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Characteristics of Effective Teaching of Mathematics

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KLJUČNE BESEDE: matematika, učinkovit pouk, prepričanja učiteljev, uspešnost učencev

POVZETEK – Sinonimi dobrega ali kakovostnega poučevanja se pogosto uporabljajo, ko govorimo o učinkovitem pouku matematike. Ni univerzalne definicije dobrega ali učinkovitega poučevanja matematike in pogledi na te koncepte so v veliki meri odvisni od izobraževalnih tradicij in vrednot v različnih državah, pa tudi od prepričanj učiteljev matematike. Pojem učinkovitega poučevanja je pomemben, saj pomembno vpliva na izobraževalne politike in oblikovalske odločitve. Namen tega prispevka je problematizirati vprašanje učinkovitega pouka matematike, določiti značilnosti učinkovitega pouka matematike in opredeliti, kako učitelji dojemajo učinkovito poučevanje matematike. Način poučevanja, ki ga uporablja učitelj matematike, je pokazatelj tistega, kar se mu zdi najpomembnejše. Učitelji so ključni pri učnem in izobraževalnem napredku učencev, zato jih je treba usposobiti za kakovostno in učinkovito poučevanje. Pouk matematike je učinkovit, ko čim bolj spodbuja uspešnost učencev. Kulturne norme vplivajo na izvajanje učinkovitega poučevanja.

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ABSTRACT – When referring to effective mathematics teaching, the terms good teaching and quality teaching are often used. There is no universal definition of what constitutes good or effective teaching of mathematics, and views on these concepts are largely dependent on the educational traditions and values in different countries, as well as on the beliefs of mathematics teachers. The notion of effective teaching is important because it significantly influences educational policies and teaching design decisions. The aim of this paper is to problematize the issue of effective mathematics teaching, to determine the features of effective mathematics teaching and how teachers perceive effective mathematics teaching. The way a mathematics teacher teaches is an indication of what he or she considers to be most important. Teachers are critical determinants of students' learning and educational progress, so they must be trained to deliver quality and effective lessons. Mathematics teaching is effective when it promotes students' performance as best as possible. Nevertheless, cultural norms influence the way effective features are implemented.

1 Introduction

In light of the increasing importance of international comparative studies such as TIMSS and PISA, mathematics teachers' knowledge and the impact it has on the development of student knowledge have become of particular interest. Thus, students' achievement in mathematics has become the focus of educational policies in countries around the world that use PISA and TIMSS results to identify problems in the education system and improve the quality of teaching. The mathematical literacy of the individual, examined by PISA, can serve as a starting point for reflecting on the quality of mathematics teaching. *Mathematical literacy* is the students' ability to analyze, logically infer, and effectively convey their ideas as they formulate, solve, and interpret

solutions to mathematical problems in different situations (OECD, 2013). But critics of this understanding of mathematics teaching outcomes argue that mathematics teaching should provide students more than what is examined by PISA. The focus of PISA is not the successful adoption of a particular subject, but the students' ability to function as individuals and engaged citizens in the real world after compulsory education (Niss, 2015). Moreover, the concept of mathematical literacy focuses on mathematics as a tool for solving non-mathematical issues, while the concept of mathematical proficiency is much broader – it focuses on what it means to master mathematics in general, including the ability to solve both mathematical and non-mathematical problems (Jablonka & Niss, 2014). The concept of mathematical proficiency includes five interlocking parts of knowledge and personal characteristics of the individual: conceptual understanding, procedural knowledge, strategic competence, adaptive reasoning and productive character. Mathematical proficiency goes beyond mastering mathematics (Kilpatrick, Swafford & Findell, 2001); this conceptualization seeks to encompass what is required to learn mathematics, and what characterizes the individual who has been able to learn it. Effective mathematics teaching should enable students to work successfully in purely mathematical structures, for example studying mathematical phenomena such as the irrationality of numbers, and, at a higher level, to understand the role of definitions and theorems in mathematics (Niss, 2015). In this way, mathematical literacy becomes an integral part of mathematical proficiency. Therefore, one can observe the existing teaching through the prism of mathematical proficiency and identify the parts of teaching which need to be improved in order for students to develop mathematical proficiency.

When referring to effective mathematics teaching, the terms *good teaching* and *quality teaching* are often used. There is no universal definition of good or effective teaching of mathematics, and views on these concepts are mainly dependent on educational traditions and values in different countries, as well as on the beliefs of mathematics teachers (e.g. Cai et al. 2009; Jaworski, 1999). Therefore, there is no straightforward answer to the question *What is effective teaching of mathematics?* The response will also depend on the person being asked the question; teachers, scientists, politicians, parents, all have their own vision of what good mathematics teaching is and what it is not. However, the notion of quality teaching is important because it significantly influences educational policies, teaching design decisions, and research on student learning.

The aim of this paper is to problematize the issue of effective mathematics teaching and by reviewing the relevant and available literature, to answer the following research questions: How do teachers perceive effective mathematics teaching? What are the features of effective mathematics teaching? Do the existing features of effective teaching overlap with teachers' beliefs about effective mathematics teaching?

The countries analyzed in this paper were selected because of their differences in size, geographical location, economic and technological progress, and also their differing educational performance in PISA and TIMSS surveys. An additional criterion was the availability of existing literature (limited to English, German and Croatian).

