Representations induced from the Zelevinsky segment and discrete series in the half-integral case

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Abstract

Let G_n denote either the group $SO(2n+1, F)$ or $Sp(2n, F)$ over a non-archimedean local field of characteristic different than two. We study parabolically induced representations of the form $\langle \Delta \rangle \rtimes \sigma$, where $\langle \Delta \rangle$ denotes the Zelevinsky segment representation of the general linear group attached to the segment Δ, and σ denotes a discrete series representation of G_n. We determine composition factors of $\langle \Delta \rangle \rtimes \sigma$ in the case when $\Delta = [\nu^a \rho, \nu^b \rho]$ where a is half-integral.

1 Introduction

Let F denote a non-archimedean local field of characteristic different than two, and let G_n denote either symplectic or special odd orthogonal group defined over F.

In this paper we continue our investigation, initiated in [12], on the structure of the induced representations of the form $\langle \Delta \rangle \rtimes \sigma$, where $\langle \Delta \rangle$ stands for a Zelevinsky segment representation of the general linear group attached to the segment Δ and σ stands for a discrete series representation of G_n.

We determine complete composition series of the induced representations $\langle \Delta \rangle \rtimes \sigma$ in the case when $\Delta = [\nu^a \rho, \nu^b \rho]$ for half-integers a and b. We note that representations of the form $\langle \Delta \rangle$ have been introduced in [25] and play a

MSC2000: primary 22E35; secondary 22E50, 11F70

Keywords: classical p-adic groups, Zelevinsky segment, discrete series, composition factors

fundamental role in the classification of the unitary dual of the general linear group, provided in [22]. Thus, it is of particular interest to study representations of G_n obtained by unitary parabolic induction from representations having the Zelevinsky segment on the general linear group part and prominent members of the unitary dual of $G_{n'}$, $n' < n$, such as discrete series or tempered representations, on the classical group part. Knowledge on the structure of the composition series of induced representations of such type should have a deep impact on better understanding of the unitary dual of classical p-adic groups.

It is rather well-known, and proved in detail in [12, Section 3], that for $\Delta = [\nu^a\rho, \nu^b\rho]$ such that $2a \notin \mathbb{Z}$ the induced representation $\langle\Delta\rangle \rtimes \sigma$ is irreducible. Also, by other irreducibility results obtained in [12], we may restrict our attention to the case when ρ is a self-contragredient representation. Moreover, for a self-contragredient representation ρ and a half-integer a we have two main cases to study. To describe them, let us denote by $(\mathrm{Jord}(\sigma), \sigma_{cusp}, \epsilon_\sigma)$ the admissible triple corresponding to σ by the classification of discrete series ([14, 16]). The first main case is $\mathrm{Jord}_\rho(\sigma) \neq \emptyset$ and all elements of $\mathrm{Jord}_\rho(\sigma)$ are even integers. The second main case is $\mathrm{Jord}_\rho(\sigma) = \emptyset$ and $\nu^{\frac{1}{2}}\rho \rtimes \sigma_{cusp}$ reduces. Although the composition series happen to be rather similar in both cases, it turns out that a description of irreducible tempered subquotients is much complicated in the first case.

To identify irreducible subquotients of studied induced representations, we adopt the strategy used in [12], combined with further adjustment of the methods used in [11] and [17] to the Zelevinsky segment case. The crucial step in our description is the characterisation of irreducible tempered subquotients of $\langle\Delta\rangle \rtimes \sigma$ in the half-integral case, which is provided in detail in Section 3. The main tool used to accomplish this is the calculation of the Jacquet modules using the structural formula, but one also has to use several key ingredients from classifications of discrete series and tempered representations, such as the results on the Jacquet modules of tempered representations ([24, Section 4]) and embeddings of strongly positive representations and discrete series coming from slightly different approaches used in [8] and [9], which both also hold in the classical group case. Along the way, we also obtain a subrepresentation result for irreducible tempered subquotients of $\langle\Delta\rangle \rtimes \sigma$ (Theorem 5.2).

When having at hand a description of the irreducible tempered subquotients, we are able to identify a general form of non-tempered irreducible subquotients. Following the subrepresentation version of the Langlands classification, we obtain very restrictive conditions which lead to rather short composition series. A complete description of the composition series, together with the calculation of multiplicities, is then provided using a case-by-case considerations. Our results show that in the half-integral case the induced representation $\langle\Delta\rangle \rtimes \sigma$ is a multiplicity one representation, which can be of length at most three. Also, using our description of the composition series and irreducibility results from [12], one can directly obtain a class of unitarizable representations appearing in the ends of complementary series.

Let us now describe the content of the paper in more detail. In the following section we introduce the notation and present some preliminaries. The third section presents the technical heart of the paper. It is devoted to the determination of necessary and sufficient conditions under which $\langle\Delta\rangle \rtimes \sigma$ contains an irreducible tempered subquotient in the $\mathrm{Jord}_\rho(\sigma) \neq \emptyset$ case. Based on the obtained conditions, in the fourth section we obtain a description of the composition series in $\mathrm{Jord}_\rho(\sigma) \neq \emptyset$ case. In the fifth section we complete our description by settling the remaining case when $\mathrm{Jord}_\rho(\sigma) = \emptyset$ and $\nu^{\frac{1}{2}}\rho \rtimes \sigma_{cusp}$ reduces.

The author would like to thank Goran Muić for his suggestion to study this subject.

This work has been supported in part by Croatian Science Foundation under the project IP-2018-01-3628.

2 Preliminaries

Let F denote a non-archimedean local field of the characteristic different from two. We first describe the groups that we consider.

Let $J_n = (\delta_{i,n+1-j})_{1\leq i,j\leq n}$ denote the $n \times n$ matrix, where $\delta_{i,n+1-j}$ stands for the Kronecker symbol. For a square matrix g, we denote by g^t its transposed matrix, and by g^τ its transposed matrix with respect to the second diagonal. In what follows, we shall fix one of the series of classical groups

$$Sp(n,F) = \left\{ g \in GL(2n,F) : \begin{pmatrix} 0 & -J_n \\ J_n & 0 \end{pmatrix} g^t \begin{pmatrix} 0 & -J_n \\ J_n & 0 \end{pmatrix} = g^{-1} \right\},$$

or

$$SO(2n+1,F) = \left\{ g \in GL(2n+1,F) : g^\tau = g^{-1} \right\}$$

and denote by G_n the rank n group belonging to the series which we fixed. Also, let $GL(m, F)$ denote the general linear group of rank m over F.

The set of standard parabolic subgroups will be fixed in a usual way, i.e., we fix a minimal F-parabolic subgroup in the classical group G_n consisting of upper-triangular matrices in the usual matrix realization of the classical group. Then the Levi factors of standard parabolic subgroups have the form $M = GL(n_1, F) \times \cdots \times GL(n_k, F) \times G_{n'}$. If δ_i is a representation of $GL(n_i, F)$ and τ a representation of $G_{n'}$, the normalized parabolically induced representation $\mathrm{Ind}_M^{G_n}(\delta_1 \otimes \cdots \otimes \delta_k \otimes \tau)$ will be denoted by $\delta_1 \times \cdots \times \delta_k \rtimes \tau$. We use a similar notation to denote a parabolically induced representation of $GL(m, F)$.

By $\mathrm{Irr}(G_n)$ we denote the set of all irreducible admissible representations of G_n. Let $R(G_n)$ denote the Grothendieck group of admissible representations of finite length of G_n and define $R(G) = \oplus_{n \geq 0} R(G_n)$. In a similar way we define $\mathrm{Irr}(GL(n, F))$ and $R(GL) = \oplus_{n \geq 0} R(GL(n, F))$. We note that in $R(G)$ we have $\pi \rtimes \sigma = \widetilde{\pi} \rtimes \sigma$ and $\pi_1 \times \pi_2 \rtimes \sigma = \pi_2 \times \pi_1 \rtimes \sigma$.

For $\sigma \in \mathrm{Irr}(G_n)$ and $1 \leq k \leq n$ we denote by $r_{(k)}(\sigma)$ the normalized Jacquet module of σ with respect to the maximal parabolic subgroup $P_{(k)}$ having the Levi subgroup equal to $GL(k, F) \times G_{n-k}$. We identify $r_{(k)}(\sigma)$ with its semisimplification in $R(GL(k, F)) \otimes R(G_{n-k})$ and consider

$$\mu^*(\sigma) = 1 \otimes \sigma + \sum_{k=1}^{n} r_{(k)}(\sigma) \in R(GL) \otimes R(G).$$

Let ν stand for a composition of the determinant mapping with the normalized absolute value on F. Let $\rho \in R(GL)$ denote an irreducible cuspidal unitarizable representation. By a segment Δ we mean a set of the form $[\rho, \nu^m \rho] := \{\rho, \nu\rho, \ldots, \nu^m \rho\}$, where m stands for a non-negative integer. The induced representation $\rho \times \nu\rho \times \cdots \times \nu^m \rho$ has a unique irreducible subrepresentation ([25]), which is denoted by $\langle \Delta \rangle$ and called the Zelevinsky segment representation.

The induced representation $\nu^m \rho \times \nu^{m-1} \rho \times \cdots \times \rho$ also contains a unique irreducible subrepresentation, denoted by $\delta(\Delta)$. Representation $\delta(\Delta)$ is essentially square-integrable, and, by [25], every irreducible essentially square-integrable representation in $R(GL)$ can be obtained in this way.

Since in $R(G)$ we have $\langle [\nu^a \rho, \nu^b \rho] \rangle \rtimes \sigma = \langle [\nu^{-b} \widetilde{\rho}, \nu^{-a} \widetilde{\rho}] \rangle \rtimes \sigma$, we may assume $a + b \geq 0$.

We frequently use the following structural formulas, obtained in [5, Theorem 1.4] and in [23]:

Theorem 2.1. *Let $\rho \in Irr(GL(m, F))$ be a cuspidal representation and $k, l \in \mathbb{R}$ such that $k + l \in \mathbb{Z}_{\geq 0}$. Let σ denote an admissible representation of finite length of G_n. Write $\mu^*(\sigma) = \sum_{\pi,\sigma'} \pi \otimes \sigma'$. Then we have:*

$$\mu^*(\langle[\nu^{-k}\rho, \nu^l\rho]\rangle \rtimes \sigma) = \sum_{i=0}^{k+l+1} \sum_{j=0}^{i} \sum_{\pi,\sigma'} \langle[\nu^{-l}\widetilde{\rho}, \nu^{-i+k}\widetilde{\rho}]\rangle \times \langle[\nu^{-k}\rho, \nu^{j-k-1}\rho]\rangle \times \pi \otimes \langle[\nu^{j-k}\rho, \nu^{i-k-1}\rho]\rangle \rtimes \sigma',$$

$$\mu^*(\delta([\nu^{-k}\rho, \nu^l\rho]) \rtimes \sigma) = \sum_{i=-k-1}^{l} \sum_{j=i}^{l} \sum_{\pi,\sigma'} \delta([\nu^{-i}\widetilde{\rho}, \nu^k\widetilde{\rho}]) \times \delta([\nu^{j+1}\rho, \nu^l\rho]) \times \pi \otimes \delta([\nu^{i+1}\rho, \nu^j\rho]) \rtimes \sigma'.$$

Let us take a moment to recall the subrepresentation version of the Langlands classification for general linear groups.

For every irreducible essentially square-integrable representation $\delta \in R(GL)$, there is a unique $e(\delta) \in \mathbb{R}$ such that $\nu^{-e(\delta)}\delta$ is unitarizable. We note that $e(\delta([\nu^a\rho, \nu^b\rho])) = (a + b)/2$. Suppose that $\delta_1, \delta_2, \ldots, \delta_k$ are irreducible essentially square-integrable representations such that $e(\delta_1) \leq e(\delta_2) \leq \ldots \leq e(\delta_k)$. Then the induced representation $\delta_1 \times \delta_2 \times \cdots \times \delta_k$ has a unique irreducible (Langlands) subrepresentation, denoted by $L(\delta_1, \delta_2, \ldots, \delta_k)$, which appears with multiplicity one in the composition series of $\delta_1 \times \delta_2 \times \cdots \times \delta_k$. Every irreducible representation $\pi \in R(GL)$ is isomorphic to some $L(\delta_1, \delta_2, \ldots, \delta_k)$ and, for a given π, the representations $\delta_1, \delta_2, \ldots, \delta_k$ are unique up to a permutation.

We also use the subrepresentation version of the Langlands classification for classical groups, since it happens to be more appropriate for our Jacquet module considerations. We realize a non-tempered irreducible representation π of G_n as the unique irreducible (Langlands) subrepresentation of an induced representation of the form $\delta_1 \times \delta_2 \times \cdots \times \delta_k \rtimes \tau$, where τ is an irreducible tempered representation of some G_t, and $\delta_1, \delta_2, \ldots, \delta_k \in R(GL)$ are irreducible essentially square-integrable representations such that $e(\delta_1) \leq e(\delta_2) \leq \cdots \leq e(\delta_k) < 0$. In this case, we write $\pi = L(\delta_1, \delta_2, \ldots, \delta_k, \tau)$.

We will use the following result ([6, Lemma 5.5]) several times.

Lemma 2.2. *Suppose that $\pi \in R(G_n)$ is an irreducible representation, λ an irreducible representation of the Levi subgroup M of G_n, and π a subrepresentation of $Ind_M^{G_n}(\lambda)$. If $L > M$, then there is an irreducible subquotient ρ of $Ind_M^L(\lambda)$ such that π is a subrepresentation of $Ind_L^{G_n}(\rho)$.*

Now we recall the Mœglin-Tadić classification of discrete series. We note that this classification now holds unconditionally, due to [1], [15, Théorème 3.1.1] and [3, Theorem 7.8]. Every discrete series representation in G_n is uniquely described by the following three invariants: the partial cuspidal support, the Jordan block and the ϵ-function.

The partial cuspidal support of a discrete series representation σ of G_n is an irreducible cuspidal representation σ_{cusp} of some G_m such that σ is a subrepresentation of $\pi \rtimes \sigma_{cusp}$ for some irreducible admissible representation $\pi \in R(GL)$.

The Jordan block of σ, which we denote by $\mathrm{Jord}(\sigma)$, is a set of all pairs (x, ρ) where $\rho \in R(GL)$ is an irreducible cuspidal unitarizable self-contragredient representation and x is a positive integer such that the following two conditions are satisfied:

(i) x is even if and only if $L(s, \rho, r)$ has a pole at $s = 0$. The local L-function $L(s, \rho, r)$ is the one defined by Shahidi ([20], [21]), where r is the exterior square representation of the standard representation on $\mathbb{C}^{n_\rho}$ of $GL(n_\rho, \mathbb{C})$ if G_n is a symplectic group and r is the symmetric-square representation of the standard representation on $\mathbb{C}^{n_\rho}$ of $GL(n_\rho, \mathbb{C})$ if G_n is an odd-orthogonal group.

(ii) The induced representation $\delta([\nu^{-(x-1)/2}\rho, \nu^{(x-1)/2}\rho]) \rtimes \sigma$ is irreducible.

The Jordan triples are triples of the form $(\mathrm{Jord}, \sigma', \epsilon)$ where

(1) σ' is an irreducible cuspidal representation of some G_n.

(2) Jord is a finite set (possibly empty) of pairs (x, ρ), where $\rho \in R(GL)$ is an irreducible self-contragredient cuspidal representation and x is a positive integer which is even if and only if the local L-function $L(s, \rho, r)$ has a pole at $s = 0$. For such an irreducible representation ρ we define $\mathrm{Jord}_\rho = \{x : (x, \rho) \in \mathrm{Jord}\}$. If $\mathrm{Jord}_\rho \neq \emptyset$ and $x \in \mathrm{Jord}_\rho$, let us write $x_- = \max\{y \in \mathrm{Jord}_\rho : y < x\}$, if it exists.

(3) ϵ is a function defined on a subset of $\mathrm{Jord} \cup (\mathrm{Jord} \times \mathrm{Jord})$ and attains the values 1 and -1. If $(x, \rho) \in \mathrm{Jord}$, then $\epsilon(x, \rho)$ is not defined if and only if x is odd and $(y, \rho) \in \mathrm{Jord}(\sigma')$ for some positive integer y. Next, ϵ is defined on a pair $(x, \rho), (y, \rho') \in \mathrm{Jord}$ if and only if $\rho \cong \rho'$ and $x \neq y$.

It follows from the compatibility conditions, which can be found in [8] and [16], that it is enough to know the value of the ϵ-function on the consecutive pairs $(x_-, \rho), (x, \rho)$ and on the minimal element of Jord_ρ (if it is defined on elements, not only on pairs).

Suppose that, for the Jordan triple $(\mathrm{Jord}, \sigma', \epsilon)$, there is $(x, \rho) \in \mathrm{Jord}$ such that $\epsilon((x_-, \rho), (x, \rho)) = 1$. If we put $\mathrm{Jord}' = \mathrm{Jord} \setminus \{(x_-, \rho), (x, \rho)\}$ and consider the restriction ϵ' of ϵ to $\mathrm{Jord}' \cup (\mathrm{Jord}' \times \mathrm{Jord}')$, we obtain a new Jordan triple $(\mathrm{Jord}', \sigma', \epsilon')$, and we say that such Jordan triple is subordinated to $(\mathrm{Jord}, \sigma', \epsilon)$.

We say that the Jordan triple $(\mathrm{Jord}, \sigma', \epsilon)$ is a triple of alternated type if $\epsilon((x_-, \rho), (x, \rho)) = -1$ holds whenever x_- is defined and there is an increasing bijection $\phi_\rho : \mathrm{Jord}_\rho \to \mathrm{Jord}'_\rho(\sigma')$, where $\mathrm{Jord}'_\rho(\sigma')$ equals $\mathrm{Jord}_\rho(\sigma') \cup \{0\}$ if a is even and $\epsilon(\min(\mathrm{Jord}_\rho), \rho) = 1$ and $\mathrm{Jord}'_\rho(\sigma')$ equals $\mathrm{Jord}_\rho(\sigma')$ otherwise.

We say that the Jordan triple $(\mathrm{Jord}, \sigma', \epsilon)$ dominates the Jordan triple $(\mathrm{Jord}', \sigma', \epsilon')$ if there is a sequence of Jordan triples $(\mathrm{Jord}_i, \sigma', \epsilon_i)$, $0 \leq i \leq k$, such that $(\mathrm{Jord}_0, \sigma', \epsilon_0) = (\mathrm{Jord}, \sigma', \epsilon)$, $(\mathrm{Jord}_k, \sigma', \epsilon_k) = (\mathrm{Jord}', \sigma', \epsilon')$ and $(\mathrm{Jord}_i, \sigma', \epsilon_i)$ is subordinated to $(\mathrm{Jord}_{i-1}, \sigma', \epsilon_{i-1})$ for $i \in \{1, 2, \ldots, k\}$. The Jordan triple $(\mathrm{Jord}, \sigma', \epsilon)$ is called an admissible triple if it dominates a triple of alternated type.

