.

Since $L(\delta_m, \delta_{m+1}, \ldots, \delta_k, \tau)$ is a subrepresentation of $\delta_m \times \delta_{m+1} \times \cdots \times \delta_k \rtimes \tau$, Frobenius reciprocity and transitivity of Jacquet modules imply that there is some irreducible constituent $\delta_m \otimes \sigma'$ of $\mu^*(L(\delta_m, \delta_{m+1}, \ldots, \delta_k, \tau))$. Thus, $\mu^*(\langle[\nu^{a+s}\rho, \nu^{b-t}\rho]\rangle \rtimes \sigma) \geq \delta_m \otimes \sigma'$ and there are $0 \leq j \leq i \leq b-t-a-s+1$ and an irreducible constituent $\pi \otimes \sigma''$ of $\mu^*(\sigma)$ such that

$$\delta_m \leq \langle[\nu^{-b+t}\widetilde{\rho}, \nu^{-i-a-s}\widetilde{\rho}]\rangle \times \langle[\nu^{a+s}\rho, \nu^{j+a+s-1}\rho]\rangle \times \pi.$$

Since $e(\delta_m) < 0$, square-integrability of σ implies that we can not have both $i = b-t-a-s+1$ and $j = 0$. Since $\nu^{a+s+x}\rho$ and $\nu^{-b+t+y}\widetilde{\rho}$ do not appear in the cuspidal support of σ for $x, y \in \mathbb{Z}$, it follows that

$$\delta_m \leq \langle[\nu^{-b+t}\widetilde{\rho}, \nu^{-i-a-s}\widetilde{\rho}]\rangle \times \langle[\nu^{a+s}\rho, \nu^{j+a+s-1}\rho]\rangle.$$

Since δ_m is irreducible essentially square-integrable, the assumption that $a_m < b_m$ implies that we have either $\nu^{a+s}\rho = \nu^{-b+t-1}\widetilde{\rho}$ or $\nu^{a+s}\rho = \nu^{-b+t+1}\widetilde{\rho}$, a contradiction. Thus, $a_i = b_i$ for all i.

A repeated application of Lemma 2.3 now implies that τ is an irreducible subquotient of $\langle[\nu^{a+s}\rho, \nu^{b-t}\rho]\rangle \rtimes \sigma$, for some $s, t \in \mathbb{Z}_{\geq 0}$. Since $a_i < 0$ for all i, Lemma 3.1 shows that $\tau \cong \rho \rtimes \sigma$ if a is an integer, and $\tau \cong \sigma$ otherwise. Note that the induced representation $\rho \rtimes \sigma$ is irreducible since $\rho \not\cong \widetilde{\rho}$.

Consequently, if $a \geq 0$, $L(\delta_1, \ldots, \delta_k, \tau)$ is a unique irreducible subrepresentation of the induced representation

$$\nu^{-b}\widetilde{\rho} \times \nu^{-b+1}\widetilde{\rho} \times \cdots \times \nu^{c}\widetilde{\rho} \rtimes \tau,$$

where c stands for $-a$ if $a > 0$, and $c = -1$ if $a = 0$.

On the other hand, if $a < 0$, $L(\delta_1, \ldots, \delta_k, \tau)$ is a unique irreducible subrepresentation of the induced representation

$$\nu^{-b}\widetilde{\rho} \times \nu^{-b+1}\widetilde{\rho} \times \cdots \times \nu^{-b+\lceil b\rceil-1}\widetilde{\rho} \times \nu^{a}\rho \times \nu^{a+1}\rho \times \cdots \times \nu^{a-\lceil a\rceil}\rho \rtimes \tau.$$

Note that the assumption of the proposition implies that $\nu^{b-s}\widetilde{\rho} \times \nu^{a+t}\rho \cong \nu^{a+t}\rho \times \nu^{b-s}\widetilde{\rho}$ for $s, t \in \mathbb{Z}$.

This shows that all irreducible subquotients of $\langle\Delta\rangle \rtimes \sigma$ are mutually isomorphic, and Lemma 3.2 implies that such an irreducible subquotient appears in the composition series of $\langle\Delta\rangle \rtimes \sigma$ with multiplicity one, so $\langle\Delta\rangle \rtimes \sigma$ is irreducible. $\square$

Proposition 3.4. *If $Jord_\rho(\sigma) \neq \emptyset$, $2a \in \mathbb{Z}$, but $2a + 1 - x \notin 2\mathbb{Z}$, for $x \in Jord_\rho(\sigma)$, then $\langle\Delta\rangle \rtimes \sigma$ is irreducible.*

Proof. Similarly as in the proof of the previous proposition, we determine all irreducible subquotients of $\langle\Delta\rangle \rtimes \sigma$. Suppose that $L(\delta_1, \delta_2, \ldots, \delta_k, \tau) \leq \langle\Delta\rangle \rtimes \sigma$ and let $\delta_i \cong \delta([\nu^{a_i}\rho_i, \nu^{b_i}\rho_i])$. We prove that for all $i = 1, 2, \ldots, k$ we have $a_i = b_i$ and $\rho_i \cong \rho$.

Let us first assume that there is some $j \in \{1, 2, \ldots, k\}$ such that $a_j \leq b_j - 2$ and let $m = \min\{j : a_j \leq b_j - 2\}$. By Lemma 2.6, there are non-negative integers s, t such that

$\mu^*(\langle[\nu^{a+s}\rho,\nu^{b-t}\rho]\rangle \rtimes \sigma)$ contains an irreducible constituent of the form $\delta_m \otimes \sigma'$. Thus, there are $0 \leq j \leq i \leq b-t-a-s+1$ and an irreducible constituent $\pi \otimes \sigma''$ of $\mu^*(\sigma)$ such that

$$\delta_m \leq \langle[\nu^{-b+t}\rho,\nu^{-i-a-s}\rho]\rangle \times \langle[\nu^{a+s}\rho,\nu^{j+a+s-1}\rho]\rangle \times \pi,$$

but this implies that $\nu^{b_m}\rho$ appears in the cuspidal support of σ and $b_m = a + x$, for some $x \in \mathbb{Z}$, a contradiction.

Let us now assume that there is some $i \in \{1, 2, \ldots, k\}$ such that $a_i = b_i - 1$. Note that this implies $a \leq 0$. Let us prove that then we have $a_j = b_j - 1$ for all $j \geq i$. Suppose, on the contrary, that there is a positive integer n such that $a_n = b_n$ and $a_{n-1} = b_{n-1} - 1$.

Then we have $\mu^*(\langle[\nu^{a_{n-1}+1}\rho,\nu^{-a_{n-1}}\rho]\rangle \rtimes \sigma) \geq \delta([\nu^{a_{n-1}}\rho,\nu^{a_{n-1}+1}\rho]) \otimes L(\delta_n,\ldots,\delta_k,\tau)$ and $\mu^*(\langle[\nu^{a_{n-1}+2}\rho,\nu^{-a_{n-1}-1}\rho]\rangle \rtimes \sigma) \geq \nu^{a_n}\rho_n \otimes L(\delta_{n+1},\ldots,\delta_k,\tau)$. Consequently, $\rho_n \cong \rho$ and $a_n \in \{a_{n-1}+1, a_{n-1}+2\}$. If $a_n = a_{n-1}+1$, we have the following embeddings and isomorphisms:

$$\begin{aligned} L(\delta_{n-1},\delta_n,\ldots,\delta_k,\tau) &\hookrightarrow \delta([\nu^{a_{n-1}}\rho,\nu^{a_{n-1}+1}\rho]) \rtimes L(\delta_n,\ldots,\delta_k,\tau) \\ &\hookrightarrow \delta([\nu^{a_{n-1}}\rho,\nu^{a_{n-1}+1}\rho]) \times \nu^{a_{n-1}+1}\rho \rtimes L(\delta_{n+1},\ldots,\delta_k,\tau) \\ &\cong \nu^{a_{n-1}+1}\rho \times \delta([\nu^{a_{n-1}}\rho,\nu^{a_{n-1}+1}\rho]) \rtimes L(\delta_{n+1},\ldots,\delta_k,\tau) \\ &\hookrightarrow \nu^{a_{n-1}+1}\rho \times \nu^{a_{n-1}+1}\rho \times \nu^{a_{n-1}}\rho \rtimes L(\delta_{n+1},\ldots,\delta_k,\tau). \end{aligned}$$

By Lemma 2.2 or [15, Lemma 3.2], there is an irreducible representation τ' such that $L(\delta_{n-1},\delta_n, \ldots,\delta_k,\tau)$ is a subrepresentation of $\nu^{a_{n-1}+1}\rho \times \nu^{a_{n-1}+1}\rho \rtimes \tau'$. Thus, we have $\mu^*(\langle[\nu^{a_{n-1}+1}\rho,\nu^{-a_{n-1}}\rho]\rangle \rtimes \sigma) \geq \nu^{a_{n-1}+1}\rho \times \nu^{a_{n-1}+1}\rho \otimes \tau'$, and using the structural formula and square-integrability of σ it can be easily seen that this is impossible. Consequently, $a_n = a_{n-1}+2$ and $L(\delta_{n+1},\ldots,\delta_k,\tau) \leq \langle[\nu^{a_{n-1}+3}\rho,\nu^{-a_{n-1}-1}\rho]\rangle \rtimes \sigma$. This implies $\rho_{n+1} \cong \rho$ and $a_{n+1} = b_{n+1} = a_{n-1}+3$, since $e(\delta_n) \leq e(\delta_{n+1})$. Continuing in this way, we get that τ is an irreducible subquotient of $\langle[\nu^{a_{n-1}+c}\rho,\nu^{-a_{n-1}-1}\rho]\rangle \rtimes \sigma$, for $c \in \mathbb{Z}$ such that $a_{n-1}+c \leq \frac{1}{2}$. This contradicts Lemma 3.1, so we have $a_j = b_j - 1$ for all $j \geq i$. We note that this is possible only if $a \in \mathbb{Z}$.

It follows from Lemma 2.6 that there is an y such that $2y \in \mathbb{Z}_{\geq 0}$ and we have $L(\delta_k,\tau) \leq \langle[\nu^{-y+1}\rho,\nu^{y}\rho]\rangle \rtimes \sigma$. Then we have $\rho_k \cong \rho$, $a_k = -y$ and $\tau \leq \langle[\nu^{-y+2}\rho,\nu^{y-1}\rho]\rangle \rtimes \sigma$. By Lemma 3.1, this is possible only if $y = 1$. Using [17, Theorem 2.1(ii)], we deduce that $\delta([\rho,\nu\rho]) \rtimes \sigma$ is irreducible and isomorphic to $L(\delta_k,\tau)$, so $\mu^*(L(\delta_k,\tau))$ contains an irreducible constituent of the form $\nu\rho \otimes \tau''$. On the other hand, it follows directly from the structural formula that $\mu^*(\langle[\rho,\nu\rho]\rangle \rtimes \sigma)$ does not contain an irreducible constituent of the form $\nu\rho \otimes \tau''$, so $a_i = b_i$ for all $i = 1, 2, \ldots, k$. Note that $2a + 1 - x \notin 2\mathbb{Z}$ for $x \in \mathrm{Jord}_\rho(\sigma)$, together with [15, Lemma 3.6], implies that $\mu^*(\sigma)$ does not contain an irreducible constituent of the form $\nu^z\rho \otimes \tau'$ for $z \in \mathbb{Z}$.

