On representations induced from the Zelevinsky segment and a tempered representation in the half-integral case

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Abstract

Let G_n denote either the group $SO(2n+1, F)$ or $Sp(2n, F)$ over a non-archimedean local field of characteristic zero. We determine the reducibility criteria for a parabolically induced representation of the form $\langle[\nu^a\rho, \nu^b\rho]\rangle \rtimes \tau$, where $\langle[\nu^a\rho, \nu^b\rho]\rangle$ denotes the Zelevinsky segment representation of the general linear group attached to the segment $[\nu^a\rho, \nu^b\rho]$, with a half-integral, and τ denotes an irreducible tempered representation of G_n.

1 Introduction

We study the structure of parabolically induced representations of the form $\langle[\nu^a\rho, \nu^b\rho]\rangle \rtimes \tau$, where $\langle[\nu^a\rho, \nu^b\rho]\rangle$ denotes the Zelevinsky segment representation of the general linear group attached to the segment $[\nu^a\rho, \nu^b\rho]$, and τ denotes an irreducible tempered representation of either odd special orthogonal or symplectic group over a non-archimedean local field of characteristic zero.

Zelevinsky segment representations and irreducible tempered representations both present prominent members of the unitary duals. Furthermore, the Zelevinsky segment representations belong to the class of the essentially

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Speh representations, which are the basic building blocks in the unitary dual of the general linear group. On the other hand, irreducible tempered representations play a fundamental role in the Langlands classification for the classical groups, and have been independently classified in [5] and [17], based on the work of Goldberg ([2]) and on the Mœglin-Tadić classification of discrete series ([10, 12]).

The main aim of our investigation is to provide the reducibility criteria for the induced representations of the form $\langle[\nu^a\rho, \nu^b\rho]\rangle \rtimes \tau$, with a half-integral and τ irreducible tempered. This is a natural continuation of our previous work on the reducibility and composition factors of representations induced from the Zelevinsky segment representation and a discrete series of the classical p-adic group ([7, 8]). It appears that the reducibility criteria mostly rely on a deeper knowledge on the structure of irreducible tempered subquotients.

Our work builds on the methods introduced in [13, 14], and further developed in [7]. But, as expected, the tempered case happens to be much more involved then the discrete series one. To obtain more precise information regarding the irreducible tempered subquotients of $\langle[\nu^a\rho, \nu^b\rho]\rangle \rtimes \tau$, for an irreducible tempered representation τ, in most of the cases we use the reduction to the discrete series case, and then follow the results of [8]. We note that proofs appearing in [8] can also be directly applied to the case of positive integral a, so we also cover that case.

The main strategy used in [13, 14] is rather straightforward, and relies on the determination of conditions under which the induced representation contains at least two mutually non-isomorphic irreducible subquotients. We mostly follow this strategy, in a similar way as in [7], but we also obtain more precise results on the existence of irreducible tempered subquotients and on the general form of irreducible non-tempered subquotients. This enables us to provide a complete and uniform description of the reducibility criteria in the cases considered.

For the convenience of the reader, we cite the less technical version of our reducibility criteria here.

Theorem 1.1. *Let ρ denote an irreducible cuspidal representation of the general linear group and let τ stand for an irreducible tempered representation of either symplectic or odd special orthogonal group over a non-archimedean local field of characteristic zero. Let a, b denote real numbers such that $b - a$ is a non-negative integer. If $2a \notin \mathbb{Z}$, then the induced representation*

$\langle[\nu^a\rho,\nu^b\rho]\rangle \rtimes \tau$ is irreducible.

If $2a$ is a positive integer, $a \geq 1$, and $\rho \ncong \widetilde{\rho}$, then $\langle[\nu^a\rho,\nu^b\rho]\rangle \rtimes \tau$ reduces if and only if the Jacquet module of τ with respect to the appropriate parabolic subgroup contains an irreducible constituent of the form $\delta([\nu^{-a+1}\widetilde{\rho},\nu^{a-1}\widetilde{\rho}]) \otimes \pi$. If $2a$ is a positive integer, $a \geq 1$, and $\rho \cong \widetilde{\rho}$, then $\langle[\nu^a\rho,\nu^b\rho]\rangle \rtimes \tau$ reduces if and only if one of the following holds:

(1) The Jacquet module of τ with respect to the appropriate parabolic subgroup contains an irreducible constituent of the form $\delta([\nu^{-a+1}\rho,\nu^{a-1}\rho]) \otimes \pi$.

(2) There is a $c \in \{a, a+1, \ldots, b\}$ such that $\langle[\nu^a\rho,\nu^c\rho]\rangle \rtimes \tau$ contains an irreducible tempered subquotient.

(3) There is a $c \in \{a, a+1, \ldots, b\}$ and an irreducible tempered representation τ' such that $\langle[\nu^a\rho,\nu^c\rho]\rangle \rtimes \tau$ contains a unique irreducible subrepresentation of $\delta([\nu^{-c}\rho,\nu^{c-1}\rho]) \rtimes \tau'$.

If $a = \frac{1}{2}$, then $\langle[\nu^a\rho,\nu^b\rho]\rangle \rtimes \tau$ reduces if and only if $\rho \cong \widetilde{\rho}$ and one of the following holds:

(1) There is a $c \in \{\frac{1}{2}, \frac{3}{2}, \ldots, b\}$ such that $\langle[\nu^{\frac{1}{2}}\rho,\nu^c\rho]\rangle \rtimes \tau$ contains an irreducible tempered subquotient.

(2) There is a $c \in \{\frac{3}{2}, \frac{5}{2}, \ldots, b\}$ and an irreducible tempered representation τ' such that $\langle[\nu^{\frac{1}{2}}\rho,\nu^c\rho]\rangle \rtimes \tau$ contains a unique irreducible subrepresentation of $\delta([\nu^{-c}\rho,\nu^{c-1}\rho]) \rtimes \tau'$.