By [14] and [16], or also [8], there is a one-to-one correspondence between the set of all discrete series in $R(G)$ and the set of all admissible triples $(\mathrm{Jord}, \sigma', \epsilon)$ given by $\sigma = \sigma_{(\mathrm{Jord}, \sigma', \epsilon)}$, such that $\sigma_{cusp} = \sigma'$ and $\mathrm{Jord}(\sigma) = \mathrm{Jord}$.

Throughout the paper, the admissible triple corresponding to a discrete series $\sigma \in R(G)$ will be denoted by $(\mathrm{Jord}(\sigma), \sigma_{cusp}, \epsilon_\sigma)$.

Triples of the alternated type correspond to so-called strongly positive representations. We say that $\sigma \in R(G)$ is strongly positive if for every embedding $\sigma \hookrightarrow \nu^{a_1}\rho_1 \times \cdots \times \nu^{a_k}\rho_k \rtimes \sigma_{cusp}$, where $\rho_1, \ldots, \rho_k, \sigma_{cusp}$ are irreducible cuspidal unitary representations, we have $a_i > 0$ for all i.

Let $\rho \in R(GL)$ denote an irreducible cuspidal unitarizable representation, and let $\sigma \in R(G)$ stand for a discrete series. It is proved in [12, Proposition 3.3] that $\langle [\nu^x\rho, \nu^y\rho] \rangle \rtimes \sigma$ is irreducible if $2x \notin \mathbb{Z}$.

Let a, b denote half-integers such that $b - a \geq 0$. By the irreducibility criterion provided in [12, Section 3], the induced representation $\langle [\nu^a\rho, \nu^b\rho] \rangle \rtimes$

σ is irreducible unless ρ is self-contragredient and one of the following holds:

(1) $\mathrm{Jord}_\rho(\sigma) \neq \emptyset$ and $\mathrm{Jord}_\rho(\sigma)$ consists of even integers,

(2) $\mathrm{Jord}_\rho(\sigma) = \emptyset$ and $\nu^{\frac{1}{2}}\rho \rtimes \sigma_{cusp}$ reduces.

3 Tempered subquotients in $\mathrm{Jord}_\rho(\sigma) \neq \emptyset$ case

Throughout this section we assume that $\Delta = [\nu^a\rho, \nu^b\rho]$, where $\rho \in R(GL)$ is an irreducible self-contragredient cuspidal unitary representation, a and b are half-integers such that $a + b \geq 0$, and $\sigma \in R(G)$ is a discrete series such that $\mathrm{Jord}_\rho(\sigma) \neq \emptyset$ and all elements of $\mathrm{Jord}_\rho(\sigma)$ are even.

In this section we provide necessary and sufficient conditions under which the induced representation $\langle\Delta\rangle \rtimes \sigma$ contains an irreducible tempered subquotient.

Lemma 3.1. *Let c, d denote half-integers such that $\frac{3}{2} \leq c \leq d$.*

(1) If $2c-1 \notin Jord_\rho(\sigma)$ and $x \in Jord_\rho(\sigma)$ for all $x \in \{2c+1, 2c+3, \ldots, 2d+1\}$, then the Jacquet module of σ with respect to the appropriate parabolic subgroup contains an irreducible representation of the form $\nu^c\rho \otimes \nu^{c+1}\rho \otimes \nu^{c+2}\rho \otimes \cdots \otimes \nu^d\rho \otimes \pi$.

(2) If $x \in Jord_\rho(\sigma)$ for all $x \in \{2c+1, 2c+3, \ldots, 2d+1\}$, $(2c+1)_-$ is defined and $\epsilon_\sigma(((2c+1)_-, \rho), (2c+1, \rho)) = 1$, then the Jacquet module of σ with respect to the appropriate parabolic subgroup contains an irreducible representation of the form $\nu^c\rho \otimes \nu^{c+1}\rho \otimes \nu^{c+2}\rho \otimes \cdots \otimes \nu^d\rho \otimes \pi$.

(3) If $c < d$, the Jacquet module of σ with respect to the appropriate parabolic subgroup does not contain an irreducible representation of the form $\nu^c\rho \otimes \nu^{c+1}\rho \otimes \cdots \otimes \nu^{d-1}\rho \otimes \nu^d\rho \times \nu^d\rho \otimes \pi$. Also, $\mu^*(\sigma)$ does not contain an irreducible representation of the form $\nu^d\rho \times \nu^d\rho \otimes \pi$.

Lemma 3.2. *Let c denote a non-negative half-integer such that $2c + 1 \notin Jord_\rho(\sigma)$. Then, for a non-negative integer k, the Jacquet module of σ with respect to the appropriate parabolic subgroup does not contain an irreducible representation of the form $\nu^{c-k}\rho \otimes \nu^{c-k+1}\rho \otimes \cdots \otimes \nu^c\rho \otimes \pi$.*

Proposition 3.3. *Suppose that $a > \frac{1}{2}$. If $\langle\Delta\rangle \rtimes \sigma$ contains a discrete series subquotient then $x \in Jord_\rho(\sigma)$ for all $x \in \{2a-1, 2a+1, \dots, 2b-1\}$, $\epsilon_\sigma((x_-, \rho), (x, \rho)) = -1$ for all $x \in \{2a+1, 2a+3, \dots, 2b-1\}$ and $2b+1 \notin Jord_\rho(\sigma)$. Furthermore, if $\langle\Delta\rangle \rtimes \sigma$ contains a discrete series subquotient then it contains a unique discrete series subquotient, which is a subrepresentation.*

Proof. Suppose that $\langle\Delta\rangle \rtimes \sigma$ contains a discrete series subquotient, and let us denote such a subquotient by σ_{ds}. Using cuspidal support considerations, in a similar way as in [4, Subsection 4.2.1, page 216] we deduce that $2a-1 \in \mathrm{Jord}_\rho(\sigma)$ and $2b+1 \notin \mathrm{Jord}_\rho(\sigma)$. Note that $\mathrm{Jord}(\sigma_{ds}) = \mathrm{Jord}(\sigma) \cup \{(2b+1, \rho)\} \setminus \{(2a-1, \rho)\}$. In the rest of the proof we may assume that $a < b$.

Suppose that $\{2a+1, 2a+3, \dots, 2b-1\}$ is not a subset of $\mathrm{Jord}_\rho(\sigma)$, and let us denote by $x_{\max}$ the largest element of $\{2a+1, 2a+3, \dots, 2b-1\}$ which does not appear in $\mathrm{Jord}_\rho(\sigma)$. First part of Lemma 3.1 implies that the Jacquet module of σ_{ds} with respect to the appropriate parabolic subgroup contains an irreducible representation of the form $\nu^{\frac{x_{\max}+1}{2}}\rho \otimes \nu^{\frac{x_{\max}+3}{2}}\rho \otimes \cdots \otimes \nu^b\rho \otimes \pi$. Using the structural formula and Lemma 3.2, we deduce that the Jacquet module of $\langle\Delta\rangle \rtimes \sigma$ with respect to the appropriate parabolic subgroup does not contain an irreducible representation of the form $\nu^{\frac{x_{\max}+1}{2}}\rho \otimes \nu^{\frac{x_{\max}+3}{2}}\rho \otimes \cdots \otimes \nu^b\rho \otimes \pi$, a contradiction.

In the same way, using the second part of Lemma 3.1, we also conclude that $\epsilon_{\sigma_{ds}}((x_-, \rho), (x, \rho)) = -1$ for all $x \in \{2a+1, 2a+3, \dots, 2b+1\}$. Applying [24, Lemma 8.1] several times, we obtain an embedding

$$\sigma_{ds} \hookrightarrow \nu^a\rho \times \nu^{a+1}\rho \times \cdots \times \nu^b\rho \rtimes \sigma',$$

where σ' is a discrete series such that $x \in \mathrm{Jord}_\rho(\sigma')$ for all $x \in \{2a-1, 2a+1, \dots, 2b-1\}$, $\epsilon_{\sigma'}((x_-, \rho), (x, \rho)) = -1$ for all $x \in \{2a+1, 2a+3, \dots, 2b-1\}$ and $2b+1 \notin \mathrm{Jord}_\rho(\sigma')$. From Lemma 2.2 follows that there is an irreducible subquotient π of $\nu^a\rho \times \nu^{a+1}\rho \times \cdots \times \nu^b\rho$ such that $\sigma_{ds} \hookrightarrow \pi \rtimes \sigma'$. Since $\epsilon_{\sigma_{ds}}((x_-, \rho), (x, \rho)) = -1$ for all $x \in \{2a+1, 2a+3, \dots, 2b+1\}$, it follows at once that $\pi \cong \langle\Delta\rangle$.

Frobenius reciprocity implies that $\nu^a\rho \otimes \nu^{a+1}\rho \otimes \cdots \otimes \nu^b\rho \otimes \sigma'$ is contained in the Jacquet module of σ_{ds} with respect to the appropriate parabolic subgroup. Since $2b+1 \notin \mathrm{Jord}_\rho(\sigma)$, using Lemma 3.2 and Theorem 2.1 we obtain that the only irreducible representation of the form $\nu^a\rho \otimes \nu^{a+1}\rho \otimes \cdots \otimes \nu^b\rho \otimes \sigma'$ appearing in the Jacquet module of $\langle\Delta\rangle \rtimes \sigma$ with respect to the appropriate parabolic subgroup is $\nu^a\rho \otimes \nu^{a+1}\rho \otimes \cdots \otimes \nu^b\rho \otimes \sigma$, which appears there with multiplicity one. Consequently, $\sigma' \cong \sigma$, so $\epsilon_\sigma((x_-, \rho), (x, \rho)) = -1$ for all

$x \in \{2a+1, 2a+3, \ldots, 2b-1\}$. Also, σ_{ds} is a subrepresentation of $\langle \Delta \rangle \rtimes \sigma$, and the previous multiplicity one result implies that $\langle \Delta \rangle \rtimes \sigma$ contains a unique discrete series subquotient. $\square$

From Proposition 3.3 and [12, Lemma 4.4] we obtain the following criterion.

Theorem 3.4. *Suppose that $a > \frac{1}{2}$. Then $\langle \Delta \rangle \rtimes \sigma$ contains a discrete series subquotient if and only if $x \in Jord_\rho(\sigma)$ for all $x \in \{2a-1, 2a+1, \ldots, 2b-1\}$, $\epsilon_\sigma((x_-, \rho), (x, \rho)) = -1$ for all $x \in \{2a+1, 2a+3, \ldots, 2b-1\}$ and $2b+1 \notin Jord_\rho(\sigma)$. Furthermore, if $\langle \Delta \rangle \rtimes \sigma$ contains a discrete series subquotient then it contains a unique discrete series subquotient, which is a subrepresentation.*

Lemma 3.5. *Suppose that $\sigma_{ds} \in R(G)$ is a discrete series such that $\langle [\nu^{\frac{1}{2}}\rho, \nu^c \rho] \rangle \rtimes \sigma_{ds}$ contains a strongly positive discrete series σ_{sp} for some $c \geq \frac{1}{2}$. Then there is an irreducible representation $\pi \in R(G)$ such that σ_{sp} is a subrepresentation of $\nu^{\frac{1}{2}}\rho \rtimes \pi$.*

Proof. Since σ_{sp} is strongly positive, from the cuspidal support follows that σ_{ds} also has to be strongly positive. In the same way as in the proof of Proposition 3.3 we deduce that $x \in \mathrm{Jord}_\rho(\sigma_{ds})$ for all $x \in \{2, 4, \ldots, 2c-1\}$ and $2c+1 \notin \mathrm{Jord}_\rho(\sigma_{ds})$.

By the classification of strongly positive discrete series, given in [9], there is an ordered k-tuple $(\sigma_1, \sigma_2, \ldots, \sigma_k)$ of strongly positive discrete series representations such that $\sigma_1 \cong \sigma_{ds}$, there are no twists of ρ in the cuspidal support of σ_k, and for $i = 1, 2, \ldots, k-1$ there are half-integers a_i, b_i, $\frac{1}{2} \leq a_i \leq b_i$, such that σ_i is a unique irreducible subrepresentation of $\delta([\nu^{a_i}\rho, \nu^{b_i}\rho]) \rtimes \sigma_{i+1}$. Also, for $i = 1, 2, \ldots, k-2$ we have $a_{i+1} = a_i + 1$, $b_{i+1} \geq b_i + 1$ and $\nu^{a_{k-1}}\rho \rtimes \sigma_{cusp}$ reduces, where σ_{cusp} stands for the partial cuspidal support of σ_{sp}.

Using [16, Proposition 2.1], we deduce that either $k = 1$ and $\nu^c \rho \rtimes \sigma_{cusp}$ reduces, or $k \geq 2$ and $a_1 = c + 1$. Since a strongly positive discrete series is completely determined by its cuspidal support, it now follows directly from the classification of strongly positive discrete series that σ_{sp} is a unique irreducible subrepresentation of $\nu^{\frac{1}{2}}\rho \times \nu^{\frac{3}{2}}\rho \times \cdots \times \nu^c \rho \rtimes \sigma_{ds}$. Now Lemma 2.2 finishes the proof. $\square$

To deal with the next case, we need the following preparing result.

Lemma 3.6. *Suppose that $\sigma_{ds} \in R(G)$ is a discrete series representation and let $\rho \in R(GL)$ denote an irreducible self-contragredient representation. Suppose that for $x \in Jord_\rho(\sigma_{ds})$ we have $x \in 2\mathbb{Z}$. Then the following holds:*

(1) *Let* $c = \min(Jord_\rho(\sigma_{ds}))$ *and suppose that* $\epsilon_{\sigma_{ds}}(c,\rho) = 1$. *Then there is a discrete series* σ_1 *such that* σ_{ds} *is a subrepresentation of* $\delta([\nu^{\frac{1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho]) \rtimes \sigma_1$.

(2) *Let* c *denote a positive half-integer such that* $2c+1 < \min(Jord_\rho(\sigma_{ds}))$. *Then in* $R(G)$ *we have*

$$\delta([\nu^{\frac{1}{2}}\rho, \nu^{c}\rho]) \rtimes \sigma_{ds} = L(\delta([\nu^{-c}\rho, \nu^{-\frac{1}{2}}\rho]), \sigma_{ds}) + \sigma_2,$$

where σ_2 *is a discrete series subrepresentation of* $\delta([\nu^{\frac{1}{2}}\rho, \nu^{c}\rho]) \rtimes \sigma_{ds}$.

Proof. For the first part of the lemma, it follows from the definition of the ϵ-function that there is an irreducible representation $\pi_1 \in R(G)$ such that σ_{ds} is a subrepresentation of $\delta([\nu^{\frac{1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho]) \rtimes \pi_1$. Suppose that π_1 is not a square-integrable representation. Let us first assume that π_1 is tempered. Then there is an $x \geq 0$, an irreducible representation $\rho' \in R(GL)$ and an irreducible tempered representation $\tau \in R(G)$ such that π_1 is a subrepresentation of $\delta([\nu^{-x}\rho', \nu^{x}\rho']) \rtimes \tau$. If $\rho' \not\cong \rho$ or $\rho' \cong \rho$ and $x \geq \frac{c-1}{2}$, we have

$$\sigma_{ds} \hookrightarrow \delta([\nu^{\frac{1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho]) \times \delta([\nu^{-x}\rho', \nu^{x}\rho']) \rtimes \tau \cong \delta([\nu^{-x}\rho', \nu^{x}\rho']) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho]) \rtimes \tau,$$

contradicting the square-integrability criterion. If $\rho' \cong \rho$ and $x < \frac{c-1}{2}$, we have an embedding

$$\sigma_{ds} \hookrightarrow \nu^{x}\rho \times \delta([\nu^{\frac{1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho]) \times \delta([\nu^{-x}\rho, \nu^{x-1}\rho]) \rtimes \tau,$$

which contradicts the minimality of c by [16, Lemma 3.6].

Let us now assume that π_1 is non-tempered. Then there are x, y such that $x+y < 0$, an irreducible representation $\rho' \in R(GL)$ and an irreducible representation $\pi_2 \in R(G)$ such that π_1 is a subrepresentation of $\delta([\nu^{x}\rho', \nu^{y}\rho']) \rtimes \pi_2$. In the same way as before, we conclude that $\rho' \cong \rho$, $y = -\frac{1}{2}$, and that σ_{ds} has to be contained in the kernel of the intertwining operator

$$\delta([\nu^{\frac{1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho]) \times \delta([\nu^{x}\rho, \nu^{-\frac{1}{2}}\rho]) \rtimes \pi_2 \to \delta([\nu^{x}\rho, \nu^{-\frac{1}{2}}\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^{\frac{c-1}{2}}\rho]) \rtimes \pi_2.$$

Thus, σ_{ds} is a subrepresentation of $\delta([\nu^{x}\rho, \nu^{\frac{c-1}{2}}\rho]) \rtimes \pi_2$, for $x < 0$. Using [14, Remarque 3.2] we conclude that there are x', $0 < x' < \frac{c-1}{2}$, and a discrete series $\sigma'_{ds} \in R(G)$ such that σ_{ds} is a subrepresentation of $\delta([\nu^{-x'}\rho, \nu^{\frac{c-1}{2}}\rho]) \rtimes \sigma'_{ds}$. From [16, Proposition 2.1] now follows that $2x'+1 \in \mathrm{Jord}_\rho(\sigma_{ds})$. Since

$2x' + 1 < c$, this contradicts the minimality of c. Consequently, π_1 has to be square-integrable and the first part of the lemma is proved.