If $a \in \mathbb{Z}$ and $a \leq 0$, let $\tau = \rho \rtimes \sigma$, otherwise let $\tau = \sigma$. Note that if $a \in \mathbb{Z}$, then $\rho \rtimes \sigma$ is irreducible. If $a = 0$, let $c = -1$, and if $a > 0$ let $c = -a$. If $a \geq 0$, let $\pi = L(\nu^{-b}\rho,\nu^{-b+1}\rho,\ldots,\nu^{c}\rho,\tau)$, otherwise let

$$\pi = L(\nu^{-b}\rho,\nu^{-b+1}\rho,\ldots,\nu^{a-1}\rho,\nu^{a}\rho,\nu^{a}\rho,\nu^{a+1}\rho,\nu^{a+1}\rho,\ldots,\nu^{a-\lceil a\rceil}\rho,\nu^{a-\lceil a\rceil}\rho,\tau).$$

Lemmas 2.6 and 3.1 imply that every irreducible subquotient of $\langle\Delta\rangle \rtimes \sigma$ is isomorphic to π. Lemma 3.2 shows that π appears in $\langle\Delta\rangle \rtimes \sigma$ with multiplicity one, so the induced representation $\langle\Delta\rangle \rtimes \sigma$ is irreducible. $\square$

Proposition 3.5. *Assume that $\rho \cong \widetilde{\rho}$ and $Jord_\rho(\sigma) = \emptyset$. Let use denote by s_ρ a unique non-negative real number such that $\nu^{s_\rho}\rho \rtimes \sigma_{cusp}$ reduces. Then $\langle\Delta\rangle \rtimes \sigma$ reduces if and only if $a \leq s_\rho$, $2a$ is an integer, and $2a+1$ is even if and only if $\nu^{\frac{1}{2}}\rho \rtimes \sigma_{cusp}$ reduces.*

Proof. The assumption $\mathrm{Jord}_\rho(\sigma) = \emptyset$ implies that $s_\rho \in \{0, \frac{1}{2}\}$. If either $2a \notin \mathbb{Z}$ or $a - s_\rho \notin \mathbb{Z}$, one can prove in the same way as in the proof of Propositions 3.3 and 3.4 that $\langle\Delta\rangle \rtimes \sigma$ is irreducible. If $a > s_\rho$, from the cuspidal support of the induced representation $\langle[\nu^a\rho, \nu^{a+l}\rho]\rangle \rtimes \sigma$ we deduce that it does not contain an irreducible tempered subquotient. Now one can obtain, in the same way as in the proof of Proposition 3.3, that every irreducible subquotient of $\langle\Delta\rangle \rtimes \sigma$ is isomorphic to $L(\nu^{-b}\rho, \nu^{-b+1}\rho, \ldots, \nu^{-a}\rho, \sigma)$. Lemma 3.2 shows that $L(\nu^{-b}\rho, \nu^{-b+1}\rho, \ldots, \nu^{-a}\rho, \sigma)$ appears in the composition series of $\langle\Delta\rangle \rtimes \sigma$ with multiplicity one, so $\langle\Delta\rangle \rtimes \sigma$ is irreducible.

Let us now assume that $a \leq s_\rho$, $2a \in \mathbb{Z}$ and $a - s_\rho \in \mathbb{Z}$. There are two cases to consider. First, assume that $a \in \mathbb{Z}$ and write $\rho \rtimes \sigma = \tau_1 + \tau_2$ in $R(G)$, where τ_1 and τ_2 are mutually non-isomorphic irreducible tempered representations. A repeated application of Lemmas 2.3 and 2.4 shows that both irreducible representations

$$L(\nu^{-b}\rho, \ldots, \nu^{a-1}\rho, \nu^a\rho, \nu^a\rho, \ldots, \nu^{-1}\rho, \nu^{-1}\rho, \tau_i), i = 1, 2,$$

are contained in the composition series of $\langle\Delta\rangle \rtimes \sigma$, so $\langle\Delta\rangle \rtimes \sigma$ reduces.

Let us now assume that $a \notin \mathbb{Z}$ and let σ' denote a discrete series subrepresentation of $\nu^{\frac{1}{2}}\rho \rtimes \sigma$ constructed in the proof of Lemma 6.1 of [17]. A repeated application of Lemmas 2.3, 2.4 and 2.5 shows that both irreducible representations

$$L(\nu^{-b}\rho, \ldots, \nu^{a-1}\rho, \nu^a\rho, \nu^a\rho, \ldots, \nu^{-\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho, \sigma),$$

and

$$L(\nu^{-b}\rho, \ldots, \nu^{a-1}\rho, \nu^a\rho, \nu^a\rho, \ldots, \nu^{-\frac{3}{2}}\rho, \nu^{-\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho, \sigma')$$

are contained in $\langle\Delta\rangle \rtimes \sigma$, so $\langle\Delta\rangle \rtimes \sigma$ reduces. $\square$

We note that, in the case when $\mathrm{char} F = 0$, the last three propositions also follow from [6, Theorem 10.5] and [23, Theorem 13.2].

4. Main reducibility criterion in the $a \geq 1$ case

Let $\rho \in R(GL)$ stand for an irreducible self-contragredient cuspidal representation, and let $a, b \in \mathbb{R}$ such that $a + b \in \mathbb{Z}_{\geq 0}$ and $2a, 2b \in \mathbb{Z}$. Let $\Delta = [\nu^a\rho, \nu^b\rho]$. Let $\sigma \in R(G)$ denote a discrete series representation.

From now on, we assume that $\mathrm{Jord}_\rho(\sigma) \neq \emptyset$ and that for $x \in \mathrm{Jord}_\rho(\sigma)$ we have $2a+1-x \in 2\mathbb{Z}$.

In this section we obtain the irreducibility criterion for $\langle\Delta\rangle \rtimes \sigma$ in the $a \geq 1$ case.

Lemma 4.1. *Let $c, d \in \mathbb{R}$ such that $c + d \in \mathbb{Z}_{\geq 0}$ and for $x \in Jord_\rho(\sigma)$ we have $2c + 1 - x \in 2\mathbb{Z}$. Suppose that $c \geq 1$ and that one of the following holds:*

(1) $2c-1 \notin Jord_\rho(\sigma)$,
(2) $x \in Jord_\rho(\sigma)$ *for all* $x \in \{2c-1, 2c+1, \dots, 2d+1\}$, *and* $\epsilon_\sigma((x_-, \rho), (x, \rho)) = -1$ *for all* $x \in \{2c+1, 2c+3, \dots, 2d+1\}$.

Then $\langle [\nu^c\rho, \nu^d\rho] \rangle \rtimes \sigma$ *does not contain an irreducible tempered subquotient.*

Proof. If $2c-1 \notin \mathrm{Jord}_\rho(\sigma)$, the claim of the lemma can be obtained using the cuspidal support considerations, following the similar reasoning as in [4, Subsection 4.2.1], before [4, Proposition 4.6].

Let us now assume that $x \in \mathrm{Jord}_\rho(\sigma)$ for all $x \in \{2c-1, 2c+1, \dots, 2d+1\}$. It follows from the cuspidal support considerations that every irreducible tempered subquotient τ of $\langle [\nu^c\rho, \nu^d\rho] \rangle \rtimes \sigma$ can be written as a subrepresentation of an induced representation of the form $\delta([\nu^{-d}\rho, \nu^d\rho]) \rtimes \tau'$, for some irreducible tempered representation τ'. Thus, if $\tau \le \langle [\nu^c\rho, \nu^d\rho] \rangle \rtimes \sigma$, then $\mu^*(\langle [\nu^c\rho, \nu^d\rho] \rangle \rtimes \sigma) \ge \delta([\nu^{-d}\rho, \nu^d\rho]) \otimes \tau'$. By the structural formula, there are $0 \le j \le i \le d-c+1$ and an irreducible constituent $\pi \otimes \sigma'$ of $\mu^*(\sigma)$ such that

$$\delta([\nu^{-d}\rho, \nu^d\rho]) \le \langle [\nu^{-d}\rho, \nu^{-i-c}\rho] \rangle \times \langle [\nu^c\rho, \nu^{j+c-1}\rho] \rangle \times \pi.$$

It follows that the Jacquet module of π with respect to the appropriate parabolic subgroup contains an irreducible constituent of the form $\nu^d\rho \otimes \pi'$. Transitivity of Jacquet modules and [24, Proposition 7.2] now imply that $\epsilon_\sigma(((2d+1)_-, \rho), (2d+1, \rho)) = 1$, a contradiction. □

Proposition 4.2. *Suppose that* $a \ge 1$ *and that one of the following holds:*
(1) $2a-1 \notin Jord_\rho(\sigma)$,
(2) $x \in Jord_\rho(\sigma)$ *for all* $x \in \{2a-1, 2a+1, \dots, 2b+1\}$, *and* $\epsilon_\sigma((x_-, \rho), (x, \rho)) = -1$ *for all* $x \in \{2a+1, 2a+3, \dots, 2b+1\}$.

