Discrete series and the essentially Speh representations

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Abstract

Let π denote an essentially Speh representation of the general linear group over a non-archimedean local field or its separable quadratic extension, and let σ_c denote an irreducible cuspidal representation of either symplectic, special odd-orthogonal, or unitary group. We determine when the induced representation $\pi \rtimes \sigma_c$ contains a discrete series subquotient. We also identify all discrete series subquotients.

1 Introduction

Let F denote a non-archimedean local field F, and let F' stand either for F or for its separable quadratic extension. Let ρ denote an irreducible cuspidal representation of the general linear group over F'. For a real number a, a non-negative integer k and a positive integer n, a unique irreducible subrepresentation of the induced representation

$$\delta([\nu^a\rho, \nu^{a+k}\rho]) \times \delta([\nu^{a+1}\rho, \nu^{a+k+1}\rho]) \times \cdots \times \delta([\nu^{a+n-1}\rho, \nu^{a+k+n-1}\rho])$$

is called the essentially Speh representation, and we denote by $S(a, k, n, \rho)$. We emphasize that representations of such a form play a fundamental role in the identification of the unitary representations of the general linear group ([16, Theorem 7.5]).

Let us denote by σ_c an irreducible cuspidal representation of either symplectic, special odd-orthogonal, or unitary group. Since the essentially Speh

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representations play a prominent role in the unitary dual of the general linear group, it is of particular interest to have a better understanding of the composition factors of parabolically induced representation $S(a, k, n, \rho) \rtimes \sigma_c$.

Recently, a complete description of the composition series in the case $a \geqslant \frac{1}{2}$ has been given in [3]. We note that, although [3] deals with the symplectic and special odd-orthogonal group, the unitary group case can be handled in the exactly same way. There is still not much known about the composition factors of an induced representation of the form $S(a, k, n, \rho) \rtimes \sigma_c$ in the case $a < \frac{1}{2}$. Our aim is to tackle this problem by determining the existence of discrete series subquotients in a very concise way. Besides being interesting by itself, the existence of such subquotients usually presents one of the crucial steps towards the description of all irreducible composition factors.

In the following theorem we summarize our main results.

Theorem 1.1. *Let ρ denote an irreducible cuspidal representation of the general linear group over a non-archimedean local field, and let σ_c stand for an irreducible cuspidal representation of either symplectic, special odd-orthogonal, or unitary group. Let a stand for a real number, let k denote a non-negative integer, and let n stand for the positive one. If $a + k \leqslant 0$ and $a + n - 1 \geqslant 0$, the induced representation $S(a, k, n, \rho) \rtimes \sigma_c$ does not contain a discrete series subquotient. Otherwise we can assume that $a + k > 0$. Then $S(a, k, n, \rho) \rtimes \sigma_c$ contains a discrete series subquotient if and only if ρ is F'/F-selfdual, for a unique non-negative α such that $\nu^\alpha \rho \rtimes \sigma_c$ reduces we have $a - \alpha \in \mathbb{Z}$, $-\frac{k}{2} < a$, and either $0 < a + n - 1 = \alpha$ or $a + n - 1 \leqslant \alpha$.*

Furthermore, if $S(a, k, n, \rho) \rtimes \sigma_c$ contains a discrete series subquotient, then it contains a discrete series subrepresentation.

We note that in the case $n = 1$ analogous results have been obtained in [14], while in the case $k = 0$ and a half-integral analogous results follow from [9, Section 3]. Condition $-\frac{k}{2} < a$ makes a natural sense, since then all representations $\delta([\nu^a \rho, \nu^{a+k} \rho]), \ldots, \delta([\nu^{a+n-1} \rho, \nu^{a+k+n-1} \rho])$ have positive central characters.

In Propositions 3.1, 3.4, and 3.7 we provide an explicit description of all discrete series subquotients of $S(a, k, n, \rho) \rtimes \sigma_c$.

In the following section we present some preliminaries, while in the third section we obtain our main results, using a case-by-case consideration. Our approach is based on the calculation of embeddings and Jacquet modules of discrete series representations, using the Mœglin-Tadić classification, and covers symplectic, special odd-orthogonal, and unitary groups over non-archimedean local fields of arbitrary characteristic. In the case of symplectic and special odd-orthogonal groups over a non-archimedean local field of characteristic zero, it seems that analogous results could also be obtained using the LLC approach to the classification of discrete series, given in [19].

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2 Preliminaries

Through the paper, we denote by F a non-archimedean local field. We will fix one of the following series $\{G_n\}$ of classical groups over F.

In the odd orthogonal group case, we fix an anisotropic orthogonal vector space Y_0 over F of odd dimension and consider the Witt tower based on Y_0. For n such that $2n+1 \geqslant \mathrm{dim} Y_0$, there is exactly one space V_n in the tower of dimension $2n+1$. Let G_n stand for the special orthogonal group of this space. If V_n stands for the symplectic space of dimension $2n$ in the corresponding Witt tower, we denote by G_n the symplectic group of this space. We also consider the unitary groups $U(n, F'/F)$, where F' stands for a separable quadratic extension of F. There is also an anisotropic unitary space Y_0 over F', and the Witt tower of unitary spaces V_n based on Y_0. We denote by G_n the unitary group of the space V_n of dimension either $2n+1$ or $2n$.

We fix a minimal parabolic subgroup in G_n and consider only the standard parabolic subgroups with respect to this fixed minimal parabolic subgroup. When working with the unitary groups, we let F' denote a separable quadratic extension of F, otherwise let F' denote F. We fix one of the series $\{G_n\}$ as above. For representations δ_i of $GL(n_i, F')$, $i = 1, 2, \ldots, k$, and representation τ of $G_{n'}$, we denote by $\delta_1 \times \cdots \times \delta_k \rtimes \tau$ the representation parabolically induced by $\delta_1 \otimes \cdots \otimes \delta_k \otimes \tau$. We use a similar notation to denote a parabolically induced representation of $GL(m, F')$.

By $\mathrm{Irr}(G_n)$ we denote the set of all irreducible admissible representations of G_n. Let $R(G_n)$ denote the Grothendieck group of admissible representations of finite length of G_n and define $R(G) = \oplus_{n\geqslant 0} R(G_n)$. In a similar way we define $\mathrm{Irr}(GL(n, F'))$ and $R(GL) = \oplus_{n\geqslant 0} R(GL(n, F'))$.

Let n' be the Witt index of V_n if V_n is symplectic or even-unitary group, and $n' = n - \frac{1}{2}(\dim_{F'}(Y_0) - 1)$ otherwise. For $\sigma \in \mathrm{Irr}(G_n)$ and $0 \leqslant k \leqslant n'$ we denote by $r_{(k)}(\sigma)$ the normalized Jacquet module of σ with respect to the parabolic subgroup $P_{(k)}$ having the Levi subgroup equal to $GL(k, F') \times G_{n-k}$. We identify $r_{(k)}(\sigma)$ with its semisimplification in $R(GL(k, F')) \otimes R(G_{n-k})$ and consider

$$\mu^*(\sigma) = 1 \otimes \sigma + \sum_{k=1}^{n'} r_{(k)}(\sigma) \in R(GL) \otimes R(G).$$

For $\pi \in \mathrm{Irr}(GL(n, F'))$ we define $m^*(\pi) = \sum_{k=0}^{n}(r_{(k)}(\pi)) \in R(GL) \otimes R(GL)$, where $r_{(k)}(\pi)$ denotes the normalized Jacquet module of π with respect to the standard parabolic subgroup having the Levi factor equal to $GL(k, F') \times GL(n-k, F')$. We identify $r_{(k)}(\pi)$ with its semisimplification, and then extend m^* linearly to the whole of $R(GL)$.

We denote by ν the composition of the determinant mapping with the normalized absolute value on F'. Let $\rho \in R(GL)$ denote an irreducible cuspidal representation. By a segment we mean a set of the form $[\rho, \nu^m \rho] := \{\rho, \nu\rho, \ldots, \nu^m\rho\}$, for a non-negative integer m. By [20], the induced representation $\nu^m\rho \times \nu^{m-1}\rho \times \cdots \times \rho$ has a unique irreducible subrepresentation, denoted by $\delta([\rho, \nu^m\rho])$, which is essentially square-integrable. For every irreducible essentially square-integrable representation $\delta \in R(GL)$, there is a unique $e(\delta) \in \mathbb{R}$ such that $\nu^{-e(\delta)}\delta$ is unitarizable. Note that $e(\delta([\nu^a\rho, \nu^b\rho])) = (a+b)/2$. Suppose that $\delta_1, \delta_2, \ldots, \delta_k$ are irreducible essentially square-integrable representations such that $e(\delta_1) \leqslant e(\delta_2) \leqslant \ldots \leqslant e(\delta_k)$. Then the induced representation $\delta_1 \times \delta_2 \times \cdots \times \delta_k$ has a unique irreducible subrepresentation, which we denote by $L(\delta_1, \delta_2, \ldots, \delta_k)$. This irreducible subrepresentation is called the Langlands subrepresentation, and it appears with multiplicity one in the composition series of $\delta_1 \times \delta_2 \times \cdots \times \delta_k$. Every irreducible representation $\pi \in R(GL)$ is isomorphic to some $L(\delta_1, \delta_2, \ldots, \delta_k)$ and, for a given π, the representations $\delta_1, \delta_2, \ldots, \delta_k$ are unique up to a permutation ([2, 15]).

The essentially Speh representations are irreducible representations of the form $L(\delta_1, \delta_2, \ldots, \delta_n)$, where $\delta_i \cong \delta([\nu^{a+i-1}\rho, \nu^{b+i-1}\rho])$, for $i = 1, 2, \ldots, n$, real numbers a and b such that $b - a$ is a nonnegative integer, and an irreducible cuspidal representation ρ of $GL(n_\rho, F')$.