2 Teachers' attitudes and influence on mathematics teaching

The way a teacher understands and experiences mathematics influences his or her teaching of mathematics (Ernest, 1989). Specifically, this understanding has an influence on two other closely related beliefs – how mathematics should be taught and how mathematics should be learned. There exists evidence showing a significant correlation between these two beliefs about the learning and teaching of mathematics (e.g. Speer, 2005). A teacher who perceives mathematics as a set of procedures and rules to follow, teaches students algorithms and emphasizes practicing similar tasks until students become proficient in performing operations. A teacher who perceives mathematics as an interconnected network of concepts, properties, and relationships among these concepts, encourages students to use different strategies to solve the problems, explain and interpret those results, and make their own conclusions. The first conception of mathematics denotes the traditional view of mathematics, while the second conception denotes the modern view of mathematics. However, the research also shows that some teachers experience a dichotomy between understanding mathematics and teaching mathematics (e.g. Speer, 2005; Raymond, 1997). For instance, in the qualitative research of Croatian mathematics teachers, Jukić Matić and Glasnović Gracin (2017) found that some teachers with a modern approach to mathematics used traditional ways of teaching.

2.1 Effective teaching from a cultural standpoint

Cai & Wang (2010) emphasize the importance of a cultural context that gives the teacher the tools to work with, and creates habits and assumptions about teaching mathematics effectively. Therefore, to determine what effective teaching involves, and how teachers envision it, one also needs to look at the cultural context from which the teacher comes. The idea that teachers' beliefs and their understanding of effective teaching have an influence on the teaching practice is not new (Cai, 2005; Miao & Reynolds, 2018; Perry, Wong & Howard, 2006; Stigler & Hiebert, 1999). Teachers draw on their cultural beliefs as a normative framework of values and goals that guide teaching (Rogoff, 2003). The way a mathematics teacher teaches is an indication of what he or she considers to be most important, influencing how students learn mathematics (e.g. Cai, 2004).

The following text examines teachers from different parts of the world and their understanding of what constitutes effective teaching of mathematics. The term *East Asia* refers to countries or education systems such as China, Hong Kong, Japan, Korea, Taiwan and Singapore, while the *West* relates to countries in North America, Europe and Australia. Various researchers claim that East and West are cultural, not geographical, boundaries with the Confucian tradition in the East and the Greek/Latin/Christian culture in the West (e.g. Leung, Park, Shimizu & Xu, 2015). Although this separation is not entirely justified, it is useful in the context of this study. For example, the Confucian tradition holds the belief that learning is hard work and that learning should not be fun (Leung, Park, Shimizu & Xu, 2015). The main characteristic of the Confucian tradition is a social orientation, opposed to an individual orientation, which is usually found in Western societies. Teaching a whole class, where the teacher has a dominant

role, is considered very important in East Asian countries, unlike the individualization present in Western countries, which emphasizes independence and individualism in learning (Leung, 2001). The difference is also notable when one looks at how teachers view mathematics. For Chinese teachers, the real beauty of mathematics lies in the possibilities of generalization and logical connection, so any solution that does not lead to generality should be rejected. American teachers emphasize the pragmatic side of mathematics: as long as something works, students can choose all kinds of representations of mathematical objects and all strategies (Cai & Wang, 2010).

Furthermore, research on European traditions of education emphasizes the diversity of approaches to mathematics education within Europe. There is a significant difference between approaches coming from the UK on the one hand, and Scandinavian countries and continental Europe on the other (Kaiser & Blömeke, 2013). Differences are evident in the type of mathematical knowledge a student needs to acquire, the role of argumentation and proof in mathematics teaching, and the expected interactions among students during mathematical activities. Although there are differences, the approach to students is similar, so it is justified to place them in the same category – the category of Western countries (Kaiser & Blömeke, 2013).

On the other hand, Russia, Lithuania and Estonia are singled out because of the great influence of the Russian mathematical school on Lithuania and Estonia throughout history. Karp & Zvavich (2011) state that within the Russian mathematical school, the individual work of students with theoretical textbook materials was highly valued, and this was one of the most important goals of the teaching practice – to help the student develop the skills necessary for working with the textbook. The authors also state that the teaching of mathematics had to be constructed in such a way that the work of the class as a whole would help individual understanding.

2.1.1 The United States of America

Teachers in the US encourage a mathematical understanding that is focused on connecting and applying mathematical knowledge in real contexts (Cai & Wang, 2010). For US teachers, memory can only come after understanding. They believe that students should be encouraged to explore the relationships between mathematics and their own life experiences by providing extensive real-life examples and tactile teaching experiences. The focus is on student engagement by listening to students carefully and creating group discussions frequently. In general, US teachers see effective teaching as a process of creating challenges and guiding students to explore, create their own knowledge, and use it independently.

2.1.2 Australia

In a survey conducted in Australia, the sample consisted of teachers who were characterized as effective teachers in the education system (Perry, 2007). Teachers emphasized challenges in mathematics teaching and topics relevant to students as important features of effective teaching. According to their understanding, an effective math teacher is well-prepared in terms of what students want to learn, but also allows students to explore when opportunities arise. The teachers also emphasized the importance of structuring lessons and establishing routines because students need routines. The im-

pact of productive pedagogy is evident in teachers' comments about explicit teaching, children's awareness of what is expected of them and what they will learn, and the importance of a meaningful inquiry.

2.1.3 European countries

The study conducted by Kaiser & Vollstedt (2007) found that many German teachers believe that students should memorize basic algorithms, such as calculating percentages or applying formulas to solve equations. However, they also emphasize the importance of both the understanding of formulas and the ability of students to derive formulas themselves.