If $a < 0$, $2a \in \mathbb{Z}$, $a \notin \mathbb{Z}$ and $-a \leq b$, then $\langle[\nu^a\rho,\nu^b\rho]\rangle \rtimes \tau$ reduces if and only if $\rho \cong \widetilde{\rho}$ and there is a $c \in \{\frac{1}{2}, \frac{3}{2}, \ldots, b\}$ such that $\langle[\nu^{\frac{1}{2}}\rho,\nu^c\rho]\rangle \rtimes \tau$ reduces.

The above criteria is explicitly given in terms of the classifications of irreducible tempered representations and discrete series representations in Theorems 4.5 and 5.3, and also summarized in Theorem 6.6.

In the following section we present some preliminaries. Also, we briefly collect some results on the structure of tempered representations, and provide an adjustment of some technical results from [7]. The third section is the technical core of the paper. Using the Jacquet modules method and embeddings of tempered representations, in that section we obtain necessary and sufficient conditions under which $\langle[\nu^a\rho,\nu^b\rho]\rangle \rtimes \tau$ contains an irreducible tempered subquotient for $a \le \frac{1}{2}$, a half-integral. Also, under a technical assumption which does not effect the reducibility criteria, we obtain similar conditions for $a \ge 1$. This enables us to provide an explicit description of the reducibility points, using a case-by-case consideration. In the Section 4 we study the case $a \ge 1$, while in the Section 5 we describe the case $a = \frac{1}{2}$. We close the paper with the investigation of the case a negative and half-integral, which we reduce to the previously considered $\frac{1}{2}$-case, followed by a summary of our results.

2 Preliminaries

Let F denote a non-archimedean local field of the characteristic zero. We first describe the groups that we consider.

Let $J_n = (\delta_{i,n+1-j})_{1\le i,j\le n}$ denote the $n \times n$ matrix, where $\delta_{i,n+1-j}$ stands for the Kronecker symbol. For a square matrix g, we denote by g^t its transposed matrix, and by g^τ its transposed matrix with respect to the second diagonal. In what follows, we shall fix one of the series of classical groups

$$Sp(n,F) = \left\{ g \in GL(2n,F) : \begin{pmatrix} 0 & -J_n \\ J_n & 0 \end{pmatrix} g^t \begin{pmatrix} 0 & -J_n \\ J_n & 0 \end{pmatrix} = g^{-1} \right\},$$

or

$$SO(2n+1,F) = \left\{ g \in GL(2n+1,F) : g^\tau = g^{-1} \right\}$$

and denote by G_n the rank n group belonging to the series which we fixed. Also, let $GL(m,F)$ denote the general linear group of rank m over F.

The set of standard parabolic subgroups will be fixed in a usual way, i.e., we fix a minimal F-parabolic subgroup in the classical group G_n consisting of upper-triangular matrices in the usual matrix realization of the classical group. Then the Levi factors of standard parabolic subgroups have the form $M = GL(n_1,F)\times\cdots\times GL(n_k,F)\times G_{n'}$. If δ_i is a representation of $GL(n_i,F)$, for $i = 1,\ldots,k$, and τ a representation of $G_{n'}$, the normalized parabolically induced representation $\mathrm{Ind}_M^{G_n}(\delta_1\otimes\cdots\otimes\delta_k\otimes\tau)$ will be denoted by $\delta_1\times\cdots\times\delta_k \rtimes \tau$. We use a similar notation to denote a parabolically induced representation of $GL(m,F)$.

By $\mathrm{Irr}(G_n)$ we denote the set of all irreducible admissible representations of G_n. Let $R(G_n)$ denote the Grothendieck group of admissible representations of finite length of G_n and define $R(G) = \oplus_{n\ge 0} R(G_n)$. In a similar way we define $\mathrm{Irr}(GL(n,F))$ and $R(GL)=\oplus_{n\geq 0}R(GL(n,F))$. We note that in $R(G)$ we have $\pi\rtimes\tau=\widetilde{\pi}\rtimes\tau$ and $\pi_1\times\pi_2\rtimes\tau=\pi_2\times\pi_1\rtimes\tau$.

For $\sigma\in\mathrm{Irr}(G_n)$ and $1\leq k\leq n$ we denote by $r_{(k)}(\sigma)$ the normalized Jacquet module of σ with respect to the maximal parabolic subgroup $P_{(k)}$ having the Levi subgroup equal to $GL(k,F)\times G_{n-k}$. We identify $r_{(k)}(\sigma)$ with its semisimplification in $R(GL(k,F))\otimes R(G_{n-k})$ and consider

$$\mu^*(\sigma)=1\otimes\sigma+\sum_{k=1}^{n}r_{(k)}(\sigma)\in R(GL)\otimes R(G).$$

Let ν stand for a composition of the determinant mapping with the normalized absolute value on F. Let $\rho\in R(GL)$ stand for an irreducible cuspidal representation. By a segment in $R(GL)$ we mean a set of the form $[\rho,\nu^m\rho]:=\{\rho,\nu\rho,\ldots,\nu^m\rho\}$, where m stands for a non-negative integer. The induced representation $\rho\times\nu\rho\times\cdots\times\nu^m\rho$ has a unique irreducible subrepresentation ([18]), which is denoted by $\langle[\rho,\nu^m\rho]\rangle$ and called the Zelevinsky segment representation.

The induced representation $\nu^m\rho\times\nu^{m-1}\rho\times\cdots\times\rho$ contains a unique irreducible subrepresentation, denoted by $\delta([\rho,\nu^m\rho])$. Representation $\delta([\rho,\nu^m\rho])$ is essentially square-integrable, and, by [18], every irreducible essentially square-integrable representation in $R(GL)$ can be obtained in this way.