Let us now prove the second part of the lemma. Discrete series σ_2 is constructed in [19, Section 6]. From [19, Lemma 2.2] we deduce that $\delta([\nu^{\frac{1}{2}}\rho, \nu^c\rho]) \rtimes \sigma_{ds}$ contains a unique irreducible non-tempered subquotient, $L(\delta([\nu^{-c}\rho, \nu^{-\frac{1}{2}}\rho]), \sigma_{ds})$. Now, following the same arguments as in the proof of [17, Theorem 2.1] we obtain that every irreducible tempered subquotient of $\delta([\nu^{\frac{1}{2}}\rho, \nu^c\rho]) \rtimes \sigma_{ds}$ is a subrepresentation. But, it is easy to see, using the structural formula and $2c+1 < \min(\mathrm{Jord}_\rho(\sigma_{ds}))$, that $\delta([\nu^{\frac{1}{2}}\rho, \nu^c\rho]) \otimes \sigma_{ds}$ is contained in $\mu^*(\delta([\nu^{\frac{1}{2}}\rho, \nu^c\rho]) \rtimes \sigma_{ds})$ with multiplicity one. Thus, $\delta([\nu^{\frac{1}{2}}\rho, \nu^c\rho]) \rtimes \sigma_{ds}$ contains a unique irreducible subrepresentation and the lemma is proved. $\square$

Proposition 3.7. *Suppose that* $a = \frac{1}{2}$. *If* $\langle\Delta\rangle \rtimes \sigma$ *contains a discrete series subquotient then* $x \in Jord_\rho(\sigma)$ *for all* $x \in \{2, 4, \ldots, 2b-1\}$, $\epsilon_\sigma((x_-, \rho), (x, \rho)) = -1$ *for all* $x \in \{4, 6, \ldots, 2b-1\}$, $2b+1 \notin Jord_\rho(\sigma)$, *and if* $b > \frac{1}{2}$ *then* $\epsilon_\sigma(2, \rho) = -1$. *Furthermore, if* $\langle\Delta\rangle \rtimes \sigma$ *contains a discrete series subquotient then it contains a unique discrete series subquotient, which is a subrepresentation.*

Proof. Let us denote a discrete series subquotient of $\langle\Delta\rangle \rtimes \sigma$ by σ_{ds}. In the same way as in the proof of Proposition 3.3 we deduce that $x \in \mathrm{Jord}_\rho(\sigma)$ for all $x \in \{2, 4, \ldots, 2b-1\}$, $\epsilon_{\sigma_{ds}}((x_-, \rho), (x, \rho)) = -1$ for all $x \in \{4, 6, \ldots, 2b-1\}$, and $2b+1 \notin \mathrm{Jord}_\rho(\sigma)$. Also, $\mathrm{Jord}(\sigma_{ds}) = \mathrm{Jord}(\sigma) \cup \{(2b+1, \rho)\}$.

If $b = \frac{1}{2}$, from Lemma 3.6 we deduce that $\epsilon_{\sigma_{ds}}(2, \rho) = 1$. Let us now assume that $b > \frac{1}{2}$.

By the classification obtained in [14] and [16], there is an ordered k-tuple $(\sigma_1, \sigma_2, \ldots, \sigma_k)$ of discrete series representations such that $\sigma_1 \cong \sigma_{ds}$, σ_k is strongly positive, and for every $i = 1, 2, \ldots, k-1$ there are $(a_i, \rho), (b_i, \rho_i) \in \mathrm{Jord}(\sigma_i)$ such that $a_i = (b_i)_-$ in $\mathrm{Jord}_{\rho_i}(\sigma_i)$ and

$$\sigma_i \hookrightarrow \delta([\nu^{-\frac{a_i-1}{2}}\rho_i, \nu^{\frac{b_i-1}{2}}\rho_i]) \rtimes \sigma_{i+1}.$$

Note that if $(2, \rho) \in \mathrm{Jord}(\sigma_i)$, for some $i \in \{1, 2, \ldots, k\}$, then $\epsilon_{\sigma_{ds}}(2, \rho) = \epsilon_{\sigma_i}(2, \rho)$.

If $k = 1$, from Lemma 3.5 follows that $\epsilon_{\sigma_{ds}}(2, \rho) = 1$.

Suppose that $k \geq 2$. It follows that $\mu^*(\langle\Delta\rangle \rtimes \sigma) \geq \delta([\nu^{-\frac{a_1-1}{2}}\rho_1, \nu^{\frac{b_1-1}{2}}\rho_1]) \otimes \sigma_2$. Thus, there are $0 \leq j \leq i \leq b + \frac{1}{2}$ and an irreducible constituent $\pi \otimes \sigma'$

of $\mu^*(\sigma)$ such that

$$\delta([\nu^{-\frac{a_1-1}{2}}\rho_1, \nu^{\frac{b_1-1}{2}}\rho_1]) \leq \langle[\nu^{-b}\rho, \nu^{-\frac{1}{2}-i}\rho]\rangle \times \langle[\nu^{\frac{1}{2}}\rho, \nu^{j-\frac{1}{2}}\rho]\rangle \times \pi$$

and $\sigma_2 \leq \langle[\nu^{\frac{1}{2}+j}\rho, \nu^{i-\frac{1}{2}}\rho]\rangle \rtimes \sigma'$. Since $b > \frac{1}{2}$ and σ is square-integrable, it follows that $j = 0$ and $i \in \{b - \frac{1}{2}, b + \frac{1}{2}\}$. If $i = b + \frac{1}{2}$, then $\pi \cong \delta([\nu^{-\frac{a_1-1}{2}}\rho_1, \nu^{\frac{b_1-1}{2}}\rho_1])$ and, by [18, Theorem 2.3], σ' is a discrete series such that $\mathrm{Jord}(\sigma) = \mathrm{Jord}(\sigma') \cup \{(a_1, \rho_1), (b_1, \rho_1)\}$. Since $\sigma_2 \leq \langle\Delta\rangle \rtimes \sigma'$, it follows that if $\rho_1 \cong \rho$ then $a_1 > 2b + 1$.

On the other hand, if $i = b - \frac{1}{2}$, then $\rho_1 \cong \rho$, $a_1 = 2b + 1 = (b_1)_-$ in $\mathrm{Jord}_\rho(\sigma_{ds})$ and $\pi \cong \delta([\nu^{-b+1}\rho, \nu^{\frac{b_1-1}{2}}\rho])$. Since in $\mathrm{Jord}_\rho(\sigma)$ then we have $(b_1)_- = 2b - 1$, again from [18, Theorem 2.3] we conclude that σ' is a discrete series such that $\mathrm{Jord}(\sigma) = \mathrm{Jord}(\sigma') \cup \{(2b - 1, \rho), (b_1, \rho)\}$. Now, $\sigma_2 \leq \langle[\nu^{\frac{1}{2}}\rho, \nu^{b-1}\rho]\rangle \rtimes \sigma'$.

Repeating this procedure, we deduce that either there is $c \geq \frac{1}{2}$ and a discrete series σ'' such that $\sigma_k \leq \langle[\nu^{\frac{1}{2}}\rho, \nu^{c}\rho]\rangle \rtimes \sigma''$, or there is an $i \in \{2, 3, \ldots, k-1\}$ and a discrete series σ'' such that $\sigma_i \leq \nu^{\frac{1}{2}}\rho \rtimes \sigma''$.

If $\sigma_k \leq \langle[\nu^{\frac{1}{2}}\rho, \nu^{c}\rho]\rangle \rtimes \sigma''$ for some $c \geq \frac{1}{2}$, from Lemma 3.5 we obtain that $\epsilon_{\sigma_k}(2, \rho) = 1$. If $\sigma_i \leq \nu^{\frac{1}{2}}\rho \rtimes \sigma''$ for some $i \in \{2, 3, \ldots, k-1\}$, from Lemma 3.6 we get $\epsilon_{\sigma_i}(2, \rho) = 1$. In any case, it follows that $\epsilon_{\sigma_{ds}}(2, \rho) = 1$.

Consequently, σ_{ds} is a subrepresentation of $\nu^{\frac{1}{2}}\rho \rtimes \sigma'_{ds}$, where $\mathrm{Jord}(\sigma_{ds}) = \mathrm{Jord}(\sigma'_{ds}) \cup \{(2, \rho)\}$. In the same way as in the proof of Proposition 3.3 we obtain that σ_{ds} is a subrepresentation of $\langle\Delta\rangle \rtimes \sigma'$, for a discrete series σ' such that $x \in \mathrm{Jord}_\rho(\sigma')$ for all $x \in \{2, 4, \ldots, 2b-1\}$, $\epsilon_{\sigma'}((x_-, \rho), (x, \rho)) = -1$ for all $x \in \{4, 6, \ldots, 2b-1\}$ and $2b + 1 \notin \mathrm{Jord}_\rho(\sigma')$.

If $\epsilon_{\sigma'}(2, \rho) = 1$, then σ' is a subrepresentation of $\nu^{\frac{1}{2}}\rho \rtimes \sigma''$, for some irreducible representation σ''. Since $\langle\Delta\rangle \times \nu^{\frac{1}{2}}\rho$ is irreducible, we obtain an embedding $\sigma_{ds} \hookrightarrow \nu^{\frac{1}{2}}\rho \times \nu^{\frac{1}{2}}\rho \times \langle[\nu^{\frac{3}{2}}\rho, \nu^{b}\rho]\rangle \rtimes \sigma''$. Thus, $\mu^*(\sigma_{ds})$ contains an irreducible constituent of the form $\nu^{\frac{1}{2}}\rho \times \nu^{\frac{1}{2}}\rho \otimes \pi$. Consequently, $\mu^*(\nu^{\frac{1}{2}}\rho \rtimes \sigma'') \geq \nu^{\frac{1}{2}}\rho \times \nu^{\frac{1}{2}}\rho \otimes \pi$, and using transitivity of Jacquet modules we easily deduce that this contradicts [13, Lemma 8.2]. Thus, $\epsilon_{\sigma'}(2, \rho) = -1$.

The rest of the proof now follows on the lines of the proof of Proposition 3.3. $\square$

From Proposition 3.7 and [12, Lemma 5.10] we obtain the following criterion.

Theorem 3.8. *Suppose that* $a = \frac{1}{2}$*. Then* $\langle\Delta\rangle \rtimes \sigma$ *contains a discrete series subquotient if and only if* $x \in Jord_\rho(\sigma)$ *for all* $x \in \{2, 4, \dots, 2b-1\}$*,* $\epsilon_\sigma((x_-, \rho), (x, \rho)) = -1$ *for all* $x \in \{4, 6, \dots, 2b-1\}$*,* $2b+1 \notin Jord_\rho(\sigma)$*, and if* $b > \frac{1}{2}$ *then* $\epsilon_\sigma(2, \rho) = -1$*. Furthermore, if* $\langle\Delta\rangle \rtimes \sigma$ *contains a discrete series subquotient then it contains a unique discrete series subquotient, which is a subrepresentation.*

The following result completes the criteria for discrete series subquotients.

Theorem 3.9. *Suppose that* $a < 0$*. Then* $\langle\Delta\rangle \rtimes \sigma$ *does not contain a discrete series subquotient.*

Proof. Suppose, on the contrary, that there is a discrete series subquotient σ_{ds} of $\langle\Delta\rangle \rtimes \sigma$. From the cuspidal support of $\langle\Delta\rangle \rtimes \sigma$ follows that σ_{ds} is not a strongly positive representation. Also, $-a < b$ and, using cuspidal support considerations in a similar way as in [4, Subsection 4.1.1], we conclude that $-2a+1, 2b+1 \notin \mathrm{Jord}_\rho(\sigma)$. In the same way as in the proof of Proposition 3.3 we obtain that $x \in \mathrm{Jord}_\rho(\sigma)$ for all $x \in \{2, 4, \dots, -2a-1\} \cup \{-2a+3, -2a+5, \dots, 2b-1\}$. Note that $\mathrm{Jord}(\sigma_{ds}) = \mathrm{Jord}(\sigma) \cup \{(-2a+1, \rho), (2b+1, \rho)\}$.

Now we proceed in the same way as in the proof of Proposition 3.7 and take an ordered k-tuple $(\sigma_1, \sigma_2, \dots, \sigma_k)$ of discrete series representations such that $\sigma_1 \cong \sigma_{ds}$, σ_k is strongly positive, and for every $i = 1, 2, \dots, k-1$ there are $(a_i, \rho), (b_i, \rho_i) \in \mathrm{Jord}(\sigma_i)$ such that $a_i = (b_i)_-$ in $\mathrm{Jord}_{\rho_i}(\sigma_i)$ and

$$\sigma_i \hookrightarrow \delta([\nu^{-\frac{a_i-1}{2}}\rho_i, \nu^{\frac{b_i-1}{2}}\rho_i]) \rtimes \sigma_{i+1}.$$

Note that if $(2, \rho) \in \mathrm{Jord}(\sigma_i)$, for some $i \in \{1, 2, \dots, k\}$, then $\epsilon_{\sigma_j}(2, \rho) = \epsilon_{\sigma_i}(2, \rho)$, for all $j \in \{1, 2, \dots, i\}$. Obviously, $k \geq 2$.

Since $\mu^*(\langle\Delta\rangle \rtimes \sigma) \geq \delta([\nu^{-\frac{a_1-1}{2}}\rho_1, \nu^{\frac{b_1-1}{2}}\rho_1]) \otimes \sigma_2$, there are $0 \leq j \leq i \leq b-a+1$ and an irreducible constituent $\pi \otimes \sigma'$ of $\mu^*(\sigma)$ such that

$$\delta([\nu^{-\frac{a_1-1}{2}}\rho_1, \nu^{\frac{b_1-1}{2}}\rho_1]) \leq \langle[\nu^{-b}\rho, \nu^{-i-a}\rho]\rangle \times \langle[\nu^{a}\rho, \nu^{j+a-1}\rho]\rangle \times \pi$$

and $\sigma_2 \leq \langle[\nu^{a+j}\rho, \nu^{a+i-1}\rho]\rangle \rtimes \sigma'$. Using cuspidal support considerations, we obtain that if $-b \leq -i-a$, then $\langle[\nu^{-b}\rho, \nu^{-i-a}\rho]\rangle$ has to be an essentially square-integrable representation, and a similar argument applies for $\langle[\nu^{a}\rho, \nu^{j+a-1}\rho]\rangle$. It directly follows that $i \in \{b-a, b-a+1\}$ and $j \in \{0, 1\}$.

If $i = b-a+1$ and $j = 0$, in the same way as in the proof of Proposition 3.7 we conclude that $\sigma_2 \leq \langle\Delta\rangle \rtimes \sigma'$, for a discrete series σ' such that $\mathrm{Jord}(\sigma) =$

$\mathrm{Jord}(\sigma') \cup \{(a_1, \rho_1), (b_1, \rho_1)\}$. Similarly, if $i = b - a$ and $j = 0$, we conclude that $\sigma_2 \leq \langle[\nu^a \rho, \nu^{b-1}\rho]\rangle \rtimes \sigma'$, where σ' is a discrete series such that $\mathrm{Jord}(\sigma) = \mathrm{Jord}(\sigma') \cup \{(2b-1, \rho), (b_1, \rho)\}$. Note that this possibility does not appear when $-a = b - 1$.

Let us now consider the case $(i, j) = (b - a + 1, 1)$. Then we have $\pi \cong \delta([\nu^{a+1}\rho, \nu^{\frac{b_1-1}{2}}\rho])$. If $-a = b - 1$, this is impossible because then we have $\frac{b_1-1}{2} = b$, but $2b + 1 \notin \mathrm{Jord}_\rho(\sigma)$. If $-a < b - 1$, it follows that $\frac{b_1-1}{2} < b$.

If $a < -\frac{1}{2}$, in $\mathrm{Jord}_\rho(\sigma)$ we have $(b_1)_- = -2a + 1$. This leads to $\sigma_2 \leq \langle[\nu^{a+1}\rho, \nu^b\rho]\rangle \rtimes \sigma'$, where σ' is a discrete series such that $\mathrm{Jord}(\sigma) = \mathrm{Jord}(\sigma') \cup \{(-2a - 1, \rho), (b_1, \rho)\}$. But this is impossible since then $b_1 \notin \mathrm{Jord}_\rho(\sigma')$ and $\frac{b_1-1}{2} < b$, so $\langle[\nu^{a+1}\rho, \nu^b\rho]\rangle \rtimes \sigma'$ does not contain a discrete series subquotient.

If $a = -\frac{1}{2}$, since in $\mathrm{Jord}_\rho(\sigma_{ds})$ we have $(b_1)_- = -2a + 1$, it follows that $b_1 = \min(\mathrm{Jord}_\rho(\sigma))$ and $\epsilon_\sigma(2, \rho) = 1$, by [24, Proposition 7.4]. From Lemma 3.6 we get that σ' is a discrete series such that $\mathrm{Jord}(\sigma) = \mathrm{Jord}(\sigma') \cup \{(b_1, \rho)\}$. Now we have $\sigma_2 \leq \langle[\nu^{\frac{1}{2}}\rho, \nu^b\rho]\rangle \rtimes \sigma'$, which is impossible since $(2, \rho), (b_1, \rho) \notin \mathrm{Jord}(\sigma')$.

Thus, if $j = 1$, then $i = b - a$ and the square-integrability of σ implies that $-a = b - 1$. If $a < -\frac{1}{2}$, in the same way as before we conclude that $\sigma_2 \leq \langle[\nu^{a+1}\rho, \nu^{b-1}\rho]\rangle \rtimes \sigma'$, for a discrete series σ' such that $\mathrm{Jord}(\sigma) = \mathrm{Jord}(\sigma') \cup \{(-2a - 1, \rho), (b_1, \rho)\}$. If $a = -\frac{1}{2}$, then $\sigma_2 \leq \nu^{\frac{1}{2}}\rho \rtimes \sigma'$, for a discrete series σ' such that $\mathrm{Jord}(\sigma) = \mathrm{Jord}(\sigma') \cup \{(b_1, \rho)\}$, and $b_1 = \min(\mathrm{Jord}_\rho(\sigma))$.

Repeating the same arguments, we conclude that if $\sigma_l \leq \langle[\nu^c\rho, \nu^d\rho]\rangle \rtimes \sigma'$, for some $l \in \{1, 2, \ldots, k - 1\}$, for $0 < -c < d$ and a discrete series σ', then there is an ordered pair $(i, j) \in \{(0, 0), (1, 0), (1, 1)\}$ and a discrete series σ'' such that $\sigma_{l+1} \leq \langle[\nu^{c+j}\rho, \nu^{d-i}\rho]\rangle \rtimes \sigma''$. Furthermore, if $(i, j) = (1, 1)$, then $-c = d - 1$.