Teachers in England believe that effective mathematics teaching should be real-life related, differentiated and focused primarily on the student (Miao & Reynolds, 2018). The central question of the teaching process is *what to teach* and *what to teach to whom*. Thus, teachers seek to differentiate tasks, adapt them to students' abilities, and utilize several differentiation strategies, such as self-differentiation or pre-assessment.

In France, teachers often emphasize the importance of reasoning and mind coaching in teaching mathematics (Pepin, 1999; Kaiser & Vollstedt, 2007). They believe that definitions, theorems and formulas must be used visibly so the theoretical aspect of mathematics is evident. Teachers emphasize that adherence to strictly prescribed procedures and routines is important, but this is not always sufficient to solve complex problems (Kaiser, Hino & Knipping, 2006). Teachers find teaching effective if it provides students with the skills needed for precise performance and enables them to discover mathematical concepts themselves (Kaiser & Vollstedt, 2007).

Swedish teachers view effective mathematics teaching as an individual interaction with students, which builds on students' interests and on mathematics from everyday situations. Teachers emphasize the significant role of adaptation to the students and their needs. This means they need to be able to follow each student's learning path and help the individual move forward in content (Hemmi & Ryve, 2014).

Finnish teachers view mathematics teaching as whole-class teaching that must include activities that are constantly repeated, such as mental arithmetic and homework on a daily basis (Hemmi & Ryve, 2014). In particular, Finnish teachers are considered to be proactive classroom leaders who clarify the goals of the lesson, value formative assessment, and take differences among students into consideration. Effective teaching, according to Finnish teachers, is characterized by well-designed lesson planning with clearly defined goals and precise instruction.

2.1.4 Baltic countries and Russia

A study by Kardanova, Ponomaryova, Safuanov and Osin (2014) examined how teachers in Russia, Latvia and Estonia perceive effective and high-quality mathematics teaching. Russian teachers either prefer a modern understanding of mathematics teaching, where students actively build their knowledge, or combine the features of traditional and contemporary mathematics teaching. Estonian teachers' beliefs of good teaching combine an understanding of teaching as an active creation of knowledge and as a transfer of knowledge, i.e. a modern and traditional approach to teaching. These approaches, i.e. traditional and modern constructivist, are seen as complementary rather

than opposing. But there are some aspects of these approaches that Estonian teachers do not find effective. For example, teachers do not advocate the memorizing of rules and specific task-solving procedures, and are not keen on open learning, discussions with students, or working in small groups. The results of the above study showed that although Latvian teachers support a modern understanding of mathematics teaching, their orientation toward teaching itself is more traditional than modern. In their teaching practice, they often use traditional approaches to learning (Šapkova, 2014).

2.1.5 East Asian countries

Teachers in China encourage the development of mathematical understanding by linking abstract parts of knowledge (Cai & Wang, 2010). They believe that memorization can come before or after understanding. Moreover, memorization before understanding can serve as an indirect step towards conceptual knowledge. Chinese teachers see effective teaching as teacher-led, with a clear and coherent structure, where the existing knowledge should be transferred accurately. The teachers see the teaching of the whole class as their main goal, whereby they provide equal opportunities for every student (Miao & Reynolds, 2018).

Teachers in Hong Kong consider mathematics a practical, logical, useful, and thought-provoking subject (Wong, 2007). Abstract thinking is considered to be one of the major goals of learning mathematics, and the teacher is the one who builds a path for students to move from concrete to abstract thinking. Learning and teaching for understanding are valuable, but teachers also see the role of remembering, practicing, and using concrete experiences in enhancing mathematics understanding. Teachers believe that a good relationship with students is also an indicator of effective teaching.

South Korean teachers believe that effective mathematics teaching incorporates lessons based on basic mathematical concepts and their connections (Pang, 2009). Korean teachers often focus on the meaningful development of mathematical content, rather than on adapting their approach to students (Pang, 2012). Highly individualized forms of teaching are less valued. One of the main features of effective teaching is a teacher with a high-quality knowledge of mathematics (Pang, 2009). When given a list of effective teaching features, elementary school teachers chose building curricula by selecting content as the most important feature.

According to Hsieh, Wang and Chen (2020), several different curriculum reforms have influenced the beliefs of Taiwanese mathematics teachers. Nowadays, teachers consider that effective teaching encourages the utilization of concrete materials and situations related to everyday life. Although formalism, symbolic representation, quick challenges and demands for a quality performance are still present, the emphasis on these factors has been significantly reduced.

Kaur (2009) investigated the characteristics of good mathematics teaching in Singapore and found that teaching is focused on both the teacher and the students. The teacher leads the teaching, but the emphasis is on the students' understanding of mathematics. Teachers believe that careful selection of examples that vary in complexity, from low to high, is a very important aspect of teaching, as well as supervising students' understanding. The students' knowledge of mathematics is enhanced and deepened by a thorough examination of their individual work (e.g. homework). The teachers also

highlight summarization of the key parts of a topic as being an important feature of effective teaching.

Japanese mathematics teachers form their teaching around problem-solving, where the main output is the mathematical concept or procedure (Corey, Peterson, Lewis & Bukarau, 2010). The teachers value detailed lesson planning as a key feature of effective mathematics teaching. The planning involves a thorough and detailed lesson flow, with clear goals (not just one, but several), the development of the lesson according to the overall learning path for a particular topic, the intellectual engagement of the students, and the adaptation of the lesson to the students' needs. Moreover, Japanese teachers consider that an effective teacher continuously reassesses enacted lessons and reflects on possible improvements.

