

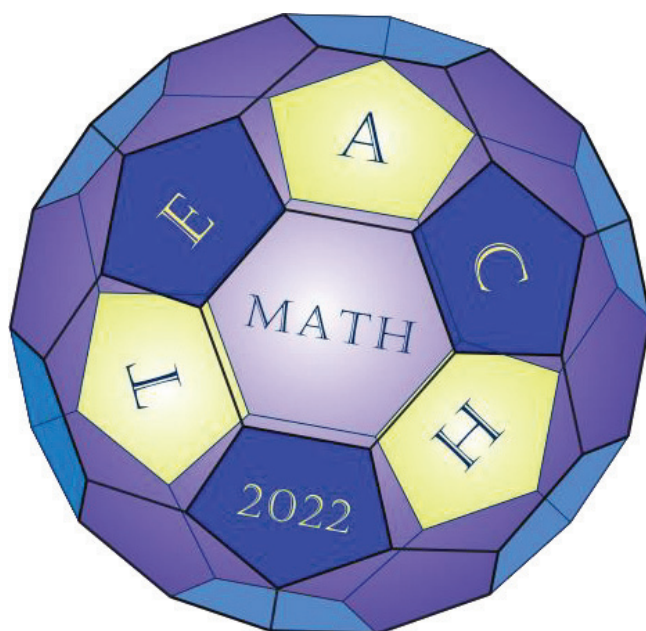
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Faculty of Education



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ADVANCES IN RESEARCH ON TEACHING MATHEMATICS

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ELEMENT

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ADVANCES IN RESEARCH ON TEACHING MATHEMATICS

NAPREDAK U ISTRAŽIVANJU NASTAVE MATEMATIKE

monograph

Editors:

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Osijek, 2022

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A word from the Editorial Board

Teaching mathematics has historically advanced since educating selected professionals in mastering particular mathematical skills and drilling the general population with rudimentary arithmetic skills. Teaching is a practice of disseminating knowledge conformed to various social, cultural, scientific and pedagogical conditions. It assumes different forms – engaging in the classroom discourse as a social practice, presenting mathematical knowledge appropriately and truthfully, moderating students learning through discussion, inquiry and evaluation, and others. The role of the teacher moved away from a transmitter of knowledge toward roles of facilitator of learning, manager of tools and discourse, and researcher in the study process (Lerman, 2020).

Research on teaching mathematics develops through different theoretical frameworks and focuses on different aspects of teaching, but the underlying question remains: *How can we advance mathematics education by advancing teaching?* Educational interventions have to be informed by the research in mathematics education (Chevallard, 1980). Implementing a new curriculum, exercising different teaching strategies, introducing alternative representations and technology needs to be accompanied by reflection on the outcomes of the teaching-learning process.

Studies about mathematical knowledge inform the educational practice, regardless of the scale. Revealing students' conceptions about a mathematical notion adverts teachers to consider those particular aspects in their teaching. Similarly, classroom assessment for learning contributes to the teaching-learning process two-fold; students utilize the (formative) assessment results to enhance their learning and knowledge, and teachers use them to modify their teaching. The stakeholders should draw on the results of the external state examinations and large scale international studies when framing educational policies, but keeping in mind the social, cultural and pedagogical conditions in the country (Artigue & Winsløw, 2010). Researchers and educators should disseminate the practice-oriented results from the studies on teachers and teaching to advance professional practice through in-service teacher conferences and pre-service teacher education. Teachers' and students' attitudes about mathematics have a significant role in the teaching-learning process (Philipp, 2007). Raising students' interest meliorates their attitudes and motivation, which in turn has a positive impact on learning outcomes. The former can be accomplished by implementing different teaching strategies and learning activities, using contextually appropriate and relevant content, engaging in popularization activities, and mathematical modelling. The Covid-19 lockdown brought new challenges for teachers and their practice (Ní Fhloinn & Fitzmaurice, 2021). Emergency remote teaching was organized differently but mainly utilized resources from the numerable assortment of digital, technological and web-based tools. Though implementing technology under unprecedented circumstances in teaching was prompt and inevitable, it provided invaluable experience for further questioning and designing deliberate and purposeful technology integration in mathematics education.

Advancing in teaching mathematics puts to the fore the issue of effective teaching in the sense of practice that enhances learning outcomes. What are the premises of effective mathematics teaching? How does students' knowledge shape mathematics teaching? Which teaching methods and strategies are the most effective in mathematics education and related to particular mathematical notions? How can using technology support effective mathematics teaching? Answering these and many other questions is the focus of research in teaching mathematics.

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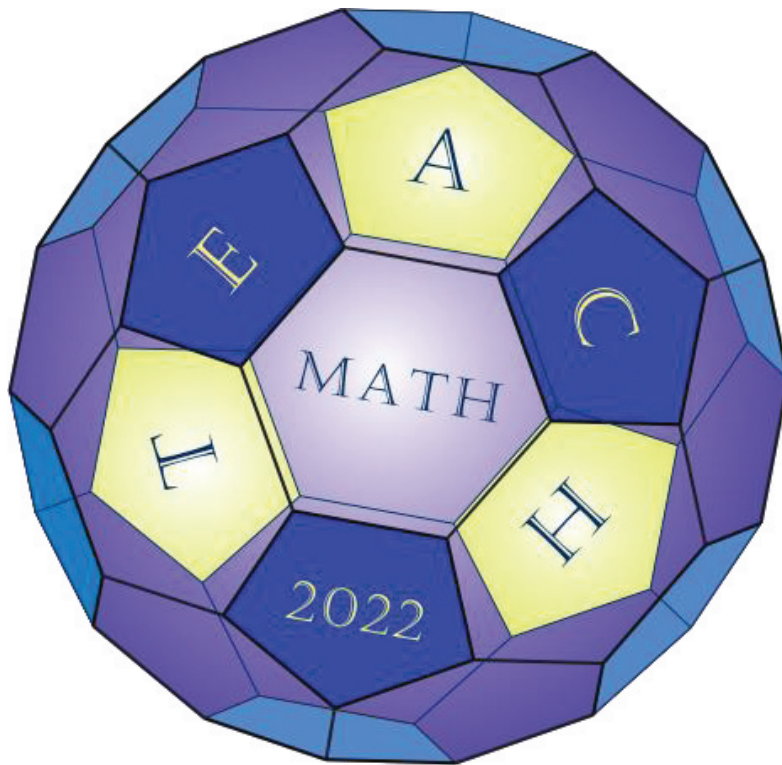
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ADVANCES IN RESEARCH ON TEACHING MATHEMATICS

monograph



Preface

Research presented in the papers collected for this Monograph deals with various aspects of teaching mathematics on different educational levels and across several European countries. The Monograph consists of three chapters.

In the first chapter of the Monograph, the authors wrote about students' knowledge and representations of knowledge in teaching mathematics. The first paper examines types of errors in students dealing with different aspects of fractions. The author cautions for teaching directed towards comprehension to reduce misconceptions and recognize mathematical concepts in a real-life context. The results of recent studies focussing on typical misconceptions when dealing with different aspects of fractions are present in the paper. In the second paper, the author compared geometry tasks in university entrance examinations across a period and found the content of the tasks decreasing in complexity. Throughout his survey, the author considered the commonly used five components of spatial intelligence as well as the mathematical content of the problems. The authors of the third paper in this chapter raise an interesting issue of managing dichotomies in mathematics education. They discuss and advise mindfully combining different teaching approaches to produce the best results depending on the educational context. Using the example of introducing students to the mathematical structure of division, they demonstrate different ways of knowledge transfer, the role and meaning of repetition, and the logic of including concrete representations. The fourth paper reports on teachers' use of didactic manipulatives in primary mathematics education. In the final paper in this chapter, the authors conducted an experiment showing that visual representation facilitated problem-solving.

The second chapter contains papers focused on teaching practice. The author of the first paper presents a theoretically supported method for introducing arithmetic operations in primary education. The concurrent method of teaching multiplication and division in elementary school is discussed. The second paper investigates interactions in children's mathematical lessons. The authors found that the different classroom setting encourages mathematical communication and connecting pieces of mathematical knowledge. In the third paper, the authors describe the construction process and use of a novel game for mathematics education. It advantages students' knowledge by describing and connecting different ideas encountered across the mathematics curriculum. The authors of the final paper in this chapter propose football-related tasks and activities across all grades of primary education. They argue for the popularization of mathematics by connecting to interesting and familiar realistic topics.

The content of the third and the last chapter of the Monograph relates to technology in teaching mathematics. The first paper reports on prospective teachers'

familiarity and attitudes about participating in the eTwinning programme. The following papers address the consequences of online education during Covid-19 lockdown. The authors of the first paper examined the differences in grades obtained after onsite and online learning and the author of the second paper examined students' attitudes and satisfaction concerning an online course. The final paper presents a result of mathematical modelling of population density data used for computer mobility simulations.

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1.

Students Knowledge and Representations in Teaching Mathematics



An Analysis of Students' Misconceptions as a Method of Improving the Teaching and Learning of Mathematics Through Better Comprehension: The Case of Fractions

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Abstract. Students' misconceptions in mathematics can be addressed at various levels, one of which is recognizing errors at a deep, discursive level, which requires a diagnosis of the reasons for observed errors and the involvement of students in dialogue, explanation, and justification of sounder mathematical reasoning. Four typical error types identified in the diagnosis of students' misconceptions are presented in the paper: modelling, prototype, overgeneralization, and process-object errors. Recognizing error type is crucial for planning the kind of mathematics instruction that will result in a higher level of comprehension. We will emphasize the rule of productive tasks that trigger cognitive conflict and promote students' articulation, reformulation, reflection, and resolution of their thinking process. We present the results of recent studies on fractions focussing on typical misconceptions when dealing with different aspects of fractions. An analysis of errors gives us insight into the reasons for students' misconceptions, which may be related to a lack of conceptual understanding of fractions and the failure to recognize the mathematical context in a contextual, real-life situation. Research findings remind us of the importance of careful planning by teachers in order to promote the development of higher levels of comprehension in mathematics. Teacher should consider both the complexity of the mathematical concept and the importance of the associated mathematical context using an appropriate selection of realistic and contextual problems.

Keywords: misconceptions, fractions, modelling error, prototype error, overgeneralization, process-object error

1. Introduction

Designers of teaching plans have long recognized the importance of analysing subject-matter in order to facilitate learning through the appropriate selection, organization, and sequencing of knowledge. The didactics literature of mathematics frequently explores different forms of knowledge: for example, instrumental, relational, conceptual, procedural, algorithmic, formal, visual, intuitive, implicit, explicit, elementary, advanced, knowing that, knowing why, and knowing how (Tirosh, 1999). The most established classification of knowledge is conceptual and procedural knowledge (Hiebert, 1986). Conceptual knowledge refers to the understanding of the concepts (the object of knowledge), while procedural knowledge refers to the rules (the action on the objects of knowledge). Fischbein (1999), presents a very basic classification of knowledge, distinguishing between intuitive and formal knowledge. He argues that knowledge of intuitive interpretations is crucial for teachers and didactic researchers because formal knowledge often collides with intuitive interpretations, which are resistant by nature and often contradict scientifically established ideas. Recent research, which pursues the importance of developing higher levels of thinking and reasoning as well as relating mathematics to real life situations, not only promotes these categories of knowledge but also complements them with other categories, including the following:

- contextual knowledge, which relates to everyday problems in the real world (Rittle-Johnson & Koedinger, 2005);
- principled-conceptual knowledge where students are able to invent procedures that are mathematically appropriate and recognize that their knowledge can be applied in a range of different contexts (Van den Heuvel-Panhuizen, 2003);
- knowing-to-act knowledge (an extension of what is called knowing about – knowing what, knowing why, and knowing how – but does not automatically develop the awareness that enables students to apply this knowledge in new situations) is the type of knowledge that requires sensitivity to situational features and some awareness of the moment so that relevant knowledge can be accessed when appropriate (Mason & Spence, 1999).

These recent extensions of the types of knowledge correspond with the ideas of Dubinsky (2001) who believes that the process of acquiring mathematical knowledge can be understood as a structured sequence, starting with the interpretation of problems through the process of internalization and ending with the process of the externalization of concepts needed to solve a problem.

Based on the different types of knowledge in education, we are also able to identify different types of errors that students tend to make during the learning process. Donaldson (1963) and Orton (1983) explain three types of errors that are frequently in students' learning of mathematics: (a) structural errors that are due to the failure to recognize the relationships involved in a problem; (b) arbitrary errors that occur because students do not consider the constraints of the given information or situation, and; (c) executive errors that consist of students not performing the

proper steps even though they understand the principals involved. In their research study, Avital and Libeskind (1978) similarly mention the following three types of difficulties faced by students: conceptual difficulties, mathematical difficulties, and technical problems. Despite the different names, parallels with Donaldson's classifications are easily made.

Among recent research studies, Riccomini's classification (2016) should be mentioned, which distinguishes between: (1) conceptual errors that occur when students have misconceptions and poor understanding of mathematical concepts, procedures, and applications; (2) procedural errors that occur when students perform steps in the wrong order (e.g., regrouping, decimal position, equivalent fraction); (3) factual errors, also known as arithmetic errors, that occur when students do not recognize signs, digits, or place values, or when they use incorrect formulas, and; (4) fluency errors that occur when students do not pay attention or work too quickly when solving mathematics problems (e.g., counting incorrectly, writing the wrong number, or not following directions). According to Makhubele, Nkhoma, and Luneta (2015), the primary sources of errors are insufficient prior knowledge, faulty reasoning, insufficient procedural and conceptual knowledge, educators, faulty schema, and insufficient knowledge of content. Errors are classified as follows: slips, conceptual errors, and procedural errors.

Thus, we see that all of these different classifications of error types are related. On the one hand, there are errors that have no obvious developmental or conceptual explanation (students make slips when they misread, misremember facts, skip some steps in the procedure, miscalculate, etc.). On the other hand, there are also many significant, developmental errors that relate to misunderstandings of the concepts.

The second types of errors – misunderstandings of concepts – are the focus of this paper so we will begin by taking a closer look at the definition of conceptual errors. Maryati and Priatna (2018) state that a conceptual error can be defined as a definition that is incorrect with respect to a concept, the use of an incorrect concept, the classification of an incorrect concept, and the relationship of incorrect concepts. The learning of mathematics is cumulative, meaning that newly acquired knowledge is related to previously acquired knowledge. Thus, if a student is unable to “assimilate” and “accommodate” new knowledge, a gap in concept learning occurs, and this in turn leads to mathematical errors or misconceptions (Roselizawati Hj Sarwadi & Shahrill, 2014). Misconceptions can also be caused by the cognitive development of students not matching the concept being learned, the thinking limitations of students, and the inability of students to understand concepts, and finally the interest of students in learning concepts. According to Kmetič (2013) misconceptions can occur as an incomplete or incorrect conception of a concept. In the case of a complex concept, a mistake may be due to a deficiency in the conceptual schema, or, in contrast, the conceptual schema may not be fully constructed or it may have broken down in the context of a learning situation that is incomprehensible to students.

Thus, there are various reasons for misconceptions. The role of teachers in the learning process is to be able to judge what kind of error has occurred and how

deep the misunderstanding is. We can distinguish between two levels of teacher assessment of the knowledge of students:

- the superficial level or recognizing the error: here the teacher identifies the error and offers a correct alternative (the focus is on procedure which is more suitable for recognizing procedural errors). The superficial level does not produce insight into the nature of student errors to the extent that teachers cannot know with certainty whether the errors are procedural, conceptual, or simply a fluency error.
- the deeper level: this requires a diagnosis of the reasons for the observed errors, and then that the teacher engage in dialog, explanations, justifications of reasoning, and teaching strategies that address or oppose the underlying reasoning and subsequent miscomprehension of students (Ryan & Williams, 2007).

While some errors are not obviously related to misconceptions or conceptual limitations, some clearly are, and may relate to limitations in students' basic mathematical concepts. These are often associated with student's knowledge at certain points in their conceptual development. Understanding such errors, gaining insight into student's reasoning and the reasons for errors they make, are crucial for teaching and learning with understanding. Errors, in fact, provide the teacher with insight into the conceptual structures that students are building and can therefore provide clues to appropriate interventions: specifically, through the design of productive tasks, argumentation, and cognitive conflict in discussions.

Additional analysis and classification of conceptual errors contributes to a more efficient process for diagnosing problems. The following is a summary of some of the most common reasons for conceptual errors as summarized by Ryan and Williams (2007).

2. Conceptual errors

Ryan and Williams (2007) describe four types of misconceptions:

1. modelling
2. prototypes
3. overgeneralization
4. process, object, and structure misconceptions.

By classifying conceptual errors into these categories, we give teachers the ability to interpret and better understand learner behaviour in mathematics.

2.1. Modelling

Sometimes a mathematical task or problem is represented in a way that contradicts a child's mental image of the concept: that is, the mathematical way of represent-

ing a concept does not match the child's non-mathematical way of representing it. When we talk about modelling, we are referring to the way in which mathematics is connected to the everyday world – how the real world is represented by mathematics. When a child commits a modelling error, it means that his or her own model of the situation is at odds with the mathematical model that is expected in the academic context.

Some students easily recognize that a mathematical mindset is required: that is, that a task must be solved in a specific, mathematical way, but some students do not have this sense of “mathematical play”. Most teachers take the ability to decode the mathematical mindset for granted, consequently mathematical rules remain invisible (e.g., the relationship between Dienes cubes for ones, tens, and hundreds).

Modelling errors refer to different types of representations: mathematical language, symbolic, graphic, or concrete. Discrepancies between mathematical and non-mathematical meaning can occur in all types of representation.

The following is an example of a terminological modelling error: how much do we get if we divide six by one half? Intuitively – and this is tied to everyday speech about halves – we conclude that to divide by half means to divide into halves. In this case, mathematical language is interpreted in terms of everyday language: six halves leading to a result of three, which is contrary to the correct mathematical solution $6 : \frac{1}{2} = 12$. It follows, therefore, that students need to learn the rules of discourse that school mathematics require, the transition from informal and contextual language to formal and mathematical language. It is a matter of pedagogical judgment to decide when to allow informal language and when to introduce the formal, mathematical language.

In general, everyday contexts provide an essential resource for building mathematical comprehension. However, all too often we are unaware of the limitations of external representations in representing mathematical ideas.

As an example of modelling errors in concrete or graphic representations, let us consider how the cake model is often used to represent fractions. If we are interested in how many halves we get when we cut a cake, or if we are interested in how many halves we get from six cakes, dividing cakes is a helpful model, but if we are wondering how much is six divided by a half, you cannot divide six whole cakes by halves to get the result twelve. Thus, the representation of fractions using a cake model turns out to be incomplete. Rarely can a model be completely sufficient on its own to develop a concept; usually teachers must flexibly alternate and connect different representations of the concept. When dealing with numbers up to 100, the graphic model of 100 squares can be a useful model for representing the size of a number and its decimal structure, but it fails when comparing and sorting numbers by size for which the number line model is more efficient.

The use of models (metaphors, contexts, concrete representations) gives meaning to mathematics by making connections between mathematics and what is already known. But these concretizations of mathematics always entail limitations: if

the context of the model were an exact match for the mathematics, then it would *be* the mathematics. Thus, errors and misconceptions often reveal the gap between the use of a model, context, or metaphor, and the mathematics being taught (Wartofsky 1979, Black, 1993).

2.2. Prototypes

People do not usually learn new concepts deductively but more naturally learn through examples that illustrate the concepts: that is, inductively. In doing so, we often use prototypes or typical representatives of a certain concept. This can lead to misunderstandings about the concept as we fail to recognize the concept in different circumstances and the breadth and variety of the concept is often overlooked.

If we ask students how they imagine a rectangle, they will most likely choose a shape where one side is longer than the other, with a longer horizontal side. It is unlikely that a student will choose the image of a square because it is generally not perceived as a rectangle. This is a prototype error. Most people naturally develop concepts informally and intuitively in a prototypical manner and therefore often fail to recognize an atypical representative of the concept.

Furthermore, misconceptions can be reinforced by inadequate instruction that does not feature an expansion of the understanding of the concept that includes atypical representatives. With inductive reasoning, mathematical definitions are generated from the observation of a diverse set of examples and counterexamples. These provide the basis for generalizations that capture the common features of individual cases. Thus, there may be a divergence between the intuitive perception of a concept and its mathematical definition generated by the use of strictly defined mathematical terms.





	Triangle	Non-triangle
Intuitive		
Non-intuitive		

Figure 1. The classification of shapes from the mathematical and psychological points of view.

Take the question: what shape is a triangle? To answer the question, we must create a rational set of examples, and examine both the mathematical and

psychological points of view. The mathematical aspect of the problem is based on the mathematical definition of the concept and divides the set of shapes into examples and counterexamples. The psychological aspect of the problem is based on non-mathematical knowledge about shapes, which can be either intuitive or non-intuitive (Tsamir & Tirosh, 2010).

Prototypes are narrowed down to a subset of a set of shapes, which is presented in the upper left quadrant. A prototype error can have two possible consequences:

- the failure to perceive triangles belonging to a non-intuitive group of triangles (lower left quadrant),
- perceiving the group of non-intuitive non-triangles as triangles (lower right quadrant).

An important task of teaching mathematics, therefore, is to recognize and use the prototypicality of an intuitive notion in the first place. Students must be given a rational set of examples containing atypical representatives of the concept with irrelevant properties (side length, angle size), and a set of counterexamples illustrating the relevant missing properties of the concept (the type of boundary line).

2.3. Overgeneralization

Unlike the prototype error, where there is too narrow an understanding of a concept and the necessity of generalizing to all relevant cases, overgeneralization is a type of error where generalization is extended to cases where the rule/method no longer applies. The following are two groups of typical overgeneralizations: the first has to do with the preference zone related to natural numbers compared to other sets of numbers, and the second has to do with the preference zone related to the use of addition compared to other arithmetic operations.

Let us consider the first kind of overgeneralization. When doing arithmetic with natural numbers, students often learn “rules” that make it easier for them to calculate, but that are only valid in the context of arithmetic with natural numbers. At a certain point in the teaching of mathematics, we simplify mathematics to make arithmetic easier for students, adapting mathematical rules to the developmental level of the students. For example, students often maintain the belief that “multiplication makes numbers bigger, division makes numbers smaller, or, for example, when we multiply by 10, 100, or 1000, we just add 1, 2 or 3 zeros to the end of the number.” These rules, which initially make arithmetic easier, can later create a misconception in the form of an overgeneralization error: for example, when students transfer the rules established for natural numbers to calculations of fractions, decimals, and negative numbers. Students often ignore what is not familiar to them. If they ignore the $(-)$ in front of negative numbers, then -5 becomes 5; they ignore the dot in decimals and 1.2 becomes 12; they ignore the sign % and 30 % becomes 30. Moreover, this misconception leads to miscalculations:

students calculate $1.2 + 3$ as if there were no decimal point and it results in 15, or $1.6 \cdot 100 = 1.600$ as they simply add 2 zeros. We may infer that students prefer natural numbers to other number sets, and these errors are typical of restricting their perspective to natural numbers.

A more challenging example of this type of error is reading decimals as a pair of integers separated by a dot. This rule does not cause an error, for example, with the calculation $1.2 + 3.4 = 4.6$, but leads to cognitive conflicts when students see the number 2.5 as smaller than the number 2.36, because 5 is smaller than 36, or when the addition of 1.23 and 2.4 results in 3.27.

The second type of overgeneralization error involves the use of an addition strategy when a multiplication strategy is required. This results in cancelling fractions by subtracting the same number from the numerator and denominator: so $3/5$ becomes $2/4$.

Later in the mathematics curriculum, a similar problem occurs when teaching the power of numbers. Students tend to calculate powers by multiplying the base by a degree, rather than multiplying the base as many times as the degree dictates. Again, this shows the preference for adding equal numbers instead of multiplying equal numbers.

In conclusion, generalizations that are introduced as general rules can, at a certain level of mathematics instruction, become overgeneralizations when they are extended to new sets of numbers or operations. This problem could be avoided if we recognized the point when extending the range of numbers or new arithmetic operations becomes a critical moment in the lesson that needs special attention, and make students aware of the situation by asking them suitable questions such as: is this rule still correct? When can you no longer use this rule? Why not? How would you explain this?

2.4. Process, object, and structure errors

These misconceptions are most easily illustrated by the example of counting when the process of counting and the result of counting – the object – overlap. In this situation, it is necessary to reconcile, on the one hand, a process that follows the order and one-to-one principle, and, on the other hand, the final state that is determined by the cardinality principle. The counting process is only completed by determining the final state of counting, i.e. the numerosity of the counted set. Grey et al. (1999) describe the differences between elementary and advanced mathematics, and argue that the difference between students who succeed in both elementary and advanced mathematics and students who fail, is rooted in the ability to switch flexibly between viewing a symbol as a process and viewing a symbol as a concept.

Learning mathematics involves many such process-object relationships, and failures in this context often indicate a students' inability to complete process-object verification (Grey & Tall, 1994). Students must understand which perspective to

adopt in a given mathematical task and be able to switch between the two. In addition to counting, understanding the equal sign also falls into the category of process-object misconceptions. Students first perceive an equal sign as a process: it is an instruction to perform an arithmetic process, such as $3 + 2 = ?$ (with the equal sign being interpreted as “it makes”). Later, the equal sign takes on an additional meaning, namely as an element of balance in a number sentence, and is interpreted as: it is the same as. In the beginning, students may have problems understanding the meaning of sentences like $___ = 5 - 3$. Later, when they are confronted with negative numbers, this problem manifests itself in a misunderstood sentence like -5 , $-(-5)$ or $3 - (-5)$. Students must switch between two different meanings of the same sign: minus as an object defining negative numbers, and minus as a sign for the operation of subtracting.

Another process-object error occurs in measurement. Students may have difficulty recognizing the relationship between a number that indicates a measurement and the process of measurement that produces it. For example, the label 8 on the ruler represents the object of measurement, but the process of measurement actually involves 9 numbers, not 8, because we start at the label 0 on the ruler and we count 8 intervals, not 8 points, from the starting point. Later, this misconception can lead to further misunderstandings when calculating with the ruler. For example, students may conclude that $12 - 5 = 8$ because they are counting down the 5 digits and not the 5 intervals between the numbers.

Recognizing a process-object misconception suggests that reification must take place: the use of the appropriate model, context, and name of an object. Sometimes just the grammatical shift from verb to noun helps students to notice the distinction between process and object (e.g., process – add, object – addition or sum).

The choice of context based on the increase of the original quantity can support the dynamics of the process (e.g. There were three children on the playground; two more came. How many children are there now?). However, if we want to focus on the object, we can choose another kind of additional situation that is more static (e.g. Maja has three red and two yellow marbles. Kaja has the same number of marbles as Maja. How many marbles does Kaja have?). A graphic representation that includes a representation of the left and right sides of the equal sign can help the equal sign be recognized as an object (it is the same as):

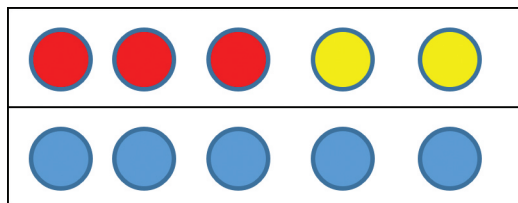


Figure 2. An example of graphical representation for static additional situation.

3. Analysis of fractional misconceptions

In this section, conceptual errors that concern the mathematical concept of the fraction are analysed.

Research in the field of understanding fractions (Clarke, Roche & Mitchell 2007; Pantziara & Philippou 2012; Hodnik Čadež & Manfreda Kolar, 2018; Manfreda Kolar, Janežič & Hodnik Čadež, 2015) demonstrates that many students have a weak conceptual understanding of fractions. The main reason for students' difficulties in learning fractions lies in the complex notion of the fraction itself (Empson & Levi 2011; Kieren 1993; Lamon 1999; Pantziara & Philippou 2012; Steffe & Olive 2010). Fractions can be interpreted in different ways, specifically through the following five subconstructs: the part-whole subconstruct; fractions as positions on a number line – measure subconstruct; fractions as the result of a division – quotient subconstruct; fractions as operators – operator subconstruct, and; fractions as ratios – ratio subconstruct (Behr et al. 1992; Carraher 1996; Kieren 1976). Recent research shows that fraction misconceptions are also related to inadequate mental schemas for fraction-based reasoning (Hackenberg & Tillema 2009; Norton 2008; Steffe 2002, Hackenberg 2007; Steffe & Olive 2010), that is, how students operate with fraction units and how they coordinate these units to give meaning to fractions.

Additional difficulties that students have understanding fractions are explored in light of the types of conceptual errors presented here. We focused on several examples from our recent research in this area. The examples are taken from the following research papers on fractions:

- HODNIK, T., MANFREDA KOLAR, V. (2018) “How fifth-grade pupils think about fractions: the reliance on part-whole subconstructs”. *Educational Studies in Mathematics*;
- MANFREDA KOLAR, V., JANEŽIČ, A., HODNIK, T. (2015) “Diagnosing students' difficulties in understanding the concept of fraction”. In: Novotná, J., Moraová, H. (Eds.). *Developing mathematical language and reasoning : proceedings*;
- MANFREDA KOLAR, V., HODNIK, T. (2021). “Mathematical literacy from the perspective of solving contextual problems”. *European Journal of Educational Research*.

We were interested in finding out to what extent the categorization of conceptual errors described above is useful for clarifying and analysing students' errors in the process of constructing a mathematical concept: do the presented conceptual error groups cover the full spectrum of conceptual errors in the chosen mathematical concept, or do some of the conceptual errors remain outside this categorization?

What follows is a presentation of selected cases from our research. We will briefly present the background of each case: the study sample, task, research objective, and the performance of participants. Our interpretation of the results is

complemented with an analysis of the nature of these kinds of misconceptions by Ryan and Williams (2007).

Case 1 (from Hodnik, Manfreda Kolar, 2018):

Sample: 90 fifth-graders from the age of ten to eleven from four public primary schools in Slovenia.

Task:

1. Which shape is divided into quarters? Mark YES or NO.
In cases you mark with YES, color one quarter.

YES NO YES NO YES NO YES NO

YES NO YES NO YES NO

Research objective: to investigate whether participants understand dividing a whole into equal parts as dividing a whole into congruent parts.

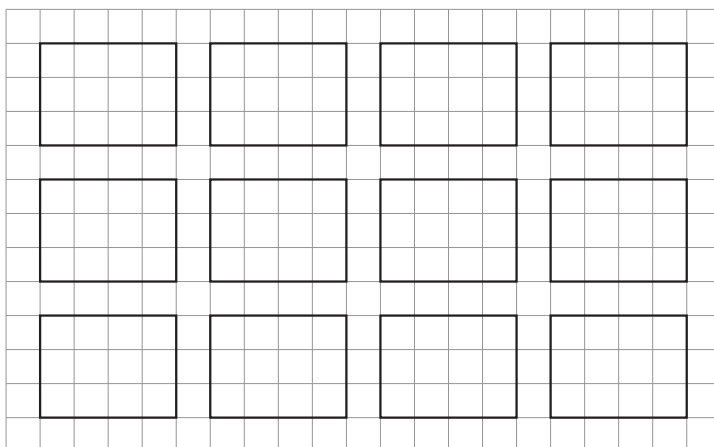
Analysis of the type of error: the task allowed us to evaluate whether participants understand Steffe and Olive's (2010) part-whole concept. The results show that only one among 90 participants managed to solve the problem entirely correctly, i.e. to recognize all of the divisions of the whole into quarters. We categorized the solutions of the other 89 participants according to the type of mistake they made. Analysis of the incorrect solutions revealed that participants mostly marked examples where the whole was divided into congruent parts (83 %). Therefore, the participants' misconceptions mostly arose from the (mis)understanding that dividing a whole into quarters is the same as dividing it into four congruent parts. Their understanding of dividing the whole into equal parts is narrowed to concrete examples, and thus this misconception can be classified in the group of prototype errors.

Conclusion: prototype error.

Case 2 (from Hodnik, Manfreda Kolar, 2018):

Sample: 90 fifth-graders aged ten to eleven from four public primary schools in Slovenia.

Task: colour one half of a rectangle in as many ways as possible.



Research objective: to investigate whether participants understand dividing a whole into equal parts as dividing the whole into congruent parts.

Analysis of the type of error: this time we used a more open-ended task where participants could present their own solutions for dividing the whole. We categorized their solutions according to the following characteristics:

- Type A solution: division into two congruent parts with a straight line.
- Type B solution: division into two congruent parts with a curved line.
- Type C solution: division into two congruent parts with a line consisting of several segments.
- Type D solution: division into two non-congruent parts with equal areas.
- Type E solution: a half is represented as a set of discrete parts of an area.

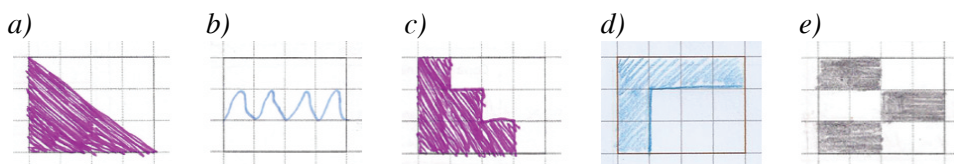


Figure 3. Examples of solutions for categories A, B, C, D, and E.

An analysis of these categories in the framework of Ryan and Williams's types of conceptual errors shows that the categories increase from the most prototypical to the least prototypical solutions. Category A includes the most typical representatives of the division of a whole where the parts overlap exactly. The same is true

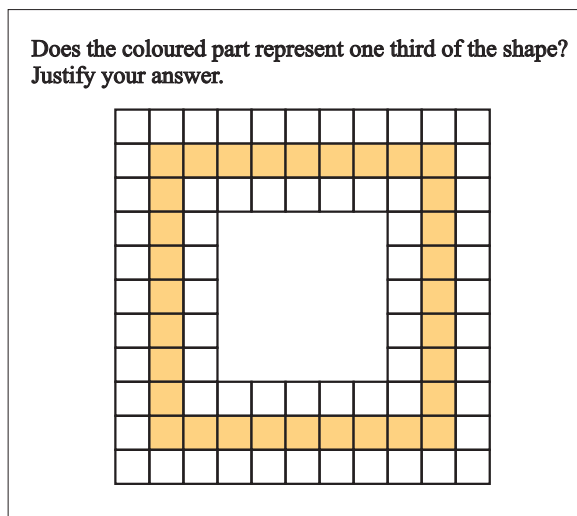
for examples B and C, except that the type of line used to divide the whole into two identical parts is less prototypical. Examples D and E, however, correspond to a non-prototypical division of the whole because the halves are not congruent (example D) or because the parts of the whole are not presented together (example E). Of the 90 participants, 58.9 % included at least one of these two types of solutions (D or E) in their responses. Thus, we can conclude that these participants succeeded in generalizing their understanding of the fraction concept and the part-whole aspect of the concept to non-prototypical cases, while the understanding of 42.1 % of the participants was connected to the division of a whole into two equal parts, that is, prototypical divisions of a whole where the resulting parts are congruent.

Conclusion: prototype error.

Case 3 (from Hodnik, Manfreda Kolar, 2018):

Sample: 90 fifth-graders aged ten to eleven from four public primary schools in Slovenia.

Task:



Research objective: to analyse participants' method of reasoning. Namely, do they realize that the whole is not divided into parts of equal area, but, despite that, the coloured part represents one third of the whole?

Analysis of the type of error: the results show that 49 % of the solutions were incorrect. Many of the participants focused on comparing the three parts. Instead of answering the question asked – “Is one part one-third of the whole?” – they answered the question – “Is the whole divided into three equal parts?” These appear to be intuitive interpretations of the question that do not respond to the actual wording of the question. This is consistent with the findings of Tirosh and Stavy (1999) who demonstrate that students' responses to a given task are often determined by the specific wording of the task and also by a repertoire of intuitive rules.

Let us now consider types of errors using Ryan and Williams's error types. An analysis of textbooks shows that students are usually confronted with two types of demands in part-whole tasks: either they have to determine whether the whole is divided into equal parts, or they have to determine a part of the whole that is already divided into equal parts. The following assumption follows very quickly from these examples: if the whole is divided into three congruent parts, then one part is a third. If the whole is not divided into congruent parts, then one part is not a third. This conclusion is a reflection of the prototypical approach to learning. If the whole is not divided into congruent parts, then we do not generally question the relationship between a particular part and the whole, but are interested in a comparison between parts of the whole. We thus conclude that one possible reason for incorrect answers in Case 3 is a non-prototypical formulation of the question.

However, this is not the only possible interpretation. Incorrect task solutions may also be associated with the process-object type of error. The task requires the students to switch between two perspectives: the process of dividing a whole and the result of dividing a whole – a part of the whole. These two perspectives may lead to conflicting conclusions. Observing the process, participants may conclude that the whole was not divided into thirds. In contrast, observing the finished object, participants may conclude that the coloured part represents a third of the whole.

Conclusion: prototype or process-object error.

Case 4 (from Manfreda Kolar, V., Janežič, A., Hodnik, T. (2015)):

Sample:

- 50 third-year elementary school student teachers,
- 24 fourth-year mathematics student teachers,
- 22 primary school students, grade 8.

Task:

The drawn rectangle presents $\frac{3}{4}$ of a whole. Draw $1\frac{1}{4}$ of a whole.



Research objective: to investigate students' understanding of fractions greater than one.

Analysis of the type of error: the most common error in this task was the incorrect division of a given rectangle representing $\frac{3}{4}$. Participants divided it into four equal parts instead of three (18 % of eighth-grade students, 6 % of elementary-school student teachers, and 17 % of mathematics student teachers). Participants reinterpreted the meaning of the rectangle: from $\frac{3}{4}$ to $\frac{4}{4}$. This error occurs when presenting fractions greater than one.

Hackenberg (2007) describes this as a problem of coordinating units. Student with a sufficient level of comprehension understand that the quantity, for example,

of four-thirds is represented by four units, such that three units represent the whole and the fourth third goes beyond the whole. In contrast, students without this level of comprehension will relabel four thirds as four fourths: i.e. they think of a third iterated four times as four fourths and each part is transformed into $\frac{1}{4}$ (Manfreda Kolar, Janežič & Hodnik, 2015).

In the context of Ryan and William’s error types, we consider this error a combination of two error types: the first error type may be attributed to the choice of models representing the fractions, and the second to the prototypical method of thinking. The area models of rectangular shapes are very useful for representing fractions up to one, but they become inadequate as we move above one. Students use the area model in such a way that they divide it into many smaller equal parts, and then a particular part of the whole is represented by combining the appropriate part of the smaller parts.

However, when we want to illustrate fractions greater than one, this type of reasoning fails because we need more than one whole to show the final value of the fraction. The second error type associated with this error is, as stated above, the prototype error. A typical fraction task on a regional model requires the solver to determine what is a part of the whole, given the whole. The converse task requires the opposite: the solver must determine the whole, given a part of the whole. Prototypical thinking that the model should be divided into the number of parts represented by the denominator of the fraction must be replaced by a new method of thinking: specifically, that the model representing a part of the whole must be divided into the number of parts represented by the numerator of the fraction.

Conclusion: modelling and prototype errors.

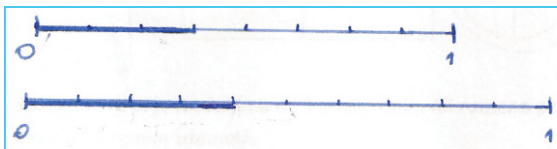
Case 5 (from Manfreda Kolar, V., Janežič, A., Hodnik, T. (2015)):

Sample:

- 50 third-year elementary-school student teachers,
- 24 fourth-year mathematics student teachers,
- 22 eighth-grade elementary school students.

Task: Show the values of the fraction as you wish and circle the bigger one: $\frac{3}{8}$ and $\frac{4}{10}$.

a)



b)

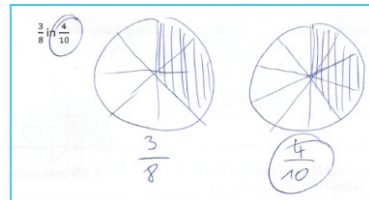


Figure 4. Students’ incorrect solutions when comparing two fractions.

Research objective: to investigate participants’ understanding of the importance of the solid whole when comparing fractions.

Analysis of the type of error: a deeper analysis of participants' incorrect solutions when comparing two fractions revealed the two most common errors: comparing two fractions with using an additive method of comparing fractions (see Figure 4a) and using an inadequate graphic representation with the awareness that it is the same whole (see Figure 4b).

The additive method of comparing fractions was used by 9 % of eighth-grade students, by 20 % of elementary-school student teachers (20 %), and by 4 % of mathematics student teachers. Inadequate graphical representations were used by 5 % of eighth-grade students, 36 % of elementary-school student teachers, and 29 % of mathematics student teachers. Each of these two groups of errors can be assigned to one of Ryan and William's error types.

First, we examine the error of using the additive method. This means that participants compare fractions with different denominators by adding the units (usually of area), which cause the whole to grow. Where does this pre-treatment method come from? Before children enter school, they are first exposed to addition and subtraction situations, and only later are they introduced to multiplication as an abbreviation of adding equal numbers. Given the power of counting strategies, it is not surprising that fractions are often introduced in the context of counting equal parts. According to Gould (2012), a strong emphasis on counting shaded parts of different shapes can lead to very restrictive ideas about what fractions are. Many children relate fractions to two separate counts: counting shaded parts, which represents a numerator, and counting all parts, which represents a denominator. Therefore, they understand fractions as an additive relationship rather than as a multiplicative relationship between quantities. This biased additive approach to comparing fractions can be defined as a type of overgeneralizing error.

Inadequate graphic representations (Figure 4) point to a different type of error, namely a modelling error. The circular shape (Figure 4b) turns out to be insufficient to identify the answer to the question as the two fractions are very close in value. A rectangular shape that allows division into 40 parts, which is the common denominator of the two fractions, would be a much more appropriate choice.

Conclusion: overgeneralization and modelling error.

Case 6 (from Manfreda Kolar, V., Hodnik, T. (2021)).

Sample: 72 sixth-graders aged eleven to twelve from three public elementary schools in Slovenia.

Task:

Mina, Kaja, and Lara have just come home from school. Mina comes into the kitchen and finds a full tray of freshly baked biscuits on the table, along with a note from her mother: "Distribute the cookies fairly among yourselves." Mina eats a third of the biscuits from the tray and leaves. Then her sister Kaja enters. Not knowing that Mina was already in the kitchen, she reads her mother's message and eats a third of the biscuits from the tray. Finally, Lara enters. Unaware that Mina and Kaja were already in the kitchen, she eats a third of the biscuits from the tray.

When her mother comes home, she finds eight untouched biscuits on the tray. How many biscuits did their mother bake?


Research objective: to analyse participants' strategies in solving a contextual problem on fractions.

Analysis of the type of error: the problem was solved incorrectly by the majority of participants (85 %). Their incorrect solutions include a wider range of misconceptions about fractions. We present some of these misconceptions in detail.

Figure 5a shows that participants convert the part $\frac{1}{3}$ of the whole into the rational number $\frac{1}{3}$. They are confusing two different aspects of fractions: fractions as divisions of a given whole and fractions as positions on a number line (Manfreda Kolar and Hodnik, 2021). The error is classified as an overgeneralization error because the rule of natural numbers prevails over rational numbers, and the participants disregard the fact that some data in the task refer to natural numbers while others refer to fractions.

Another misconception can be seen in the Figure 5b. Here participants assign the role of a third to the number eight, instead of two thirds, while at the same time disregarding the fact that the whole changes during the solution process. Therefore, all three girls receive the same part. The solution indicates that the students divided the whole into $\frac{3}{3} + \frac{1}{3}$. Thus, the remaining eight actually represents a quarter of the initial sum. According to Hackenberg (2007), students had difficulty coordinating units. They think of a third iterated four times as four quarters, and each part is considered to be transformed into a quarter. In accordance with Ryan and Williams (2007), this error can be defined as a modelling error (see explanation for Example 4). However, we believe that there is something more to it that cannot be explained only by the four types of errors discussed above. Let's analyse a similar case.

In the Figure 5c, we can see that participants attribute the wrong meaning to the number eight and solve the problem as if there were one-third instead of two-thirds. On the other hand, the three equations in the solution process indicate that the participants are aware that the whole changes during the problem solving process and therefore perform a backward solution process in three successive steps (Manfreda Kolar and Hodnik, 2021). Therefore, we can conclude that the main reason for the wrong solution is the misunderstanding of the context of the problem: participants misinterpret the given data and assign the wrong meaning to the remainder of the cookies. It is difficult to judge which of Ryan and Williams's types of error most closely corresponds to the situations of misunderstanding the context or failing to recognize the mathematical content.

a) $\frac{1}{3} + \frac{2}{3} + \frac{3}{3} = \frac{6}{3} + 8 =$ 

b) $8 \cdot 4 = 32$

c) $8 \cdot 3 = 24$
 $24 \cdot 3 = 52$
 $52 \cdot 3 = 156$

Figure 5. Participants' incorrect solutions to "the biscuits problem".

4. Conclusion

The investigation and analysis of the above cases shows that exploring the background of students' reasoning and their difficulties with mathematical understanding helps us, in almost all cases, to identify the type of conceptual error in Ryan and Williams's error categories. However, it should be noted that not all cases are clear-cut. In some cases, we were able to associate errors with different types of misconceptions, and in some cases we were not able to accurately identify the type of error at all.

Let us investigate the possible reasons for these unspecified cases. Selected examples of problems involving fractions were analysed according to the type of knowledge represented by the task. Here we refer to Gagne's knowledge taxonomy.