Since σ_k is strongly positive, it is not a subquotient of an induced representation of the form $\langle[\nu^c\rho, \nu^d\rho]\rangle \rtimes \sigma'$, for $0 < -c < d$ and σ' a discrete series. Thus, there is an $m \in \{1, 2, \ldots, k - 1\}$ and discrete series representations σ', σ'' such that $\sigma_m \leq \langle[\nu^{-\frac{1}{2}}\rho, \nu^{\frac{3}{2}}\rho]\rangle \rtimes \sigma'$ and $\sigma_{m+1} \leq \nu^{\frac{1}{2}}\rho \rtimes \sigma''$. From Lemma 3.6 follows that $\epsilon_{\sigma_{m+1}}(2, \rho) = 1$ and, consequently, $\epsilon_{\sigma_m}(2, \rho) = 1$, so $\mu^*(\sigma_m)$ contains an irreducible constituent of the form $\nu^{\frac{1}{2}}\rho \otimes \pi$. But, since $\langle[\nu^{-\frac{1}{2}}\rho, \nu^{\frac{3}{2}}\rho]\rangle \rtimes \sigma'$ contains a discrete series subquotient, we have $2 \notin \mathrm{Jord}_\rho(\sigma')$ and it follows from the structural formula that $\mu^*(\langle[\nu^{-\frac{1}{2}}\rho, \nu^{\frac{3}{2}}\rho]\rangle \rtimes \sigma')$ does not contain $\nu^{\frac{1}{2}}\rho \otimes \pi$, a contradiction. $\square$

To determine irreducible tempered subquotients, we need the following result, which is [2, Théorème 0.1] or [7, Lemma 1.3.3].

Lemma 3.10. *Let a, b denote positive half-integers such that $a < b$, and let $\rho \in R(GL)$ denote an irreducible cuspidal unitarizable representation. Then the induced representation $\delta([\nu^{-b}\rho, \nu^b\rho]) \times \langle[\nu^a\rho, \nu^{b-1}\rho]\rangle$ is irreducible and isomorphic to $\langle[\nu^a\rho, \nu^{b-1}\rho]\rangle \times \delta([\nu^{-b}\rho, \nu^b\rho])$.*

Proposition 3.11. *Suppose that $a > \frac{1}{2}$. If $\langle\Delta\rangle \rtimes \sigma$ contains an irreducible tempered subquotient which is not square-integrable then $x \in Jord_\rho(\sigma)$ for all $x \in \{2a-1, 2a+1, \ldots, 2b+1\}$, $\epsilon_\sigma((x_-, \rho), (x, \rho)) = -1$ for all $x \in \{2a+1, 2a+3, \ldots, 2b-1\}$ and $\epsilon_\sigma((2b-1, \rho), (2b+1, \rho)) = 1$. Furthermore, if $\langle\Delta\rangle \rtimes \sigma$ contains an irreducible tempered subquotient then it contains a unique irreducible tempered subquotient, which is a subrepresentation.*

Proof. Let us denote an irreducible tempered subquotient of $\langle\Delta\rangle \rtimes \sigma$ by τ. Applying the structure formula as in the proof of [12, Lemma 4.1], it directly follows that $2b+1 \in \mathrm{Jord}_\rho(\sigma)$. Also, τ is a subrepresentation of an induced representation of the form $\delta([\nu^{-b}\rho, \nu^b\rho]) \rtimes \sigma_1$, where σ_1 is a discrete series such that $\mathrm{Jord}(\sigma) = \mathrm{Jord}(\sigma_1) \cup \{(2a-1, \rho), (2b+1, \rho)\}$.

Since $\mu^*(\langle\Delta\rangle \rtimes \sigma) \geq \delta([\nu^{-b}\rho, \nu^b\rho]) \otimes \sigma_1$, from Theorem 2.1 we obtain that $\mu^*(\sigma) \geq \delta([\nu^{-b+1}\rho, \nu^b\rho]) \otimes \sigma_2$. Now [24, Proposition 7.2] implies $2b-1 \in \mathrm{Jord}_\rho(\sigma)$ and $\epsilon_\sigma((2b-1, \rho), (2b+1, \rho)) = 1$. Also, from [18, Theorem 2.3] follows that σ_2 is a discrete series representation such that σ is a subrepresentation of $\delta([\nu^{-b+1}\rho, \nu^b\rho]) \rtimes \sigma_2$, and σ_1 is an irreducible subquotient of $\langle[\nu^a\rho, \nu^{b-1}\rho]\rangle \rtimes \sigma_2$. Proposition 3.3 implies that $x \in \mathrm{Jord}_\rho(\sigma_2)$ for all $x \in \{2a-1, 2a+1, \ldots, 2b-3\}$ and $\epsilon_{\sigma_2}((x_-, \rho), (x, \rho)) = -1$ for all $x \in \{2a+1, 2a+3, \ldots, 2b-3\}$. From [16, Proposition 2.1, Lemma 5.1] or [8, Proposition 3.1, Theorem 3.15], we conclude that also $x \in \mathrm{Jord}_\rho(\sigma)$ for all $x \in \{2a-1, 2a+1, \ldots, 2b-3\}$ and $\epsilon_\sigma((x_-, \rho), (x, \rho)) = -1$ for all $x \in \{2a+1, 2a+3, \ldots, 2b-3\}$. It remains to show that $\epsilon_\sigma((2b-3, \rho), (2b-1, \rho)) = -1$, and in the rest of the proof we can assume that $a < b$.

Note that $\mu^*(\tau)$ does not contain an irreducible constituent of the form $\nu^b\rho \times \nu^b\rho \otimes \pi$.

From Lemma 3.10 and Theorem 3.4 we obtain an embedding

$$\tau \hookrightarrow \langle[\nu^a\rho, \nu^{b-1}\rho]\rangle \times \delta([\nu^{-b}\rho, \nu^b\rho]) \rtimes \sigma_2.$$

By Lemma 2.2, there is an irreducible subquotient τ' of $\delta([\nu^{-b}\rho, \nu^b\rho]) \rtimes \sigma_2$ such that τ is a subrepresentation of $\langle[\nu^a\rho, \nu^{b-1}\rho]\rangle \rtimes \tau'$.

Since $2b+1 \notin \mathrm{Jord}_\rho(\sigma_2)$, in $R(G)$ we have $\delta([\nu^{-b}\rho, \nu^b\rho]) \rtimes \sigma_2 = \tau_1 + \tau_{-1}$, for mutually non-isomorphic irreducible tempered representations τ_1 and τ_{-1}.

Since $2b-3 \in \mathrm{Jord}_\rho(\sigma_2)$, by [24, Lemma 4.1] there is a unique $i \in \{1,-1\}$ such that τ_i can be written as a subrepresentation of an induced representation of the form $\delta([\nu^{b-1}\rho, \nu^b\rho]) \times \delta([\nu^{b-1}\rho, \nu^b\rho]) \rtimes \pi$, for irreducible π. If $\tau' \cong \tau_i$, we have the following embeddings and isomorphisms:

$$\begin{aligned}
\tau &\hookrightarrow \langle[\nu^a\rho, \nu^{b-1}\rho]\rangle \times \delta([\nu^{b-1}\rho, \nu^b\rho]) \times \delta([\nu^{b-1}\rho, \nu^b\rho]) \rtimes \pi \\
&\hookrightarrow \langle[\nu^a\rho, \nu^{b-2}\rho]\rangle \times \nu^{b-1}\rho \times \delta([\nu^{b-1}\rho, \nu^b\rho]) \times \delta([\nu^{b-1}\rho, \nu^b\rho]) \rtimes \pi \\
&\cong \langle[\nu^a\rho, \nu^{b-2}\rho]\rangle \times \delta([\nu^{b-1}\rho, \nu^b\rho]) \times \delta([\nu^{b-1}\rho, \nu^b\rho]) \times \nu^{b-1}\rho \rtimes \pi \\
&\hookrightarrow \langle[\nu^a\rho, \nu^{b-2}\rho]\rangle \times \nu^b\rho \times \nu^b\rho \times \nu^{b-1}\rho \times \nu^{b-1}\rho \times \nu^{b-1}\rho \rtimes \pi \\
&\cong \nu^b\rho \times \nu^b\rho \times \langle[\nu^a\rho, \nu^{b-2}\rho]\rangle \times \nu^{b-1}\rho \times \nu^{b-1}\rho \times \nu^{b-1}\rho \rtimes \pi,
\end{aligned}$$

which is impossible. Thus, $\tau' \cong \tau_{-i}$.

Since $\nu^{-b}\rho \rtimes \sigma_2$ is irreducible by [19, Theorem 6.1], we have

$$\begin{aligned}
\tau' &\hookrightarrow \delta([\nu^{-b+1}\rho, \nu^b\rho]) \times \nu^{-b}\rho \rtimes \sigma_2 \\
&\cong \delta([\nu^{-b+1}\rho, \nu^b\rho]) \times \nu^b\rho \rtimes \sigma_2 \\
&\cong \nu^b\rho \times \delta([\nu^{-b+1}\rho, \nu^b\rho]) \rtimes \sigma_2.
\end{aligned}$$

It follows from Lemma 2.2 that there is an irreducible subquotient σ_3 of $\delta([\nu^{-b+1}\rho, \nu^b\rho]) \rtimes \sigma_2$ such that τ' is a subrepresentation of $\nu^b\rho \rtimes \sigma_3$. Also, since $\mu^*(\tau')$ contains an irreducible constituent of the form $\nu^b\rho \times \nu^b\rho \otimes \pi$, we get that $\mu^*(\sigma_3) \geq \nu^b\rho \otimes \pi'$, for some irreducible π'. From [17, Theorem 2.1] we easily obtain that σ_3 has to be a discrete series subrepresentation of $\delta([\nu^{-b+1}\rho, \nu^b\rho]) \rtimes \sigma_2$. In $R(G)$ we have $\delta([\nu^{-b+1}\rho, \nu^{b-1}\rho]) \rtimes \sigma_2 = \tau'_1 + \tau'_{-1}$, for mutually non-isomorphic irreducible tempered representations τ'_1 and τ'_{-1}, and there is a unique $j \in \{1,-1\}$ such that σ_3 is a subrepresentation of $\nu^b\rho \rtimes \tau'_j$.

Using [24, Lemma 4.1] again, we see that there is a unique $k \in \{1,-1\}$ such that τ'_k can be written as a subrepresentation of an induced representation of the form $\nu^{b-1}\rho \times \nu^{b-1}\rho \rtimes \pi$, for some irreducible $\pi \in R(G)$.

If $j = k$, from $\tau' \hookrightarrow \nu^b\rho \times \nu^b\rho \times \nu^{b-1}\rho \times \nu^{b-1}\rho \rtimes \pi$ and $2b-1 \notin \mathrm{Jord}_\rho(\sigma_2)$ we get $\tau' \hookrightarrow \delta([\nu^{b-1}\rho, \nu^b\rho]) \times \delta([\nu^{b-1}\rho, \nu^b\rho]) \rtimes \pi$, a contradiction. Thus, $j = -k$.

Let us prove that $\epsilon_{\sigma_3}((2b-3,\rho),(2b-1,\rho)) = -1$ (we note that this also follows from the proof of [8, Theorem 3.15], but we include a proof here, for the sake of completeness). Suppose that $\epsilon_{\sigma_3}((2b-3,\rho),(2b-1,\rho)) = 1$.

Then $\mu^*(\sigma_3)$ contains $\delta([\nu^{-b+2}\rho, \nu^{b-1}\rho]) \otimes \sigma_4$, for some discrete series σ_4. Since $\sigma_3 \leq \nu^b\rho \rtimes \tau_j'$, using the structural formula and $2b-1, 2b+1 \notin \mathrm{Jord}_\rho(\sigma_2)$, we obtain that $\sigma_4 \leq \nu^b\rho \rtimes \pi'$, for an irreducible representation $\pi' \in R(G)$ such that $\mu^*(\tau_j') \geq \delta([\nu^{-b+2}\rho, \nu^{b-1}\rho]) \otimes \pi'$. Obviously, $\pi' \leq \nu^{b-1}\rho \rtimes \sigma_2$. Using [24, Theorem 8.2], we deduce that the Jacquet module of σ_4 with respect to the appropriate parabolic subgroup contains $\nu^b\rho \otimes \nu^{b-1}\rho \otimes \sigma_2$.

It is now easy to see that the Jacquet module of τ_j' with respect to the appropriate parabolic subgroup contains $\delta([\nu^{-b+2}\rho, \nu^{b-1}\rho]) \otimes \nu^{b-1}\rho \otimes \sigma_2$. Transitivity of the Jacquet modules implies that there is an irreducible constituent $\pi_1 \otimes \sigma_2$ of $\mu^*(\tau_j')$ such that the Jacquet module of π_1 with respect to the appropriate parabolic subgroup contains $\delta([\nu^{-b+2}\rho, \nu^{b-1}\rho]) \otimes \nu^{b-1}\rho$. Since $2b-1, 2b+1 \notin \mathrm{Jord}_\rho(\sigma_2)$, we deduce that $\pi_1 \cong \delta([\nu^{-b+2}\rho, \nu^{b-1}\rho]) \times \nu^{b-1}\rho$, and it can now be directly seen that τ_j' can be written as a subrepresentation of an induced representation of the form $\nu^{b-1}\rho \times \nu^{b-1}\rho \rtimes \pi$, for some irreducible $\pi \in R(G)$, a contradiction.

From $\tau \hookrightarrow \langle[\nu^a\rho, \nu^{b-1}\rho]\rangle \times \nu^b\rho \rtimes \sigma_3$, Lemma 2.2 and the fact that $\mu^*(\tau)$ does not contain an irreducible constituent of the form $\nu^b\rho \times \nu^b\rho \otimes \pi$, we obtain $\tau \hookrightarrow \langle[\nu^a\rho, \nu^b\rho]\rangle \rtimes \sigma_3$. Consequently, the Jacquet module of τ with respect to the appropriate parabolic subgroup contains $\nu^a\rho \otimes \nu^{a+1}\rho \otimes \cdots \otimes \nu^b\rho \times \nu^b\rho \otimes \tau_j'$.

Since τ is an irreducible subquotient of $\langle\Delta\rangle \rtimes \sigma$, using the structural formula and the third part of Lemma 3.1 we deduce that $\mu^*(\nu^b\rho \rtimes \sigma)$ contains $\nu^b\rho \times \nu^b\rho \otimes \tau_j'$. Thus, $\mu^*(\sigma)$ contains $\nu^b\rho \otimes \tau_j'$ and it follows from [24, Section 7] that σ is a unique irreducible subrepresentation of $\nu^b\rho \rtimes \tau_j'$. Thus, $\sigma \cong \sigma_3$. Consequently, $\epsilon_\sigma((2b-3, \rho), (2b-1, \rho)) = -1$ and τ is a subrepresentation of $\langle[\nu^a\rho, \nu^b\rho]\rangle \rtimes \sigma$.

Since $\mu^*(\sigma)$ contains $\delta([\nu^{-b+1}\rho, \nu^b\rho]) \otimes \sigma_2$ with multiplicity one and σ_1 is the unique discrete series subquotient of $\langle[\nu^a\rho, \nu^{b-1}\rho]\rangle \rtimes \sigma_2$, $\delta([\nu^{-b}\rho, \nu^b\rho]) \otimes \sigma_1$ appears with multiplicity one in $\mu^*(\langle\Delta\rangle \rtimes \sigma)$, so τ is the unique irreducible tempered subquotient of $\langle\Delta\rangle \rtimes \sigma$. □

From Proposition 3.11 and [12, Lemma 4.6] we obtain

Theorem 3.12. *Suppose that $a > \frac{1}{2}$. Then $\langle\Delta\rangle \rtimes \sigma$ contains an irreducible tempered subquotient which is not square-integrable if and only if $x \in Jord_\rho(\sigma)$ for all $x \in \{2a-1, 2a+1, \ldots, 2b+1\}$, $\epsilon_\sigma((x_-, \rho), (x, \rho)) = -1$ for all $x \in \{2a+1, 2a+3, \ldots, 2b-1\}$ and $\epsilon_\sigma((2b-1, \rho), (2b+1, \rho)) = 1$. Furthermore, if $\langle\Delta\rangle \rtimes \sigma$ contains an irreducible tempered subquotient then it contains a unique irreducible tempered subquotient, which is a subrepresentation.*

In the same way as in the proof of Lemma 3.6, we obtain the following result.

Proposition 3.13. *Suppose that $a = b = \frac{1}{2}$. Then $\langle\Delta\rangle \rtimes \sigma = \nu^{\frac{1}{2}}\rho \rtimes \sigma$ contains an irreducible tempered subquotient which is not square-integrable if and only if $(2,\rho) \in$ Jord(σ) and $\epsilon_\sigma(2,\rho) = 1$. In that case, in $R(G)$ we have $\nu^{\frac{1}{2}}\rho \rtimes \sigma = L(\nu^{-\frac{1}{2}}\rho, \sigma) + \tau$, where τ is an irreducible tempered subrepresentation of $\nu^{\frac{1}{2}}\rho \rtimes \sigma$.*

Proposition 3.14. *Suppose that $a = \frac{1}{2}$ and $a < b$. If $\langle\Delta\rangle \rtimes \sigma$ contains an irreducible tempered subquotient which is not square-integrable then $x \in$ Jord$_\rho(\sigma)$ for all $x \in \{2, 4, \ldots, 2b+1\}$, $\epsilon_\sigma((x_-,\rho),(x,\rho)) = -1$ for all $x \in \{4, 6, \ldots, 2b-1\}$, $\epsilon_\sigma((2b-1,\rho),(2b+1,\rho)) = 1$ and $\epsilon_\sigma(2,\rho) = -1$. Furthermore, if $\langle\Delta\rangle \rtimes \sigma$ contains an irreducible tempered subquotient then it contains a unique irreducible tempered subquotient, which is a subrepresentation.*

Proof. We will comment only the case $b = \frac{3}{2}$, since the case $b > \frac{3}{2}$ can be handled in the same way as in the proof of Proposition 3.11, using Theorem 3.8.

Suppose that $b = \frac{3}{2}$ and let us denote by τ an irreducible tempered but not square-integrable subquotient of $\langle\Delta\rangle \rtimes \sigma$. Then τ is a subrepresentation of $\delta([\nu^{-\frac{3}{2}}\rho, \nu^{\frac{3}{2}}\rho]) \rtimes \sigma_1$, for a discrete series σ_1. Using Frobenius reciprocity and the structural formula we obtain that $\mu^*(\sigma) \geq \delta([\nu^{-\frac{1}{2}}\rho, \nu^{\frac{3}{2}}\rho]) \otimes \sigma_2$, where σ_2 is a discrete series such that $\mathrm{Jord}(\sigma) = \mathrm{Jord}(\sigma_2) \cup \{(2,\rho),(4,\rho)\}$ and σ_1 is a subquotient of $\nu^{\frac{1}{2}}\rho \rtimes \sigma_2$.