3 Effective teacher – effective teaching of mathematics?

Teacher effectiveness is often measured by students' achievement on standardized tests. However, Knight et al. (2015) state that this perspective limits the qualities of effective teachers to skills that can be measured by achievement tests. Their recommendation is to define quality in terms of cognitive resources and performance, and to focus on the *quality of teaching*. These considerations place the focus on student performance, rather than on the characteristics of the teacher himself or herself. In describing the characteristics of effective teachers, Knight et al. (2015) suggest that the focus should be on the affective components of the teacher. An effective teacher would, therefore, show compassion, honesty and respect, enthusiasm, motivation, the right attitude toward teaching, would interact with students in and out of the classroom, and reflect on their teaching.

On the other hand, a teacher's effectiveness can also be assessed in the light of the expectations held by students. Martin and Rimm-Kaufman (2015) point out that ideal mathematics learning is not a passive process of remembering and using standard algorithms. Instead, students engage in reasoning, problem-solving, mathematical discussions with the teacher and other students to explore mathematical problems. A study by Rivkin, Hanushek & Kain (2005) found that students who are exposed to high-quality teaching have more significant and continuous achievement than their peers who are exposed to lower-quality teaching. Cruickshank and Haefele (2001) noted that good teachers, at various moments, were called ideal, analytical, dedicated, competent, professional, thoughtful, and respected.

If we view cognitive resources as indicators of quality, then it makes sense to focus on the teacher's knowledge. Research has shown that there exists a relation between a teacher's knowledge and students' mathematical achievement (e.g. Ball et al., 2008; Baumert et al., 2010; Hattie, 2003). Teachers need to know the mathematical content they teach from a more advanced perspective. Then, they need to know how the material connects to other mathematical domains that precede and follow the level at which they teach. Teachers who have a greater and deeper mathematical knowledge are more effective in teaching than those with superficial knowledge (Marshall & Sorto, 2012). More often such teachers encourage students to make inferences, assumptions and solve

problems, while at the same time they can diagnose misconceptions and errors more accurately and correct them more successfully (Kilpatrick et al. 2001).

The knowledge for teaching mathematics differs significantly from the mathematical knowledge possessed by experts in other disciplines related to mathematics (Hill, Ball & Schilling, 2008). This knowledge is called *mathematical knowledge for teaching* (Hill & Ball, 2004) or *mathematics teacher's specialized knowledge* (Carrillo-Yañez et al., 2016). The knowledge which an effective mathematics teacher possesses consists of several interconnected subdomains: knowledge of mathematical topics, knowledge of the structure of mathematics, knowledge of mathematical practices, knowledge of mathematics teaching, knowledge of features of mathematics learning and knowledge of mathematical learning standards (Carrillo-Yañez et al., 2016).

In other words, mathematics teachers need to integrate pedagogical and mathematical knowledge into a meaningful whole. They need to know not only the content they are supposed to teach, but also how students learn, develop their mathematical knowledge and how their knowledge is structured; how to relate different representations of mathematical concepts; how to make students' understanding of mathematics visible; how to diagnose student errors and misconceptions; how to develop procedural knowledge from a conceptual knowledge of mathematics (Hill, Ball & Schilling, 2008). Moreover, the teachers need to evaluate which curricular resources are useful for teaching mathematical concepts, what metaphors and scenarios are appropriate, and how to use mathematical language appropriately.

However, knowledge alone is not enough for effective performance. Possessing specialized knowledge for teaching is in some ways static, which is why Brown (2009) proposed a dynamic construct – pedagogical design capacity. Brown perceives pedagogical design capacity as a skill necessary to identify and mobilize existing resources to craft effective teaching episodes. Pedagogical design capacity contains the specialized knowledge of mathematics teachers within itself. This construct is useful for evaluating a teacher's performance; not just whether the teacher has specialized knowledge for teaching mathematics, but also how he or she uses it in teaching.

How can the quality of mathematics teachers be viewed in the context of East and West? The Eastern approach puts teaching practice at the forefront and views the teacher's knowledge as a part of integrated expertise, therefore in a holistic way. Cai et al. (2009) point out that Asian teachers are oriented toward a thorough understanding of mathematical content and planning a well-structured lesson. In Western countries, the teacher's knowledge is an important part of teacher competence, but it is viewed as a stand-alone component of teacher expertise (Kaiser & Li, 2011). Other components of teacher expertise are beliefs and teacher performance. Thus, teachers from the US and Europe tend to be more oriented toward students as individuals, placing them in the center of the teacher's actions; student learning is viewed individually and is a major goal of classroom activities.

4 Features of effective teaching

It has long been debated whether teaching should be entirely teacher-centered or student-centered. There is evidence that both approaches are effective in achieving some outcomes of mathematics teaching (e.g. teacher progress in classroom discussions, achievement of mathematical norms in the classroom, learning of specific topics). However, high-quality research does not support the exclusive use of any approach, that is, that teaching should be entirely student- or teacher-focused (e.g. Clarke, 2006; Kilpatrick et al., 2001; Wong, 2004). For example, Gersten et al. (2009) recommend a 10-minute direct teaching of elementary school students so they can quickly arrive at arithmetic facts, but the authors also propose that teachers allow students to solve problems in a group, to collaborate and create joint problem-solving strategies.