For an irreducible smooth representation $\pi \in R(GL)$, let $\widetilde{\pi}$ stand for the contragredient representation of π. If $F = F'$, we say that π is F'/F-selfdual if $\pi \cong \widetilde{\pi}$. If $F \neq F'$, we denote by θ the non-trivial F-automorphism of F', let $\check{\widetilde{\pi}}$ denote the representation $g \mapsto \widetilde{\pi}(\theta(g))$, and say that the representation π is F'/F-selfdual if $\pi \cong \check{\widetilde{\pi}}$.

Through the paper we fix an irreducible cuspidal representation $\sigma_c \in R(G)$ and an irreducible cuspidal representation $\rho \in R(GL)$.

By the classification of discrete series representations ([10, 13]), which now holds unconditionally due to [1], [12, Théorème 3.1.1] and [5, Theorem 7.8], a discrete series representation $\sigma \in G_n$ is uniquely described by an admissible triple which consists of the Jordan block $\mathrm{Jord}(\sigma)$, the partial cuspidal support σ_{cusp}, and the ϵ-function ϵ_σ.

The partial cuspidal support of σ is an irreducible cuspidal representation $\sigma_{cusp} \in R(G)$ such that there is an irreducible representation $\pi \in R(GL)$ and an embedding $\sigma \hookrightarrow \pi \rtimes \sigma_{cusp}$.

The Jordan block of σ is set of all ordered pairs (x, ρ), where x is a positive integer and $\rho \in R(GL)$ is an irreducible F'/F-selfdual cuspidal representation, such that the induced representation $\delta([\nu^{-\frac{x-1}{2}}\rho, \nu^{\frac{x-1}{2}}\rho]) \rtimes \sigma$ is irreducible, and $\delta([\nu^{-\frac{x-1}{2}-m}\rho, \nu^{\frac{x-1}{2}+m}\rho]) \rtimes \sigma$ reduces for some positive integer m. The ϵ-function ϵ_σ is defined on a subset of $\mathrm{Jord}(\sigma) \cup \mathrm{Jord}(\sigma) \times \mathrm{Jord}(\sigma)$, and attains values on $\{1, -1\}$.

For an irreducible F'/F-selfdual cuspidal representation $\rho \in R(GL)$ we write $\mathrm{Jord}_\rho(\sigma) = \{x : (x, \rho) \in \mathrm{Jord}(\sigma)\}$. If $\mathrm{Jord}_\rho(\sigma) \neq \emptyset$ and $x \in \mathrm{Jord}_\rho(\sigma)$, denote $x_- = \max\{y \in \mathrm{Jord}_\rho(\sigma) : y < x\}$, if it exists. We note that to define the ϵ-function on the elements of $\mathrm{Jord}(\sigma) \times \mathrm{Jord}(\sigma)$, it is enough to define the ϵ-function on the elements of the form $((x_-, \rho), (x, \rho))$. Also, to define the ϵ-function on the elements of the form (x, ρ), it is enough to define it either on $(\min(\mathrm{Jord}_\rho), \rho)$ or on $(\max(\mathrm{Jord}_\rho), \rho)$.

Let us recall some properties of the ϵ-functions which are commonly used in the paper, following [14, Section 2]. If $\epsilon_\sigma((x_-, \rho), (x, \rho)) = 1$, there is a discrete series σ' such that $\mathrm{Jord}(\sigma') = \mathrm{Jord}(\sigma) \backslash \{(x, \rho), (x_-, \rho)\}$, σ is a subrepresentation of $\delta([\nu^{-\frac{x_--1}{2}}\rho, \nu^{\frac{x-1}{2}}\rho]) \rtimes \sigma'$. If in $\mathrm{Jord}_\rho(\sigma)$ we have $x = y_-$ and $z = (x_-)_-$, then

$$\epsilon_{\sigma'}((z, \rho), (y, \rho)) = \epsilon_\sigma((z, \rho), (x_-, \rho)) \cdot \epsilon_\sigma((x, \rho), (y, \rho)). \tag{1}$$

If in $\mathrm{Jord}_\rho(\sigma)$ we have $x = y_-$ and $\epsilon_\sigma(x, \rho)$ is defined, then

$$\epsilon_\sigma(x_-, \rho) = \epsilon_{\sigma'}(y, \rho) \cdot \epsilon_\sigma((x, \rho), (y, \rho)). \tag{2}$$

For more details on these invariants and the notion of the admissible triple, we refer the reader to [10, 13] and [14, Section 2].

Let us briefly note that notions of the Jordan blocks and ϵ-function from the Mœglin-Tadić classification are transferred to the work of Arthur. The Jordan blocks are precisely the L-parameters of the discrete series of group G_n, and for a discrete L-parameter ϕ of G_n, there is σ belonging to the corresponding L-packet such that we have

$$\phi = \oplus_{(a,\rho)\in \mathrm{Jord}(\sigma)} \rho \otimes V_a,$$

where V_a stands for the unique irreducible a-dimensional representation of $SL(2, \mathbb{C})$. This can be seen in [11, Theorem 1.3.1] and, for the unitary case, we refer reader to [4, Sections 7, 8]. Details about the compatibility of the ϵ-functions can be found in [19].

Basic building blocks in the Mœglin-Tadić classification of discrete series are the strongly positive representations. An irreducible representation $\sigma \in R(G)$ is called strongly positive if for every embedding

$$\sigma \hookrightarrow \nu^{s_1}\rho_1 \times \nu^{s_2}\rho_2 \times \cdots \times \nu^{s_k}\rho_k \rtimes \sigma_{cusp},$$

where $\rho_i \in R(GL(n_{\rho_i}, F'))$, $i = 1, 2, \ldots, k$, are irreducible cuspidal unitary representations and $\sigma_{cusp} \in R(G)$ is an irreducible cuspidal representation, we have $s_i > 0$ for each i. By the classification, they are parametrized by the admissible triples of alternated type. This implies that $\epsilon_\sigma((x_-, \rho), (x, \rho)) = -1$ for all $x \in \mathrm{Jord}_\rho$.

Suppose that σ_{sp} is a strongly positive discrete series such that every element of its cuspidal support belongs to the set $\{\nu^x\rho, \sigma_c : x \in \mathbb{R}\}$. By [8, Theorem 1.2], which also covers the classical group case, we have the following description of σ_{sp}: If ρ is not F'/F-selfdual or $\rho \rtimes \sigma_c$ reduces, we have $\sigma_{sp} \cong \sigma_c$. If ρ is F'/F-selfdual and $\nu^\alpha\rho \rtimes \sigma_c$ reduces for $\alpha > 0$, then there are $a_1, a_2, \ldots, a_{\lceil\alpha\rceil}$, where $\lceil\alpha\rceil$ denotes the smallest integer which is not smaller than α, such that $-1 < a_1 < a_2 < \cdots < a_{\lceil\alpha\rceil}$, $a_i - \alpha \in \mathbb{Z}$ for $i = 1, 2, \ldots, \lceil\alpha\rceil$, $a_1 \geqslant \alpha - \lceil\alpha\rceil$, and σ_{sp} is a unique irreducible subrepresentation of

$$\delta([\nu^{\alpha-\lceil\alpha\rceil+1}\rho, \nu^{a_1}\rho]) \times \delta([\nu^{\alpha-\lceil\alpha\rceil+2}\rho, \nu^{a_2}\rho]) \times \cdots \times \delta([\nu^{\alpha}\rho, \nu^{a_{\lceil\alpha\rceil}}\rho]) \rtimes \sigma_c.$$

Also, if $a_i \geqslant \alpha - \lceil\alpha\rceil + i$, then $2a_i + 1 \in \mathrm{Jord}_\rho(\sigma_{sp})$.

This directly implies the following result:

Lemma 2.1. *Let $\rho \in R(GL)$ denote an irreducible F'/F-selfdual cuspidal representation such that $\nu^{\alpha}\rho \rtimes \sigma_c$ reduces for $\alpha > 0$. Suppose that $\sigma_{sp} \in R(G)$ is a strongly positive discrete series such that every element of its cuspidal support belongs to the set $\{\nu^x\rho, \sigma_c : x \in \mathbb{R}\}$. Let k denote a positive integer, $k \leqslant \lceil\alpha\rceil$, and suppose that there is an $x \in \mathbb{R}$ such that $\nu^x\rho$ appears exactly k times in the cuspidal support of σ_{sp}. We denote the largest such x by $x^{(k)}_{\max}$. Then $2x^{(k)}_{\max} + 1 \in Jord_{\rho}(\sigma_{sp})$.*

We frequently use the following immediate consequence of the discrete series classification, which can also be deduced from [14, Section 2]:

Lemma 2.2. *Let $\sigma \in R(G)$ denote a discrete series such that every element of its cuspidal support belongs to the set $\{\nu^x\rho, \sigma_c : x \in \mathbb{R}\}$, for an irreducible F'/F-selfdual cuspidal representation $\rho \in R(GL)$. Then there exists a non-negative integer m and an ordered $(m+1)$-tuple $(\sigma_1, \sigma_2, \ldots, \sigma_{m+1})$ of discrete series representations in $R(G)$ such that*

1. *$\sigma \cong \sigma_1$,*
2. *σ_{m+1} is strongly positive,*
3. *for $i \in \{1, 2, \ldots, m\}$ there are $x_i, y_i \in Jord_{\rho}(\sigma_i)$ such that $(y_i)_- = x_i$, y_i is the minimal $y \in Jord_{\rho}(\sigma_i)$ such that y_- is defined and $\epsilon_{\sigma_i}((y_-, \rho), (y, \rho)) = 1$, and σ_i is a subrepresentation of $\delta([\nu^{-\frac{x_i-1}{2}}\rho, \nu^{\frac{y_i-1}{2}}\rho]) \rtimes \sigma_{i+1}$.*

Let us provide a technical result which happens to be particularly useful in our investigation.