It follows that τ is a subrepresentation of $\nu^{\frac{1}{2}}\rho \times \delta([\nu^{-\frac{3}{2}}\rho, \nu^{\frac{3}{2}}\rho]) \rtimes \sigma_2$, and there is an irreducible subquotient τ_1 of $\delta([\nu^{-\frac{3}{2}}\rho, \nu^{\frac{3}{2}}\rho]) \rtimes \sigma_2$ such that τ is a subrepresentation of $\nu^{\frac{1}{2}}\rho \rtimes \tau_1$. Since $\mu^*(\tau)$ does not contain an irreducible constituent of the form $\nu^{\frac{3}{2}}\rho \times \nu^{\frac{3}{2}}\rho \otimes \pi$, using [24, Lemma 4.4] we deduce that τ_1 is a unique irreducible tempered subrepresentation of $\delta([\nu^{-\frac{3}{2}}\rho, \nu^{\frac{3}{2}}\rho]) \rtimes \sigma_2$ which does not contain $\delta([\nu^{\frac{1}{2}}\rho, \nu^{\frac{3}{2}}\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^{\frac{3}{2}}\rho]) \otimes \sigma_2$ in the Jacquet module with respect to the appropriate parabolic subgroup.

Since $(2,\rho) \notin \mathrm{Jord}(\sigma_2)$, it follows in the same way as in the proof of Proposition 3.11 that τ_1 is a subrepresentation of $\nu^{\frac{3}{2}}\rho \rtimes \sigma_3$, for some discrete series subrepresentation σ_3 of $\delta([\nu^{-\frac{1}{2}}\rho, \nu^{\frac{3}{2}}\rho]) \rtimes \sigma_2$. From the classification of discrete series follows that σ_3 is a subrepresentation of $\nu^{\frac{3}{2}}\rho \rtimes \tau_2$, for an irreducible tempered subrepresentation τ_2 of $\delta([\nu^{-\frac{1}{2}}\rho, \nu^{\frac{1}{2}}\rho]) \rtimes \sigma_2$. Using $\mu^*(\tau_1) \not\geq \delta([\nu^{\frac{1}{2}}\rho, \nu^{\frac{3}{2}}\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^{\frac{3}{2}}\rho]) \otimes \sigma_2$ we get that $\mu^*(\tau_2) \not\geq \nu^{\frac{1}{2}}\rho \times \nu^{\frac{1}{2}}\rho \otimes \sigma_2$.

Let us prove that $\epsilon_{\sigma_3}(2,\rho) = -1$ (we note that this also follows from the proof of [8, Lemma 4.9], but we include a proof here, for the sake of completeness). Suppose that $\epsilon_{\sigma_3}(2,\rho) = 1$. Then there is an irreducible representation $\pi \in R(G)$ such that $\mu^*(\sigma_3) \geq \nu^{\frac{1}{2}}\rho \otimes \pi$. The square-integrability criterion implies that π is a discrete series. Using $(2,\rho) \notin \mathrm{Jord}(\sigma_2)$ again, we deduce that $\pi \leq \nu^{\frac{3}{2}}\rho \rtimes \pi'$ for an irreducible subquotient π' of $\nu^{\frac{1}{2}}\rho \rtimes \sigma_2$ such that $\mu^*(\tau_2) \geq \nu^{\frac{1}{2}}\rho \otimes \pi'$. It follows from [13, Lemma 8.3] that $\mu^*(\pi) \geq \nu^{\frac{3}{2}}\rho \otimes \pi''$ for some discrete series π''. Since $(4,\rho) \notin \mathrm{Jord}(\sigma_2)$ we obtain that $\pi' \cong \pi''$, i.e., π' is also a discrete series representation. By the second part of Lemma 3.6, π' is a subrepresentation of $\nu^{\frac{1}{2}}\rho \rtimes \sigma_2$. Thus, the Jacquet module of τ_2 with respect to an appropriate parabolic subgroup contains $\nu^{\frac{1}{2}}\rho \otimes \nu^{\frac{1}{2}}\rho \otimes \sigma_2$. Transitivity of Jacquet modules implies that there is an irreducible constituent $\pi_1 \otimes \sigma_2$ of $\mu^*(\tau_2)$ such that the Jacquet module of π_1 with respect to the appropriate parabolic subgroup contains $\nu^{\frac{1}{2}}\rho \otimes \nu^{\frac{1}{2}}\rho$. Since τ_2 is a subrepresentation of $\delta([\nu^{-\frac{1}{2}}\rho, \nu^{\frac{1}{2}}\rho]) \rtimes \sigma_2$ and $(2,\rho) \notin \mathrm{Jord}(\sigma_2)$, it directly follows that $\pi_1 \cong \nu^{\frac{1}{2}}\rho \times \nu^{\frac{1}{2}}\rho$, a contradiction.

Since $\tau \hookrightarrow \nu^{\frac{1}{2}}\rho \times \nu^{\frac{3}{2}}\rho \rtimes \sigma_3$, in the same way as in the proof of Proposition 3.11 we deduce $\tau \hookrightarrow \langle \Delta \rangle \rtimes \sigma_3$ and $\sigma \cong \sigma_3$, so the proposition is proved. □

From Proposition 3.14 and [12, Lemma 5.11] we obtain

Theorem 3.15. *Suppose that $a = \frac{1}{2}$ and $a < b$. Then $\langle \Delta \rangle \rtimes \sigma$ contains an irreducible tempered subquotient which is not square-integrable if and only if $x \in \mathrm{Jord}_\rho(\sigma)$ for all $x \in \{2, 4, \ldots, 2b+1\}$, $\epsilon_\sigma((x_-,\rho),(x,\rho)) = -1$ for all $x \in \{4, 6, \ldots, 2b-1\}$, $\epsilon_\sigma((2b-1,\rho),(2b+1,\rho)) = 1$ and $\epsilon_\sigma(2,\rho) = -1$. Furthermore, if $\langle \Delta \rangle \rtimes \sigma$ contains an irreducible tempered subquotient then it contains a unique irreducible tempered subquotient, which is a subrepresentation.*

Proposition 3.16. *If $a < 0$, then $\langle \Delta \rangle \rtimes \sigma$ does not contain an irreducible tempered subquotient.*

Proof. Suppose, on the contrary, that there is an irreducible tempered subquotient τ of $\langle \Delta \rangle \rtimes \sigma$. By Theorem 3.9, τ is not a discrete series representation. Two possibilities will be studied separately.

Let us first assume that τ is a subrepresentation of an induced representation of the form $\delta([\nu^{-x}\rho_1, \nu^x\rho_1]) \times \delta([\nu^{-y}\rho_2, \nu^y\rho_2]) \rtimes \sigma'$, for a discrete series σ'. From the cuspidal support of $\langle \Delta \rangle \rtimes \sigma$ we deduce that $\rho_1 \cong \rho_2 \cong \rho$, $\{x, y\} = \{-a, b\}$, $x \neq y$, and $-2a+1, 2b+1 \in \mathrm{Jord}_\rho(\sigma)$. Then there is an irreducible tempered subrepresentation τ' of $\delta([\nu^{-b}\rho, \nu^b\rho]) \rtimes \sigma'$ such that $\mu^*(\tau) \geq \delta([\nu^a\rho, \nu^{-a}\rho]) \otimes \tau'$. From $\mu^*(\langle\Delta\rangle \rtimes \sigma)$ we deduce that $\mu^*(\sigma) \geq \delta([\nu^{a+1}\rho, \nu^{-a}\rho]) \otimes \sigma_1$, for a discrete series σ_1 such that $-2a+1 \notin \mathrm{Jord}_\rho(\sigma_1)$ and $\langle[\nu^{a+1}\rho, \nu^b\rho]\rangle \rtimes \sigma_1 \geq \tau'$. From Theorems 3.9 and 3.15 we get that $\langle[\nu^{a+1}\rho, \nu^b\rho]\rangle \rtimes \sigma_1$ does not contain an irreducible tempered subquotient, a contradiction.

Inspecting the cuspidal support of $\langle\Delta\rangle \rtimes \sigma$ we obtain that the only other possibility is that τ is a subrepresentation of an induced representation of the form $\delta([\nu^{-x}\rho, \nu^x\rho]) \rtimes \sigma'$, where $x \in \{-a, b\}$ and σ' is a discrete series. Also, $2x+1 \in \mathrm{Jord}_\rho(\sigma)$, and if $-a < b$ then $2y+1 \notin \mathrm{Jord}_\rho(\sigma)$ for y such that $\{x, y\} = \{-a, b\}$.

If $x = -a < b$, $x = b > -a+1$ or $x = b = -a > \frac{1}{2}$, we get a contradiction in the same way as in the previously considered case.

If $x = b = -a = \frac{1}{2}$, it follows that τ is a subrepresentation of $\delta([\nu^{-\frac{1}{2}}\rho, \nu^{\frac{1}{2}}\rho]) \rtimes \sigma'$, for a discrete series σ' which is a subquotient of $\nu^{\frac{1}{2}}\rho \rtimes \sigma_1$, where σ_1 is a discrete series such that $\mu^*(\sigma) \geq \nu^{\frac{1}{2}}\rho \otimes \sigma_1$. From Lemma 3.6 we obtain an embedding $\tau \hookrightarrow \nu^{\frac{1}{2}}\rho \times \nu^{\frac{1}{2}}\rho \times \nu^{-\frac{1}{2}}\rho \rtimes \sigma_1$, which is impossible since $\mu^*(\tau)$ does not contain an irreducible constituent of the form $\nu^{\frac{1}{2}}\rho \times \nu^{\frac{1}{2}}\rho \otimes \pi$.

It remains to consider the case $x = b = -a+1$. First, assume that $-a > \frac{1}{2}$. Since $-2a+1 \notin \mathrm{Jord}_\rho(\sigma)$, using the structural formula we deduce that $\mu^*(\sigma) \geq \delta([\nu^{-b+2}\rho, \nu^b\rho]) \otimes \sigma_1$, in $\mathrm{Jord}_\rho(\sigma)$ we have $(2b+1)_- = -2a-1$ and σ_1 is a discrete series representation. Since $\sigma' \leq \langle[\nu^{a+1}\rho, \nu^{b-1}\rho]\rangle \rtimes \sigma_1$ and $a+1 < 0$, we get a contradiction with Theorem 3.9. Let us now assume that $-a = \frac{1}{2}$. Then $\min(\mathrm{Jord}_\rho(\sigma)) = 4$ and $\epsilon_\sigma(4, \rho) = 1$. Also, we easily see that σ' is a subquotient of $\nu^{\frac{1}{2}}\rho \rtimes \sigma_1$, for a discrete series σ_1 such that σ is a subrepresentation of $\delta([\nu^{-\frac{1}{2}}\rho, \nu^{\frac{3}{2}}\rho]) \rtimes \sigma_1$. By Lemma 3.6, σ' is a subrepresentation of $\nu^{\frac{1}{2}}\rho \rtimes \sigma_1$ and we have

$$\tau \hookrightarrow \delta([\nu^{-\frac{3}{2}}\rho, \nu^{\frac{3}{2}}\rho]) \times \nu^{\frac{1}{2}}\rho \rtimes \sigma_1 \cong \nu^{\frac{1}{2}}\rho \times \delta([\nu^{-\frac{3}{2}}\rho, \nu^{\frac{3}{2}}\rho]) \rtimes \sigma_1,$$

so $\mu^*(\tau)$ contains an irreducible constituent of the form $\nu^{\frac{1}{2}}\rho \otimes \pi$. But, since $2 \notin \mathrm{Jord}_\rho(\sigma)$, $\mu^*(\langle[\nu^{-\frac{1}{2}}\rho, \nu^{\frac{3}{2}}\rho]\rangle \rtimes \sigma)$ does not contain $\nu^{\frac{1}{2}}\rho \otimes \pi$, a contradiction. This ends the proof. □

4 Composition series in $\mathrm{Jord}_\rho(\sigma) \neq \emptyset$ case

In this section we again assume that $\Delta = [\nu^a\rho, \nu^b\rho]$, where $\rho \in R(GL)$ is an irreducible self-contragredient representation, a and b are half-integers such that $a+b \geq 0$, and $\sigma \in R(G)$ is a discrete series such that $\mathrm{Jord}_\rho(\sigma) \neq \emptyset$ and all elements of $\mathrm{Jord}_\rho(\sigma)$ are even integers.

We determine complete composition series of the induced representation $\langle \Delta \rangle \rtimes \sigma$. To obtain candidates for non-tempered irreducible subquotients, we need the following two results.

Proposition 4.1. *Suppose that $L(\delta_1, \delta_2, \ldots, \delta_k, \tau)$ is an irreducible non-tempered subquotient of $\langle \Delta \rangle \rtimes \sigma$, and let $\delta_i \cong \delta([\nu^{a_i}\rho_i, \nu^{b_i}\rho_i])$ for $i = 1, 2, \ldots, k$. Then for all $i = 1, 2, \ldots, k$ we have $\rho_i \cong \rho$ and $b_i - a_i \in \{0, 1\}$.*

Proof. Suppose, on the contrary, that there is some $j \in \{1, 2, \ldots, k\}$ such that $b_j - a_j \geq 2$. Let us denote the minimal such j by $j_{\min}$. Using [12, Lemma 2.6] we deduce that there are c and d, $a \leq c \leq \frac{1}{2}$, $-\frac{1}{2} \leq d \leq b$, such that $L(\delta_{j_{\min}}, \delta_{j_{\min}+1}, \ldots, \delta_k, \tau)$ is an irreducible subquotient of $\langle [\nu^c\rho, \nu^d\rho] \rangle \rtimes \sigma$. Since $L(\delta_{j_{\min}}, \delta_{j_{\min}+1}, \ldots, \delta_k, \tau)$ is a subrepresentation of $\delta_{j_{\min}} \rtimes L(\delta_{j_{\min}+1}, \ldots, \delta_k, \tau)$, it follows that $\mu^*(\langle [\nu^c\rho, \nu^d\rho] \rangle \rtimes \sigma)$ contains $\delta_{j_{\min}} \otimes L(\delta_{j_{\min}+1}, \ldots, \delta_k, \tau)$.

Using the structural formula, we deduce that there are $0 \leq j \leq i \leq d - c + 1$ and an irreducible constituent $\pi \otimes \sigma'$ of $\mu^*(\sigma)$ such that

$$\delta([\nu^{a_{j_{\min}}}\rho_{j_{\min}}, \nu^{b_{j_{\min}}}\rho_{j_{\min}}]) \leq \langle [\nu^{-d}\rho, \nu^{-c-i}\rho] \rangle \times \langle [\nu^c\rho, \nu^{c+j-1}\rho] \rangle \times \pi,$$

and $L(\delta_{j_{\min}+1}, \ldots, \delta_k, \tau) \leq \langle [\nu^{c+j}\rho, \nu^{c+i-1}\rho] \rangle \rtimes \sigma'$. Since σ is square-integrable and $b_{j_{\min}} - a_{j_{\min}} \geq 2$, it follows that $\rho_{j_{\min}} \cong \rho$, $c = -d+1 \leq -\frac{1}{2}$, $i = d - c$ and $j = 1$. Thus, $a_{j_{\min}} = -d$, $b_{j_{\min}} = d-1$, and $\pi \cong \delta([\nu^{-d+2}\rho, \nu^{d-1}\rho])$. Note that this implies $(2d-1, \rho) \in \mathrm{Jord}(\sigma)$.

There are two possibilities to consider. Let us first assume that $d \geq \frac{5}{2}$. Then we have $-d+2 < 0$ and, by the classification of discrete series and [18, Theorem 2.3], σ' is a discrete series such that $(2d-1, \rho), (2d-3, \rho) \notin \mathrm{Jord}(\sigma')$. By Theorem 3.9 and Proposition 3.16, $j_{\min} < k$ and, since $(2d-3, \rho) \notin \mathrm{Jord}(\sigma')$, we have $\delta_{j_{\min}+1} \in \{\nu^{c+1}\rho, \nu^c\rho, \delta([\nu^c\rho, \nu^{c+1}\rho])\}$. From $e(\delta_{j_{\min}}) = -\frac{1}{2}$ and $e(\delta_{j_{\min}}) \leq e(\delta_{j_{\min}+1})$, we obtain that the only possibility is $c = -\frac{3}{2}$ and $\delta_{j_{\min}+1} \cong \nu^{-\frac{1}{2}}\rho$. This implies $\delta_{j_{\min}} \times \delta_{j_{\min}+1} \cong \delta_{j_{\min}+1} \times \delta_{j_{\min}}$, so $\mu^*(L(\delta_{j_{\min}}, \delta_{j_{\min}+1}, \ldots, \delta_k, \tau)) \geq \nu^{-\frac{1}{2}}\rho \otimes \pi'$, for some irreducible π'. But, it can be seen at once that $\mu^*(\langle [\nu^{-\frac{3}{2}}\rho, \nu^{\frac{5}{2}}\rho] \rangle \rtimes \sigma)$ does not contain $\nu^{-\frac{1}{2}}\rho \otimes \pi'$, a contradiction.

It remains to consider the case $d = \frac{3}{2}$. Then $\epsilon_\sigma(2, \rho) = 1$ and σ' is a discrete series such that $(2, \rho) \notin \mathrm{Jord}(\sigma')$. Also, $L(\delta_{j_{\min}+1}, \ldots, \delta_k, \tau) \leq \nu^{\frac{1}{2}}\rho \rtimes \sigma'$, and in the appropriate Grothendieck group we have $\nu^{\frac{1}{2}}\rho \rtimes \sigma' = \sigma + L(\nu^{-\frac{1}{2}}\rho, \sigma')$. If $L(\delta_{j_{\min}+1}, \ldots, \delta_k, \tau) \cong \sigma$, from $\sigma \hookrightarrow \nu^{\frac{1}{2}}\rho \rtimes \sigma'$ we get that

$L(\delta_{j_{\min}+1}, \dots, \delta_k, \tau)$ is a subrepresentation of $\nu^{\frac{1}{2}}\rho \times \nu^{\frac{1}{2}}\rho \times \delta([\nu^{-\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho]) \rtimes \sigma'$, which is impossible since $\mu^*(\langle[\nu^{-\frac{1}{2}}\rho, \nu^{\frac{3}{2}}\rho]\rangle \rtimes \sigma)$ does not contain an irreducible constituent of the form $\nu^{\frac{1}{2}}\rho \times \nu^{\frac{1}{2}}\rho \otimes \pi'$. On the other hand, if $L(\delta_{j_{\min}+1}, \dots, \delta_k, \tau) \cong L(\nu^{-\frac{1}{2}}\rho, \sigma')$, a similar commutativity argument shows that $L(\delta_{j_{\min}+1}, \dots, \delta_k, \tau)$ is a subrepresentation of $\nu^{-\frac{1}{2}}\rho \times \nu^{\frac{1}{2}}\rho \times \nu^{-\frac{1}{2}}\rho \times \nu^{-\frac{3}{2}}\rho \rtimes \sigma'$. But, it can be easily seen that $\nu^{-\frac{1}{2}}\rho \otimes \nu^{\frac{1}{2}}\rho \otimes \nu^{-\frac{1}{2}}\rho \otimes \nu^{-\frac{3}{2}}\rho \otimes \sigma'$ is not contained in the Jacquet module of $\langle[\nu^{-\frac{1}{2}}\rho, \nu^{\frac{3}{2}}\rho]\rangle \rtimes \sigma$ with respect to the appropriate parabolic subgroup, a contradiction.