Table 1. Cases of fractions according to knowledge type.

Case	Student objective	Knowledge type according to Gagne
1	To recognize if the whole is divided into quarters	Conceptual
2	To mark one half of the whole	Conceptual
3	To recognize a part of the whole	Conceptual
4	To draw a fraction which is greater than one	Conceptual
5	To compare the values of two fractions	Conceptual
6	To solve a contextual problem about fractions	Problem

From Table 1, we may conclude that the problem of identifying the error type occurred in only one case, which was actually not a conceptual task but a mathematical problem (in accordance with Gagne).

In cases of taxonomically pure tasks of the conceptual type, it is easier to identify the error type than in more complex tasks where there is an interplay of different types of knowledge and the identification of the reasons is multifaceted. Errors can be classified into those related to limitations in the comprehension of the concept of fractions (four types of errors according to Ryan and Williams), and those related to difficulties in activating mathematical competences (i.e. where students do not know how to adapt the appropriate mathematical concept to a contextual problem) (Manfreda Kolar & Hodnik, 2021).

In our opinion, the latter task is of a more complex nature and errors in its solution cannot be reduced to the above four categories of conceptual errors. Relying on Rittle-Johnson and Koedinger (2005), we refer to a deficit in both contextual and conceptual knowledge. Research findings remind us of the importance of careful planning in the development of student comprehension by teachers who need to take into account both the complexity of the mathematical concept and the importance of the associated mathematical context with the appropriate selection of realistic, contextual problems.

In conclusion, we will answer some of the questions that emerged in the analysis of the students' misconceptions:

1. What are the benefits of analysing the causes of students' errors and identifying their misconceptions in relation to a particular mathematical concept?

The formative assessment of students' comprehension is one of the most important components of instruction. If we are to improve the quality of instructional practices, it is critical that the information teachers receive about students' (mis)comprehension be clearly defined, accurate, and applicable. It is useful when such information helps teachers to identify the reasons for students' difficulties, and enables teachers to plan additional instruction to improve students' understanding. In this sense, insight into the reasons for learning difficulties indicate guidelines for further instruction, the organization of teaching, and activities at beginning of instruction. Therefore, it is also useful to ask the following question:

2. What is the best way to organize lessons to get the best insight into the students' way of thinking?

The learning environment in which the teaching and learning of mathematics takes place needs to be encouraging for students in the sense that it allows for interaction between all participants and broader discussions about mathematics. Cooney (1999) argues that the focus on mathematics should change to one that encourages attention to context and reflection, creating a more student-centred classroom in which discussion plays a crucial role.

The term discussion refers to mathematical dialog in which different viewpoints are explored based on mathematical reasoning. According to Ryan and Williams (2007), mathematical dialog should involve a cycle of articulation, reformulation, reflection, and resolution. There are several key requirements for such dialog to occur:

- a) Teachers must choose productive tasks that include the possibility of discussion and argumentation, namely tasks that encourage the expression diverse student opinions.
- b) Students have the opportunity to communicate and share their views. There should be both an articulation and rearticulation stage: "I think this because. . ." (articulation); "I listened to what X said and now I think this because. . ." (reformulation).
- c) There must be criteria to assess what constitutes a good mathematical argument. During this stage, teachers guide students by asking questions such as: is this a good argument? How can you show that this is correct? Is there a more convincing argument? What is the best argument?
- d) Students should be given the opportunity to reflect on the discussion: What did you think before? What do you think now? What made you change our mind?
- e) The resolution phase comes after discussion of different views and reflection on the discussion: "I now think this because. . .".

Thus, if we want to improve students' comprehension of mathematical concepts, it is important to find suitable tasks, representations, models, and contexts, namely ones that trigger cognitive conflict in students with a lower level of comprehension. We believe that the examples of problems presented in this article

meet the criteria of a productive tasks. Specifically, incorrect solutions allow to identify students' typical misconceptions. It is up to individual teachers' discretion to design the mathematics lesson in the most suitable manner. Students may solve the tasks individually, and teachers later analyse their written answers to find possible explanations for students' errors. This is an indirect way of gaining insight into students' thinking as teachers make judgments on the reasons for students' difficulties based on written work but has no direct insight into the actual reasons for their problems. However, we believe that it is much more effective if teachers use methods that allow them to gain direct insight into students' reasoning (rather than relying only on written solutions). This can be achieved by guiding students in a mathematical dialog that follows the stages described above, concluding with the reflection and solution phase. This method is based on engaging students and developing learning through increased understanding, actively involving students in the process of transforming their thinking, and encouraging teachers to engage with students' thinking directly through mathematical reasoning.

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Analiza napačnih predstav učencev kot metoda za izboljšanje poučevanja in učenja matematike z boljším razumevanjem: Primer ulomkov

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Povzetek. Napačne predstave učencev pri matematiki se lahko obravnavajo na različnih ravneh, ena izmed njih je prepoznavanje napak na globlji, diskurzivni ravni, ki zahteva diagnosticiranje razlogov za opažene napake in vključevanje učencev v dialog, razlago in utemeljevanje matematičnega razmišljanja. V prispevku so predstavljene štiri tipične vrste napačnih predstav učencev: napake pri modeliranju, prototipu, pretiranem posploševanju in procesno – objektna napaka. Prepoznavanje vrste napake je ključnega pomena za načrtovanje pouka matematike na način, ki bo usmerjen k višjim ravnam razumevanja. Poudarili bomo pomen produktivnih nalog, ki sprožajo kognitivni konflikt in spodbujajo učenca k artikuliranju, preoblikovanju, reflektiranju in razreševanju konfliktov lastnega miselnega procesa. Predstavljamo rezultate nedavnih študij o ulomkih, ki se osredotočajo na tipične napačne predstave pri obravnavanju različnih vidikov ulomkov. Analiza napak nam daje vpogled v razloge za napačne predstave učencev, ki so lahko povezane bodisi s pomanjkanjem konceptualnega razumevanja ulomkov bodisi z nezmožnostjo prepoznavanja matematičnega konteksta v realistični situaciji. Ugotovitve raziskav nas opozarjajo na pomen skrbnega učiteljevega načrtovanja pouka za spodbujanje razvoja višjih ravni razumevanja pri matematiki. Učitelj mora z ustrežno izbiro realističnih problemov naglasiti tako zapletenost matematičnega koncepta kot tudi pomen z njim povezanega matematičnega konteksta.

Ključne besede: napačne predstave, ulomki, produktivna naloga, napaka modeliranja, prototipska napaka, pretirano posploševanje, procesno-objektna napaka

Spatial Geometry in Technical University Entrance Examinations in Hungary 1980 – 85 vs. 2015 – 2020

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Abstract. One of the outstanding challenges in technical higher education is to continue the development of visuospatial skills that has been started in primary and secondary education. In this paper the present state has been compared with the recent past by comparing the spatial geometry problems occurring in the technical University entrance examinations in both periods.

The hypothesis we set up is that the spatial abilities and geometric knowledge of incoming students are constantly narrowing. In order to decide the question, the utmost, macro-level, standardized, quantitative, summative, knowledge-level exam waiting for high school students, which is the University entrance exam/Matura has been investigated as a valid indicator of the secondary education in Mathematics.

In the study the quantity and weight of the spatial geometry problems in the two periods are compared and the visuospatial ability components required for their solution are identified, together with their quantity. The Mathematical complexity of the occurring problems has also been discussed.

As final result it is proved that the level of spatial geometry problems in University entrance examinations decreased significantly: their solution requires either one spatial intelligence factor less, or 2 mathematical subareas less than 35 years earlier.

Keywords: visuospatial skills, higher education, spatial geometry, Matura, teaching Mathematics

1. Introduction and personal motivation

As an enthusiastic teacher in university Mathematics and Geometry education, from the beginning I have placed great emphasis on ensuring that the methods and tools used during the classes should help as many students as possible. I have been teaching engineering students for more than 25 years, and in this period I have devoted myself primarily to the development of spatial skills, through the probably most suitable, traditional method, that is by teaching Descriptive Geometry, in all levels, at different Universities.

In the last decades we carried out several studies, using different test tools – (Bölskei et al., 2012), (Babály & Bölskei, 2017), (Babály et al., 2013) just to name a few and concluded that some factors of the visuospatial thinking can be developed effectively by Descriptive geometry. These results were confirmed later by other national and international studies, as well (Bölskei et al., 2013), (Kovács & Németh, 2014), etc.

However, by the time passing, we have the feeling that in order to bring the students to the required level of 3D thinking we need more and more effort and despite the huge exertion the results hardly meet the expectations. We believe that the reasons are to be found in secondary school education, therefore we formulate our hypothesis: the spatial abilities and geometric knowledge of incoming University students are constantly narrowing.

In order to decide the question we need an appropriate tool that helps to measure the mentioned factors and that will be the graduation (= Matura). Assessment is one of the basic concepts in the educational learning theory, which performs a feedback function in the examined pedagogical process. Its role is to measure the effectiveness of the teaching of a particular teaching unit and to observe whether the results meet the preliminary expectations. Adequate evaluation thus serves as a mirror for the whole pedagogical process. The utmost, macro-level, standardized, quantitative, summative, knowledge-level exam waiting for high school students is the graduation, which has also been used as a University entrance examination in Hungary since 2005. Therefore, its use as an indicator of the secondary education process is valid. Prior to 2005, University entrance examinations served as a Matura for those who passed them. Hence, our research compares the written entrance examination tasks in Mathematics between 1980 and 85 (my student age – “old school”) with the advanced level written graduation problems in Mathematics in the period 2015–20 (age of my recent students – “new school”).

2. Comparison of the Matura in “Old vs. New school”

Once again, by Matura we mean the secondary school exit exam, with other words graduation. It is very similar to the German Abitur, that is the Maturity Diploma, and to Baccalaureate Diploma which is the common name of the same document in other countries. State Matura has a rather long tradition in Hungary, because the first time it was organized in 1851. Graduation happens usually after 12 or 13 years

of schooling – it depends on the length of the secondary school education period – in some schools the pupils learn a foreign language for one year in the 9th class. The Matura, as a document, contains the grades achieved in the final exam and it is very important, because it formally enables the students to go to a University; without having it, you cannot enter a University. In Hungary, the graduation exams are obligatory from Hungarian literature and grammar, Mathematics, History, and one foreign language. The last – fifth – subject is of the student's choice. The only requirement is that the student had to study it for at least 2 years.

In the “old school”, more precisely, till 2005, those students, who wanted to apply to a University had to pass a unified, centrally coordinated university entrance examination (usually from two subjects, e.g. for technical Universities from Math and Physics), which served them as normal Matura. For the other – usually three – subjects they had to attend the ordinary Matura.

In the year 2005, the new system of the two-level Matura was introduced. That means that from each subject it is possible to take basic and advanced level graduation. In parallel the separately organized University entrance examinations were canceled. The advantage of the new system is being uniform, standardized and more controlled (centrally in advanced level, locally in basic level). By the original idea the advanced level graduation gave the permission to apply for University studies but the University lobby had made it to be optional, because they preferred to maximize the number of students (and hence, the income). Therefore, for most majors, for a long time the Universities didn't require the advanced level graduation, but it had changed recently: since 2020 one advanced level graduation is already needed for the University admission (but just one. . .). Therefore, it makes sense to consider the advanced level Matura as a new generation of the University entrance exams.

If we compare the “old school” with the new one we can observe that in the old one the oral part weighted much (50 %) and was more subjective, because it was organized separately by the Universities themselves, with University teachers. In the new system the written part of an advanced level Math exam scores 115 points, whilst the oral one just 35. The latter one must be done in selected high schools (so-called centers) in front of dedicated high school teachers.

In the “old school” the problem sheet contained 8 tasks – 5 obligatory and easier to solve ones, and 3 more difficult ones, which were different for general and specialized high schools. Each task covered one single problem, just very rarely more. In the “new school” the students have 4 obligatory tasks, which are easier to solve and 5 more difficult ones, but only 4 of them are allowed to be solved, one must be selected and omitted. The tasks – mostly contextual problems – always have more sub items, which are connected to very different chapters of Mathematics, and not to each other. A profound analysis of the Hungarian Maturation system and its change can be found in (Csapodi, 2017).

As to “old school”, the written part of the central entrance examinations was organized for all applicants by the state, at the same time, generally in great lecture halls of Universities. The majors were classified into “technical” and “economy” groups and these two had to solve different problem sheets. In the following we

restrict ourselves just to entrance exams in Mathematics to technical Universities, because we focus on engineering tertiary education. In addition, in each year extra (replacement) exams were also organized centrally for those who could not attend the main exam. The number of the problem sheets for these extra exams varies yearly from 2 to 4. Here the application for technical Universities was also possible, therefore we consider them in the survey. As a last possibility to enter technical Universities we mention the entrance examinations to abroad, into the COMECON countries (CCCP, DDR, Czechoslovakia, Bulgaria, etc). This exam was held much earlier, in January each year, and was centrally organized too. All in all, in the examined 6 year period 33 task sheets were considered, with 282 problems in total. From them 165 tasks belonged to the obligatory and easier to solve part and 117 problems were more complicated and belonged to the optional part. (The examinations are collected in (Scharnitzky, 1985 and 1987).)

In the “new school” the system is more transparent. Independently from the type of high schools the applicants attend or the type of University they want to apply for, the same task sheets must be solved. In advanced level the written part is held in some selected secondary schools (centers) for everyone at the same time. No replacement exams are organized in the spring period. For those, who couldn’t attend or want to improve the achievement, it is possible to repeat the Matura in the autumn, in October. This means that in the examined 6 years period we have less, just 12 task sheets, with 108 tasks in total. From them 48 tasks belong to the obligatory and easier to solve part, and 60 tasks are more complicated and partly optional. (The problem sheets can be found in [7].)

3. The first result and the compulsion to refinement

The method of the investigation was to carefully select the spatial problems from the whole list and to record their number and the points they score. In the “old school” usually one task equals one problem, but in the new one a spatial task often worths just $1/2$ or $1/3$ problem, because the other sub items of the same task are not spatial ones.

As first results we concluded the following.

Result I.: The weight of spatial problems is approximately the same (=) in 1980–85 and in 2015–2020.

Table 1. Weight of spatial problems in 1980–85 vs. 2015–20.

	1980–85	relation	2015–20
Weight in total points	8.80 %	=	9.06 %
Weight in number of problems	7.91 %	=	7.88 %

Result II.: The weight of spatial problems (expressed in the number of problems) become greater in the obligatory section and become much less in the optional section from 1980–85 to 2015–20.

Table 2. Weight of spatial problems in obligatory and in optional sections separately in 1980–85 vs. 2015–20.

	1980–85	relation	2015–20
Weight in obligatory section	5.00 %	<	7.98 %
Weight in complicated problems section	12.75 %	≫	7.22 %

Result I. seems to be comforting for those who worried about the declension of the graduation and contradicted the hypothesis we set. However, close examination of the issue revealed a more basic question. Namely what kind of problem can be considered as “spatial”? The question can be surprising, since all of us who teach Mathematics know this or don’t we? Very likely the answer would be, that if a problem deals with three-dimensional objects, spatial relations, determination of angles, distance, surface area, volume, etc. then it is obviously a spatial one.

But what to do with the following Example?

Matura 2017 Mai, Problem 6 a: Given a circular cylindrical barrel with volume of 200 liter. Its height is 80 cm. How much plate do we need to make such a barrel if it is open on the top and we must calculate with 12 % of waste?

In order to solve the task we have to know a volume and a surface area formula plus we need some basic skills in computing percent value. The solution is pure algebraic and no real spatial manipulation is needed!

This and quite a few other problems highlight the need of defining the class of spatial problems. In our opinion a real spatial problem needs spatial abilities, without their application the solution is merely magic with formulas. All this is supported by the history of spatial intelligence, whose cognitive approach focuses on those mental processes that are related to solving spatial problems. By a very popular definition of Hungarian researchers (Séra et al., 2002) spatial ability is “the ability of solving spatial problems by using the perception of two and three dimensional shapes and the understanding of the perceived information and relations” and this is exactly what we require in Mathematics. Hence, in the following by a spatial problem we mean a Mathematical problem that uses at least one factor of spatial intelligence.

The number and description of the factors of the visuospatial intelligence depends on the researchers and their preferences. Throughout this survey we used the commonly used five components of spatial intelligence from (Sorby, 1999). These are (1) spatial perception: reception and interpretation of visual stimuli – e.g. understanding a given figure of bodies; (2) visualization: complex, multi-step manipulation in which an optimal strategy must be sought – e.g. imagine and draw an appropriate figure from text; (3) mental rotation: rotation of three dimensional solids mentally; (4) spatial orientation: the ability of entering into a given spatial situation – e.g. choosing a new viewpoint; (5) spatial relations: the ability of recognizing the relations between the parts of a solid – e.g. analyze the section of a solid/folding-unfolding.

4. Results and their interpretation

If we accept the argumentation above and omit the pseudo-spatial problems from the list, then the number and weight of the spatial problems have been reduced significantly, but mostly in the new era. The reformulation of the first results are the following.

Result 1: The weight of spatial problems was greater in 1980–85 than in 2015–2020.

Table 3. Weight of real spatial problems in 1980–85 vs. 2015–20.

	1980–85	relation	2015–20
Weight in total points	7.51 %	>	6.30 %
Weight in number of problems	6.93 %	>	5.56 %

Result 2: The weight of spatial problems become greater in the obligatory section and become much less in the optional section from 1980–85 to 2015–20.

Table 4. Weight of real spatial problems in obligatory and in optional sections separately in 1980–85 vs. 2015–20.

	1980–85	relation	2015–20
Weight in obligatory section	3.89 %	<	6.61 %
Weight in complicated problems section	11.16 %	≫	4.17 %

These results already support the starting hypothesis and verify that spatial geometry is significantly repressed. What is more, the second result points out that the remaining problems are mostly easier and hardly any of them belong to the complicated part of the task sheets. Here we note again that one task from the optional part must be left by the applicant, so they can easily avoid any spatial problem if they wish!

Result 3: Among real spatial problems of entrance examinations the average number of the required factors of spatial intelligence was greater (2.15) in 1980–85 than in 2015–20 (1.67).

Result 4: In the “new school” the “Visualization” factor of spatial intelligence has been reduced to a large extent, “Spatial relations” measurably; but “Spatial perception” has gained greater meaning. There is no significant change in the other two components.

Table 5. Frequency of spatial skills’ components in 1980–85 vs. 2015–20.

	1980–85	relation	2015–20
“Visualization”	100 %	≫	44 %
“Spatial relations”	80 %	>	56 %
“Spatial perception”	10 %	<	44 %

Result 4 describes the structural changes concerning spatial ability factors. In the “old school” in every case “Visualization” was an obligatory part of the solution, because apart from some very difficult problems, no figures were added to the text. In the “new school” figures are given more often.

In the following the Mathematical content of the problems will be discussed. For each problem we collected elements of the Mathematical apparatus which were necessary to solve them. In our terminology the Mathematical apparatus consists of definitions, theorems, processes and their application. The list of these components is the following: knowing volume formula; knowing surface formula; knowledge from plane geometry as trigonometric functions, Pythagorean theorem, congruence, similarity, law of sines/cosines, area formulas; definition of mutual positions of space elements (parallel, perpendicular, their angle); determination of intersection of planes; plane section of a body; proof; vectors in space; vector product; scalar product; solving inequality; solving equation; solving system of equations; series; combinatorics.

Result 5: The mathematical apparatus needed to solve the real spatial geometry problems, with other words, the Mathematical complexity of the problem, has been reduced from 1980–85 to 2015–20.

Table 6. Number of Mathematical components in 1980–85 vs. 2015–20.

	1980–85	relation	2015–20
Average number of Mathematical units needed for solving spatial problems	4.35	>	3.56

Result 6: From 1980–85 to 2015–20 the application of eight teaching items totally disappeared (law of sin/cos, planar congruence, line of intersection of planes, plane section of a body, proof, scalar product, vector product, 3D vectors), one subarea suffered great reduction (mutual position of spatial elements: 45 % > 25 %); whilst one subarea appeared newly (combinatorics and bodies) and two others gained greater meaning (volume formulas: 45 % < 67 %; area of planar figures: 40 % < 67 %). In the obligatory part of written exams determination of the area turned out to be the leader (14 % \ll 100 %), in addition the application of trigonometric functions reduced (71 % > 41 %).

The last two results prove that not only the spatial intelligence, but also the Mathematical background has shrunk and changed: instead of orientation in the space and qualitative approach, computation and quantitative approach is of the greatest importance.

If we examine the set of shapes occurring in the tasks we obtain an interesting result too.

Result 7: The average number of bodies appearing in the problems has been reduced from 1.45 to 1.11. Mostly the problems about pyramids and the cube decreased (50 % > 22 % and 30 % > 11 %, respectively). The truncated pyramid totally disappeared from the tasks but as new objects we can find the circular cylinder and the cuboid. Separately, in the obligatory part we observed that in the “old

school” the cone, truncated cone and pyramid were the most common (29 % each), in the new one however the cylinder and pyramid are the most popular (40–40 %).

This last result hints at another important change. While in the “old school” very often two bodies were connected, considered and compared to each other (e.g. a cube and its circumscribed sphere), in the new generation of problems we usually have just one object and a very specific question about it.

At the end of this section we try to express all what we have found in a single metric. Therefore, we introduce a subjective index number for characterizing the difficulty of a spatial problem with respect to all the aspects we have investigated. Let D (difficulty) be a number composed from the above mentioned factors in the following way.

$D = 2x$ number of spatial intelligence factors + number of used items from Mathematics + number of body types involved.

With the double weight for spatial intelligence factors we express their importance in the research, as a fundamental property in developing visuospatial skills.

Result 8: In the “old school” the average D index equals 10.10, while in the new one it is just 8.01.

This means that a recent spatial problem requires either one spatial intelligence factor less, or 2 mathematical subareas less, or 1 mathematical subarea less and deals just with one body instead of more.

5. Conclusion

By the results above it has been proved that from 1980–85 to 2015–20 both the spatial and the mathematical complexity of the examined problems of technical University entrance exams has been reduced significantly. Newly, it seems to be enough to understand the given figures and reiterate the volume and area formulas.

In this situation the development of spatial abilities seems questionable, because balancing factors are hardly to be found. We just mention here, that in the last decades the subject “Drawing” – which also has an important role in the foundation of 3D thinking – intrinsically disappeared from secondary school and was reduced in elementary education in Hungary. Therefore the “education through art” way of developing spatial abilities is no option. We can only trust that the virtual environments of computer games provide enough support to possess the minimum level of spatial abilities.

As we mentioned in the introductory part, the admission requirements of the majors are influenced by the University lobby. Here we only mention the case of an architect for which no advanced level Matura from Math is prescribed. You as an applicant are allowed to redeem it by Physics or History or even Hungarian language and literature. . . After this we can be sure that a lot of students have even less geometric knowledge than deduced from the research above.

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Térgeometria az egyetemi felvételi vizsgákon Magyarországon 1980 – 85 vs. 2015 – 20

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Absztrakt. A műszaki felsőoktatás egyik kitüntetett feladata az általános és középiskolai nevelésben megkezdett vizuális-téri képességfejlesztés továbbvitele. Jelen előadás célja az, hogy a műszaki egyetemi matematikai felvételi feladatok tükrében hasonlítsa össze a jelent a közelmúlttal, mégpedig a vizsgákban elforduló térgeometriai feladatok összehasonlításával.

A cél, annak a kérdésnek a vizsgálata, hogy a kutatás alátámasztja-e, azt a felsőoktatásban eltöltött több, mint 25 év oktatási tapasztalat nyomán megfogalmazódó szubjektív véleményt, hogy a bekerülő hallgatók téri képességei és geometriai ismeretei folyamatosan szűkülnek?

Az értékelés a tanítás-tanulási folyamat egyik alapfogalma, mely a vizsgált pedagógiai folyamatban visszacsatolási funkciót lát el. Szerepe az, hogy megvizsgáljuk, hogy egy adott tanítási egység tanítása mennyire sikeres, megfelelnek-e az előzetes várakozásoknak az eredmények? A jó értékelés tehát tükröként szolgál a teljes pedagógiai folyamat számára. A középiskolai diákokra váró legnagyobb, makroszintű, standardizált, kvalitatív, szummatív, tudásszint mérő vizsga az érettségi, mely Magyarországon 2005 óta egyúttal egyetemi felvételiként is szolgál. Fentiek indokolják, hogy a középiskolai oktatási folyamat indikátoraként használjuk.

A kutatás a fentiekben megfogalmazott kérdés vizsgálatára céljából az 1980 és 85 közötti központi matematika írásbeli felvételiket hasonlíttja össze a 2015–20 közötti emelt szintű írásbeli érettségi feladatokkal, kitérve az egyes időszakok matematika tanítási jellemzőire és az időközbeni változásokra.

A tanulmányban a térgeometriai feladatok mennyiségét és súlyát vetjük össze a két időszakban, illetve azonosítjuk a megoldásukhoz szükséges vizuális-téri képesség elemeket is és összehasonlítjuk ezek mennyiségét és komplexitását.

Kulcsszavak: vizuális-téri képességek, felsőoktatás, térgeometria, érettségi, a matematika tanítása

Beyond Dichotomies in Mathematics Teaching

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Abstract. In school instruction, including mathematics teaching, we have long observed that heterogeneous social and educational processes result in “uncertainty” among teachers with regard to their professional knowledge and to processes in which theories and concepts, rather than being applied reflectively, are transformed into popular “ideas” and “solutions” that lead to pedagogical errors. Among such errors are falsely understood conceptual dichotomies.

We will focus on the following dichotomies: abstract/concrete, repetition in learning/learning as fun, and the transfer of knowledge/the independent discovery of knowledge. A common feature of all such dichotomies is that they each contain one part that is *a priori* negatively marked.

We demonstrate why teaching must take place beyond dichotomies and show the complexity of the pedagogical process with reference to the goals established in lesson plans, professional and pedagogical knowledge, and individual examples in the field of teaching and learning mathematics. The operation of the aforementioned dichotomies is primarily placed in the context of simplistically perceived constructivism. Constructivism, as defined, for example, by one of its founders, Glasersfeld, is not prescriptive but descriptive. It does not instruct the teacher on how to act in the classroom, but rather encourages him/her to think about how to lead the learning process, which, given its complexity, can clearly not be fully controlled by the teacher. For instance, the idea that “students create their own knowledge”, which is a widespread understanding of constructivism, omits basic assumptions about knowledge and learning; namely, that knowledge is a social construct.

Based on the example of introducing students to the mathematical structure of division, we demonstrate different ways of knowledge transfer, the role and meaning of repetition, and the logic of including concrete representations. In the conclusion, we point out that in education we should not turn knowledge into a means of asserting “new” teaching approaches. It is wrong to take forms and methods of work that are a means of instruction and transform them into the goal of teaching.

Keywords: teaching mathematics, dichotomies in education, mathematical structure, repetition in education, transmission of knowledge, constructivism

1. Preface

In school instruction, including mathematics teaching, we have long observed that heterogeneous social and educational processes result in “uncertainty” among teachers with regard to their professional knowledge and to processes in which theories and concepts, rather than being applied reflectively, are transformed into popular “ideas” and “solutions” that lead away from quality pedagogical work. Our starting point is the thesis that the quality professional knowledge of teachers and its application in mathematics teaching requires the permanent assertion of the teacher’s autonomy of thinking and his/her opposition to various “external” ideologies and pedagogical errors.

It is not, of course, possible to exhaust all of the pedagogical errors that constantly arise in the school field in this presentation. Instead, we will focus on the following dichotomies: abstract/concrete, repetition in learning/learning as fun, and the transfer of knowledge/the independent discovery of knowledge. A common feature of all such dichotomies is that they each contain one part that is *a priori* negatively marked as something “traditional”, “old-fashioned”, “overly demanding” and “bad”.

We demonstrate why teaching must take place beyond dichotomies and show the complexity of the pedagogical process with reference to the goals established in lesson plans, professional and pedagogical knowledge, and individual examples in the field of teaching and learning mathematics.

As we know, one of the most prominent themes within the mathematics education research literature has been emergent theories of learning and knowing. The most familiar theoretical framework within these discussions is constructivism and the operation of the aforementioned dichotomies is primarily placed in the context of simplistically perceived constructivism. Constructivism is hardly a unified discourse and it has been subject to many divergent interpretations. We interact almost daily with teachers who claim to have embraced constructivism in their teaching, but whose interpretations of the framework appear to extend little beyond the assertion that individuals construct their own knowledge. The idea that “students create their own knowledge” omits basic assumptions about knowledge and learning; namely, that knowledge is a social construct (the goals and standards of knowledge cannot, therefore, be defined by the student and, taken as a whole, instruction is the transfer of knowledge) and that learning is the formation (construction) of internal representations (structures) on the basis of external representations and the constant intertwining and supplementing of both.

Such trivialised versions of constructivism fit very well with recent waves of classroom prescriptions regarding group learning (emphasised in socio-constructivism), varied teaching approaches, child-centred learning, the learning styles framework, etc., which present themselves as the opposite to “traditional teaching”. The term *traditional* within such understandings is, however, never clearly defined, which is precisely what allows the ideological use of the term “traditional” and establishes dichotomies.

A great deal of criticism of the constructivist approach has emerged. Kirschner et al. (2006), for instance, have posited that constructivism promotes a teaching style with unguided or minimally guided instruction. The authors suggest that when students learn with minimal instruction they become “lost and frustrated” (p. 6) and that minimally guided approaches, as practiced through constructivist approaches, ignore empirical studies demonstrating that unguided instruction is not effective in learning environments.

2. Transfer of knowledge vs. the independent discovery of knowledge

The so-called progressive pedagogy of the twentieth century (Egan, 2002) and the now specifically understood constructivism contrast education as “the transfer of knowledge” with education as “the independent discovery of knowledge”.

In Slovenia, debates about the opposition of so-called “modern” and “traditional” schooling have been going on for at least four decades, if not longer (cf. Kovač Šebart, Krek & Kovač 2004). Within these frameworks, we can, a long way back, find a rhetoric of “less and less” (“traditional”) and “more and more” (“modern”) with the purpose of promoting that which is supposedly “new” and “modern”. An example from the early 1980s is: “The teacher’s duty is less and less the implanting of knowledge and more and more the encouraging of thinking; if we disregard their formal functions, teachers must become, and are in fact becoming, to an increasing extent advisors, interlocutors, people who help to denude opposing opinions, rather than dispense absolute truths” (Vzgoja in izobraževanje danes in jutri [Education Today and Tomorrow] 1980, pp. 43–44). The teacher’s role should, therefore, move away from the authoritative mediation of information and towards “diagnosing pupils’ needs”, motivating and encouraging learning and verifying the acquired knowledge (cf.: op. cit., p. 59). The teacher is becoming less and less a leader in educational work, the main agent in all efforts towards the formation of the pupils, and is turning into an advisor who supervises, encourages and teaches how the child should learn.

When “the independent discovery of knowledge” (“modern”) and “the transfer of knowledge” (“traditional”) are dichotomously opposed in this way, we must ask ourselves the following question: Can teaching and learning really be the independent discovery of knowledge? All knowledge is specifically a human product. There is therefore nothing natural in knowledge: knowledge is a socially established construct that essentially exists outside the individual as a “social reality”. The knowledge we pass on to students at school has exactly the same character. The social production of knowledge does, of course, involve a series of social processes at different levels. It is first created and validated in processes within individual scientific disciplines, which, on the basis of previously established concepts and specific methodologies, determine what truth is. The field of “determining truth” in science is followed by processes in society in which the acquired knowledge is established into socially valid truths and becomes a convention, a “social agreement”. Finally, a certain part of knowledge enters the school curriculum. Knowledge is thus formalised and validated in lesson plans in the form of goals and standards of

knowledge. It is through these processes of verification that knowledge is objectified. It is also objective in the sense that it exists in the form of human discourse and formalised categories that establish truth outside individuals. The discursive and formal nature of knowledge enables its transmission, and this applies fully to the knowledge that is transferred from the older to the younger generation in the school system.

In contemporary teaching, the fact that knowledge is a social construct is unfortunately often overlooked or at least not sufficiently considered. The teacher is a representative of society and a mediator of “social knowledge”. As a mediator of objective, socially existing knowledge “outside the student”, the teacher must be aware that her/his teaching as a whole is the transmission of prior socially established knowledge. In other words, the transmission (transfer) of socially existing knowledge is the *universal* foundation of teaching, which determines all possible teaching as well as education in general. Therefore, the same is true for the student and learning: although the student’s individual learning is a process of acquiring knowledge that is open and unknown to both the teacher and the student, the knowledge to be acquired is already known and is therefore not something that the student is able to discover at all. This is due to the fact that knowledge is already objectively socially established prior to this individual process, that the goals and standards of knowledge contained in curricula are known and determined, that textbooks are already written, etc. And finally, a student can demonstrate acquired knowledge only by demonstrating it to teachers and *at last to the society*, to the Other, that is in a place of recognizing it.

Frontal work, which is one of the forms of knowledge transfer – and is often mistakenly labelled “traditional teaching” and then reduced to this label – has, as a phenomenon, the appearance of “transmission”, of “direct transfer”. This appearance, which is perhaps more blurred in other forms of knowledge transfer, can create the impression and understanding that in a form such as student group work, or in individual methods of work, the teaching process does not involve the transmission of knowledge. In fact, all forms and methods of work are determined by the universal fact that education is the transmission of socially existing knowledge and the goals and standards of knowledge. In all cases, we pass on knowledge to the student, who endeavours to acquire it. Therefore, the forms and methods of work are without exception the transmission of knowledge. Essentially, every instance of the teacher’s teaching and every instance of the student’s learning is the transmission of knowledge. Forms and methods of work are merely different methods or different paths by which we achieve one and the same goal: the transfer of knowledge. As a particular form of knowledge transfer and acquisition, frontal work is simply the form that most visibly embodies the transmission of knowledge as a fundamental and universal feature of education.

The transfer (transmission) of knowledge can be achieved in different ways. One of the possible divisions, adapted from Alexander (2008) and with some added examples of their application to teaching mathematics concepts, is presented below.

— *Teaching as transmission* views education primarily as a process of instructing children to absorb, replicate and apply basic information and skills (e.g.,

- symbols in geometry, arithmetic, measuring, standard units, etc.).
- *Teaching as initiation* views education as a means of providing access to, and passing on from one generation to the next, the culture's stock of high-status knowledge (e.g., Euclidian geometry).
 - *Teaching as negotiation* reflects the Deweyan idea that teachers and students jointly discover elements of knowledge and understanding (which are already known to teachers) rather than teachers relating to students as the only authoritative source of knowledge (e.g., problem solving).
 - *Teaching as facilitation* guides the teacher by principles that are developmental (Piagetian) rather than cultural or epistemological. The teacher respects and nurtures individual differences, and waits until children are ready to move (e.g., mastering numbers up to 100 before beginning to calculate up to 100).
 - *Teaching as acceleration*, in contrast, implements the Vygotskian principle that education is planned and guided acculturation rather than "natural" development, and that the teacher seeks to outpace development rather than follow it (e.g., if a student is still operating at the level of counting all of the objects when adding, the teacher can, for example, encourage the child to count from the first addend on by covering the first addend).
 - *Teaching as technique* is relatively neutral in its stance to society, knowledge and the child. Here the important issue is the efficiency of teaching, carefully graduated tasks, economic use of time and space, regular assessment, and clear feedback (e.g., teaching algorithm for long division).

3. Concrete vs. abstract

As Egan demonstrates, so-called progressive education has created the series of notions about teaching and learning first promoted by Herbert Spencer: "the child must be active not passive; learning occurs best through play in the early years; new knowledge must be connected with what children already know and will thus initially be concerned with the local, the concrete, and the simple; learning should be pleasurable and not forced; and so on" (Egan 2002, pp. 42–43). If we place these ideas in the rhetorical discourse of the exclusionary opposition between "less and less" and "more and more", this rhetoric causes one side of the dichotomy to be positively valued and the other negatively valued. "Less" and "more" are value signifiers and operate semantically such that they implicitly place ideas in a black and white perspective. The discourse offers the notion of "the correct passage" in one direction. Precisely because this is not stated literally, in the final analysis it can implicitly communicate the message that that of which there should be "less" is not necessary in the pedagogical process and, moreover, is not desirable, without the need to seek the reasons for such a radical viewpoint or to reflect on the consequences of exclusion. The form of the discursive logic of exclusion determines, or better yet precludes, thinking about the content and the real scope of "new" ideas. Such discourse is not a knowledge discourse and its effect is sheer belief in that which is placed on the "right side" within the ideological opposition.

When we are “within” this discourse, we begin to believe, for example, that in the dichotomous opposition of abstract and concrete knowledge, it is right to teach with “as many” concrete examples as possible. The fact is, however, that all knowledge is also abstract, and the relationship between the abstract and the concrete is much more complex. Let us examine a specific example.

There is no doubt that the use of various illustrations, of concrete material, is very prevalent in the teaching of mathematics. In this regard, a key role is played by the works of Dienes, Piaget, Bruner, Skemp and Cobb, who advocate the importance of concrete representations in learning mathematics, justifying the handling of concrete material by claiming that it allows the student to progress to abstract thinking. Illustrations are structured so that they represent mathematical ideas that are fundamentally abstract. The transparency of a teaching aid (intended) for teaching a particular concept depends on the student: the aid does not in itself bear the meaning of mathematical ideas. Some teachers use teaching aids for the purpose of upgrading instruction without reflecting on how the inclusion of these aids affects changes in mathematics teaching. A study by Moyer (2001) showed that teachers described the time spent handling tools as “fun mathematics”, while the time spent learning procedures, solving problems in textbooks and algorithms was described as “real mathematics”. Moreover, teachers also offer aids to students as a reward for appropriate behaviour. Thus, teachers often associate concrete material in the classroom with a reward system.

Teachers’ understanding of the use of teaching aids is based on their understanding of mathematical concepts. Teachers with a lack of understanding of mathematics do not establish connections between mathematical ideas, but use concrete material as part of their instruction, as “a necessary” component. Another problem we see in handling concrete material is that the concrete representation does not lead to, does not ensure, adequate support in teaching certain content (Hodnik, 2020). There may be no connection at all between representation and the concept (e.g., in written subtraction based on the rule of difference where concrete representation does not make sense) or only an aspect between them (in learning the procedure of addition to 20, for example, we must first be able to distinguish it from the introduction of the concept of the sum, which can be introduced by counting both addends, illustrated with concrete material; the latter does not mean addition to 20).

Take, for example, the concept of “the remainder in division”. What role do concrete illustrations play in teaching this concept? The teacher can first present division with concrete material in which the division does not elicit and introduce the concept of the remainder. However, the repetition of this specific activity has no significance for the execution of the process of division that does not work out. Learning the procedure must be based on a knowledge of multiplication, that is, a knowledge of determining the closest multiple smaller than the divisor. It is on this basis that the quotient and the remainder can be determined. Returning to a concrete representation, if the student has not mastered the process of determining the remainder in division, the representation has virtually no meaning. What should be done in the case of such problems is to return to the repetition of multiplication (we can, of course, present again the case in which the division does not work out

with concrete material, but this will not help in mastering the procedure if there is no knowledge of multiplication). The same happens when adding up to 100. We present the concept of addition to the student with concrete material, typically base ten blocks, with which we present both addends. When we combine them, the student determines the sum by counting, so s/he does not calculate. Learning the process of calculation requires a different treatment, and the idea that it is necessary to return to a concrete representation that somehow solves everything is too simplistic.

4. Repetition in learning vs. learning is fun

Spencer's idea that learning should be pleasurable and not forced has established the belief today that "real" or "good" learning is only that which is fun, while, on the other hand, repetition in learning is excluded and thrown in the bin labelled "bad", and should therefore be avoided.

Thus, repetition, which is one of the key elements of education, has come into disrepute, so to speak. In contrast, if we consider the structure of the act in education it becomes clear that the teacher can and must design acts in teaching that are of high quality in terms of structure and content, appropriate to the particular context, and can be assumed to *gradually* lead to achieving the goal. Nevertheless, in specific individual cases, neither the teacher nor the student can, in principle, know in advance how many times the act will have to be repeated in order for its goal to be achieved. Realisation always brings something more or less than was envisioned. The act realises something other than it intended according to its will. It requires the acceptance of unknown consequences in advance (Krek, 2021).

Perhaps we could even say that pedagogy behaves in a Hegelian way and respects and accepts the principle of the failure of the act as a silent precondition of education. The processes in education are planned, they have established goals, yet at the same time they take place with the silent assumption that the act will fail. The act really has failed, but it is precisely in this failure that the truth comes to light. This is the basic driving force of Hegel's dialectic, and at the same time a precondition of the processes of education. The act has failed, the process moves forward; all teaching and learning is a dialectical process of "success in failure", driven to achieve the established goal by acts that have failed (Krek, 2020). It is precisely the acceptance of the silent assumption of the failure of the act that is a condition for parents and teachers to be patient, to be persistent, and to teach children and students to persevere and repeat the act until the goal is achieved. We could say that the other, articulated or visible side of the principle of the failure of the act is the principle of perseverance in repetition, until the repetition ends in the achieved goal: the act is done.

If the student repeats a particular method of calculation or, for example, a method of solving a certain type of textual task, followed by tasks that are unlike the previous ones (they are variations of previous tasks), such tasks offer the student an opportunity to approach a different (new) mathematical thinking and determine

the pattern in the repeated tasks. Students are not aware of the pattern in the repeated tasks until they are faced with a different task (Watson & Mason, 2006). In other words, this means that the variation or difference in the tasks enables the experience of understanding this difference precisely because the difference enters the stability of the repetitions. This can be illustrated by the example of the repetition of quotients in calculations within multiplication table, which allows the student to recognise the difference in tasks in which the division does not work out. Another example is the solving of textual problems in which the data for the solution is just right. If the student repeats the solving of such tasks to the extent that we can say that s/he knows how to solve them, s/he will recognise the difference in textual tasks in which, for example, one or more pieces of information for solving the task is superfluous. Without acts of repetition, each variation of the task enters as an arbitrary problem, which does not support understanding.

5. The remainder in division: An example of constructing a concept and placing it in a mathematical structure. The role of repetition, transmission and concrete representations.

Within the framework of this argument, we can explain the construction of mathematical concepts, especially arithmetic concepts, and explain the importance of concrete representations, transmission, repetition and the teacher's well-considered leading of instruction based on the interweaving of the approaches we have outlined. For the purpose of presenting a development of the concept "remainder in division", we will refer to four levels of mathematical knowledge (van den Heuvel-Panhuizen, 2003): 1) intuitive knowledge (also identified as informal knowledge, the solver is confronted with personally relevant problems to solve, associated with the out-of-school knowledge of students); 2) the concrete level of knowledge (representations and manipulatives, the use of different symbols and manipulatives that play a crucial role in assisting students to build their initial understanding of the concept); 3) computational or procedural knowledge (students' skills in applying procedures for solving tasks, consisting of the formal language or symbol representation system, the algorithms or rules for completing mathematical tasks); and 4) principled-conceptual knowledge (students master the mathematical idea and their knowledge can be applied in a variety of different contents/contexts).

In the case of introducing division with a remainder, which in mathematics instruction is placed in the broader area of arithmetic content, we will demonstrate the construction of a mathematical structure for division, suggest different ways of knowledge transfer, highlight the role and importance of transmission and repetition, determine when and how the inclusion of concrete representations is justified, and explain the sense in which the student cannot construct knowledge on his/her own.

In the illustration below (Figure 1), we schematically present the placement of the content of division with a remainder in the broader mathematical structure of division. It can be immediately established that only part of this structure is shown,

as division is placed much more broadly in the structure referred to as arithmetic in the school curriculum.

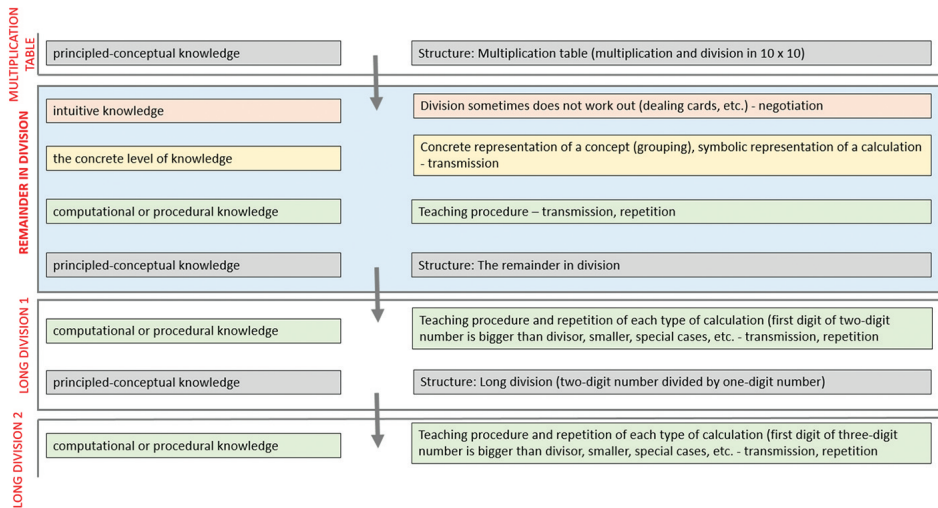


Figure 1. Introduction to the structure of remainder in division.