Proposition 2.3. *Let $\sigma \in R(G)$ denote a discrete series representation such that every representation of the general linear group appearing in its cuspidal support is the twist of the same irreducible F'/F-selfdual cuspidal representation ρ. Let us denote the partial cuspidal support of σ by σ_c, and suppose that $\nu^{\alpha}\rho \rtimes \sigma_c$ reduces for $\alpha > 0$. To σ we attach an ordered $(m+1)$-tuple $(\sigma_1, \sigma_2, \ldots, \sigma_{m+1})$ of discrete series representations as in Lemma 2.2, and let $x_i, y_i \in Jord_{\rho}(\sigma_i)$ be such that $x_i = (y_i)_-$ and σ_i is a subrepresentation of $\delta([\nu^{-\frac{x_i-1}{2}}\rho, \nu^{\frac{y_i-1}{2}}\rho]) \rtimes \sigma_{i+1}$, for $i = 1, \ldots, m$. Let $k_1 = |\{i : 1 \leqslant i \leqslant m, \frac{x_i-1}{2} \geqslant \alpha\}|$ and $k_2 = |\{j : 1 \leqslant j \leqslant m, \frac{y_j-1}{2} \geqslant \alpha\}|$. Let $z_1, z_2, \ldots, z_{\lceil\alpha\rceil}$ be such that σ_{m+1} is a unique irreducible subrepresentation of*

$$\delta([\nu^{\alpha-\lceil\alpha\rceil+1}\rho, \nu^{z_1}\rho]) \times \cdots \times \delta([\nu^{\alpha}\rho, \nu^{z_{\lceil\alpha\rceil}}\rho]) \rtimes \sigma_c, \quad (3)$$

and let $k_3 = |\{i : 1 \leqslant i \leqslant \lceil\alpha\rceil, z_i \geqslant \alpha\}|$. Then $k_1 + k_2 + k_3 \geqslant 2m$.

Proof. We note that it follows from [12, Théorème 3.1.1] and [5, Theorem 7.8] that 2α is an integer.

Since $\frac{x_i-1}{2} \geqslant 0$ and $\frac{x_i-1}{2} - \alpha \in \mathbb{Z}$ for all $i = 1, 2, \ldots, m$, if $\alpha = \frac{1}{2}$ we have $k_1 + k_2 = 2m$. Thus, we can assume that $\alpha \geqslant 1$. If $\sigma_{m+1} \cong \sigma_c$, using a description of $\mathrm{Jord}_\rho(\sigma_c)$ and $x_i \notin \mathrm{Jord}_\rho(\sigma_c)$ for $i = 1, 2, \ldots, m$, we obtain that $\frac{x_i-1}{2} \geqslant \alpha$ for $i = 1, 2, \ldots, n$. This again gives $k_1 + k_2 = 2m$.

It remains to consider the case of non-cuspidal σ_{m+1}. Let S stand for the set $\{\frac{x_1-1}{2}, \ldots, \frac{x_m-1}{2}, \frac{y_1-1}{2}, \ldots, \frac{y_m-1}{2}, z_1, \ldots, z_{\lceil\alpha\rceil}\}$. Since the sets $\{\frac{x_1-1}{2}, \ldots, \frac{x_m-1}{2}\}$, $\{\frac{y_1-1}{2}, \ldots, \frac{y_m-1}{2}\}$, and $\{z_1, \ldots, z_{\lceil\alpha\rceil}\}$ are mutually disjoint and for $x \in S$ we have $x - \alpha \in \mathbb{Z}$, it follows that there are at most $\lfloor\alpha\rfloor$ elements in S which are smaller than α, where $\lfloor\alpha\rfloor$ stands for the largest integer which is not larger than α. If $k_1 + k_2 = 2m$, there is nothing to prove.

Suppose that $k_1 + k_2 < 2m$ and let $l = 2m - k_1 - k_2$. Let us denote by $x_{\min}$ the smallest element of the set $\{\frac{x_1-1}{2}, \ldots, \frac{x_m-1}{2}\}$. Note that $x_{\min} < \alpha$ and $2x_{\min} + 1 \notin \mathrm{Jord}_\rho(\sigma_{m+1})$. This implies that $z_{\lceil x_{\min}\rceil+1} \geqslant \alpha - \lceil\alpha\rceil + x_{\min} + 1$, so $z_j \geqslant \alpha - \lceil\alpha\rceil + j$ for $j = x_{\min} + 1, x_{\min} + 2, \ldots, \lceil\alpha\rceil$, i.e., at least $\alpha - x_{\min}$ segments appearing in (3) are nonempty.

Since l elements of $\{\frac{x_1-1}{2}, \ldots, \frac{x_m-1}{2}, \frac{y_1-1}{2}, \ldots, \frac{y_m-1}{2}\}$ are less than α, and the smallest one of them equals $x_{\min}$, at most $\lfloor\alpha\rfloor - l - x_{\min}$ elements of $\{z_{\lceil x_{min}\rceil+1}, \ldots, z_{\lceil\alpha\rceil}\}$ can be less than α. Consequently, at least

$$\lceil\alpha\rceil - x_{\min} - (\lfloor\alpha\rfloor - l - x_{\min}) = \lceil\alpha\rceil - \lfloor\alpha\rfloor + l$$

elements of $\{z_{x_m+1}, \ldots, z_{\lceil\alpha\rceil}\}$ are greater than or equal to α. Using $\lceil\alpha\rceil - \lfloor\alpha\rfloor \geqslant 0$ we deduce that $k_3 \geqslant l$, so $k_1 + k_2 + k_3 \geqslant 2m$ and the proposition is proved. □

In the rest of the paper, we fix a real number a, and non-negative integers k and n. By $S(a, k, n, \rho)$ we denote the essentially Speh representation

$$L(\delta([\nu^a\rho, \nu^{a+k}\rho]), \delta([\nu^{a+1}\rho, \nu^{a+k+1}\rho]), \ldots, \delta([\nu^{a+n-1}\rho, \nu^{a+k+n-1}\rho])).$$

We take a moment to explicitly state the formula for the Jacquet modules of $S(a, k, n, \rho) \rtimes \sigma_c$, which present our main tool for the investigation of discrete series subquotients. It is completely based on the Tadić's structural formula ([17, Theorem 5.4]) and a description of the Jacquet modules of a ladder representation ([7, Theorem 2.1]). Let $Lad(S(a, k, n, \rho))$ denote the set of all ordered n-tuples $(x_1, x_2, \ldots, x_n)$ of real numbers such that $x_i < x_{i+1}$ for $i = 1, 2, \ldots, n-1$, $x_i - a \in \mathbb{Z}$ and $a + i - 2 \leqslant x_i \leqslant a + k + i - 1$ for

$i = 1, 2, \ldots, n$. Let $Lad(S(a,k,n,\rho))'$ stand for the set of all ordered pairs $((x_1, \ldots, x_n), (y_1, \ldots, y_n)) \in Lad(S(a,k,n,\rho)) \times Lad(S(a,k,n,\rho))$ such that $x_i \leqslant y_i$ for $i = 1, 2, \ldots, n$. Suppose that ρ is F'/F-selfdual. We have

$$\mu^*(S(a,k,n,\rho) \rtimes \sigma_c) = \tag{4}$$
$$\sum_{Lad(S(a,k,n,\rho))'} L(\delta([\nu^{-x_n}\rho, \nu^{-a-n+1}\rho]), \ldots, \delta([\nu^{-x_1}\rho, \nu^{-a}\rho])) \times$$
$$L(\delta([\nu^{y_1+1}\rho, \nu^{a+k}\rho]), \ldots, \delta([\nu^{y_n+1}\rho, \nu^{a+k+n-1}\rho])) \otimes$$
$$L(\delta([\nu^{x_1+1}\rho, \nu^{y_1}\rho]), \ldots, \delta([\nu^{x_n+1}\rho, \nu^{y_n}\rho])) \rtimes \sigma_c.$$

3 Discrete series

By the Mœglin-Tadić classification, if $S(a,k,n,\rho) \rtimes \sigma_c$ contains a discrete series then ρ is F'/F-selfdual. Thus, in what follows we assume that ρ is F'/F-selfdual and denote by α a unique non-negative real number such that $\nu^{\alpha}\rho \rtimes \sigma_c$ reduces.

Again, by the Mœglin-Tadić classification, we can assume that $a - \alpha$ is an integer.

If both $a + k \leqslant 0$ and $a + n - 1 \geqslant 0$, then $\mu^*(S(a,k,n,\rho) \rtimes \sigma_c)$ does not contain an irreducible constituent of the form $\nu^x\rho \otimes \pi$ for $x > 0$, so $S(a,k,n,\rho) \rtimes \sigma_c$ does not contain a discrete series subquotient, since this would contradict the square-integrability criterion. Thus, $a + k > 0$ or $a + n - 1 < 0$. Since in the appropriate Grothendieck group we have

$$S(\widetilde{a,k,n},\rho) \rtimes \sigma_c = S(-a-k-n+1, k, n, \rho) \rtimes \sigma_c, \tag{5}$$

we can assume that $a + k > 0$.