Consequently, $b_i - a_i \in \{0, 1\}$ for all $i = 1, 2, \dots, k$ and [12, Lemma 2.6] implies that $\rho_i \cong \rho$ for all i. $\square$

Proposition 4.2. *Suppose that $L(\delta_1, \delta_2, \dots, \delta_k, \tau)$ is an irreducible non-tempered subquotient of $\langle\Delta\rangle \rtimes \sigma$, and let $\delta_i \cong \delta([\nu^{a_i}\rho, \nu^{b_i}\rho])$ for $i = 1, 2, \dots, k$. Suppose that there is a $j \in \{1, 2, \dots, k\}$ such that $b_j - a_j = 1$, and let $l = a+b$. Then the following holds:*

(1) $a < 0$ and $-a < b$,

(2) $a_l = b_l - 1 = a - 1$ and for $i < l$ we have $a_i = b_i = -b + i - 1$,

(3) there is a positive half-integer x, $x \leq -a$, such that $\langle[\nu^{\frac{1}{2}}\rho, \nu^{x}\rho]\rangle \rtimes \sigma$ contains an irreducible tempered subquotient,

(4) if for $j' \in \{j+1, j+2, \dots, k\}$ we have $b_{j'} - a_{j'} = 1$, then $b_i - a_i = 1$ for all $i \in \{j+1, j+2, \dots, j'-1\}$.

Proof. Let us denote by $j_{\min}$ the minimal j such that $b_j - a_j = 1$. It follows directly from [12, Lemma 2.6] that $a < 0$. Suppose that $-a = b$. Using [12, Lemma 2.6] again, we obtain that $j_{\min} \geq 2$, $L(\delta_{j_{\min}-1}, \delta_{j_{\min}}, \dots, \delta_k, \tau)$ is an irreducible subquotient of $\langle[\nu^{-c}\rho, \nu^{c}\rho]\rangle \rtimes \sigma$, for some c such that $\frac{3}{2} \leq c \leq b$, and $L(\delta_{j_{\min}}, \delta_{j_{\min}+1}, \dots, \delta_k, \tau)$ is an irreducible subquotient of $\langle[\nu^{-c+1}\rho, \nu^{c}\rho]\rangle \rtimes \sigma$.

It follows that $\delta_{j_{\min}-1} \cong \nu^{-c}\rho$ and $\delta_{j_{\min}} \cong \delta([\nu^{-c}\rho, \nu^{-c+1}\rho])$. Now a simple commutativity argument, together with the Frobenius reciprocity, implies that $\mu^*(L(\delta_{j_{\min}-1}, \delta_{j_{\min}}, \dots, \delta_k, \tau)) \geq \nu^{-c+1}\rho \otimes \pi$, for some irreducible π. It follows from the structural formula and the square-integrability criterion that $\mu^*(\langle[\nu^{-c}\rho, \nu^{c}\rho]\rangle \rtimes \sigma)$ does not contain an irreducible constituent of the form $\nu^{-c+1}\rho \otimes \pi$, a contradiction. Thus, $-a < b$.

If for all $i \in \{j_{\min}, j_{\min}+1, \ldots, k\}$ we have $b_i - a_i = 1$, then a repeated application of [12, Lemma 2.6] shows that τ is an irreducible subquotient of $\nu^{\frac{1}{2}}\rho \rtimes \sigma$.

Suppose that there is an $i \in \{j_{\min}+1, \ldots, k\}$ such that $b_i - a_i = 0$, and let us denote the minimal such i by $i_{\min}$. Again, then there is some c, $\frac{3}{2} \leq c \leq b$, such that $L(\delta_{i_{\min}-1}, \delta_{i_{\min}}, \ldots, \delta_k, \tau)$ is an irreducible subquotient of $\langle[\nu^{-c+1}\rho, \nu^{c}\rho]\rangle \rtimes \sigma$, $\delta_{i_{\min}-1} \cong \delta([\nu^{-c}\rho, \nu^{-c+1}\rho])$ and $L(\delta_{i_{\min}}, \ldots, \delta_k, \tau)$ is an irreducible subquotient of $\langle[\nu^{-c+2}\rho, \nu^{c-1}\rho]\rangle \rtimes \sigma$. Note that we also have $c \leq -a+1$. It follows that $b_{i_{\min}} \in \{-c+2, -c+1\}$. If $b_{i_{\min}} = -c+1$, a simple commutativity argument and Frobenius reciprocity imply that $\mu^*(L(\delta_{i_{\min}-1}, \delta_{i_{\min}}, \ldots, \delta_k, \tau)) \geq \nu^{-c+1}\rho \times \nu^{-c+1}\rho \otimes \pi$, which is impossible since $\mu^*(\langle[\nu^{-c+1}\rho, \nu^{c}\rho]\rangle \rtimes \sigma)$ does not contain an irreducible constituent of the form $\nu^{-c+1}\rho \times \nu^{-c+1}\rho \otimes \pi$. Thus, $b_{i_{\min}} = -c+2$ and $L(\delta_{i_{\min}+1}, \ldots, \delta_k, \tau)$ is an irreducible subquotient of $\langle[\nu^{-c+3}\rho, \nu^{c-1}\rho]\rangle \rtimes \sigma$. Since for $i < j$ we have $e(\delta_i) \leq e(\delta_j)$, a repeated application of [12, Lemma 2.6] shows that for $i \in \{i_{\min}, i_{\min}+1, \ldots, k\}$ we have $a_i = b_i$ and that τ is an irreducible subquotient of $\langle[\nu^{\frac{1}{2}}\rho, \nu^{c-1}\rho]\rangle \rtimes \sigma$.

If $b > -a+1$, then we have $a_1 = b_1$ and $a_1 \in \{-b, a\}$. If $a_1 = a$, then $L(\delta_2, \delta_3, \ldots, \delta_k, \tau)$ is an irreducible subquotient of $\langle[\nu^{a+1}\rho, \nu^{b}\rho]\rangle \rtimes \sigma$, and using $e(\delta_1) \leq e(\delta_2)$ we deduce that $a_2 = b_2 = a+1$. Repeating the same arguments, we obtain that $a_i = b_i$ for all $i = 1, 2, \ldots, k$, a contradiction. Thus, if $b > -a+1$ we have $a_1 = b_1 = -b$.

Proceeding in the same way, we get that $a_i = b_i = -b+i-1$ for $i = 1, 2, \ldots, l-1$, and $L(\delta_l, \delta_{l+1}, \ldots, \delta_k, \tau)$ is an irreducible subquotient of $\langle[\nu^{a}\rho, \nu^{-a+1}\rho]\rangle \rtimes \sigma$. Suppose that $l < j_{\min}$. Then $a_l = b_l$ and $a_l \in \{a, a-1\}$. If $a_l = a$, in the same way as before we conclude that $a_i = b_i$ for all $i = 1, 2, \ldots, k$, a contradiction. If $a_l = a-1$, it follows that $L(\delta_{l+1}, \ldots, \delta_k, \tau)$ is an irreducible subquotient of $\langle[\nu^{a}\rho, \nu^{-a}\rho]\rangle \rtimes \sigma$, where $l+1 \leq j_{\min}$, and we have already seen that this is impossible. Consequently, $l = j_{min}$ and $a_l = b_l - 1 = a - 1$. This finishes the proof. $\square$

The following proposition will be useful when proving that an irreducible representation of certain type is a subquotient of $\langle\Delta\rangle \rtimes \sigma$.

Proposition 4.3. *Let c denote a positive half-integer, and suppose that $c \geq \frac{3}{2}$. Suppose that $L(\delta_1, \delta_2, \ldots, \delta_k, \tau)$ is an irreducible subquotient of $\langle[\nu^{-c+1}\rho, \nu^{c}\rho]\rangle \rtimes \sigma_{ds}$, for some discrete series σ_{ds}, and $\delta_i \cong \delta([\nu^{a_i}\rho, \nu^{-c+i}\rho])$, $a_i \in \{-c+i-1, -c+i\}$, for $i = 1, 2, \ldots, k$. Then $L(\delta([\nu^{-c-1}\rho, \nu^{-c}\rho]), \delta_1, \ldots, \delta_k, \tau)$ is an irreducible subquotient of $\langle[\nu^{-c}\rho, \nu^{c+1}\rho]\rangle \rtimes \sigma_{ds}$.*

Proof. Since $L(\delta([\nu^{-c-1}\rho,\nu^{-c}\rho]),\delta_1,\ldots,\delta_k,\tau)$ is a subrepresentation of $\nu^{-c}\rho\times\nu^{-c-1}\rho\rtimes L(\delta_1,\ldots,\delta_k,\tau)$, from Lemma 2.2 follows that there is an irreducible subquotient π of $\nu^{-c-1}\rho\rtimes L(\delta_1,\ldots,\delta_k,\tau)$ such that $L(\delta([\nu^{-c-1}\rho,\nu^{-c}\rho]),\delta_1,\ldots,\delta_k,\tau)$ is a subrepresentation of $\nu^{-c}\rho\rtimes\pi$.

The Jacquet module of $L(\delta([\nu^{-c-1}\rho,\nu^{-c}\rho]),\delta_1,\ldots,\delta_k,\tau)$ with respect to the appropriate parabolic subgroup contains $\nu^{-c}\rho\otimes\nu^{-c-1}\rho\otimes L(\delta_1,\ldots,\delta_k,\tau)$. Since τ is tempered and $\delta_i\cong\delta([\nu^{a_i}\rho,\nu^{-c+i}\rho])$, it follows directly from Theorem 2.1 that $\mu^*(\pi)$ does not contain an irreducible constituent of the form $\nu^{-c}\rho\otimes\pi'$. Thus, we obtain that $\mu^*(\pi)$ contains $\nu^{-c-1}\rho\otimes L(\delta_1,\ldots,\delta_k,\tau)$. Since $L(\nu^{-c-1}\rho,\delta_1,\ldots,\delta_k,\tau)$ is a subrepresentation of $\nu^{-c-1}\rho\rtimes L(\delta_1,\ldots,\delta_k,\tau)$ and it can be easily seen that $\mu^*(\nu^{-c-1}\rho\rtimes L(\delta_1,\ldots,\delta_k,\tau))$ contains $\nu^{-c-1}\rho\otimes L(\delta_1,\ldots,\delta_k,\tau)$ with multiplicity one, it follows that $\pi\cong L(\nu^{-c-1}\rho,\delta_1,\ldots,\delta_k,\tau)$.

Also, $\pi\leq\nu^{-c-1}\rho\times\langle[\nu^{-c+1}\rho,\nu^{c}\rho]\rangle\rtimes\sigma_{ds}$. In $R(G)$ we have

$$\begin{aligned}\nu^{-c-1}\rho\times\langle[\nu^{-c+1}\rho,\nu^{c}\rho]\rangle\rtimes\sigma_{ds}&=\langle[\nu^{-c+1}\rho,\nu^{c}\rho]\rangle\times\nu^{c+1}\rho\rtimes\sigma_{ds}\\&=\langle[\nu^{-c+1}\rho,\nu^{c+1}\rho]\rangle\rtimes\sigma_{ds}+\\&+L(\nu^{-c+1}\rho,\ldots,\nu^{c-1}\rho,\delta([\nu^{c}\rho,\nu^{c+1}\rho]))\rtimes\sigma_{ds}.\end{aligned}$$

From $\mu^*(\pi)\geq\nu^{-c-1}\rho\otimes L(\delta_1,\ldots,\delta_k,\tau)$, we conclude that π is a subquotient of $\langle[\nu^{-c+1}\rho,\nu^{c+1}\rho]\rangle\rtimes\sigma_{ds}$.

Consequently, $L(\delta([\nu^{-c-1}\rho,\nu^{-c}\rho]),\delta_1,\ldots,\delta_k,\tau)$ is a subquotient of $\nu^{-c}\rho\times\langle[\nu^{-c+1}\rho,\nu^{c+1}\rho]\rangle\rtimes\sigma_{ds}$ and, since $\mu^*(L(\delta([\nu^{-c-1}\rho,\nu^{-c}\rho]),\delta_1,\ldots,\delta_k,\tau))\geq\nu^{-c}\rho\otimes\pi$ in the same way as before we deduce that $L(\delta([\nu^{-c-1}\rho,\nu^{-c}\rho]),\delta_1,\ldots,\delta_k,\tau)$ is a subquotient of $\langle[\nu^{-c}\rho,\nu^{c+1}\rho]\rangle\rtimes\sigma_{ds}$. □

We are now ready to provide a description of the composition series, using a case-by-case consideration.

Theorem 4.4. *Suppose that $a\geq\frac{3}{2}$. Let us denote by x_1 the minimal positive even integer such that $x_1\geq 2a+1$ and $x_1\notin Jord_\rho(\sigma)$. Also, let us denote by x_2 the minimal $x\in Jord_\rho(\sigma)$ such that $x\geq 2a+1$ and $\epsilon_\sigma((x_-,\rho),(x,\rho))=1$, if such x exists. Otherwise, let $x_2=x_1+1$. Let $x_{\min}$ denote $\min\{\frac{x_1-1}{2},\frac{x_2-1}{2}\}$, and let τ denote the unique irreducible tempered subquotient of $\langle[\nu^{a}\rho,\nu^{x_{\min}}\rho]\rangle\rtimes\sigma$.*

(1) If $2a-1\notin Jord_\rho(\sigma)$ or $2a-1\in Jord_\rho(\sigma)$ and $x_{\min}>b$, then we have

$$\langle\Delta\rangle\rtimes\sigma=L(\nu^{-b}\rho,\nu^{-b+1}\rho,\ldots,\nu^{-a}\rho,\sigma).$$

(2) If $2a-1 \in Jord_\rho(\sigma)$ and $x_{\min} \le b$, then in $R(G)$ we have

$$\langle \Delta \rangle \rtimes \sigma = L(\nu^{-b}\rho, \nu^{-b+1}\rho, \ldots, \nu^{-a}\rho, \sigma) + L(\nu^{-b}\rho, \nu^{-b+1}\rho, \ldots, \nu^{-x_{\min}-1}\rho, \tau).$$

Proof. First part of the theorem follows from [12, Theorem 4.8].

For the second part, from the proof of [12, Proposition 4.7] follows that both $L(\nu^{-b}\rho, \nu^{-b+1}\rho, \ldots, \nu^{-a}\rho, \sigma)$ and $L(\nu^{-b}\rho, \nu^{-b+1}\rho, \ldots, \nu^{-x_{\min}-1}\rho, \tau)$ are subquotients of $\langle \Delta \rangle \rtimes \sigma$. Let us prove that there are no other irreducible subquotients.

By Theorems 3.4 and 3.12, any other irreducible subquotient has to be non-tempered. Let $L(\delta_1, \delta_2, \ldots, \delta_k, \tau')$ denote a non-tempered irreducible subquotient of $\langle \Delta \rangle \rtimes \sigma$, and $\delta_i \cong \delta([\nu^{a_i}\rho_i, \nu^{b_i}\rho_i])$ for $i = 1, 2, \ldots, k$. From Proposition 4.1 we get $a_i = b_i$ for all i. Now [12, Lemma 2.6] implies that τ' is a subquotient of $\langle [\nu^a\rho, \nu^c\rho] \rangle \rtimes \sigma$ for some $c \le b$ and $\delta_i \cong \nu^{-b+i-1}\rho$ for $i = 1, 2, \ldots, b-c$. Theorems 3.4 and 3.12 imply $c \in \{a-1, x_{min}\}$. Thus, every irreducible subquotient of $\langle \Delta \rangle \rtimes \sigma$ is isomorphic either to $L(\nu^{-b}\rho, \nu^{-b+1}\rho, \ldots, \nu^{-a}\rho, \sigma)$ or to $L(\nu^{-b}\rho, \nu^{-b+1}\rho, \ldots, \nu^{-x_{\min}-1}\rho, \tau)$.

It is easy to see that both $\nu^{-b}\rho \otimes \nu^{-b+1}\rho \otimes \cdots \otimes \nu^{-a}\rho \otimes \sigma$ and $\nu^{-b}\rho \otimes \nu^{-b+1}\rho \otimes \cdots \otimes \nu^{-x_{\min}-1}\rho \otimes \tau$ appear with multiplicity one in the Jacquet module of $\langle \Delta \rangle \rtimes \sigma$ with respect to the appropriate parabolic subgroup. Consequently, both $L(\nu^{-b}\rho, \nu^{-b+1}\rho, \ldots, \nu^{-a}\rho, \sigma)$ and $L(\nu^{-b}\rho, \nu^{-b+1}\rho, \ldots, \nu^{-x_{\min}-1}\rho, \tau)$ appear in the composition series of $\langle \Delta \rangle \rtimes \sigma$ with multiplicity one and the theorem is proved. □

The following theorem can be proved following the lines of the proof of the previous one, so we skip the proof.

Theorem 4.5. *Suppose that $a = \frac{1}{2}$.*

(1) If $2 \notin Jord_\rho(\sigma)$ or $\epsilon_\sigma(2, \rho) = 1$, in $R(G)$ we have

$$\langle \Delta \rangle \rtimes \sigma = L(\nu^{-b}\rho, \nu^{-b+1}\rho, \ldots, \nu^{-\frac{1}{2}}\rho, \sigma) + L(\nu^{-b}\rho, \nu^{-b+1}\rho, \ldots, \nu^{-\frac{3}{2}}\rho, \tau),$$

where τ is the unique irreducible tempered subquotient of $\nu^{\frac{1}{2}}\rho \rtimes \sigma$.