The remainder in division is content that requires certain prior knowledge from the student at the level of principled-conceptual knowledge of the multiplication table 10×10 . Acquiring this knowledge obviously requires a great deal of repetition; concrete illustrations can only serve to introduce the notation of equal addends in the form of a product in the sense that students determine the value of the product on the basis of the addition of the displayed addends (graphic illustrations could play exactly the same role, as it is actually only a case of counting or adding up to 100, which students should master before dealing with multiplication tables). In the topic of the remainder in division, we could proceed from intuitive knowledge, that is, from situations that are familiar to the student from everyday life. When the division does not work out, we propose a negotiation method of instruction (according to Alexander, 2008), in which teachers and students jointly discuss (“create”) elements of knowledge and understanding, as the situation from everyday life is in the foreground. In Figure 1, we mention the example of dealing cards. When constructing concrete knowledge, it is important to show the concept of the remainder in division with the help of concrete representations, taking into account mathematical principles. This means that we start from division, which involves dividing elements into sets of equal size (grouping) and presenting the remainder or pointing out that the division may not work out. In Figure 1, we propose a transmission approach (the process of instructing children to absorb, replicate and apply basic information and skills) because it is otherwise impossible for the student to derive the formal rules in division of this kind and appropriately translate the process into a symbolic notation. (Some might even dispute this, saying that the student can invent or suggest a way of notation; this is, of course, true, but it is necessary to ask why this intermediate step would be necessary or meaningful).

When the student acquires concrete knowledge in the described example, s/he is still unable to perform the procedure of determining the remainder in division. When acquiring procedural knowledge, which is essential for further mathematical operations in this situation, we proceed in such a way that the student writes down multiples of the divisor for the given division and finds the closest one that is smaller than the dividend. The quotient and the remainder are determined in this way. In Figure 1, when teaching this procedure, we again propose transmission and highlight repetition that enables students to determine the pattern in the repeated tasks. We should reiterate that repeating a concrete division situation that does not work out fails to equip the student to perform this process of division. In other words, further concrete illustration of the concept of division with a remainder, or constantly returning to a concrete illustration, does not represent a solution that enables the student to advance his/her knowledge if the s/he does not know the process of division. The student must repeat the process of determining the remainder of the division until s/he masters it, that is, until s/he achieves principled-conceptual knowledge. This is a condition for further introducing the student to the process of written division with a one-digit divisor (i.e., the dividend is more than ten times the divisor, e.g., $78 : 5$), where the student applies principled-conceptual knowledge of division with a remainder and progresses to the principled-conceptual knowledge in written division with a one-digit divisor. The student acquires this knowledge only on condition that s/he repeats the process. In Figure 1, we do not propose concrete illustrations or specifically highlight concrete knowledge with regard to this content, as the procedure of written division with a single-digit divisor differs from the previously treated division only in the procedure, and even this differs only slightly from the previous case. The acquired knowledge is again a condition for further advancement in the student's knowledge of written division. In Figure 1, we point out that the next step is written division of a three-digit dividend by a one-digit divisor, with regard to which it is important that the teacher also acquaints the students with specific examples of this division, e.g., $506 : 5$, $134 : 5$. What can the student construct him/herself in the process that we have presented? Certainly not knowledge of written division. In certain situations, the teacher may allow the student to reflect on arithmetic procedures before the actual instruction. The teacher could, for example, ask the student how s/he would calculate $78 : 5$ before presenting the calculation in the way foreseen by the curriculum, that is, the conventional mathematical way, and discuss different procedures with the students. What would this achieve? As we have already mentioned, it would primarily enable the teacher an insight into the student's understanding of arithmetic operations or his/her numerical conceptualisations. For instance, the student could suggest subtracting the number 5 from 78 and thus determine the quotient and the remainder; s/he could divide the dividend into 5 equal addends in several steps ($10 + 10 + 10 + 10 + 10 = 50$; $3 + 3 + 3 + 3 + 3 = 15$, $2 + 2 + 2 + 2 + 2 = 10$, we conclude that the result is 15 ($10 + 3 + 2$) and with a remainder of 3); s/he could try to determine the quotient by the inverse operation, by multiplication (how many times 5 is 75?), and so on. Situations of this kind in mathematics can be defined as problem solving, which is characterised by the student developing a certain solution strategy that is not yet known or that s/he cannot recall (cf. Hodnik and Manfreda Kolar, 2015). Fosnot and Dolks (2001) would call such forms of

thinking mathematisation, which is characteristic of socio-constructivism. In their work, they gave many examples of students' own creation of procedures, but it is not clear how these methods are "translated" into a formal algorithm and how less successful students cope with these kinds of activities.

Some psychologists criticise constructivism because dominant students control interactions in the classroom while average students might be ignored (Gupta, 2011). These critics contend that the dominant group drives the whole class towards their thinking while leaving other students behind, thus constructivist teaching overlooks the development of many students' skills because the activity is led by the few (Alanazi, 2016).

From this we can deduce that only on the basis of the knowledge acquired in the teaching process, the student forms various arithmetic procedures that have little to do with formal mathematical knowledge, that is, they are not defined as goals in the mathematics curriculum in the area of division. They do, however, have the potential role of allowing the teacher to verify the student's knowledge of numbers and arithmetic operations, thus enabling the student to establish connections with already acquired knowledge his his/her own way. We should especially emphasise the fact that the structure of the selected mathematical content must be clear and must be coherently presented to all students. We can set challenges of this kind for more successful students, but it is necessary to consider what they are capable of and how much these challenges benefit less successful students.

The described example of introducing the mathematical structure of division is intended primarily to illustrate that in teaching mathematics it is crucial to equip the student with knowledge that gives him/her certainty and to establish clear teacher expectations regarding the knowledge defined in the curriculum. In the Figure 1, we highlight the teaching approaches or working methods that are fundamental for specific sub-content concerning division from the point of view of achieving mathematical goals. Although differences between students may lead the teacher to use other approaches/working methods, from the perspective of our understanding of quality teaching these cannot replace the approaches/working methods we suggest.

This example of teaching reminder in division does not neglect the findings of the cognitive theory of learning, nor is it behaviourist, as it could be superficially described; rather, it represents an intertwining of different findings in the field of education in the broadest sense of the word. Not least, by emphasising the importance of repetition in introducing the student to the mathematical structure, we have reiterated the fact that learning mathematics is a task that requires attention, the observance and implementation of rules, and perseverance. As such, it is worth mentioning that this also concerns the area of moral education. Mathematics is especially suited to accustoming the student to structured thinking, to persisting with repetition until a robustness in knowledge is achieved; thus, it is a subject that directs students away from permissiveness. It represents a shift away from the lenience that the student can bring from the home environment, but it does, of course, require an authoritative attitude and appropriate behaviours on the part of the teacher. Other subjects that by their nature are not as hierarchical as mathematics no doubt allow more flexibility and offer more opportunities than mathematics

to develop other competencies, such as the ability to discuss, to work in groups, to experiment, etc. We do not claim that this is impossible in mathematics instruction; however, it is always necessary to adapt the methods and forms of work to the *learning* objectives, and not to undermine the achievement of objectives in mathematical knowledge due to “desirable” methods and forms of work. Such an approach transforms methods and forms of work that should merely be a means into teaching objectives.

6. Conclusion

In conclusion, we should point out that we have not provided a rule shaping mathematical structure in general, nor do we not claim that the process of establishing mathematical structure is linear; we are aware that there are individual differences between students. Nonetheless, the presented example of forming a structure clearly shows that a carefully considered introduction to a mathematical structure, combining various teaching approaches, is crucial in mathematics teaching. It is certainly not appropriate to blindly follow current or popular approaches that are not – and cannot be – justified from the point of view of teaching mathematical structure, which is undoubtedly the fundamental goal, merely because “they are popular”.

It is worth pointing out again that one of the common mistakes that occurs among teachers due to a *belief in the concrete*, which establishes a poorly considered pedagogical ideology, is that concrete illustrations in practice completely unjustifiably take the place of a “saviour” of virtually all learning problems.

Since it is realistic to assume that the “creativity” of pedagogical “innovators” will not simply disappear one day, the pursuit of quality in teaching and in students’ knowledge demands professional didactic knowledge from teachers and their courage to “step aside”, that is, an ethics that rejects misguided pedagogical dichotomies that result in teachers who, under the slogans of the “creativity”, “independent discovery” and “innovation” of the student, deviate from structured mathematical knowledge.

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Povzetek. Pri pouku v šoli in tudi pri poučevanju matematike že dalj časa opažamo, da heterogeni družbeni in edukacijski procesi vodijo v učiteljevo “negotovost” strokovnega znanja in v procese, v katerih se teorije in koncepte namesto reflektirane uporabe pretvarja v popularne “pojme” in “rešitve”, ki pedagoško delo učiteljev vodijo stran od kakovosti. Izhajamo iz teze, da kakovostno strokovno znanje učiteljev in njegova uporaba pri poučevanju matematike zahteva permanentno uveljavljanje učiteljeve avtonomije mišljenja in njegovo zoperstavljanje različnim “zunanjim” ideologijam in “pedagoškim zmotam”.

V teoretičnem prispevku seveda ne moremo izčrpati vseh pedagoških zmot, ki nenehno vznikajo v šolskem polju, posvetili se bomo naslednjim dihotomijam: abstraktno/konkretno, ponavljanje v učenju/učenje kot zabava in prenos znanja/samostojno odkrivanje znanja. Skupna poteza vseh takšnih dihotomij je, da je v vsaki posebej en del a priori vrednotno označen negativno: označen je kot nekaj “tradicionalnega”, “preživelega”, “slabega”, “prezahtevnega”.

V prispevku pokažemo, zakaj mora poučevanje potekati onstran dihotomij, in, izhajajoč iz v učnih načrtih postavljenih ciljev, strokovnega in pedagoškega znanja, na posameznih primerih na področju poučevanja in učenja matematike pokažemo kompleksnost pedagoškega procesa. Delovanje prej navedenih dihotomij postavljamo predvsem v kontekst poenostavljeno dojetega konstruktivizma. Konstruktivizem, kot ga denimo opredeli eden od utemeljitev Glasersfeld, ni predpisujoč, ampak opisujoč. Ne podaja navodil učitelju, kako naj ravna v razredu, spodbuja ga k razmisleku, kako voditi proces učenja, za katerega je glede na njegovo kompleksnost jasno, da ga ne more popolnoma nadzorovati. Denimo, razširjeno razumevanje konstruktivizma “učenci kreirajo lastno znanje” izpušča temeljne predpostavke o znanju in o učenju, namreč da je znanje družbeni konstrukt (zato ciljev in standardov znanja ne more opredeliti sam učenec in pouk kot celota je prenos znanja) in da je učenje oblikovanje (konstruiranje) notranjih reprezentacij (struktur) na osnovi zunanjih reprezentacij in nenehna prepletanja ter dopolnjevanja obojih.

Ker je realistično predpostaviti, da “kreativnost” pedagoških “inovatorjev” ne bo preprosto nekega dne izginila, prizadevanja za kakovost v poučevanju in v znanju učencev od učiteljev zahtevajo strokovno

didaktično znanje in pogum “koraka v stran”, to je etiko, ki zavrača zgrešene pedagoške dihotomije, katerih posledica je poučevanje učiteljev, ki pod gesli “ustvarjalnosti”, “samostojnega odkrivanja”, “inovativnosti” učenca odmika od strukturiranega matematičnega znanja.

Ključne besede: poučevanje matematike, “pedagoške zmote”, dihotomije, konstruktivizem

Didactic Manipulatives in Primary Mathematics Education in Croatia

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Abstract. Didactic materials can be used to introduce abstract mathematical concepts. Various studies found positive effects of using manipulatives in mathematics teaching and learning. Teachers' knowledge and attitudes towards manipulatives influence the profound and helpful implementation of manipulatives in the didactic process. There is a variety of ready-made and self-made manipulatives that can be utilized in mathematics education. Following that perspective, we questioned what role manipulatives play in Croatian primary education. For that purpose, we conducted research with teachers on the use of didactic materials in mathematics classroom teaching. Results showed that teachers use all sorts of manipulatives. They acknowledged the benefits of using manipulatives, and most of them claimed they used manipulatives often in the didactic process. However, some answers indicate there is a lack of knowledge about the characteristics of manipulatives and the implementation of manipulatives in the didactic process.

Keywords: didactic manipulatives, mathematics teacher, primary mathematics education

1. Introduction

For decades, educators and theorists have researched how small children acquire knowledge. Piaget's theory of cognitive development stated that children in the concrete operations phase (7–10 years of age) understand abstract mathematical concepts through concrete models which make mathematical problems familiar and perceptible. Bruner described three stages of representation that occur through learning: enactive (concrete), iconic (representation), and symbolic (abstract) stage (Johnson et al., 2021). Thus, children build abstract knowledge through interaction with concrete, real-world objects and using manipulatives seems like

an inevitable part of primary mathematics education. The first person who used concrete objects for learning mathematics was Friedrich Froebel, who created manoeuvrable objects – Froebel gifts. Many educators continued creating different manipulatives to help students develop conceptual understanding, such as Montessori, Cuisenaire, Dienes, etc.

Different studies explored the ways and benefits of using manipulatives in mathematics education. Though using different representations, moving between them and connecting manipulatives, images and abstract notions helps develop mathematics understanding, teachers' knowledge and attitudes had a major impact on the positive effect of using manipulatives in practice. In the Croatian context, the concretization principle is one of the key educational principles – hence in this study, we explored Croatian primary school teachers' use of didactic materials in mathematics classroom teaching.

1.1. Manipulatives in mathematics education

Mathematical manipulatives are artefacts that students use to grasp mathematical concepts or processes and perform activities drawing on perceptual evidence (Bartolini & Martignone, 2014). We can distinguish between concrete and virtual manipulatives, historic-cultural and “artificial” manipulatives. The former are tools created in the history of civilisation to solve mathematical or everyday problems, and the latter are tools designed for particular mathematics education aims. Sometimes the difference between historic-cultural and artificial manipulatives is vague. For example, the Slavonic abacus used as any other kind of abaci is a historic-cultural manipulative from 1820, while Froebel gifts from around 1840 are artificial concrete manipulatives because they were designed purposely for kindergarten activities and education (Bartolini & Martignone, 2014). Concrete manipulatives are physical artefacts that provide intentionally designed sensory information, and virtual manipulatives are digital artefacts that simulate physical objects. Virtual manipulatives are usually in the form of applets or apps, and they support visual representations, actions performed on them, and symbolic representations to focus on mathematical concepts (Moyer-Packenham & Westenskow, 2013).

Manipulatives are not mathematical concepts, nor do they represent a concept by themselves. There must be an interplay between the abstract notion and concrete manipulative, that is, we must have some idea about the mathematical notion to properly relate it to the manipulative. Different physical actions with different manipulatives engage different mental actions. Rote actions with a particular manipulative, such as explaining the procedure step by step, repeating, and then practising, need not develop meaningful structures about a concept because there is no need for mental effort (McNeil & Jarvin, 2007). The role of the teacher is to instruct students to purposefully use manipulative, aid them to reflect on its use and develop meaningful representations of a concept (Bartolini & Martignone, 2014; Puchner et al., 2008; Sarama & Clements, 2009; Uttal et al., 1997). The low or high-level instructions refer to the level of teacher's guidance in students' interaction with manipulatives. Instructional guidance includes demonstration, hints,

follow up questions, feedback, resolving problems, and others. Free interaction with manipulatives makes students more curious and engaged, but low guidance does not necessarily lead to learning because students do not spontaneously focus on important features of manipulatives. Teacher support is a significant segment of developing conceptual understanding (Carbonneau & Marley, 2015). Using manipulatives in the classroom depends on the teacher's attitude towards manipulatives; pondering if their use is effective and finding it time-consuming had teachers using them inappropriately (Golafshani, 2013).

Sarama and Clements (2009) differentiate three kinds of relationships with knowledge, Sensory-Concrete, Abstract, or Integrated-Concrete. The former is related to children being able to reason mathematically only in a concrete situation, and the latter is related to the complex interaction between different (physical, pictorial, abstract) representations of mathematical concepts. Uttal et al. (1997) argued that using manipulatives falls under the dual-representation model, that is, a manipulative is an object in its own right and it is a symbol representing some other object. Students either understand the relationship with the other object or use manipulatives as objects with particular interest. In the latter, students do double work, they learn to use manipulatives separately from the intended outcome and they learn the formal content related to the outcome. The fact that a manipulative, as a particular object, can represent something else is not evident. Moyer (2001) noted several 'transpositions' between manipulatives and mathematical concepts. At first, the teacher interprets the manufacturers' intention with the manipulative, then organizes their instruction to impose the mathematical relationship on the manipulative. This meaning is not transparent to students, and they must connect their available knowledge with the suggested relationship.

1.2. Effectiveness of using manipulatives in mathematics education

Studies about using manipulatives in mathematics education yielded different, even contrasting, results. Carbonneau et al. (2013) performed a meta-analysis on 55 studies that compared the instructions with concrete manipulatives and those using only abstract notation. In general, instruction with manipulatives had a small to moderate positive impact on learning mathematics. However, that effect depended on different characteristics, such as instructional guidance, mathematical topic, developmental age, and perceptual features of the manipulatives. For example, authors found that using manipulatives was most effective in instruction with children 7–11 years of age, that a high level of guidance had a positive impact on retention, and that perceptually rich manipulatives had a negative effect on problem-solving. Moyer-Packenham and Westenskow (2013) performed a meta-analysis on 66 studies about using virtual manipulatives. They found a positive impact on teaching with virtual manipulatives or a combination of virtual and concrete compared to instruction with a textbook. Studies also showed positive effects of the concrete-representational-abstract approach in special education (Bouck & Park, 2018).

Bartolini and Martignone (2014) highlighted their concerns about using manipulatives. They questioned what is the role of instruction in making connections

between concrete and symbols, and what are the opportunities of implementing manipulatives at different educational levels as means of representation of concepts. Moyer (2001) found that teachers used manipulatives with different pedagogical and didactical purposes and they differentiated between formal instruction as ‘real’ and instruction with manipulatives as ‘fun math’. Teachers used manipulatives separately from the main part of the lesson when covering the content, to motivate students, reward or discipline them. The focus was on reinforcing the learning of formal content rather than supporting and giving meaning to the content. In another study, students in the same classroom had free access to manipulatives. Moyer and Jones (2004) noted that students became selective and argumentative in their choice of manipulatives and demonstrated different approaches in problem-solving activities. Puchner et al. (2008) observed lessons performed by teachers who participated in a professional development programme. In their study, teachers also included manipulatives at the end of the lesson and as an activity for itself, in particular, students had to interpret content using manipulatives after the content was already covered formally. The authors argued that the issue were the pedagogical choices concerning teaching particular content, that is, how to teach (and learn) content with a manipulative.

Thoughtful integration of manipulatives is a prerequisite to its positive effect on learning outcomes. Laski et al. (2015) listed four principles of effective instruction with manipulatives.

- 1) Using a manipulative over a long time allows building a deeper understanding of its representation, that is, developing an Integrated-Concrete relationship with knowledge. When children are familiar with the representation they can use the manipulative purposefully without additional cognitive load. The more experience children have with manipulatives, and as early as possible, the better conceptual understanding will be developed later (McNeil & Jarvin, 2007).
- 2) Moving from transparent concrete representations towards more abstract representations eases the transfer of knowledge. Carbonneau et al. (2013) found that the effect of using manipulatives was small on the transfer learning outcome, and Laski et al. (2015) argue that this “concreteness fading” can enlarge that effect.
- 3) Avoiding perceptually rich manipulatives moves students away from playfully engaging with manipulatives toward giving them a mathematical meaning. This is also related to the dual-representation model and how students focus on the manipulative as an object in its own right. Students do not easily convert an external representation of manipulative into an internal one (Puchner et al., 2008).
- 4) Explaining explicitly how a manipulative is related to the notion directs focus to the mathematically relevant features of the manipulatives that would otherwise be inconceivable. The transparency of the manipulative, that is, that the manipulative is an appropriate and effective representation of a concept, shows when students use it within the intended outcome. Understanding through manipulatives occurs when using them as a tool in problem-solving (Puchner et al., 2008).

Uttal et al. (1997) raised similar concerns about using manipulatives in mathematics education. A teacher must bear in mind that the inclusion of manipulatives does not ease understanding or improve education per se, nor that the representation of the concept would be obvious to students. They must choose the most appropriate manipulative and modify their instruction when using manipulatives to make it effective. Moyer (2001) emphasised that both teachers' knowledge about using manipulatives and their attitudes about mathematics education influence how they use manipulatives in their practice.

Moyer-Packenham and Westenskow (2013) listed five affordances of using virtual manipulatives: the design constraints students to focus on relevant features of virtual objects compared to the mathematical notion, students engage in exploration and developing creative strategies and solutions, virtual manipulatives can simultaneously activate various representation (dynamic, pictorial, symbolic, verbal, gestural), the virtual environment can make the use accurate and efficient (time-consuming) compared to the static environment, there is motivation and intrinsic engagement to learn.

2. Methodology

We administered an online questionnaire with primary school teachers. The questionnaire consisted of seven demographic questions and 15 questions of a different type. Two questions were closed questions related to teachers' practice, about attending professional conferences in Q.1, and teaching advanced mathematics groups in Q.2. There were two multiple-choice questions about different manipulatives that teachers use; they chose among ready-made, self-made and virtual manipulatives in Q.4 and common ready-made manipulatives in Q.5. One question was an open question about self-made manipulatives – what manipulatives teachers made themselves and the purpose of that manipulative. Four questions were multiple-choice questions with an option to provide an additional answer about sources of information on manipulatives in Q.7, advantages of using manipulatives in mathematics education in Q.8 and disadvantages in Q.9, and motivation to use manipulatives more often in mathematics education in Q.15. Six questions were four-point Likert scale questions about the level of pupils' satisfaction with using manipulatives in Q.10, frequency of using manipulatives in mathematics education in general in Q.3, particular domain in Q.11, a particular part of a lesson in Q.12, a particular type of work in Q.13, and with different purposes in Q.14.

The questionnaire was produced as a Google form and shared with professional contacts in primary schools and alumni from the Faculty of Education in Osijek. We collected 116 answers with three male participants and 94 participants from the region Slavonija and Baranja. The answers were exported from Google form to an Excel file, and data were further analysed using Statistica. Exploratory and descriptive statistics were appropriate to analyse the majority of questions, including cross-tabulation, diagrams, and frequencies. Questions Q.3 and Q.11–14 were clustered using a weighted pair-group average with Euclidean distance. The distance between two clusters is then the average distance between all pairs of

objects in different clusters where the size of the respective clusters is used as a weight. Participants' answers to questions Q.6–9 were appropriate for qualitative interpretation concerning participants' understanding of the use of manipulatives in mathematics education.

3. Results

3.1. Declared frequency of using manipulatives

Participants in our study declared they use manipulatives in mathematics education, and half of them use manipulatives often. They had different experiences regarding years of practice, teaching mathematics groups and attending professional conferences (Table 1). In particular, 50 % of participants with less than 10 years of practice; 56,7 % of participants who had between 12 and 26 years of practice, and 46,3 % of participants with more than 27 years of practice declared they use manipulatives often. The frequency at first increases and then decreases with the years of experience.

Table 1. Declared frequencies of using manipulatives in Q.3 compared to participants' practice elicited in demographic data, and questions Q.1, Q.2 and Q.4

	Frequency of using manipulatives			
	Rarely	Sometimes	Often	Total
Years of practice				
0–10		19	19	38
12–26	1	15	21	37
27–42	3	19	19	41
Mathematics group				
No	1	26	20	47
Yes	3	27	39	69
Professional conferences				
Never		1		1
Rarely		2	2	4
Sometimes		14	7	21
Often	4	36	50	90
Type of manipulatives				
Ready-made	3	48	55	106
Self-made	3	35	44	82
Virtual	1	33	54	88
Total	4	53	59	116

The majority of participants lead mathematics groups and often participate in professional conferences. In the former group, 56,5 % of participants, and in the latter group, 55,5 % of participants declared they use manipulatives often. On the other hand, 42,5 % of participants who do not lead the mathematics group stated

they use manipulatives often, and 44,4 % of participants who participate in professional conferences often declared they use manipulatives sometimes or rarely. The two criteria do not seem to be related to the frequency of using manipulatives.

Participants in our study declared they use different manipulatives, ready-made, self-made, and virtual. The dominant source of information about manipulatives, regardless of the type, were web resources (Figure 1). Participants chose peer-supported sources more often than professional literature sources. In particular, participants mainly relied on professional conferences and colleagues' recommendations. They found information across sources mainly about ready-made manipulatives, then self-made and virtual manipulatives. Professional literature was an exceptional source of information about self-made manipulatives, as were web sources about virtual manipulatives.

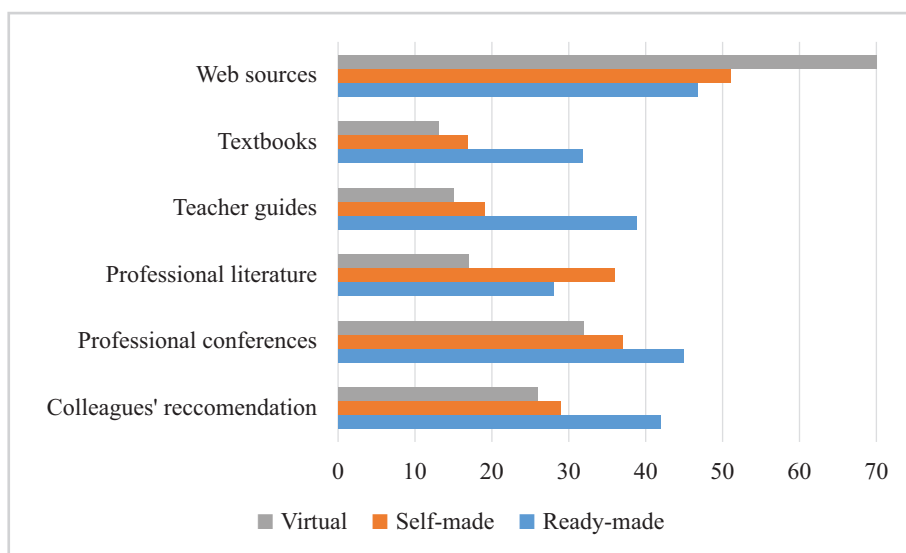


Figure 1. Number of participants who use particular source of information for particular type of manipulative.

Among ready-made manipulatives, the most common ($N > 20$) were models of solids, tangram puzzles, abacus, model of solids with nets, and a balance scale. We noted that some participants use self-made versions of the ready-made manipulatives. Some participants ($10 \leq N \leq 20$) selected Cuisenaire rods, Unifix cubes, Geoboard from the ready-made manipulatives or mentioned various counters and number lines as self-made manipulatives. Few participants ($N \leq 10$) selected Dienes blocks or Montessori manipulatives from the ready-made manipulatives or mentioned they used dominos, multiplication table, bead string, models for decimal values, and models of geometric figures as self-made manipulatives. Though many participants selected they used virtual manipulatives, we noted few mentions of Geogebra or copyrighted software for online learning and some mentions of applications for creating quizzes. When listing the self-made manipulatives, 36 participants included various (confusing) resources (such as applications, games,

posters, sheets) that are not manipulatives in the sense that they concretize abstract mathematical notions (see Appendix A).

3.2. Purpose of using manipulatives

Almost all participants estimated that pupils' satisfaction with using manipulatives is high ($N = 66$ very high, $N = 46$ high, and $N = 8$ medium), and so did all participants who declared they used manipulatives often (Table 2). Regardless of declared frequency and pupils' satisfaction, the majority of participants acknowledged that manipulatives "facilitate understanding of abstract content" by concretization and "increase pupils' motivation". Only the participants who estimated high pupils' satisfaction selected that manipulatives "develop pupils' creativity and divergent thinking". Around half of the participants selected "developing pupils' positive attitude towards learning" and "improving pupils' motor skills" as advantages of using manipulatives. Over half of participants selected "manipulatives are expensive" and "additional preparation is required" as disadvantages of using manipulatives. Participants who estimated pupils' satisfaction as lower selected "manipulatives are expensive". Participants who used manipulatives sometimes chose that "activities with manipulatives last too long" and "additional education is required" as disadvantages more often than participants who used manipulatives often.

Table 2. Declared frequencies of using manipulatives in Q.3 compared to participants' choices on advantages and disadvantages of using manipulatives in Q.9 and Q.10.

Frequency of using manipulatives	Rarely		Sometimes		Often	
Estimated pupils' satisfaction	Lower	Higher	Lower	Higher	Higher	Total
8. Selected advantages						
Concretization	2	2	3	46	58	111
Motivation	1	2	5	40	52	100
Creativity	0	2	0	35	52	89
Attitude	0	1	2	29	34	66
Motor skills	1	1	1	24	29	56
9. Selected disadvantages						
Expensiveness	2	1	5	26	40	74
Preparation	1	2	2	29	35	69
Time consuming	1	0	1	15	16	33
Education	0	0	1	13	12	26
Total	2	2	6	47	59	116

Participants declared they used manipulatives most often in curriculum domains Shape and Space, Number and Measure, which is in line with the manipulatives participants listed. Closest to the declared frequency of using manipulatives was the frequency of the purposes "demonstrating a new concept" or "pupils with learning difficulties in their work". Participants also declared to use manipulatives (more than sometimes) in the main and the introduction part of the lesson, in group

work and when demonstrating a new procedure. Participants declared they rarely (less than sometimes) used manipulatives for the content in curriculum domains Algebra and Function, and Data, Statistics and Probability, in individual work, and with the purpose “pupils create new ideas and content” (see Table 3).

Table 3. Descriptive statistics in questions Q.3, Q.11-14 about frequency of using manipulatives on the scale 0 – never, 1 – rarely, 2 – sometimes, 3 – often¹.

Descriptive Statistics	Mean	Median	Mode	Freq. of mode	Lower Quartile	Upper Quartile	Std. Dev.
3. Frequency of using manipulatives	2,47	3	3	59	2	3	0,57
11. Number	2,39	2	3	57	2	3	0,69
11. Algebra and Function	1,75	2	2	58	1	2	0,94
11. Shape and Space	2,55	3	3	76	2	3	0,7
11. Measure	2,35	3	3	59	2	3	0,77
11. Data, Statistics and Probability	1,57	2	2	50	1	2	0,93
12. Introduction part	2,09	2	2	44	2	3	0,85
12. Main part	2,28	2	2	60	2	3	0,66
12. Closing part	1,94	2	2	59	2	3	0,87
13. Frontal work	1,93	2	2	47	1	3	0,88
13. Pair work	1,99	2	2	53	1	3	0,81
13. Group work	2,12	2	2	54	2	3	0,76
13. Individual work	1,79	2	2	47	1	2	0,83
14. Demonstrating Concept	2,35	2	3	55	2	3	0,71
14. Demonstrating Procedure	2,07	2	2	64	2	3	0,74
14. Problem Solving	1,93	2	2	58	1	2	0,74
14. Exploring Content	1,99	2	2	65	2	2	0,74
14. Creating Content	1,69	2	2	49	1	2	0,88
14. Pupils with Learning Difficulties	2,32	2,5	3	58	2	3	0,82

Questions about using manipulatives in the practice were clustered into three groups. The first group can be described as collaborative explorative activities, the second as frontal demonstration activities and the third as individual creative

¹The questions were: Q.3 How often do you use manipulatives in mathematics education? Q.11. How often do you use manipulatives in mathematics education with respect to curriculum domains? Q.12. How often do you use manipulatives in mathematics education with respect to parts of a lesson? Q.13. How often do you use manipulatives in mathematics education with respect to types of work? Q.14. How often do you use manipulatives in mathematics education with respect to its purpose?

activities (Figure 2). The question about the purpose “Pupils with learning difficulties use manipulatives in their work” was clustered with the second group of questions. Comparing declared frequencies across the questions, participants used manipulatives often as frontal demonstration activities and for assisting pupils with learning difficulties; and rarely as individual creative activities.

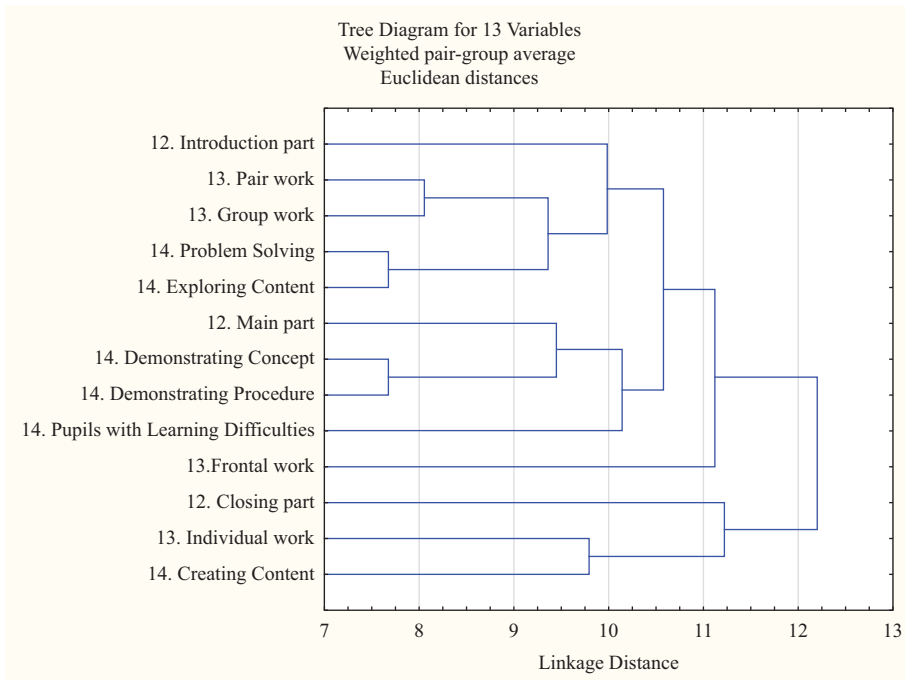


Figure 2. Clusters in questions about using manipulatives in mathematics education.

We examined participants' answers in open questions about the purpose of the self-made manipulatives they have created and used ($N = 66$). The participants had a different focus, in particular

- 36 participants wrote about different aspects of a mathematics lesson, that is, introducing, demonstrating, practising;
- 29 participants wrote explicitly about concretization by using manipulatives;
- 14 participants wrote about raising pupils' interest and motivation;
- 5 participants wrote about serving gifted pupils or pupils with learning difficulties.

In particular, among participants who wrote about concretization, nine listed other confusing resources; 22 participants who focused on the lesson listed confusing resources; almost all ($N = 12$) participants who wrote about raising interest listed various games.

Using the classification given above for 66 participants, we retrospectively examined their other declared characteristics (Table 4). Experienced teachers fo-

cused on the lesson and listed confusing resources, while novice teachers focused on the concretization purpose of manipulatives and had fewer mentions of confusing resources. Majority of participants who used manipulatives and attended professional conferences often were focused on the lesson and listed confusing resources.

Table 4. Classification from open questions compared to participants' practice elicited in demographic data, and questions Q.1 and Q.3.

		Lesson focus	Concretisation	Interest focus	Special education	Confusing resources	Total
Years of practice	0–10	7	12	2	3	5	18
	12–26	12	7	6	1	11	20
	27–42	17	10	6	1	20	28
Freq. of attending prof. conferences	Rarely					1	1
	Sometimes	8	4	3	2	5	13
	Often	28	25	11	3	30	52
Freq. of using manipulatives	Rarely		1	1		2	2
	Sometimes	13	13	4	1	11	26
	Often	23	15	9	4	23	38
Total		36	29	14	5	36	66

Participant 1. had six years of practice, attended professional conferences often and led a mathematics group. She sometimes used manipulatives and used all types of them. She listed appropriate self-made manipulatives (finger tracing on sand, sticks and figures for numbers, Lego cubes for arithmetic, money) and described their purpose across domains (relationship between numbers, recognizing geometric shape, estimating and measuring and applying computations).

Participant 2. had three years of practice, attended professional conferences sometimes and led a mathematics group. She often used manipulatives and used all types of them. She listed appropriate self-made manipulatives (egg cartons for decimal units, cardboard rods for numbers up to 20, a bundle of 10 sticks as ten, models of geometric figures and solids) and described their purpose as “visual representation of mathematical problems and problem-solving”.

Participant 3. had 21 years of practice, attended professional conferences often and led a mathematics group. She often used manipulatives and used self-made and virtual manipulatives. She listed all sorts of resources, including appropriate self-made manipulatives (dominos, plastic corks, cubes) and confusing resources (puzzles, games, sheets, virtual games and quizzes). As a purpose of manipulatives, she wrote “I use games in teaching mostly for motivation, repetition and practice of covered content and automatization of calculation.”

Participant 4. had 22 years of practice, attended professional conferences often and led a mathematics group. She often used manipulatives and used self-made and virtual manipulatives. She mainly listed confusing resources (games, puzzles, board games) and a couple of resources that could be used as counters (plastic

corks and cubes). As a purpose of manipulatives, she wrote “I often use games in mathematics in introduction and closing part of the lesson. Sometimes I prepare thematic challenges and escape rooms.”

4. Discussion

Teachers participating in our study declared they use manipulatives often with students at the age particularly sensible to instruction with manipulatives. However, some findings suggest that they did not effectively use manipulatives in mathematics teaching and learning. First, they mainly used manipulatives as a tool to demonstrate a new concept in the introductory part of the lesson. In this way, the relationship between concept and manipulative can be made transparent to students but does not support or facilitate their understanding of the concept. The latter comes from the prolonged and coherent use of manipulatives by connecting representations. Second, participants in our study rarely used manipulatives to support students’ own creative and problem-solving work. Similar to participants in other studies, they related using manipulatives with motivating and entertaining students in the lesson. Hence, students do not have an opportunity to use manipulatives purposefully, focusing on the appropriate feature of the manipulative relevant for a given context.

The major issue that emerged from this study is what primary school teachers in Croatia regard as manipulative. Only a few participating teachers selected manipulatives that are affirmed concretization of particular mathematical concepts, such as Dienes blocks, Cuisenaire rods, Geoboard, or listed self-made manipulatives that represent mathematical concepts, such as bead string, egg cartons. They also listed various resources that are not manipulatives intended to represent mathematical concepts but means and tools to modify the pedagogical organization of a lesson. An important aspect of effective use of manipulatives is “concreteness fading” – moving between representations to facilitate the transition from the symbolic-concrete to integrated-concrete knowledge, and to abstract knowledge. Artefacts that participants selected or mentioned in our study do not support such development.

The results suggest several constraints on using manipulatives in primary mathematics education in Croatia. Participants in our study noted that manipulatives are expensive, hence they might not have any manipulatives at their disposal let alone that each student has a set of manipulatives for individual work. Additionally, they recognized that using manipulatives in mathematics education requires additional preparation and is time-consuming. Participants are aware of the benefits of using manipulatives and the importance of proper instruction when using manipulatives. Many teachers in our study underlined their focus on the lesson structure. The demanding organization and implementation might disrupt their usual lesson structure; hence they choose to include manipulatives in the least challenging way. For example, the most frequently chosen manipulative were models of solids. Note that composing and decomposing, and comparing and transforming figures and solids is beyond the curriculum requirements, so the teachers require only one set of models

which they can show and share around the class to facilitate recognition. Another example of frequently chosen manipulative were tangrams. They are a valuable model for figure recognition, transformations, and exploration of the relationship between perimeter and area. Following participants' other answers, they might utilize them only as puzzles to entertain students.

Around a quarter of participants in our study acknowledged a need for additional education about using manipulatives. Participating teachers often attend professional conferences, but the results of our study suggest that professional conferences and literature do not provide quality support for using manipulatives in mathematics education. Teachers rely on web sources and peer support to obtain information about manipulatives. Several issues about primary mathematics education in Croatia arise following the results of our study. The mathematics lessons appear teacher-centred, and the concepts are frontally exposed. Mathematical notions, properties and procedures are seldom represented with manipulatives, and the focus is on covering prescribed content and exercising in the abstract environment. Teachers tend to occupy and maintain students' attention with content-light technology and games instead of contextual and engaging problems.

5. Conclusion

Studies in mathematics education suggest that using manipulatives is effective as it increases students' conceptual knowledge and is beneficial as it creates a positive attitude towards learning mathematics. Understanding through manipulatives happens when students are motivated and challenged to use manipulatives as tools to solve mathematical problems and create new ideas rather than enacting already taught concepts and procedures. However, studies suggest that the effectiveness of using manipulatives depends on teachers' practice. The results of our study align with previous research. Puchner et al. (2008) argued using manipulatives should be carefully and meticulously included in teachers' training and professional development.

We noted that participants showed a positive attitude towards using manipulatives, but their knowledge about what are manipulatives and when and how to use them, seemed incomplete. We suggest developing a peer-supported network to share the experience of purposeful use of manipulatives in different settings. Educators should invest in professional conferences and literature as sources for disseminating knowledge about manipulatives and aim-oriented activities with manipulatives. A practice-oriented education about (cheap) self-made manipulatives could compensate for the expensiveness and inaccessibility of manipulatives in primary schools in Croatia. The manipulatives alone are not a solution to educational issues, but training pre-service and in-service teachers for effective use of manipulatives includes a shift in the practice toward student-centred lessons, problem-solving activities and relational knowledge of mathematical concepts.

Appendix A

List of manipulatives with corresponding number of participants using them

MANIPULATIVES IN NUMBER DOMAIN

Ready-made manipulatives

- Abacus ($N = 51$)
- Balance scale ($N = 24$)
- Cuisenaire rods ($N = 15$)
- Unifix cubes ($N = 12$)
- Dienes blocks ($N = 6$)
- Bead stairs – Montessori ($N = 4$)
- Number cards – Montessori ($N = 3$)
- Multiplication and Division board – Montessori ($N = 2$)
- Numerical rods – Montessori ($N = 1$)

Self-made manipulatives

- Counters (corks, sticks, beads, cards, cubes) ($N = 16$)
- Number line ($N = 15$)
- Dominos ($N = 5$)
- Multiplication table ($N = 5$)
- Bead string ($N = 4$)
- Decimal values models (egg carton, bundle of sticks) ($N = 4$)
- Finger tracing of numerals ($N = 2$)
- Cards with values and numbers ($N = 1$)
- Others: cardboard number rods, table of decimal values, Lego cubes, sandpaper numbers

MANIPULATIVES IN SHAPE AND SPACE DOMAIN

Ready-made manipulatives

- Models of solids ($N = 111$)
- Tangram puzzle ($N = 70$)
- Models of solids with net ($N = 30$)
- Geoboard ($N = 16$)

Self-made manipulatives

- Geometric figures ($N = 9$)
- Solids from kinetic sand or sticks ($N = 1$)

OTHER MANIPULATIVES

Virtual manipulatives

- Geogebra ($N = 2$)
- Matific ($N = 2$)

Other manipulatives

- A virtual material ($N = 5$)
- Everyday objects (wool, string, figures) ($N = 4$)
- Paper money ($N = 3$)
- Dice ($N = 3$)
- Hula hoop for multiplication ($N = 2$)
- Images ($N = 2$)
- Diagrams ($N = 2$)
- Meter ($N = 2$)
- Other: cube with unit volume, cards with notions, materials for sequencing, materials for grouping, tables

Artefacts and resources

- Games ($N = 14$)
- Puzzles ($N = 9$)
- Cards with tasks ($N = 7$)
- Tombola ($N = 7$)
- Memory ($N = 6$)
- Playing cards ($N = 5$)
- Posters ($N = 5$)
- Bingo ($N = 4$)
- Sudoku ($N = 3$)
- Kahoot quizzes ($N = 3$)
- Wordwall quizzes ($N = 2$)
- Powerpoint presentations ($N = 2$)
- Exercise sheets ($N = 2$)
- Other: music wands, postcard, body, Pseudodabar, crosswords, objects for measurement

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Didaktički materijali u razrednoj nastavi matematike u Hrvatskoj

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Sažetak. Didaktički materijali mogu poslužiti za uvođenje apstraktnih matematičkih pojmova. Različita istraživanja su pokazala pozitivne učinke korištenja didaktičkih materijala u poučavanju i učenju matematike. Znanje i stavovi učitelja o didaktičkim materijalima utječu na smislenu i korisnu uporabu materijala u nastavi matematike. Postoje mnogi gotovi materijali, dok neke materijale u svrhu korištenja u matematičkom obrazovanju učitelji i samostalno izrađuju. Uzimajući to u obzir, istražili smo koja je uloga didaktičkih materijala u razrednoj nastavi u Hrvatskoj. U tu svrhu, proveli smo istraživanje s učiteljima o uporabi didaktičkih materijala u razrednoj nastavi matematike. Rezultati su pokazali da učitelji koriste različite vrste didaktičkih materijala. Istaknuli su prednosti korištenja didaktičkih materijala te naveli da ih koriste često u nastavi matematike. Ipak, odgovori nekih ispitanika ukazuju na nepoznavanje karakteristika didaktičkog materijala te načina uključivanja didaktičkih materijala u nastavni proces.

Ključne riječi: didaktički materijali, razredna nastava matematike, učitelj matematike

Visualization in the Teaching and Learning of Mathematics

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Abstract. Teaching mathematics requires the use of various external representations, concrete (objects) and visual (picture) models. The activity is much more efficient if pupils employ and combine several representations at the same time.

This paper focuses on a pedagogical experiment assessing the problem solving skills of elementary school children and university students. The experimental and control groups solved the problems using various visual representations. We hypothesized that using different representations facilitates problem solving to various extent. The hypothesis was proved as those participants who used cards were more likely to devise strategies and achieved better compared to the participants who used virtual cards.