In the case $a > 0$, a complete description of the composition series of $S(a,k,n,\rho) \rtimes \sigma_c$ is a special case of the results of the first author, given in [3]. In particular, if $a > 0$, then a discrete series subquotient of $S(a,k,n,\rho) \rtimes \sigma_c$ has to be strongly positive, and the following result is a consequence of [3, Theorem 3.1]:

Proposition 3.1. *If $a > 0$, then $S(a,k,n,\rho) \rtimes \sigma_c$ contains a discrete series subquotient if and only if $\alpha > 0$ and $a + n - 1 = \alpha$. Furthermore, if $a > 0$, $\alpha > 0$ and $a + n - 1 = \alpha$, then $S(a,k,n,\rho) \rtimes \sigma_c$ contains a unique discrete series subquotient, which appears with multiplicity one, and is also the unique* *irreducible subrepresentation of both $S(a,k,n,\rho) \rtimes \sigma_c$ and $\delta([\nu^a\rho, \nu^{a+k}\rho]) \times \cdots \times \delta([\nu^{\alpha}\rho, \nu^{a+k+n-1}\rho]) \rtimes \sigma_c$.*

In what follows, we discuss the case $a \leqslant 0$.

Proposition 3.2. *Suppose that $a + k > 0$, $a \leqslant 0$, and $a + n - 1 > 0$. If $S(a,k,n,\rho) \rtimes \sigma_c$ contains a discrete series subquotient, then $\alpha > 0$, $a - \alpha \in \mathbb{Z}$, $-\frac{k}{2} < a$ and $a + n - 1 = \alpha$.*

Proof. Suppose that $S(a,k,n,\rho) \rtimes \sigma_c$ contains a discrete series subquotient σ_{ds}. We have already seen that $a - \alpha \in \mathbb{Z}$. Since $a \leqslant 0$ and $a + k > 0$, the cuspidal support of σ_{ds} either contains ρ, or contains $\nu^{\frac{1}{2}}\rho$ at least twice. Thus, σ_{ds} is not a strongly positive discrete series. We attach to σ_{ds} an ordered $(m+1)$-tuple of discrete series representations $(\sigma_1, \sigma_2, \ldots, \sigma_{m+1})$ as in Lemma 2.2.

If $\alpha \in \mathbb{Z}$, then ρ appears m times in the cuspidal support of σ_{ds}, so $m = -a + 1$. If $\alpha \notin \mathbb{Z}$, then $\nu^{\frac{1}{2}}\rho$ appears $2\lceil -a \rceil + 1$ times in the cuspidal support of $S(a,k,n,\rho) \rtimes \sigma_c$. Since $\nu^{\frac{1}{2}}\rho$ appears at most once in the cuspidal support of σ_{m+1}, in this case we get $m = \lceil -a \rceil$. Now it can be directly seen that in both cases holds $m = \lceil -a + \frac{1}{2} \rceil$.

Since $\mu^*(S(a,k,n,\rho) \rtimes \sigma_c)$ contains an irreducible constituent of the form $\nu^x\rho \otimes \pi$, $x > 0$, only for $x = a + k$, it follows that $y_1 = 2(a+k) + 1$. Also, it directly follows that for $y \in \mathrm{Jord}_\rho(\sigma_{ds})$ such that $y \neq y_1$ and y_- is defined we have $\epsilon_{\sigma_{ds}}((y_-, \rho), (y, \rho)) = -1$.

If $y_2 \neq y_1 + 2$, using an embedding $\sigma_2 \hookrightarrow \nu^{\frac{y_2-1}{2}}\rho \times \delta([\nu^{-\frac{x_2-1}{2}}\rho, \nu^{\frac{y_2-3}{2}}\rho]) \rtimes \sigma_3$ and a simple commuting argument, we obtain that σ_{ds} is a subrepresentation of an induced representation of the form $\nu^{\frac{y_2-1}{2}}\rho \rtimes \pi$. Now the Frobenius reciprocity implies $\mu^*(S(a,k,n,\rho) \rtimes \sigma_c) \geqslant \nu^{\frac{y_2-1}{2}}\rho \otimes \pi$, which is impossible. Thus, $y_2 = y_1 + 2$, and repeating the same arguments we deduce that $y_{i+1} = y_i + 2$ for all $i = 1, 2, \ldots, m-1$.

It follows at once that for $i = 1, 2, \ldots, m-1$ we have $x_i \geqslant x_{i+1} + 2$. If for some $i \in \{1, 2, \ldots, m-1\}$ we have $x_i \neq x_{i+1} + 2$, [18, Lemma 8.1] implies that $\mu^*(\sigma_{ds})$ contains an irreducible constituent of the form $\nu^{\frac{x_i-1}{2}}\rho \otimes \pi$, which is impossible. Thus, for $i = 1, 2, \ldots, m-1$ we have $x_i = x_{i+1} + 2$.

Since for $\alpha \in \mathbb{Z}$ we have $x_m \geqslant 1$, it follows that $x_1 \geqslant 2(m-1) + 1$ and $\frac{x_1-1}{2} \geqslant -a$. Similarly, for $\alpha \notin \mathbb{Z}$ we have $x_m \geqslant 2$, so it follows that $x_1 \geqslant 2(m-1) + 2$ and $\frac{x_1-1}{2} \geqslant -a$. In any case, from $y_1 > x_1$ we obtain $-a < a + k$, i.e. $-\frac{k}{2} < a$.

Using $a+n-1>0$, and inspecting the cuspidal support of σ_{ds}, we conclude that $m<n$, so $\frac{y_m-1}{2}=a+k+m-1<a+k+n-1$ implies that $\nu^{a+k+n-1}\rho$ appears in the cuspidal support of σ_{m+1}, so σ_{m+1} is a non-cuspidal strongly positive discrete series. Consequently, from the classification of strongly positive discrete series follows $\alpha>0$ and $2(a+k+n-1)+1\in \mathrm{Jord}_\rho(\sigma_{m+1})$. Thus, there is an $y\in \mathrm{Jord}_\rho(\sigma_m)$ such that in $\mathrm{Jord}_\rho(\sigma_m)$ we have $y_-=y_m$. If $x_m\neq \min(\mathrm{Jord}_\rho(\sigma_m))$, there is a $x\in \mathrm{Jord}_\rho(\sigma_m)$ such that $(x_m)_-=x$. Since $x,y\in \mathrm{Jord}_\rho(\sigma_{m+1})$ and σ_{m+1} corresponds to an admissible triple of the alternated type, we have $\epsilon_{\sigma_{m+1}}((x,\rho),(y,\rho))=-1$. Using $\epsilon_{\sigma_m}((x_m,\rho),(y_m,\rho))=1$ and (1), we obtain

$$\epsilon_{\sigma_m}((x,\rho),(x_m,\rho))\cdot\epsilon_{\sigma_m}((y_m,\rho),(y,\rho))=-1,$$

so we have either $\epsilon_{\sigma_m}((x,\rho),(x_m,\rho))=\epsilon_{\sigma_{ds}}((x,\rho),(x_m,\rho))=1$ or $\epsilon_{\sigma_m}((y_m,\rho),(y,\rho))=\epsilon_{\sigma_{ds}}((y_m,\rho),(y,\rho))=1$. Since $y_1\notin \mathrm{Jord}_\rho(\sigma_m)$, we have noted earlier that both of these equalities must be equal to -1, and we obtain $x_m=\min(\mathrm{Jord}_\rho(\sigma_m))$. If $x_m>2$, [18, Lemma 8.1] implies that $\mu^*(\sigma_{ds})$ contains an irreducible constituent of the form $\nu^{\frac{x_m-1}{2}}\rho\otimes\pi$, which is impossible. This implies that for $i=1,2,\dots,m$ we have $x_i=-2a+1-2(i-1)=-2(a+i)+3$.

Let $a_1,a_2,\dots,a_{\lceil\alpha\rceil}$ be such that σ_{m+1} is a unique irreducible subrepresentation of

$$\delta([\nu^{\alpha-\lceil\alpha\rceil+1}\rho,\nu^{a_1}\rho])\times\delta([\nu^{\alpha-\lceil\alpha\rceil+2}\rho,\nu^{a_2}\rho])\times\cdots\times\delta([\nu^{\alpha}\rho,\nu^{a_{\lceil\alpha\rceil}}\rho])\rtimes\sigma_c.$$

We have noted that $\nu^{a+k+n-1}\rho$ appears in the cuspidal support of σ_{m+1}, so $a_{\lceil\alpha\rceil}=a+k+n-1$. Also, since $\nu^{\alpha-\lceil\alpha\rceil+1}\rho$ appears in the cuspidal support of σ_{m+1}, we have $a_1\geqslant\alpha-\lceil\alpha\rceil+1$. Thus, $2a_i+1\in \mathrm{Jord}_\rho(\sigma_{ds})$ for $i=1,2,\dots,\lceil\alpha\rceil$.

Since in $\mathrm{Jord}_\rho(\sigma_m)$ we have $(y_m)_-=x_m$ and $x_m=\min(\mathrm{Jord}_\rho(\sigma_m))$, in $\mathrm{Jord}_\rho(\sigma_{ds})$ we have $(2a_1+1)_-=y_m$. In the same way as in the first part of the proof we deduce that $y_m=2a_1-1$ and that for $i=1,2,\dots,\lceil\alpha\rceil$ we have $a_{i+1}=a_i+1$.