We frequently use the following structural formulas, obtained in [3, Theorem 1.4] and in [16, Theorems 5.4, 6.5]:

Theorem 2.1. *Let $\rho\in R(GL)$ denote an irreducible cuspidal representation and $k,l\in\mathbb{R}$ such that $k+l$ is a non-negative integer. Let $\sigma\in R(G)$ denote an irreducible admissible representation of finite length. Write $\mu^*(\sigma)=\sum_{\pi,\sigma'}\pi\otimes\sigma'$. Then we have:*

$$\begin{aligned}\mu^*(\langle[\nu^{-k}\rho,\nu^{l}\rho]\rangle\rtimes\sigma)=\sum_{i=0}^{k+l+1}\sum_{j=0}^{i}\sum_{\pi,\sigma'}&\langle[\nu^{-l}\widetilde{\rho},\nu^{-i+k}\widetilde{\rho}]\rangle\times\langle[\nu^{-k}\rho,\nu^{j-k-1}\rho]\rangle\times\pi\\&\otimes\langle[\nu^{j-k}\rho,\nu^{i-k-1}\rho]\rangle\rtimes\sigma',\end{aligned}$$

$$\begin{aligned}\mu^*(\delta([\nu^{-k}\rho,\nu^{l}\rho])\rtimes\sigma)=\sum_{i=-k-1}^{l}\sum_{j=i}^{l}\sum_{\pi,\sigma'}&\delta([\nu^{-i}\widetilde{\rho},\nu^{k}\widetilde{\rho}])\times\delta([\nu^{j+1}\rho,\nu^{l}\rho])\times\pi\\&\otimes\delta([\nu^{i+1}\rho,\nu^{j}\rho])\rtimes\sigma'.\end{aligned}$$

We omit $\langle[\nu^x\rho,\nu^y\rho]\rangle$ and $\delta([\nu^x\rho,\nu^y\rho])$ if $x>y$.

Let us take a moment to recall the subrepresentation version of the Langlands classification for general linear groups.

For every irreducible essentially square-integrable representation $\delta \in R(GL)$, there is a unique $e(\delta) \in \mathbb{R}$ such that $\nu^{-e(\delta)}\delta$ is unitarizable. We note that $e(\delta([\nu^a\rho, \nu^b\rho])) = (a+b)/2$. Suppose that $\delta_1, \delta_2, \ldots, \delta_k$ are irreducible essentially square-integrable representations such that $e(\delta_1) \leq e(\delta_2) \leq \ldots \leq e(\delta_k)$. Then the induced representation $\delta_1 \times \delta_2 \times \cdots \times \delta_k$ has a unique irreducible (Langlands) subrepresentation, denoted by $L(\delta_1, \delta_2, \ldots, \delta_k)$, which appears with multiplicity one in the composition series of $\delta_1 \times \delta_2 \times \cdots \times \delta_k$. Every irreducible representation $\pi \in R(GL)$ is isomorphic to some $L(\delta_1, \delta_2, \ldots, \delta_k)$ and, for a given π, the representations $\delta_1, \delta_2, \ldots, \delta_k$ are unique up to a permutation.

We also use the subrepresentation version of the Langlands classification for classical groups, and realize a non-tempered irreducible representation π of G_n as the unique irreducible (Langlands) subrepresentation of an induced representation of the form $\delta_1 \times \delta_2 \times \cdots \times \delta_k \rtimes \tau$, where τ is an irreducible tempered representation of some G_t, and $\delta_1, \delta_2, \ldots, \delta_k \in R(GL)$ are irreducible essentially square-integrable representations such that $e(\delta_1) \leq e(\delta_2) \leq \cdots \leq e(\delta_k) < 0$. In this case, we write $\pi = L(\delta_1, \delta_2, \ldots, \delta_k; \tau)$.

In the following theorem we gather some results from [2] and [15, Section 1] on irreducible tempered representations in $R(G)$.

Theorem 2.2. *(1) Suppose that $\delta_1, \ldots, \delta_k \in R(GL)$ and $\sigma_1 \in R(G)$ are discrete series. Let us denote by m the number of mutually nonisomorphic representations δ_i such that $\delta_i \rtimes \sigma_1$ reduces. The induced representation $\delta_1 \times \cdots \times \delta_k \rtimes \sigma_1$ is a direct sum of 2^m mutually nonisomorphic irreducible tempered representations.*

(2) For an irreducible tempered representation $\tau_1 \in R(G)$, exist discrete series $\delta_1, \ldots, \delta_k \in R(GL)$ and $\sigma_1 \in R(G)$ such that τ_1 is a subrepresentation of $\delta_1 \times \cdots \times \delta_k \rtimes \sigma_1$. If $\delta'_1, \ldots, \delta'_k \in R(GL)$ and $\sigma'_1 \in R(G)$ are discrete series such that τ_1 is a subrepresentation of $\delta'_1 \times \cdots \times \delta'_l \rtimes \sigma'_1$, then $\sigma_1 \cong \sigma'_1$ and the sequence $\delta'_1, \ldots, \delta'_l$ can be obtained from the sequence $\delta_1, \ldots, \delta_k$ by permuting and taking contragredients.

(3) Suppose that an irreducible tempered representation $\tau_1 \in R(G)$ is a subrepresentation of $\delta_1 \times \cdots \times \delta_k \rtimes \sigma_1$, for discrete series $\delta_1, \ldots, \delta_k \in R(GL)$ and $\sigma_1 \in R(G)$. Let $\delta \in R(GL)$ stand for a discrete series. The induced

representation $\delta \rtimes \tau_1$ reduces if and only if $\delta \notin \{\delta_1, \ldots, \delta_k, \widetilde{\delta_1}, \ldots, \widetilde{\delta_k}\}$ and $\delta \rtimes \sigma$ reduces.

(4) Let $\delta \in R(GL)$ denote a discrete series, and let $\tau_1 \in R(G)$ denote an irreducible tempered representation. If $\delta \rtimes \tau_1$ reduces, then it is a direct sum of two mutually nonisomorphic tempered representations.