Keywords: unusual problems, representations, problem solving, mathematical model, strategies

1. Introduction

Our previous research investigated the problem solving skills of economics majors, on a smaller sample ($N = 57$) (Debrenti, 2015). Research participants were asked to solve a logic problem, using three different types of representation. The first group used the traditional method, i.e. pencil and paper; the second group used picture cards, carrying out concrete operations; while the third group solved the problem with the help of a computer game.

Results show that the traditional method was the least efficient (62 %). Students using the computer game were slightly more efficient (64 %), while the picture card method resulted in a 100 % success rate.

Recently, due to the COVID-19 pandemic, all age groups had no choice but to use computing devices as for the past one and a half year teaching was done online. In our current research we set out to investigate whether a few years of extra “digital

proximity” would yield different results. Furthermore, we wished to test not only economics majors but also other majors (such as graphic arts, for whom visualization is essential), but most of all teacher training students and elementary school children. We also chose to investigate problem solving by age groups focusing on picture representation as a means of solving problems.

The research was based on three mathematical logic problems. Given the complexity of the problems, participants need not only logical thinking for solving the problems but also a high level of comprehension.

Our pedagogical experiment examines which method, – the card based or the computer game based – proves to be more efficient for solving the same logic problem.

2. Theoretical background

2.1. Problem and problem solving

A mathematical problem is a problem with a higher level of difficulty. These are “accurately formulated practical or theoretical questions, which can be solved through reasoning and mathematical operations” (Olosz & Olosz, 2000, p. 205).

Curiosity, volition and determination contribute to problem solving. The more curious we are, the more difficult problems we can solve (Pólya, 1971). Consequently, in order to be more efficient and to motivate children, we should not only develop problem solving skills but also increase their curiosity.

Pólya identifies two types of individuals as regards problem solving: the experienced and the inexperienced. They use different types of reasoning. When solving problems everyone makes guesses to a certain extent, experienced individuals, however, try to guess part of the solution, they have an idea, an anticipation of the solution, and they rely on previously gained knowledge. Nonetheless, Pólya claims that both experienced and the less well-versed people can reach the correct answer (Pólya, 1971).

Schoenfeld (1985) believes that the two individuals should be treated very differently. According to him, the difficulty of the problem does not lie in the text, or the problem itself but in the individual. People are at different levels, thus the same problem can pose different difficulties to different individuals. Some might find a problem complicated, while others do not find it challenging at all (András et al., 2010).

Nevertheless, a problem is a problem for someone until they wish to solve it, until they are interested in it and try to do their best in order to find the answer to it. If they pass on the task, it will no longer be their problem but that person’s problem to whom they have passed it on (Tuzson, 2014).

2.2. Representations

Most information transmitted to our brain is visual, consequently pictures and objects play an important role. Numerous psychological studies have attested to the fact that illustration facilitates a better understanding of various concepts (Debrenti & Zakota, 2014).

Mathematics often requires the representation of the elements of various structures. In communication we mostly rely on external representations (Lesh, Post & Behr, 1987, as cited in Debrenti, 2013), which can be objects, i.e. material; pictures, i.e. visual; and symbolic, which encompass the spoken and written language, as well as symbols. On the other hand, when we reason about a mathematical concept, we use internal representation, which cannot be directly researched or accessed.

Cognitive psychologists have formulated two hypotheses on representations. The first establishes a relationship between internal and external representations as we can make deductions on internal representations based on the manipulation of external representations. The second hypothesis states that internal representations are interconnected, and this relationship can be stimulated by establishing a connection between external representations (Ambrus, 2000).

The key to successful problem solving is to apply the most appropriate representation method and to switch with ease to geometric representation when solving algebraic problems (Ambrus, 2000). In our opinion, the duties of teachers should include raising children with good problem solving skills; consequently they must find the most appropriate representation method.

Traditional didactics prescribes visual representation and the use of objects in elementary education, and the focus on symbolic representations in secondary education. International literature in the field of mathematics didactics now claims that visual representations should be used in secondary education and in higher education as well (Ambrus, 2000). A number of studies highlight the fact that mathematics taught in schools is many times completely different from everyday problem solving. They point out that with the use of solely the symbolic level children will not be able to create the transfer effect (Dobi, 1998, as cited in Debrenti, 2014).

According to Dienes, mathematics should build on children's activities, games and concrete experiences. Mathematics learning should be a joyful experience, children should be drawn to this subject while using tools that enhance efficiency (Dienes, 2015).

“The involvement of hands and activities in reasoning is not a sign of lower order thinking, on the contrary, it means adapting to the opportunity to utilize abstraction in the topic in question.” (Szendrei, 2005, p. 319). What can be solved by hand should be done using concrete objects not with the help of computers as this method develops children's senses, memory, imagination and spatial abilities. We share the view that the online space will never be a perfect substitute for concrete, hands-on tools.

One of the fundamental principles of Varga Tamás' complex mathematics teaching method was the use of realia in the classroom in order to gain practical experience. One of his aims was to formulate mathematical models for real-world problems and to put these into practice (Pálfalvi, 2007).

2.3. The digital world and today's generations

Digitalization has evolved rapidly in the past twenty years, so much so that today we cannot imagine life without digital tools and the internet.

Benedek (2008) defines digital pedagogy as a traditional or constructive pedagogy, i.e. the teaching-learning method, in which both the teacher and the learner use digital tools.

Due to the worldwide spread of the coronavirus epidemic (SARS-CoV-2) in March 2020, there was a shift to online learning (Szűts, 2020). As a result, even students who have never used computing devices before had to familiarize themselves with the digital sphere. Teachers had to use various tools, online games, and create exercises to develop children's abilities.

Young people today socialize mostly in the digital sphere. As a result, the traditional school system is facing a serious challenge (Tapscott, 2001, as cited in Jakab, 2019). Digitalization has many negative effects, such as the deterioration of calligraphy and spelling skills; or the use of calculators instead of improving arithmetic skills. It is noteworthy to mention that the advantage of digital tools is that it can turn passive, monotonous learning into interactive and productive learning (Jakab, 2019). We believe that if learning is more interactive, children are more enthusiastic and curious, which has a positive effect on problem solving (Pólya, 1971).

Szűts (2020) states that smart tools, internet communication and digital content should only be used in education if they prove to be more efficient than traditional tools. There is a huge discrepancy between the digital competences of children and those of the teacher.

3. Participants, methodology

3.1. Research aims, objectives

In what follows we will present three mathematical logic problems, which elementary school children and university students solved in two different ways: online and offline. Our research sets out to answer the question whether concrete, hands-on (card based) problem solving (i.e. offline) is more efficient than virtual (computer based) problem solving (i.e. online on the computer).

Another question we aimed to answer is whether there is a significant difference between the result of students majoring in different fields of study, namely teacher training, graphic art and economics.

We found it important to compare the results of teacher trainees and 9–11 year-old elementary school children. We believe that there will be a significant difference between these results as the former have better problem solving skills than their future pupils.

3.2. Hypotheses

We formulated three main hypotheses.

The first hypothesis states that hands-on (card based) problem solving will prove to be more efficient than the online game (computer based) problem solving, both in the case of digital natives, i.e. elementary school students, and undergraduates.

The second hypothesis proposes that teacher trainees will achieve significantly better results than elementary school students.

The third hypothesis predicts that there will not be significant differences between the achievements of undergraduates majoring in different fields of study.

3.3. Research participants

A total of 227 individuals were involved in the research, 103 (9–11 year-old) elementary school students (49 online, 54 offline) and 124 undergraduates (47 online, 77 offline). Elementary school students came from 3 different schools, in two different counties from Romania (Arad and Bihor). The majority of undergraduates studied at Partium Christian University, however, there were also teacher trainees involved from Babeş-Bolyai University (Satu Mare, Romania) and the University of Nyíregyháza (Hungary).

3.4. Research methodology

Given the current epidemiological situation, the research was carried out in two different forms: face to face and online. All elementary school students were supervised during the test. For online testing children used the school's informatics lab. We also supervised and monitored the majority of undergraduates, apart from participants from other universities, who were monitored by their teachers.

The face-to-face tests were also supervised in person. Participants were given the same instructions and approximately the same timeframe, i.e. 1 hour and 20 minutes.

The measurement instrument used contained three logic problems of different type. The first was a puzzle-like game, the second assessed two-dimensional spatial skills, while the third was a logic problem, which could most easily be solved using the truth-table method. Comprehension skills were essential for all three tasks. The virtual space was designed by a student in the 11th grade using JavaScript, PHP,

HTML and CSS to create a website for the three games involved in the research test. The programme made screenshots of the solutions at the end of the game. Since this was a universal online platform, there was no need for installation, or high recommended system requirements, consequently the website could be easily accessed in school informatics labs or on home computers. For face-to-face testing we personally selected the picture cards to be used. These were printed, laminated, cut out and put into envelopes. Each envelope contained one game. Face-to-face testing required more extensive preparation than online testing.

Research participants were given clear instructions. The complexities of the texts were pointed out every time, and subjects were asked to read the text multiple times if needed.

Criteria for assessment were laid down in such a way that in many cases it allowed for partial grading. Each problem was worth 100 points. In order to calculate individuals' results we had to analyse each screenshot.

3.5. Problems and their solutions

The hexagon problem. The first problem asks for matching seven hexagon-shaped pieces following the rules. According to the literature of the field this type of problem is a transformation:

Place the small hexagons into the large shape so that adjacent triangles contain the same number (triangles are considered adjacent if they share a side). The hexagons cannot be rotated (Marchis, 2013).

The figure below (fig. 1) presents the seven hexagons and the larger image that had to be constructed similarly to a puzzle.

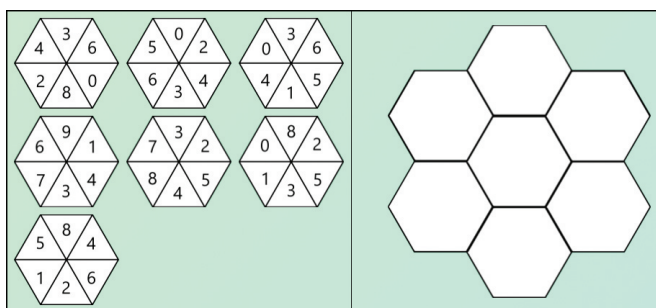


Figure 1. The online representation of the hexagon game.

Different strategies could be used for solving this problem. Some participants tried to find the middle piece, others started from the sides, and there were also students who solved the problem using the trial and error method. The only correct solution is shown in fig. 2.

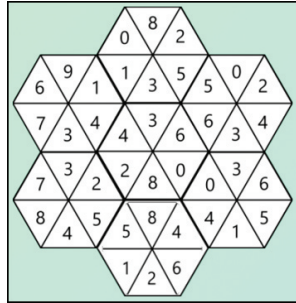


Figure 2. The correct solution of the hexagon game.

In the case of this problem there was no point in partial grading. Students either solved the problem correctly or they didn't. Thus, correct answers were worth 100 points, wrong answers were worth zero points.

The cake problem. The second problem, i.e. the cake problem, is the perfect example of manipulating plane shapes:

The figure represents a lattice cake consisting of 20 equal-size squares. Five friends wish to share the cake in such manner that each of them gets a differently shaped four-square piece. Could you help them out? (Matlap 8, 308)

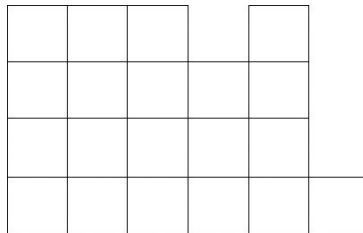


Figure 3. The squares to be filled in the cake problem.

There is only one correct solution to the problem:

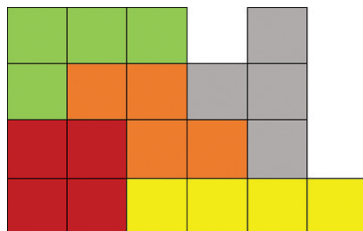


Figure 4. The correct solution presented in the Matlap.

The thirds task is a well-known logic problem: *There are 5 houses in 5 different colours. Each house is inhabited by a person of a different nationality. Each owner prefers a certain beverage, has a different hobby and keeps a certain pet. NO OWNER drinks THE SAME beverage, has the same pet, or has the same hobby as their neighbour. What we know is:*

1. The Brit lives in a red house.
2. The Swede has a dog.
3. The Dane drinks tea.
4. The German plays the piano.
5. The Norwegian lives in the first house.
6. The green house owner drinks coffee.
7. The owner who plays golf likes juice.
8. The owner of the yellow house plays football in his free time.
9. The owner who dances has a parrot.
10. The man who lives in the middle house drinks tea.
11. The owner who plays board games lives next to the one who has a cat.
12. The man who has a horse lives next to the one who plays football.
13. The Norwegian lives next door to the blue house.
14. The owner who plays board games is the neighbour of the one who drinks water.
15. The green house is next to the white house, on the left.

Fig. 5 presents the solution, the houses and the pictures adapted by us for this logic problem. Each item placed correctly was worth 4 points. Since there are 5 houses, in 5 different orders, the maximum point number was 100 ($4 \times 5 \times 5$), which was subsequently transformed into percentage.



Figure 5. The solution for the adapted Einstein's problem.

Computers automatically measured the time, and once students clicked the "Completed" button, the solution was saved in the database in the form of an image. Participants could also use the "Start again" button.

4. Results

4.1. Results of the hexagon game

The results of the hexagon game proved to be quite successful. Wrong solutions resulted from comprehension problems. Though strong emphasis was placed on reading the problem carefully, many of the students, including undergraduates, did not conform to the rules of the problem. For example, many participants rotated the pieces.

The majority (145 individuals, 63.87 %) solved the problem correctly. Only 82 participants (36.12 %) did not manage to solve it. From the 131 offline participants 72 individuals (54.96 %) achieved accurate results, while 59 students (45.03 %) did not solve the problem correctly. From the 96 online participants 73 individuals (76.04 %) achieved accurate results, while 23 students (23.95 %) did not solve the problem correctly. On the whole, almost half of the offline participants solved the problem correctly, while about half did not solve it. As regards online participants there were much more correct results (76.04 %) than incorrect ones.

One third of the elementary school students tested offline (18 individuals, i.e., 33.33 %) managed to solve the problem with cards, while two thirds were unsuccessful (36 individuals, 66.66 %). Exactly 46 (87.76 %) of the elementary school students who worked online managed to solve the problem, and only 6 (12.24 %) of them failed. Elementary school students achieved significantly better results online than with cards: $t(103) = 6.67, p < 0.05$.

Exactly 54 (i.e., 70.13 %) of the undergraduates tested offline managed to solve the problem with cards, while the rest were unsuccessful (23 individuals, 29.87 %). Respectively, 30 (63.83 %) of the undergraduates who worked online managed to solve the problem and 17 individuals (36.17 %) failed. Undergraduates achieved better results using picture cards than working online.

The averages of correct solutions are given in percentage in the diagram (fig. 6) below.

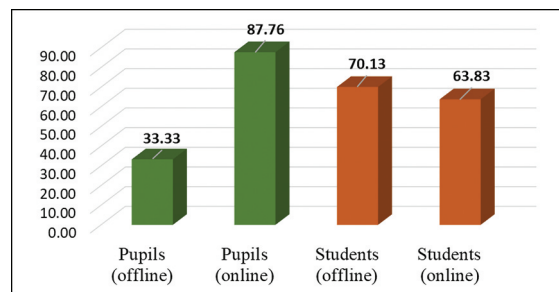


Figure 6. The diagram representing the averages of correct solutions for the hexagon problem (in percentage).

As the diagram shows elementary school students achieved better results online.

The results of the undergraduates show less difference between the averages of the two methods. The F -test and t -test confirm that there is no significant difference between the two averages: $t(124) = 0.70, p < 0.05$.

When comparing the offline groups we found that adults achieved significantly better results than children. The F -test and t -test confirm the significant difference: $t(131) = 4.44, p < 0.05$.

On the other hand, children achieved better results online than adults did (t -value is significant: $t(96) = 2.83, p < 0.05$).

4.2. Results of the cake problem

Almost all participants interpreted the problem correctly. What posed a challenge was finding the perfect solution. Students had to fit five shapes into the grid, i.e., to cut the cake into five pieces. This is the reason why the highest possible score was 5, which was then converted into percent. In the light of this, if someone achieved 5/5 they were given 100 %, 4/5 was 80 %, and so on. Out of the 227 research participants 35 (6 elementary school students and 29 adults) managed to find the correct solution. The table below (table 1) presents the proportions in details:

Table 1. Results of the cake problem.

	Pupils (offline)	Pupils (online)	Students (offline)	Students (online)	Total
100 %	5	1	24	5	35
80 %	18	22	37	31	108
60 %	12	24	15	9	60
40 %	15	2	9	1	18
20 %	1	0	0	0	1
0 %	3	0	1	1	5

The diagram (fig. 7) based on the table perfectly illustrates that the majority achieved 60–80 % results. It also shows that perfect results were achieved among undergraduates who worked offline.

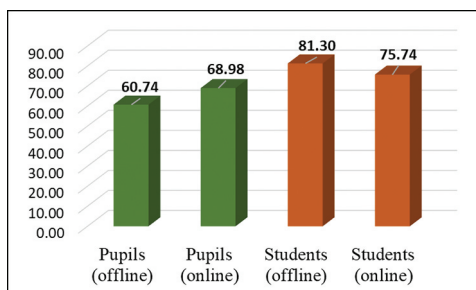


Figure 7. The averages of strict results for the cake problem.

If we compare the averages presented in table 7, we find less variation. The results achieved by the two groups of elementary school students and two groups of undergraduates are neck and neck.

The F -test and t -test were carried out on the averages. There is a significant difference in the results of the two groups of elementary school students: online respondents had a significantly better result than offline respondents: $t(103) = 2.08$, $p < 0.05$. When comparing the groups of undergraduates we found that offline respondents had a significantly better result than online respondents: $t(124) = 1.78$, $p < 0.05$.

We also compared the two large groups of offline participants and found that there is a significant difference between them: $t(131) = 5.59$, $p < 0.05$, in favour of the undergraduates. Comparing the differences between online results, we find that the F -test and t -test yield a significant difference: $t(96) = 2.27$, $p < 0.05$ also in favour of the undergraduates.

4.3. Results of the logic problem

The last task was a complex logic problem based on comprehension. Only 41 (18.06 %) of the participants solved the problem correctly, while 34 (14.97 %) subjects failed or gave up. The other students were partly successful, managing to solve the problem to varying degrees.

Some of the students used very interesting problem solving methods. There were participants who placed the picture cards in the form of a chart, other children used the space on the floor, saying they needed more room to process the task.

The table below (table 2) presents detailed data on the results achieved by different groups. It is thought – provoking that more than half of the participants (149 subjects, i.e., 65.63 %) did not even achieve 50 %.

Table 2. Results of the logic problem.

	Pupils (offline)	Pupils (online)	Total Pupils	Students (offline)	Students (online)	Total Students	Total
100 %	1	1	2	30	9	39	41
75 % – 99 %	1	1	2	9	2	11	13
50 % – 74 %	5	2	7	14	3	17	24
25 % – 49 %	27	16	43	18	7	25	68
1 % – 24 %	19	24	43	3	1	4	47
0 %	1	5	6	3	25	28	34

A strange phenomenon can be observed, namely that both the best and the worst results were achieved by undergraduates. According to the diagram, adults proved to be successful with the hands-on (card based) method but this does not hold for their achievements online.

The diagram shows that the majority of elementary school students achieved 50 %, while adults achieved over 25 % or 100 %.

The averages achieved within groups are shown in fig. 8 below.

As the diagram shows the best results were obtained by undergraduates offline. F -test and t -test show that this result is significantly better than the results of the undergraduates working online: $t(124) = 5.45, p < 0.05$. The results of undergraduates working offline do not only exceed the results of undergraduates online but also those of elementary school students working offline. The difference is statistically significant: $t(131) = 7.71, p < 0.05$.

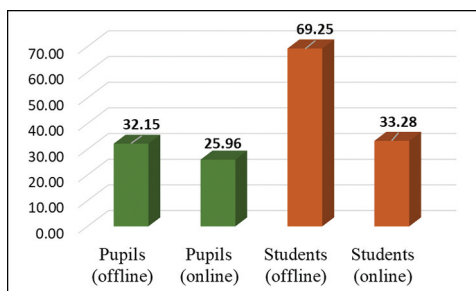


Figure 8. The average results of the logic problem.

F -test and t -test were also carried out on the results of the two groups of elementary school students. The results show no significant difference: $t(103) = 1.62, p < 0.05$ between the achievements of the two groups. Nonetheless, elementary school students were more efficient when using the hands-on approach.

A second statistical calculation was made comparing and contrasting the achievement of elementary school students working online and undergraduates working online. F -test and t -test show no significant difference: $t(96) = 1.11, p < 0.05$ between the achievements of the two groups.

Our first hypothesis stated that hands-on (card based) problem solving would prove to be more efficient than the online game (computer based) problem solving, both in the case of elementary school students and undergraduates. The findings are surprising as elementary school students achieved better online with almost all problems. Apart from the logic problem there is a significant difference in all cases. The logic problem is an exception as students working offline achieved better results, however the difference is not statistically significant. Thus, it can be concluded that the hypothesis has not been confirmed as regards elementary school students.

Undergraduates achieved better results using the hands-on approach in all three tasks, however the difference was statistically significant only in the case of the cake problem and the logic problem. As regards university students our hypothesis has been confirmed.

Teacher trainees are responsible for future generations. We hypothesized that these majors would achieve better results than elementary school students. The

hypothesis was tested for each problem in part. Results were turned into averages, then compared and contrasted.

The diagram below (fig. 9) shows that elementary school students working online achieved better results on the hexagon problem. Conversely, when using the offline method adults achieved better.

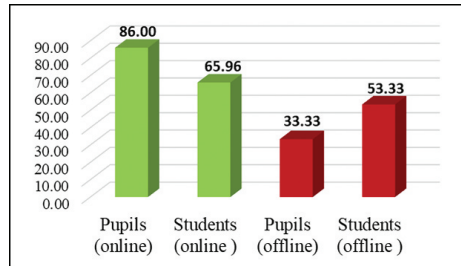


Figure 9. The averages of teacher trainees and elementary school students for the hexagon problem.

Since all undergraduates participating online were teacher trainees, no further calculations were made. The result of the significance test is the same as with our first hypothesis: elementary school students achieved significantly better than teacher trainees.

Offline results were also subjected to statistical tests. Findings show that teacher trainees achieved significantly better than elementary school students, the t -value was: $t(84) = 1.80, p < 0.05$.

The results of the cake problem are shown in the diagram (fig. 10) below. It is immediately obvious that teacher trainees working online, as well as offline achieved better results than elementary school students.

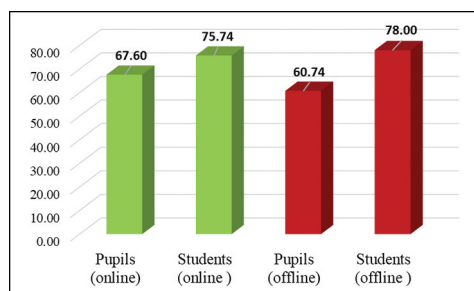


Figure 10. The averages of teacher trainees and elementary school students for the cake problem.

The calculations mentioned in the first hypothesis proved the significant difference in favour of teacher trainees when working online. When analysing offline results, we found that teacher trainees achieved better again, compared to elementary school students: $t(96) = 3.49, p < 0.05$.

The logic problem posed challenges to many participants. However, teacher trainees working offline achieved far better results than all the other participants.

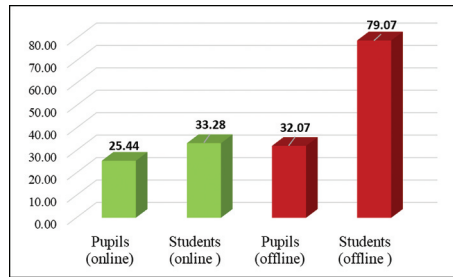


Figure 11. Averages of teacher trainees and elementary school students for the Einstein problem.

Referring back again to the calculations in the first hypothesis, it can be stated that teacher trainees working online achieved better than elementary school students, however the difference is not statistically significant. In contrast, comparing offline results we found that teacher trainees attained significantly better results than elementary school students. We obtained: $t(96) = 8.96, p < 0.05$.

Calculations show that teacher trainees achieved significantly better than elementary school students in almost all cases. Exceptions were the hexagon problem solved online, in the case of which children attained significantly better results, as well as the online logic problem, in the case of which there was no significant difference between the two groups.

The third hypothesis examines the three types of adult participants: teacher trainees (47 online, 30 offline), graphic art majors (28) and economics majors (19). Since teacher trainees were tested both online and offline, they were divided into two categories.

The results are shown in table 3.

Table 3. Summary table for the results of adult groups.

	Hexagon problem	Cake problem	Logical problem
Teacher trainees (online) (N = 47)	65.96	75.74	33.28
Teacher trainees (offline) (N = 30)	53.33	78.00	79.07
Fine Arts Graphics (N = 28)	71.43	83.57	61.86
Economics (N = 19)	94.74	83.16	64.63

The diagram shows that in the hexagon problem economics majors achieved the best results (94.74). Graphic art majors came off second best (71.43).

Though there was no striking difference, we compared the groups achieving the highest and lowest results for the cake problem. These were the graphic art majors with an average of 83.57 and the teacher trainees working online with a 75.74 average.

In the case of the last problem teacher trainees who solved the problem offline achieved better results, 79.07.

We used ANOVA to compare the differences between the student groups. Results from the single factor analysis shows that there are no significant differences between the achievements of undergraduates majoring in different fields of study.

Table 4. Results from the single factor analysis.

SUMMARY						
<i>Groups</i>	<i>Count</i>	<i>Sum</i>	<i>Average</i>	<i>Variance</i>		
Teacher trainees (online)	3	174.98	58.33	494.56		
Teacher trainees (offline)	3	210.4	70.13	211.96		
Fine Arts Graphics	3	216.85	72.29	118.43		
Economics	3	242.53	80.842	230.60		
ANOVA						
<i>Source of Variation</i>	<i>SS</i>	<i>df</i>	<i>MS</i>	<i>F</i>	<i>P-value</i>	<i>F crit</i>
Between Groups	775.32	3	258.44	0.98	0.45	4.066
Within Groups	2111.12	8	263.89			
Total	2886.44	11				

5. Conclusion

Our first hypothesis stated that hands-on (card based) problem solving will prove to be more efficient than the online game problem solving, both in the case of elementary school students and undergraduates. The findings are surprising as elementary school students achieved better online with almost all problems. Apart from the logic problem there is a significant difference in all cases.

The second hypothesis was confirmed. Teacher trainees achieved significantly better results than elementary school students in almost all cases. Exceptions were the hexagon problem solved online, in the case of which children attained significantly better results, as well as the online logic problem, in the case of which there was no significant difference between the two groups.

The third hypothesis, which predicted no significant difference between the results of the groups of undergraduates, was confirmed. Economics and graphic art

majors were mostly neck and neck, achieving better results than teacher trainees, except in the case of the logic problem, where teacher trainees working offline achieved better results.

We experienced many interesting things during the investigation carried out on elementary school students and undergraduates. We got to know new people, we discovered different ways of thinking, and we found this research an edifying experience. We began to think about the fact that changes in the world might lead to changes in children's perception and problem solving abilities. If this is confirmed, new methods and tools might have to be devised for qualitative development.

We realized that it wasn't logic that posed challenges at all times but rather comprehension and lack of experience. Adults find it much easier to think systematically. They can identify unnecessary and important information with more ease than children who learned to read not long ago.

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Vizuális reprezentációk szerepe a matematika tanítása és tanulása során

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Kivonat. Többféle külső reprezentáció, konkrét (tárgyi) és vizuális (képi) modell alkalmazása szükséges a matematika tanításakor, a tevékenység sokkal hatékonyabb, ha a tanuló többféle reprezentációt párhuzamosan használ és összekapcsolja azokat.

Az előadásban egy pedagógiai kísérletről számolok be, amely során elemi osztályos tanulók és tanítóképzésben résztvevő hallgatók problémamegoldó képességét mértük, a feladatokat különböző vizuális reprezentációkat használva oldotta meg a kísérleti csoport, illetve a kontroll csoport. A hipotézisünk az volt, hogy a különböző reprezentációk alkalmazása eltérő mértékben lehet segítségükre a megoldásban. A kísérlet során ez be is bizonyosodott, hisz azok, akik kártyákkal dolgozhattak, nagyobb eséllyel dolgoztak ki stratégiát, jobban teljesítettek, mint azok, akik virtuális kártyákkal dolgozhattak.

Kulcsszavak: szokatlan feladatok, reprezentációk, problémamegoldás, matematikai modell, stratégiák

2. Improving Methods and Strategies in Teaching Mathematics



The “Concurrent” Method of Teaching Multiplication and Division

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Abstract. This paper proposes the concurrent method of teaching multiplication and division in elementary school. The term “concurrent” implies the basic characteristic of the method which is to simultaneously introduce both arithmetic operations. The concurrent method supports making connections between arithmetic operations and ease memorization of the table of multiplication and division. The theoretical rationale for the concurrent method is based on Vergnaud’s theoretical perspective and the principle of connections. This paper aims to describe the method, contrasting it with the traditional method. The method was implemented within a textbook series for primary grades of elementary school and key features of the implemented model are presented.

Keywords: teaching method, curriculum design, multiplication, division, textbook

1. Introduction

This paper provides a rationale for introducing an alternative to the standard approach to teaching multiplication and division and describes this approach. The challenging parts of implementing a new mathematics teaching approach in the school curriculum are to influence teachers’ beliefs about what is an appropriate mathematics teaching and to change their well-established practices. Traditionally, teachers often teach using conventional methods without considering the logical thinking, critical and creative aspects of the subject matter (Treffers, 1991). Confrey and coauthors identify multiplication and division as one of eight major research areas of research on rational numbers (Confrey & Maloney Nguyen, 2008). The struggle to meet the challenge of changing accustomed practice related to the introduction of multiplication and division is an important element of this paper.

2. Theoretical perspective on Multiplication and Division

We consider here Vergnaud's perspective on multiplication and division as arithmetic operations. Vergnaud (1988) utilizes the context of the conceptual field of multiplicative structures to identify multiplication and division. It consists of "all situations that can be analyzed as simple and multiple proportion problems and for which one usually needs to multiply or divide" (Vergnaud, 1988, p. 141). He links the multiplicative structure to various mathematical concepts: multiplication, division, fraction, ratio, linear function, vector spaces, etc.

Within multiplicative structures, three classes of problems are identified:

- 1) isomorphism of measures,
- 2) product of measures,
- 3) multiple proportions.

Isomorphism of measures covers all situations where there is a direct proportion between two measures spaces, regardless of the types of numbers involved. This class of problems encloses seven types of problems: equal groups, equal measures, rate, measure conversion, multiplicative comparison, part/whole, multiplicative change. Multiplication and division problems can be seen as special cases of problems involving two ratios, so-called rule-of-three problems. The second class of problems, products of measures, covers situations where two-measure spaces are mapped into third. For example, "3 children each have 4 oranges. How many oranges do they have altogether?" (Greer, 1992, p. 280) This category encloses Cartesian products and rectangular area. The third class of problems, multiple proportions refers to a product of measure and may be decomposed into simpler problems already mentioned in previous classes. Here is an example of a multiple proportions problem.

A cost of a coffee in a student cafeteria is \$2. Five days a week Lara had a coffee in the cafeteria for 18 weeks. What was the total amount of money she spent on the coffee?

Greer (1992) presents exemplary context situations for each class of problems. For the first seven classes of problems, it is possible to formulate "multiplication problem", "division (by multiplier)", and "division (by multiplicand)" but for the last three there are only two variants, "multiplication problem" and "division problem". The classifications by Usiskin and Bell (1983) also matched classifications of multiplication and division. They distinguished the Size change, Acting across, and Rate factor as types of multiplication contextual problems and Ratio, Rate, Rate Divisor, Size Change Divisor, and Recovering Factor as types of division contextual problems. "Division possesses three other use classes; each related to a use class of multiplication, rate divisor to rate factor, size change to size change, and divisor to recovering factor to acting across" (Usiskin & Bell, 1983, p. 267). Later Schwartz proposed that not all classes of problems should be introduced simultaneously. He suggests that when dealing with integers, only six problem situations should be

modeled: equal groups, multiplicative comparison, a rectangular array, rectangular area, and Cartesian product.

Zweng provided a simple classification of division, distinguishing two types of division “measurement division” and “partition division” (Zweng, 1972, p. 623). It was explained that in a measurement problem, the pupil is asked to find how many subsets containing a given number of elements can be formed from a set. The total number of objects in the set is also known. Measurement problems include contexts of sets, length, area, capacity, weight, and so on.

Mary had 30 pieces of fudge that she was putting into small boxes. If she put 6 pieces in a box, how many boxes could she fill?

In problems of the partition type, the number of objects in a set is given. The set is to be separated into a known number of subsets and the problem is to determine how many objects are in each of the subsets, e.g. “sharing equally”.

Bill gave 30 baseball cards to 6 of his friends. If he gave each friend the same number of cards, how many cards did each boy receive? (Zweng, 1983, p. 624)

Zweng (1972) pointed out that since the curriculum is not often clear about the two interpretations of division, the students have difficulty in analyzing problem situations. The difficulty is compounded by the fact that the formal representation of the division in the two types of problems is identical. Kaput (1985, p. 45) presents an example of the problem: “cutting a 30-foot ribbon into pieces 8 feet long – how many pieces? – how long is each piece? He points that the formal description of the computation in each case is “30 divided by 8 equals n ” but the referents of the elements are very different in the two cases. The differences show up in the interpretation of the results. It was shown that students often confuse feet with pieces or vice versa (Kaput, 1985).

One important issue is to consider how much pupils engage in studying relations between multiplication and division. The study of Le Fevre and coauthors (1996) shows that solution of division problems, facilitated the solution of multiplication problems more often than the reverse, “recasting” problems of division as multiplication and vice versa.

$$56 : 8 = ? \implies 8 \cdot ? = 56$$

$$7 \cdot 6 = ? \implies ? : 7 = 6$$

Another study showed that pupils do not draw upon inversion reasoning as a logical shortcut to help them solve problems (Mc Crink et al., 2007). Such findings present a call for inventing a didactical approach to overcome difficulties in this domain.

3. Multiplication and Division in Mathematics Curriculum in Serbia

The aim of learning the subject of mathematics in the 2018 school curriculum is that the student, masters mathematical concepts, knowledge, and skills, develops

the basics of abstract and critical thinking, positive attitudes towards mathematics, ability to communicate in mathematical language and scripts and applies acquired knowledge and skills in further education and problem solving as well as to form the basis for further development of mathematical concepts (MNPT, 2018). Multiplication and division are introduced in the second grade of elementary school.

Each in-school introduced arithmetic operation is linked to an implicit, unconscious, and primitive intuitive model (Fischbein, 1987). This model, Fischbein believed, mediates identification of the operation needed to solve a problem with two items of numerical data. In the Serbian curriculum, the arithmetic operation of multiplication is initiated as repeated addition (addition of equal sums). It is by Fischbein's perspective that addition is a primitive model for multiplication (Fischbein et al., 1985). The model imposes its constraints on the process of solving problems involving multiplication and division.

Collectively, the product is the number of elements of the union of disjoint equipotent sets. Various examples are observed that represent a multiplicative scheme, i.e. situations are described (in words, sets, visually $m \times n$ diagrams) when we have n elements in each of m places. By analogy, the arithmetic operation of division is formed by observing the breaking (disassembling) of sets with a finite number of elements into disjoint equipotent subsets (in cases when this is possible). The division table is built upon the relationship between multiplication and division. For example, $8 \cdot 3 = 24$, so $24 : 3 = 8$. After forming the division table, the notion of divisibility of a number is formed, it is determined which numbers are divisible by a given number, i.e. which number is the holder of given numbers. For example, number 12 is divisible by numbers 1, 2, 3, 4, 6, and 12, i.e. number 12 is the container of numbers 1, 2, 3, 4, 6, and 12. Incomplete induction through examples and analysis of pictorial representations leads to the rules of changing the place of factors, combining factors, multiplication, and division of sums and differences by numbers as well as multiplication by numbers 1 and 0, division of zero, and division by 1, without symbolic notation of rules. Based on the stated rules, we move on to the cases of non-variable multiplication (i.e. division) by a one-digit number. E.g. see (1) and (2).

$$12 \cdot 3 = (10 + 2) \cdot 3 = 10 \cdot 3 + 2 \cdot 3 = 30 + 6 = 36 \quad (1)$$

$$48 : 3 = (30 + 18) : 3 = 30 : 3 + 18 : 3 = 10 + 6 = 16 \quad (2)$$

In the second grade, only the "oral" calculation procedure is performed. One of the important goals is to remember the multiplication and division table up to 100, which is why enough time and diverse activities should be provided for practicing and gaining computational security.

The stated goals in teaching arithmetic in the new curriculum are to develop the ability to count, memorize multiplication tables and apply knowledge in solving simple textual problems with up to two operations. These goals drive the teachers to apply the mechanistic approach, practicing symbols and emphasizing the application of algorithm (Treffers, 1991). For instance, learning multiplication table first and division successively does not suit the expectation of making strong connections between operations. This model focuses on memorizing abilities rather than

on the pupils’ understanding. Whenever pupils forget some part of the multiplication table they do not have anything to fall back on. Then they develop a corrupt procedure. The weakness of such an approach becomes apparent when the time for solving equations with multiplication or division comes. After the “guessing” method of solving simple equations, pupils learn to solve equations with multiplication and division based on connections of those two operations which are visually presented in graph schema (Figure 1).

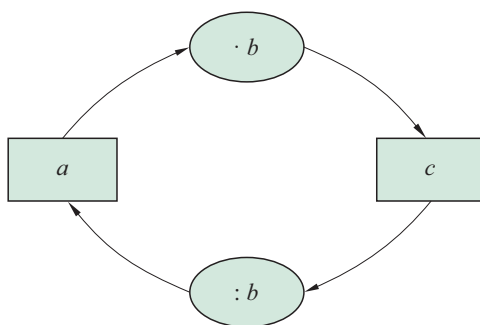


Figure 1. Graph schema connecting multiplication and division.

The school curriculum in Serbia does not specify which classification of the problems of multiplication and division is officially established. Although the curriculum does not specify which types of problems of multiplication are mandatory for mathematics instructions, in the case of division, it is stated that both “measurement division” and “partition division” problems should be studied during the instructions (Zweng, 1983).

4. Concurrent method of teaching multiplication and division

The concurrent method of teaching is built upon the principle of connection. The main idea is to point and emphasize connections between inverse operations of multiplication and division in building up a multiplication/division table. It also relies on realistic contexts which make connections “visible” and therefore easier to apprehend. The 2018 curriculum made it available by advocating introducing division as the inverse operation to multiplication, instead of the traditional approach of introducing a contextual problem to encourage the pupils to use their understanding of repeated subtraction as analog to division.

The “concurrent” method is based on a premise that two inverse arithmetic operations should be simultaneously introduced. This method also indicates concurrent use of multiple representations and model situations for each operation hinting at bonds between operations. Each representation may be used for both operations. The “concurrent” method supports making connections between multiplication and division as inverse operations and eases memorization of the table of multiplication and division. Establishing connections between different representations of an arithmetic operation contributes to a better understanding. Recognizing

that the same representation serves both operations helps the creation of bonds which will contribute in process of problem-solving. Linking bonds between multiplication and division provides support for conceptual and procedural knowledge. Finally, establishing connections between operations strengthens the mathematical structure, calculation proficiency, and problem-solving.

Characteristics of the concurrent method prototype can most easily be understood when considering a particular model of the approach as it was implemented in the development of a textbook series “*Matematika*” for grades 2 to 4 authored by J. Milinković (2019, 2020, 2021) Characteristics of the concurrent approach prototype were operationalized in quality aspects: connections, common representations for multiplication and division, multiplication/division table, common realistic contexts for both operations, textual problems emphasizing inversibility of operations. Ensuring the creation of strong connections between the conceptual understanding of multiplication and division is achieved through the order and manner of realization of teaching units. First, alternating teaching units are dedicated to the formation of the concept of multiplication and division. Second, the concept of division is developed by analogy with the development of the concept of multiplication. Shavelson and Salomon (1985, p. 4) argue explicitly for multiple representations: “The power of representing information in more than one symbol system (representation) lies in the ability to: (a) provide a more complete picture of a phenomenon than any single symbol system can; (b) of the chances of linking new information to the learner’s preferred mode of learning (i.e., to the learner’s preferred symbolic representation); and (c) cultivate cognitive skills in translating or shifting among symbolic representations.” We expect that use of multiple representations will bring better understanding and calculation skills. It is noticeable that in the approach we emphasize that the same representation or the same problem situation can be analyzed from the aspect of multiplication and division.



Figure 2. External representations of the concept of multiplication (Milinković, 2019).

The operations may be seen as modeling a variety of situations. The top left context (Figure 2) belongs to the equal groups class of problems (Vergnaud, 1988). We could have described the nests with birds situation by saying “There are two nests with three birds in each nest.” But we could also say “There are six birds split into two nests so each nest has the same number of birds.” The multiplicative scheme explicitly points that these operations are inverse. In the first case the creation of a union of sets with an equal number of elements and in the second the division of the set into a certain number of subsets with an equal number of elements. The terminology involving “set” could be avoided. Simply stated by

Greer, when “there is a number of groups of objects having the same number in each group” normally constitutes a child’s earliest encounter with an application for multiplication.

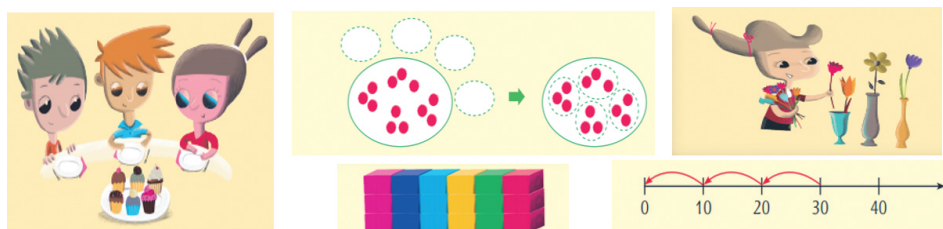


Figure 3. External representations of the concept of division (Milinković, 2019).

Building up the *Multiplication-division table* is for the most part built simultaneously. Within the same unit, pupils study multiplication by 7 and division by 7. The line of this reasoning is shown in (3).

$$7 \cdot 3 = 21, \quad \text{so} \quad 21 : 3 = 7 \tag{3}$$

The *multiplication table* (Figure 4) does not support making connections between multiplication and division facts. It is why memorizing division by numbers 1 to 10 requires additional effort. In contrast, the *Multiplication-division table* (Figure 4) reflects the connection between multiplication and division, highlighting the inversibility of these operations.

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Figure 4. Multiplication table (left) and Multiplication-division table (right).

The *Multiplication-division table* prompts reasoning based on connections of operations and makes memorization of the table easier. For example, because four times six is twenty four, twenty four divided by four is six, and twenty four divided by six is four. It is expected that in later grades, pupils using concurrent method show more confidence in solving equations with multiplication and division and mathematical modeling of problem situations corresponding to multiplicative scheme and invariance of operations. Simple mathematical tasks, as well as various problem situations, may contribute to strengthening connections (Figure 5).

Discover the rule and fill the picture with the missing numbers.

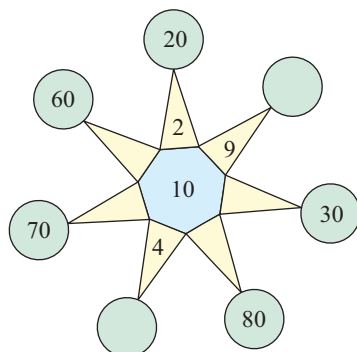


Figure 5. Mathematical task strengthening connections between operations (Milinković, 2019, p. 72).

5. Conclusions

In this paper, attention has been directed toward theoretical perspectives on multiplication and division. Particular attention has been placed on Fischbein's theory of primitive models and Vergnaud's classification of multiplicative situations. Characteristics of the concurrent model approach were operationalized in quality aspects: connections, shared representations for multiplication and division, multiplication/division table, common realistic contexts for both operations, textual problems illustrating connections, and invariance of multiplication and division. We believe that school practice will show advantages of the concurrent method for efficient learning of multiplication and division. The results of the implementation should be empirically evaluated in the future.

Acknowledgements

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“Simultani” metod podučavanja množenja i deljenja

Jasmina Milinković

Učiteljski fakultet, Univerzitet u Beogradu, Srbija

Apstrakt. U ovom radu predlažemo “simultanu” metodu podučavanja množenja i deljenja u osnovnoj školi. Izraz “simultano” podrazumeva osnovnu karakteristiku metode koja je istovremeno uvođenje obe aritmetičke operacije. Simultana metoda podržava uspostavljanje veza između aritmetičkih operacija i olakšava pamćenje tabele množenja i deljenja. Teorijsko obrazloženje simultane metode zasniva se na principu veza kao osnovnom principu matematičkog obrazovanja. Metoda je primenjena u inovativnom udžbeniku za 2. razred osnovne škole. Cilj ovog rada je da opiše metodu, suprotstavljajući je tradicionalnoj metodi. Date su smernice za primenu simulatne metode i predstavljen je model ove metode.