Anthony and Walshaw (2009) point out that effective teaching of mathematics implies the belief that all students, regardless of age, can develop positive mathematical identities and become successful in mathematics. They also believe that effective teaching is based on interpersonal respect and empathy and that such teaching responds to the multiplicity of cultural heritage, thought processes and realities found in classrooms. Further, they claim that effective teaching is focused on optimizing desirable learning outcomes, like mathematical proficiency, and is committed to enhancing a range of social outcomes within the mathematics classroom, such as contributing to the holistic development of students for productive citizenship.

However, Krainer (2005) points out that teaching is a complex process. The different stakeholders in education, from students to policy makers, should be co-constructors of the norms that determine quality mathematics teaching. Krainer also believes that norm-building should not be based only on teachers' beliefs, or rigorously imposed by researchers, but the result of negotiation and decision-making – in the classroom, development teams, teacher education programs, publications that describe the vision and goals of teaching. Researchers' suggestions should include norms for quality mathematics teaching that have emerged from research and are supported by evidence. But the standards created must serve as a starting point for discussion and agreement between scholars, teachers, and policy makers.

Country-specific organizations, associations and institutions (e.g. Australia (Sullivan, 2011), USA (NCTM, 2014), Singapore (MOE, 2019), China (MOEPRC, 2011; Huang & Li, 2014), Germany (MSW, 2008), UK (NCETM, 2007), Canada (Sinay & Nahornick, 2016)) have produced documents that contain lists of features of effective mathematics teaching.

Although the documents do not have exactly the same number of features, the descriptions reveal common ideas, based on theoretical assumptions and the long-term results of qualitative and quantitative research among mathematics teachers and students in elementary and high schools. In this paper, a synthesis of the features of effective mathematics teaching is provided from the above-mentioned documents.

Features of effective mathematics teaching include the following recommendations:

- *Determine the mathematical goals that guide learning.* The mathematics that students learn in school requires clear goals. It is also important to set goals for the whole learning path of a particular topic. Those goals should be used to guide the teaching process.
- *Build on the knowledge that students bring to the classroom*. Effective teachers should assess and use students' prior knowledge and tailor teaching to students' needs. Students' new knowledge needs to be built on prior experience, both mathematically and experientially, by linking to stories that provide context and reason for learning.
- *Implement tasks that promote reasoning and problem-solving*. Students need to be involved in problem-solving and in the discussion of tasks that promote mathematical reasoning, allow multiple entry points and diverse problem-solving strategies. Students should be encouraged to do the activities offering them rich and challenging tasks, which:
    - Enable decision making;
    - Involve students examining assumptions, proving, explaining, reflecting, interpreting;
    - Promote discussion and communication;
    - Encourage originality and discovery;
    - Raise What if? and What if not? questions;
    - Contain an opportunity for surprise.
- *Use and connect mathematical representations*. Teachers should encourage students to represent mathematical content in various ways (words, drawings, diagrams, graphs, lists, tables, numbers, symbols, etc.) and to make connections to deepen their understanding of mathematical concepts and procedures, as well as tools for problem-solving.
- *Facilitate meaningful mathematical discussions*. Classroom discussions on mathematical topics are an important tool for knowledge building and developing understanding. Teachers should facilitate discussion with and among students to build a common understanding of mathematical ideas, analyzing and comparing student approaches and arguments.
- *Expose students to common misconceptions and mistakes*. Learning activities should expose students' current thinking, should make students aware of inconsistencies in their knowledge, and should create opportunities for students to correct misconceptions.
- *Differentiate challenges*. Teachers need to assist those students who need additional support and need to provide challenges to those who are ready for them.
- *Build procedural knowledge from conceptual understanding*. Procedural knowledge needs to be built with conceptual understanding so that over time students become skilled and flexible in using procedures and algorithms while solving contextual and mathematical problems. Procedural knowledge can be developed by the short daily practice of mental processes and, through practice, the transfer of learned skills can be strengthened and encouraged.

- *Encourage collaboration*. The teacher should encourage collaboration among students. Effective teachers use collaborative work in small groups to help students discuss important ideas. This has positive effects on learning, social skills and self-esteem.
- *Use technology*. Effective teachers use technology to present mathematical concepts in dynamic, visually appealing ways that motivate students and encourage them to work. Technology is also useful for collaborative learning.

Unlike in documents originating from Western countries, for example, the US or the UK, the term direct teaching is highlighted in the Singapore Mathematics Curriculum (MOE, 2019), which explicitly states that: "The engagement phase can include one or more of the following: activity-based learning; inquiry-based learning; direct instruction." (p. 37). A typical and socially accepted feature of Chinese mathematics teaching today is teacher-centered and student-focused teaching, which is under the control of the teacher most of the time, and in which a significant amount of time is spent on teacher-student interaction and students' independent work (Cao, Dong & Li, 2018).

Therefore, one may ask what the basis of the judgment of good or efficient mathematics teaching is. This question was raised by Jaworski (1999), who asked in what ways forms of teaching practice, recognized as good or effective, reflect the theoretical propositions suggested by researchers. Furthermore, Jaworski points out that a too narrow understanding of theoretical perspectives results in "ridiculous stances, such as, for example, the so-called 'constructivist' pedagogy in which the teacher never engages in explanation or exposition because it would be understood as 'transmission teaching'" (p. 200). Moreover, Hiebert and Grouws (2007) reviewed research evidence on the impact of some teaching forms and strategies on student learning conducted in the United States since the 1930s. They found that learning features that include basic skills development and conceptual understanding do not fit neatly into categories such as exposure or discovery, direct teaching versus research-based teaching, student or teacher-focused teaching, traditional teaching versus reform-based teaching. In fact, the effectiveness or inefficiency of specific methods and strategies depend on what the learning goals are (McNaught & Grouws, 2007).