Using embeddings $\sigma_i\hookrightarrow\delta([\nu^{-\frac{x_i-1}{2}}\rho,\nu^{\frac{y_i-1}{2}}\rho])\rtimes\sigma_{i+1}$ for $i=1,2,\dots,m$ and the Frobenius reciprocity, we get that the Jacquet module of σ_{ds} with respect to the appropriate parabolic subgroup contains an irreducible representation of the form

$$\nu^{\frac{y_1-1}{2}}\rho\otimes\nu^{\frac{y_2-1}{2}}\rho\otimes\cdots\otimes\nu^{\frac{y_m-1}{2}}\rho\otimes\nu^{a_1}\rho\otimes\nu^{a_2}\rho\otimes\cdots\otimes\nu^{a_{\lceil\alpha\rceil}}\rho\otimes\pi=$$
$$\nu^{a+k}\rho\otimes\nu^{a+k+1}\rho\otimes\cdots\otimes\nu^{a+k+m-1}\rho\otimes\nu^{a+k+m}\rho\otimes\cdots\otimes\nu^{a+k+n-1}\rho\otimes\pi.$$

It follows that $\lceil\alpha\rceil = a+k+n-1-(a+k+m)+1 = n-m$, so $n = \lceil\alpha\rceil + m$. Using $m = \lceil -a+\frac{1}{2}\rceil$ we deduce

$$a+n-1 = a+\lceil\alpha\rceil + m - 1 = a + \lceil\alpha\rceil + \left\lceil -a+\frac{1}{2}\right\rceil - 1.$$

If $a \in \mathbb{Z}$, then $\alpha \in \mathbb{Z}$, so $\lceil\alpha\rceil = \alpha$, $\lceil -a+\frac{1}{2}\rceil = -a+1$ and $a+n-1 = a+\alpha-a+1-1 = \alpha$. If $a \notin \mathbb{Z}$, then $\alpha \notin \mathbb{Z}$, $2a, 2\alpha \in \mathbb{Z}$, so $\lceil\alpha\rceil = \alpha+\frac{1}{2}$, $\lceil -a+\frac{1}{2}\rceil = -a+\frac{1}{2}$. Thus, we again have $a+n-1 = a+\alpha+\frac{1}{2}-a+\frac{1}{2}-1 = \alpha$. This ends the proof. $\square$

Theorem 3.3. *Suppose that $a+k>0$, $a \leqslant 0$, and $a+n-1>0$. Then $S(a,k,n,\rho) \rtimes \sigma_c$ contains a discrete series subquotient if and only if $\alpha > 0$, $a-\alpha \in \mathbb{Z}$, $-\frac{k}{2} < a$ and $a+n-1 = \alpha$.*

Proof. The necessity part of the proof follows from the previous proposition.

Let us now assume that $\alpha > 0$, $a-\alpha \in \mathbb{Z}$, $-\frac{k}{2} < a$ and $a+n-1 = \alpha$. Let $l = \lceil -a+\frac{1}{2}\rceil$ and note that then we have $a+l>0$, $a-l \leqslant 0$, and $a+l = \alpha - \lceil\alpha\rceil + 1$. We denote by σ_1 the unique irreducible subrepresentation of

$$\begin{gathered}\delta([\nu^{a+l}\rho, \nu^{a+k+l}\rho]) \times \delta([\nu^{a+l+1}\rho, \nu^{a+k+l+2}\rho]) \times \cdots \times \\ \times \delta([\nu^{a+n-1}\rho, \nu^{a+k+n-1}\rho]) \rtimes \sigma_c = \\ \delta([\nu^{\alpha-\lceil\alpha\rceil+1}\rho, \nu^{a+k+l}\rho]) \times \delta([\nu^{\alpha-\lceil\alpha\rceil+2}\rho, \nu^{a+k+l+1}\rho]) \times \cdots \times \\ \times \delta([\nu^{\alpha}\rho, \nu^{a+k+n-1}\rho]) \rtimes \sigma_c.\end{gathered}$$

Note that σ_1 is strongly positive. Since $S(a+l,k,n-l,\rho) \rtimes \sigma_c$ is a subrepresentation of the induced representation above, it also has a unique irreducible subrepresentation which is isomorphic to σ_1.

We inductively define discrete series representations $\sigma_2, \sigma_3, \ldots, \sigma_{l+1}$ such that, for $i = 2, 3, \ldots, l+1$, σ_i is a unique irreducible subrepresentation of

$$\delta([\nu^{a+l-i+1}\rho, \nu^{a+k+l-i+1}\rho]) \rtimes \sigma_{i-1}$$

such that $\epsilon_{\sigma_i}((2(a+k+l-i+1)+1,\rho),(2(a+k+l-i+1)+3,\rho)) = -1$.

Since σ_2 is a subrepresentation of $\delta([\nu^{a+l-1}\rho, \nu^{a+k+l-1}\rho]) \rtimes \sigma_1$ and we have $\min(\mathrm{Jord}_\rho(\sigma_2)) = 2(-a-l+1)+1$, it directly follows that $\epsilon_{\sigma_2}((x_-,\rho),(x,\rho)) = 1$ only for $x = 2(a+k+l-1)+1$, and $\mu^*(\sigma_2)$ contains an irreducible constituent of the form $\nu^{a+k+l-1}\rho \otimes \pi$, by [18, Proposition 7.2]. If $\mu^*(\sigma_2)$ contains an irreducible constituent of the form $\nu^x\rho\otimes\pi$ for $x > a+k+l-1$, using $(2x+1)_- = 2x-1$ and [18, Proposition 7.2] we deduce $\epsilon_{\sigma_2}((2x-1,\rho),(2x+1,\rho)) = 1$, a contradiction.

If $a \notin \mathbb{Z}$, then $a+l = \frac{1}{2}$, and using $\epsilon_{\sigma_2}((2,\rho),(2(a+k+l-1)+1,\rho)) = 1$, $\epsilon_{\sigma_2}(2(a+k+l-1)+1,\rho),(2(a+k+l-1)+3,\rho)) = -1$, $\epsilon_{\sigma_1}(2(a+k+l-1)+3,\rho) = 1$, and the property (2) of the ϵ-function, we obtain $\epsilon_{\sigma_2}(2,\rho) = -1$. This implies that $\mu^*(\sigma_2)$ does not contain an irreducible constituent of the form $\nu^{\frac{1}{2}}\rho\otimes\pi$, since otherwise [18, Proposition 7.4] would imply $\epsilon_{\sigma_2}(2,\rho) = 1$.

Thus, the ϵ-function ϵ_{σ_2} is completely determined and if $\mu^*(\sigma_2)$ contains an irreducible constituent of the form $\nu^x\rho\otimes\pi$, then $x = a+k+l-1$.

In a similar way, for $i \geqslant 3$, using the property (1) of the ϵ-function, $\epsilon_{\sigma_i}((2(-a-l+i-1)+1,\rho),(2(a+k+l-i+1)+1,\rho)) = 1$, $\epsilon_{\sigma_i}((2(a+k+l-i+1)+1,\rho),(2(a+k+l-i+1)+3,\rho)) = -1$, and $\epsilon_{\sigma_{i-1}}((2(-a-l+i-2)+1,\rho),(2(a+k+l-i+1)+3,\rho)) = 1$, we obtain that $\epsilon_{\sigma_i}((x_-,\rho),(x,\rho)) = 1$ only for $x = 2(a+k+l-i+1)+1$. Consequently, for $i \geqslant 3$, ϵ_{σ_i} is completely determined and $\mu^*(\sigma_i)$ contains an irreducible constituent of the form $\nu^{a+k+l-i+1}\rho\otimes\pi$.

Since $\mu^*(\sigma_2)$ does not contain an irreducible constituent of the form $\nu^{\frac{1}{2}}\rho\otimes\pi$, we get that $\mu^*(\sigma_i)$ also does not contain an irreducible constituent of such a form.

If $\mu^*(\sigma_i)$ contains an irreducible constituent of the form $\nu^x\rho\otimes\pi$ for $x \notin \{\frac{1}{2}, a+k+l-i+1\}$, using $2x+1, 2x-1 \in \mathrm{Jord}_\rho(\sigma_i)$, together with [18, Proposition 7.2], we obtain $\epsilon_{\sigma_i}((2x-1,\rho),(2x+1,\rho)) = 1$, which is impossible.

Consequently, if $\mu^*(\sigma_i)$ contains an irreducible constituent of the form $\nu^x\rho\otimes\pi$, then $x = a+k+l-i+1$.

We inductively prove that, for $i = 1,2,\ldots,l+1$, σ_i is a subrepresentation of $S(a+l-i+1,k,n-l+i-1,\rho)\rtimes\sigma_c$. We have already seen that this holds for $i=1$. Suppose that $i \in \{2,3,\ldots,l\}$ and that for $j = 1,2,\ldots,i$ we have $\sigma_j \hookrightarrow S(a+l-j+1,k,n-l+j-1,\rho)\rtimes\sigma_c$. Let us prove that σ_{i+1} embeds into $S(a+l-i,k,n-l+i,\rho)\rtimes\sigma_c$.

From embeddings

$$\begin{aligned}\sigma_{i+1} &\hookrightarrow \delta([\nu^{a+l-i}\rho,\nu^{a+k+l-i}\rho])\rtimes\sigma_i\\ &\hookrightarrow \delta([\nu^{a+l-i}\rho,\nu^{a+k+l-i}\rho])\times S(a+l-i+1,k,n-l+i-1,\rho)\rtimes\sigma_c,\end{aligned}$$

using [6, Lemma 5.5], whose proof carries directly to the unitary group case, we deduce that there is an irreducible subquotient π_1 of $\delta([\nu^{a+l-i}\rho,\nu^{a+k+l-i}\rho])\times$

$S(a+l-i+1,k,n-l+i-1,\rho)$ such that σ_{i+1} embeds into $\pi_1 \rtimes \sigma_c$. Let $\pi_1 \cong L(\delta_1,\delta_2,\ldots,\delta_m)$ where $\delta_j \cong \delta([\nu^{x_j}\rho,\nu^{y_j}\rho])$ for $j=1,2,\ldots,m$.