Cljučne reči: nastavna metoda, dizajn kurikuluma, množenje, deljenje, udžbenik

Potential of Children Math Lesson for Investigating Peer Interactions

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Abstract. In mathematics didactics, there are several studies that look at the cooperation of students with each other. The focus is on partner or group work and its conditions (Götze, 2007; Nührenbörger, 2009; Schülke, 2013; Gysin, 2017). These are 'peer situations'. Children's mathematics lessons offer a very special form of student co-creation, in which one or two students 'plan', structure, and take charge of the lesson, while the teacher participates largely in an observational way. We analyze the available data on such a children's mathematics lesson in a qualitative-interpretative way in order to identify particularities in the interaction and role allocation of the students. The leading students take on "double roles" in the present data. On the one hand, they act as teachers and their classmates perceive them as such. On the other hand, they act as equal peers. We can reconstruct at least one child question that critically-constructively questions the subject-specific procedure of written multiplication. Researchers see such children's questions as interesting and productive (Götz, 1997), but according to Ritz-Fröhlich (1992), they rarely occur in classroom instruction directed at the teacher.

Keywords: children's math lesson, children's questions, peer interaction, roles, interaction analysis, written multiplication

1. Introduction

In recent years, there is an increased focus on student-student interactions in mathematics education (Götze, 2007; Nührenbörger, 2009; Schülke, 2013; Gysin, 2017). Student-centered learning is particularly broached in the format of children's math lessons. In such a lesson, the students themselves take over the planning, structuring and executive chores of the teacher. The authors could not find any mention of children's math lessons in the literature. There are approaches such as "learning through teaching" (Renkl, 1997) or "students helping students" (Feldmann, 1980), which also focus on the students' own activity. In lessons itself, interactions take

place between the teacher and the students and between the students themselves. The extent to which interactions between “teachers” and students or between students occur in the children’s math lessons will be analysed in this paper by means of a transcript excerpt. In the following, the authors elaborate on the interactions in the classroom. The characteristics of children’s math lessons are also presented and a transcript excerpt is analysed qualitatively and interpretatively. Finally, the particularities of the children’s math lesson in terms of interaction are noted.

2. Interactions in math lessons

Mathematics education is of central importance in today’s educational culture. Math lessons are oriented quantitatively towards algorithmic solutions and qualitatively towards model-building and structure (cf. Fuchs & Kraler, 2020). These two characteristics are also reflected in the educational standards of Germany, which move from input orientation to output orientation. The general mathematical competences of the educational standards are problem solving, reasoning, representing, communicating and modelling (KMK, 2004). Mathematics education should focus on the understanding of mathematical content and not only on the learning of knowledge and skills (cf. *ibid.*). Similarly, the active participation of the students is important, as they are supposed to learn how to solve problems by mathematical devices (Matter, 2008). Mathematics teaching has a need for holistic learning, where working through textbook pages in small steps does not enhance the understanding of mathematical relationships (cf. *ibid.*). Contemporary mathematics teaching is thus about enhancing key qualifications, making use of the broad spectrum of talents and enabling cross-age learning. Thereby, the teacher has the task of stimulating and coaching the students in activating learning environments with challenging tasks (cf. *ibid.*).

Mathematics teaching, like general teaching, consists of the actions of the teacher and the students being taught. Within the framework of the lessons, they interact with each other. The interactions within the lesson can be divided into teacher-student interactions and student-student interactions. In the next section, the authors focus on the interactions at the general instructional level, which, however, also apply to mathematics instruction, as this is a component of general instruction.

2.1. Teacher-student interaction

In school lessons, students and teachers take on the central roles. Both actors also have specific preconditions concerning the lessons and the interaction. The teacher can take on different roles in the classroom, but according to Feldmann (1980) there is always a structure of conflict within it. This means that the teacher is on the one hand the representative of the state and the advocate of the interests of society and on the other hand teachers should adapt his or her teaching to the interests of the students. These interests could be individual or collective and create a new field

of tension (Feldmann, 1980). The teacher's primary role is that of a transmitter of knowledge influenced by society's expectations. The role of the learning helpers becomes apparent when the teacher focuses exclusively on individual students. In addition, teachers are always planners and organizers of lessons. The teacher creates the basis for profitable instruction. A final role of the teacher is that of the assessor. In the classroom itself, this role of the teacher occurs in classroom discussion. It can be seen that multifactorial conditions influence the interactions between teachers and students. This should be taken into account.

When teaching a primary school class himself, Hugh Mehan noticed that in the classroom there are repetitive interactions between the person teaching and the person learning (Wenzl, 2014). From his observations, he elaborated a three-step interaction pattern used by the interacting persons. The three steps of this pattern are "initiation", "reply" and "evaluation" (IRE) (Mehan, 1979). The "initiation act" emanates from the teacher in the form of a question or an instruction. The "answering act" is the task of the students and takes place in response to the question asked or the work assignment. Not always a direct answer follows a teacher's question. There are different ways to respond to the teacher's prompt. Students can not answer, they can answer partially correctly, they can answer incorrectly or they can answer without reference to the actual question (Mehan, 1979). The "evaluating act" completes the three-step process, which is again executed by the teacher and relates to the given answer (Wenzl, 2014; Schwarz et. al., 2006).

The characteristic of classroom interaction is the "evaluative act", which is only used in situations between teachers and learners. In the interaction pattern (IRE), the teacher takes two of three acts and the students one. This aspect shows the asymmetry of teacher-student interaction in teacher-centered instruction. Further research (Buttlar, 2019) shows that teacher-centered instruction takes place in at least three-step sequences and not exclusively in this three-step interaction pattern (IRE).

Finalizing on the topic of teacher-student interactions, the questions emanating from the students in class will be examined. A basic observation according to Niegemann (2004) is that students' questions are a rare occurrence in class, especially in comparison to teacher' questions. Students' questions about the subject matter have a special potential. They can arise from a knowledge deficit that has been noticed through metacognitive skills and competences. These metacognitive skills are a particularly challenging task for children and young people, as they need to be based on an understanding of their own learning (Niegemann, 2004). At this point, it is noted that not all student questions are intrinsically motivated and have the goal of real learning, but establishing student questions in the classroom can lead to better learning conditions and better adaptivity of teaching (cf. *ibid.*). Because students ask questions based on the phenomena dealt with, their interest or their wonder about the environment and as a result build up a completely different point of view on the teaching object (Ritz-Fröhlich, 1992). The primary school should create the basis for questioning and the further promotion of the development of questions (cf. *ibid.*). Social and communicative competences must also be supported at school by learning questioning and answering skills. Also children should not only learn how to answer at school, but also train basic conversation

skills, which are an important cultural technique (cf. *ibid.*). Asking questions is an important learning strategy, as it can enhance and secure the understanding of the subject matter at the same time (Niegemann, 2004). It is important that the children's questions not only find their place in the morning circle (Meister, 2012), but also find expression in subject-related situations. As already mentioned, students rarely ask questions in class compared to the teacher. However, it has been found that students ask more questions in situations with a teaching person, not a regular teacher, possibly not feeling as inhibited as in class (Niegemann, 2004).

Situations in which students feel least inhibited are student-student interactions or peer interactions. The following chapter describes these in more detail from the point of view of instruction.

2.2. Student-student interaction

Student-student interactions belong to peer interactions. These describe a special form of social interaction between equal interaction partners in contrast to teacher-student interactions which are asymmetrical as described above. In student-student interactions, and also in peer interactions, there is an approximate symmetry in the distribution of speech contributions (Kordulla, 2017). Student-student interactions occur inside and outside the classroom. In relation to classroom interactions aimed at knowledge acquisition, there is the equally important concept of peer learning. This refers to a specific form of learning that results from peer interactions. Peer learning "is based on the idea that individuals who see themselves as basically equal or on an equal standing in terms of their social status or their level of knowledge and development jointly develop ideas about an object or subject matter through processes of negotiation" (Kordulla, 2017). Therefore, in contrast to teacher-centred knowledge transfer, peer learning does not involve asymmetrical roles, but rather mutual benefit in the co-constructive acquisition of knowledge (cf. *ibid.*). In particular, the further development of one's own understanding of mathematical content benefits from peer learning (Renkl, 1997).

Partner and group work are the most frequently used methods in class in which students cooperate with each other (Götze, 2007; Schülke, 2013). The aim is to activate the different prior knowledge of a subject and to contribute to its appropriate application in finding a solution (Thiel, 2016). On the linguistic-interactive side of the group work, it becomes apparent that the students adapt their conversational style to the situation. The people addressed are no longer teachers but peers of the same age. "The conversational style is characterised by classroom and subject-specific expressions, but at the same time by everyday language style with its short contributions, rapid speaker changes and youthful turns of phrase" (Spiegel, 2018). In these group situations, phases of simultaneous speaking or joint speaking can also arise, whereby participation in the content-related discussion is not always guaranteed (cf. *ibid.*). Language therefore becomes increasingly central in the group phases and likewise preferably topic- and task-related (Morek & Quasthoff, 2018). The study by Kämäräinen (2019) shows a phenomena of role distributions in peer interactions in relation to group work in mathematics lessons.

When students work co-constructively, students find themselves in the teaching role due to the differences in the knowledge structures of the participants. In this way, approximate structures of teacher-student interactions can also occur in peer interactions.

In addition to group work, mutual help is another form of interaction between students in class. So-called helpers or mentors have the advantage that the students explain and learn among themselves on the same level as far as possible without major differences and so the “zone of next progression” (Vygotsky, 1978) can be reached more easily (Naujok et al., 2008). Here, the helping students also use the discursive practices of explaining and argumentation. Another student-student interaction in which, however, these discursive practices of explaining and arguing take a back seat is working side by side, which has only little exchange, but still leads to the exchange of information (cf. *ibid.*).

3. Children’s math lesson

The author (KB) collected the present data in the context of a cooperation with a German elementary school. The teacher occasionally lets her students from grade 3 onwards do the lessons independently. The author (KB) used the term “children’s mathematics lesson” because of the following circumstances: The teacher selects one or two students to be told the topic of the lesson. The selected students, in the following called leading students, plan, structure and lead the lesson. The teacher only gives them the topic. The chosen tasks and the implementation are left to the leading students. The teacher gives a lot of responsibility to the students, both the leading students and the rest of the students. The teacher himself observes the lesson attentively and does not intervene. The setting of the children’s mathematics lesson shows a great closeness to the concept of student-centered learning (Neumann, 2013), specifically learning between groups of students (Eronen, 2019) as well as socio-constructivist minimalist instruction (Carroll, 1990, as cited in Eronen, 2019).

Observation of several lessons by the teacher shows that most lessons expire the following pattern: Introduction usually in a sitting circle with silent impulses, partner or group work that results in content from the introduction, and a final evaluation, e.g., in a semicircle in front of the blackboard.

The videotaped children’s math lesson analyzed here is the introduction of written multiplication in 4th grade. Since the students have been attending this teacher’s class since 1st grade, they know the lesson script and the lead students will probably follow it. It is worth mentioning that this is not an exercise lesson in which technical contents are practiced and at best deepened, but that it is an introduction lesson to a complex algorithm. We have about 90 minutes of video material. The two leading students announce the timetable for the lesson on the blackboard. The students then sit in a circle for 30 minutes. As an initial task, the leading students place the task $8 \cdot 11$ in the sitting circle as well as the task $139 \cdot 51$. The students discuss $8 \cdot 11$ intensively for about 10 min. Afterwards, they begin

to deal with the task $139 \cdot 51$. Below, we evaluate the transcribed statements from this section. After the introduction, students work in small groups to self-construct new multiplication problems to be solved using written multiplication. Finally, the small groups present and discuss their solutions in an evaluation session.

Excursus on difficulties with written multiplication

Please keep in mind that this is an introductory lesson in written multiplication. Both task examples are only suitable to a limited extent for two different reasons: $8 \cdot 11 = 88$ does not show the special features of the execution of the written multiplication algorithm and $139 \cdot 51$ is already a rather demanding multiplication task that requires several carryovers and only allows a simple rollover to a limited extent. I (KB) draw on Gerster (1982, Test M2, p. 119) to show that $139 \cdot 51$ is a multiplication task in which the multiplier has several digits, the digits of the multiplier are less than or equal to 5, exactly 2 retention digits are less than or equal to 5, and adding the retention digits in each case does not lead to the tens carryover. As an introduction to written multiplication, this task already seems very demanding due to the requirements mentioned above.

4. Methodological approach

We carefully evaluate the videotaped data in an qualitative-interpretive manner. Before starting the interpretive analysis, we perform a kind of “paraphrase” of the interaction in the observed episode. We had the opportunity to watch the video of the episode – supported by the corresponding transcript – as many times as we wanted, in order to test and refute or support different alternative interpretations. The following step is to interpret the episode qualitatively (for the research approach of qualitative and interpretative analyses of mathematical interaction processes, see e.g. Krummheuer (2000).)

Research Questions

Based on the empirical data, we aim to answer the following research questions for the selected sequence:

- What are the characteristics of the interaction between the leading students and the other students in the children’s mathematics lesson?
- In which way do the students (leading students as well as other students) talk about mathematics (written multiplication)?

5. Framing of the selected sequence

The leading students open the lesson by introducing the timetable and all students meet in a circle. They put two sheets of paper in the middle of the circle with $8 \cdot 11 = 88$ and $139 \cdot 51$. The students discuss $8 \cdot 11$ among themselves by calculating $8 \cdot 12$ and $8 \cdot 10$ and pointing out the relationships between the three tasks. Pia, a student, gives the impulse that they “still have to calculate” $139 \cdot 51$ (line 90). From here on, the transcribed as well as interpreted sequence begins.

We have selected lines 90–95 for this article from the entire children’s math lesson after careful consideration. A quick note: When interpreting, please keep in mind that this is the introductory lesson to written multiplication.

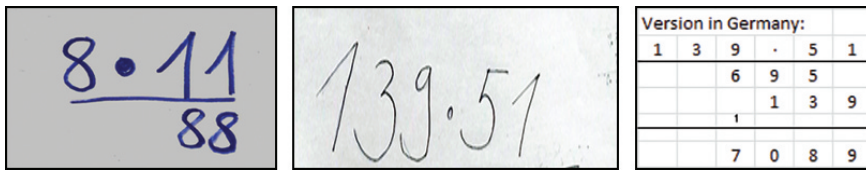


Figure 1.

Transcript

- 90 Pia Die Aufgabe müssen wir ja auch noch rechnen. (*Zeigt auf ein großes Blatt, das in der Kreismitte liegt, auf dem die Aufgabe $139 \cdot 51$ steht.*) **We still have to calculate this task.** (*Points to a large sheet of paper in the middle of the circle on which the task $139 \cdot 51$ is written.*)
- 91 L2 Probiert sie doch mal zu rechnen. (...) (Robin meldet sich.) Robin. **Give it a try to calculate it.** (...) (*Robin raises his hand.*) **Robin.**
- 92 Robin Wie habt ihr das eigentlich alles herausgefunden mit dem schriftlichen Malrechnen?
How did you actually find out all that about that written multiplication?
- 93 L2 Naja das war die Rechenart, wie wir das gemacht haben.
Well that was the calculation way how we did it.
- 94 Robin Wurde euch bei der Aufgabe geholfen?
Have you had help with the task?
- 95 L1 Nein, das haben wir Zuhause gemacht.
(4 sec. Pause)
No, we did that at home.

5.1. Analysis of the transcript

- 90 Pia Die Aufgabe müssen wir ja auch noch rechnen. (*Zeigt auf ein großes Blatt, das in der Kreismitte liegt, auf dem die Aufgabe $139 \cdot 51$ steht.*)
We still have to calculate this task. (*Points to a large sheet of paper in the middle of the circle on which the task $139 \cdot 51$ is written.*)

Pia takes on an organizational role with her statement. She indirectly declares that she considers the editing process for $8 \cdot 11$ to be finished. Her statement “we still have to do it” is an indication that this is a student interaction or peer interaction in which we interpret “we still have to do it” as another (annoying) task. With “we” she includes all present students. Thereby she points to the further present and not yet solved multiplication task $139 \cdot 51$. The “also” is a little irritating. The result of the task $8 \cdot 11$ was already on the sheet. Thus they are to solve $139 \cdot 51$ as the first task. In a “traditional” lesson, the teacher is more likely to take over Pia’s organizational role. The verb “to calculate” gives the impression that the students know how to solve the task. However, compared to $8 \cdot 11 = 88$, the children would have to figure out or try to transfer the written multiplication algorithm. It could also be that they are to determine the result. If she were to use “solve” instead of the verb “calculate”, the task would take on the status of a problem being solved rather than the syntactic processing of a task by means of calculation. The students could now find a solution to $139 \cdot 51$ using half-written multiplication or calculate a rollover, e.g. $140 \cdot 50 = 70 \cdot 100 = 7000$.

- 91 L2 Probiert sie doch mal zu rechnen. (...) (Robin meldet sich.) Robin.
Give it a try to calculate it. (...) (*Robin raises his hand.*) **Robin.**

The leading student L2 gives an impulse to action with her statement. In this there is both a relief of action and an idea of accidentally finding a solution. “Try” in some sense implies “try your luck”. It is possible that L2 has an awareness of how challenging the task $139 \cdot 51$ is. She also uses the verb “to calculate”. Possibly she picks this up from Pia or she is more concerned with applying an algorithm than really exploring a procedure. Both interpretations have to take into account that the algorithm is not visible in the task $8 \cdot 11$ and that the students do not have an orientation to a sample task on which they could understand the procedure of written multiplication. The formulation of L2 resembles more a TaS interaction than a SaS interaction, because L2 takes herself out of the activity with “they” and gives this to the SaS. At the same time, she also gives Robin the right to speak. In the context of the previous stimulus, this is more like a teacher activity. L2 seems to adopt common and familiar communicative phrases of the teacher. She probably imitates the teacher in her way of leading and guiding the students.

- 92 Robin Wie habt ihr das eigentlich alles herausgefunden mit dem schriftlichen Malrechnen?
How did you actually find out all that about that written multiplication?

Robin’s inquiry leads away from the impulses previously posed and is on a meta-level. He asks for a why and not for a how regarding the explanation. Thus, he

implicitly asks the question how someone could find out, a question of history. The solution of the task $139 \cdot 51$ thereby moves into the background. Robin asks about the understanding of the procedure and thereby questions “how did you actually find out all that”, how the two leading students could figure this out. “Actually” seems restrictive here, in the sense that he does not believe that they figured it out themselves.

93 L2 Naja das war die Rechenart, wie wir das gemacht haben.
Well that was the calculation way how we did it.

L2 responds in a somewhat dismissive way to Robin’s inquiry. “Well” is rather evasive and may also testify to an uncertainty about how to answer the question. L2 refers to the algorithm, “that was the calculation way how we did it” and thus answers on a procedural level. Here, Robin’s conceptual demand and L2’s procedural answer collide. L2 seems to want to continue working on solving the task by giving a short answer rather than talking about a meta question. Further on, Robin could ask again for a rationale for the approach or Pia takes over her organizational function again or a student starts to calculate.

94 Robin Wurde euch bei der Aufgabe geholfen?
Have you had help with the task?

Robin continues to ask, but not in a conceptual sense, but he now wants to know if someone supported them in this. In doing so, it becomes clear that he has probably previously questioned that the two leading students have figured this out themselves. With this, he may doubt their (mathematical) competence. However, he may also want to find out if he can learn this and they can show it to him, just as someone else got it to the two. If anyone has helped you to be able to perform this procedure, please help us too. Robin’s inquiry now seems to be more about procedural.

95 L1 Nein, das haben wir Zuhause gemacht.
(4 sec. Pause)
No, we did that at home.

L1 responds to Robin’s question by referring to home. This could mean that the parents supported the children and not the teacher or they only want to say that the teacher did not help and therefore name home as a different location compared to school. Perhaps it is also a kind of self-revelation of the students that there is only help with the tasks at school and not at home. By the “we” it could be that L1 and L2 each solved it at home or together outside of school. Possibly L1 and L2 have not yet solved the task themselves or cannot demonstrate or explain the procedure.

After the line-by-line interpretation, various aspects become clear, which we will take up again and summarize in the following. In these 5 lines alone, interactions between students and teachers become clear, whereby only students speak with each other. Here, L1 and L2 take on the role of a teacher as well as the role of a peer. Particularly striking is Robin’s statement in line 92. On the one hand, this question seems very unusual in a student-teacher interaction, since the child asks a

real question and, on the other hand, a teacher could understand this as questioning her mathematical competence. It is also surprising that Robin asks a question on a meta-level instead of dealing with the calculation of the given task as requested by L2. Robin is asking for an understanding of why this algorithm works rather than a procedure. If we look at the interaction in these 5 lines, we can see that line 92–95 can be seen as SaS interaction or peer interaction. Especially line 94 is an utterance that occurs in this way exclusively in peer interaction. The utterances (lines 90–91) could both be from a teacher. The utterance of L2 (line 91) seems to be known to the other students by the regular teacher, since the students do not ask any questions.

6. Conclusion

Children's math lessons are a new setting that challenges the leading students and makes high demands on all students. Since we did not find any publications explicitly on this topic, we refer to teacher-student and student-student interactions to take a closer look at this setting from an interactional perspective. Through the analyses, we can show that both teacher-student and student-student interactions occur in the children's mathematics lesson under study. In this context, the lead students sometimes take on the role of the teacher and then again the role of the classmate as peer. Similar results were shown by Kämäräinen et al. (2019) in their study. In contrast to the studies of Kämäräinen et al. (2020, 2019) and the subteaching in Tholander & Aronsson (2003), in the present children's mathematics lesson only students work with each other. The teacher observes the process and does not intervene in the spirit of socio-constructivist minimalist instruction (Carroll, 1990, as cited in Eronen, 2019), with the two leading students occasionally taking on the role of teacher. This implements a particularly pronounced form of "student-centered learning" (Neumann, 2013), as all the students take joint responsibility for the entire lesson. In the analyses, it appears that the IRE pattern (Mehan, 1979) does not occur. Peer interaction is the defining feature of this entire sequence, in which the students "jointly develop ideas about an object or subject matter through negotiation processes" (Kordulla, 2017, p. 53).

Another aspect we want to consider here is the generation of mathematical knowledge also in the context of procedural and conceptual knowledge. According to Renkl (1999), the students benefit from peer learning by further developing their own understanding of mathematical content. This is evident from the fact that the students address both procedural and conceptual knowledge in their utterances of the short analyzed sequence. It is evident that classmates (Peer, Robin) also initiate conceptual inquiries and that the students are not only concerned with the procedural implementation of the written multiplication algorithm in the children's mathematics lesson. For a detailed treatise on the interplay between procedural and conceptual knowledge, see Haapasalo and Kadujevich (2000) and Haapasalo (2003).

In traditional teaching, it would be a peculiarity if Robin would ask his child question (Ritz-Fröhlich, 1992). He thus critically and constructively questions the

subject-specific procedure of written multiplication. Researchers see such children's questions as professionally interesting and productive (Götz, 1997). In the children's mathematics lesson, it also seems that the students respond to their inquiries on as equal a level as possible and thus the children reach the "zone of next development" (Vygotsky, 1978) more easily through their cooperation (Naujok et al., 2008, p. 788).

In further investigations of this children's mathematics lesson, the focus will be on the group work and its evaluation both with regard to particularities in the execution of the written multiplication algorithm ("types of errors") and with regard to the interaction processes during the presentation of the results. It also seems interesting to look at the similarities and differences between the setting without guiding students in Kämäräinen et al. (2020, 2019) and the children's mathematics lesson with guiding students.

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Kindermathematikstunde – Untersuchung der Potentiale von Peer-Interaktionen

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Abstract. In der Mathematikdidaktik gibt es mehrere Studien, die die Zusammenarbeit von Schüler*innen untereinander betrachten. Dabei stehen Partner oder Gruppenarbeiten und deren Bedingungen im Mittelpunkt (Götze 2007, Nührenböcker 2009, Schülke 2013, Gysin 2017). Es handelt sich um 'Peer-Situationen'. Eine ganz besondere Form der Mitgestaltung von Schüler*innen bieten Kindermathematikstunden, die dadurch gekennzeichnet sind, dass ein oder zwei Schüler*innen den Unterricht 'planen', strukturieren und die Leitung der Stunde übernehmen, während die Lehrperson größtenteils beobachtend am Unterrichtsgeschehen teilnimmt. Die vorliegenden Daten zu einer solchen Kindermathematikstunde werden interpretativ-qualitativ analysiert, um Besonderheiten in der Interaktion und der Rollenverteilung der Schüler*innen herauszuarbeiten. Die leitenden Schülerinnen übernehmen in den vorliegenden Daten "Doppelrollen". Auf der einen Seite agieren sie als Lehrende und werden von ihren Mitschüler*innen auch so wahrgenommen. Auf der anderen Seite agieren sie als gleichberechtigte Peers. Es lässt sich mindestens eine Kinderfrage rekonstruieren, die kritisch-konstruktiv das fachliche Verfahren der schriftlichen Multiplikation hinterfragt. Solche Kinderfragen werden von Forschenden als fachlich interessant und produktiv angesehen (Götz 1997), treten jedoch laut Ritz-Fröhlich (1992) im Klassenunterricht, gerichtet an die Lehrperson, sehr selten auf.

Schlagwörter: Kinder-Mathematikstunde, Kinderfragen, Keer-Interaktion, Rollen, Interaktionsanalyse, Schriftliche Multiplikation

Mathduel, a Math Game for High School

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Abstract. Playing games is an essential activity for every child, the dominant way children learn. So, it is no surprise that a lot of research explores the influence that games have on learning and achievements. Research shows that the use of educational games in the classroom has a positive impact on learning, motivation, students' cognitive abilities, and students' achievements. With that in mind, it is interesting to see using educational games as a teaching strategy in Croatian high schools is minimal. MathDuel is a mathematical game designed as a teaching and learning aid in high school math education. Playing MathDuel helps students develop better conceptual knowledge by connecting different mathematical concepts in a creative and resourceful ways. With this math game, we want to enrich teaching and learning mathematics.

In this paper, we present MathDuel: the idea, rules, constructing process, and gameplay.

Keywords: game, mathematics, conceptual knowledge, math teaching, students' achievements

1. Introduction

Playing games is the essential way how children learn. For decades' researchers investigated influence that games have on children's motivation, learning, and achievements. Obioma's 1992 working paper (as cited in Azuka & Awogbemi, 2012) quoted Plato "Amusement and pleasure ought to be combined with instruction to make the subject more interesting. These things make a pupil useful to himself and more wide-awake". By using games, teachers create a positive learning environment, motivate students, and provoke mathematical discussion (Oldfield, 1991b; Bragg, 2012). In other words, using games provides an effective way of engaging students in mathematics. Oldfield (1991a) defines a mathematical game as a social activity with goals, rules, structure, and specific mathematical cognitive

objectives. Pulos & Sneider (1994) recommended adding games to the curriculum because they provide a unique opportunity to integrate cognitive, affective, and social aspects of learning. Because of that, we could easily presume that playing games is a highly-used learning strategy in education. It is interesting to note that Croatian teachers rarely use games in classrooms because they are usually considered chaotic, time-consuming, entertaining activities with no educational impact. However, most people love playing games, and since there are so many different types available (computer, board, sports, action, card, and so on), there is a game for everybody.

We believe that playing MathDuel enables students to develop better conceptual knowledge connecting different mathematical concepts in a creative and resourceful way. Since 2017 we gathered five preservice mathematics teachers in Split who worked on the construction of two editions of the game – one for primary and one for high school. Their work and contributions are presented in their master theses.

MathDuel is a mathematical, guessing game for two teams of various numbers of players. In every game, players are randomly given 25 concept cards showing mathematical notions, symbols, expressions, and drawings. Team captains are then alternating in giving one-word clues which should direct their teammates to the right cards. Based on this information, their teammates try to differentiate the cards belonging to their team from the others. The team that manages to find all their cards first wins the game. To excel in the game, captains must carefully choose clues that are relatable to as many target cards as possible, while at the same time avoiding unwanted hits on other cards. Mathematical content on cards, as well as the intended clues, are adapted to students' age.

2. Game setup and rules

The game is inspired by a famous card game *Codenames* published by Czech Games Edition in 2015. We used this word game as a basis and created a math game with similar rules (with permission of CGE). The name MathDuel (*Matoboj*) was inspired by Renaissance mathematics duels, which were held among mathematicians to distinguish better and more knowledgeable ones. In MathDuel, players split up into two teams of similar size and skill. The minimum number of players is four, with six being optimum. In the high school edition of the game, we named these teams: team algebra (orange) and team calculus (blue). Primary school edition teams are named geometry and arithmetic. Each team selects one player as their captain. Everybody settles around a table, team captains on one side and team members on the opposite side of the table. Players randomly choose 25 concept cards and place them on the table in a 5-by-5 grid. Team captains then randomly take one key card and put it in a holder in front of them, keeping it secret from other players. The colored frame indicates which team goes first. (Figure 1)

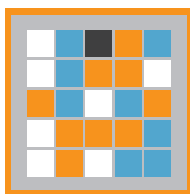


Figure 1. Example of the key card.

The colored grid on the key card corresponds to the grid on the table. Orange / blue squares denote concepts cards that the corresponding team must guess. The team that goes first has nine cards to guess, while the other team has only eight. In this way, any possible disadvantage of going second is minimized. Seven white squares represent neutral cards, and a black square represents a forbidden card. If any player mistakenly guesses this card, the opposing team instantly wins the game. (Figure 2)

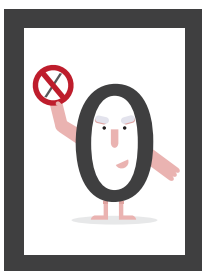


Figure 2. Dividing by zero-cover card used for a forbidden concept card.

Each team captain takes cover cards in the color of their team, orange or blue. A double-colored card belongs to the team whose color matches the colored frame on the key card placed in front of the captains. Cover cards for each team present one of the famous mathematicians who contributed to algebra or calculus. (Figure 3)



Figure 3. Great mathematicians – teams cover card.

In this way, we introduced some lesser-known mathematicians to students, thus directing them to investigate and discover stories from the history of mathematics

and mathematicians' biography. Neutral cover cards show famous scientists from other fields. (Figure 4)

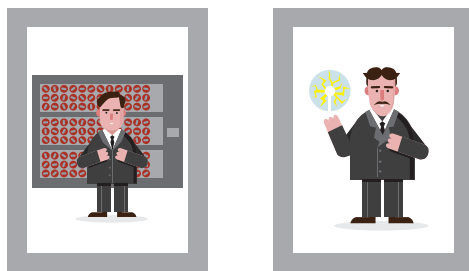


Figure 4. Famous scientist – neutral cover cards.

3. Playing the game

After setting up a game, captains take turns giving clues to their teammates, who try to identify the right cards on the grid. The team that covers all their cards first wins the game.

Captains should look at the grid carefully to come up with efficient clues for their teammates. Each clue consists of one word and one number and can be related to multiple cards from the grid. The word relates to concept cards located in the grid, and the number represents the number of cards related to the word. For example, if a captain says: “parallel, 2”, this instructs his teammates to look for two uncovered cards which are relatable to a concept of being parallel. After giving a clue, captains are not allowed to give extra hints – verbal or nonverbal. It is time for the captains' teammates to figure out which cards are connected to the provided clue. They can discuss it among themselves, but the captain must keep a straight face. After reaching the agreement, the captains' teammates indicate their decision by touching one card on the grid. If they touch the card belonging to their team, the captain will cover it with a team cover card. The team gets another guess or passes the turn. If the touched card belongs to their opponent's team, their turn is over and the opposing team's captain covers that card with one of their team cover cards. This mistake brings the other team one step closer to victory. Touching the neutral card ends the current team's turn as well, while touching the forbidden card ends the game. Once the second team has done guessing it is the first team captain's turn to give another clue. Teams alternate in this way until either one team has covered all their cards or the forbidden card was touched.

With a solid structure and clear set of rules, this game provides students with a great deal of freedom during the play. Since there is no need for the referee, a teacher has only two roles – to observe and help if asked.

4. Example of play

We will use the Figure 5 setup and simulate a couple of opening turns of a game.

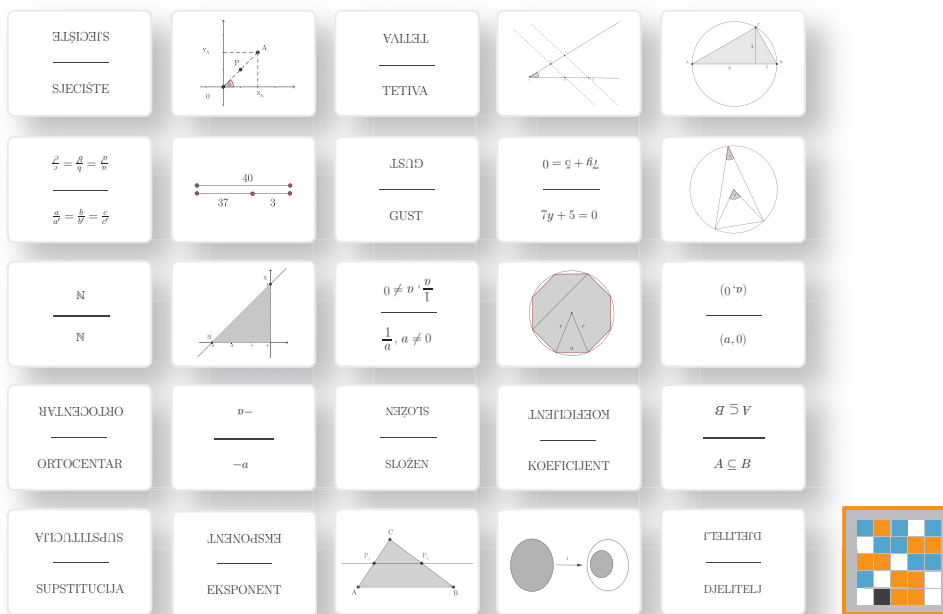


Figure 5. Example of the game setup.

According to the key card, the orange team goes first. Their captain gives a clue: “coordinate, 2”. After the discussion among themselves, the orange team members decide to point out the card in the first row, the second column. According to the key card, it is their concept card to guess, so their attempt is correct, and the card is covered with an orange cover card. Their second guess is the card in the third row, the fifth column, but that card belongs to the blue team, so they lose their turn and help the opposite team. The situation after the first turn is shown in Figure 6.

The blue team goes next. Their captain gives clue: “height, 2” (or altitude). The blue team members point out the card in the first row, the fifth column, which is a correct guess. Their second guess is the card in the fourth row, the first column, and the blue team has guessed correctly again. The blue team can continue with their turn and pick another card from the grid blindly, but it is a risky move and not recommended at all. In this example, the blue team is done guessing and passes the turn to the orange team. (Figure 7)

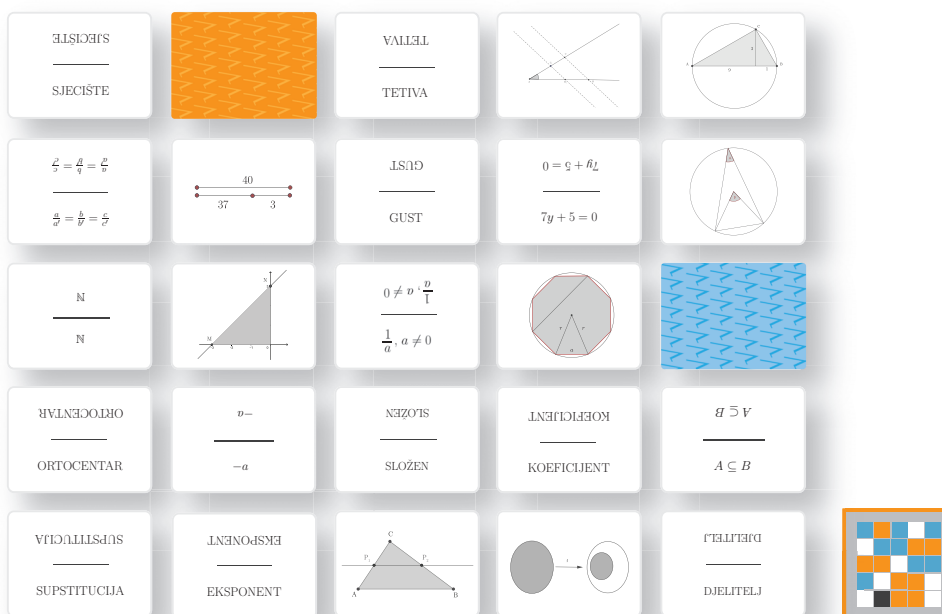


Figure 6. The situation after the first clue.

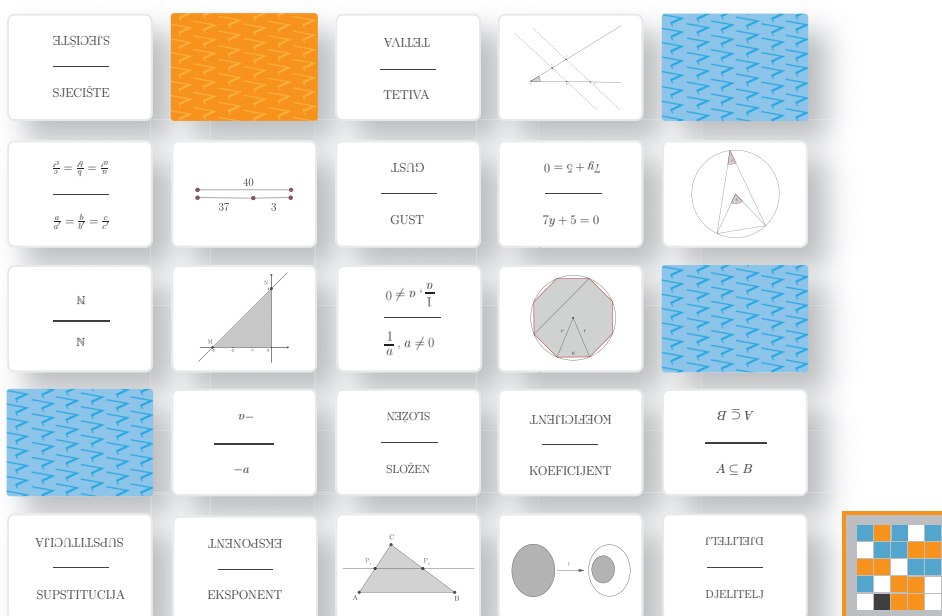


Figure 7. The situation after the second clue.

The new clue for the orange team is “twelve, 2”. The orange team points out the card in the third column, the first row, and the card in the fourth column, the third row (*složen = composite*), and both guesses are correct. Since their turn isn’t over yet and because they haven’t guessed one card from the first clue they point out the card in the third column, the second row, and that is correct. Now the orange team has exhausted all the given clues and passes their turn. Figure 8 shows the situation in the grid, and the game continues.

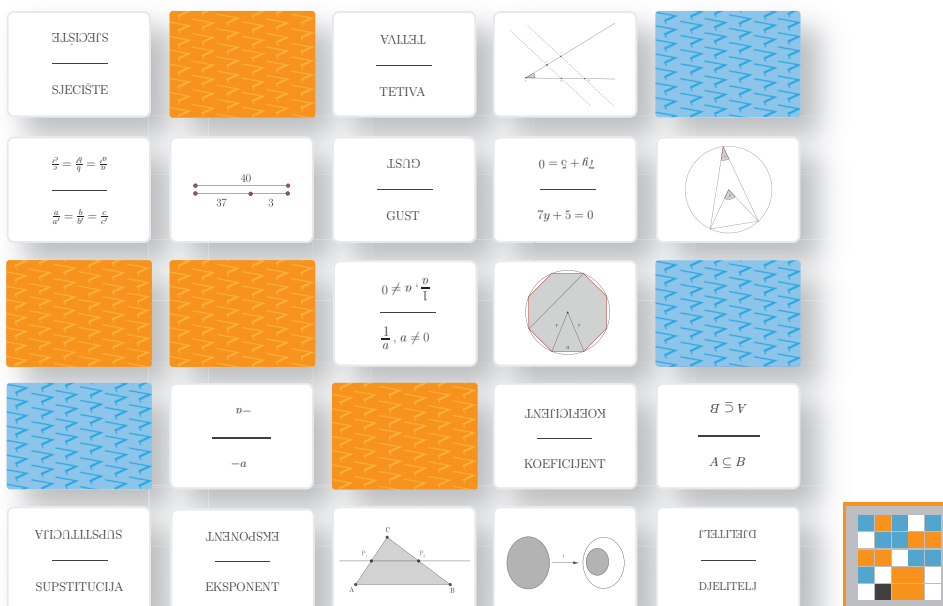


Figure 8. The situation after the third clue.

5. Construction process

We intended to make students recognize mathematical concepts and terms on the cards and use them as clues. We first created a list of key terms and topics in the new curriculum for high schools in Croatia. Then we worked on creating concept cards that contain these terms in some way. Students playing as team captains should recognize some of these concepts and use them as clues when appropriate. It is forbidden for a captain to use any word (or its stem) appearing on uncovered concept cards as a clue. Because of this, most of the content on the concept cards is given in mathematical symbols, formulas, graphs, and drawings.

The biggest challenge during the construction process was in creating concept cards that offered players a large variety of possible clues. To allow for clues that relate to more than one card we tried to make complex concept cards that contain

several key terms. During this process, a new idea emerged. We wanted to analyze the association between each content card and possible clue. To do that, we made a table containing all the content cards crosschecked with the list of key terms i.e. intended clues. (Figure 9) This allowed us to detect cards that are either too *heavy*, or too *light*, i.e. related to too many, or too few possible clues. One of our further plans is to research this database using certain combinatorial methods.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W
		SUMA (1)	SUMA (ZA)	K1	K2	K3	K4	K5	K6	K7	K8	K9	K10	K11	K12	K13	K14	K15	K16	K17	K18	K19	K20
1																							
2	SUMA (1)			2	3	7	4	3	2	2	2	2	3	3	3	3	1	4	4	4	4	10	3
3	SUMA (ZA)			0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	Prirodan	34	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	Cijeli	9	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	Racionalan	9	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7	Realan	4	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
8	Imaginaran	4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
9	Presjek	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
10	Unija	1	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
11	Aritmetička sredin	4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
12	Komutativnost	2	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0
13	Asocijativnost	2	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0
14	Distributivnost	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
15	Suprotan	2	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0

Figure 9. The table crosschecking concept cards and some possible clues.

6. Conclusion

MathDuel is a mathematical educational game we think can become efficient teaching and learning aid in mathematics education. We believe that playing MathDuel frequently during the school year can improve students' knowledge, understanding, and affection for mathematics. It is also an excellent activity for celebrating some special days, like the first or the last day of school. It is up to the teacher to find the best way to implement playing MathDuel into teaching practice. From the presented example, it is evident that players should be creative in connecting various concepts with only one word. The captains' role holds a great responsibility in finding the perfect clue for the team, considering all the cards on the grid and at the same time making sure his clue does not relate to any other card on the grid. Players should be open-minded and able to correlate mathematical content in productive ways. This game offers a different way of doing mathematics by describing, creating, and connecting various representations of mathematical concepts.

After finalizing the high school edition, we plan to finish the primary school edition, create the game website, and research the effect of the game on students' achievements, motivation, and conceptual knowledge. There are numerous ideas and possibilities ahead of us.

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Matoboj, matematička igra za srednju školu

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Sažetak. Igranje je osnovna aktivnost svakog djeteta, najvažniji način na koji djeca uče. Stoga ne iznenađuje što mnoga istraživanja istražuju utjecaj igara na učenje i postignuća učenika. Istraživanja pokazuju da korištenje obrazovnih igara u učionici pozitivno utječe na učenje, motivaciju, kognitivne sposobnosti i postignuća učenika. Imajući to na umu, zanimljivo je vidjeti da je korištenje obrazovnih igara kao strategije poučavanja u hrvatskim srednjim školama minimalno. Matoboj je matematička igra zamišljena kao pomoć u nastavi i učenju matematike u srednjoškolskom obrazovanju. Igranje Matoboja pomaže studentima razviti bolje konceptualno znanje povezivanjem različitih matematičkih pojmova na kreativan i domišljat način. Ovom matematičkom igrom želimo obogatiti nastavu i učenje matematike.

U ovom radu predstavljamo Matoboj: ideju, nastanak, pravila i kako igrati igru.

Ključne riječi: igra, matematika, konceptualno znanje, poučavanje matematike, učenička postignuća

A Football Trip Through Mathematics

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Abstract. Football (soccer) is as much popular as mathematics is not, at least in the general public and among kids in particular. However, one cannot be a football fan without using some mathematics, as will be demonstrated by examples ranging from 1st to 8th year of primary school. For each grade we give examples of incorporating football topics (of real relevance to anyone interested in football) into the Croatian mathematical curriculum. This ranges from basic arithmetic needed to follow the results of a championship up to the application of the Pythagorean theorem and basic probability to explain penalty statistics.

Keywords: primary school mathematics, football, arithmetic, geometry, probability, algebra

1. Introduction

It is a commonplace fact that football is popular, and mathematics is not, at least when considering the average child and European (and South American) surroundings. Furthermore, many mathematicians tend to consider football to be a “lower” activity, to quote one we will not name: “Football is just base instincts”. On a superficial level it might be true that football and mathematics are on opposite sides of human activities regarding mode, type of talent, usage of skills. However, the truth is – and this should be understandable to any mathematician – that football not only can be *described* using mathematics, but at least some mathematics is *essential* both for passive footballers (football-fans) and active ones. This paper aims to demonstrate that thesis by examples ranging from 1st up to 8th year of schooling in Croatia.