Kaasila & Pehkoken (2009) note that when talking about effective mathematics learning, it is reasonable to separate the development of basic skills from problem-solving competencies because those types of knowledge are developed in different ways; one is practiced to the level of automation, and the other rests on connecting conceptual knowledge. Moreover, the authors offer a definition of effective mathematics teaching: mathematics teaching is effective when it promotes students' performance as best as possible, i.e. when students' basic skills and conceptual understanding are optimally developed.

5 Conclusion

The aim of this paper was to provide an overview of effective mathematics teaching and its features as well as teacher beliefs of what constitutes effective teaching. The literature review shows that despite the fact that features of effective teaching have

been identified, it is not possible for mathematics lessons in different countries to be the same. Lessons have the cultural characteristics of the country in which they are taught. Teachers' beliefs on effective mathematics teaching also provide similar evidence. For example, in China and Hong Kong, teachers emphasize the importance of building an abstract mathematical knowledge in their teaching, while US and Australian teachers see mathematics as a useful tool for solving everyday problems. In three European countries, teachers' attitudes are completely different: French teachers' beliefs are similar to those of East Asian teachers, English teachers promote an understanding of mathematics in line with Australian and American teachers, and teachers' beliefs in Germany are somewhere in the middle. Authors of studies from the East highlight the emerging influence of the constructivist paradigm on Eastern mathematics teaching. Moreover, the constructivist paradigm of teaching changes Eastern teachers' perspectives on effective and quality teaching (e.g. Pang, 2012; Hsieh, Wang & Chen, 2020; Miao & Reynolds, 2018). Studies examining how teachers in Russia, Latvia, and Estonia perceive effective teaching have also indicated a change in their paradigm of teaching toward constructivist teaching (Kardanova et al., 2014; Šapkova, 2014). The results of this review indicate that the cultural context plays an important role in shaping effective teaching. Teachers, if they accept the constructivist paradigm, continue to design teaching in accordance with the norms of the culture in which they are located.

The characteristics of effective mathematics teaching are, to a certain extent, consistent with teachers' beliefs about what constitutes quality teaching. But the way these features are intertwined and achieved in a classroom is also conditioned by the cultural context. For example, students in East Asia are extremely successful in international mathematics studies such as TIMSS and PISA, and, on the other hand, various studies show that teaching mathematics in these countries is quite traditional (Leung, Park, Shimizu & Xu, 2015). Problem-solving stands out as a feature of effective teaching across cultures, but in East Asian countries, problem-solving takes a completely different form than problem-solving in, for example, the United States and Australia. In Western countries, when solving problems, the emphasis is on processes, while in East Asian countries, the emphasis is on the product of that solution (Corey, Peterson, Lewis & Bukarau, 2010).

Effective teaching requires a quality teacher with a high level of pedagogical design capacity to implement the features of effective teaching adequately. Also, the results of this paper can serve as a starting point for reforms and changes in mathematics teaching. The reform and improvement of teaching practice should take into account the impact of historical heritage and cultural beliefs. Teachers' beliefs about mathematics, the teaching and learning of mathematics are crucial for achieving quality teaching, so their beliefs are of great importance in reforming the educational process. Therefore, it is desirable to know how mathematics teachers see effective teaching, how it relies on the historical tradition of mathematics teaching, what is useful in that tradition, how it can be preserved, and how to fix what does not work. This approach does not impose strategies that conflict with the cultural heritage of a certain country nor does it determine specific strategies that must be used to improve mathematics teaching.

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Značilnosti učinkovitega poučevanja matematike

Zaradi vse večjega pomena mednarodnih primerjalnih študij, kot sta TIMSS ali PISA, sta znanje učiteljev matematike in njihov vpliv na razvoj znanja učencev postala še posebej zanimiva. Tako dosežki učencev iz matematike postajajo tema izobraževalnih politik v državah po vsem svetu, ki rezultate PISA in TIMSS uporabljajo bolj ali manj učinkovito za prepoznavanje težav v izobraževalnem sistemu in za izboljšanje kakovosti poučevanja. Koncept znanja matematike lahko služi kot izhodišče za razmislek o kakovosti pouka matematike. Z opazovanjem je mogoče evalvirati obstoječe poučevanje in določiti, katere dele pouka je treba izboljšati, da lahko učenci razvijejo matematično znanje.

Za učinkovito poučevanje matematike se pogosto uporabljajo sinonimi dober pouk ali kakovostno poučevanje. Univerzalne definicije dobrega ali učinkovitega poučevanja matematike ni. Pogledi na te koncepte so v veliki meri odvisni od izobraževalnih tradicij in vrednot v različnih državah, pa tudi od prepričanj učiteljev matematike (npr. Cai in sod., 2009; Jaworski, 1999). Zato odgovor na vprašanje "Kaj je učinkovito poučevanje matematike?" ni enostaven. Odvisno je tudi od osebe, na katero je naslovljeno vprašanje. Učitelji, znanstveniki, politiki, starši, vsi imajo svojo vizijo, kaj je dobro poučevanje matematike in kaj ne. Kljub temu je pojem kakovostnega poučevanja pomemben, saj pomembno vpliva na izobraževalne politike, oblikovalske odločitve in raziskave o učenju.

Učiteljeva prepričanja, znanje, mnenja in odločitve pomembno vplivajo na dogajanje v učilnici (Peterson in sod., 1989). Način, kako učitelj razume in doživlja matematiko, vpliva na njegovo poučevanje matematike (Ernest, 1989).