Since π_1 embeds into an induced representation of the form $\nu^{y_1}\rho \rtimes \pi$, it follows at once that $y_1 = a+k+l-i$. If $x_1 \neq a+l-i$, since for every x such that $\nu^x\rho$ appears in the cuspidal support of π_1, we have $x \geqslant a+l-i$ there is a $j \in \{2,3,\ldots,m\}$ such that $x_j = a+l-i$. For $j' \in \{1,2,\ldots,j-1\}$ we have $x_{j'} > x_j$, since $\nu^{a+l-i}\rho$ appears in the cuspidal support of π_1 with multiplicity one, and $e(\delta_{j'}) \leqslant e(\delta_j)$, which implies $y_{j'} < y_j$. Thus, for $j' \in \{1,2,\ldots,j-1\}$ we have $\delta_{j'} \times \delta_j \cong \delta_j \times \delta_{j'}$, and a simple commuting argument implies that σ_{i+1} is a subrepresentation of an induced representation of the form $\nu^{y_j}\rho \rtimes \pi$, which is impossible since $y_j > y_1 = a+k+l-i$. Consequently, $x_1 = a+l-i$, and π_1 is an irreducible subrepresentation of

$$\delta([\nu^{a+l-i}\rho,\nu^{a+k+l-i}\rho]) \times L(\delta_2,\delta_3,\ldots,\delta_m).$$

Thus, $m^*(\pi_1)$ contains $\delta([\nu^{a+l-i}\rho,\nu^{a+k+l-i}\rho]) \otimes L(\delta_2,\delta_3,\ldots,\delta_m)$. It can be easily seen that $\delta([\nu^{a+l-i}\rho,\nu^{a+k+l-i}\rho]) \otimes S(a+l-i+1,k,n-l+i-1,\rho)$ is a unique irreducible constituent of

$$m^*(\delta([\nu^{a+l-i}\rho,\nu^{a+k+l-i}\rho]) \times S(a+l-i+1,k,n-l+i-1,\rho))$$

of the form $\delta([\nu^{a+l-i}\rho,\nu^{a+k+l-i}\rho]) \otimes \pi$.

Thus, π_1 is an irreducible subrepresentation of $\delta([\nu^{a+l-i}\rho,\nu^{a+k+l-i}\rho]) \times S(a+l-i+1,k,n-l+i-1,\rho)$, so $\pi_1 \cong S(a+l-i,k,n-l+i,\rho)$. For $i=l$ we obtain that σ_{l+1} is a subrepresentation of $S(a,k,n,\rho) \rtimes \sigma_c$. This ends the proof. □

Proposition 3.4. *Suppose that* $a+k>0$, $a \leqslant 0$, $a+n-1=\alpha$, *and* $-\frac{k}{2} < a$. *Let*

$$\begin{aligned} Jord = {} & Jord(\sigma_c) \backslash \{(x,\rho) : (x,\rho) \in Jord(\sigma_c)\} \cup \\ & \{(2(a+k+i)+1,\rho) : i = 0,1,\ldots,n-1\} \cup \\ & \{(2i+1,\rho) : i = \alpha - \lceil \alpha \rceil + 1, \alpha - \lceil \alpha \rceil + 2, \ldots, -a\}. \end{aligned}$$

We define an admissible triple $(Jord,\sigma_c,\epsilon)$ *with* $\epsilon((2(-a)+1,\rho),(2(a+k)+1,\rho)) = 1$ *and* $\epsilon((x_-,\rho'),(x,\rho')) = -1$ *for* $(x,\rho') \in Jord$ *such that* x_- *is defined and* $(x,\rho') \neq (2(a+k)+1,\rho)$. *Also, for* $\rho' \in R(GL)$ *such that* $Jord_{\rho'}$ *is non-empty and consists of even integers, let* $\epsilon(\min(Jord_{\rho'}),\rho') = -1$. *Let* σ_1 *stand for a discrete series corresponding to* $(Jord,\sigma_c,\epsilon)$. *Then* σ_1 *is a subrepresentation of* $S(a,k,n,\rho) \rtimes \sigma_c$. *If* σ *is a discrete series subquotient of* $S(a,k,n,\rho) \rtimes \sigma_c$, *then* $\sigma \cong \sigma_1$.

Proof. It can be directly verified that $(\mathrm{Jord}, \sigma_c, \epsilon)$ is an admissible triple. In the proof of Theorem 3.3 we have constructed a discrete series subrepresentation of $S(a, k, n, \rho) \rtimes \sigma_c$ which corresponds to $(\mathrm{Jord}, \sigma_c, \epsilon)$.

Let us denote by σ a discrete series subquotient of $S(a, k, n, \rho) \rtimes \sigma_c$. Obviously, $\sigma_{cusp} \cong \sigma_c$ and σ is not strongly positive. Using cuspidal support considerations, as in [13, Section 8], we directly obtain $\mathrm{Jord}(\sigma) = \mathrm{Jord}$. Using (4) we obtain that $\mu^*(S(a, k, n, \rho) \rtimes \sigma_c)$ contains an irreducible constituent of the form $\nu^x\rho \otimes \pi$, with $x > 0$, only for $x = a + k$. From [18, Propositions 7.2, 7.4] we deduce $\epsilon_\sigma = \epsilon$. □

Let us now consider the remaining case, where $a + k > 0$, $a \leqslant 0$, and $a + n - 1 \leqslant 0$. Equality (5) enables us to assume $a + k + n - 1 \geqslant -a$.

Proposition 3.5. *Suppose $a + k > 0$, $a \leqslant 0$, and $a + n - 1 \leqslant 0$. If $S(a, k, n, \rho) \rtimes \sigma_c$ contains a discrete series subquotient, then $-\frac{k}{2} < a$.*

Proof. Let us suppose that $S(a, k, n, \rho) \rtimes \sigma_c$ contains a discrete series subquotient and $-\frac{k}{2} \geqslant a$ i.e., $-a \geqslant a + k$. Let us fix a discrete series subquotient of $S(a, k, n, \rho) \rtimes \sigma_c$ and denote it by σ_{ds}. To σ_{ds} we attach an ordered $(m+1)$-tuple of discrete series representations $(\sigma_1, \sigma_2, \ldots, \sigma_{m+1})$ as in Lemma 2.2. This leads to $y_i < y_{i+1}$ for $i = 1, 2, \ldots, m-1$. Using the Frobenius reciprocity we deduce that the Jacquet module of σ_{ds} with respect to the appropriate parabolic subgroup contains an irreducible constituent of the form

$$\nu^{\frac{y_1-1}{2}}\rho \otimes \nu^{\frac{y_2-1}{2}}\rho \otimes \cdots \otimes \nu^{\frac{y_m-1}{2}}\rho \otimes \pi.$$

If $a \in \mathbb{Z}$, then ρ appears n times in the cuspidal support of $S(a, k, n, \rho) \rtimes \sigma_c$ and, since ρ does not appear in the cuspidal support of σ_{m+1}, ρ appears m times in the cuspidal support of σ_{ds}.

If $a \notin \mathbb{Z}$, note that $\nu^{\frac{1}{2}}\rho$ appears $2n$ times in the cuspidal support of $S(a, k, n, \rho) \rtimes \sigma_c$ and, since $\nu^{\frac{1}{2}}\rho$ appears at most once in the cuspidal support of σ_{m+1}, $\nu^{\frac{1}{2}}\rho$ appears either $2m$ or $2m + 1$ times in the cuspidal support of σ_{ds}. Since σ_{ds} is an irreducible subquotient of $S(a, k, n, \rho) \rtimes \sigma_c$, we conclude $2m = 2n$, i.e., $m = n$.

Several possibilities are studied separately.

Let us first consider the case $y_1 = 2(a + k) + 1$ and $y_{i+1} = y_i + 2$ for $i = 1, \ldots, n-1$. This gives $x_{i+1} \leqslant x_i - 2$ for $i = 1, \ldots, n-1$. Also, $y_n = 2(a + k + n - 1) + 1$ and there is a $j \in \{1, \ldots, n\}$ such that $y_j = 2(-a) + 1$, so $x_1 < 2(-a) + 1$. Thus, the cuspidal support of σ_{n+1} equals

$[\nu^{\frac{x_1+1}{2}}\rho, \nu^{-a}\rho] \cup [\nu^{\frac{x_2+1}{2}}\rho, \nu^{-a-1}\rho] \cup \cdots \cup [\nu^{\frac{x_n+1}{2}}\rho, \nu^{-a-n+1}\rho] \cup \{\sigma_c\}$, so $\nu^{-a}\rho$ appears in the cuspidal support of σ_{n+1} with multiplicity one, and for an x such that $\nu^x\rho$ appears in the cuspidal support of σ_{n+1} we have $x \leqslant -a$. Now Lemma 2.1 implies $2(-a)+1 \in \mathrm{Jord}_\rho(\sigma_{n+1})$, a contradiction.