We frequently use the following result ([4, Lemma 5.5]):

Lemma 2.3. *Suppose that $\pi \in R(G_n)$ is an irreducible representation, λ an irreducible representation of the Levi subgroup M of G_n, and π is a subrepresentation of $Ind_M^{G_n}(\lambda)$. If $L > M$, then there is an irreducible subquotient ρ of $Ind_M^L(\lambda)$ such that π is a subrepresentation of $Ind_L^{G_n}(\rho)$.*

To shorten the notation, for an irreducible essentially square-integrable representation $\delta \in R(GL)$ and a positive integer m, we denote by δ^m the induced representation $\delta \times \cdots \times \delta$, where δ appears m times.

Throughout the paper, we fix an irreducible tempered representation $\tau \in R(G)$ and an irreducible cuspidal representation $\rho \in R(GL)$.

We are interested in determining when the induced representation $\langle[\nu^a\rho, \nu^b\rho]\rangle \rtimes \tau$, for real numbers a and b such that $b - a$ is a nonnegative integer. Since in $R(G)$ we have $\langle[\nu^a\rho, \nu^b\rho]\rangle \rtimes \tau \cong \langle[\nu^{-b}\widetilde{\rho}, \nu^{-a}\widetilde{\rho}]\rangle \rtimes \tau$, we can always assume that $a + b \geq 0$.

Also, from now on we fix a discrete series $\sigma \in R(G)$ such that τ can be written as a subrepresentation of $\delta^{(1)} \times \cdots \times \delta^{(r)} \rtimes \sigma$, for discrete series $\delta^{(1)}, \ldots, \delta^{(r)} \in R(GL)$. By the part (2) of Theorem 2.2, σ is unique up to an isomorphism.

By the Mœglin-Tadić classification, which now holds unconditionally, due to [1] and [11, Théorème 3.1.1], every discrete series $\sigma' \in R(G)$ corresponds to an admissible triple, consisting of the Jordan block, the partial cuspidal support, and the ϵ-function. The admissible triple corresponding to σ' will be denoted by $(\mathrm{Jord}(\sigma'), \sigma'_{cusp}, \epsilon_{\sigma'})$. For more details on these invariants we refer the reader to [10], [12], and [14, Section 1].

For a discrete series $\sigma' \in R(G)$ and an irreducible selfcontragredient cuspidal representation $\rho' \in R(GL)$ we write $\mathrm{Jord}_{\rho'}(\sigma') = \{x : (x, \rho') \in \mathrm{Jord}(\sigma')\}$. If $\mathrm{Jord}_{\rho'}(\sigma') \neq \emptyset$ and $x \in \mathrm{Jord}_{\rho'}(\sigma')$, denote $x_- = \max\{y \in \mathrm{Jord}_{\rho'}(\sigma') : y < x\}$, if it exists.

We note that, by the classification of discrete series, $\delta([\nu^{-x_i}\rho_i, \nu^{x_i}\rho_i]) \rtimes \sigma'$ reduces if and only if $\rho_i \cong \widetilde{\rho_i}$, $2x_i + 1$ is of the appropriate parity, and $(2x_i + 1, \rho_i) \notin \mathrm{Jord}(\sigma')$.

Proof of the following lemma is immediate.

Lemma 2.4. *Let $\delta \in R(GL)$ denote an irreducible square-integrable representation. Then $\mu^*(\tau)$ contains an irreducible constituent of the form $\delta \otimes \pi$ if and only if there is an irreducible tempered representation $\tau' \in R(G)$ such that τ is a subrepresentation of $\delta \rtimes \tau'$. Also, if $\mu^*(\tau)$ contains an irreducible constituent of the form $\delta \otimes \pi$, then π is tempered.*

The following result can be obtained following the same lines as in the proofs of [17, Lemma 4.1, Lemma 4.4].

Lemma 2.5. *Let $\tau_1 \in R(G)$ denote an irreducible tempered representation, which is a subrepresentation of $\delta_1 \times \cdots \times \delta_k \rtimes \sigma_1$, for discrete series $\delta_1, \ldots, \delta_k \in R(GL)$, $\sigma_1 \in R(G)$. Let $\rho_1 \in R(GL)$ stand for an irreducible self-contragredient cuspidal representation, and let m and c denote positive integers such that $\delta([\nu^{-\frac{c-1}{2}}\rho_1, \nu^{\frac{c-1}{2}}\rho_1])^m \rtimes \tau_1$ reduces.*

(1) If $Jord_{\rho_1}(\sigma_1) \cap [1, c] \neq \emptyset$ and $d = \max(Jord_{\rho_1}(\sigma_1) \cap [1, c])$, then there is a unique irreducible subrepresentation τ_2 of $\delta([\nu^{-\frac{c-1}{2}}\rho_1, \nu^{\frac{c-1}{2}}\rho_1])^m \rtimes \tau_1$ which contains an irreducible constituent of the form

$$\delta([\nu^{\frac{d+1}{2}}\rho_1, \nu^{\frac{c-1}{2}}\rho_1])^{2m} \otimes \pi$$

in its Jacquet module with respect to the appropriate parabolic subgroup.

(2) If $Jord_{\rho_1}(\sigma_1) \cap [1, c] = \emptyset$ and c is even, then there is a unique irreducible subrepresentation τ_2 of $\delta([\nu^{-\frac{c-1}{2}}\rho_1, \nu^{\frac{c-1}{2}}\rho_1])^m \rtimes \tau_1$ which contains an irreducible constituent of the form

$$\delta([\nu^{\frac{1}{2}}\rho_1, \nu^{\frac{c-1}{2}}\rho_1])^{2m} \otimes \pi$$

in its Jacquet module with respect to the appropriate parabolic subgroup.

At the end of this section we obtain several useful technical results.