Cai & Wang (2010) poudarjata pomen kulturnega konteksta, ki omogoča razvoj navad in oblikovanje predpostavk o učinkovitem poučevanju matematike. Zato je treba ugotoviti, kaj vključuje učinkovito poučevanje in kako si ga učitelji predstavljajo, smiselno je proučiti tudi kulturni kontekst, iz katerega učitelj prihaja. Ideja, da prepričanja učiteljev in njihovo razumevanje vpliva na učiteljsko prakso, ni nova (Cai, 2005; Miao in Reynolds, 2018; Perry, Wong in Howard, 2006; Stigler in Hiebert, 1999). Dejansko učitelji svoje kulturno prepričanje vidijo kot normativni okvir vrednot in ciljev, ki vodijo poučevanje (Rogoff, 2003).

Izraz Vzhodna Azija se nanaša na države ali izobraževalne sisteme, kot so Kitajska, Hong Kong, Japonska, Koreja, Tajvan in Singapur, medtem ko se Zahod nanaša na države Severne Amerike, Evrope in Avstralije. Različni raziskovalci označujejo, da Vzhod in Zahod predstavljata kulturno in ne geografsko mejo, s konfucijsko tradicijo na Vzhodu in grško/latinsko/krščansko kulturo na Zahodu (npr. Leung in sod., 2015). Čeprav ta ločitev ni povsem upravičena, je uporabna v okviru te študije. Konfucijanski pristop na primer verjame, da je učenje trdo delo in da učenje ne sme biti zabavno (Leungin sod., 2015). Glavna značilnost konfucijanskega pristopa je družbena usmerjenost, ki je nasprotna individualni usmeritvi, ki jo običajno najdemo v zahodnih družbah. Poučevanje celega razreda, kjer ima učitelj prevladujočo vlogo, se v vzhodnoazijskih državah šteje za zelo pomembno, za razliko od individualizacije, ki je prisotna v zahodnih državah in poudarja neodvisnost ter individualizem pri učenju (Leung, 2001). Razlika je opazna tudi, če pogledamo, kako učitelji gledajo na matematiko. Za kitajske učitelje je resnična lepota

matematike v možnostih posploševanja in logične povezanosti, zato je treba zavrniti vsako rešitev, ki ne vodi do splošnosti. Ameriški učitelji poudarjajo pragmatično plat matematike: dokler nekaj deluje, lahko učenci izberejo vse vrste predstavitev matematičnih konceptov in vse strategije (Cai in Wang, 2010).

Ko gre za učinkovitost učitelja, se ta lastnost pogosto meri z dosežkom učencev na standardiziranih testih. Vendar pa Knight in sod. (2015) navajajo, da takšna perspektiva omejuje lastnosti učinkovitih učiteljev samo na veščine, ki jih je mogoče meriti s preizkusi znanja. Njihovo priporočilo je, da se določi kakovost v smislu kognitivnih virov in uspešnosti ter se osredotoči na kakovost poučevanja, ki je povezana z učenjem učencev, ne pa na lastnosti učitelja samega. Kako si lahko ilustrirate kakovost učiteljev matematike v kontekstu Vzhoda in Zahoda? Vzhodni pristop v ospredje postavlja učiteljsko prakso in učiteljevo znanje obravnava kot del celostnega strokovnega znanja, torej na celovit način. Cai in sod. (2009) poudarjajo, da so azijski učitelji usmerjeni k temeljitemu razumevanju matematičnih vsebin in načrtovanju dobro strukturiranega pouka. V zahodnih državah je znanje učitelja pomemben del učiteljeve kompetence, vendar ga obravnavamo kot samostojno komponento strokovnega znanja učiteljev (Kaiser in Li, 2011). Druge komponente strokovnega znanja učiteljev so prepričanja in uspešnost učiteljev. Tako so učitelji iz ZDA in Evrope bolj usmerjeni k učencem kot posameznikom, zato jih postavljajo v središče učiteljevega delovanja; učenje učencev se obravnava individualno in je glavni cilj dejavnosti v učilnici.

Zelo dolgo se je razpravljalo o tem, ali naj bo poučevanje v celoti usmerjeno v učitelje ali učence. Za vsak pristop poučevanja matematike obstajajo dokazi o njegovi učinkovitosti (npr. napredek učitelja v razpravah v učilnici, doseganje matematičnih norm v učilnici, učenje določenih tem). Raziskovalci ne podpirajo izključne uporabe posameznega pristopa, tj. da mora biti poučevanje v celoti usmerjeno v učence ali učitelje (npr. Clarke, 2006; Kilpatrick in sod., 2001; Wong, 2004). Anthony in Walshaw (2009) poudarjata, da učinkovito poučevanje matematike pomeni, da lahko vsi učenci, ne glede na starost, razvijejo pozitivno matematično identiteto in postanejo uspešni pri matematiki. Prav tako verjameta, da učinkovito poučevanje temelji na medosebnem spoštovanju in empatiji ter vključuje množico kulturne dediščine, miselnih procesov in realnosti, ki obstajajo v učilnici.