To achieve this, we analysed Croatian curriculum documents, in particular the Croatian mathematics curriculum (MZOS, 2006).¹ In Croatia, primary schooling lasts eight years. The first four years of schooling are called “classroom education”, meaning that one teacher teaches all topics, while the second four are called “subject education”, meaning that the subjects get more specialised and teachers are specifically educated for each of them. After analysing the mathematical content for each year of primary schooling (eight years in Croatia), we designed various tasks suitable both for usage in teaching mathematics in school and for popularisation purposes. Our tasks are designed in such a way that they are suitable for illustrating specific mathematical contents from the curriculum, as examples and “classic” problems, but with the aim to be easily generalised in interdisciplinary explorative and inquiry learning. Finally, we performed a mathematical analysis of all our problems and examples to ensure their relevance, adequacy and, obviously, correctness, after which we made our final selection to be presented in this paper (Watson & Ohtani, 2015).

2. Football maths, from 1st up to 8th grade

2.1. First grade

In the 1st grade of Croatian schools, children are taught about the following mathematical topics (NN, 2006):

- Basic shapes and relations in space;
- Notation, comparison, addition and subtraction in the set of integers 1 to 20.

All of these have connections to football, and the arithmetic of integers 1 to 20 is essential for watching a football match (e.g., when a team scores a goal, their number of goals increases by 1, when a team wins, their number of points in the league increases by 3). We give a, real, example of using this level of arithmetic in football context.

Example 1. In the 10th round of the Turkish *Süper Lig*, season 2020/21, the İstanbul derby match Fenerbahçe-Beşiktaş was played. Before that match the situation was the following (SofaScore, 2021): Beşiktaş had missed one of the previous league matches, while Fenerbahçe had played all the previous rounds. Of the played matches, Beşiktaş won 4 and Fenerbahçe 6, but 2 of Fenerbahçe’s and only 1 of Beşiktaş’s matches ended in a draw. In their match against each other, Fenerbahçe scored 3 goals and received 6 yellow cards, while Beşiktaş scored 4 goals, received 4 yellow and 1 red card.

¹ Recently, in 2017, a new Croatian national curriculum was adopted, but regarding the analysed content it does not differ much from the 2006 one. We chose to use the “old” one because the mathematical topics in each grade are presented in a way (and order) better suitable to follow in a paper which aims to be readable on its own.

Let's analyse the data given in the description above. Since $3 < 4$, Beşiktaş scored more goals and won the match, while Fenerbahçe scored less and lost.

Every team begins the match with 11 players, but due to the red card Beşiktaş ended the match with $11 - 1 = 10$ players.

This was a match of the 10th round so generally both teams should have played $10 - 1 = 9$ matches in the league before this. However, Beşiktaş missed a round, so before this match they had played $9 - 1 = 8$ matches.

Now we consider that a match that is neither won or drawn was lost, and we can calculate the numbers of lost matches before the derby. For Beşiktaş, of the previously played 8 matches, they lost $8 - 4 - 1 = 4 - 1 = 3$ matches, and of the 9 matches they played previously, Fenerbahçe lost $9 - 6 - 2 = 1$ match.

A win brings 3 points, a draw brings 1 point, a loss brings 0 points. So, before this match Beşiktaş had $3 + 3 + 3 + 3 = 12$ points from wins,² 1 from draws, 0 from losses, so they entered the match with $12 + 1 + 0 = 13$ points. The analogous calculation for Fenerbahçe ($3 + 3 + 3 + 3 + 3 + 3 = 18$, $1 + 1 = 2$, 0 ; $18 + 2 + 0 = 20$) shows that Fenerbahçe entered the match with 20 points in the league table.

Since Fenerbahçe lost (gained 0 points), after the match they still had $20 + 0 = 20$ points, while Beşiktaş won, obtaining 3 more points, and ended up with $13 + 3 = 16$ points.

2.2. Second grade

In the 2nd grade of Croatian schools the following mathematical topics are discussed (NN, 2006):

- Notation, comparison, addition and subtraction of integers up to 100;
- Multiplication and division with single digit numbers.

Obviously, the examples like the one given for the 1st grade can now be extended to matches played later in various championships, when the total numbers of points go up to, but rarely exceed, 100. Also, calculations of total numbers of points can be now be simplified (e.g., instead of $3 + 3 + 3 + 3 = 12$, now one can calculate $4 \cdot 3 = 12$). Instead of giving a similar example to the last one, we give a different type of problem with solution.

Example 2. Table 1 is the table of the top 5 clubs in the final ranking of the Turkish *Süper Lig*, season 2020/21 (SofaScore, 2021), with some data missing. The abbreviations have the following meanings: W – number of wins, D – number of draws, L – number of lost matches, PTS – total number of points. Each club played a total of 40 matches. Fill the gaps!

² Note that multiplication is not yet taught in 1st grade.

Table 1. The top 5 clubs in the Turkish Süper Lig, season 20/21.

Ranking	Club	W	D	L	PTS
1.	Beşiktaş	26	6	8	
2.	Galatasaray		6		84
3.	Fenerbahçe	25	7		
4.	Trabzonspor		14		71
5.	Sivasspor				65

The first row is easiest to fill: Since every win brings 3 points and every draw brings 1, while losses bring 0 points, Beşiktaş obtained $26 \cdot 3 = 78$ points from wins, $6 \cdot 1 = 6$ points from draws and 0 from losses, thus they ended the season with $78 + 6 + 0 = 84$ points.

In the second row we know the total (84) and that $6 \cdot 1 = 6$ points of these were from draws. Thus, Galatasaray won $84 - 6 = 78$ points by wins, i.e. they won $78/3 = 26$ matches. The remaining $40 - 26 - 6 = 8$ matches they lost.

The third row is only on the first glance different from the first one, since the number of lost matches is not given, but they don't bring any points anyway. We calculate: $25 \cdot 3 = 75$, $7 \cdot 1 = 7$, so Fenerbahçe ended the season with $75 + 7 = 82$ points. The number of their lost matches is $40 - 25 - 7 = 15 - 7 = 8$.

The fourth row, Trabzonspor's results, is calculated similarly: $14 \cdot 1 = 14$, so $71 - 14 = 57$ points they had from won matches, thus they won $57/3 = 19$ matches. Finally, they lost $40 - 14 - 19 = 26 - 19 = 7$ matches.

In contrast to the first four rows, the task of completing the fifth one has a non-unique solution. First, we note that $65/3 = 21$ (plus remainder). This can be discovered e.g. by multiplying $20 \cdot 3 = 60 < 65$, $21 \cdot 3 = 63 < 65$, but $22 \cdot 3 = 66 > 65$. We conclude: Sivasspor won *at most* 21 matches. Now one can enumerate (in higher grades one could pose the problem to determine the minimal number of wins using systems of linear equations) the possibilities:

- If they won 21 matches, they had 63 points from wins, i.e. $65 - 63 = 2$ points from draws, so one possibility to fill the row is 21; 2; $40 - 21 - 2 = 19 - 2 = 17$.
- If they won 20 matches, they had 60 points from wins, i.e. $65 - 60 = 5$ points from draws, so the second possibility to fill the row is 20; 5; $40 - 20 - 5 = 20 - 5 = 15$.
- Etc.
- If they won 13 matches, they had $13 \cdot 3 = 39$ points from wins, i.e. $65 - 39 = 26$ points from draws, so another possibility to fill the row is 13; 26; $40 - 13 - 26 = 27 - 26 = 1$.
- But, if they won only 12 matches, they would have had $12 \cdot 3 = 36$ points from wins, i.e. $65 - 36 = 29$ points from draws, but this would mean that they played $12 + 29 = 41$ at least 41 + at least zero, i.e. more than 40 matches, which is impossible.

So, without additional information, the fifth row can be filled in nine different ways, one for each value of W between 13 and 21.

2.3. Third grade

In the 3rd grade of Croatian schools the following mathematical topics are discussed (NN, 2006):

- Notation, comparison, addition and subtraction of integers up to 1000,
- Multiplication and division with single digit numbers, with 10 and 100,
- Straight lines, segments and circles,
- Measuring length, volume, mass.

Obviously, the arithmetic problems from the 1st and 2nd grade are still applicable, but there are not many situations when one needs arithmetic with integers larger than 100, maybe 110 or 120, in a football setting, so this type of problems would generally stay at 2nd grade level and be more suitable for repeating old than for introducing new mathematical contents.

But, even if in the 1st grade one could have used football examples of geometric shapes (ball as sphere, football field as rectangle, various straight and curved lines on a football field), in the 3rd grade the explorations of lines found on a football field can be much more extensive, including measuring lengths.

Example 3. Fig. 1 shows the diagram of a football field. It is easy to find many pairs of mutually perpendicular as well as parallel lines in it. One can also use the central circle to explain the difference between the circle line and the circular disk: A footballer running only along the line obviously does not come to positions in its interior, and painting just the line is simpler (and uses less colour) than if one would have to paint the full circle.

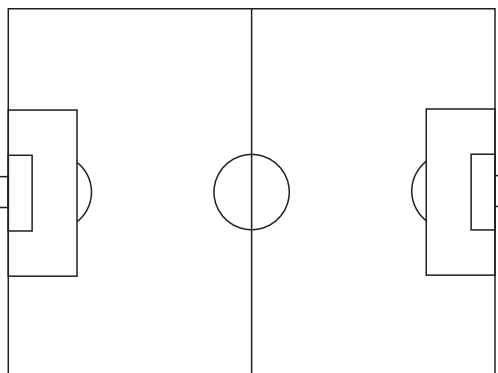


Figure 1. Diagram of a football field.

In this grade, students only learn to express lengths with integer values, so one could either use the original, British Imperial, measures (given in yards and feet) and express lengths given in yards in feet or inches, and lengths given in feet in inches, knowing that $1 \text{ yd} = 3 \text{ ft} = 36 \text{ in}$, or one could stick to the few integer measures in meters and express them in decimeters and centimeters – the width of a football field must be (IFAB, 2020) between 45 m and 90 m (for international matches: between 64 m and 75 m), while the length must be between 90 m and 120 m (for international matches: between 100 m and 110 m). For example, the Allianz Arena (Bayern München stadium) has, like most football fields of well-known teams, width 68 m and length 105 m, so the students could calculate (by multiplication with 10 and 100, also a part of the curriculum for this grade), that is the same as width 680 dm (or 6800 cm) and length 1050 dm (or 10 500 cm).

2.4. Fourth grade

In the 4th grade of Croatian schools the following mathematical topics are discussed (NN, 2006):

- Notation, comparison, addition and subtraction of integers up to 1 000 000,
- Multiplication and division with two-digit numbers,
- Angles, triangles, rectangles, squares, rectangular cuboids and cubes,
- Perimeter, area, volume.

Like for the 3rd grade, we choose to illustrate the application of football to maths teaching in the 4th grade by a geometry example.

Example 4. Determine the area difference between the smallest and largest allowed football field (IFAB, 2020). What is the area per player of a team for the most common $105 \text{ m} \times 68 \text{ m}$ football field?

The smallest allowed football field has dimensions $90 \text{ m} \times 45 \text{ m}$ (for international matches: $100 \text{ m} \times 64 \text{ m}$), so the minimal area of a football field is $90 \text{ m} \cdot 45 \text{ m} = 4050 \text{ m}^2$ (6400 m^2 for international matches).

The largest allowed football field has dimensions $120 \text{ m} \times 90 \text{ m}$ (for international matches: $110 \text{ m} \times 75 \text{ m}$), so the maximal area of a football field is $120 \text{ m} \cdot 90 \text{ m} = 10\,800 \text{ m}^2$ (8250 m^2 for international matches).

Consequently, the area difference between the largest and smallest field is $10\,800 \text{ m}^2 - 4050 \text{ m}^2 = 6750 \text{ m}^2$ (1850 m^2 for international matches).

The most common football field has area $105 \text{ m} \cdot 68 \text{ m} = 7140 \text{ m}^2$. Each team has 11 players, so per capita this is $7140 \text{ m}^2 / 11 \approx 649 \text{ m}^2$ ($649 \cdot 11 = 7139$, $650 \cdot 11 = 7150$).

2.5. Fifth grade

With the 5th grade of primary schooling in Croatia, the teaching of various subjects, including mathematics, becomes more specialised and taught by teachers who obtained not a general pedagogical degree, but a specific degree in their subject (i.e., masters in mathematics education in our case).

In the 5th grade of Croatian schools the following mathematical topics are discussed (NN, 2006):

- Arithmetic in the set positive integers,
- Prime and composite numbers, common factors and multiples,
- Sets of points in the plane,
- Fractions (as a notion, no arithmetic with fractions yet!),
- Arithmetic with decimal fractions.

This is the first year of teaching mathematics in which not all topics can be reasonably connected with football, specifically, there is no football-relevant maths problem one could think of (at least, we, the authors, can't think of any) that includes prime and composite numbers, factorisations, common factor and multiples. On the other hand, the other mathematical topics of this grade now allow for a much bigger variety of examples which relate to football. We give an example that combines the topic of arithmetic with decimal fractions with the topic of sets of points in the plane.

Example 5. The official football rules (IFAB, 2020) define that the penalty point should be 12 yd from the midpoint of the goal line and the goal dimensions are 8 yd by 8 ft. In meters, it is said that the penalty points is 11 m from the midpoint of the goal line, and that the goal is 7.32 m wide and 2.44 m high. Knowing that 1 yd = 0.9144 m and 1 ft = 0.3048 m, check if the measures given in yd/ft and m are truly equal. Express the metric distance in centimeters. How many kilometers/miles (1 mile = 1.609344 km) would a player warming up running right along the edge of a standard (105 m × 68 m) football field run in one round?

Since 1 yd is 0.9144 m, 12 yd is $12 \cdot 0.9144 \text{ m} = 10.9728 \text{ m}$, i.e. the penalty point is a bit less (almost 3 cm less) than 11 m away from the midpoint of the goal line. The distance is 1097.28 cm exactly, which – due to measures smaller than 1 cm being negligible in this context – can be rounded to 1097 cm.

Similarly, the width of a goal is $8 \text{ yd} = 8 \cdot 0.9144 \text{ m} = 7.3152 \text{ m} = 731.52 \text{ cm}$ (almost half a centimetre less than the official $7.32 \text{ m} = 7320 \text{ cm}$, but can be rounded to it), and its height is $8 \text{ ft} = 8 \cdot 0.3048 \text{ m} = 2.4384 \text{ m} = 243.84 \text{ cm}$ (again, slightly less than the official $2.44 \text{ m} = 244 \text{ cm}$, but the difference is negligible in this setting).

The second question relates to the notion of perimeter. The perimeter of the standard football field is $2 \cdot 105 \text{ m} + 2 \cdot 68 \text{ m} = 346 \text{ m} = 0.346 \text{ km}$. So, a player running along the edge would run 0.346 km in one round. Since

$1.609344 \text{ km} = 1 \text{ mile}$, dividing by 1.609344 gives $1 \text{ km} = 1/1.609344 \text{ mile}$, i.e. $0.346 \text{ km} = 0.346/1.609344 \text{ mile} = 0.215 \text{ mile}$ (when rounded according to the rule for significant digits).

2.6. Sixth grade

In the 6th grade of Croatian schools the following mathematical topics are discussed (NN, 2006):

- Arithmetic with fractions,
- Triangles and quadrilaterals,
- Integers (positive, zero and negative),
- Linear equations with one unknown.

Again, all of the topics can be reasonably connected to football. From the point of mathematics, one topic stands out – negative integers. Namely, due to all interested in football knowing that ties in a championship table are resolved by calculating the goal difference (goals scored minus goals received), football is now a context in which one meets negative numbers regularly.

Example 6. A full championship table includes not only numbers of wins, draws and lost matches (W, D, L), and points (PTS), but also goals scored (goals for, GF), goals received (goals against, GA) and goal difference (GD). Calculate the GD column for the final table (Fig. 2) of the German *Bundesliga*, season 2020/21 (Fussballdaten, 2021). If GD were the main criterion for ranking football teams, instead of ranking them by PTS, how would the order of the teams be in the table?

Ranking	Club	W	D	L	GF	GA	GD	PTS
1.	Bayern München	24	6	4	99	44		78
2.	RB Leipzig	19	8	7	60	32		65
3.	Borussia Dortmund	20	4	10	75	46		64
4.	Wolfsburg	17	10	7	61	37		61
5.	Eintracht Frankfurt	16	12	6	69	53		60
6.	Bayer Leverkusen	14	10	10	53	39		52
7.	Union Berlin	12	14	8	50	43		50
8.	Borussia Mönchengladbach	13	10	11	64	56		49
9.	VfB Stuttgart	12	9	13	56	55		45
10.	SC Freiburg	12	9	13	52	52		45
11.	Hoffenheim	11	10	13	52	54		43
12.	Mainz 05	10	9	15	39	56		39
13.	Augsburg	10	6	18	36	54		36
14.	Hertha	8	11	15	41	52		35
15.	Bielefeld	9	8	17	26	52		35
16.	Köln	8	9	17	34	60		33
17.	Werder Bremen	7	10	17	36	57		31
18.	Schalke 04	3	7	24	25	86		16

Figure 2. The (incomplete) final ranking of the Bundesliga, season 20/21.

For example, Bayern München had $GF = 99$ and $GA = 44$, so their GD is $99 - 44 = 55$; SC Freiburg had both GF and GA equal to 52, so their GD is $52 - 52 = 0$; Werder Bremen had $GF = 36$ and $GA = 57$, so their GD was $36 - 57 = -21$; etc. The goal differences for all teams, in the order given in Fig. 2., would be 55, 28, 29, 24, 16, 14, 13, 8, 1, 0, -2 , -17 , -18 , -11 , -26 , -26 , -21 and -61 . Thus, the ranking by GD would be Bayern München, Borussia Dortmund, RB Leipzig, Wolfsburg, Eintracht Frankfurt, Bayer Leverkusen, Union Berlin, Borussia Mönchengladbach, VfB Stuttgart, SC Freiburg, Hoffenheim, Hertha, Mainz 05, Augsburg, Werder Bremen, then Bielefeld and Köln would share their position and finally last would be Schalke 04. Note that we see by this example (and a table of a different league might show this even more clearly) that even if GD has the same trend as PTS, it is not necessarily true that a team with more PTS has a bigger GD.

2.7. Seventh grade

In the 7th grade of Croatian schools the following mathematical topics are discussed (NN, 2006):

- Cartesian coordinate system and line equation,
- Proportionality and percentages,
- Basic probability and data charts,
- Polygons and circles,
- Systems of two linear equations with two unknowns.

Once more, almost all topics can be connected to football in a realistic manner, except general polygons. In fact, this is the year where, among all the primary school years, the range of connections between mathematical topics and football is the largest, so we shall give three instead of just one example appropriate for the 7th grade.

We first give an example with percentages, for two reasons: firstly, percentages tend to stay a problematic point in mathematics for all of the rest of education, and secondly, in everyday communication about football, percentages have a prominent place (e.g. ball possession, estimates of probability, . . .).

Example 7. Given the final table of the German *Bundesliga*, season 2020/21 (see example 6 and Fig. 2), one can see that 18 matches ended without goals, 37 with 1 goal, 72 with 2 goals, 70 with 3 goals, 51 with 4 goals, 32 with 5 goals, 15 with 6 goals, 9 with 7 goals and 2 with 8 goals (Fussballdaten, 2021).

Represent this data graphically. Calculate the average number of goals per game. What is the percentage of matches with more, equal or less goals than average?

To calculate anything here, we need the total number of goals scored during the season. Since obviously the total of GF is equal to GA (every goal for one team is a goal against another), the total number of goals is equal to the sum of the GF

(or alternatively, the GA) column: there were 928 goals in the *Bundesliga* 20/21 season.

Then we also need the total number of matches. Each of the 18 teams played 34 matches (twice with each of the remaining 17 teams). Since when we calculate $18 \cdot 34$ we obtain the double number of matches (each match being counted twice, once for the home and once for the away team), the total number of matches was $18 \cdot 17 = 306$.

We conclude that the average number of goals per match in the 20/21 *Bundesliga* was $928/306 = 464/153$, which is almost exactly 3 (precisely: 3.0326797...).

From the additional data given in the text of the problem, we see there were $18 + 37 + 72 = 127$ matches with less than average (i.e., less than 3 goals), 70 matches with average (3) goals, and the remaining $306 - 70 - 127 = 109$ matches were with more than 3 goals. Thus, $127/306 = 41.5\%$ matches had less than the average number of goals, $70/306 = 22.9\%$ had the average number of goals, and $109/306 = 35.6\%$ were with more than the average number of goals (note that after rounding we checked that the sum of the percentages is 100 %!).

The data given in the problem are best represented by a column chart (Fig. 3). It is a good idea to discuss that chart to demonstrate, for example, that it is a common misconception that the average is the most frequent result (note that there were more matches with 2 than with 3 goals).

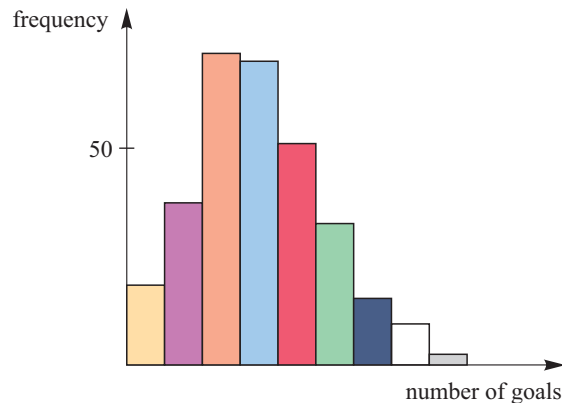


Figure 3. Numbers of matches with 0 to 8 goals in the Bundesliga 20/21 seasons.

The second 7th grade example relates to geometry of circles and conversion of measures.

Example 8. The official football rules (IFAB, 2020) state: “All balls must be: spherical; made of suitable material; of a circumference of between 68 cm (27 in) and 70 cm (28 in); between 410 g (14 oz) and 450 g (16 oz) in weight at the start of the match; of a pressure equal to 0.6 – 1.1 atmosphere ($600 - 1100 \text{ g/cm}^2$)

at sea level (8.5 lbs/sq in – 15.6 lbs/sq in)”. Express the masses in dekagram and kilogram, the circumferences in meter and decimeter. Calculate the mass and circumference spans. What is the diameter of a football?

We first calculate the measure conversions: $68\text{ cm} = 6.8\text{ dm} = 0.68\text{ m}$, $70\text{ cm} = 7\text{ dm} = 0.7\text{ m}$; $410\text{ g} = 4.1\text{ dag} = 0.41\text{ kg}$, $450\text{ g} = 4.5\text{ dag} = 0.45\text{ kg}$. The circumference span is $70\text{ cm} - 68\text{ cm} = 2\text{ cm}$, and the mass span is $450\text{ g} - 410\text{ g} = 40\text{ g}$.

We know that the perimeter of a circle, i.e. a spherical ball, is $P = \pi d$, where d is the diameter of the ball. Also, the rule says that P is between 68 cm and 70 cm, so the diameter is between $68\text{ cm}/\pi$ (approximately, rounded to a mm, 21.6 cm) and $70\text{ cm}/\pi$ (approximately 22.3 cm).

Finally, we give a more creative example appropriate for the 7th grade, which combines much of the mathematics learned in this and previous years.

Example 9. It is well known that in the early days of football the number of players per team was not standardised. Use mathematics to argue that the number of 10 field players, i.e. 10 players who are used in the whole field (the keeper usually stays near the goal, unless it is Higuaita or Neuer)³, per team is a reasonable number! Use the fact that general football statistics data suggest that on average a field player keeps the ball for 3 seconds before passing it to another player or shooting at the goal (Ludwig, 2008).

Obviously, if the number of players is too large, say 25 per team, the game would be too fast, so that would ruin the point of the game. On the other hand, if the number of players is too small, say 5, they would play slower and get tired too soon, so the game would be boring. But where between 5 and 25 is the optimum?

One possible reasoning is the following. A player keeping the ball for 3 s can, essentially (we can ignore the situation when a player is near the edge of the football field as this is a rough estimate calculation), run in any direction, so he covers a circular area. The radius of this circle is the distance he can cover in the 3 s. As for such a short time a player can run relatively fast, we can estimate his velocity as 5 m/s (Ludwig, 2008), meaning the radius of the circle is $3\text{ s} \cdot 5\text{ m/s} = 15\text{ m}$. Thus, the area a player covers at any moment of the game is approximately $\pi \cdot 15^2\text{ m}^2$, which is roughly 707 m^2 . But, we know that the area of the football field is most commonly 7140 m^2 (see Example 4). Thus, every player covers roughly 10 % of the field, i.e. we need 10 field players to efficiently cover the field.

2.8. Eighth grade

In the last year of primary schooling in Croatia, i.e. in the 8th grade, the following mathematical topics are discussed (NN, 2006):

³ The Colombian ex-goalkeeper René Higuita and the German goalkeeper Manuel Neuer are famous for their “trips” outside the penalty area.

- Squares and square roots of numbers,
- Pythagoras' theorem,
- Real numbers,
- Symmetries in the plane,
- Basic geometry of space.

Again, there is a huge choice of football-related examples appropriate for illustrating these topics, and we choose an example illustrating the usage of the theorem of Pythagoras combined with spatial reasoning (and a simple interconnection with physics) to obtain a conclusion of real relevance for any football fan or pro.

Example 10. Knowing that a goal has width 7.32 m and height 2.44 m, that the penalty kick point is 10.97 m from the midpoint of the goal line, that the reaction time of a goalie is about 0.2 s and that he jumps with speeds (The Hoops Geek, 2019) of about 2 to 2.5 m/s at most (no significant difference between the top and not so top quality goalies!), and that good penalty kickers send the ball flying very precisely with speeds up to (and sometimes even more than) 100 km/h, explain why penalty kicks mostly result in a goal, with percentages that are in general equally high in top and “low” championships (Ludwig, 2008).

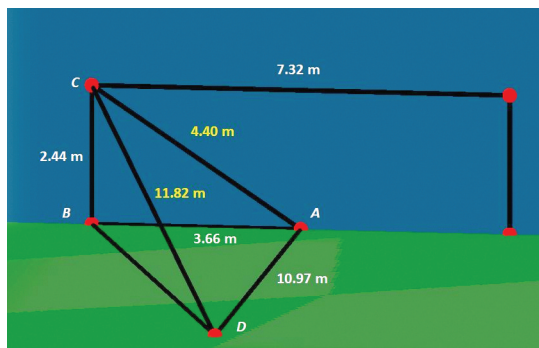


Figure 4. The basic geometry of a penalty kick.

Let's look at the situation, as it is depicted in Fig. 4. The penalty kick point D is 10.97 m away from the midpoint A of the goal line, a line DA is perpendicular to the goal line. If B is one of the bottom corners of the goal, then $|AB|$ is half the width of the goal, i.e. $7.32/2$ m = 3.66 m, and triangle DAB is a right triangle, as is triangle ABC , where C is the top corner directly above B .

Triangle DAC is a right triangle as well, with right angle at point A , so the distance $|DC|$ from the penalty kick point to upper corner C , by Pythagoras' theorem and rounded to three significant digits, equal to 11.82 m. If a top penalty kicker sends a ball flying with about 100 km/h (i.e. about 28 m/s) very precisely to upper corner, the ball will be flying about 11.82 m/28 m/s, which is approximately 0.4 s. A not so good penalty kicker will not kick that fast nor so precise, but in a not

really top league, the goalie will also neither react nor jump so fast, so essentially, the “minuses” of the kicker and the goalie will cancel themselves.

Goal height is $|BC| = 2.44$ m, thus by Pythagoras’ theorem the distance $|AC|$ is (rounded to three significant digits) 4.40 m. If instead of A we use a point A' representing the position of the goalie’s hand while waiting for the penalty kick to happen (he has to stand in the middle of the goal line, usually he stands with hands outstretched), the distance from A' to C is obviously less. Goalies are typically big, say 1.90 m height, thus A' can be estimated to be at say 1.60 height, i.e. about 2.44 m $-$ 1.60 m $=$ 1.84 m below the crossbar. For humans the arm span is approximately equal to height, thus with arms horizontally outstretched A' is about 3.66 m $-$ $1.90/2$ m $=$ 2.71 m away from post BC (see Fig. 5), so the distance $|A'C|$ is, again using the theorem of Pythagoras, about 2.84 m. Except for a goalie with significantly longer arms it cannot get much better than that (see Fig. 5: in other positions the hands, with arms outstretched, move approximately around a circle).

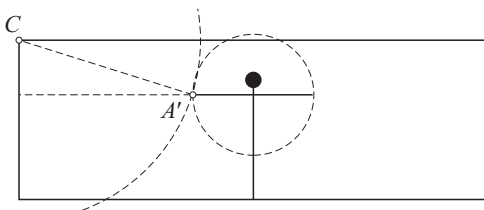


Figure 5. Diagram of a goalie waiting for a penalty kick.

We conclude that if the penalty kicker makes a precise shot to the corner C the goalie has to jump at least some 2.75 m to catch it. But, he can jump only after he notices the kick, i.e. in his reaction time (0.2 s) and then with speed at most 2.5 m/s, i.e. minimum time to reach the upper corner is about $(0.2 + 2.75/2.5)$ s $=$ 1.6 s. Even if we round it down for an extra good keeper with top reactions, it is definitely above 1 s, much more than the flight time of the ball sent precisely and with high speed to an upper corner. Thus, in a top league (a top goalie and a top kicker) or in a low one (not so good goalie with slower reaction times and less good estimate of the ball direction, but also not so good kicker giving kicking with less force and not so precise) is about the same – a goalie simply has no time to catch the ball if it flies to the upper corner.

3. Conclusion

One of the main goals of popularisation of mathematics is to make the subject more interesting for children. This is often achieved by connecting mathematical topics with more popular activities. Of course, mathematics can be connected to virtually any desired topic, but often such connections are far-fetched, complicated and/or very specific. On the other hand, one of the trends of modern teaching is increasing interdisciplinarity. Here sports in general, and more specifically football, is a natural idea to achieve all of this: sports is in general more popular than mathematics,

thus provides more motivation; there is an extensive variety of proper and still not overly complex applications of mathematics in sports; and both maths and sports belong to common core subjects.

However, to the best of our knowledge, even if there exists a reasonable number of good-quality texts about connections between maths and sports, and also specifically maths and football, not many references can be found on the topic of using sports (football) in teaching mathematics; most of the existing ones are limited to sports statistics in teaching maths. Some scattered references can be found with concrete examples and calculations suitable for teaching maths, e.g. (Ludwig, 2008) is a good reference for sports mathematics in general with clear connections to specific maths topics and (Ludwig et al., 2021) or (Parr, 1997–2021) for football maths. After checking references in English, German and Croatian, we found that the only systematic approach to teaching mathematics via football is (Mijač, 2021), which is also our main reference, and where the interested reader can find many more examples (not only for primary, but also for secondary school). We hope that this paper inspires more similar texts, and in particular we hope that we have successfully demonstrated two points:

- a maths teacher can choose a hobby or general interest to introduce and illustrate specific parts of the maths curriculum, and this being his true interest, he will teach more passionately and thus convey more enthusiasm to the students;
- football is particularly well suited for teaching mathematics in primary school as virtually any topic can be connected to football in a realistic manner.

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Sažetak. Koliko je nogomet popularan, toliko matematika to nije, bar u široj populaciji i posebno među djecom. No, nemoguće je biti ljubitelj nogometa bez korištenja bar nešto matematike, što ćemo pokazati kroz primjere osnovnoškolske matematike od 1. do 8. razreda. Za svaki razred dat ćemo primjere kako uključiti nogometne teme (koje su stvarno relevantne za svakoga koga zanima nogomet) u hrvatski osnovnoškolski matematički kurikulum. Tako će biti predstavljeni primjeri od osnovne aritmetike potrebne za praćenje rezultata prvenstava do primjena Pitagorinog poučka i osnova vjerojatnosti za objašnjenje statistika kaznenih udaraca.

Ključne riječi: osnovnoškolska matematika, nogomet, aritmetika, geometrija, vjerojatnost, algebra

3. The Role of Technology in Teaching Mathematics



eTwinning as a Potential Tool in Teacher Education

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Abstract. eTwinning is a segment of the Erasmus program which seeks to utilize new technologies in order to network schools and kindergartens. Next to its emphasis on the importance of digital competencies, eTwinning also underlines the importance of collaborative and project-based work, as well as lifelong learning in a safe environment for all participants in the online environment. eTwinning launched a pilot program Teacher Training Institutions (TTIs) in 2012 aimed to include teacher education institutions in its platform. Since 2018, the TTIs initiative has been officially available to all countries participating in the eTwinning program. Faculty of Education in Osijek has been participating in the TTIs initiative since 2014 and has carried out nine projects so far. Given that the Croatian National Curriculum Framework emphasizes the importance of moving away from traditional educational models and places emphasis on the development of critical thinking skills, creativity, innovation, communication skills, collaboration, informational and digital literacy, eTwinning offers the possibilities of new modes of work aimed to develop these competencies within its platform. This paper will present the results of a survey conducted among students of the Faculty of Education in Osijek related to their familiarity with eTwinning, opinions on advantages and disadvantages of implementing eTwinning in regular education, as well as their attitudes on the possibility of applying eTwinning in school subjects Mathematics and Computer Science.

Keywords: eTwinning, competencies, Mathematics, ICT, teacher education

1. Introduction

Schwab (2016) introduced the term fourth industrial revolution, which describes the impact of digital technology on almost all aspects of our lives, including education. Digital competence has become mandatory, especially in the world after COVID-19. Almost all advanced learning and jobs in all sectors will require some form of digital skills, but on average, 40 % of Europeans aged 16 to 74 do not have these skills (European Commission^a, 2020).

The European Commission (2019) recognized the importance of digital competence and highlighted it as one of the eight key competences for lifelong learning. Lifelong learning is important in every person's life, but especially for people working in professions that transmit knowledge and facilitate learning (European Commission, 2021). The recommendation is to support educational staff in the development of their knowledge by encouraging them to cooperate within and outside their educational institutions. For this purpose, they suggest peer learning, staff exchange and mobility, participation in networks and communities of practice (European Commission, 2019).

The Digital Education Action Plan (2021 – 2027) sets out the European Commission's vision for high-quality, inclusive and affordable digital education in Europe. It is also a call for increased cooperation at a European level to adapt education and training systems to the digital age (European Commission^b, 2020).

In order to achieve the European Education Area by 2025, investing in education, skills and competences is necessary for all Member States and should be a strategic priority for the EU. The objectives at the EU level include strengthening the Erasmus program by updating the mobility framework for the purpose of learning, inter alia, on digital mobility, including the combination of online and live exchange. Digital topics should be a priority in collaborative projects because of their future-orientedness and strategic character. It was also pointed out that the Erasmus+ program played a key role in promoting successful practice (European Commission^c, 2020).

eTwinning is a community for schools in Europe and neighbouring partner countries, launched in 2005 and funded by the European Commission under the Erasmus+ programme (eTwinning.net, 2021^a). It is an open, safe, free education network that promotes innovation in learning and teaching practices (Papadakis, 2016; Basaran, 2020). In almost 16 years of its existence, it involved teachers from 36 European and 8 neighbouring countries. More precisely, it involved more than 800 000 teachers working in more than 200 000 schools, more than 93 000 carried out projects, involving many students from all educational levels across the continent (Licht, Pateraki & Scimeca, 2020).

eTwinning promotes school collaboration in Europe through the use of Information and Communication Technologies (ICT) by providing support, tools and services for schools. eTwinning also offers opportunities for free, and continuing online Professional Development for educators. Launched in 2005 as the main action of the European Commission's eLearning Programme, eTwinning is co-funded

by the Erasmus+, the European programme for Education, Training, Youth and Sport, since 2014 (eTwinning.net, 2021^a).

In 2012, the TTI initiative was launched as part of eTwinning. The purpose of this initiative is to include eTwinning in initial teacher training. Through examples of projects launched under the TTI initiative, it has been shown that they enable students to apply 21st century skills as well as participate in international collaborative projects. The introduction of the TTI initiative in higher education results in significant value, both for the institution and for the students themselves. Since 2018, the TTIs initiative has been officially available to all countries participating in the eTwinning program. In 2020, it was renamed “eTwinning for Future Teachers”. (eTwinning.net, 2021^c).

1.1. Purpose of study

Faculty of Education in Osijek has been participating in the TTIs initiative since 2014 and has carried out nine projects so far. Projects were conducted through different courses and in different study years. Some students have been involved in eTwinning since high school, some met eTwinning for the first time as part of a college course.

Study was conducted as it was seen, as a result of literature review, that the number of studies on implementation of eTwinning in teaching and learning in Croatia, and in higher education generally, is limited.

As eTwinning combines the use of ICT and collaborative project learning and provides opportunities for professional development, we were interested in the attitudes of students studying at the Faculty of Education in Osijek on these issues, as well as their general attitude towards the implementation of eTwinning in teaching and recognizing its strengths and weaknesses.

In order for eTwinning to be successfully implemented in the teaching of any subject and at any educational level, it is necessary that the teachers have the necessary competencies in the field of ICT application and collaborative learning, as well as pedagogical competencies for using ICT to improve teaching quality and professional development. In the research, students were required to self-assess these sets of competencies as well as present own attitudes to the implementation of eTwinning in their future work.

We paid special attention to the implementation of eTwinning in the teaching of mathematics and computer science as different kind of subjects. More precisely, we were interested in students' opinion about implementation of e-Twinning in teaching and learning mathematics and computer science and, if they felt it was possible to implement e-Twinning in teaching and learning mathematics and computer science, to list their own ideas on how it might be done.

2. Literature review

Although today's generations of pupils are considered digital natives (Prensky, 2001), which means they grow up in the digital age of smartphones, computers, gaming consoles and video games, research has shown that the increase in the number of digital devices we use contributes to the development of digital skills only at the operational level, that is, increased use of digital technologies cannot be considered as a key indicator of digital competence (Van Deursen, 2010). As the European Commission (2019) stated, digital competence involves confident, critical and responsible use of, and engagement with, digital technologies for learning, at work, and for participation in society. Individuals should be able to use digital technologies to support their active citizenship and social inclusion, collaboration with others, and creativity towards personal, social or commercial goals.

Teachers and trainers have the responsibility to facilitate learners' acquisition of key competences (The Council of the European Union, 2020). Morze (2019) states that teachers are becoming organizers of independent activity where pupils are at the center of the educational process. Everyone could realize their abilities and interests, create conditions for and create an environment in which it becomes possible to develop personality, to acquire knowledge and skills necessary for life in the digital society.

Furthermore, participation in eTwinning develops 8 key competencies recommended by the European Parliament (Crişan, 2013; Gilleran & Kearney, 2014), it is recognized as a concrete tool that enables innovation in learning through ICT (Kampylis, Bocconi & Punie, 2012). By participating in eTwinning projects, educational environments are enriched with applied and innovative activities that include collaborative teamwork and positively reflect on students' academic success (Başaran et al., 2020). In addition, it empowers schools to open up to the community, to teachers and students from other countries (Gajek, 2018; Acar & Peker, 2021; Gilleran, 2019).

The teachers have a key role in eTwinning projects (Gajek, 2018), but they are still at different stages of technology adoption. Their attitudes towards technology interrelate with their pedagogical use of it (Knezek & Christensen, 2008). Integration of ICT into course environments improves the technological skills of the teacher (Başaran et al., 2020) by giving them an opportunity to get in touch with new digital tools (Alexiou, 2019). Besides that, it helps them to teach in different, innovative (Brincat, 2019) and more efficient way (Başaran et al., 2020). Importance of curricular integration of eTwinning projects is highlighted (Akdemir, 2017; Carpenter and Tanner, 2011; Kołodziejczak, 2019).

Teachers need opportunities for continuing professional development (OECD, 2019; The Council of the European Union, 2020). The continuing professional development of teachers should be perceived as a precondition to delivering quality teaching and training; teachers and trainers should therefore be encouraged to reflect on their practices and training needs (The Council of the European Union, 2020). The OECD's (2019) International Teaching and Learning Survey (TALIS) found that a significant proportion of teaching staff expressed the need to develop

competences for teaching pupils with special needs, the use of digital technologies and teaching in multilingual and multicultural classrooms. eTwinning affects the improvement of teachers' teaching skills in order to adapt it to the needs and abilities of the individual students (Kołodziejczak, 2019), ICT competence (Kołodziejczak, 2019; Gilleran, 2019) and pedagogical skills (Gilleran, 2019). eTwinning has a positive effect on professional development (Holmes, 2013) and encourages professional sharing (Başaran et al., 2020). Teachers who attended more types of continuing professional development were more likely to have engaged in collaborative and interactive training. Moreover, those teachers were more likely to perceive their professional development as useful (European Commission, 2021). Cross-border mobility, either short-term or longer-term, physical, virtual or mixed, is a powerful learning experience and a valuable opportunity in developing participants' competences (The Council of the European Union, 2020).

eTwinning offers a valuable alternative to traditional teacher training as it supports teachers to learn with collaboration and reflection on their experience with peers across regions and countries (Holmes, 2013). It contributes to teachers' professional development (Crisan, 2013), and allows them the experience of innovative teaching methods (Barorova et al., 2007; Vlada et al., 2009).

Participating in eTwinning projects is motivating and encouraging both for teachers and their pupils (Crişan, 2013). It affects the improvement of pupils' social skills, develops their entrepreneurship and creativity (Kołodziejczak, 2019; Brincat, 2019), encouraging respect for each other's views and ideas (Komninou, 2010), teamwork, technological knowledge (Başaran et al., 2020, Brincat, 2019, Gilleran, 2019), speaking and writing skills (Brincat, 2019; Akdemir, 2017). The most significant development is in the use of technology and communication technology (Başaran et al., 2020). Active participation in the lesson (Başaran et al., 2020) helps them to deepen understanding of the subject being taught (Komninou, 2010). Once they participate in an eTwinning project, it encourages them to participate in other future projects (Brincat, 2019; Alexiou, 2019).

Teachers and students experience some difficulties while conducting eTwinning projects. Most of these problems are caused by technical deficiencies of schools, lack of ICT skills and the issue of integrating eTwinning into the curriculum (Akdemir, 2017). Başaran et al. (2020) point out the inadequacy of technology use skills of younger age group pupils as another disadvantage. One solution to this problem is providing parents' support. Technical deficiencies of schools, which was most evident during the implementation phase of the eTwinning project, were noted (Akdemir, 2017; Brincat, 2019). The projects will be more efficient when this deficiency is eliminated (Başaran et al., 2020).

Besides technical deficiencies of schools, the following eTwinning implementation challenges have been noted in literature: lack of ICT skills (Akdemir, 2017; Brincat, 2019), discrepancies between eTwinning applications and curriculum (Akdemir, 2017; Brincat, 2019), negative comments from other teachers (Brincat, 2019), need for parent support (Başaran et al., 2020) and language barriers (Brincat, 2019).

Perception of future teacher students involved in the eTwinning TTI initiative is extremely positive (Prieto & Escobar, 2017). Students believe that eTwinning should be used as a pedagogical collaborative tool at all levels of education, including colleges (Prieto & Cirugeda, 2017). It turned out that the participation of future primary school students in the eTwinning project results in great satisfaction, especially getting to know the eTwinning platform and project planning with participants from other European countries (Sammarano, 2021).

3. Research methodology

Research was conducted at the Faculty of Education in Osijek, Croatia, during the academic year 2020/2021 on pre-service teachers in order to establish their familiarity with eTwinning, opinions on advantages and disadvantages of implementing eTwinning in regular education and their attitudes to the possibility of applying eTwinning in school subjects Mathematics and Computer Science. Since they have practical experience gained through professional practice in primary schools and also attend methodological exercises since second study year, students of 3rd, 4th and 5th study year were invited to participate in the research (around 220 students). Participation in the research was entirely voluntary, so only 69 of them decided to join the research and fill out the questionnaire. Participants were familiar with the purpose of the research and it was conducted anonymously, so the privacy of the participants was granted. The questionnaire was available online, so the students could access and complete it arbitrarily at any time, at the Faculty of Education or at home. To ensure that students are able to express their opinion on eTwinning even in cases when they have not heard of it before (during primary or secondary school), basic information on eTwinning is provided at the beginning of the questionnaire. The questionnaire had sets of questions and statements and was distributed in five categories:

- demographic information – 8 questions;
- digital competence – 8 questions;
- pedagogical competence – 5 questions;
- collaborative competence – 4 questions;
- a set of questions regarding pre-service teachers' attitudes to implementation of eTwinning in their future work – 10 questions.

A set of questions for three categories: digital, pedagogical and collaborative competence, was partly adopted from the 2018 eTwinning MeTP 2.0 Framework questionnaire and was translated and adjusted to Croatian language. That was part of eTwinning's ongoing pedagogical monitoring. The MeTP 2.0 questionnaire was used in the research because it was developed as a part of a large-scale monitoring activity that confirms that the involvement in eTwinning activities is linked to the improvement of teachers' perceptions about their digital, pedagogical and collaborative competencies (eTwinning.net, 2021^b). We were interested in the situation among our students, future primary school teachers, on these issues.

In regard to the gender, 97.1 % of the respondents were female, which is not unusual considering the gender structure of the students at the Faculty of Education in Osijek, and they were equally distributed by modules.

4. Results

In regard to the demographic questions, 68.1 % of participants used a computer and accessed the Internet for the first time between the age of 7 and 10 and 24.6 % of them used a computer even earlier. Results concerning students' use of ICT in teaching practice show that 97.1 % of participants use the ICT in teaching practice in the context of preparing lessons for teaching practice at the Faculty of Education. As shown in the Figure 1 below, 44.9 % of the respondents haven't heard of eTwinning and 43.5 % of the respondents have heard but don't know much about the eTwinning and the possibilities of eTwinning.