Lemma 2.6. *Suppose that $2a$ is a positive integer. Let $L(\delta_1, \ldots, \delta_k; \tau')$ stand for an irreducible non-tempered subquotient of $\langle[\nu^a\rho, \nu^b\rho]\rangle \rtimes \tau$, and let $\delta_i \cong \delta([\nu^{x_i}\rho_i, \nu^{y_i}\rho_i])$ for $i = 1, \ldots, k$. Then for all $i = 1, \ldots, k$ we have $\rho_i \cong \widetilde{\rho}$ and $x_i \in \{y_i, -y_i - 1\}$. Furthermore, we have the following:*

(1) Suppose that for every $i \in \{1, 2, \ldots, k\}$ we have $x_i = y_i$. Then $x_i = -b + i - 1$ and τ' is an irreducible subquotient of $\langle[\nu^a\rho, \nu^{b-k}\rho]\rangle \rtimes \tau$.

(2) *Suppose that there is an* $i \in \{1, 2, \ldots, k\}$ *such that* $x_i = -y_i - 1$, $x_i \neq y_i$. *Then* $i = k$, *and there is an irreducible tempered representation* $\tau_1 \in R(G)$ *such that* τ *is a subrepresentation of* $\delta([\nu^{-b+k}\widetilde{\rho}, \nu^{b-k}\widetilde{\rho}]) \rtimes \tau_1$, *and* τ' *is a subquotient of* $\langle[\nu^a\rho, \nu^{b-k}\rho]\rangle \rtimes \tau_1$.

Proof. Since $L(\delta_1, \ldots, \delta_k; \tau')$ is a subrepresentation of $\delta_1 \rtimes L(\delta_2, \ldots, \delta_k; \tau')$, Frobenius reciprocity implies that $\mu^*(\langle[\nu^a\rho, \nu^b\rho]\rangle \rtimes \tau)$ contains $\delta_1 \otimes L(\delta_2, \ldots, \delta_k; \tau')$. By the structural formula, there are i, j such that $a - 1 \leq i \leq j \leq b$, and an irreducible constituent $\delta \otimes \pi$ of $\mu^*(\tau)$ such that

$$\delta([\nu^{x_1}\rho_1, \nu^{y_1}\rho_1]) \leq \langle[\nu^{-b}\widetilde{\rho}, \nu^{-j-1}\widetilde{\rho}]\rangle \times \langle[\nu^a\rho, \nu^i\rho]\rangle \times \delta$$

and

$$L(\delta_2, \ldots, \delta_k; \tau') \leq \langle[\nu^{i+1}\rho, \nu^j\rho]\rangle \rtimes \pi.$$

Since $a > 0$ and τ is tempered, it follows that $\rho_1 \cong \widetilde{\rho}$ and $x_1 = -b$. Also, since $\delta([\nu^{x_1}\rho_1, \nu^{y_1}\rho_1])$ is an irreducible essentially square-integrable representation and $a \geq 1$ implies $a > -b+1$, we obtain that $j = b-1$ and $i = a-1$. Consequently, either $\pi \cong \tau$ or $\delta \cong \delta([\nu^{x_1+1}\widetilde{\rho}, \nu^{y_1}\widetilde{\rho}])$. Since $x_1 + y_1 < 0$ and τ is tempered, we conclude that $\pi \ncong \tau$ implies $x_1 = -y_1 - 1$. Thus, π is tempered, $x_1 \in \{y_1, -y_1 - 1\}$ and $L(\delta_2, \ldots, \delta_k; \tau')$ is an irreducible subquotient of $\langle[\nu^a\rho, \nu^{b-1}\rho]\rangle \rtimes \pi$. Repeating this procedure, we obtain that for all $i = 1, \ldots, k$ we have $\rho_i \cong \widetilde{\rho}$ and $x_i \in \{y_i, -y_i - 1\}$.

Suppose that there is an $i \in \{1, 2, \ldots, k\}$ such that $x_i = -y_i - 1$, $x_i \neq y_i$, and let $i_{\min}$ denote the minimal such i. From the first part of the proof, we deduce that $L(\delta_{i_{\min}}, \ldots, \delta_k; \tau')$ is an irreducible subquotient of $\langle[\nu^a\rho, \nu^{b-i_{\min}+1}\rho]\rangle \rtimes \tau$.

If $i_{\min} = k$, we get that there is an irreducible tempered representation $\tau_1 \in R(G)$ such that τ is a subrepresentation of $\delta([\nu^{-b+k}\widetilde{\rho}, \nu^{b-k}\widetilde{\rho}]) \rtimes \tau_1$ and τ' is a subquotient of $\langle[\nu^a\rho, \nu^{b-k}\rho]\rangle \rtimes \tau_1$.

Suppose that $i_{\min} < k$. Then there is an irreducible tempered representation $\tau_1 \in R(G)$ such that τ is a subrepresentation of $\delta([\nu^{-b+i_{\min}}\widetilde{\rho}, \nu^{b-i_{\min}}\widetilde{\rho}]) \rtimes \tau_1$ and $L(\delta_{i_{\min}+1}, \ldots, \delta_k; \tau')$ is an irreducible subquotient of $\langle[\nu^a\rho, \nu^{b-i_{\min}}\rho]\rangle \rtimes \tau_1$. In the same way as in the first part of the proof, we deduce $\delta_{i_{\min}+1} \in \{\nu^{-b+i_{\min}}\widetilde{\rho}, \delta([\nu^{-b+i_{\min}}\widetilde{\rho}, \nu^{b-i_{\min}-1}\widetilde{\rho}])\}$. If any case, we have

$$\begin{aligned} L(\delta_{i_{\min}}, \ldots, \delta_k; \tau') &\hookrightarrow \delta_{i_{\min}} \times \delta_{i_{\min}+1} \rtimes L(\delta_{i_{\min}+2}, \ldots, \delta_k; \tau') \\ &\cong \delta_{i_{\min}+1} \times \delta_{i_{\min}} \rtimes L(\delta_{i_{\min}+2}, \ldots, \delta_k; \tau'), \end{aligned}$$

which is impossible since the structural formula implies that $\mu^*(\langle[\nu^a\rho, \nu^{b-i_{\min}+1}\rho]\rangle \rtimes \tau)$ does not contain an irreducible constituent of the form $\delta_{i_{\min}+1} \otimes \pi$ for $\delta_{i_{\min}+1} \in \{\nu^{-b+i_{\min}}\widetilde{\rho}, \delta([\nu^{-b+i_{\min}}\widetilde{\rho}, \nu^{b-i_{\min}-1}\widetilde{\rho}])\}$. Thus, $i_{\min} = k$. □

The following two lemmas can be proved following exactly the same lines as in the proofs of [7, Lemma 2.3, Lemma 2.4].