Then the induced representation $\langle \Delta \rangle \rtimes \sigma$ *is irreducible.*

Proof. Using Lemma 4.1, in the same way as in the proof of Proposition 3.3 we conclude that every irreducible subquotient of $\langle \Delta \rangle \rtimes \sigma$ is isomorphic to $L(\nu^{-b}\rho, \nu^{-b+1}\rho, \dots, \nu^{-a}\rho, \sigma)$. Lemma 3.2 implies that such an irreducible representation appears in the composition series of $\langle \Delta \rangle \rtimes \sigma$ with multiplicity one, so $\langle \Delta \rangle \rtimes \sigma$ is irreducible. □

Lemma 4.3. *Let* $c \in \mathbb{R}$ *be such that for* $x \in Jord_\rho(\sigma)$ *we have* $2c+1-x \in 2\mathbb{Z}$. *Suppose that* $c > 0$, $2c-1 \in Jord_\rho(\sigma)$ *and* $2c+1 \notin Jord_\rho(\sigma)$. *Then the induced representation* $\nu^c\rho \rtimes \sigma$ *contains a unique irreducible subrepresentation which is square-integrable.*

Proof. By [17, Theorem 6.1], the induced representation $\nu^c\rho \rtimes \sigma$ reduces. It can be easily obtained from the structural formula, using $2c+1 \notin \mathrm{Jord}_\rho(\sigma)$ and [15, Lemma 3.6], that $\nu^c\rho \otimes \sigma$ appears in $\mu^*(\nu^c\rho \rtimes \sigma)$ with multiplicity one, so $\nu^c\rho \rtimes \sigma$ has a unique irreducible subrepresentation. From [17, Lemma 2.2] we deduce that $\nu^c\rho \rtimes \sigma$ contains a unique irreducible non-tempered subquotient, $L(\nu^{-c}\rho, \sigma)$. Also, from [15] we get that $\nu^c\rho \rtimes \sigma$ does not contain irreducible tempered subquotients which are not square-integrable, since $2c+1 \notin \mathrm{Jord}_\rho(\sigma)$. Thus, a unique irreducible subrepresentation of $\nu^c\rho \rtimes \sigma$ has to be square-integrable. □

Lemma 4.4. *Let* $c, d \in \mathbb{R}$ *such that* $c+d \in \mathbb{Z}_{\ge 0}$ *and for* $x \in Jord_\rho(\sigma)$ *we have* $2c+1-x \in 2\mathbb{Z}$. *Suppose that* $c \ge 1$, $x \in Jord_\rho(\sigma)$ *for all* $x \in \{2c-1, 2c+1, \dots, 2d-1\}$,

$\epsilon_\sigma((x_-,\rho),(x,\rho)) = -1$ for all $x \in \{2c+1, 2c+3, \ldots, 2d-1\}$ and $2d+1 \notin Jord_\rho(\sigma)$. Then $\langle[\nu^c\rho, \nu^d\rho]\rangle \rtimes \sigma$ contains a discrete series subrepresentation.

Proof. By Lemma 4.3, $\nu^d\rho \rtimes \sigma$ contains a unique discrete series subrepresentation, which we denote by σ_1. Let n stand for $d-c+1$. For $i = 2, 3, \ldots, n$, we inductively define discrete series σ_i by $\sigma_i \hookrightarrow \nu^{d-i+1}\rho \rtimes \sigma_{i-1}$. From [24, Lemma 8.1] we deduce that $\epsilon_{\sigma_n}((x_-,\rho),(x,\rho)) = -1$ for all $x \in \{2c+1, 2c+3, \ldots, 2d+1\}$. It follows that σ_n is not a subrepresentation of an induced representation of the form $\nu^y\rho \rtimes \sigma'$ for $y \in c+1, c+2, \ldots, d$, and σ' irreducible. Obviously, we have

$$\sigma_n \hookrightarrow \nu^c\rho \times \nu^{c+1}\rho \times \cdots \times \nu^d\rho \rtimes \sigma.$$

From Lemma 2.2 we get that there is some irreducible subquotient π of $\nu^c\rho \times \nu^{c+1}\rho \times \cdots \times \nu^d$ such that σ_n is a subrepresentation of $\pi \rtimes \sigma$. It directly follows that π can not be a subrepresentation of an induced representation of the form $\nu^y\rho \rtimes \pi'$ for $y \in \{c+1, c+2, \ldots, d\}$ and π' irreducible, so we conclude that $\pi \cong \langle[\nu^c\rho, \nu^d\rho]\rangle$ and the lemma is proved. □

To study the tempered case, we need the following result which follows from [2, Théorème 0.1] or from [7, Lemma 1.3.3].

Lemma 4.5. *Let c, d denote positive real numbers such that $2c, 2d \in \mathbb{Z}$ and $c < d$. Let $\rho \in R(GL)$ denote an irreducible cuspidal representation. Then the induced representation $\delta([\nu^{-d}\rho, \nu^d\rho]) \times \langle[\nu^c\rho, \nu^{d-1}\rho]\rangle$ is irreducible and isomorphic to $\langle[\nu^c\rho, \nu^{d-1}\rho]\rangle \times \delta([\nu^{-d}\rho, \nu^d\rho])$.*

Lemma 4.6. *Let $c, d \in \mathbb{R}$ such that $c + d \in \mathbb{Z}_{\geq 0}$ and for $x \in Jord_\rho(\sigma)$ we have $2c + 1 - x \in 2\mathbb{Z}$. Suppose that $c \geq 1$, $x \in Jord_\rho(\sigma)$ for all $x \in \{2c-1, 2c+1, \ldots, 2d+1\}$, $\epsilon_\sigma((x_-,\rho),(x,\rho)) = -1$ for all $x \in \{2c+1, 2c+3, \ldots, 2d-1\}$ and $\epsilon_\sigma((2d-1,\rho),(2d+1,\rho)) = 1$. Then $\langle[\nu^c\rho, \nu^d\rho]\rangle \rtimes \sigma$ contains an irreducible tempered subrepresentation.*

Proof. If $c = d$, the claim of the lemma follows in the same way as in the proof of Lemma 4.3. So, we may assume that $c < d$. Let us denote by σ' a discrete series representation such that σ is a subrepresentation of $\delta([\nu^{-d+1}\rho, \nu^d\rho]) \rtimes \sigma'$. It follows from [15] or [10, Theorem 3.15] that $x \in \mathrm{Jord}_\rho(\sigma')$ for all $x \in \{2c-1, 2c+1, \ldots, 2d-3\}$, $\epsilon_\sigma((x_-,\rho),(x,\rho)) = -1$ for all $x \in \{2c+1, 2c+3, \ldots, 2d-3\}$ and $2d-1 \notin \mathrm{Jord}_\rho(\sigma)$. We denote by σ'' a discrete series subrepresentation of $\langle[\nu^c\rho, \nu^{d-1}\rho]\rangle \rtimes \sigma'$ constructed in Lemma 4.4.

It is well-know ([15] or [16, Theorem 2.1]) that the induced representation $\delta([\nu^{-d+1}\rho, \nu^d\rho]) \rtimes \sigma'$ is a representation of length three, which contains two mutually non-isomorphic discrete series subrepresentations. Let use denote a discrete series subrepresentation different than σ by σ_1. Then we have $\epsilon_{\sigma_1}((2d-3,\rho),(2d-1,\rho)) = 1$. Also, if π is an irreducible subquotient of $\delta([\nu^{-d+1}\rho, \nu^d\rho]) \rtimes \sigma'$ such that $\mu^*(\pi)$ contains an irreducible constituent of the form $\nu^d\rho \otimes \pi'$, then $\pi \in \{\sigma, \sigma_1\}$.

We note that, since $\epsilon_\sigma((2d-3,\rho),(2d-1,\rho)) = -1$, it follows from the proof of [10, Theorem 3.15] that σ is a subrepresentation of the induced representation $\nu^d\rho \rtimes \tau_1$, where τ_1 is an irreducible tempered subrepresentation of $\delta([\nu^{-d+1}\rho, \nu^{d-1}\rho]) \rtimes \sigma'$ such that $\mu^*(\tau_1)$ does not contain an irreducible constituent of the form $\nu^{d-1}\rho \times \nu^{d-1}\rho \otimes \pi$. On the other hand, σ_1 is a subrepresentation of the induced representation $\nu^d\rho \rtimes \tau_2$, where τ_2 is an irreducible tempered subrepresentation of $\delta([\nu^{-d+1}\rho, \nu^{d-1}\rho]) \rtimes \sigma'$ such that $\mu^*(\tau_2)$ contains an irreducible constituent of the form $\nu^{d-1}\rho \times \nu^{d-1}\rho \otimes \pi$.

By [24, Lemma 4.1], there is a unique irreducible tempered subrepresentation τ of $\delta([\nu^{-d}\rho, \nu^{d}\rho]) \rtimes \sigma''$ such that $\mu^*(\tau)$ does not contain an irreducible constituent of the form $\nu^d\rho \times \nu^d\rho \otimes \pi$. By Lemma 4.5, we have

$$\tau \hookrightarrow \langle[\nu^c\rho, \nu^{d-1}\rho]\rangle \times \delta([\nu^{-d}\rho, \nu^d\rho]) \rtimes \sigma'.$$

Lemma 2.2 shows that there is an irreducible subquotient τ' of $\delta([\nu^{-d}\rho, \nu^d\rho]) \rtimes \sigma'$ such that $\tau \hookrightarrow \langle[\nu^c\rho, \nu^{d-1}\rho]\rangle \rtimes \tau'$. If τ' is a subrepresentation of an induced representation of the form $\delta([\nu^{d-1}\rho, \nu^d\rho]) \times \delta([\nu^{d-1}\rho, \nu^d\rho]) \rtimes \tau_1'$, we have the following embeddings and isomorphisms:

$$\begin{aligned}
\tau &\hookrightarrow \langle[\nu^c\rho, \nu^{d-1}\rho]\rangle \times \delta([\nu^{d-1}\rho, \nu^d\rho]) \times \delta([\nu^{d-1}\rho, \nu^d\rho]) \rtimes \tau_1' \\
&\hookrightarrow \langle[\nu^c\rho, \nu^{d-2}\rho]\rangle \times \nu^{d-1}\rho \times \delta([\nu^{d-1}\rho, \nu^d\rho]) \times \delta([\nu^{d-1}\rho, \nu^d\rho]) \rtimes \tau_1' \\
&\cong \langle[\nu^c\rho, \nu^{d-2}\rho]\rangle \times \delta([\nu^{d-1}\rho, \nu^d\rho]) \times \delta([\nu^{d-1}\rho, \nu^d\rho]) \times \nu^{d-1}\rho \rtimes \tau_1' \\
&\hookrightarrow \langle[\nu^c\rho, \nu^{d-2}\rho]\rangle \times \nu^d\rho \times \nu^d\rho \times \nu^{d-1}\rho \times \nu^{d-1}\rho \times \nu^{d-1}\rho \rtimes \tau_1' \\
&\cong \nu^d\rho \times \nu^d\rho \times \langle[\nu^c\rho, \nu^{d-2}\rho]\rangle \times \nu^{d-1}\rho \times \nu^{d-1}\rho \times \nu^{d-1}\rho \rtimes \tau_1',
\end{aligned}$$

and $\mu^*(\tau)$ contains an irreducible constituent of the form $\nu^d\rho \times \nu^d\rho \otimes \pi$, a contradiction. It follows that τ' is a unique irreducible subrepresentation of $\delta([\nu^{-d}\rho, \nu^d\rho]) \rtimes \sigma'$ which does not contain an irreducible constituent of the form $\delta([\nu^{d-1}\rho, \nu^d\rho]) \times \delta([\nu^{d-1}\rho, \nu^d\rho]) \otimes \tau_1'$ in the Jacquet module with respect to an appropriate parabolic subgroup.