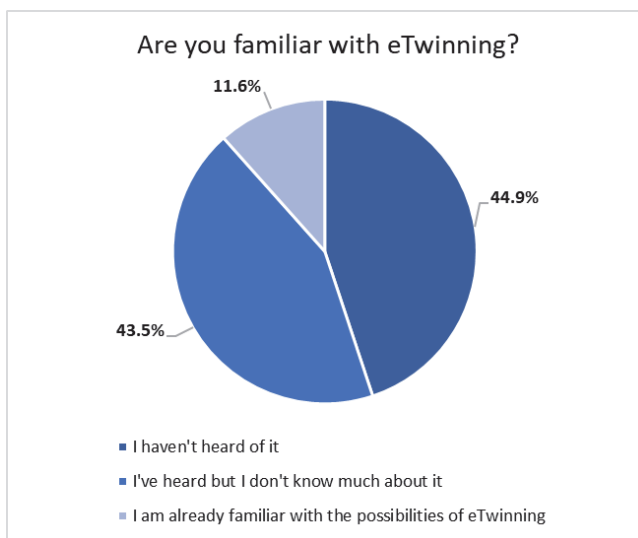


Figure 1. Answers regarding students' familiarity with eTwinning.

In the digital competence section, all of the respondents agreed about the importance of using ICT in their future practice. Precisely, 36.2 % of the respondents plan to make significant use of ICT in their teaching so that their pupils actively participate and they believe that the use of ICT benefits learning and teaching, 27.5 % of the respondents plan to use ICT in their teaching, regularly evaluate its use for teaching purposes and said that pupils' opinions will be extremely important to them when assessing their own practice and, also, 27.5 % of the respondents plan to develop their teaching through a systematic evaluation of their curricula and practices, to evaluate teaching regularly, work with others to further develop practice and regularly use ICT in teaching because its integration and proper use is one of the features of quality teaching. Only 8.8 % of the respondents plan

to sometimes integrate ICT into their teaching, but more in terms of using digital media instead of teaching its use.

Although the results above show that respondents recognize the importance of the use of ICT in their future practice, 24.6 % of the respondents don't feel confident about using a computer in teaching practice.

Results regarding students' answers on how confident they are in adapting or changing their teaching approaches, tools, and methods are shown in Table 1. It is clear from results that 73.9 % of the respondents (sometimes or often) consult different resources to update their knowledge of approaches to learning and teaching (new approaches, methods and tools).

Other results in the pedagogical section of questions show that 75.4 % of the students will evaluate their own practice and think it is good to participate in professional development activities and more than 80 % of the students will occasionally or regularly monitor new research and relevant sources to improve their skills, subject and pedagogical knowledge and teaching.

Results from the collaborative section of statements and questions show that all our respondents are willing to collaborate with their future colleagues at work and 91.3 % of them are familiar with online learning communities, plan to use someone else's activities to be inspired for their own future teaching practice, publish their own ideas from time to time or even coordinate an educational online community.

Table 1. Answers regarding the question "How confident are you to adjust or change your teaching approaches, tools and methods?".

Answers	N	%
I'd rather stick to the methods and tools I know than experiment with new ones.	2	2.9
I rarely try new approaches, methods and tools of teaching and learning in the classroom, but when I do, I do not evaluate them in order to be informed about their implementation.	4	5.8
I rarely try new approaches, methods and tools of teaching and learning in the classroom and beyond, but when I use them I evaluate them to be informed about their implementation.	12	17.4
Sometimes I consult different resources to update my knowledge of teaching and learning approaches.	39	56.5
I often try out new approaches, methods and tools for teaching and learning in the classroom and beyond and assess their impact on student learning. I share my findings with colleagues and support them in changing practices and, if necessary, lead policy change at the school level or beyond.	12	17.4

When it comes to involvement of their pupils in school in the online learning community in their future work, 95.7 % of the respondents answered that they are willing to involve their pupils in an online learning community. Results related to respondents' willingness to involve their pupils in eTwinning community are shown in the Figure 2.

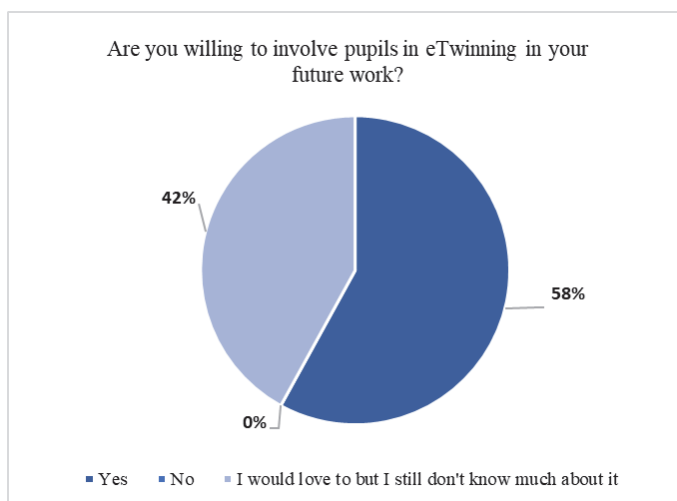


Figure 2. Answers regarding pupils involvement in eTwinning community.

Results are encouraging because they are all open to the possibility of joining the online community, more precisely, joining eTwinning community and implement eTwinning in regular education.

All of the respondents agreed, as can be seen in Table 2, that involvement in eTwinning (in the form of available online courses/webinars) provides them with opportunities for professional development and 98.6 % of them think that involving classes in eTwinning projects improves teaching and offers multiple benefits to pupils.

Table 2. Students' attitudes on involvement in eTwinning.

Statements	Agree	Disagree
Involvement in eTwinning (in the form of available online courses/webinars) provides me with opportunities for professional development.	69 (100 %)	0 (0 %)
Involving classes in eTwinning projects improves teaching and offers multiple benefits to pupils	68 (98.6 %)	1 (1.4 %)

When asked to list some advantages of involvement in eTwinning, they listed the following benefits:

- exchange of opinions,
- experiences and materials,
- improving pupils' and teachers' knowledge,
- opportunity for professional development,
- new experiences for pupils,

- diversity of ideas,
- connection with Europe,
- promoting ICT for learning purposes,
- cooperation of teachers, students, parents, principals and local authorities,
- collaborative online learning,
- advancing teachers' and students' use of ICT.

Some of the disadvantages they listed are the following:

- problems with Internet and technology,
- insufficient school equipment,
- lack of time,
- lack of social contact,
- moving away from the traditional way of teaching,
- disinterest in cooperation and improvement of practice,
- cyberbullying.

At the end of questionnaire, respondents were asked to answer the questions regarding the possibility of applying eTwinning in school subjects Mathematics and Computer Science.

In Table 3 the percentage distribution of responses to the questions “Is it possible to implement eTwinning in teaching mathematics (1st to 4th grade)?” and “Is it possible to implement eTwinning in teaching computer science (1st to 4th grade)?” is shown.

Table 3. The percentage distribution of responses to questions regarding implementation of eTwinning in teaching mathematics and computer science.

Questions	Yes	No
Is it possible to implement eTwinning in teaching mathematics (1st to 4th grade)?	45 (65.2 %)	24 (34.8 %)
Is it possible to implement eTwinning in teaching computer science (1st to 4th grade)?	57 (82.6 %)	12 (17.4 %)

When asked to list some of the ways we can do this in teaching mathematics, most of them answered “I don't know enough about it at the moment, but I think it's certainly possible” and some of them had following ideas:

- “involvement in various projects that would help students be more interested in mathematics”;
- “we can conduct small presentations on the topic of the material that students are currently working on and publish the content on the project, students can submit well-solved tasks as an example for learning, etc.”;
- “a large number of tasks for all students, different tasks that they do not have in the classic literature”;

- “in geometry”.

When asked to list some of the ways we can do this in teaching computer science, they had following ideas:

- “we can fully integrate the platform into the lessons so that students master online collaboration and generally have the experience of participating in an online project”;
- “different projects that would teach students about the dangers of the Internet, but not only about the dangers, but also about the different opportunities it offers”;
- “competitions in programming (Scratch), video production, education on digital literacy, etc.”;
- “listening to lectures on e.g. data protection on the Internet”.

We can see that a slightly higher percentage of them think that it is easier to implement eTwinning in teaching computer science than in mathematics, which they explain by the fact that computer science teaching is performed on computers anyway.

5. Discussion

All respondents in this study recognized the importance of ICT in their future work, which is in line with the recommendation of the European Commission cited in this paper (European Commission, 2019; European Commission^a, 2020; European Commission^b, 2020; European Commission^c, 2020; European Commission, 2021). In addition, the vast majority of them use ICT during lesson preparation for teaching practice at the Faculty of Education.

More than 90 % of respondents plan to integrate ICT into teaching in one of the following ways: extensively use ICT during teaching, plan to use ICT but with regular assessment of their own practice or plan to participate intensively in training to improve their knowledge of the proper use of ICT in teaching. Only 8 % of respondents plan to sometimes use digital media in teaching, but not as a teaching tool. These results are consistent with the conclusion of Knezek and Christensen (2008) that teachers are still at different stages of technology adoption and that their attitudes to technology interrelates with their pedagogical use of it. Their conclusion is confirmed by the fact that 24.6 % of respondents do not feel confident about using a computer in teaching practice. Also, a slightly higher percentage of them think that it is easier to implement eTwinning in teaching computer science than in mathematics, because the computer science teaching is performed on computers anyway, which indicates a lack of knowledge about purposeful implementation of ICT in teaching.

73.9 % of students sometimes or often consult different resources to update their knowledge of new approaches, methods and tools in teaching practice. 26.1 % of them plan to rarely or never develop their knowledge and they could benefit greatly from the integration of ICT into teaching and participating in eTwinning,

because it enables them to learn about new digital tools and to teach in different, innovative and more efficient ways, as concluded in researches of Alexiou (2019), Brincat (2019), Kamyliis, Bocconi and Punie (2012), Barorova et al. (2007), Vlada (2009) and Basaran et al. (2020). The positive is that 98.6 % of respondents in this research think that involving pupils in eTwinning improves teaching. In a European Commission report about teachers in Europe (European Commission, 2021) it is stated that teachers who attended more types of continuing professional development were more likely to perceive their professional development as useful. Holmes (2013) claims that eTwinning offers a valuable alternative to traditional teacher training as it supports teachers to learn with collaboration and reflection on their experience with peers across regions and countries. Perhaps participation in professional development through eTwinning would encourage professional development of the remaining 26.1 % of respondents who do not have ambitions to improve their knowledge.

For about 80 % of future teachers from this research, professional development is very important. All of them are willing to collaborate with their future colleagues and almost everyone plan to use someone else's activities and materials and to share their own. They will achieve these goals through eTwinning since it has a positive effect on professional development (Holmes, 2013) and encourages professional sharing (Başaran et al., 2020). All of respondents recognized the opportunity of eTwinning for professional development.

As 95.7 % of respondents are willing to involve their future pupils in one of the online learning communities, and all of them are willing to involve them in eTwinning, we can expect their future pupils would: improve their social skills, develop their entrepreneurship and creativity (Kołodziejczak, 2019; Brincat, 2019), have more respect for each other's views and ideas (Komninou, 2010), participate in teamwork and develop technological knowledge (Başaran et al., 2020, Brincat, 2019, Gilleran, 2019), improve speaking and writing skills (Brincat, 2019; Akdemir, 2017), actively participate in the lesson (Başaran et al., 2020) and adopt a deep understanding of the subject being taught (Komninou, 2010). Since participation in an eTwinning project encourages pupils to participate in other future projects (Brincat, 2019; Alexiou, 2019) the development of their knowledge and competencies will be constant and continuous.

Respondents reported that one of the disadvantages of participation in eTwinning is problems with Internet and technology as well as insufficient school equipment, which is consistent with the conclusions of Akdemir (2017), Brincat (2019) and Başaran et al. (2020). Since they noted "lack of time" as one possible disadvantage, it is obvious that curricular integration of eTwinning projects is necessary, which is consistent with the Akdemir (2017), Carpenter and Tanner (2011) and Kołodziejczak (2019) conclusions.

5.1. Research limitations

Since students are invited to participate in the research without obligation, one of the limitations of the research is certainly the fact that 30 % of the invited students

participated in it. As participation in the survey was entirely voluntary, there is a possibility that the survey included a larger number of students who had an initial positive opinion of eTwinning than would have been the case if all invited students had participated.

6. Conclusion

Teaching with the help of ICT is the future of the teaching profession. A quarter of the pre-service teachers in this study do not feel confident when using ICT for teaching purposes, and 8 % of them don't want to use it. These facts indicate the insufficient preparation of pre-service teachers for the application of ICT in teaching. Therefore, it is necessary to upgrade the curriculum of teacher education in order to enable them for pedagogically appropriate use of ICT in teaching.

All respondents recognized the possibilities of eTwinning, both for the purpose of their own professional development, and for the purpose of improving students' knowledge and competencies. As part of the respondents (26.1 %) do not plan to participate significantly in their future work in professional development and upgrade their knowledge, their involvement in eTwinning during education would give them the opportunity to meet new approaches to teaching and see opportunities to develop their own knowledge and competence.

A large number of respondents believe that it is easier to involve students in eTwinning in information sciences than in mathematics, which confirms that their competencies for the purposeful application of ICT in teaching are not sufficiently developed and that the teacher education curriculum needs to be improved.

From the answers collected from the open-ended questions, it is clear that some respondents believe that involving students in eTwinning projects requires additional time. This emphasizes the need for curricular integration of eTwinning at the level of teacher education courses. In this way, teacher education students will be introduced to eTwinning and the opportunities it offers, will realize that eTwinning can be integrated into regular teaching without additional time load, and will also develop their digital skills and skills of purposeful pedagogical use of ICT in teaching.

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eTwinning kao opcija u obrazovanju budućih učitelja

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Sažetak. eTwinning je sastavni dio Erasmus programa koji teži umrežavanju škola i vrtića uz pomoć novih tehnologija. Osim isticanja važnosti digitalne kompetencije, eTwinning ističe i važnost suradničkog i projektnog rada te cjeloživotnog učenja u sigurnom okruženju svih sudionika na internetu. Od 2012. godine eTwinning je započeo s pilot programom uključivanja ustanova za obrazovanje budućih učitelja u svoju zajednicu pod nazivom Teacher Training Institutions (TTIs), a od 2018. godine je TTIs inicijativa službeno dostupna svim zemljama koje sudjeluju u eTwinningu. Fakultet za odgojne i obrazovne znanosti u Osijeku sudjeluje u TTIs inicijativi od 2014. godine te je do sada ostvareno devet projekata. Kako je u Okviru nacionalnoga kurikulumata istaknuta važnost u odmicanju od tradicionalnog načina obrazovanja te se stavlja naglasak na razvoj kritičkog mišljenja, kreativnosti i inovativnosti, komunikacijskih vještina, suradnje, informacijske i digitalne pismenosti, eTwinning kroz svoju platformu nudi mogućnost novih oblika rada u svrhu razvoja istaknutih kompetencija. U radu će biti dani rezultati provedenog upitnika među studentima Fakulteta za odgojne i obrazovne znanosti u Osijeku o upoznatosti s eTwinningom, stavovima o prednostima i nedostacima implementacije eTwinning-a u redovnu nastavu kao i stavovima o mogućnostima primjene u kontekstu matematičkih i informatičkih predmeta.

Ključne riječi: eTwinning, kompetencije, matematika, IKT, obrazovanje učitelja

Does Online Learning Make a Difference in Students' Grades?

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Abstract. This past academic year was challenging for both students and professors. Although the Covid 19 pandemic led to a chaotic 2020, it will also greatly affect the years to come. While many universities are now making plans for future distance learning programs, it is a good time to reevaluate the pros and cons of online learning. The purpose of this paper is to determine if there is a difference in student grades between online and onsite learning by comparing exam scores in Mathematics 1 and Mathematics 2 in the academic year in which both courses were taken at the faculty with the year in which Mathematics 1 was taken at the faculty and Mathematics 2 was taken in a distance learning environment. The statistical indicators obtained suggest that the pass rate is almost the same regardless of the form of teaching and midterm exams. However, the correlation coefficient between the results of the exams of Mathematics 1 and Mathematics 2 courses is statistically significantly lower in the academic year in which the first course was taken at the faculty and the second in a distance learning environment. This suggests that there are a certain number of students who did not perform well in the online environment, but also those who were helped to perform better. Through the method of interviewing students, both models have uncovered key factors in the teaching process that are believed to be the main causes of student success or failure.

Keywords: online learning, onsite learning, grades, correlation, mathematics

1. Introduction

At the start of the 2019/2020 summer term, significant restrictions were introduced by the Covid 19 pandemic, forcing us, like the rest of the world, to switch entirely to online learning. In April 2020, the Croatian Ministry of Science and Education issued instructions for assessment and evaluation in distance education (MSE, 2020). For assessment in higher education, it is recommended to replace colloquia

with online tests consisting of assignments from the assignment database so that each student receives their assignments. It is recommended that the database of questions from which the tests are generated should be large as this will ensure less re-writing. Generating questions based on random selection and changing the order of answers with the time limit within which the test is available and providing one question per page can ensure relatively good reliability of the test.

The purpose of this paper is to investigate whether there is a difference in students' grades between online and onsite classes. To do so, grades in two consecutive courses in the academic year in which both courses were taught in a faculty setting will be compared to grades in the same courses one year later when one course was taught in a faculty setting and the other in a distance learning setting.

2. Literature on challenges in higher education for the pandemic period COVID-19 disease

To find out how students and staff of higher education institutions cope with the new realities and challenges they face in times of a pandemic COVID-19, the Agency for Science and Higher Education (ASHE) conducted a survey Challenges in higher education during a pandemic COVID-19 diseases and social isolation in June 2020 (Bezjak et al., 2020). According to the report, half of the students (50 %) think that the quality of online teaching through lectures is much worse and worse than before the quarantine. 23 % think it is as good as before and 24 % think it is better and much better than before the quarantine. 38 % of university staff consider the quality of lectures to be the same as before the quarantine, 27 % slightly worse than before the quarantine and 39 % of staff slightly better than before the quarantine. The majority of students (42 %) feel that the ability of lecturers to interact with students in a virtual environment is much worse and worse than before the quarantine, 31 % feel it is as good as before and 26 % feel it is better and much better than before the quarantine. In terms of interacting with students in the virtual environment, 44 % of staff say it was worse or much worse than before the quarantine, 24 % of staff say it was the same as before the exceptional circumstances, while 28 % of staff think it was better or much better than before. In terms of teacher workload, 46 % of staff think it was higher or much higher than before the quarantine, 27 % think it was the same, while 19 % said it was less or much less than before the quarantine.

In October 2021, the Agency for Science and Higher Education (Croatia) conducted a survey Students and the Pandemic: How Did We Survive? (Đorđević et al., 2021) on the impact of the COVID-19 pandemic on the experience of studying in the 2020/2021 academic year. The survey found that the largest percentage of participants, 47 %, fully or mostly agree that online study provides more room for unethical behaviour in testing knowledge. 29 % of participants generally disagree or strongly disagree. 24 % of participants agree that online study provides as much room for unethical behaviour as onsite study.

The first transnational survey on the impact of the pandemic COVID-19 on students in the EHEA (European Higher Education Area) was conducted by a team

of researchers from the University of Zadar (Croatia) on behalf of the European Union of Students and supported by the Institute for the Development of Education (Croatia) and the Croatian Ministry of Science and Education (Doolan et al., 2021). A total of 17116 respondents from 41 European countries completed the questionnaire. Countries with a higher number of respondents include Portugal (6652), Romania (3110), Croatia (2029) and the Czech Republic (1768). The results of the survey from ESU show that almost half of all students surveyed (47.43 %) believe that their academic performance has deteriorated since the removal of on-site classes, and more than half of the students surveyed said that their workload has increased since the transition to online teaching and learning. Only 19.04 % reported a lower workload than before, while 25.46 % noted no change.

3. Methodology

The Faculty of Engineering in Rijeka offers undergraduate and graduate university study programs in Mechanical Engineering, Naval Architecture, Electrical Engineering and Computer Engineering. Mathematics 1 is a winter semester course and Mathematics 2 is a summer semester course in the first year of all study programs, with over 300 students attending lectures and exercises. Although officially one subject is not a necessary prerequisite for enrollment in another, knowledge of the material of Mathematics 1 is necessary for successful mastery of Mathematics 2. That is, one course is a natural continuation of the other.

Until Spring 2020, when Pandemic Covid 19 began, only onsite classes were held, along with midterms and final exams. During the semester, three midterm exams were held in which 70 % of the course grade could be achieved. Because of the large number of students, four different exams, each with eight questions, had to be prepared for each midterm. It is necessary to obtain at least 50 % of the achievable points in order to qualify for the final exam, which is also held at the faculty and accounts for the remaining 30 % of the course grade. A score of at least 50 % on the final exam is required to pass the course. For many years, virtual e-learning environment in higher education Merlin, based on the Moodle platform, has been used to support faculty courses. In the pre-pandemic times, the environment was typically used for sharing course materials and information with students and faculty, and as a virtual record of students' grades.

In the summer term of the academic year 2019/2020, when the Covid 19 pandemic broke out, we had to quickly switch to distance learning, which we still practice due to restrictions associated with a large number of students enrolled. We organized distance learning in the following way. The lectures are delivered in real time via Big Blue Button, an open-source distance learning tool, and recordings are made available to students until the end of the semester. The auditory tutorials are delivered via video lessons on Merlin and consist of recorded videos of teachers solving problems using a graphics tablet, as well as tasks that students must complete independently to advance through the lesson. The distribution of points in the courses has also changed, so that under the new rules 50 % of the grade can be obtained in the midterms and the remaining 50 % in the final exams.

Perhaps the biggest challenge for teachers was designing online tests for a large number of students. The midterm exams were designed as online tests within the Merlin platform. Using formula type question, we created a database with about 40 tasks for each midterm. However, since the questions depend on a number of randomly generated parameters, we actually created more than 6000 different tasks this way. In the beginning, creating such questions was very time-consuming, but in the meantime, we have perfected and optimized the preparation process in such a way that creating an online exam takes no more time than creating an onsite exam. Final exams were still held at the Faculty.

Figure 1 shows an example of several tasks generated from a single Formula question type object on Merlin. In this way we achieved that each student gets different test, and that the evaluation is as objective as possible. For some questions, students had to upload a picture of detailed solutions, which were later reviewed by tutors.

Za svaku od ponuđenih diferencijalnih jednačbi odredite metodu kojom je moguće riješiti tu jednačbu:

- $2y' + \frac{y}{x} = x^2 \cos x$
- $y' - \frac{3y}{x} = \sin^2\left(\frac{y}{x}\right)$
- $5y' - \frac{y}{x} = y^4 \sin x$
- $4x \sin(x)y' - y^2 = 0$

Za svaku od ponuđenih diferencijalnih jednačbi odredite metodu kojom je moguće riješiti tu jednačbu:

- $4x \cos(x)y' - y^3 = 0$
- $2y' + \frac{y}{x} = x^3 \sin x$
- $y' - \frac{4y}{x} = \sin^3\left(\frac{y}{x}\right)$
- $2y' - \frac{y}{x} = y^2 \ln x$

separacija varijabli
supstitucija y=u x
supstitucija y=u v
direktna integracija

Figure 1. Example for the question type Formula on Merlin.

In this paper we use the basic techniques of descriptive statistics in the form of calculation of proportions, graphical representations and calculation of statistical indicators. We use the chi-square test to examine the statistical significance of differences in the proportions and distributions of the data. Since the studied results show normality, the study of differences between the indicators of central tendency is carried out using parametric methods, i.e. the differences in the indicator of central tendency are determined using the t -test. The analysis of correlation between variables is expressed by Pearson's correlation coefficient. Statistical data analysis was performed using the software package MS Excel.

4. Results

After the final exams, we were interested to see if and in what way the pass rate changed due to the new circumstances and the short period of adjustment to the new work regime available to students and professors. In Table 1, we can see that the pass rate for Mathematics 1 does not show statistical significance in either of the observed academic years.

Table 1. The pass rates.

	MAT 1 2018/19 onsite	MAT 1 2019/20 onsite	p	MAT 2 2018/19 onsite	MAT 2 2019/20 online	p
number of students	386	338		314	306	
≥ 50 % on midterms	230 (60 %)	197 (58 %)	0.6382	198 (63 %)	227 (74 %)	0.0092
≥ 50 % on final exam	219 (57 %)	185 (55 %)	0.6413	194 (62 %)	182 (59 %)	0.5379

In the Mathematics 2 course, a significantly greater number of students earned the right to take the final exam ($p = 0.0092$) when the midterm exams were administered online, but the final pass rate was ultimately the same ($p = 0.5379$).

The numerical indicators of descriptive statistics (Figure 2) also show that there is no significant difference between the final grade in the two observed semesters with different teaching methods, onsite and online ($p = 0.9265$ for Mathematics 1 and $p = 0.9632$ for Mathematics 2).

MAT1 2018/19 onsite		MAT1 2019/20 onsite		MAT2 2018/19 onsite		MAT1 2019/20 online	
Mean	43,0233161	Mean	43,17825	Mean	45,92723	Mean	45,83824
Standard Error	1,15294585	Standard Error	1,220125	Standard Error	1,402804	Standard Error	1,320553
Median	51,5	Median	51,125	Median	53,375	Median	54,44
Mode	60	Mode	50	Mode	60	Mode	60,33
Stand. Deviation	22,6517919	Stand. Deviation	22,43173	Stand. Deviation	24,85775	Stand. Deviation	23,10025
Sample Variance	513,103676	Sample Variance	503,1823	Sample Variance	617,9076	Sample Variance	533,6214
Kurtosis	-0,93090315	Kurtosis	-1,06172	Kurtosis	-0,90598	Kurtosis	-1,23343
Skewness	-0,08507687	Skewness	-0,00387	Skewness	-0,02312	Skewness	-0,16011
Range	95	Range	90,75	Range	94,75	Range	87,33
Minimum	5	Minimum	5,75	Minimum	5,25	Minimum	5,67
Maximum	100	Maximum	96,5	Maximum	100	Maximum	93
Sum	16607	Sum	14594,25	Sum	14421,15	Sum	14026,5
Count	386	Count	338	Count	314	Count	306

Figure 2. Numerical indicators of numerical statistics.

The distribution of final grades (Figure 3) for Mathematics 1 shows no statistical difference between the years of study analyzed ($p = 0.9882$). We have the same results for Mathematics 2 ($p = 0.1900$), but with a significantly lower p -value. Descriptively, there were slightly more C grades than D grades when

midterm exams were administered via online testing. The proportion of A grades was also descriptively smaller in the pandemic year.

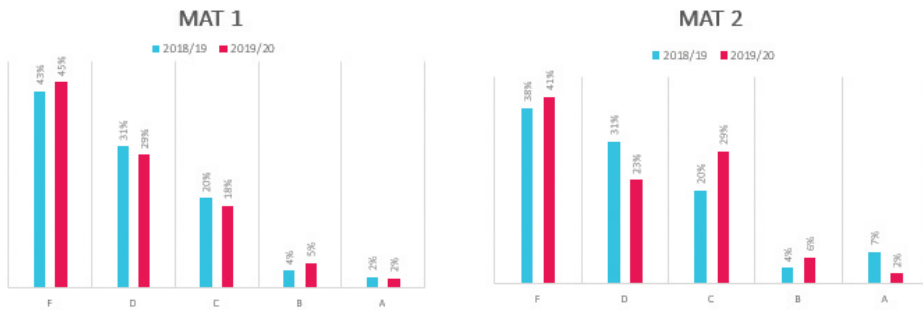


Figure 3. Distribution of final grades.

Although at first glance most indicators show that the pass rate of the online course is not significantly different from the results of the traditional class, a comparison of the final grades of students who took both courses in the same year shows a different picture. In academic years where there were only onsite courses, there is a moderate to strong correlation between final grades in both courses (Figure 4). Here we see an example for 2018/19, but this is also true for earlier years.

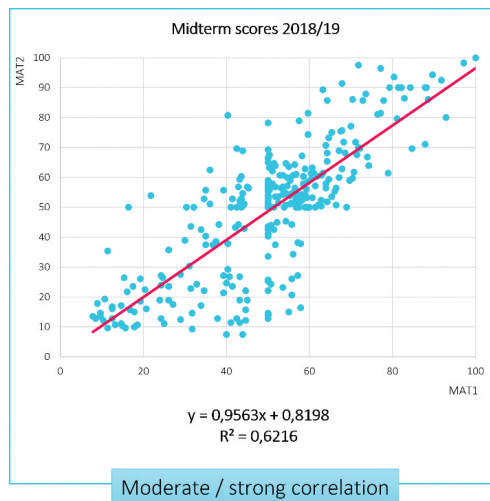


Figure 4. Correlation between midterm scores when both courses were held at the Faculty.

However, if one of the courses was taken in a faculty environment and the other in a distance learning environment, there is no correlation between the final grades. This fact is very surprising considering the results of the last few years.

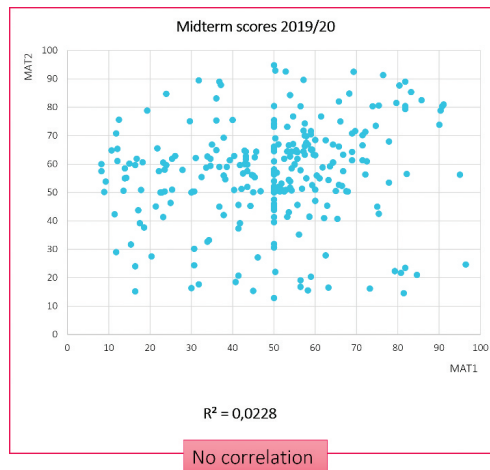


Figure 5. Correlation between midterm scores when one course was delivered onsite and the other one online.

Based on the relevant values of the t -test comparing the expected values for the distribution of paired data shown in Figure 6, at a statistical significance level of 5 %, we cannot reject the assumption that there is no difference between the mean scores of the midterm exams in two courses in the academic year 2018/2019. On the other hand, the same test conducted for the academic year in which a mixed onsite/online model was used shows that at the 5 % significance level, we can reject the assumption that the mean scores of the grades are the same. In other words, the expected grade in the case where the onsite instruction technique was used in both courses can be considered the same, while this is not true for the combined onsite/online approach.

2018/19 – onsite midterms			2019/20 – onsite and online midterms		
	Variable 1	Variable 2		Variable 1	Variable 2
Mean	50.05012531	48.68496	Mean	49.14311879	56.40147
Variance	345.9912987	509.0766	Variance	363.5792504	299.4972
Observations	285	285	Observations	273	273
Pearson Correlation	0.788416364		Pearson Correlation	0.151135305	
Hypothesized Mean Difference	0		Hypothesized Mean Difference	0	
df	284		df	272	
t Stat	1.657669632		t Stat	-5.052857768	
P(T<=t) one-tail	0.049244154		P(T<=t) one-tail	3.99823E-07	
t Critical one-tail	1.650236662		t Critical one-tail	1.650474964	
P(T<=t) two-tail	0.098488308		P(T<=t) two-tail	7.99645E-07	
t Critical two-tail	1.968352158		t Critical two-tail	1.968723847	

Figure 6. Paired t -test.

This suggests that there are a significant number of students who performed worse or better in the online environment than onsite.

We were interested in students' opinions on the difficulties and benefits of online learning that they encountered. Students expressed their thoughts in an end-

of-semester survey. For example, students indicated the key factors in the teaching process that they felt were the main causes of student success or failure. They indicated as a positive that the lectures and auditory tutorials are always available so that if they do not understand part of the lecture, they can go back and watch it again. They also indicated that students are less hesitant to ask questions in the online environment. The difficulties they mentioned with online learning were the limited ability to manage their time during the exam because they had to solve the tasks in the given order, i.e. the fact that they could not see all the tasks at the beginning of the exam and thus choose the best solution strategy based on what they had learned. They also stated that the online class was too fast paced and that they had to invest more time in learning. They cited insufficient communication and interaction with colleagues and professors as the biggest shortcoming of online learning.

As we continued with online instruction the next academic year, we took some of the students' comments into account. We kept the order of questions on the test, but split the 10-question test into two 5-question tests according to the estimated time needed to solve them, and clearly stated the types of questions in advance. We have also divided the lessons into smaller units to make them easier to follow.

5. Discussion and conclusion

The purpose of this paper was to determine if there is a difference in student grades between online and onsite courses. To do so, grades in two consecutive courses in one academic year in which both courses were taught in a faculty setting were compared to grades in the same courses one year later in which one course was taught in a faculty setting and the other in a distance learning setting.

The pass rate was very similar in both years of study observed. In the year in which one course was taught online, a greater number of students earned the right to take the final exam, but a smaller number of students passed the final exam, so the pass rate was ultimately the same. This is consistent with the study (Paul & Jefferson, 2019) comparing learning via F2F and online learning modalities when teaching an environmental science course. Russell's publication, *The No Significant Difference Phenomenon* (Russell, 2001), also supports minimal differences in outcomes between online and onsite courses. The same is claimed by the study (Summers et al., 2005) whose results state that students taking statistics online learn as much as students in a traditional onsite course and the study (Alghazo, 2005) whose results show that there is no significant difference in the effectiveness of distance education and traditional classroom education.

Although the pass rate was very similar, a comparison of the final grades of students who took both courses in the same year shows a different picture. In academic years where there were only onsite classes, there is a moderate to strong correlation between final grades in both courses. However, when one of the courses was taken in a faculty setting and the other in a distance learning setting, there is no correlation between final grades. This suggests that there are a significant number

of students who did worse or better in the online environment than onsite. We can relate this to research findings (Carnevale, 2002) that students who took an online course at Michigan State University did not do as well as students who took the same course onsite.

Our students expressed their opinions about online learning in an end-of-semester survey. They indicated as a positive that the lectures and listening exercises can be accessed at any time, so if they did not understand part of the lecture, they could go back and watch it again. They also indicated that students are less hesitant to ask questions in the online environment. This is consistent with a study (Warschauer, 1997) that advocated interaction in online environments because there is less opportunity for intimidation between individuals and also less time pressure than in onsite classes. The difficulties they cited with online learning was that they had to invest more time in learning. Mellon (2003) raises the question of whether or not the learning styles of all students are necessarily compatible with online instruction. He points out that many students seem to find it difficult to succeed in an environment where instructors cannot rely on onsite interaction to motivate and build rapport with students. Spitzer (2001) agrees, pointing out the obvious: "Fancy graphics alone cannot sustain student interest and motivation for long." Our students cite insufficient communication and interaction with colleagues and professors as the biggest shortcoming of online learning. This is consistent with a study (Northrup, 2001) that asserts that increased interaction between teachers and students (as well as between students themselves) promotes student engagement in online learning contexts. One reason for the importance of online interaction is that learners experience a "sense of community" (Rovai, 2002). Also Russell (2001) believes that "Students do not always value education primarily for its academic content. Rather, for many it is an opportunity to meet their friends and socialize".

From this we can conclude that the overall success rate in courses does not differ between the two teaching methods (onsite and online), but there is a significant percentage of students who performed worse or better. This means that, in general, neither teaching method can be considered dominant. However, it has been shown that some students perform better in the online courses while others still prefer the traditional form of teaching, which raises the question of how to individualize teaching when the pandemic stops in order to improve the quality of teaching.

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Postoji li razlika u ocjenama studenata kada se nastava održava na daljinu?

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Sažetak. Prethodna akademska godina bila je izazovna, kako za studente tako i za profesore. Pandemija uzrokovana virusom Covid 19 uzrokovala je kaotičnu 2020. godinu te će utjecati i na naredne godine. Mnoga sveučilišta rade planove za nastavak učenja na daljinu te je sada pravo vrijeme za evaluaciju nastave u online okruženju, njenih prednosti i nedostataka. Cilj ovog rada je ispitivanje postojanja razlika u uspjehu studenata kada se nastava održava u učionicama od učenja u online okruženju. Uspoređuju se ocjene kolegija Matematika I i Matematika II u akademskoj godini kada se nastava iz oba predmeta izvodila u učionicama i u godini u kojoj se nastava iz kolegija Matematika I održavala u učionici, a iz kolegija Matematika II na daljinu. Dobiveni statistički pokazatelji ukazuju da je prolaznost gotovo ista bez obzira na oblik izvođenja nastave. Međutim, koeficijent korelacije za ocjene iz kolegija Matematika I i Matematika II je statistički značajno manji kada se nastava iz prvog predmeta održavala u učionicama, a iz drugog u online okruženju, što ukazuje na postojanje određenog broja studenata koji se nisu dobro snašli u online okruženju, kao i onih kojima je isto pomoglo da ostvare bolje rezultate. Metodom intervju sa studentima došlo se do ključnih čimbenika nastavnog procesa u oba modela za koje se pretpostavlja da su glavni uzroci studentskog (ne)uspjeha.

Cljučne riječi: učenje na daljinu, nastava u učionici, ocjene, korelacija, matematika

Attitudes Towards Online Learning Among Students of the Faculty of Civil Engineering and Architecture Osijek

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Abstract. This paper presents an analysis of an online course *Mathematics for Engineers II* at the Faculty of Civil Engineering and Architecture Osijek. Due to a pandemic caused by the coronavirus SARS-CoV-2, after only two weeks of traditional classes, it was necessary to switch to an online performance that lasted until the end of the semester, without any prior preparations. Such a work brought many new challenges, and in the end, it was important to analyse the students' opinions as main actors in this process.

The paper contains a quantitative and qualitative analysis of the data collected from the students through an online survey questionnaire. It has been shown that there are some differences in attitudes towards online course according to gender. It has also been shown that students who have passed this course are more satisfied with their personal engagement and work in the online course as well as what they have learned in the online course. It also analyses students' technical requirements for participating in distance learning, satisfaction with communication and collaboration among students and teacher as well as between students themselves, and student satisfaction with the selection of tools and activities used in the online course.

Keywords: online teaching and learning, mathematics, survey questionnaire, attitudes, undergraduate students

1. Introduction and preliminaries

Everyone agrees that the 2020 situation caused by the Covid-19 virus was a milestone for education around the world. With the outbreak of the pandemic, in Croatia, as in most countries in the world, to increase health security measures at

the national level, it was decided that all education (including faculties) would be switched to online education on March 21. This happened suddenly, and it was up to the faculties and teachers to decide how they wanted to implement online teaching. Although we do not have detailed knowledge of the use at that time (which was probably very different) of any form of online instruction at our university, we do know that in the last twenty years, the use of e-learning in higher education has shown a great upward trend (Yen & Lee, 2011). Also, the level of preparation and training of teachers in higher education, that is, the level of their digital literacy for such a way of working, was very questionable. A survey conducted among university teachers in 58 countries showed a great difference in their readiness (Scherer et al., 2021). But in such a situation, there was no space or time for additional education or training as quick action was expected.

However, such a challenging situation around the world has opened up space for new research in this area. Scherer et al. (2021) analyse several elements that affect readiness for online instruction, first at the state level and second at the level of each individual teacher. It is the state that promotes, encourages, and invests in the potential for innovation in education. The cultural orientation of the state such as individualism versus collectivism, long term versus short term plans, the safety index also leaves visible traces in creating readiness for online teaching. It is clear that there are differences in academic disciplines as well. Different institutional decisions may have a positive or negative impact on online teaching. However, the final performance depends on individuals who have different teaching experiences and different levels of experience and prior knowledge of online learning and teaching (Scherer et al, 2021). Another factor that influenced the readiness and success of online teaching during the pandemic was the attitude of teachers towards the new situation. Positive perceptions of the transition to online teaching as a challenge to personal progress were found to be positively associated with better goal attainment. In contrast, perceptions of the new situation as threatening were positively related to levels of burnout and negatively related to students' ratings of the quality of instruction (Daumiller, 2021).

Different learning methods also come into play when teaching online. For example, younger students prefer interactive online activities, while older students prefer learning through pre-recorded video material (Simonds & Brock, 2014). It has been shown that pre-recorded video materials in online college courses are one of the key elements that improve student achievement performance. For example, in the *Algebra* course, scores on all tests (online and written) were statistically significantly better for students who took a redesigned course with video instruction than for students who took an older version of the same course without pre-recorded video instruction (Hegeman, 2015). Even at the high school level, watching recorded video lessons has been shown to improve mathematics achievement (Jefferson, 2021). Of course, many other elements also impact the success of online courses and student satisfaction with them. Teaching presence in an online subject has been shown to be an important predictor of undergraduate and graduate students' affective learning, cognition, and motivation (Baker, 2010). It is clear that teaching presence is easier to achieve in the synchronous form of teaching, but this does not preclude it from being achieved in the asynchronous form. In

this second case, to achieve teaching presence, the teacher is expected to answer students' questions as soon as possible, organize and monitor the implementation of all activities in the online course in a timely manner, and inform students in a timely manner of all expected commitments during the online course, i.e., promote timely and meaningful communication both horizontally (peer-to-peer) and vertically (student-to-teacher and vice versa) (Davis, 2014). In addition to the extremely important role of the teacher, clearly important to the management of a successful online course are its design and good structure, both of all teaching content and of all activities aimed at promoting teaching presence and increasing student motivation. For example, it is noted that the teaching content should be divided into smaller units and then organized into similarly structured chapters. Designing an online course in this way helps students focus on the actual content without additional distractions that a poorly organized structure might provide (Hegeman, 2015).

In addition to the traditional face-to-face and online instruction described in this paper, there are other transitional forms. Of particular note are the blended and hybrid modes (Figure 1). There is an important distinction between these two combined modes of learning: "Unlike hybrid learning, blended learning uses online instruction to complement or supplement traditional face-to-face instruction, not replace it" (Reed, 2020). Common to the forms of learning that use technology is that they require students to be more responsible and engaged, i.e., they encourage active learning (Reed, 2020). In addition, teachers are expected to understand the online teaching process, which does not operate in the same way as traditional teaching, and to have sufficiently developed skills to be able to create high quality and stimulating teaching content (Krishnan, 2016).

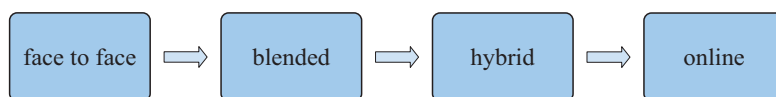


Figure 1. From face-to-face to online teaching.

The students' preferences regarding the form of teaching are not clear, and often lead to ambiguous results. For example, Davis emphasizes in his dissertation that his study found no advantage of any form of teaching (traditional, hybrid, or online) in terms of student success and satisfaction (Davis, 2014). Sometimes students prefer face-to-face teaching over hybrid teaching because they indicate that they can better understand mathematical concepts in face-to-face teaching (Krishnan, 2016). In contrast, a study by Greek scholars on a group of undergraduate students concluded that most students prefer the online form of teaching over traditional form (Vernadakis et al., 2012).

As described earlier, online delivery of learning content is very complex and involves many different approaches (Sengil et al., 2021). The purpose of this paper is to analyse students' attitudes towards different performance parameters of an online course. That is, to consider all the advantages and disadvantages that occurred during that semester in order to incorporate positive elements in further work.

2. Methodology

2.1. Research Design

This paper is primarily concerned with student attitudes toward online instruction in the *Mathematics for Engineers II* course, which was delivered in the usual manner for only the first two weeks and then abruptly changed to an online delivery that unexpectedly lasted until the end of the semester. The author of this paper has been using the online course for ten years as a supplement to regular teaching in the teaching of *Mathematics for Engineer I* and *II* at the Faculty of Civil Engineering and Architecture Osijek, J. J. Strossmayer University in Osijek to promote active learning in blended mode. Many teachers successfully use Modular Object-Oriented Dynamic Learning Environment, Moodle for short, as a very useful tool to promote active learning (Shoufan, 2020). The aforementioned online course was created using the Moodle learning management system. The online course has been systematically added to over the years by providing more and more teaching content, but also activities such as short quizzes for self-evaluation of knowledge and activities in Geogebra. By incorporating the communication and collaboration tools of the forum and wiki, students were encouraged to explore some mathematical problems (Matotek, 2020). The workshop was a good example of promoting peer evaluations, where students evaluated each other's previously created mind maps in addition to the well-prepared rubrics.

In the full transition to online teaching, the prior regular use of Moodle proved to be a great advantage. From a technical point of view, it was not necessary to form a group to inform students about the place and method of teaching, as all this was already done at the beginning of the academic year. In fact, the same group of students attended the *Mathematics for Engineers I* course with the same teacher in the first semester. Thus, the students were in a familiar environment, so it was relatively easy to set up a new communication and notification system, and the adaptation to the online environment went smoothly.

Somehow, at the time of the transition to online teaching, the Croatian capital of Zagreb was hit by a major earthquake with a magnitude of 5.5, according to the geophysical department of the Faculty of Science in Zagreb. Some students live in the area affected by the earthquake and had problems with their internet connection, which was very unstable. Moreover, the Moodle system used at that time was working very slowly due to the very large number of users at the same time. In view of the above problems, the teacher decided to use an asynchronous form of teaching. For this reason, the teacher regularly posted video lessons that he recorded herself using a screen-touch laptop. In addition to the theory, the tasks were solved on a whiteboard and a laptop screen, and the voice of the teacher explaining the process of solving the tasks was recorded. Students were given a weekly task to submit in Moodle in .jpg or .pdf format (depending on whether they had scanned or photographed the solved task in a notebook with their mobile phone). Approximately every other week, a real-time videoconference class was held (synchronous mode) in which the tasks from the previous lessons that the students had indicated as a problem for them were solved.