Lemma 2.7. *Let π stand for $L(\nu^{-c}\rho_1, \delta_1, \ldots, \delta_k; \tau_1)$, where $-c < e(\delta_1)$ and let $\rho_1 \in R(GL)$ denote an irreducible self-contragredient cuspidal representation. If $L(\delta_1, \ldots, \delta_k; \tau_1)$ is an irreducible subquotient of $\langle[\nu^d\rho_1, \nu^{c-1}\rho_1]\rangle \rtimes \tau_2$, for some irreducible tempered representation τ_2 and $-c+1 \leq d$, then π is an irreducible subquotient of $\langle[\nu^d\rho_1, \nu^c\rho_1]\rangle \rtimes \tau_2$.*

Lemma 2.8. *Let π stand for $L(\nu^{-c}\rho_1, \nu^{-c}\rho_1, \delta_1, \ldots, \delta_k; \tau_1)$, where $-c < e(\delta_1)$ and let $\rho_1 \in R(GL)$ denote an irreducible self-contragredient cuspidal representation. If $L(\delta_1, \ldots, \delta_k; \tau_1)$ is an irreducible subquotient of $\langle[\nu^{-c+1}\rho_1, \nu^{c-1}\rho_1]\rangle \rtimes \tau_2$, for some irreducible tempered representation τ_2, then π is an irreducible subquotient of $\langle[\nu^{-c}\rho_1, \nu^c\rho_1]\rangle \rtimes \tau_2$.*

3 On tempered subquotients

Throughout this section, a and b denote real numbers such that $b - a$ is a non-negative integer. If $a \leq 0$, we assume that $-a \leq b$ and either $2a \notin \mathbb{Z}$ or a is half-integral (i.e., $2a \in \mathbb{Z}$ but $a \notin \mathbb{Z}$).

In this section we determine when the induced representation $\langle[\nu^a\rho, \nu^b\rho]\rangle \rtimes \tau$ contains an irreducible tempered subquotients, under a technical assumption on τ in the case $a \geq 1$.

The next lemma follows directly from [9, Section 8].

Lemma 3.1. *Suppose that $\sigma' \in R(G)$ is a discrete series, and let $\rho' \in R(GL)$ denote an irreducible self-contragredient cuspidal representation. Let c, d be such that $1 \leq c \leq d$, $d - c, 2c \in \mathbb{Z}$.*

(1) If $2c - 1 \notin Jord_{\rho'}(\sigma')$ and $x \in Jord_{\rho'}(\sigma')$ for all $x \in \{2c+1, 2c+3, \ldots, 2d+1\}$, then the Jacquet module of σ' with respect to the appropriate parabolic subgroup contains an irreducible representation of the form $\nu^{c+1}\rho' \otimes \nu^{c+2}\rho' \otimes \cdots \otimes \nu^d\rho' \otimes \pi$.

(2) *If $x \in Jord_{\rho'}(\sigma')$ for all $x \in \{2c+1, 2c+3, \dots, 2d+1\}$, $(2c+1)_-$ is defined and $\epsilon_{\sigma'}(((2c+1)_-, \rho'), (2c+1, \rho')) = 1$, then the Jacquet module of σ' with respect to the appropriate parabolic subgroup contains an irreducible representation of the form $\nu^{c+1}\rho' \otimes \nu^{c+2}\rho' \otimes \cdots \otimes \nu^{d}\rho' \otimes \pi$.*

(3) *The Jacquet module of σ' with respect to the appropriate parabolic subgroup does not contain an irreducible representation of the form $\nu^{c}\rho' \otimes \nu^{c+1}\rho' \otimes \cdots \otimes \nu^{d-1}\rho' \otimes \nu^{d}\rho' \times \nu^{d}\rho' \otimes \pi$.*

Lemma 3.2. *Suppose that either $2a \notin \mathbb{Z}$ or ρ is not a self-contragredient representation. Then $\langle[\nu^a\rho, \nu^b\rho]\rangle \rtimes \tau$ does not contain an irreducible tempered subquotient.*

Proof. Inspecting the cuspidal support of a tempered representation, we obtain that if $\langle[\nu^a\rho, \nu^b\rho]\rangle \rtimes \tau$ contains an irreducible tempered subquotient, then $2a \in \mathbb{Z}$. It remains to consider the case $2a \in \mathbb{Z}$ and $\rho \not\cong \widetilde{\rho}$. Since twists of ρ do not appear in the cuspidal support of a discrete series, we obtain that if $\langle[\nu^a\rho, \nu^b\rho]\rangle \rtimes \tau$ contains an irreducible tempered subquotient and $\rho \not\cong \widetilde{\rho}$, then $a = -b$. Suppose that τ is a subrepresentation of $\delta([\nu^{-b}\rho, \nu^b\rho])^m \rtimes \tau'$, for an irreducible tempered representation τ' such that τ' is not a subrepresentation of an induced representation of the form $\delta([\nu^{-b}\rho, \nu^b\rho]) \rtimes \pi$. Then an irreducible tempered subquotient of $\langle[\nu^{-b}\rho, \nu^b\rho]\rangle \rtimes \tau$ has to be a subrepresentation of an induced representation of the form $\delta([\nu^{-b}\rho, \nu^b\rho])^{m+1} \rtimes \pi$, but it follows directly from the structural formula that $\mu^*(\langle[\nu^{-b}\rho, \nu^b\rho]\rangle \rtimes \tau)$ does not contain an irreducible constituent of the form $\delta([\nu^{-b}\rho, \nu^b\rho])^{m+1} \otimes \pi$. If τ is not a subrepresentation of an induced representation of the form $\delta([\nu^{-b}\rho, \nu^b\rho]) \rtimes \pi$, it follows that an irreducible tempered subquotient of $\langle[\nu^{-b}\rho, \nu^b\rho]\rangle \rtimes \tau$ is a subrepresentation of an induced representation of the form $\delta([\nu^{-b}\rho, \nu^b\rho]) \rtimes \pi'$, but $\mu^*(\langle[\nu^{-b}\rho, \nu^b\rho]\rangle \rtimes \tau)$ does not contain an irreducible constituent of the form $\delta([\nu^{-b}\rho, \nu^b\rho]) \otimes \pi'$. This ends the proof. □

The following proposition can be proved following the same lines as in the proof of [7, Proposition 3.3] and using Lemma 3.2, details being left to the reader.