Also, $\tau' \hookrightarrow \delta([\nu^{-d+1}\rho, \nu^d\rho]) \times \nu^{-d}\rho \rtimes \sigma'$ and, since $\nu^{-d}\rho \rtimes \sigma' \cong \nu^d\rho \rtimes \sigma'$ by [17, Theorem 6.1], a commuting argument shows $\tau' \hookrightarrow \nu^d\rho \times \delta([\nu^{-d+1}\rho, \nu^d\rho]) \rtimes \sigma'$. Thus, by Lemma 2.2, there is an irreducible subquotient π of $\delta([\nu^{-d+1}\rho, \nu^d\rho]) \rtimes \sigma'$ such that $\tau' \hookrightarrow \nu^d\rho \rtimes \pi$. Since $\tau' \hookrightarrow \nu^d\rho \times \nu^d\rho \times \delta([\nu^{-d+1}\rho, \nu^{d-1}\rho]) \rtimes \sigma'$, it follows that $\mu^*(\pi) \geq \nu^d\rho \otimes \pi_1$, for some irreducible π_1. Consequently, $\pi \in \{\sigma, \sigma_1\}$.

Suppose that $\pi \cong \sigma_1$. Then, using [24, Corollary 4.2], we get $\pi \hookrightarrow \nu^d\rho \times \nu^{d-1}\rho \times \nu^{d-1}\rho \rtimes \pi'$, for some irreducible π'. This leads to an embedding $\tau' \hookrightarrow \nu^d\rho \times \nu^d\rho \times \nu^{d-1}\rho \times \nu^{d-1}\rho \rtimes \pi'$, and there is an irreducible subquotient π_2 of $\nu^d\rho \times \nu^d\rho \times \nu^{d-1}\rho \times \nu^{d-1}\rho$ such that $\tau' \hookrightarrow \pi_2 \rtimes \pi'$. Since τ' is a subquotient of $\delta([\nu^{-d}\rho, \nu^d\rho]) \rtimes \sigma'$, we get that $\mu^*(\tau')$ does not contain an irreducible constituent of the form $\nu^{d-1}\rho \otimes \pi''$. This implies $\pi_2 \cong \delta([\nu^{d-1}\rho, \nu^d\rho]) \times \delta([\nu^{d-1}\rho, \nu^d\rho])$, which is impossible.

Consequently, $\pi \cong \sigma$ and $\tau' \hookrightarrow \nu^d\rho \rtimes \sigma$. This shows that

$$\tau \hookrightarrow \langle[\nu^c\rho, \nu^{d-1}\rho]\rangle \times \nu^d\rho \rtimes \sigma.$$

It follows that there is an irreducible subquotient π_1 of $\langle[\nu^c\rho, \nu^{d-1}\rho]\rangle \times \nu^d\rho$ such that $\tau \hookrightarrow \pi_1 \rtimes \sigma$. Thus,

$$\pi_1 \in \{\langle[\nu^c\rho, \nu^d\rho]\rangle, L(\nu^c\rho, \nu^{c+1}\rho, \dots, \nu^{d-2}\rho, \delta([\nu^{d-1}\rho, \nu^d\rho]))\}.$$

Suppose that $\pi_1 \cong L(\nu^c\rho, \nu^{c+1}\rho, \ldots, \nu^{d-2}\rho, \delta([\nu^{d-1}\rho, \nu^d\rho]))$. Since $\sigma \hookrightarrow \nu^d\rho \rtimes \tau_1$, using the commuting argument we obtain the following embeddings and isomorphisms:

$$\begin{aligned}\tau &\hookrightarrow L(\nu^c\rho, \nu^{c+1}\rho, \ldots, \nu^{d-2}\rho, \delta([\nu^{d-1}\rho, \nu^d\rho])) \times \nu^d\rho \rtimes \tau_1\\ &\hookrightarrow \nu^c\rho \times \nu^{c+1}\rho \times \cdots \times \nu^{d-2}\rho \times \delta([\nu^{d-1}\rho, \nu^d\rho])) \times \nu^d\rho \rtimes \tau_1\\ &\cong \nu^c\rho \times \nu^{c+1}\rho \times \cdots \times \nu^{d-2}\rho \times \nu^d\rho \times \delta([\nu^{d-1}\rho, \nu^d\rho])) \rtimes \tau_1\\ &\hookrightarrow \nu^c\rho \times \nu^{c+1}\rho \times \cdots \times \nu^{d-2}\rho \times \nu^d\rho \times \nu^d\rho \times \nu^{d-1}\rho \rtimes \tau_1\\ &\cong \nu^d\rho \times \nu^d\rho \times \nu^c\rho \times \nu^{c+1}\rho \times \cdots \times \nu^{d-2}\rho \times \nu^{d-1}\rho \rtimes \tau_1.\end{aligned}$$

This implies that $\mu^*(\tau) \geq \nu^d\rho \times \nu^d\rho \otimes \pi_1'$, for some irreducible π_1', a contradiction. Thus, $\pi_1 \cong \langle[\nu^c\rho, \nu^d\rho]\rangle$ and the lemma is proved. $\square$

Proposition 4.7. *Suppose that $a \geq 1$, $2a-1 \in Jord_\rho(\sigma)$, and that one of the following holds:*

(1) there is an $x \in \{2a+1, 2a+3, \ldots, 2b+1\}$ such that $x \notin Jord_\rho(\sigma)$,

(2) there is an $x \in \{2a+1, 2a+3, \ldots, 2b+1\}$ such that $x \in Jord_\rho(\sigma)$ and $\epsilon_\sigma((x_-, \rho), (x, \rho)) = 1$.

Then the induced representation $\langle\Delta\rangle \rtimes \sigma$ reduces.

Proof. A repeated application of Lemma 2.3 shows that $L(\nu^{-b}\rho, \nu^{-b+1}\rho, \ldots, \nu^{-a}\rho, \sigma)$ is an irreducible subquotient of $\langle\Delta\rangle \rtimes \sigma$.

Let us denote by x_1 the minimal $x \in \{2a+1, 2a+3, \ldots, 2b+1\}$ such that $x \notin \mathrm{Jord}_\rho(\sigma)$, if such x exists. Otherwise, let $x_1 = 2b+3$.

Also, let us denote by x_2 the minimal $x \in \{2a+1, 2a+3, \ldots, 2b+1\}$ such that $x \in \mathrm{Jord}_\rho(\sigma)$ and $\epsilon_\sigma((x_-, \rho), (x, \rho)) = 1$, if such x exists. Otherwise, let $x_2 = 2b+3$.

Let $x_{\min}$ stand for $\min\{\frac{x_1-1}{2}, \frac{x_2-1}{2}\}$. By conditions of the lemma, we have $a \leq x_{\min} \leq b$. Previous two lemmas show that there is an irreducible tempered subrepresentation τ of $\langle[\nu^a\rho, \nu^{x_{\min}}\rho]\rangle \rtimes \sigma$. Note that $\tau \not\cong \sigma$. A repeated application of Lemma 2.3 implies that $L(\nu^{-b}\rho, \nu^{-b+1}\rho, \ldots, \nu^{-x_{\min}-1}\rho, \tau)$ is an irreducible subquotient of $\langle\Delta\rangle \rtimes \sigma$, different than $L(\nu^{-b}\rho, \nu^{-b+1}\rho, \ldots, \nu^{-a}\rho, \sigma)$. Consequently, $\langle\Delta\rangle \rtimes \sigma$ reduces. $\square$

As a consequence of Propositions 4.2 and 4.7, we obtain our first main result.

Theorem 4.8. *Suppose that $a \geq 1$. The induced representation $\langle\Delta\rangle \rtimes \sigma$ is irreducible if and only if one of the following holds:*

(1) $2a-1 \notin Jord_\rho(\sigma)$,

(2) $x \in Jord_\rho(\sigma)$ for all $x \in \{2a-1, 2a+1, \ldots, 2b+1\}$, and $\epsilon_\sigma((x_-, \rho), (x, \rho)) = -1$ for all $x \in \{2a+1, 2a+3, \ldots, 2b+1\}$.

5. Main reducibility criterion in the $a \leq \frac{1}{2}$ case

In this section we obtain the reducibility criterion for $\langle\Delta\rangle \rtimes \sigma$ in the $a \leq \frac{1}{2}$ case. We continue with the notation introduced in the previous section.

We will first handle the case $a \in \mathbb{Z}$. In this case, we assume that $\mathrm{char} F = 0$, since we will use [15, Theorem 13.1].

In the same way as in the proof of Lemma 4.1, we get the following result.

Lemma 5.1. *Let $c, d \in \mathbb{R}$ such that $c + d \in \mathbb{Z}_{\geq 0}$ and for $x \in Jord_\rho(\sigma)$ we have $2c + 1 - x \in 2\mathbb{Z}$. Suppose that $c \in \mathbb{Z}$, $c \leq 0$, $x \in Jord_\rho(\sigma)$ for all $x \in \{1, 3, \ldots, 2d + 1\}$ and $\epsilon_\sigma((x_-, \rho), (x, \rho)) = -1$ for all $x \in \{3, 5, \ldots, 2d + 1\}$. Then $\langle[\nu^c\rho, \nu^d\rho]\rangle \rtimes \sigma$ contains an irreducible tempered subquotient if and only if $c = d = 0$.*

Proposition 5.2. *Suppose that $a \in \mathbb{Z}$, $a \leq 0$, $x \in Jord_\rho(\sigma)$ for all $x \in \{1, 3, \ldots, 2b + 1\}$ and $\epsilon_\sigma((x_-, \rho), (x, \rho)) = -1$ for all $x \in \{3, 5, \ldots, 2b + 1\}$. Then the induced representation $\langle\Delta\rangle \rtimes \sigma$ is irreducible.*

Proof. We have already seen that there are no irreducible tempered subquotients of $\langle\Delta\rangle \rtimes \sigma$. Let $L(\delta_1, \delta_2, \ldots, \delta_k, \tau)$ denote a non-tempered irreducible subquotient of $\langle\Delta\rangle \rtimes \sigma$, where $\delta_i = \delta([\nu^{a_i}\rho_i, \nu^{b_i}\rho_i])$, for $i = 1, 2, \ldots, k$. First we prove that for every $i = 1, 2, \ldots, k$ we have $b_i - a_i \leq 1$. Suppose, on the contrary, that there is some i such that $b_i - a_i \geq 2$ and let us denote the minimal such i by n. As before, there are $c \geq a$ and $0 < d \leq b$ such that $L(\delta_n, \delta_{n+1}, \ldots, \delta_k, \tau)$ is a subquotient of $\langle[\nu^c\rho, \nu^d\rho]\rangle \rtimes \sigma$. Thus, $\mu^*(\langle[\nu^c\rho, \nu^d\rho]\rangle \rtimes \sigma) \geq \delta_n \otimes L(\delta_{n+1}, \ldots, \delta_k, \tau)$. Structural formula implies that there are $0 \leq j \leq i \leq d - c + 1$ and an irreducible constituent $\pi \otimes \sigma'$ of $\mu^*(\sigma)$ such that

$$\delta([\nu^{a_n}\rho_n, \nu^{b_n}\rho_n]) \leq \langle[\nu^{-d}\rho, \nu^{-i-c}\rho]\rangle \times \langle[\nu^c\rho, \nu^{j+c-1}\rho]\rangle \times \pi.$$

Since $b_n - a_n \geq 2$, we get $\rho_n \cong \rho$ and $\pi \cong \delta([\nu^y\rho, \nu^{b_n}\rho])$ for some $y \geq a_n$. Since $a_n + b_n < 0$ and square-integrability of σ implies $y + b_n > 0$, we obtain that $-d + 2 = y < b_n$, which implies $\epsilon_\sigma(((2d + 1)_-, \rho), (2d + 1, \rho)) = 1$, a contradiction.