Let us now assume that $y_1 = 2(-a-n+1)+1$ and $y_{i+1} = y_i + 2$ for $i = 1, \ldots, n-1$. Note that $y_n = 2(-a)+1$. It follows that $x_i \geqslant x_{i+1}+2$ for $i = 1, \ldots, n-1$ and $x_i < y_1$ for $i = 1, \ldots, n$. If there is an $i \in \{1, \ldots, n-1\}$ such that $x_i > x_{i+1}+2$, then [18, Lemma 8.1] implies that $\mu^*(\sigma_{ds})$ contains an irreducible constituent of the form $\nu^{\frac{x_i-1}{2}}\rho \otimes \pi$, so $x_i = 2(a+k)+1$ which is impossible since $-a \geqslant a+k \geqslant -a-n+1$ implies that $y_j = 2(a+k)+1$ for some $j \in \{1, \ldots, n\}$. Thus, $x_i = x_{i+1}+2$ for $i = 1, \ldots, n-1$, and the cuspidal support of σ_{n+1} equals $[\nu^{\frac{x_n+1}{2}}\rho, \nu^{a+k}\rho] \cup [\nu^{\frac{x_{n-1}+1}{2}}\rho, \nu^{a+k+1}\rho] \cup \cdots \cup [\nu^{\frac{x_1+1}{2}}\rho, \nu^{a+k+n-1}\rho] \cup \{\sigma_c\}$. It follows that $\nu^{-a}\rho$ appears $2a+k+n$ times in the cuspidal support of σ_{n+1}, since it appears in the segments $[\nu^{\frac{x_{2a+k+n}+1}{2}}\rho, \nu^{-a}\rho], [\nu^{\frac{x_{2a+k+n-1}+1}{2}}\rho, \nu^{-a+1}\rho], \ldots, [\nu^{\frac{x_1+1}{2}}\rho, \nu^{a+k+n-1}\rho]$. Also, if $z > -a$, then $\nu^z\rho$ can appear at most $2a+k+n-1$ times in the cuspidal support of σ_{n+1}, since it can appear only in the segments $[\nu^{\frac{x_{2a+k+n-1}+1}{2}}\rho, \nu^{-a+1}\rho], \ldots, [\nu^{\frac{x_1+1}{2}}\rho, \nu^{a+k+n-1}\rho]$. Lemma 2.1 implies $2(-a)+1 \in \mathrm{Jord}_\rho(\sigma_{n+1})$, a contradiction.

Finally, let us assume that there is an $r \in \{1, \ldots, n-1\}$ such that $y_1 = 2(-a-n+1)+1$, $y_2 = 2(-a-n+2)+1, \ldots, y_r = 2(-a-n+r)+1$, $y_{r+1} = 2(a+k)+1$, $y_{r+2} = 2(a+k+1)+1, \ldots, y_n = 2(a+k+n-r-1)+1$. This implies $-a-n+r < a+k$, so $-a < a+k+n-r$, and there is a $j \in \{r+1, \ldots, n\}$ such that $y_j = 2(-a)+1$. Note that we have

$$x_r < \ldots < x_1 < y_1 = 2(-a-n+1)+1 < y_{r+1} = 2(a+k)+1 \leqslant 2(-a)+1.$$

The cuspidal support of σ_{n+1} equals

$$[\nu^{\frac{x_1+1}{2}}\rho, \nu^{a+k+n-1}\rho] \cup [\nu^{\frac{x_2+1}{2}}\rho, \nu^{a+k+n-2}\rho] \cup \cdots \cup [\nu^{\frac{x_r+1}{2}}\rho, \nu^{a+k+n-r}\rho] \cup$$
$$[\nu^{\frac{x_{r+1}+1}{2}}\rho, \nu^{-a}\rho] \cup [\nu^{\frac{x_{r+2}+1}{2}}\rho, \nu^{-a-1}\rho] \cup \cdots \cup [\nu^{\frac{x_n+1}{2}}\rho, \nu^{-a-n+r+1}\rho] \cup \{\sigma_c\}.$$

Observe that $\nu^{-a}\rho$ appears $r+1$ times in the cuspidal support of σ_{n+1}, since it appears in the segment $[\nu^{\frac{x_{r+1}+1}{2}}\rho, \nu^{-a}\rho]$ and in the segments $[\nu^{\frac{x_r+1}{2}}\rho, \nu^{a+k+n-r}\rho]$, ..., $[\nu^{\frac{x_1+1}{2}}\rho, \nu^{a+k+n}\rho]$, since $x_i < 2(-a)+1$ for $i = 1, \ldots, r$. On the other hand, if $z > -a$, then $\nu^z\rho$ can appear at most r times in the cuspidal support of σ_{n+1}, since it does not appear in the segments $[\nu^{\frac{x_{r+1}+1}{2}}\rho, \nu^{-a}\rho], \ldots,$

$[\nu^{\frac{x_n+1}{2}}\rho, \nu^{-a-n+r+1}\rho]$, and it can appear at most once in each of the segments $[\nu^{\frac{x_r+1}{2}}\rho, \nu^{a+k+n-r}\rho], \ldots, [\nu^{\frac{x_1+1}{2}}\rho, \nu^{a+k+n}\rho]$. Lemma 2.1 implies $2(-a)+1 \in \mathrm{Jord}_\rho(\sigma_{n+1})$, a contradiction. □

Theorem 3.6. *If $a+k>0$, $a \leqslant 0$, and $a+n-1 \leqslant 0$, then $S(a,k,n,\rho) \rtimes \sigma_c$ contains a discrete series subquotient if and only if $-\frac{k}{2} < a$ and $a+n-1 \leqslant -\alpha$.*

Proof. Let us first suppose that $S(a,k,n,\rho) \rtimes \sigma_c$ contains a discrete series subquotient σ_{ds}, $a+k>0$, $a \leqslant 0$, and $a+n-1 \leqslant 0$. Proposition 3.5 implies $-\frac{k}{2} < a$. If $\alpha = 0$ we obviously have $a+n-1 \leqslant -\alpha$, so we can assume that $\alpha > 0$. To σ_{ds} we attach an ordered $(m+1)$-tuple $(\sigma_1, \sigma_2, \ldots, \sigma_{m+1})$ of discrete series representations as in Lemma 2.2. In the proof of Proposition 3.5 we have seen that $m=n$. Thus, from the Proposition 2.3 follows that $\nu^\alpha\rho$ appears in the cuspidal support of σ_{ds} at least $2n$ times. Equality of the cuspidal supports of σ_{ds} and $S(a,k,n,\rho) \rtimes \sigma_c$ implies that for every $i \in \{0,1,\ldots,n-1\}$ the segment $[\nu^{a+i}\rho, \nu^{a+k+i}\rho]$ contains both α and $-\alpha$, which implies $a+n-1 \leqslant -\alpha$.

Conversely, suppose that $-\frac{k}{2} < a$ and $a+n-1 \leqslant -\alpha$. By the classification of discrete series, the induced representation $\delta([\nu^{a+n-1}\rho, \nu^{a+k+n-1}\rho]) \rtimes \sigma_c$ contains two mutually non-isomorphic discrete series subrepresentations. We fix one of them, denote it by σ_1, and inductively define discrete series representations $\sigma_2, \sigma_3, \ldots, \sigma_n$ where, for $i = 2,3,\ldots,n$, σ_i is a discrete series subrepresentation of

$$\delta([\nu^{a+n-i}\rho, \nu^{a+k+n-i}\rho]) \rtimes \sigma_{i-1}$$

such that $\epsilon_{\sigma_i}((2(a+k+n-i)+1,\rho),(2(a+k+n-i)+3,\rho)) = -1$.

We inductively prove that, for $i = 1,2,\ldots,n$, σ_i is a subrepresentation of $S(a+n-i,k,i,\rho) \rtimes \sigma_c$. This obviously holds for $i=1$. Suppose that $i \in \{2,3,\ldots,n-1\}$ and that for $j = 1,2,\ldots,i$ we have $\sigma_j \hookrightarrow S(a+n-j,k,j,\rho) \rtimes \sigma_c$.

We prove that σ_{i+1} embeds into $S(a+n-i-1,k,i+1,\rho) \rtimes \sigma_c$. Using embeddings $\sigma_{i+1} \hookrightarrow \delta([\nu^{a+n-i-1}\rho, \nu^{a+k+n-i-1}\rho]) \rtimes \sigma_i$ and $\sigma_i \hookrightarrow S(a+n-i,k,i,\rho) \rtimes \sigma_c$, together with [6, Lemma 5.5], we get that there is an irreducible subquotient π_1 of $\delta([\nu^{a+n-i-1}\rho, \nu^{a+k+n-i-1}\rho]) \times S(a+n-i,k,i,\rho)$ such that σ_{i+1} embeds into $\pi_1 \rtimes \sigma_c$. Let $\pi_1 \cong L(\delta_1, \delta_2, \ldots, \delta_m)$, where $\delta_j \cong \delta([\nu^{x_j}\rho, \nu^{y_j}\rho])$ for $j = 1,2,\ldots,m$. It follows that

$$m^*(\delta([\nu^{a+n-i-1}\rho, \nu^{a+k+n-i-1}\rho]) \times S(a+n-i,k,i,\rho))$$

contains an irreducible constituent of the form $\nu^{y_1}\rho \otimes \pi$, so $y_1 \in \{a+k+n-i-1, a+k+n-i\}$. Since σ_{i+1} embeds into an induced representation of the form $\nu^{y_1}\rho \rtimes \pi'$, using $\epsilon_{\sigma_{i+1}}((2(a+k+n-i-1)+1, \rho), (2(a+k+n-i-1)+3, \rho)) = -1$ and [18, Proposition 7.2], we obtain $y_1 = a+k+n-i-1$. Now, following exactly the same lines as in the proof of Theorem 3.3, we deduce $\pi_1 \cong S(a+n-i-1, k, i+1, \rho)$.