Proposition 3.3. *Suppose that $2a \notin \mathbb{Z}$. Then the induced representation $\langle[\nu^a\rho, \nu^b\rho]\rangle \rtimes \tau$ is irreducible.*

From now on, we assume that $2a \in \mathbb{Z}$. In the rest of this section we handle the case $\rho \cong \widetilde{\rho}$.

Lemma 3.4. *Suppose that τ is a subrepresentation of $\delta([\nu^{-c}\rho_1, \nu^c\rho_1])^m \rtimes \tau_1$, for a non-negative c, positive m, an irreducible cuspidal $\rho_1 \in R(GL)$, and an irreducible tempered $\tau_1 \in R(G)$ such that $\mu^*(\tau_1)$ does not contain an irreducible constituent of the form $\delta([\nu^{-c}\rho_1, \nu^c\rho_1]) \otimes \pi$. Suppose that one the following holds:*

(1) $a \geq 1$ and $(c, \rho_1) \notin \{(a-1, \rho), (b, \rho)\}$,

(2) $a \leq 0$ and $(c, \rho_1) \notin \{(-a, \rho), (b, \rho)\}$,

(3) $a = \frac{1}{2}$ and $(c, \rho_1) \neq (b, \rho)$.

If $\langle[\nu^a\rho, \nu^b\rho]\rangle \rtimes \tau$ contains an irreducible tempered subquotient, then $\langle[\nu^a\rho, \nu^b\rho]\rangle \rtimes \tau_1$ contains an irreducible tempered subquotient.

Proof. Suppose that $\langle[\nu^a\rho, \nu^b\rho]\rangle \rtimes \tau$ contains an irreducible tempered subuqotient τ'. Using the cuspidal support considerations, together with the assumptions of the lemma, we deduce that τ' is a subrepresentation of $\delta([\nu^{-c}\rho_1, \nu^c\rho_1])^m \rtimes \tau''$, for an irreducible tempered representation τ'' such that $\mu^*(\tau'')$ does not contain an irreducible constituent of the form $\delta([\nu^{-c}\rho_1, \nu^c\rho_1]) \otimes \pi$. Thus, $\mu^*(\langle[\nu^a\rho, \nu^b\rho]\rangle \rtimes \tau)$ contains $\delta([\nu^{-c}\rho_1, \nu^c\rho_1])^m \otimes \tau''$. By the structural formula, there are i, j such that $a-1 \leq i \leq j \leq b$ and an irreducible constituent $\delta_1 \otimes \pi_1$ of $\mu^*(\tau)$ such that

$$\delta([\nu^{-c}\rho_1, \nu^c\rho_1])^m \leq \langle[\nu^{-b}\rho, \nu^{-j-1}\rho]\rangle \times \langle[\nu^a\rho, \nu^i\rho]\rangle \times \delta_1$$

and

$$\tau'' \leq \langle[\nu^{i+1}\rho, \nu^j\rho]\rangle \rtimes \pi_1.$$

If $\nu^{-b}\rho \notin [\nu^{-c}\rho_1, \nu^c\rho_1]$, we obtain $j = b$. If $\nu^{-b}\rho \in [\nu^{-c}\rho_1, \nu^c\rho_1]$ and $j \leq b-1$, from the assumptions of the lemma clearly follows that δ_1 embeds into an induced representation of the form $\delta([\nu^{-c}\rho_1, \nu^{-b-1}\rho_1]) \times \delta_2$, contradicting the temperedness of τ. In the same way we conclude that $i = a-1$. Thus, there is an irreducible constituent of $\mu^*(\tau)$ of the form $\delta([\nu^{-c}\rho_1, \nu^c\rho_1])^m \otimes \pi$ such that $\langle[\nu^a\rho, \nu^b\rho]\rangle \rtimes \pi$ contains τ''. It is easy to conclude, using the structural formula, that the unique irreducible constituent of the form $\delta([\nu^{-c}\rho_1, \nu^c\rho_1])^m \otimes \pi$ appearing in $\mu^*(\tau)$ is $\delta([\nu^{-c}\rho_1, \nu^c\rho_1])^m \otimes \tau_1$, and the lemma is proved. □

Theorem 3.5. *Suppose that a is half-integral and $a < 0$. Then $\langle[\nu^a\rho, \nu^b\rho]\rangle \rtimes \tau$ does not contain an irreducible tempered subquotient.*

Proof. By Lemmas 3.2 and 3.4, it suffices to consider the case $\rho \cong \widetilde{\rho}$ and $\tau \hookrightarrow \delta_1 \times \cdots \times \delta_k \rtimes \sigma$, for $\delta_i \in \{\delta([\nu^a\rho, \nu^{-a}\rho]), \delta([\nu^{-b}\rho, \nu^b\rho])\}$. We note that the case $\tau \cong \sigma$ is covered by [8, Theorem 3.9].