Consequently, for all $i = 1, 2, \ldots, k$ we have $b_i - a_i \leq 1$. In the same way as in the proof of Proposition 3.4, using the fact that $\delta([\rho, \nu\rho]) \rtimes \sigma$ is irreducible by [17, Theorem 5.1], one can prove that $a_i = b_i$ for all $i = 1, 2, \ldots, k$. In consequence, every irreducible subquotient of $\langle\Delta\rangle \rtimes \sigma$ is isomorphic to

$$L(\nu^{-b}\rho, \nu^{-b+1}\rho, \ldots, \nu^{a-1}\rho, \nu^a\rho, \nu^a\rho, \ldots, \nu^{-1}\rho, \nu^{-1}\rho, \rho \rtimes \sigma). \tag{1}$$

Note that $\rho \rtimes \sigma$ is irreducible since $1 \in \mathrm{Jord}_\rho(\sigma)$. Lemma 3.2 implies that the irreducible representation (1) appears in the composition series of $\langle\Delta\rangle \rtimes \sigma$ with multiplicity one, so $\langle\Delta\rangle \rtimes \sigma$ is irreducible. □

Lemma 5.3. *Let $d \in \mathbb{Z}_{\geq 0}$ such that for $x \in Jord_\rho(\sigma)$ we have $2d + 1 - x \in 2\mathbb{Z}$. Suppose that $2d + 1 \notin Jord_\rho(\sigma)$, $1, 3, \ldots, 2d - 1 \in Jord_\rho(\sigma)$, and $\epsilon_\sigma((x_-, \rho), (x, \rho)) = -1$ for all $x \in \{3, 5, \ldots, 2d - 1\}$. Then $\langle[\rho, \nu^d\rho]\rangle \rtimes \sigma$ contains an irreducible tempered subrepresentation.*

Proof. We may assume that $d > 0$, otherwise the claim of the lemma is trivial.

By Lemma 4.4, there is a discrete series subrepresentation of $\langle[\nu\rho, \nu^d\rho]\rangle \rtimes \sigma$, and let us denote such a representation by σ'. Since $1 \notin \mathrm{Jord}_\rho(\sigma')$, induced representation $\rho \rtimes \sigma'$ reduces and it is a direct sum of two mutually non-isomorphic irreducible tempered representations. By [24, Lemma 4.7], there is a unique irreducible subrepresentation τ of $\rho \rtimes \sigma'$ which is not a subrepresentation of an induced representation of the form $\nu\rho \rtimes \sigma''$, for some irreducible σ''. Thus,

$$\tau \hookrightarrow \rho \rtimes \sigma' \hookrightarrow \rho \times \langle[\nu\rho, \nu^d\rho]\rangle \rtimes \sigma,$$

so, by Lemma 2.2 there is an irreducible subquotient π of $\rho \times \langle[\nu\rho, \nu^d\rho]\rangle$ such that $\tau \hookrightarrow \pi \rtimes \sigma$. Since π is not a subrepresentation of an induced representation of the form $\nu\rho \times \pi'$, it follows at once that $\pi \cong \langle[\rho, \nu^d\rho]\rangle$. $\square$

Lemma 5.4. *Let $d \in \mathbb{Z}_{\geq 0}$ such that for $x \in Jord_\rho(\sigma)$ we have $2d+1-x \in 2\mathbb{Z}$. Suppose that $1, 3, \ldots, 2d+1 \in Jord_\rho(\sigma)$, $\epsilon_\sigma((x_-, \rho), (x, \rho)) = -1$ for all $x \in \{3, 5, \ldots, 2d-1\}$, and $\epsilon_\sigma((2d-1, \rho), (2d+1, \rho)) = 1$. Then $\langle[\rho, \nu^d\rho]\rangle \rtimes \sigma$ contains an irreducible tempered subrepresentation.*

Proof. If $d = 0$, there is nothing to prove.

If $d = 1$, let us denote by τ_1 the irreducible tempered subrepresentation of $\nu\rho \rtimes \sigma$ constructed in [17, Section 6]. Note that τ_1 is a subrepresentation of the induced representation of the form $\delta([\nu^{-1}\rho, \nu\rho]) \rtimes \sigma'$, for a discrete series σ' such that σ is a subrepresentation of $\delta([\rho, \nu\rho]) \rtimes \sigma'$. Thus, $1, 3 \notin \mathrm{Jord}_\rho(\sigma')$. Also, by the classification of discrete series, there is an irreducible tempered subrepresentation τ' of $\rho \rtimes \sigma'$ such that σ is a subrepresentation of $\nu\rho \rtimes \tau'$. Note that we also have $\tau_1 \hookrightarrow \nu\rho \times \nu\rho \rtimes \tau'$. Using [15, Theorem 13.1(i)], we deduce that $\rho \rtimes \tau_1$ reduces, and it can be easily seen that it is a length two representation such that the unique irreducible constituent of $\mu^*(\rho \rtimes \tau_1)$ of the form $\nu\rho \times \nu\rho \otimes \pi$ is $\nu\rho \times \nu\rho \otimes \rho \rtimes \tau'$, and it appears in $\mu^*(\rho \rtimes \tau_1)$ with multiplicity one. Let us denote by τ_2 the unique irreducible tempered subrepresentation of $\rho \rtimes \tau_1$ which does not contain $\nu\rho \times \nu\rho \otimes \rho \rtimes \tau'$ in the Jacquet module with respect to the appropriate parabolic subgroup. Then $\tau_2 \hookrightarrow \rho \times \nu\rho \rtimes \sigma$ and, by Lemma 2.2, there is an irreducible subquotient π of $\rho \times \nu\rho$ such that $\tau_2 \hookrightarrow \pi \rtimes \sigma$. From $\epsilon_\sigma((1, \rho), (3, \rho)) = 1$ we deduce that $\sigma \hookrightarrow \nu\rho \rtimes \tau_3$ for an irreducible tempered representation τ_3, which gives $\pi \not\cong \delta([\rho, \nu\rho])$, since otherwise we would have $\tau_2 \hookrightarrow \nu\rho \times \nu\rho \times \rho \rtimes \tau'$. Consequently, $\pi \cong \langle[\rho, \nu\rho]\rangle$.

Now we consider the case $d \geq 2$. Since $\epsilon_\sigma((2d-1, \rho), (2d+1, \rho)) = 1$, there is a discrete series representation σ' such that $\sigma \hookrightarrow \delta([\nu^{-d+1}\rho, \nu^d\rho]) \rtimes \sigma'$. By [15, Proposition 2.1, Lemma 5.1] or [10, Proposition 3.1, Theorem 3.15] we have $\{1, 3, \ldots, 2d-3\} \subseteq \mathrm{Jord}_\rho(\sigma')$ and $\epsilon_{\sigma'}((x_-, \rho), (x, \rho)) = -1$ for $x \in \{3, 5, \ldots, 2d-3\}$. We denote by σ'' a discrete series subrepresentation of $\langle[\nu\rho, \nu^{d-1}\rho]\rangle \rtimes \sigma'$ obtained in Lemma 4.4.

The induced representation $\delta([\nu^{-d}\rho, \nu^d\rho]) \rtimes \sigma''$ is a direct sum of two irreducible mutually non-isomorphic tempered representations, and by [24, Lemma 4.1] exactly one of them does not contain an irreducible constituent of the form $\nu^d\rho \times \nu^d\rho \otimes \pi$ in the Jacquet module with respect to the appropriate parabolic subgroup. Let us denote such a subrepresentation by τ_1. Now it can be proved in the same way as in the proof of Lemma 4.6 that $\tau_1 \hookrightarrow \langle[\nu\rho, \nu^d\rho]\rangle \rtimes \sigma$.

Let us denote by σ''_1 a discrete series such that $\sigma'' \hookrightarrow \nu\rho \rtimes \sigma''_1$, constructed in [24, Lemma 8.1]. Since

$$\delta([\nu^{-d}\rho, \nu^d\rho]) \rtimes \sigma'' \hookrightarrow \delta([\nu^{-d}\rho, \nu^d\rho]) \times \nu\rho \rtimes \sigma''_1 \cong \nu\rho \times \delta([\nu^{-d}\rho, \nu^d\rho]) \rtimes \sigma''_1,$$

both irreducible subrepresentations of $\delta([\nu^{-d}\rho, \nu^d\rho]) \rtimes \sigma''$ contain an irreducible representation of the form $\nu\rho \otimes \pi$ in the Jacquet module with respect to the appropriate parabolic subgroup. It is easy to see that π is an irreducible subquotient of $\delta([\nu^{-d}\rho, \nu^d\rho]) \rtimes \sigma''_1$, which is a length two representation. Let us denote by τ_1' a unique irreducible tempered subrepresentation of $\delta([\nu^{-d}\rho, \nu^d\rho]) \rtimes \sigma_1''$ such that $\mu^*(\tau_1) \geq \nu\rho \otimes \tau_1'$.

Using [15, Theorem 13.1(i)] and arguments similar to those used in the $d = 1$ case, we conclude that $\rho \rtimes \tau_1$ is a length two representation and, since $\rho \rtimes \tau_1'$ is irreducible, $\nu\rho \otimes \rho \rtimes \tau_1'$ is the unique irreducible constituent of $\mu^*(\rho \rtimes \tau_1)$ of the form $\nu\rho \otimes \pi$. Let us denote by τ_2 an irreducible tempered subrepresentation of $\rho \rtimes \tau_1$ which does not contain an irreducible representation of the form $\nu\rho \otimes \pi$ in the Jacquet module with respect to the appropriate parabolic subgroup. From $\tau_2 \hookrightarrow \rho \times \langle[\nu\rho, \nu^d\rho]\rangle \rtimes \sigma$ we conclude in the same way as before that $\tau_2 \hookrightarrow \langle[\rho, \nu^d\rho]\rangle \rtimes \sigma$. □

Proposition 5.5. *Suppose that $a \in \mathbb{Z}$ and $a < 0$. Suppose that one of the following holds:*
(1) there is an $x \in \{1, 3, \ldots, 2b+1\}$ such that $x \notin Jord_\rho(\sigma)$,
(2) there is an $x \in \{1, 3, \ldots, 2b+1\}$ such that $x \in Jord_\rho(\sigma)$ and $\epsilon_\sigma((x_-, \rho), (x, \rho)) = 1$.
Then $\langle\Delta\rangle \rtimes \sigma$ reduces.