Organizacije, združenja in ustanove so za posamezne države pripravile dokumente, ki vsebujejo sezname značilnosti učinkovitega poučevanja matematike. V tem prispevku ponujamo sintezo značilnosti učinkovitega pouka matematike iz strokovnih dokumentov:

- *Določitev matematičnih ciljev, ki vodijo učenje: Matematika, ki se jo učenci učijo v šoli, zahteva jasne cilje. Pomembno je tudi izbrati cilje iz učnega načrta za celotno učno pot določene teme. Te cilje je treba uporabiti za vodenje učnega procesa.*
- *Nadgradnja znanja, ki ga učenci prinesejo v učilnico: Učinkoviti učitelji bi morali oceniti in uporabiti predhodno znanje učencev in prilagajati pouk njihovim potrebam. Novo znanje učencev je treba graditi na predhodnih izkušnjah, tako matematično kot izkustveno, s povezovanjem z zgodbami, ki zagotavljajo kontekst in razlog za učenje.*
- *Izvajanje nalog, ki spodbujajo sklepanje in reševanje problemov: Učenci morajo biti vključeni v reševanje problemov in diskusijo o nalogah, ki spodbujajo matematično sklepanje in omogočajo raznolike strategije reševanja problemov. Učence je treba spodbujati k izvajanju dejavnosti in jim ponujati raznolike in zahtevne naloge, ki omogočajo odločanje, vključujejo učence v preučevanje predpostavk, dokazovanje, razla-*

go, razmišljanje, razlago, spodbujajo razpravo in komunikacijo, spodbujajo izvirnost in odkritje, zastavljajo vprašanja "Kaj če?" in "Kaj če ne?", vsebujejo priložnost za nepredvideno rešitev.

- *Uporabljanje in povezovanje matematičnih predstavitev: Učitelji naj učence spodbujajo k raznolikemu predstavljanju matematičnih vsebin (besede, risbe, diagrami, grafi, seznami, tabele, številke, simboli itd.) in k povezovanju znanja s ciljem poglabljanja razumevanja matematičnih konceptov in postopkov ter orodij za reševanje problema.*
- *Olajševanje smiselnih matematičnih razprav: Razprave v učilnici o matematičnih temah so pomembno orodje za gradnjo znanja in razvijanje razumevanja. Učitelji bi morali olajšati razpravo z učenci in razpravo med njimi, da bi zgradili skupno razumevanje matematičnih idej, analizirali in primerjali učenčeve pristope in argumente.*
- *Opozarjanje učencev na pogoste napake: Učne dejavnosti bi morale v ospredje postaviti razmišljanje učencev, učence soočiti s pomanjkljivostmi v njihovem znanju in ustvarjati priložnosti za odpravo napačnih predstav.*
- *Razlikovanje izzivov: Učitelji morajo pomagati tistim učencem, ki potrebujejo dodatno podporo, in izzive ponuditi tistim učencem, ki so na to pripravljeni.*
- *Gradnja postopkovnega znanja iz konceptualnega razumevanja: Proceduralno znanje je treba graditi s konceptualnim razumevanjem, tako da bodo učenci sčasoma postali usposobljeni za uporabo postopkov in algoritmov pri reševanju kontekstualnih in matematičnih problemov. Proceduralno znanje lahko razvijemo s kratko dnevno vadbo miselnih procesov, s pomočjo prakse pa lahko okrepimo in spodbudimo prenos naučenih veščin.*
- *Spodbujanje sodelovanja: Učitelj naj spodbuja sodelovanje med učenci. Učinkoviti učitelji načrtujejo delo učencev v majhnih skupinah, da jim pomagajo razpravljati o pomembnih idejah. To ima pozitivne učinke na učenje, socialne veščine in samozavest.*
- *Uporaba tehnologije: Učinkoviti učitelji uporabljajo tehnologijo za predstavitev matematičnih konceptov na dinamične, vizualno privlačne načine, ki motivirajo učence in jih spodbujajo k delu. Tehnologija je uporabna tudi za sodelovalno učenje.*

Značilnosti učinkovitega pouka matematike so do neke mere skladne s prepričanjem učiteljev o tem, kaj je kakovostno poučevanje. Toda način, kako se te lastnosti prepletajo in dosegajo v učilnici, je pogojen tudi s kulturnim kontekstom. Študenti v Vzhodni Aziji so na primer izjemno uspešni v mednarodnih študijah matematike, kot sta TIMSS in PISA, po drugi strani pa razne študije kažejo, da je poučevanje matematike v teh državah precej tradicionalno (Leung in sod., 2015). Reševanje problemov je značilno za učinkovito poučevanje med kulturami, v vzhodnoazijskih državah ima reševanje problemov popolnoma drugačno obliko kot reševanje problemov na primer v ZDA in Avstraliji. V zahodnih državah je poudarek na procesih pri reševanju problemov, medtem ko je v vzhodnoazijskih državah poudarek na izdelku te rešitve (Corey in sod., 2010).

Za učinkovito poučevanje je potreben kakovosten učitelj z visoko stopnjo pedagoškega znanja, da lahko ustrezno izvaja pouk z značilnostmi učinkovitega poučevanja. Tudi rezultati tega prispevka lahko služijo kot izhodišče za reforme in spremembe pri pouku matematike. Pri reformi in izboljšanju učne prakse bi bilo treba upoštevati vpliv zgodovinske dediščine in kulturnih prepričanj. Stališča učiteljev o matematiki, poučevanju in učenju matematike so ključnega pomena za doseganje kakovostnega pouka, zato je njihovo prepričanje zelo pomembno pri oblikovanju smernic za reforme izobraževal-

nega procesa. Zato je zaželeno vedeti, kako učitelji matematike pojmujejo učinkovito poučevanje, kako se opirajo na zgodovinsko tradicijo pouka matematike, kaj je koristno v tej tradiciji, kako jo lahko ohranimo in kako lahko popravimo tisto, kar ne deluje.

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