(2) If $\epsilon_\sigma(2, \rho) = -1$, let us denote by x_1 the minimal positive even integer such that $x_1 \notin Jord_\rho(\sigma)$, and by x_2 the minimal $x \in Jord_\rho(\sigma)$ such that $\epsilon_\sigma((x_-, \rho), (x, \rho)) = 1$, if such x exists. Otherwise, let $x_2 = x_1 + 1$. Let $x_{\min}$ denote $\min\{\frac{x_1-1}{2}, \frac{x_2-1}{2}\}$, and let τ denote the unique irreducible tempered subquotient of $\langle [\nu^{\frac{1}{2}}\rho, \nu^{x_{\min}}\rho] \rangle \rtimes \sigma$.

(i) *If* $x_{\min} > b$*, we have* $\langle\Delta\rangle \rtimes \sigma = L(\nu^{-b}\rho, \nu^{-b+1}\rho, \ldots, \nu^{-\frac{1}{2}}\rho, \sigma)$.

(ii) *If* $x_{\min} \le b$*, then in* $R(G)$ *we have*

$$\langle\Delta\rangle \rtimes \sigma = L(\nu^{-b}\rho, \nu^{-b+1}\rho, \ldots, \nu^{-\frac{1}{2}}\rho, \sigma) + \\ + L(\nu^{-b}\rho, \nu^{-b+1}\rho, \ldots, \nu^{-x_{\min}-1}\rho, \tau).$$

Let us now complete our description in the $\mathrm{Jord}_\rho(\sigma) \neq \emptyset$ case.

Theorem 4.6. *Suppose that* $a < 0$.

(1) *Suppose that* $2 \notin \mathit{Jord}_\rho(\sigma)$ *or* $\epsilon_\sigma(2,\rho) = 1$*. Let* τ *denote the unique irreducible tempered subquotient of* $\nu^{\frac{1}{2}}\rho \rtimes \sigma$*. If* $-a = b$*, in* $R(G)$ *we have*

$$\langle\Delta\rangle \rtimes \sigma = L(\nu^{a}\rho, \nu^{a}\rho, \ldots, \nu^{-\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho, \sigma) + \\ + L(\nu^{a}\rho, \nu^{a}\rho, \ldots, \nu^{-\frac{3}{2}}\rho, \nu^{-\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho, \tau).$$

If $-a < b$*, in* $R(G)$ *we have*

$$\langle\Delta\rangle \rtimes \sigma = L(\nu^{-b}\rho, \ldots, \nu^{a-1}\rho, \nu^{a}\rho, \nu^{a}\rho, \ldots, \nu^{-\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho, \sigma) + \\ + L(\nu^{-b}\rho, \ldots, \nu^{a-1}\rho, \nu^{a}\rho, \nu^{a}\rho, \ldots, \nu^{-\frac{3}{2}}\rho, \nu^{-\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho, \tau) + \\ + L(\nu^{-b}\rho, \ldots, \nu^{a-2}\rho, \delta([\nu^{a-1}\rho, \nu^{a}\rho]), \ldots, \delta([\nu^{-\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho]), \tau).$$

(2) *If* $\epsilon_\sigma(2,\rho) = -1$*, let us denote by* x_1 *the minimal positive even integer such that* $x_1 \notin \mathit{Jord}_\rho(\sigma)$*, and by* x_2 *the minimal* $x \in \mathit{Jord}_\rho(\sigma)$ *such that* $\epsilon_\sigma((x_-,\rho),(x,\rho)) = 1$*, if such* x *exists. Otherwise, let* $x_2 = x_1 + 1$*. Let* $x_{\min}$ *denote* $\min\{\frac{x_1-1}{2}, \frac{x_2-1}{2}\}$ *and let* τ *denote the unique irreducible tempered subquotient of* $\langle[\nu^{\frac{1}{2}}\rho, \nu^{x_{\min}}\rho]\rangle \rtimes \sigma$*. If* $x_{\min} > b$*, we have*

$$\langle\Delta\rangle \rtimes \sigma = L(\nu^{-b}\rho, \nu^{-b+1}\rho, \ldots, \nu^{a-1}\rho, \nu^{a}\rho, \nu^{a}\rho, \ldots, \nu^{-\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho, \sigma).$$

If $x_{\min} \le b$ *and* $x_{\min} > -a$*, in* $R(G)$ *we have*

$$\langle\Delta\rangle \rtimes \sigma = L(\nu^{-b}\rho, \nu^{-b+1}\rho, \ldots, \nu^{a-1}\rho, \nu^{a}\rho, \nu^{a}\rho, \ldots, \nu^{-\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho, \sigma) + \\ + L(\nu^{-b}\rho, \nu^{-b+1}\rho, \ldots, \nu^{-x_{\min}-1}\rho, \nu^{a}\rho, \ldots, \nu^{-\frac{1}{2}}\rho, \tau).$$

If $x_{\min} \le b$ *and* $-a = b$*, in* $R(G)$ *we have*

$$\langle\Delta\rangle \rtimes \sigma = L(\nu^{a}\rho, \nu^{a}\rho, \ldots, \nu^{-\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho, \sigma) + \\ + L(\nu^{a}\rho, \nu^{a}\rho, \ldots, \nu^{-x_{\min}-1}\rho, \nu^{-x_{\min}-1}\rho, \nu^{-x_{\min}}\rho, \ldots, \nu^{-\frac{1}{2}}\rho, \tau).$$

If $x_{\min} \leq -a$ and $-a < b$, in $R(G)$ we have

$$\begin{gathered}\langle \Delta \rangle \rtimes \sigma = L(\nu^{-b}\rho, \dots, \nu^{a-1}\rho, \nu^{a}\rho, \nu^{a}\rho, \dots, \nu^{-\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho, \sigma) + \\ + L(\nu^{-b}\rho, \nu^{-b+1}\rho, \dots, \nu^{a-1}\rho, \nu^{a}\rho, \nu^{a}\rho, \dots, \nu^{-x_{\min}-1}\rho, \nu^{-x_{\min}-1}\rho, \\ \nu^{-x_{\min}}\rho, \dots, \nu^{-\frac{1}{2}}\rho, \tau) + \\ + L(\nu^{-b}\rho, \dots, \nu^{a-2}\rho, \delta([\nu^{a-1}\rho, \nu^{a}\rho]), \dots, \delta([\nu^{-x_{\min}-1}\rho, \nu^{-x_{\min}}\rho]), \\ \nu^{-x_{\min}+1}\rho, \dots, \nu^{-\frac{1}{2}}\rho, \tau).\end{gathered}$$

Proof. We will comment only the second part of the theorem, since the first one can be proved in the same way but more easily. By Proposition 3.16, there are no irreducible tempered subquotients of $\langle \Delta \rangle \rtimes \sigma$.

If $x_{\min} > b$, the claim follows from [12, Theorem 5.13].

Now we assume that $x_{\min} \leq b$. By Proposition 3.16 there are no irreducible tempered subquotients. Suppose that $L(\delta_1, \delta_2, \dots, \delta_k, \tau')$ is an irreducible non-tempered subquotient of $\langle \Delta \rangle \rtimes \sigma$ such that $\delta_i \cong \nu^{a_i}\rho_i$, $a_i < 0$, for all $i = 1, 2, \dots, k$. By [12, Lemma 2.6], $\rho_i \cong \rho$ for all $i = 1, 2, \dots, k$ and there are c and d such that $a \leq c \leq \frac{1}{2}$, $c - 1 \leq d \leq b$ such that

$$\begin{gathered}L(\delta_1, \delta_2, \dots, \delta_k, \tau') \cong \\ L(\nu^{-b}\rho, \dots, \nu^{d-1}\rho, \nu^{-a}\rho, \nu^{-a+1}\rho, \dots, \nu^{c-1}\rho, \tau'),\end{gathered}$$

if $a \geq -d$, or

$$\begin{gathered}L(\delta_1, \delta_2, \dots, \delta_k, \tau') \cong \\ L(\nu^{-b}\rho, \dots, \nu^{a-1}\rho, \nu^{a}\rho, \nu^{a}\rho, \dots, \nu^{-d-1}\rho, \nu^{-d-1}\rho, \nu^{-d}\rho, \nu^{-d+1}\rho, \dots, \nu^{c-1}\rho, \tau'),\end{gathered}$$

if $a < -d$ (here we omit the part $\nu^{-b}\rho, \dots, \nu^{a-1}\rho$ if $-a = b$), and τ' is subquotient of $\langle [\nu^{c}\rho, \nu^{d}\rho] \rangle \rtimes \sigma$. Results obtained in the previous section imply that $(c, d) \in \{(\frac{1}{2}, -\frac{1}{2}), (\frac{1}{2}, x_{\min})\}$. Consequently, if $b \geq x_{\min} > -a$ we have

$$\begin{gathered}L(\delta_1, \delta_2, \dots, \delta_k, \tau') \in \{L(\nu^{-b}\rho, \nu^{-b+1}\rho, \dots, \nu^{a-1}\rho, \nu^{a}\rho, \nu^{a}\rho, \dots, \nu^{-\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho, \sigma), \\ L(\nu^{-b}\rho, \nu^{-b+1}\rho, \dots, \nu^{-x_{\min}-1}\rho, \nu^{a}\rho, \nu^{a+1}\rho, \dots, \nu^{-\frac{1}{2}}\rho, \tau)\},\end{gathered}$$

if $x_{\min} \leq -a = b$ we have

$$\begin{gathered}L(\delta_1, \delta_2, \dots, \delta_k, \tau') \in \{L(\nu^{a}\rho, \nu^{a}\rho, \dots, \nu^{-\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho, \sigma), \\ L(\nu^{a}\rho, \nu^{a}\rho, \dots, \nu^{-x_{\min}-1}\rho, \nu^{-x_{\min}-1}\rho, \nu^{-x_{\min}}\rho, \dots, \nu^{-\frac{1}{2}}\rho, \tau)\},\end{gathered}$$

and if $x_{\min} \leq -a < b$ we have

$$L(\delta_1, \delta_2, \ldots, \delta_k, \tau') \in \{L(\nu^{-b}\rho, \ldots, \nu^{a-1}\rho, \nu^{a}\rho, \nu^{a}\rho, \ldots, \nu^{-\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho, \sigma),$$
$$L(\nu^{-b}\rho, \ldots, \nu^{a-1}\rho, \nu^{a}\rho, \nu^{a}\rho, \ldots, \nu^{-x_{\min}-1}\rho, \nu^{-x_{\min}-1}\rho, \nu^{-x_{\min}}\rho, \ldots, \nu^{-\frac{1}{2}}\rho, \tau)\}.$$

If $x_{\min} > -a$, it follows from [12, Proposition 5.12] that both

$$L(\nu^{-b}\rho, \ldots, \nu^{a-1}\rho, \nu^{a}\rho, \nu^{a}\rho, \ldots, \nu^{-\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho, \sigma)$$

and

$$L(\nu^{-b}\rho, \nu^{-b+1}\rho, \ldots, \nu^{-x_{\min}-1}\rho, \nu^{a}\rho, \ldots, \nu^{-\frac{1}{2}}\rho, \tau)$$

are irreducible subquotients of $\langle\Delta\rangle \rtimes \sigma$.

If $x_{\min} \leq -a = b$, it follows from [12, Proposition 5.12] that both

$$L(\nu^{a}\rho, \nu^{a}\rho, \ldots, \nu^{-\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho, \sigma)$$

and

$$L(\nu^{a}\rho, \nu^{a}\rho, \ldots, \nu^{-x_{\min}-1}\rho, \nu^{-x_{\min}-1}\rho, \nu^{-x_{\min}}\rho, \ldots, \nu^{-\frac{1}{2}}\rho, \tau)$$

are irreducible subquotients of $\langle\Delta\rangle \rtimes \sigma$ (we note that in situations like this we omit the part $\nu^{a}\rho, \nu^{a}\rho, \ldots, \nu^{-x_{\min}-1}\rho, \nu^{-x_{\min}-1}\rho$ if $x_{\min} = a$).

If $x_{\min} \leq -a$ and $-a < b$, it follows from [12, Proposition 5.12] that both

$$L(\nu^{-b}\rho, \ldots, \nu^{a-1}\rho, \nu^{a}\rho, \nu^{a}\rho, \ldots, \nu^{-\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho, \sigma)$$

and

$$L(\nu^{-b}\rho, \ldots, \nu^{a-1}\rho, \nu^{a}\rho, \nu^{a}\rho, \ldots, \nu^{-x_{\min}-1}\rho, \nu^{-x_{\min}-1}\rho, \nu^{-x_{\min}}\rho, \ldots, \nu^{-\frac{1}{2}}\rho, \tau)$$

are irreducible subquotients of $\langle\Delta\rangle \rtimes \sigma$.

Let us first consider the case $x_{\min} > -a$. It can be easily seen, by a repeated application of the structural formula, that

$$\nu^{-b}\rho \otimes \nu^{-b+1}\rho \otimes \cdots \otimes \nu^{a-1}\rho \otimes \nu^{a}\rho \times \nu^{a}\rho \otimes \cdots \otimes \nu^{-\frac{1}{2}}\rho \times \nu^{-\frac{1}{2}}\rho \otimes \sigma$$

appears with multiplicity one in the Jacquet module of $\langle\Delta\rangle \rtimes \sigma$ with respect to the appropriate parabolic subgroup. Thus, $L(\nu^{-b}\rho, \nu^{-b+1}\rho, \ldots, \nu^{a-1}\rho, \nu^{a}\rho, \nu^{a}\rho, \ldots, \nu^{-\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho, \sigma)$ appears in the composition series of $\langle\Delta\rangle \rtimes \sigma$ with multiplicity one. In a similar way we conclude that the multiplicity of

$$\nu^{-b}\rho \otimes \nu^{-b+1}\rho \otimes \cdots \otimes \nu^{x_{\min}-1}\rho \otimes \nu^{a}\rho \otimes \cdots \otimes \nu^{-\frac{1}{2}}\rho \otimes \tau$$

equals the multiplicity of τ in the composition series of $\langle[\nu^{\frac{1}{2}}\rho, \nu^{x_{\min}}\rho]\rangle \rtimes \sigma$. Theorem 3.15 implies that $L(\nu^{-b}\rho, \nu^{-b+1}\rho, \ldots, \nu^{-x_{\min}-1}\rho, \nu^{a}\rho, \ldots, \nu^{-\frac{1}{2}}\rho, \tau)$ also appears in the composition series of $\langle\Delta\rangle \rtimes \sigma$ with multiplicity one. By Propositions 4.1 and 4.2, this ends the proof in the considered case.

Now we turn our attention to the case $x_{\min} \leq -a$. In the same way as in the previously considered case we conclude that if $b = -a$ then $L(\nu^{a}\rho, \nu^{a}\rho, \ldots, \nu^{-\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho, \sigma)$ appears in the composition series of $\langle\Delta\rangle \rtimes \sigma$ with multiplicity one, and if $-a < b$ then $L(\nu^{-b}\rho, \ldots, \nu^{a-1}\rho, \nu^{a}\rho, \nu^{a}\rho, \ldots, \nu^{-\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho, \sigma)$ appears in the composition series of $\langle\Delta\rangle \rtimes \sigma$ with multiplicity one.

If $-a = b$, let $\pi' = L(\nu^{a}\rho, \nu^{a}\rho, \ldots, \nu^{-x_{\min}-1}\rho, \nu^{-x_{\min}-1}\rho, \nu^{-x_{\min}}\rho, \ldots, \nu^{-\frac{1}{2}}\rho, \tau)$, and if $-a < b$ let $\pi' = L(\nu^{-b}\rho, \ldots, \nu^{a-1}\rho, \nu^{a}\rho, \nu^{a}\rho, \ldots, \nu^{-x_{\min}-1}\rho, \nu^{-x_{\min}-1}\rho, \nu^{-x_{\min}}\rho, \ldots, \nu^{-\frac{1}{2}}\rho, \tau)$.

Let us now prove that π' also appears in the composition series of $\langle\Delta\rangle \rtimes \sigma$ with multiplicity one. If $-a = b$, we calculate the multiplicity of

$$\nu^{a}\rho \times \nu^{a}\rho \otimes \cdots \otimes \nu^{-x_{\min}-1}\rho \times \nu^{-x_{\min}-1}\rho \otimes L(\nu^{-x_{\min}}\rho, \ldots, \nu^{-\frac{1}{2}}\rho, \tau)$$

in the Jacquet module of $\langle\Delta\rangle \rtimes \sigma$ with respect to the appropriate parabolic subgroup, and if $-a < b$ we calculate the multiplicity of

$$\nu^{-b}\rho \otimes \cdots \otimes \nu^{a-1}\rho \otimes \nu^{a}\rho \times \nu^{a}\rho \otimes \cdots \otimes \nu^{-x_{\min}-1}\rho \times \nu^{-x_{\min}-1}\rho \otimes L(\nu^{-x_{\min}}\rho, \ldots, \nu^{-\frac{1}{2}}\rho, \tau)$$

in the Jacquet module of $\langle\Delta\rangle \rtimes \sigma$ with respect to the appropriate parabolic subgroup. Using the structural formula, in both cases we obtain that the desired multiplicity equals the multiplicity of $L(\nu^{-x_{\min}}\rho, \ldots, \nu^{-\frac{1}{2}}\rho, \tau)$ in the composition series of the induced representation $\langle[\nu^{-x_{\min}}\rho, \nu^{x_{\min}}\rho]\rangle \rtimes \sigma$.

We have already seen that both $L(\nu^{-x_{\min}}\rho, \nu^{-x_{\min}}\rho, \ldots, \nu^{-\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho, \sigma)$ and $L(\nu^{-x_{\min}}\rho, \ldots, \nu^{-\frac{1}{2}}\rho, \tau)$ are irreducible subquotients of $\langle[\nu^{-x_{\min}}\rho, \nu^{x_{\min}}\rho]\rangle \rtimes \sigma$. Since both representations $\langle[\nu^{-x_{\min}}\rho, \nu^{x_{\min}}\rho]\rangle$ and σ are unitarizable, both $L(\nu^{-x_{\min}}\rho, \nu^{-x_{\min}}\rho, \ldots, \nu^{-\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho, \sigma)$ and $L(\nu^{-x_{\min}}\rho, \ldots, \nu^{-\frac{1}{2}}\rho, \tau)$ have to be subrepresentations of $\langle[\nu^{-x_{\min}}\rho, \nu^{x_{\min}}\rho]\rangle \rtimes \sigma$.