Proof. Let us denote by x_1 the minimal $x \in \{1, 3, \ldots, 2b+1\}$ such that $x \notin \mathrm{Jord}_\rho(\sigma)$, if such x exists. Otherwise, let $x_1 = 2b+3$. Also, let us denote by x_2 the minimal $x \in \{3, 5, \ldots, 2b+1\}$ such that $x \in \mathrm{Jord}_\rho(\sigma)$ and $\epsilon_\sigma((x_-, \rho), (x, \rho)) = 1$, if such x exists. Otherwise, let $x_2 = 2b+3$.

Let $x_{\min}$ stand for $\min\{\frac{x_1-1}{2}, \frac{x_2-1}{2}\}$. By conditions of the lemma, we have $0 \leq x_{\min} \leq b$.

If $x_{\min} = 0$, then it follows that $1 \notin \mathrm{Jord}_\rho(\sigma)$, so in $R(G)$ we can write $\rho \rtimes \sigma = \tau_1 + \tau_2$, where τ_1 and τ_2 are mutually non-isomorphic irreducible tempered representations. A repeated application of Lemmas 2.3 and 2.4 shows that both irreducible representations

$$L(\nu^{-b}\rho, \ldots, \nu^{a-1}\rho, \nu^a\rho, \nu^a\rho, \ldots, \nu^{-1}\rho, \nu^{-1}\rho, \tau_i), i = 1, 2,$$

are contained in $\langle\Delta\rangle \rtimes \sigma$, so $\langle\Delta\rangle \rtimes \sigma$ reduces.

Let us now assume that $x_{\min} \geq 1$. Previous two lemmas show that there is an irreducible tempered subrepresentation τ of $\langle[\rho, \nu^{x_{\min}}\rho]\rangle \rtimes \sigma$.

Thus, we have $L(\nu^{-1}\rho, \tau) \hookrightarrow \nu^{-1}\rho \times \langle[\rho, \nu^{x_{\min}}\rho]\rangle \rtimes \sigma$, and Lemma 2.2 implies that there is an irreducible subquotient π of $\nu^{-1}\rho \times \langle[\rho, \nu^{x_{\min}}\rho]\rangle$ such that $L(\nu^{-1}\rho, \tau) \hookrightarrow \pi \rtimes \sigma$. By [25, Theorem 7.1],

$$\pi \in \{\langle[\nu^{-1}\rho, \nu^{x_{\min}}\rho]\rangle, L(\delta([\nu^{-1}\rho, \rho]), \nu\rho, \ldots, \nu^{x_{\min}}\rho)\}.$$

If $x_{\min} > 1$, since $\mu^*(L(\nu^{-1}\rho, \tau)) \geq \nu^{-1}\rho \otimes \tau$, it follows that $\pi \cong \langle[\nu^{-1}\rho, \nu^{x_{\min}}\rho]\rangle$, since $L(\delta([\nu^{-1}\rho, \rho]), \nu\rho, \ldots, \nu^{x_{\min}}\rho)$ is a subrepresentation of $\delta([\nu^{-1}\rho, \rho]) \times \langle[\nu\rho, \nu^{x_{\min}}\rho]\rangle$ and $\mu^*(\delta([\nu^{-1}\rho, \rho]) \times \langle[\nu\rho, \nu^{x_{\min}}\rho]\rangle \rtimes \sigma)$ does not contain $\nu^{-1}\rho \otimes \tau$.

Let us now assume that $x_{\min} = 1$. It follows from proofs of previous two lemmas that there is an irreducible tempered representation τ_1 such that $\tau \hookrightarrow \rho \rtimes \tau_1$ and $\tau_1 \hookrightarrow \nu\rho \rtimes \sigma$. Thus, $L(\nu^{-1}\rho, \tau) \hookrightarrow \nu^{-1}\rho \times \rho \rtimes \tau_1$, and there is an irreducible subquotient π of $\nu^{-1}\rho \times \rho$ such that $L(\nu^{-1}\rho, \tau) \hookrightarrow \pi \rtimes \tau_1$. Since τ_1 is tempered and $\mu^*(L(\nu^{-1}\rho, \tau)) \geq \nu^{-1}\rho \otimes \tau$, it easily follows that $\pi \cong \langle[\nu^{-1}\rho, \rho]\rangle$. Thus, we have

$$L(\nu^{-1}\rho, \tau) \hookrightarrow \langle[\nu^{-1}\rho, \rho]\rangle \times \nu\rho \rtimes \sigma,$$

and, by Lemma 2.2, there is an irreducible subquotient π' of $\langle[\nu^{-1}\rho,\rho]\rangle \times \nu\rho$ such that $L(\nu^{-1}\rho,\tau) \hookrightarrow \pi' \rtimes \sigma$. By [25, Theorem 7.1], $\pi' \in \{\langle[\nu^{-1}\rho,\nu\rho]\rangle, L(\nu^{-1}\rho,\delta([\rho,\nu\rho]))\}$. Suppose that $\pi' \cong L(\nu^{-1}\rho,\delta([\rho,\nu\rho]))$. Let us first consider that case when σ is not a subrepresentation of an induced representation of the form $\nu\rho \rtimes \sigma'$. Then $\mu^*(\tau)$ does not contain an irreducible constituent of the form $\nu\rho \otimes \sigma'$.

We have

$$L(\nu^{-1}\rho,\tau) \hookrightarrow \nu^{-1}\rho \times \nu\rho \times \rho \rtimes \sigma,$$

and the Jacquet module of $L(\nu^{-1}\rho,\tau)$ with respect to an appropriate parabolic subgroup contains $\nu^{-1}\rho \otimes \nu\rho \otimes \rho \otimes \sigma$. Since τ is tempered, transitivity of Jacquet modules implies that the Jacquet module of τ with respect to an appropriate parabolic subgroup contains $\nu\rho \otimes \rho \otimes \sigma$, which is impossible. So in this case we have $\pi' \cong \langle[\nu^{-1}\rho,\nu\rho]\rangle$.

Let us now assume that σ is a subrepresentation of an induced representation of the form $\nu\rho \rtimes \sigma'$. Then $\mu^*(\tau)$ contains an irreducible constituent of the form $\nu\rho \otimes \sigma'$, but it does not contain an irreducible constituent of the form $\nu\rho \times \nu\rho \otimes \sigma'$.

Using a simple commutativity argument, we obtain

$$L(\nu^{-1}\rho,\tau) \hookrightarrow \nu^{-1}\rho \times \nu\rho \times \nu\rho \times \rho \rtimes \sigma',$$

and the Jacquet module of $L(\nu^{-1}\rho,\tau)$ with respect to an appropriate parabolic subgroup contains $\nu^{-1}\rho \otimes \nu\rho \times \nu\rho \otimes \rho \otimes \sigma$. Since τ is tempered, transitivity of Jacquet modules implies that the Jacquet module of τ with respect to an appropriate parabolic subgroup contains $\nu\rho \times \nu\rho \otimes \rho \otimes \sigma$, which is impossible. It follows that in this case we again have $\pi' \cong \langle[\nu^{-1}\rho,\nu\rho]\rangle$.

Consequently, $\mu^*(L(\nu^{-1}\rho,\tau))$ does not contain an irreducible constituent of the form $\rho \otimes \tau'$.

Note that $L(\nu^{-x_{\min}}\rho,\nu^{-x_{\min}+1}\rho,\ldots,\nu^{-1}\rho,\tau)$ is a subrepresentation of $\langle[\nu^{-x_{\min}}\rho,\nu^{-1}\rho]\rangle \rtimes \tau$, since $\nu^{-x_{\min}}\rho \times \nu^{-x_{\min}+1}\rho \times \cdots \times \nu^{-1}\rho \rtimes \tau$ has a unique irreducible subrepresentation. Thus, we have

$$L(\nu^{-x_{\min}}\rho,\nu^{-x_{\min}+1}\rho,\ldots,\nu^{-1}\rho,\tau) \hookrightarrow \langle[\nu^{-x_{\min}}\rho,\nu^{-1}\rho]\rangle \times \langle[\rho,\nu^{x_{\min}}\rho]\rangle \rtimes \sigma.$$

From Lemma 2.2 follows that there is an irreducible subquotient π of $\langle[\nu^{-x_{\min}}\rho,\nu^{-1}\rho]\rangle \times \langle[\rho,\nu^{x_{\min}}\rho]\rangle$ such that $L(\nu^{-x_{\min}}\rho,\nu^{-x_{\min}+1}\rho,\ldots,\nu^{-1}\rho,\tau)$ is a subrepresentation of $\pi \rtimes \sigma$. The only irreducible subquotients of $\langle[\nu^{-x_{\min}}\rho,\nu^{-1}\rho]\rangle \times \langle[\rho,\nu^{x_{\min}}\rho]\rangle$ are $\langle[\nu^{-x_{\min}}\rho,\nu^{x_{\min}}\rho]\rangle$ and

$$L(\nu^{-x_{\min}}\rho,\nu^{-x_{\min}+1}\rho,\ldots,\nu^{-2}\rho,\delta([\nu^{-1}\rho,\rho]),\nu\rho,\ldots,\nu^{x_{\min}}\rho). \tag{2}$$

A commuting argument shows that the representation (2) is a subrepresentation of an induced representation of the form $\rho \times \pi'$. But, since $L(\nu^{-x_{\min}}\rho,\nu^{-x_{\min}+1}\rho,\ldots,\nu^{-1}\rho,\tau)$ is also a subrepresentation of $\langle[\nu^{-x_{\min}}\rho,\nu^{-2}\rho]\rangle \rtimes L(\nu^{-1},\tau)$, it follows at once that $\mu^*(L(\nu^{-1},\tau))$ contains an irreducible constituent of the form $\rho \otimes \tau'$, a contradiction. Consequently, $\pi \cong \langle[\nu^{-x_{\min}}\rho,\nu^{x_{\min}}\rho]\rangle$.