Consequently, σ_n is a subrepresentation of $S(a, k, n, \rho) \rtimes \sigma_c$. □

Proposition 3.7. *Suppose that* $a+k > 0$, $a \leqslant 0$, $a+n-1 \leqslant -\alpha$, *and* $-\frac{k}{2} < a$. *Let* $Jord = Jord(\sigma_c) \cup \{(2(a+k+i)+1, \rho), (2(-a-i)+1, \rho) : i = 0, 1, \ldots, n-1\}$. *We define two admissible triples* $(Jord, \sigma_c, \epsilon_1)$ *and* $(Jord, \sigma_c, \epsilon_{-1})$ *with* $\epsilon_{\pm 1}((2(-a)+1, \rho), (2(a+k)+1, \rho)) = 1$, $\epsilon_{\pm 1}((x_-, \rho'), (x, \rho')) = -1$ *for* $(x, \rho') \in Jord$ *such that* x_- *is defined and* $(x, \rho') \notin \{(2(a+k)+1, \rho), (2(-a-n+1)+1, \rho)\}$. *Also, for* $j \in \{1, -1\}$ *let*

- $\epsilon_j((2\alpha-1, \rho), (2(-a-n+1)+1, \rho)) = j$ *if* $\alpha \geqslant 1$,
- $\epsilon_j(2(-a-n+1)+1, \rho) = j$ *if* $\alpha = \frac{1}{2}$, *and* $\epsilon_j(2, \rho) = -1$ *if* $\alpha > \frac{1}{2}$ *and* $\alpha \notin \mathbb{Z}$,
- $\epsilon_j(2(a+k+n-1)+1, \rho) = j$ *if* $\alpha = 0$,

and let $\epsilon_{\pm 1}(\min(Jord_{\rho'}), \rho') = -1$ *for* $\rho' \in R(GL)$ *such that* $Jord_{\rho'}$ *is non-empty, consists of even integers, and* $\rho' \not\cong \rho$. *For* $j \in \{1, -1\}$, *we denote by* σ_j *discrete series corresponding to the admissible triple* $(Jord, \sigma_c, \epsilon_j)$. *Then* σ_1 *and* σ_{-1} *are subrepresentations of* $S(a, k, n, \rho) \rtimes \sigma_c$. *If* σ *is a discrete series subquotient of* $S(a, k, n, \rho) \rtimes \sigma_c$, *then there is a* $j \in \{1, -1\}$ *such that* $\sigma \cong \sigma_j$.

Proof. It can be directly seen that both triples $(\text{Jord}, \sigma_c, \epsilon_1)$ and $(\text{Jord}, \sigma_c, \epsilon_{-1})$ are admissible. In the proof of Theorem 3.6 we have constructed discrete series subrepresentations of $S(a, k, n, \rho) \rtimes \sigma_c$ which correspond to $(\text{Jord}, \sigma_c, \epsilon_{\pm 1})$.

We note that $\text{Jord}_\rho(\sigma_c) = \emptyset$ exactly when $\alpha \in \{0, \frac{1}{2}\}$.

Let σ be a discrete series subquotient of $S(a, k, n, \rho) \rtimes \sigma_c$. It is easy to see that $\text{Jord}(\sigma) = \text{Jord}$, $\sigma_{cusp} \cong \sigma_c$, and that σ is not strongly positive.

By the formula (4), $\mu^*(S(a, k, n, \rho) \rtimes \sigma_c)$ contains an irreducible constituent of the form $\nu^x \rho' \otimes \pi$ only for $(x, \rho') \in \{(a+k, \rho), (-a-n+1, \rho)\}$. Using [18, Propositions 7.2, 7.4], we conclude that $\epsilon_\sigma((x_-, \rho'), (x, \rho')) = -1$ for $(x, \rho') \in \text{Jord}$ such that x_- is defined and $(x, \rho') \notin \{(2(a+k)+1, \rho), (2(-a-n+1)+1, \rho)\}$, and that $\epsilon_\sigma(\min(\text{Jord}_{\rho'}), \rho') = -1$ for $\rho' \in R(GL)$ such that $\rho' \not\cong \rho$, $\text{Jord}_{\rho'}$ is non-empty and consists of even integers. If additionally we have

$\epsilon_\sigma((2(-a)+1,\rho),(2(a+k)+1,\rho)) = 1$, it follows at once that $\epsilon_\sigma \in \{\epsilon_1, \epsilon_{-1}\}$. Since $(\mathrm{Jord}(\sigma), \epsilon_\sigma, \sigma_c)$ is not of the alternated type, in cases $\mathrm{Jord}_\rho(\sigma_c) = \varnothing$ and $\epsilon_\sigma((2\alpha-1,\rho),(2(-a-n+1)+1,\rho)) = -1$, for $\mathrm{Jord}_\rho(\sigma_c) \neq \varnothing$, it easily follows that $\epsilon_\sigma((2(-a)+1,\rho),(2(a+k)+1,\rho)) = 1$. Namely, if $\mathrm{Jord}_\rho(\sigma_c) = \varnothing$, we have $\min(\mathrm{Jord}_\rho) = 2(-a-n+1)+1$, so $\epsilon_\sigma((x_-,\rho), (x,\rho)) = 1$ can hold only for $x = 2(a+k)+1$. In the second case, the ϵ-function ϵ_σ attains value -1 on all pairs of the form $((x_-,\rho'), (x,\rho'))$ for $(x,\rho') \neq (2(a+k)+1,\rho)$.

It remains to consider the case $\alpha \geqslant 1$ and $\epsilon_\sigma((2\alpha-1,\rho),(2(-a-n+1)+1,\rho)) = 1$. Let us show that then we have $\epsilon_\sigma = \epsilon_1$.

We denote by σ' a discrete series such that σ embeds into $\delta([\nu^{-\alpha+1}\rho, \nu^{-a-n+1}\rho]) \rtimes \sigma'$. If $2\alpha - 1 > \min(\mathrm{Jord}_\rho)$, from $\epsilon_\sigma((2\alpha-3,\rho),(2\alpha-1,\rho)) = \epsilon_\sigma((2(-a-n+1)+1,\rho),(2(-a-n+2)+1,\rho)) = -1$ and (1) implies $\epsilon_{\sigma'}((2\alpha-3,\rho),(2(-a-n+2)+1,\rho)) = 1$. Continuing in this way, we deduce that there is a discrete series σ'' such that σ is a subrepresentation of

$$\delta([\nu^{-\alpha+1}\rho, \nu^{-a-n+1}\rho]) \times \cdots \times \delta([\nu^{-\alpha+m}\rho, \nu^{-a-n+m}\rho]) \rtimes \sigma'', \tag{6}$$

where $m = \min\{\lfloor\alpha\rfloor, n\}$. The Frobenius reciprocity implies that the Jacquet module of σ with respect to the appropriate parabolic subgroup contains

$$\begin{gathered}\delta([\nu^{-\alpha+1}\rho, \nu^{-a-n+1}\rho]) \otimes \delta([\nu^{-\alpha+2}\rho, \nu^{-a-n+2}\rho]) \otimes \cdots \otimes \\ \delta([\nu^{-\alpha+m}\rho, \nu^{-a-n+m}\rho]) \otimes \sigma''.\end{gathered} \tag{7}$$

Since σ is a subquotient of $S(a,k,n,\rho) \rtimes \sigma_c$, a repeated application of the formula (4) implies that σ'' is an irreducible subquotient of

$$\begin{gathered}L(\delta([\nu^{a}\rho, \nu^{a+k}\rho]), \ldots, \delta([\nu^{a+n-m-1}\rho, \nu^{a+k+n-m-1}\rho]), \\ \delta([\nu^{\alpha-m+1}\rho, \nu^{a+k+n-m}\rho]), \ldots, \delta([\nu^{\alpha}\rho, \nu^{a+k+n-1}\rho])) \rtimes \sigma_c,\end{gathered} \tag{8}$$

and that the multiplicity of (7) in the Jacquet module of $S(a,k,n,\rho) \rtimes \sigma_c$ with respect to the appropriate parabolic subgroup equals the multiplicity of σ'' in (8).

If $m = n$, it follows that σ'' is a discrete series subquotient of $S(\alpha-n+1, k+n-1, n, \rho) \rtimes \sigma_c$, with $\alpha - n + 1 > 0$. By Proposition 3.1, this induced representation contains a unique discrete series subquotient, which appears there with multiplicity one. Thus, the Jacquet module of $S(a,k,n,\rho) \rtimes \sigma_c$ with respect to the appropriate parabolic subgroup contains a unique constituent of the form (7), which appears with multiplicity one, and has to appear in the Jacquet module of σ_1 with respect to the appropriate parabolic subgroup since $\epsilon_1((2\alpha - 1, \rho), (2(-a - n + 1) + 1, \rho)) = 1$. Since both σ and σ_1 are subquotients of $S(a, k, n, \rho) \rtimes \sigma_c$, it follows that $\epsilon_\sigma = \epsilon_1$.

If $m < n$, σ'' is a non-strongly positive discrete series subquotient of (8), so $\mu^*(\sigma'')$ contains an irreducible constituent of the form $\delta([\nu^x \rho', \nu^y \rho']) \otimes \pi$, where $x \leqslant 0$ and $x + y > 0$. Since $m = \lfloor \alpha \rfloor$, using (4) we deduce that $(y, \rho') = (a + k, \rho)$. Thus, $\epsilon_{\sigma''}((2(-a) + 1, \rho), (2(a + k) + 1, \rho)) = 1$ and σ'' is a subrepresentation of an induced representation of the form $\delta([\nu^a \rho, \nu^{a+k} \rho]) \rtimes \pi$. Since $a < -\alpha + 1$ and $a + k > -a - n + m$, embedding (6) can be used to obtain that σ is also a subrepresentation of an induced representation of the form $\delta([\nu^a \rho, \nu^{a+k} \rho]) \rtimes \pi$. This implies $\epsilon_\sigma((2(-a) + 1, \rho), (2(a + k) + 1, \rho)) = 1$ so $\epsilon_\sigma = \epsilon_1$ and the proof follows. □

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