If $\mu^{*}(\sigma)$ does not contain an irreducible constituent of the form $\nu^{x}\rho \otimes \pi$, for $\frac{1}{2} \leq x \leq x_{\min}$, directly from the structural formula we obtain that $\langle[\nu^{-x_{\min}}\rho, \nu^{x_{\min}}\rho]\rangle \otimes \sigma$ appears in $\mu^{*}(\langle[\nu^{-x_{\min}}\rho, \nu^{x_{\min}}\rho]\rangle \rtimes \sigma)$ with multiplicity two. Thus, $\langle[\nu^{-x_{\min}}\rho, \nu^{x_{\min}}\rho]\rangle \rtimes \sigma$ has at most two irreducible subrepresentations and in this case $L(\nu^{-x_{\min}}\rho, \ldots, \nu^{-\frac{1}{2}}\rho, \tau)$ appears in the composition series of $\langle[\nu^{-x_{\min}}\rho, \nu^{x_{\min}}\rho]\rangle \rtimes \sigma$ with multiplicity one.

If $\mu^{*}(\sigma)$ contains an irreducible constituent of the form $\nu^{x}\rho \otimes \pi$, from $\tau \leq \langle[\nu^{\frac{1}{2}}\rho, \nu^{x_{\min}}\rho]\rangle \rtimes \sigma$ and Theorems 3.8 and 3.15, we obtain that $\mu^{*}(\sigma) \geq$

$\nu^{x_{\min}}\rho \otimes \tau'$ and that $\mu^*(\sigma)$ does not contain an irreducible constituent of the form $\nu^y\rho \otimes \pi$ for $y < x_{\min}$. Moreover, from [13, Section 8] we conclude that τ' is tempered, $\nu^{x_{\min}}\rho \otimes \tau'$ is a unique irreducible constituent of the form $\nu^{x_{\min}}\rho \otimes \pi$ appearing in $\mu^*(\sigma)$ and it appears there with multiplicity one. Also, σ is a subrepresentation of $\nu^{x_{\min}}\rho \rtimes \tau'$ and we have $\langle[\nu^{-x_{\min}}\rho, \nu^{x_{\min}}\rho]\rangle \rtimes \sigma \hookrightarrow \langle[\nu^{-x_{\min}}\rho, \nu^{x_{\min}}\rho]\rangle \times \nu^{x_{\min}}\rho \rtimes \tau'$. Let us calculate the multiplicity of $\langle[\nu^{-x_{\min}}\rho, \nu^{x_{\min}}\rho]\rangle \times \nu^{x_{\min}}\rho \otimes \tau'$ in $\mu^*(\langle[\nu^{-x_{\min}}\rho, \nu^{x_{\min}}\rho]\rangle \rtimes \sigma)$. By Theorem 2.1, there are $0 \leq j \leq i \leq 2x_{\min} + 1$ and an irreducible constituent $\delta \otimes \pi$ of $\mu^*(\sigma)$ such that

$$\langle[\nu^{-x_{\min}}\rho, \nu^{x_{\min}}\rho]\rangle \times \nu^{x_{\min}}\rho \leq \langle[\nu^{-x_{\min}}\rho, \nu^{x_{\min}-i}\rho]\rangle \times \langle[\nu^{-x_{\min}}\rho, \nu^{j-x_{\min}-1}\rho]\rangle \times \delta.$$

It follows that either $i = 2x_{\min} + 1$ or $j = 0$.

Let us first assume that $i = 2x_{\min}+1$. If $j < i$, then $\delta \cong \langle[\nu^{j-x_{\min}}\rho, \nu^{x_{\min}}\rho]\rangle \times \nu^{x_{\min}}\rho$, which is impossible since $\mu^*(\sigma)$ does not contain an irreducible constituent of the form $\nu^{j-x_{\min}}\rho \otimes \pi'$ for $j < 2x_{\min}$ and $\mu^*(\sigma)$ does not contain an irreducible constituent of the form $\nu^{x_{\min}}\rho \times \nu^{x_{\min}}\rho \otimes \pi'$ by the third part of Lemma 3.1. Consequently, $j = i$ and $\delta \otimes \pi \cong \nu^{x_{\min}}\rho \otimes \tau'$.

Let us now assume that $j = 0$. If $i > 0$ we get a contradiction in the same way as in the previously considered case. Thus, $i = 0$ and $\delta \otimes \pi \cong \nu^{x_{\min}}\rho \otimes \tau'$.

Again, it follows that $\langle[\nu^{-x_{\min}}\rho, \nu^{x_{\min}}\rho]\rangle \rtimes \sigma$ has at most two irreducible subrepresentations and $L(\nu^{-x_{\min}}\rho, \ldots, \nu^{-\frac{1}{2}}\rho, \tau)$ appears in the composition series of $\langle[\nu^{-x_{\min}}\rho, \nu^{x_{\min}}\rho]\rangle \rtimes \sigma$ with multiplicity one.

By Propositions 4.1 and 4.2, it remains to consider the case $x_{\min} \leq -a$ and $-a < b$. Suppose that $L(\delta_1, \delta_2, \ldots, \delta_k, \tau')$ is an irreducible subquotient of $\langle\Delta\rangle \rtimes \sigma$, $\delta_i \cong \delta([\nu^{a_i}\rho_i, \nu^{b_i}\rho_i])$ for $i \in \{1, 2, \ldots, k\}$ and there is a $j \in \{1, 2, \ldots, k\}$ such that $a_j < b_j$. From Proposition 4.1 we have $\rho_i \cong \rho$ for all i, and $a_j = b_j - 1$. Let $j_{\min}$ stand for the minimal j such that $a_j = b_j - 1$.

Then, by Proposition 4.2, we have $a_{j_{\min}} = b_{j_{\min}} - 1 = a - 1$ and $a_i = b_i = -b + i - 1$ for $i = 1, 2, \ldots, j_{\min} - 1$. Also, $j_{\min} = a + b$.

If $a_i = b_i - 1$ for all $i = j_{\min}, \ldots, k$, a repeated application of [12, Lemma 2.6] shows that τ' is an irreducible tempered subquotient of $\nu^{\frac{1}{2}}\rho \rtimes \sigma$, which contradicts Theorem 3.8 and Proposition 3.13. Thus, there is an $m \in \{j_{\min} + 1, \ldots, k\}$ such that $a_{m-1} = b_{m-1} - 1$ and $a_m = b_m$. From [12, Lemma 2.6] follows that $L(\delta_{m-1}, \delta_m, \ldots, \delta_k, \tau')$ is an irreducible subquotient of $\langle[\nu^{a_{m-1}+1}\rho, \nu^{-a_{m-1}}\rho]\rangle \rtimes \sigma$, and that $L(\delta_m, \delta_{m+1}, \ldots, \delta_k, \tau')$ is an irreducible subquotient of $\langle[\nu^{a_{m-1}+2}\rho, \nu^{-a_{m-1}-1}\rho]\rangle \rtimes \sigma$. Thus, $a_m \in \{a_{m-1}+1, a_{m-1}+2\}$.

If $a_m = a_{m-1} + 1$, we have the following embeddings and isomorphisms:

$$\begin{aligned} L(\delta_{m-1}, \dots, \delta_k, \tau') &\hookrightarrow \delta([\nu^{a_{m-1}}\rho, \nu^{a_{m-1}+1}\rho]) \times \nu^{a_{m-1}+1}\rho \rtimes L(\delta_{m+1}, \dots, \delta_k, \tau') \\ &\cong \nu^{a_{m-1}+1}\rho \times \delta([\nu^{a_{m-1}}\rho, \nu^{a_{m-1}+1}\rho]) \rtimes L(\delta_{m+1}, \dots, \delta_k, \tau') \\ &\hookrightarrow \nu^{a_{m-1}+1}\rho \times \nu^{a_{m-1}+1}\rho \times \nu^{a_{m-1}}\rho \rtimes L(\delta_{m+1}, \dots, \delta_k, \tau'). \end{aligned}$$

Consequently, $\mu^*(L(\delta_{m-1}, \delta_m, \dots, \delta_k, \tau'))$ contains an irreducible constituent of the form $\nu^{a_{m-1}+1}\rho \times \nu^{a_{m-1}+1}\rho \otimes \pi$, but $\mu^*(\langle[\nu^{a_{m-1}+1}\rho, \nu^{-a_{m-1}}\rho]\rangle \rtimes \sigma)$ does not contain such an irreducible constituent. We conclude that $a_m = a_{m-1} + 2$. This implies that $L(\delta_{m+1}, \delta_{m+2}, \dots, \delta_k, \tau')$ is an irreducible subquotient of $\langle[\nu^{a_{m-1}+3}\rho, \nu^{-a_{m-1}-1}\rho]\rangle \rtimes \sigma$. It follows that $a_{m+1} = b_{m+1}$, and, using $e(\delta_m) \leq e(\delta_{m+1})$, we deduce that $a_{m+1} = a_{m-1} + 3 = a_m + 1$. Repeating the same arguments, we obtain that $a_{j+1} = a_j + 1$ for $j \in \{m+1, m+2, \dots, k-1\}$ and that τ' is an irreducible tempered subquotient of $\langle[\nu^c\rho, \nu^{-a_{m-1}-1}\rho]\rangle \rtimes \sigma$ for $c = -a_k + 1$. Since $a_k \leq -\frac{1}{2}$, the results obtained in the previous section directly imply that $c = \frac{1}{2}$, $a_k = -\frac{1}{2}$, and $-a_{m-1} - 1 = x_{\min}$.

Consequently,

$$\begin{aligned} L(\delta_1, \delta_2, \dots, \delta_k, \tau') \cong L(&\nu^{-b}\rho, \dots, \nu^{a-2}\rho, \delta([\nu^{a-1}\rho, \nu^a\rho]), \dots, \\ &\delta([\nu^{-x_{\min}-1}\rho, \nu^{-x_{\min}}\rho]), \nu^{-x_{\min}+1}\rho, \dots, \nu^{-\frac{1}{2}}\rho, \tau). \end{aligned} \tag{1}$$

We have already seen that $L(\nu^{-x_{\min}+1}\rho, \dots, \nu^{-\frac{1}{2}}\rho, \tau)$ is an irreducible subquotient of $\langle[\nu^{-x_{\min}+1}\rho, \nu^{x_{\min}}\rho]\rangle \rtimes \sigma$. A repeated application of Proposition 4.3 implies that

$$L(\delta([\nu^{a-1}\rho, \nu^a\rho]), \dots, \delta([\nu^{-x_{\min}-1}\rho, \nu^{-x_{\min}}\rho]), \nu^{-x_{\min}+1}\rho, \dots, \nu^{-\frac{1}{2}}\rho, \tau)$$

is an irreducible subquotient of $\langle[\nu^a\rho, \nu^{-a+1}\rho]\rangle \rtimes \sigma$. Finally, a repeated application of [12, Lemma 2.3] shows that the representation (1) is an irreducible subquotient of $\langle\Delta\rangle \rtimes \sigma$.

It can be easily verified, using the structural formula and results obtained in the third section, that

$$\begin{gathered} \nu^{-b}\rho \otimes \dots \otimes \nu^{a-2}\rho \otimes \delta([\nu^{a-1}\rho, \nu^a\rho]) \otimes \dots \otimes \delta([\nu^{-x_{\min}-1}\rho, \nu^{-x_{\min}}\rho]) \otimes \\ \otimes \nu^{-x_{\min}+1}\rho \otimes \dots \otimes \nu^{-\frac{1}{2}}\rho \otimes \tau \end{gathered}$$

appears with multiplicity one in the Jacquet module of $\langle\Delta\rangle \rtimes \sigma$ with respect to the appropriate parabolic subgroup. Consequently, $L(\nu^{-b}\rho, \dots, \nu^{a-2}\rho, \delta([\nu^{a-1}\rho, \nu^a\rho]), \dots, \delta([\nu^{-x_{\min}-1}\rho, \nu^{-x_{\min}}\rho]), \nu^{-x_{\min}+1}\rho, \dots, \nu^{-\frac{1}{2}}\rho, \tau)$ appears with multiplicity one in the composition series of $\langle\Delta\rangle \rtimes \sigma$ and the theorem is proved. $\square$

5 Composition series in $\mathrm{Jord}_\rho(\sigma)=\emptyset$ case

In this section we complete our description by considering the remaining case.

Let $\Delta=[\nu^a\rho,\nu^b\rho]$, where $\rho\in R(GL)$ is an irreducible self-contragredient representation, and a,b are half-integers such that $a+b\geq 0$. Let $\sigma\in R(G)$ stand for a discrete series such that $\mathrm{Jord}_\rho(\sigma)=\emptyset$ and $\nu^{\frac{1}{2}}\rho\rtimes\sigma_{cusp}$ reduces.

Proposition 5.1. *The induced representation $\langle\Delta\rangle\rtimes\sigma$ contains an irreducible tempered subquotient if and only if $\langle\Delta\rangle\cong\nu^{\frac{1}{2}}\rho$.*

Proof. Let τ stand for an irreducible tempered subquotient of $\langle\Delta\rangle\rtimes\sigma$. If $a\leq-\frac{1}{2}$, then it follows from the cuspidal support of $\langle\Delta\rangle\rtimes\sigma$ and classifications of discrete series and tempered representations that there are $x<0$ and y such that $x+y\geq 0$, and an irreducible tempered representation τ' such that τ is a subrepresentation of $\delta([\nu^x\rho,\nu^y\rho])\rtimes\tau'$. Thus, $\mu^*(\langle\Delta\rangle\rtimes\sigma)$ contains $\delta([\nu^x\rho,\nu^y\rho])\otimes\tau'$, which is impossible since $y>0$ and $\mu^*(\sigma)$ does not contain an irreducible constituent of the form $\nu^z\rho\otimes\pi'$.

It follows that $a\geq\frac{1}{2}$. If $a>\frac{1}{2}$, it follows from [12, Proposition 3.5] that $\langle\Delta\rangle\rtimes\sigma$ is irreducible and that it does not contain an irreducible tempered subquotient. Consequently, $a=\frac{1}{2}$. From the cuspidal support of $\langle\Delta\rangle\rtimes\sigma$ we deduce that τ is square-integrable, and there is an ordered k-tuple $(\sigma_1,\sigma_2,\ldots,\sigma_k)$ of discrete series representations such that $\tau\cong\sigma_1$, σ_k is strongly positive, and for $i=1,2,\ldots,k-1$ there are a_i,b_i such that $a_i\leq 0$, $a_i+b_i>0$, $\rho_i\in R(GL)$, $\rho_i\not\cong\rho$, such that σ_i is a subrepresentation of $\delta([\nu^{a_i}\rho_i,\nu^{b_i}\rho_i])\rtimes\sigma_{i+1}$.

By [10, Lemma 3.5], σ_k is completely determined by its cuspidal support, and the classification of strongly positive discrete series provided in [9] implies that σ_k is a unique irreducible subrepresentation of $\delta([\nu^{\frac{1}{2}}\rho,\nu^b\rho])\rtimes\sigma_{sp}$, for strongly positive discrete series σ_{sp} which does not contain twists of ρ in the cuspidal support. Standard commuting argument shows that τ is a subrepresentation of an induced representation of the form $\delta([\nu^{\frac{1}{2}}\rho,\nu^b\rho])\rtimes\pi$, for some irreducible representation π such that $\mu^*(\pi)$ does not contain an irreducible constituent of the form $\nu^z\rho\otimes\pi'$. From the structural formula follows at once that this is possible only if $b=\frac{1}{2}$.

Conversely, if $\langle\Delta\rangle\cong\nu^{\frac{1}{2}}\rho$, then the claim follows in the same way as in the proof of Lemma 3.6. $\square$

From Lemma 3.6 and previous proposition we conclude that if $\langle\Delta\rangle\rtimes\sigma$ contains an irreducible tempered subquotient then it contains a unique irreducible tempered subquotient, which is a subrepresentation. Using results obtained in the Section 3 we get the following subrepresentation theorem.

Theorem 5.2. *Let x, y denote half-integers such that $x + y \geq 0$, let $\rho \in R(GL)$ denote an irreducible cuspidal unitarizable representation and let $\sigma_{ds} \in R(G)$ denote a discrete series representation. If the induced representation $\langle [\nu^x \rho, \nu^y \rho] \rangle \rtimes \sigma_{ds}$ has an irreducible tempered subquotient, then it has an irreducible tempered subrepresentation.*

The following theorem completes our determination of the composition series. It can be proved using Proposition 5.1 and the same methods as in the previous section, detailed verification being left to the reader.

Theorem 5.3. *Let us denote by σ_{ds} a unique discrete series subrepresentation of $\nu^{\frac{1}{2}} \rho \rtimes \sigma$.*

(1) If $a \geq \frac{3}{2}$, then $\langle \Delta \rangle \rtimes \sigma = L(\nu^{-b}\rho, \ldots, \nu^{-a}\rho, \sigma)$.

(2) If $a = \frac{1}{2}$, in $R(G)$ we have

$$\langle \Delta \rangle \rtimes \sigma = L(\nu^{-b}\rho, \ldots, \nu^{-\frac{1}{2}}\rho, \sigma) + L(\nu^{-b}\rho, \ldots, \nu^{-\frac{3}{2}}\rho, \sigma_{ds}).$$

(3) If $a \leq -\frac{1}{2}$ and $-a = b$, in $R(G)$ we have

$$\begin{aligned} \langle \Delta \rangle \rtimes \sigma = {} & L(\nu^{a}\rho, \nu^{a}\rho, \ldots, \nu^{-\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho, \sigma) + \\ & + L(\nu^{a}\rho, \nu^{a}\rho, \ldots, \nu^{-\frac{3}{2}}\rho, \nu^{-\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho, \sigma_{ds}). \end{aligned}$$

(4) If $a \leq -\frac{1}{2}$ and $-a < b$, in $R(G)$ we have

$$\begin{aligned} \langle \Delta \rangle \rtimes \sigma = {} & L(\nu^{-b}\rho, \ldots, \nu^{a-1}\rho, \nu^{a}\rho, \nu^{a}\rho, \ldots, \nu^{-\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho, \sigma) + \\ & + L(\nu^{-b}\rho, \nu^{-b+1}\rho, \ldots, \nu^{a-1}\rho, \nu^{a}\rho, \nu^{a}\rho, \ldots, \nu^{-\frac{3}{2}}\rho, \nu^{-\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho, \sigma_{ds}) + \\ & + L(\nu^{-b}\rho, \ldots, \nu^{a-2}\rho, \delta([\nu^{a-1}\rho, \nu^{a}\rho]), \ldots, \delta([\nu^{-\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho]), \sigma_{ds}). \end{aligned}$$

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