If $x_{\min} \geq -a$, let π' denote

$$L(\nu^{-b}\rho,\ldots,\nu^{-x_{\min}-1}\rho,\nu^{a}\rho,\ldots,\nu^{-1}\rho,\tau).$$

If $x_{\min} \leq -a$, let π' denote

$$L(\nu^{-b}\rho, \dots, \nu^{a-1}\rho, \nu^{a}\rho, \nu^{a}\rho, \dots, \nu^{-x_{\min}-1}\rho, \nu^{-x_{\min}-1}\rho, \nu^{-x_{\min}}\rho, \dots, \nu^{-1}\rho, \tau).$$

Since $L(\nu^{-x_{\min}}\rho, \dots, \nu^{-1}\rho, \tau)$ is contained in $\langle[\nu^{-x_{\min}}\rho, \nu^{x_{\min}}\rho]\rangle \rtimes \sigma$, it can be seen, using Lemmas 2.3 and 2.4, that π' is an irreducible subquotient of $\langle\Delta\rangle \rtimes \sigma$. Using the same methods, it can be seen that

$$L(\nu^{-b}\rho, \dots, \nu^{a-1}\rho, \nu^{a}\rho, \nu^{a}\rho, \dots, \nu^{-1}\rho, \nu^{-1}\rho, \rho \rtimes \sigma)$$

is also an irreducible subquotient of $\langle\Delta\rangle \rtimes \sigma$, so $\langle\Delta\rangle \rtimes \sigma$ reduces. □

Propositions 5.2 and 5.5 imply our second main result.

Theorem 5.6. *Suppose that $a \in \mathbb{Z}$ and $a \leq 0$. The induced representation $\langle\Delta\rangle \rtimes \sigma$ is irreducible if and only if $x \in Jord_\rho(\sigma)$ for all $x \in \{1, 3, \dots, 2b+1\}$ and $\epsilon_\sigma((x_-, \rho), (x, \rho)) = -1$ for all $x \in \{3, 5, \dots, 2b+1\}$.*

It remains to handle the half-integral case. In the rest of this section we do not use the assumption that $\mathrm{char} F = 0$.

Lemma 5.7. *Let $c, d \in \mathbb{R}$ such that $c + d \in \mathbb{Z}_{\geq 0}$ and for $x \in Jord_\rho(\sigma)$ we have $2c + 1 - x \in 2\mathbb{Z}$. Suppose that $c \notin \mathbb{Z}$, $c \leq \frac{1}{2}$, $x \in Jord_\rho(\sigma)$ for all $x \in \{2, 4, \dots, 2d+1\}$, $\epsilon_\sigma((x_-, \rho), (x, \rho)) = -1$ for all $x \in \{4, 6, \dots, 2d+1\}$, and $\epsilon_\sigma(2, \rho) = -1$. Then $\langle[\nu^c\rho, \nu^d\rho]\rangle \rtimes \sigma$ does not contain an irreducible tempered subquotient.*

Proof. If $d > \frac{1}{2}$, claim of the lemma follows in the same way as in the proof of Lemma 4.1. If $d = \frac{1}{2}$, then $c \in \{-\frac{1}{2}, \frac{1}{2}\}$, and every irreducible tempered subquotient of $\langle[\nu^c\rho, \nu^d\rho]\rangle \rtimes \sigma$ is a subrepresentation of an induced representation of the form $\delta([\nu^{-\frac{1}{2}}\rho, \nu^{\frac{1}{2}}\rho]) \rtimes \tau$, for an irreducible tempered representation τ. This gives $\mu^*(\langle[\nu^c\rho, \nu^d\rho]\rangle \rtimes \sigma) \geq \delta([\nu^{-\frac{1}{2}}\rho, \nu^{\frac{1}{2}}\rho]) \otimes \tau$. Using the structural formula, we obtain that $\mu^*(\sigma)$ contains an irreducible constituent of the form $\nu^{\frac{1}{2}}\rho \otimes \tau'$, which contradicts $\epsilon_\sigma(2, \rho) = -1$ by [24, Proposition 7.4]. □

Proposition 5.8. *Suppose that $a \notin \mathbb{Z}$, $a \leq \frac{1}{2}$, $x \in Jord_\rho(\sigma)$ for all $x \in \{2, 4, \dots, 2b+1\}$, $\epsilon_\sigma((x_-, \rho), (x, \rho)) = -1$ for all $x \in \{4, 6, \dots, 2b+1\}$, and $\epsilon_\sigma(2, \rho) = -1$. Then the induced representation $\langle\Delta\rangle \rtimes \sigma$ is irreducible.*

Proof. Lemma 5.7 shows that there are no irreducible tempered subquotients of $\langle\Delta\rangle \rtimes \sigma$. Let $L(\delta_1, \delta_2, \dots, \delta_k, \tau)$ denote a non-tempered irreducible subquotient of $\langle\Delta\rangle \rtimes \sigma$, where $\delta_i = \delta([\nu^{a_i}\rho_i, \nu^{b_i}\rho_i])$, for $i = 1, 2, \dots, k$. Suppose that there is some i such that $b_i - a_i \geq 2$ and let us denote the minimal such i with n. We have already seen that then there are $c \geq a$ and $d \leq b$ such that $L(\delta_n, \delta_{n+1}, \dots, \delta_k, \tau)$ is a subquotient of $\langle[\nu^c\rho, \nu^d\rho]\rangle \rtimes \sigma$. Thus, $\mu^*(\langle[\nu^c\rho, \nu^d\rho]\rangle \rtimes \sigma) \geq \delta_n \otimes L(\delta_{n+1}, \dots, \delta_k, \tau)$. Using the structural formula and square-integrability of σ, following the same lines as in the proof of Proposition 5.2, we deduce that $\rho_n \cong \rho$, $a_n = -d$, and $\mu^*(\sigma)$ contains an irreducible constituent of the form $\delta([\nu^{-d+2}\rho, \nu^{d-1}\rho]) \otimes \sigma'$. If $d - 1 \geq \frac{3}{2}$, this contradicts $\epsilon_\sigma((x_-, \rho), (x, \rho)) = -1$ for all $x \in \{4, 6, \dots, 2b+1\}$. On the other hand, if $d - 1 = \frac{1}{2}$, this contradicts $\epsilon_\sigma(2, \rho) = -1$. So, for all $i = 1, 2, \dots, k$ we have $b_i - a_i \leq 1$.

Let us now suppose that there is an $i \in \{1, 2, \ldots, k\}$ such that $b_i - a_i = 1$. Following the same lines as in the proof of Proposition 3.4, we get that then $b_j - a_j = 1$ for all $j \in \{i, i+1, \ldots, k\}$, and that τ is an irreducible subquotient of $\langle [\nu^c \rho, \nu^d \rho] \rangle \rtimes \sigma$, for $c \leq \frac{1}{2} \leq d < b$, contradicting the previous lemma.

Consequently, $a_i = b_i$ for all $i = 1, 2, \ldots, k$. Using the previous lemma we deduce that every irreducible subquotient of $\langle \Delta \rangle \rtimes \sigma$ is isomorphic to

$$L(\nu^{-b}\rho, \nu^{-b+1}\rho, \ldots, \nu^{a-1}\rho, \nu^{a}\rho, \nu^{a}\rho, \ldots, \nu^{-\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho, \sigma). \tag{3}$$

From Lemma 3.2 we conclude that the irreducible representation (3) appears in the composition series of $\langle \Delta \rangle \rtimes \sigma$ with multiplicity one, so $\langle \Delta \rangle \rtimes \sigma$ is irreducible. $\square$

The following lemma is a special case of the results of [17, Section 6] or [9, Lemma 3.2(2)].

Lemma 5.9. *Suppose that for $x \in Jord_\rho(\sigma)$ we have $x \in 2\mathbb{Z}$ and that one of the following holds:*

(1) $2 \notin Jord_\rho(\sigma)$,

(2) $2 \in Jord_\rho(\sigma)$ and $\epsilon_\sigma(2, \rho) = 1$.

Then $\nu^{\frac{1}{2}}\rho \rtimes \sigma$ reduces and contains a unique irreducible subrepresentation, which is tempered. Furthermore, if $2 \notin Jord_\rho(\sigma)$ then such a representation is square-integrable.

Lemma 5.10. *Suppose that for $x \in Jord_\rho(\sigma)$ we have $x \in 2\mathbb{Z}$. Let $d > \frac{1}{2}$ such that for $x \in Jord_\rho(\sigma)$ we have $2d + 1 - x \in 2\mathbb{Z}$. Suppose that $2, 4, \ldots, 2d-1 \in Jord_\rho(\sigma)$, $2d+1 \notin Jord_\rho(\sigma)$, $\epsilon_\sigma((x_-, \rho), (x, \rho)) = -1$ for all $x \in \{4, 6, \ldots, 2d-1\}$, and $\epsilon_\sigma(2, \rho) = -1$. Then $\langle [\nu^{\frac{1}{2}}\rho, \nu^d \rho] \rangle \rtimes \sigma$ contains an irreducible discrete series subrepresentation.*

Proof. By Lemma 4.4, there is a discrete series subrepresentation of $\langle [\nu^{\frac{3}{2}}\rho, \nu^d \rho] \rangle \rtimes \sigma$, and let us denote such a representation by σ'. Since $\epsilon_\sigma(2, \rho) = -1$, it follows that $\mu^*(\sigma')$ does not contain an irreducible constituent of the form $\delta([\nu^{\frac{1}{2}}\rho, \nu^{\frac{3}{2}}\rho]) \otimes \pi$, and [24, Lemma 8.1] implies $\epsilon_{\sigma'}(4, \rho) = -1$.

Since $2 \notin \mathrm{Jord}_\rho(\sigma')$, Lemma 5.9 implies that there is a discrete series subrepresentation σ'' of $\nu^{\frac{1}{2}} \rtimes \sigma'$. Let us prove that $\epsilon_{\sigma''}((2, \rho), (4, \rho)) = -1$. Otherwise, $\mu^*(\sigma'') \geq \delta([\nu^{-\frac{1}{2}}\rho, \nu^{\frac{3}{2}}\rho]) \otimes \sigma_1$, for some irreducible representation σ_1. Consequently, $\mu^*(\nu^{\frac{1}{2}} \rtimes \sigma') \geq \delta([\nu^{-\frac{1}{2}}\rho, \nu^{\frac{3}{2}}\rho]) \otimes \sigma_1$. Using the structural formula and square-integrability of σ', we deduce that $\mu^*(\sigma') \geq \delta([\nu^{\frac{1}{2}}\rho, \nu^{\frac{3}{2}}\rho]) \otimes \sigma_2$, for some irreducible σ_2, contradicting $\epsilon_{\sigma'}(4, \rho) = -1$.

We have $\sigma'' \hookrightarrow \nu^{\frac{1}{2}} \times \langle [\nu^{\frac{3}{2}}\rho, \nu^d \rho] \rangle \rtimes \sigma$ and, by Lemma 2.2, there is an irreducible subquotient π of $\nu^{\frac{1}{2}} \times \langle [\nu^{\frac{3}{2}}\rho, \nu^d \rho] \rangle$ such that $\sigma'' \hookrightarrow \pi \rtimes \sigma$. From $\epsilon_{\sigma''}((2, \rho), (4, \rho)) = -1$ we easily obtain $\pi \cong \langle [\nu^{\frac{1}{2}}\rho, \nu^d \rho] \rangle$ and the lemma is proved. $\square$

The following result can be proved in the same way as Lemma 4.6.