2.2. Instrumentation

At the end of the semester, a survey questionnaire was conducted in Moodle. Students completed the questionnaire voluntarily and anonymously. The questionnaire consisted of a part that collected demographic data and a part that investigated students' attitudes towards the online course held. The second part contained several open-ended questions, several multiple-choice questions, and one question that used a 5-point Likert scale (from 1 – *I strongly disagree* to 5 – *I strongly agree*) for 20 items.

In data processing, the program for statistical data processing SPSS 25 was used. Data processing included quantitative processing but also qualitative data processing. Descriptive statistics methods and non-parametric Mann-Whitney test were used for quantitative data analysis. The significance level chosen was $\alpha = 0.05$. For the reliability of the questionnaire, the Cronbach-alpha coefficient was calculated, and it was 0.857. Excel was used for visual representation of data on Likert scale.

3. Result and discussion

Before presenting and discussing the results, we would like to point out some of the methodological limitations observed in this research. Although the questionnaire was filled out by a large percentage of students who completed the online course, it was found that not all Likert scale items were answered by all respondents. In fact, the difference in the number of items answered varied greatly by item. In the future, the submission of the entire questionnaire should be made conditioned on the response to each item.

Some inconsistencies appeared in the responses regarding students' preferences for the type of instruction, i.e., traditional, or online. One possible reason for this is the construction of the items.

3.1. Demographic properties of students

The conducted survey was completed by $N = 47$ students out of 78 first-enrolled students in the first year (60.26 %). However, the actual percentage of students who completed the survey is even higher because, unfortunately, a certain number of students dropped out by the end of the year.

Table 1 provides an overview of the demographic characteristics of the respondents by gender, type of study and type of graduated high school. It is noted that there were about 15 % more female respondents. The ratio between the number of full-time students and the number of part-time students who responded to the survey is 3.2, which is only slightly lower than their ratio for the number of first-enrolled students, which is 3.3. The questionnaire on the type of graduated high school offered more choices than those listed in Table 1. Namely, gymnasiums

were divided into mathematics, general education, and other. However, since no respondent indicated a mathematical gymnasium, and only a small number indicated the other two choices, all types of gymnasiums were considered as one group for further data processing.

Table 1. Demographic properties of students.

	Property	f	%		Property	f	%
Gender	Female	27	57.4	Type of graduated high school	Gymnasium	8	17
	Male	20	42.6		Civil engineering school	21	44.7
Type of study	Full-time students	36	76.6		Other vocational schools	16	34
	Part-time students	11	23.4		Other	2	4.3

3.2. Descriptive statistic

The survey questionnaire also contained 20 questions on a 5-point Likert scale (from 1 – *I strongly disagree* to 5 – *I strongly agree*). Tables 2–6 show all 20 questions with the corresponding mean score. It should be noted that the number N , i.e., the number of responses, varies considerably from question to question (a certain number of students did not answer all questions). In the analysing the responses, the items were grouped into 5 categories based solely on their similarity in content: Communication (Table 2), Self-Evaluation (Table 3), Teaching (Table 4), Technical Limitations (Table 5), and Preferences (Table 6). The frequencies of all responses (except for the missing responses) were examined and presented in a diverging stacked bar chart. By presenting the data in this way, we can best highlight the difference between positive and negative responses, with neutral responses assigned to positive or negative group depending on the question (Pirrone, 2020).

In a follow up there is an analysis of the responses by category.

3.2.1. Communication in the e-course

From Table 2, it can be seen that item (C3) “I addressed a friend on other social networks” had a very high mean of 4.46, indicating that students communicated extensively with each other about course content outside of Moodle. There were no students who disagreed with the statement on this item. The lowest mean score of 3.00 out of all 20 items came from this category (C1): “I used the opportunity to ask questions on the forum”. Figure 2 shows that only 34 % of students used such an option. The low level of activity in the optional forums is much more common than one might expect (Matotek, 2020). This suggests that students were

much more relaxed in communicating with each other without the presence of the teacher.

Table 2. Descriptive statistics of items C1 – C4.

	N	Minimum	Maximum	Mean	Std. Deviation	Variance
C1 – I used the opportunity to ask questions on the forum	44	1	5	3.00	1.08	1.16
C2 – I asked the teacher all the questions	39	1	5	3.21	1.06	1.12
C3 – I addressed a friend on other social networks	24	3	5	4.46	0.72	0.52
C4 – I am satisfied with the communication with the teacher	26	1	5	4.15	1.05	1.10

However, almost 50 % of the students turned to the teacher (C2) because, in addition to the forum, they could also communicate with the teacher by e-mail and through the Moodle messaging system, which some of them obviously liked better. Regardless of the mode of communication, 80 % of the students were satisfied with the communication achieved with the teacher in the online course (C4).

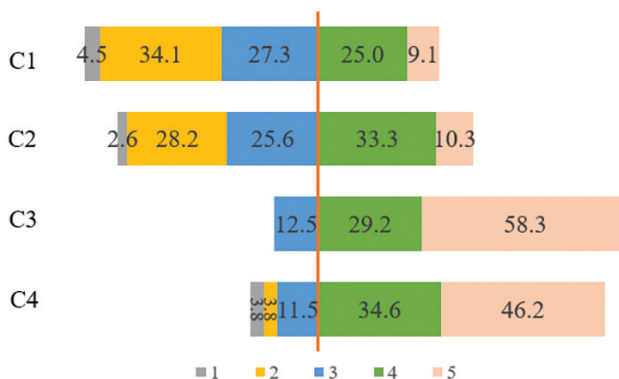


Figure 2. Frequencies (in percentages) of the Communication category in the e-course.

Students were also asked about the reasons why they were not active on the forums. 70 % of them answered that they would rather ask their questions directly to friends, 48 % of them said that they were uncomfortable asking questions in public in a group, 23 % of them preferred asking their questions directly to the teacher rather than on the forum, 18 % did not have time, and 14 % learned enough by reading others' messages. Only 5 % of the participants said that they did not see any benefit in asking questions on the forum.

3.2.2. Activity self-evaluation

In the next observed category, items related to self-evaluation of one's own activities during the semester in the mentioned e-course are singled out (Table 3).

Table 3. Descriptive statistics of items S1 – S4.

	<i>N</i>	Minimum	Maximum	Mean	Std. Deviation	Variance
S1 – I am satisfied with what I learned in online teaching	43	2	5	3.84	1.02	1.04
S2 – I am satisfied with my personal engagement and work on the e-course	36	2	5	4.06	0.98	0.97
S3 – I gladly joined the course every time	35	2	5	4.14	0.91	0.83
S4 – I performed my duties regularly	34	1	5	4.15	1.08	1.16

Looking at the item frequencies in Figure 3, we can see that none of the students chose the option of being completely dissatisfied, even for three of the four items. The only statement with which only one student was completely dissatisfied was, “I performed my duties regularly” (S4).

In the online course, special attention was paid to some other interesting and stimulating optional content and activities related to mathematics, but not to the teaching material. Probably, such a working approach also contributed to the fact that as many as 82 % of the students every time gladly joined the course (S3). Further, 67 % of the students are satisfied with what they learned in the online course (S1), while 72 % of them are satisfied with their engagement in the e-course (S2).

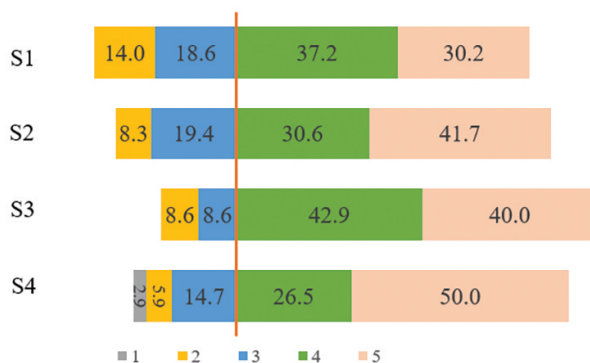


Figure 3. Frequencies (in percentages) of category Self-evaluation.

3.2.3. About how to teach online

The percentage of those who gladly attended the course is probably related to the students' opinion that the course contained enough stimulating activities, because as many as 83 % of the students agreed with this statement (T4). The item "Teacher responded to messages regularly" (T3) has the highest mean of 4.58 in the entire questionnaire (Table 4). At the same time, it is the only item that all students agree or completely agree with. Given the importance of providing timely information for the functioning of this form of teaching, the author decided to put this item in this category (teaching) and not in the category Communication, as someone might suggest.

Table 4. Descriptive statistics of items T1 – T4.

	<i>N</i>	Minimum	Maximum	Mean	Std. Deviation	Variance
T1 – I am satisfied with the way the material was presented	31	1	5	4.03	1.05	1.10
T2 – I am satisfied with the pace of work	38	1	5	3.89	1.18	1.39
T3 – The teacher responded to messages regularly	12	4	5	4.58	0.52	0.27
T4 – The course contained enough stimulating activities	31	2	5	4.26	0.89	0.80

Furthermore, a large percentage of students were satisfied with the pace of work (T2, 74 %) and the way the material was presented (T1, 77 %) (Figure 4). Let us add that the pace of work was slightly slower than it would be in traditional classes. Experience shows that it is very important to determine the pace of work that students will be able to follow.

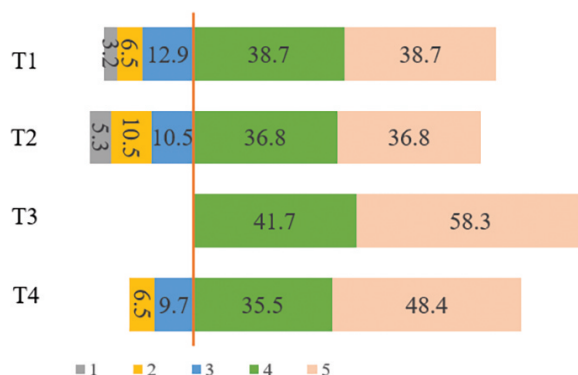


Figure 4. Frequencies (in percentages) of the category Teaching.

Additionally, students were asked to rate the usefulness of some of the activities for learning. The Likert scale was used, where a value of 1 represents *completely useless* and a value of 5 represents *completely useful*. From Figure 5, we can conclude that all the activities offered were useful to students, but by far the most useful learning materials were video materials, which none of the students rated as useless.

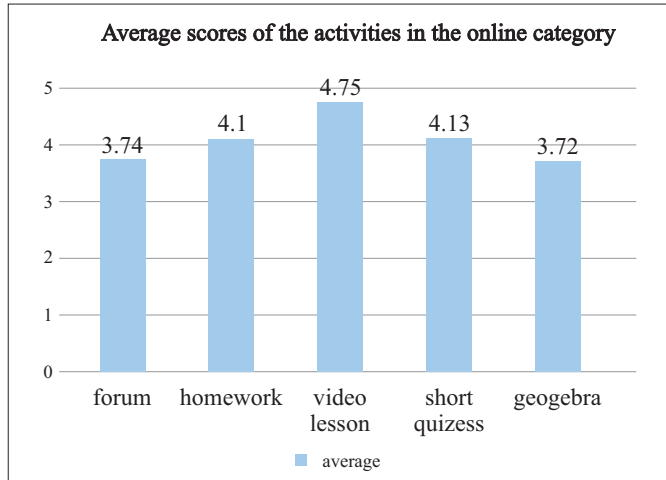


Figure 5. Average scores for some activities in the online course.

3.2.4. Technical limitations

Since some of the technical difficulties in the online class had already been described, a category on technical problems with three items was included in the questionnaire. The mean score of all three items was 3.72 (the median was 3.71), which confirmed that various technical difficulties were occasionally encountered (Table 5). It was necessary to be aware of this throughout the semester and to retrieve records of meeting deadlines in submitting assignments with a dose of tolerance and without disrupting the planned dynamics of the activities.

Table 5. Descriptive statistics of items L1 – L3.

	N	Minimum	Maximum	Mean	Std. Deviation	Variance
L1 – I had satisfactory working conditions at home	26	2	5	3.77	1.07	1.15
L2 – The internet connection at home was satisfactory	35	1	5	3.71	1.10	1.21
L3 – I did not have any technical difficulties	38	1	5	3.68	1.12	1.25

In this category, it is much more important to observe the percentage of those students who did not have satisfactory working conditions. From Figure 6, it can be seen that in the first item: “I had satisfactory working conditions at home” (L1), as many as 19 % of those students who answered this question disagreed. The problem also referred to the sharing of a computer among several household members, inadequate space to attend classes... For the remaining two technical questions, one person disagreed at all that the internet connection was satisfactory and that there were no difficulties, and disagreement was expressed by 14 % (L2) and 18 % (L3) of the students.

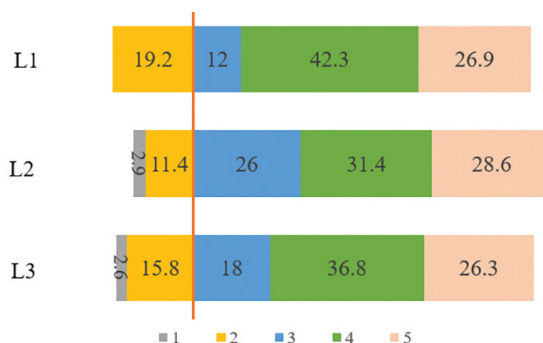


Figure 6. Frequencies (in percentages) of the category technical limitations.

Most students used laptops to follow online course (73 %), 16 % of students used a desktop computer, while as many as 11 % of students followed course only via smartphone, which is considered an inadequate solution.

3.2.5. Students' preferences in relation to traditional and online teaching

Although the observed e-course scored well on all the previous items, we were also interested in students' opinions about the performance of this course compared to traditional performance. Five items were observed in this group, with mean scores ranging from 3.10 to 4.11 (Table 6).

The mean of 3.10 in item (P5) “If I were to take this course again, I would prefer it to be delivered online rather than in the traditional manner” as well as the frequency distribution (in which the percentage of 41 % of students agreeing with the statement is exactly equal to the percentage of those disagreeing) do not indicate a preference for one form of instruction. Similarly, item (P2), “I spent more time learning in traditional classes than in online classes”. The mean of this item is 3.14 and a look at the frequencies in Figure 7 shows that the online course has only a slight preference.

In contrast, traditional instruction is given a more significant preference over online instruction in no less than three items. The range of agreement is between 71 % and 82 %, while disagreement is between 3 % and 10 %. Students believe that the instructions in traditional classes would help them more, that the explanation

Table 6. Descriptive statistics of items P1 – P4.

	N	Minimum	Maximum	Mean	Std. Deviation	Variance
P1 – If the classes were traditional as in the 1st semester, I would certainly have mastered the material better	30	2	5	4.10	0.92	0.85
P2 – I spent more time learning in traditional classes than in online classes	43	1	5	3.14	1.19	1.41
P3 – Instructions in traditional classes helped me more in my studies	28	2	5	4.11	0.79	0.62
P4 – I found it easier to understand the explanations of mathematical concepts in traditional classes	28	1	5	4.00	0.98	0.96
P5 – If I were to take this course again, I would prefer it to be delivered online rather than in the traditional manner	39	1	5	3.10	1.21	1.46



Figure 7. Frequencies (in percentages) of the category Preferences.

of mathematical concepts would be easier to understand in traditional classes, and that they would master the material better in traditional classes. It is unclear why, if they recognize traditional instruction as a form of teaching in which the information is more accessible and understandable to them, they do not choose such a form of

instruction as the primary form which they wish to take the course. Since inconsistencies in responses were observed in this category, it can be assumed that one of the causes is the construction of the items themselves, which confused students in their responses.

3.3. The influence of students' personal characteristics on their attitudes

In addition to descriptive analysis, nonparametric tests were used in the data analysis. The significance level chosen was $\alpha = 0.05$, meaning that if the observed p was less than 0.05, we rejected the null hypotheses and accepted the alternative hypotheses in the order indicated. The nonparametric Mann-Whitney test shows us that there is a statistically significant difference in the two items C3 and S3 with respect to gender (Table 7).

Table 7. Mann-Whitney test for means regarding gender.

	C3	S3
Mann-Whitney U	38.00	92.00
Asymp. Sig. (2-tailed)	0.03	0.03

From the analysis of the mean ranks by gender for the observed items (Table 8), it can be concluded that female students preferred to ask to each other for help and join the course more than male students.

Table 8. Mean rank by gender.

	Gender	N	Mean Rank	Sum of Ranks
C3	1: Female	13	15.08	196
	2: Male	11	9.45	104
S3	1: Female	19	21.16	402
	2: Male	16	14.25	228

Furthermore, it was found that students who had passed course *Mathematics for Engineers I* course in the previous semester were more satisfied with their own engagement in the online *Mathematics for Engineers II* course and with what they had learned in the online course (Table 9 and Table 10).

Table 9. Mann-Whitney test for mean values regarding passing the exam in Mathematics II.

	S1	S2
Mann-Whitney U	106.00	61.00
Asymp. Sig. (2-tailed)	0.01	0.06

Table 10. Mean rank regarding passing the exam in Mathematics I.

	Passed exam Mathematics I	N	Mean Rank	Sum of Ranks
S1	no	30	19.03	571
	yes	13	28.85	375
S2	no	25	15.44	386
	yes	11	25.45	280

From Table 11, we can see that the Man-Whitney test has shown that the p -value is less than 0.05 for all the three items in the category of Technical Limitations with respect to the grouping variable of types of study (full-time or part-time). This means that there are significant differences between these two groups. From Table 12, it can be concluded that in all the three items related to technical or working conditions, it was shown that part-time students had more difficulties than full-time students.

Table 11. Mann-Whitney test for mean values regarding the study type.

	L1	L2	L3
Mann-Whitney U	26.00	55.00	61.00
Asymp. Sig. (2-tailed)	0.01	0.03	0.03

Table 12. Mean rank regarding the study type.

	Study type	N	Mean Rank	Sum of Ranks
L1	part-time	7	7.71	54
	full-time	19	15.63	297
L2	part-time	8	11.38	91
	full-time	27	19.96	539
L3	part-time	8	12.13	97
	full-time	30	21.47	644

3.4. Qualitative analysis of student comments

The open-ended responses were examined in the qualitative analysis. Interestingly, a large percentage of students answered these questions even though they were not mandatory. The first question asked, "List the advantages and disadvantages of online math instruction. How would you improve it, would you expand some activities, reduce some, add something...?" The second question asked about suggestions for improving communication within the online course. The first question was answered by 49 % of the students, the second by 30 % of the students. There is a long list of their comments in which they emphasized the good points much more than the bad points. They highlighted the video lessons as good sites because they could watch, pause, and repeat them at a time that suited them. They liked the synchronous video calls very much, but some of them think that there should have

been more of them. The most common complaint they stated was poor internet connection. For some, this type of learning was suitable because there was no time pressure, and they could manage their time as they wanted. Attitudes in favouring one form of teaching over another (live or online) showed inconsistencies here as well as in previous analyses.

“...since everything is stored online, it’s harder to learn, whereas in the classroom it’s easier because we can interrupt you and ask you a question...”

“It was much harder to follow the material online than in the classroom.”

“The online course were better because they were more accessible.”

Another positive aspect that stood out in student comments was the emphasis on timeliness of answers and the professor’s high level of engagement.

“The good thing was that the professor was fully committed to her course.”

“We did not lack for teaching and learning materials, let alone contact with the professor, as she answered all questions regularly and always gave good advice. I personally was very satisfied with the online course!”

“I like the way and approach to the material; everything was set up regularly and was very thorough. You handled an unexpected situation well. The only drawback was perhaps the lack of assignments that were solved.”

In a previous analysis, we found that most of them were very satisfied with the communication during the online course, both horizontal and vertical, which the comments confirm.

“The good points are that we greatly improved our own communication, first with the professor and then with our colleagues.”

“Everything was great.”

4. Conclusion

With the sudden transition to online teaching in most parts of the world caused by the COVID-19 pandemic, it has become clear that teacher readiness is essential for such adaptation. There are many factors that influence this readiness, but also success in creating online course and ultimately student satisfaction with this type of instruction. Some of these elements were analysed in detail in this paper.

Students engaged in intensive exchanges about course content outside of Moodle without teacher supervision. Although they asked few questions on the forum, they communicated with the teacher through other channels and were overall satisfied with the communication achieved within the online course (80 % of them). In addition, 83 % of the students felt that the online course contained enough stimulating activities. A large percentage of students were gladly joining the online course (82 %) and were satisfied with what they learned. Students indicated that of all the activities in the online course, the combination of recorded video lessons

and synchronous instruction via video call was the most useful. Despite students' overall positive attitudes toward the way of conducting the online course, the results do not allow us to draw conclusions about students' preferences between online and traditional teaching and learning. Furthermore, student comments indicate that this mode of learning requires their greater engagement and involvement but that they also appreciate the professor's engagement.

Part-time students were found to have more technical difficulties following on-line instruction than full-time students. Some significant gender differences were also found. Female students were more likely to contact their friends on social media and prefer to take an online course than their male peers. Greater satisfaction with personal engagement, work, and learning in the online course was expressed by students who had passed the *Mathematics for Engineers I* course in the previous winter semester.

Great benefits of using recorded video lessons were also noted. They can play multiple roles in future classes, for example in cases where students miss class due to illness, but also as an important element in blended teaching that will encourage students to participate more actively and responsibly.

Considering that this academic year was also entirely online throughout the second semester, it would be interesting to examine student preferences regarding the type of instruction in light of their different learning styles. Certainly, close attention should be paid to exam results and pass rates. It would be important to examine whether there is a significant difference in exam passing rates with respect to the mode of teaching and learning.

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Stavovi studenata Građevinskog i arhitektonskog fakulteta Osijek o provedenoj online nastavi

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Sažetak. U ovom radu je predstavljena analiza provedenog online kolegija iz *Matematike za inženjere II* na Građevinskom i arhitektonskom fakultetu Osijek. Uslijed pandemije uzrokovane korona virusom SARS-CoV-2 nakon samo dva tjedna redovne nastave moralo se u potpunosti prijeći na online izvedbu koja je potrajala do kraja semestra, a bez ikakvih prethodnih priprema. Takav rad je donio puno novih izazova te je na kraju bilo bitno analizirati mišljenje studenata kao glavnih dionika tog procesa.

U radu je opisana kvantitativna i kvalitativna analiza podataka prikupljenih online anketnim upitnikom među studentima. Pokazano je da postoje neke razlike u stavovima prema izvedenoj online nastavi s obzirom na spol. Također, uočeno je da su studenti koji su položili taj kolegij zadovoljniji osobnim angažmanom i radom na online kolegiju kao i naučenim u online nastavi. Nadalje, u radu su analizirane tehničke mogućnosti studenata za praćenje nastave na daljinu, zadovoljstvo ostvarenom komunikacijom i kolaboracijom studenata sa nastavnikom te studenata međusobno kao i zadovoljstvo izborom alata i aktivnosti korištenih u online kolegiju.

Gljučne riječi: online nastava, matematika, anketni upitnik, stavovi, student

Transformation of Population Density Data in the Republic of Croatia

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Abstract. Mathematical modeling is used to create data model suitable for computer simulations in this paper. The paper presents the transformation of population density data of the Republic of Croatia from two sources. The first source was the *Mid-year total population estimate of the Republic of Croatia, by counties, 2019* and the second source was a digitized map of the Republic of Croatia with 21 complexly connected territorial units (20 counties and the City of Zagreb). Data from these two sources have been transformed into a simple data model of 16 zones with uniform population density, of different sizes, proportional to the number of inhabitants therein.

Data transformation is carried out in five steps. The first step is determining the optimal circular sector on a digitized map. The next step is dividing circular sector into an arbitrary number of matching circular sectors (16 in the paper). The third step is determining the intersection of each with county surfaces by counting common pixels. The following step is adding the inhabitants of the counties to the zones in proportion to their share in the area of the county. The final step is creating a new system of uniform density zones, of different sizes, and proportional to the population therein.

Keywords: digitized map, pixel, data transformation, intersection, ratios

1. Introduction

The geographical shape of the Republic of Croatia on the map and the territorial-political county division (local name: *županija*) are a consequence of historical and economic events. Counties population density varies. Layout and connectivity of counties are complicated. In times of geographical isolation, such as this one with the measures applied during the Covid-19 pandemic, the only possible connections among citizens become those that pass through the territory of the Republic of Croatia by road or sea.

Reshaping historical layout into simple model with different zones would simplify mobility simulations and models.

Counties population density properties will be adapted to the new model.

A census is scheduled for this year, so the latest official *Croatian bureau of statistics* estimate will be used.

2. Initial data

According to the population estimate for 2019 by counties, Croatia had the following population composition: Zagreb (1) – 309169; Krapina-Zagorje (2) – 124517; Sisak-Moslavina (3) – 145904; Karlovac (4) – 115484; Varaždin (5) – 166112; Koprivnica-Križevci (6) – 106367; Bjelovar-Bilogora (7) – 106258; Primorje-Gorski kotar (8) – 282730; Lika-Senj (9) – 44625; Virovitica-Podravina (10) – 73641; Požega-Slavonia (11) – 66256; Slavonski Brod-Posavina (12) – 137487; Zadar (13) – 168213; Osijek-Baranja (14) – 272673; Šibenik-Knin (15) – 99210; Vukovar-Sirmium (16) – 150985; Split-Dalmatia (17) – 447747; Istria (18) – 209573; Dubrovnik-Neretva (19) – 121816; Međimurje (20) – 109232; City of Zagreb (21) – 807254.



Figure 1. Counties of the Republic of Croatia.

The Republic of Croatia has a complex internal structure. Territorial units have a different number of connections to the other units, which vary from 1 to 7, depending on the individual territorial unit.

Structural representation is shown in Figure 2.

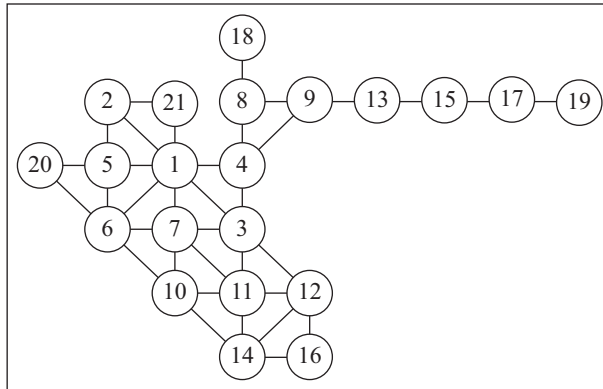


Figure 2. Connectivity between territorial units.

The initial data model is called **Data model A**, developed for further analysis. It is constructed of values which represent, earlier mentioned, individual county population and connections between them, shown as lines in Figure 2.

For example, A_4 has value 115484 and it is connected to A_1 , A_3 , A_8 and A_9 .

Modeling and building simulations for solving migration problems by using Data model A is highly complex.

3. Methodology and results

Digitized map used in this paper, shown in Figure 1, is 2201×2151 pixels, portable network graphics image (PNG). Boutell et. al. (1997) describe PNG, as extensible file format for the lossless, portable, well-compressed storage of raster images.

Colors are represented by RGB sample data with 8-bit color depth. Color space is defined as:

$$CS = \{(r, g, b) : r, g, b \in [0, 255] \cap \mathbf{Z}\}.$$

Each county is represented in different color $c_i \in CS$, where $i = 1, 2, 3, \dots, 21$.

For example, light yellow color used in the image to represent county 16 (Vukovar-Sirmium) is $c_{16} = (255, 255, 132)$.

Digitized map, shown in Figure 1, is created using coordinate system of transverse Mercator (Gauss-Krüger) projection – abbreviated HTRS96/TM, with a mean meridian of $16^\circ 30'$ and a linear scale on the mean meridian of 0.9999 respecting the *Decision on determining the official geodetic dates and planar cartographic projections of the Republic of Croatia* (NN 110/2004) and *Correction of the Decision on Determining Official Geodetic Dates and Plane Cartographic Projections of the Republic of Croatia* (NN 117/2004).

The definition of pixel has changed throughout history since William F. Schreiber used it in 1965. (Lyon, 2006). A pixel is the smallest element of a picture. It can only be one color at a time. In the picture it is defined by x -position, y -position, and color. Digital image is a set of pixels.

The map, shown in Figure 1, can be written as:

$$M = \{(x, y, c) : x \in [1, 2201] \cap \mathbf{N}, y \in [1, 2151] \cap \mathbf{N}, c \in CS\},$$

while counties are:

$$MA_i = \{(x, y, c) : x \in [1, 2201] \cap \mathbf{N}, y \in [1, 2151] \cap \mathbf{N}, c \in CS, c = c_i\} \subseteq M$$

where $i = 1, 2, 3, \dots, 21$.

Croatia, in the same image, is:

$$MA = \bigcup_{i=1}^{21} MA_i.$$

Population density in Data model A is not uniform and varies from 0.28 (county 9) to 44.48 (county 21) inhabitants per pixel.

In order to create a new connectivity optimized data model, data transformation is carried out in four steps presented in chapters 3.1. to 3.4.

3.1. Determining optimal circular sector on a digitized map

Substeps are:

Determining the smallest rectangle containing the Republic of Croatia figure, side of which lies on a geographical parallel.

Using photo editing software – shown as green lines in Figure 3.

Determining the intersection of the diagonals of a rectangle.

Using dynamic mathematics software – diagonals shown as dashed green and point marked as “E” in Figure 3.

Intersection is not contained within the Republic of Croatia figure.

Constructing rays from intersection of the diagonals of a rectangle which lies on tangential lines on the border of the Republic of Croatia from intersection of diagonals.

Using dynamic mathematics software – tangential lines shown as dashed red; rays shown as dashed blue in Figure 3.

The radius is arbitrary. The only restriction is that the circular sector must contain the entire Republic of Croatia.

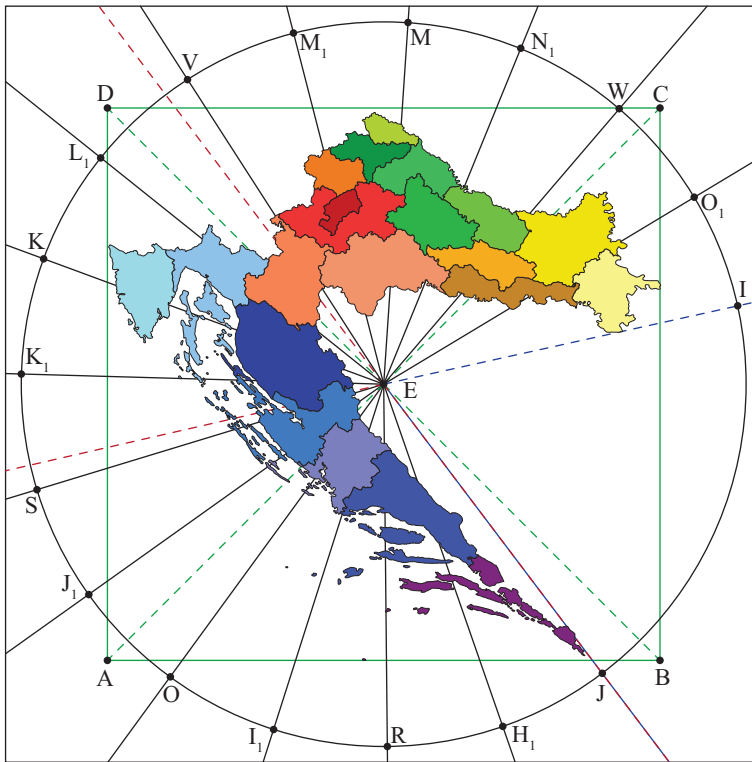


Figure 3. Geometrical construction.

3.2. Dividing circular sector into an arbitrary number of matching circular sectors on a digitized map

Dividing angle into number two raised to the power of arbitrary natural number using geometric tools is possible. By measuring and using mathematical interpolation angle can be divided into an arbitrary number of sectors within any margin error. The number of calculations can be huge. The same applies to a circular sector.

$$\log_2 21 = 4.39231742278 < 4.5.$$

Therefore, 16 is the exemplar number of matching circular sectors.

The circular sector is divided into halves, then halves into quarters, and so on, until 16 of matching circular sectors are constructed. Sectors are numbered counterclockwise S_1, S_2, \dots, S_{16} .

Painting sectors in different colors reduces the number of colors used as shown in Figure 4.

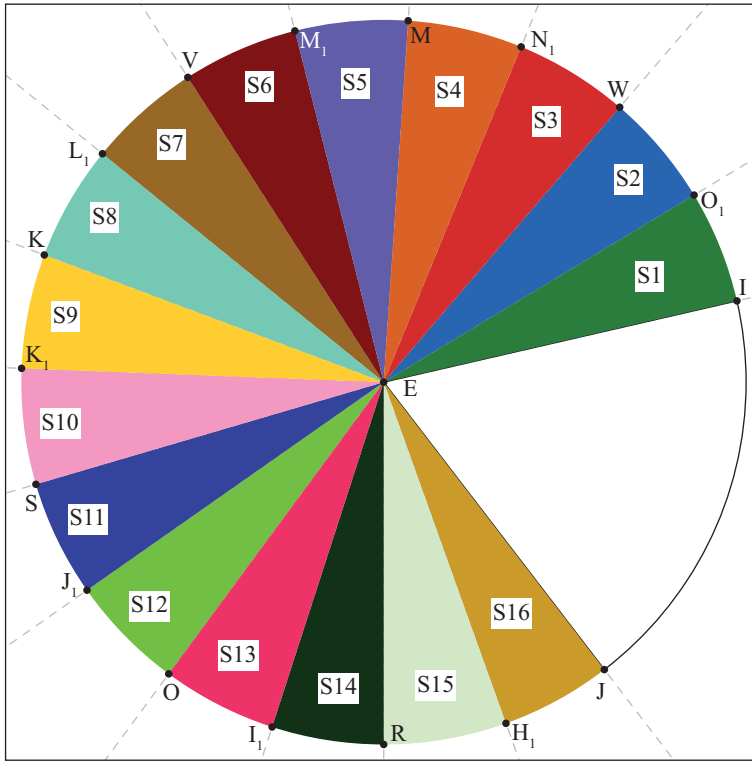


Figure 4. Painted circular sectors.

Each circular sector is represented in different color $sc_i \in CS$, where $i = 1, 2, 3, \dots, 16$.

For example, brown color used in the image to represent circular sector 7 is:

$$sc_7 = (152, 102, 0).$$

Sectors can be written as:

$$S_i = \{(x, y, c) : x \in [1, 2201] \cap \mathbf{N}, y \in [1, 2151] \cap \mathbf{N}, c \in CS, c = sc_i\} \subseteq M$$

where $i = 1, 2, 3, \dots, 16$.

3.3. Determining the number of elements for intersections in each constructed matching circular sector with each county surface on a digitized map by counting common pixels

Intersections between each constructed circular sector (S_j) with each county surface (MA_i) on digitized map are sets of pixels $MB_{i,j}$

$$MB_{i,j} = MA_i \cap S_j.$$

Let $D = (d_{ij})$ is 21×16 matrix.

$d_{ij} = n(MB_{ij})$, where n is number of pixels of the set MB_{ij} .

Using photo editing software, applying Magic Wand tool software marks and counts the number of pixels within uniform shape and entered data in Table 1.

Table 1. Intersection between circular sectors and counties in number of pixels (matrix D).

County (i)	Circular sector (j)																Σ
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	
1				1823	43962	42018											87803
2					22763	13277											36040
3			7613	38469	53687	31074											130843
4						20608	67490	17711									105809
5				2364	34035												36399
6				37352	13223												50575
7			17498	57721													75219
8							21686	66233	12784	172							100875
9							178	59173	71284	24749	1555						156939
10		7576	45294	5836													58706
11		26048	24492	2054													52594
12	12802	29560	15616														57978
13									431	24130	50663	23737					98961
14	5696	115642	1857														123195
15											243	18590	43875	19762	372		82842
16	71496	358															71854
17													488	31588	64341	31638	128055
18								55748	26369								82117
19															9871	36584	46455
20				3552	17239												20791
21					9436	8713											18149
Σ	89994	179184	112370	149171	194345	115690	89354	198865	110868	49051	52461	42327	44363	51350	74584	68222	1622199

Merging sets MB_{ij} by this rule:

$$MC_j = \bigcup_{i=1}^{21} MB_{ij}, \quad 1 \leq j \leq 16$$

new territorial division, named *zones*, is created. Zones are, essentially, parts of the Republic of Croatia defined by circular sector.

The formula calculates the share in the county area and the number of inhabitants that corresponds to the ratio of that county population from the initial data, presented in chapter 2 of this paper.

Data is displayed in Table 2.

All zone sums are presented here: Zone 1 – 193198; Zone 2 – 369123; Zone 3 – 162020; Zone 4 – 248771; Zone 5 – 986720; Zone 6 – 638515; Zone 7 – 134493; Zone 8 – 364068; Zone 9 – 124130; Zone 10 – 48535; Zone 11 – 86850; Zone 12 – 62611; Zone 13 – 54250; Zone 14 – 134115; Zone 15 – 251299; Zone 16 – 206555.

The number of connections between zones is in range from 1 to 2.

Optimal data structure, with minimal number of connections, is illustrated in Figure 5, giving 16 (2^4) zones as closest to number of political territorial units (21).

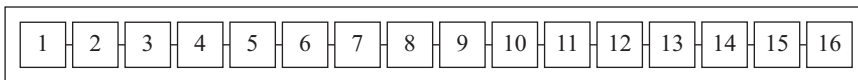


Figure 5. Optimal connectivity.

Analog to Data model A, **Data model B** is defined by the number of inhabitants in zones and connectivity between zones presented in Figure 5.

For example, Z_6 has population 638515 and it is connected to Z_5 and Z_7 .

Population density in Data model B is not uniform and varies from 0.99 (Zone 10) to 5.52 (Zone 6) inhabitants per pixel.

Distribution of population in zones as a share of total population is shown in Figure 6.

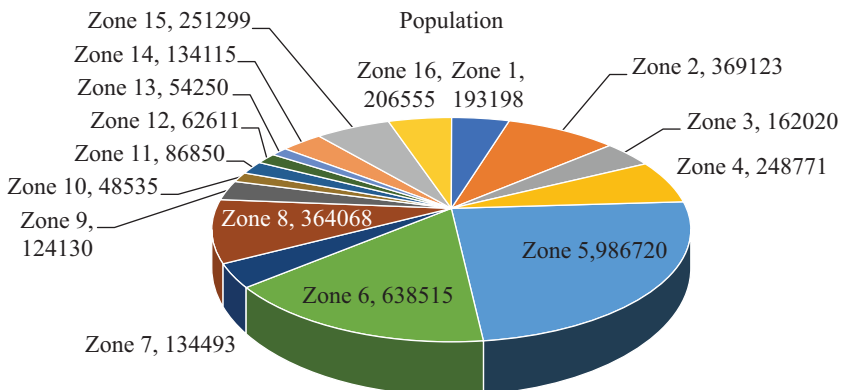


Figure 6. Zones share in total population.

Zones can be graphically presented as a digital memory map with 16 congruent rectangles with variable population density. The example in which the digital memory map has 4800 columns is illustrated in Figure 7.

Z ₁ (193198)	Z ₂ (369123)	Z ₃ (162020)	Z ₄ (248771)	Z ₅ (986720)	Z ₆ (638515)	Z ₇ (134493)	Z ₈ (364068)
Z ₉ (124130)	Z ₁₀ (48535)	Z ₁₁ (86850)	Z ₁₂ (62611)	Z ₁₃ (54250)	Z ₁₄ (134115)	Z ₁₅ (251299)	Z ₁₆ (206555)

Figure 7. Digital memory map of Croatia with congruent zones and variable population density.

Prototype simulator of spreading covid-19 infection in the Republic of Croatia, created by author of this paper, used 4800 columns in the software implementation.

In order to create a new uniform density data model, data transformation is carried out in the final step presented in chapter 3.5.

3.5. Forming new system of zones of uniform population density, of variable sizes, proportional to the population therein

Using zones share in the new total population digital memory map of the Republic of Croatia can be presented in rectangular shape, where zones are represented in different sizes to obtain the same density per zone considering the population in the zones.

Table 3. Zone borders for uniform density transformation.

Zone	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Upper border	228	664	855	1149	2313	3067	3226	3656	3803	3860	3963	4037	4101	4259	4556	4800
Lower border	1	229	665	856	1150	2314	3068	3227	3657	3804	3861	3964	4038	4102	4260	4557

Table 3 lists the columns borders obtained by recalculating the data in Figure 7.

The example in which the digital memory map has 4800 columns is illustrated in Figure 8.

The updated software implementation of prototype simulator that shows spreading covid-19 infection in the Republic of Croatia used the same number of columns.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
---	---	---	---	---	---	---	---	---	----	----	----	----	----	----	----

Figure 8. Zones in uniform density model.

Data model C is defined by zones of uniform density, variable sizes, proportional to the population therein with connectivity presented in Figure 5.

For example, on the digital memory map with 4800 columns, Zone 7 is presented in rectangular shape from column 3068 to column 3226, connected to Zone 6 and Zone 8.

Steps 1 and 2 can be performed on a printed geographic map using geometric tools and following analog procedure to digital. Step 3 can be adapted by cutting printed map and measuring weight of every section and replacing number of pixels in table 1 with masses. Calculations in steps 4 and 5 can be conducted manually or using table calculators.

4. Transformation model

Transformation model from *Data model A* to *Data model C* is shown in Figure 9.

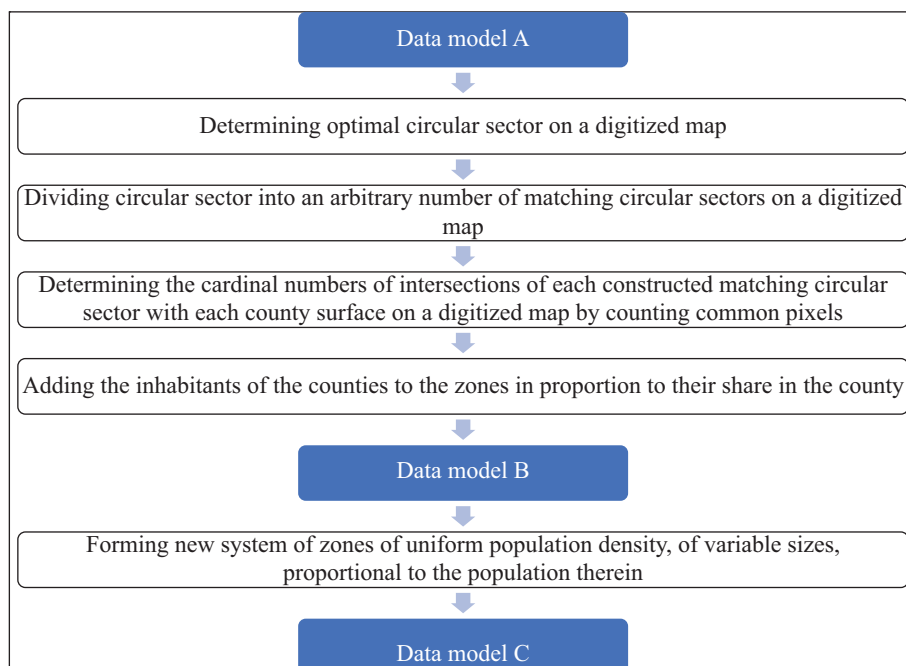


Figure 9. Transformation model.

5. Discussion and conclusion

The Republic of Croatia shape on the geographical map is unusual, similar to a horseshoe.

Territorial division into counties is complex. Those unique characteristics open up the possibility of data transformation explained in previous chapters.

Weisner and Ladyman (2019) have identified the features of complexity. Complexity of the initial model opposed to the final model brought out in this paper is greater. Numerosity of elements ($21 > 16$) and numerosity of interactions ($35 > 15$)

are significantly reduced moving from the initial to the final model. Other measuring elements mentioned by Weisner and Ladyman (2019) remained equal.

This is a new transformation model that transforms data model built from public documents into a new data model which is applicable to solving population migration and information migration problems.

The model is adaptable to similar geographical situations. Furthermore, by modifying number of zones to 2^x where $x > 4, x \in \mathbf{N}$, it becomes more accurate.

Building simulations by using the final model requires less time and reduces the possibility of errors significantly.

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Transformacija podataka o naseljenosti Republike Hrvatske

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Sažetak. U ovom radu matematičko modeliranje koristi se za stvaranje podatkovnog modela prikladnog za računalne simulacije. Isti donosi prikaz transformacije podataka o gustoći naseljenosti Republike Hrvatske iz dva izvora: procjena ukupnog stanovništva Republike Hrvatske po županijama sredinom godine 2019. i digitalizirane karte Republike Hrvatske s 21 kompleksno povezanom teritorijalnom cjelinom (20 županija i grad Zagreb).

Podaci iz ova dva izvora transformirani su u jednostavan model od 16 zona jednolike gustoće naseljenosti, različitih veličina, razmjernih broju stanovnika u njima.

Transformacija podataka odvija se u pet koraka. Prvi je korak određivanje optimalnog kružnog isječka na digitaliziranoj karti. Sljedeći je korak podjela kružnog isječka na proizvoljan broj sukladnih kružnih isječaka (16). Treći je korak određivanje presjeka svakoga s ploham županija prebrojavanjem zajedničkih piksela. U četvrtom koraku razmjerno udjelu u površini županije zonama se pridružuju stanovnici županija. U završnom koraku stvara se novi sustav zona jednolike gustoće, različitih veličina, razmjernan populaciji u njima.

Ključne riječi: digitalizirana mapa, piksel, transformacija podataka, presjek skupova, omjeri

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