Lemma 5.11. *Suppose that for $x \in Jord_\rho(\sigma)$ we have $x \in 2\mathbb{Z}$. Let $d > \frac{1}{2}$ such that for $x \in Jord_\rho(\sigma)$ we have $2d + 1 - x \in 2\mathbb{Z}$. Suppose that $2, 4, \ldots, 2d+1 \in Jord_\rho(\sigma)$, $\epsilon_\sigma((x_-, \rho), (x, \rho)) = -1$ for all $x \in \{4, 6, \ldots, 2d-1\}$, $\epsilon_\sigma((2d-1, \rho), (2d+1, \rho)) = 1$, and $\epsilon_\sigma(2, \rho) = -1$. Then $\langle [\nu^{\frac{1}{2}}\rho, \nu^d \rho] \rangle \rtimes \sigma$ contains an irreducible tempered subrepresentation.*

Proposition 5.12. *Suppose that $a \notin \mathbb{Z}$, $a \leq \frac{1}{2}$. Suppose that one of the following holds:*
(1) $2 \notin Jord_\rho(\sigma)$,
(2) $2 \in Jord_\rho(\sigma)$ and $\epsilon_\sigma(2,\rho) = 1$,
(3) $2 \in Jord_\rho(\sigma)$, $\epsilon_\sigma(2,\rho) = -1$, and there is $x \in \{2,4,\ldots,2b+1\}$ such that $x \notin Jord_\rho(\sigma)$,
(4) $2 \in Jord_\rho(\sigma)$, $\epsilon_\sigma(2,\rho) = -1$, and there is $x \in \{2,4,\ldots,2b+1\}$ such that $x \in Jord_\rho(\sigma)$ and $\epsilon_\sigma((x_-,\rho),(x,\rho)) = 1$.
Then $\langle \Delta \rangle \rtimes \sigma$ reduces.

Proof. It can be seen, using a repeated application of Lemmas 2.3 and 2.4, that

$$L(\nu^{-b}\rho, \ldots, \nu^{a-1}\rho, \nu^{a}\rho, \nu^{a}\rho, \ldots, \nu^{-\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho, \sigma),$$

is a subquotient of $\langle \Delta \rangle \rtimes \sigma$.

Let us first assume that either $2 \notin \mathrm{Jord}_\rho(\sigma)$, or $2 \in \mathrm{Jord}_\rho(\sigma)$ and $\epsilon_\sigma(2,\rho) = 1$. Lemma 5.9 implies that there is an irreducible tempered subrepresentation τ of $\nu^{\frac{1}{2}}\rho \rtimes \sigma$. Using a repeated application of Lemmas 2.3, 2.4 and 2.5 we deduce that

$$L(\nu^{-b}\rho, \ldots, \nu^{a-1}\rho, \nu^{a}\rho, \nu^{a}\rho, \ldots, \nu^{-\frac{3}{2}}\rho, \nu^{-\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho, \tau),$$

is a subquotient of $\langle \Delta \rangle \rtimes \sigma$, so we have obtained a reducibility in this case.

Let us now assume that $2 \in \mathrm{Jord}_\rho(\sigma)$ and $\epsilon_\sigma(2,\rho) = -1$.

We denote by x_1 the minimal $x \in \{2,4,\ldots,2b+1\}$ such that $x \notin \mathrm{Jord}_\rho(\sigma)$, if such x exists. Otherwise, let $x_1 = 2b+3$. Also, let us denote by x_2 the minimal $x \in \{4,6,\ldots,2b+1\}$ such that $x \in \mathrm{Jord}_\rho(\sigma)$ and $\epsilon_\sigma((x_-,\rho),(x,\rho)) = 1$, if such x exists. Otherwise, let $x_2 = 2b+3$.

Let $x_{\min}$ stand for $\min\{\frac{x_1-1}{2}, \frac{x_2-1}{2}\}$. Conditions of the lemma, together with the assumption $2 \in \mathrm{Jord}_\rho(\sigma)$, imply $\frac{3}{2} \leq x_{\min} \leq b$.

By Lemmas 5.10 and 5.11, there is an irreducible tempered subrepresentation τ of $\langle [\nu^{\frac{1}{2}}\rho, \nu^{x_{\min}}\rho] \rangle \rtimes \sigma$.

In the same way as in the proof of Lemma 2.5, we get that $L(\nu^{-\frac{1}{2}}\rho, \tau)$ is a subrepresentation of $\langle [\nu^{-\frac{1}{2}}\rho, \nu^{x_{\min}}\rho] \rangle \rtimes \sigma$. Using the structural formula and $\epsilon_\sigma(2,\rho) = -1$, it is easy to deduce that $\mu^*(L(\nu^{-\frac{1}{2}}\rho, \tau))$ does not contain an irreducible constituent of the form $\nu^{\frac{1}{2}}\rho \otimes \tau'$. Now we are in position to repeat the same arguments as in the proof of Proposition 5.5, and obtain

$$L(\nu^{-x_{\min}}\rho, \nu^{-x_{\min}+1}\rho, \ldots, \nu^{-\frac{1}{2}}\rho, \tau) \hookrightarrow \langle [\nu^{-x_{\min}}\rho, \nu^{x_{\min}}\rho] \rangle \rtimes \sigma. \tag{4}$$

If $x_{\min} \geq -a$, let π' denote

$$L(\nu^{-b}\rho, \ldots, \nu^{-x_{\min}-1}\rho, \nu^{a}\rho, \ldots, \nu^{-\frac{1}{2}}\rho, \tau).$$

If $x_{\min} \leq -a$, let π' denote

$$L(\nu^{-b}\rho, \ldots, \nu^{a-1}\rho, \nu^{a}\rho, \nu^{a}\rho, \ldots, \nu^{-x_{\min}-1}\rho, \nu^{-x_{\min}-1}\rho, \nu^{-x_{\min}}\rho, \ldots, \nu^{-\frac{1}{2}}\rho, \tau).$$

Starting from the embedding (4), and using Lemmas 2.3 and 2.4, it can be seen that π' is an irreducible subquotient of $\langle \Delta \rangle \rtimes \sigma$. Consequently, $\langle \Delta \rangle \rtimes \sigma$ also reduces in this case, and the proposition is proved. □

From Propositions 5.8 and 5.12 we obtain our final main result.

Theorem 5.13. *Suppose that $a \notin \mathbb{Z}$ and $a \leq \frac{1}{2}$. The induced representation $\langle \Delta \rangle \rtimes \sigma$ is irreducible if and only if $x \in Jord_\rho(\sigma)$ for all $x \in \{2, 4, \dots, 2b+1\}$, $\epsilon_\sigma((x_-, \rho), (x, \rho)) = -1$ for all $x \in \{4, 6, \dots, 2b+1\}$, and $\epsilon_\sigma(2, \rho) = -1$.*

References

[1] J. Arthur, *The endoscopic classification of representations. Orthogonal and symplectic groups*, vol. 61 of American Mathematical Society Colloquium Publications, American Mathematical Society, Providence, RI, 2013.

[2] I. Badulescu, E. Lapid, and A. Mínguez, *Une condition suffisante pour l'irréductibilité d'une induite parabolique de $GL(m, D)$*, Ann. Inst. Fourier (Grenoble), 63 (2013), pp. 2239–2266.

[3] W. T. Gan and L. Lomelí, *Globalization of supercuspidal representations over function fields and applications*, J. Eur. Math. Soc., 20 (2018), pp. 2813–2858.

[4] M. Hanzer, *The generalized injectivity conjecture for classical p-adic groups*, Int. Math. Res. Not., (2010), pp. 195–237.

[5] M. Hanzer and G. Muić, *On an algebraic approach to the Zelevinsky classification for classical p-adic groups*, J. Algebra, 320 (2008), pp. 3206–3231.

[6] C. Jantzen, *On supports of induced representations for symplectic and odd-orthogonal groups*, Amer. J. Math., 119 (1997), pp. 1213–1262.

[7] ——, *Jacquet modules of p-adic general linear groups*, Represent. Theory, 11 (2007), pp. 45–83.

[8] ——, *Tempered representations for classical p-adic groups*, Manuscripta Math., 145 (2014), pp. 319–387.

[9] ——, *Jacquet modules and irrreducibility of induced representations for classical p-adic groups*, Manuscripta Math., 156 (2018), pp. 23–55.

[10] Y. Kim and I. Matić, *Discrete series of odd general spin groups*, preprint, (2017).

[11] I. Matić, *Composition factors of a class of induced representations of classical p-adic groups*, Nagoya Math. J., 227 (2017), pp. 16–48.

[12] ——, *Representations induced from the Zelevinsky segment and discrete series in the half-integral case*, preprint, (2019).

[13] C. Mœglin, *Sur la classification des séries discrètes des groupes classiques p-adiques: paramètres de Langlands et exhaustivité*, J. Eur. Math. Soc., 4 (2002), pp. 143–200.

[14] C. Mœglin, *Paquets stables des séries discrètes accessibles par endoscopie tordue; leur paramètre de Langlands*, in Automorphic forms and related geometry: assessing the legacy of I. I. Piatetski-Shapiro, vol. 614 of Contemp. Math., Amer. Math. Soc., Providence, RI, 2014, pp. 295–336.

[15] C. Mœglin and M. Tadić, *Construction of discrete series for classical p-adic groups*, J. Amer. Math. Soc., 15 (2002), pp. 715–786.

[16] G. Muić, *Composition series of generalized principal series; the case of strongly positive discrete series*, Israel J. Math., 140 (2004), pp. 157–202.

[17] ——, *Reducibility of generalized principal series*, Canad. J. Math., 57 (2005), pp. 616–647.

[18] ——, *Reducibility of standard representations*, Pacific J. Math., 222 (2005), pp. 133–168.

[19] F. Shahidi, *A proof of Langlands' conjecture on Plancherel measures; complementary series for p-adic groups*, Ann. of Math. (2), 132 (1990), pp. 273–330.

[20] ——, *Twisted endoscopy and reducibility of induced representations for p-adic groups*, Duke Math. J., 66 (1992), pp. 1–41.

[21] M. Tadić, *Classification of unitary representations in irreducible representations of general linear group (non-Archimedean case)*, Ann. Sci. École Norm. Sup. (4), 19 (1986), pp. 335–382.

[22] ——, *Structure arising from induction and Jacquet modules of representations of classical p-adic groups*, J. Algebra, 177 (1995), pp. 1–33.

[23] ———, *On reducibility of parabolic induction*, Israel J. Math., 107 (1998), pp. 29–91.
[24] ———, *On tempered and square integrable representations of classical p-adic groups*, Sci. China Math., 56 (2013), pp. 2273–2313.
[25] A. V. Zelevinsky, *Induced representations of reductive p-adic groups. II. On irreducible representations of $GL(n)$*, Ann. Sci. École Norm. Sup. (4), 13 (1980), pp. 165–210.

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