Suppose, on the contrary, that $\langle[\nu^a\rho, \nu^b\rho]\rangle \rtimes \tau$ contains an irreducible tempered subquotient τ'. First we consider the case $-a = b$. Using the cuspidal support considerations, we deduce that τ' is a subrepresentation of $\delta([\nu^{-b}\rho, \nu^b\rho])^{k+1} \rtimes \sigma'$, for a discrete series $\sigma' \in R(G)$ such that $\mathrm{Jord}(\sigma) = \mathrm{Jord}(\sigma')$. If $2b+1 \in \mathrm{Jord}_\rho(\sigma)$, the induced representation $\delta([\nu^{-b}\rho, \nu^b\rho])^{k+1} \rtimes \sigma'$ is irreducible, so $\mu^*(\tau')$ contains an irreducible constituent of the form $(\nu^b\rho)^{2k+2} \otimes \pi$, but it follows from the structural formula, description of τ, and Lemma 3.1(3) that $\mu^*(\langle[\nu^{-b}\rho, \nu^b\rho]\rangle \rtimes \tau)$ does not contain an irreducible constituent of such a form.

If $2b+1 \notin \mathrm{Jord}_\rho(\sigma)$, then $\mu^*(\sigma)$ does not contain an irreducible constituent of the form $\nu^b\rho \otimes \pi$, and the structural formula implies that $\mu^*(\langle[\nu^{-b}\rho, \nu^b\rho]\rangle \rtimes \tau)$ does not contain $\delta([\nu^{-b}\rho, \nu^b\rho])^{k+1} \otimes \sigma'$, contradicting the Frobenius reciprocity.

Let us now consider the case $-a \neq b$. There is no loss of generality in assuming that $\delta_i \cong \delta([\nu^a\rho, \nu^{-a}\rho])$ for $i = 1, 2, \ldots, l$, and $\delta_i \cong \delta([\nu^{-b}\rho, \nu^b\rho])$ for $i = l+1, l+2, \ldots, k$, for some $l \in \{0, 1, \ldots, k\}$.

We consider several possibilities separately.

- $-2a+1 \notin \mathrm{Jord}_\rho(\sigma)$, $2b+1 \notin \mathrm{Jord}_\rho(\sigma)$.

In this case, τ' is a subrepresentation of $\delta_1 \times \cdots \times \delta_k \rtimes \sigma'$, for a discrete series $\sigma' \in R(G)$ such that $\mathrm{Jord}(\sigma') = \mathrm{Jord}(\sigma) \cup \{(2b+1, \rho), (-2a+1, \rho)\}$. This implies that $\delta_1 \times \cdots \times \delta_k \rtimes \sigma'$ is irreducible. Note that $\mu^*(\sigma)$ does not contain an irreducible constituent of the form $\nu^x\rho \otimes \pi$ for $x \in \{-a, b\}$.

If $\mu^*(\sigma')$ contains an irreducible constituent of the form $\nu^{-a}\rho \otimes \pi$, it follows that $\mu^*(\tau')$ contains an irreducible constituent of the form $(\nu^{-a}\rho)^{2l+1} \otimes \pi$, which is impossible since $\mu^*(\langle[\nu^a\rho, \nu^b\rho]\rangle \rtimes \tau)$ does not contain an irreducible constituent of such a form. Thus, $\mu^*(\sigma')$ does not contain an irreducible constituent of the form $\nu^{-a}\rho \otimes \pi$, so there is an irreducible tempered subrepresentation τ'' of $\delta([\nu^{a+1}\rho, \nu^{-a-1}\rho])^l \times \delta([\nu^{-b}\rho, \nu^b\rho])^{k-l} \rtimes \sigma'$ such that $\mu^*(\tau') \geq (\nu^{-a}\rho)^{2l} \otimes \tau''$. It follows that there is an irreducible tempered subrepresentation τ_1 of $\delta([\nu^{a+1}\rho, \nu^{-a-1}\rho])^l \times \delta([\nu^{-b}\rho, \nu^b\rho])^{k-l} \rtimes \sigma$ such that $\langle[\nu^{-a}\rho, \nu^b\rho]\rangle \rtimes \tau_1$ contains τ''. Following the same lines as in the proof of Lemma 3.4 we deduce that there is an irreducible tempered subrepresentation τ''' of $\delta([\nu^{-b}\rho, \nu^b\rho])^{k-l} \rtimes \sigma'$ and an irreducible tempered subrepresentation τ_2 of $\delta([\nu^{-b}\rho, \nu^b\rho])^{k-l} \rtimes \sigma$ such that $\langle[\nu^{-a}\rho, \nu^b\rho]\rangle \rtimes \tau_2$ contains τ'''. Repeating the same arguments as before we conclude that $\langle[\nu^{-a}\rho, \nu^b\rho]\rangle \rtimes \sigma$ contains σ', which contradicts [8, Theorem 3.9].

- $-2a+1 \notin \mathrm{Jord}_\rho(\sigma)$, $2b+1 \in \mathrm{Jord}_\rho(\sigma)$.

In this case, there is a discrete series $\sigma' \in R(G)$ with the property $\mathrm{Jord}(\sigma') = \mathrm{Jord}(\sigma) \cup \{(-2a+1, \rho)\} \setminus \{(2b+1, \rho)\}$ and an irreducible tempered subrepresentation τ'' of $\delta([\nu^{-b}\rho, \nu^b\rho])^{k-l+1} \rtimes \sigma'$ such that τ' is isomorphic to $\delta([\nu^a\rho, \nu^{-a}\rho])^l \rtimes \tau''$. In the same way as in the previously considered case we obtain that $\mu^*(\sigma')$ does not contain an irreducible constituent of the form $\nu^{-a}\rho \otimes \pi$, and that there is an irreducible tempered subrepresentation τ_1 of $\delta([\nu^{-b}\rho, \nu^b\rho])^{k-l} \rtimes \sigma$ such that $\langle[\nu^{-a}\rho, \nu^b\rho]\rangle \rtimes \tau_1$ contains τ''. Since $\mu^*(\tau'') \geq \delta([\nu^{-b}\rho, \nu^b\rho])^{k-l+1} \otimes \sigma'$, using the structural formula, together with Lemma 3.1(3), we deduce that there is an irreducible constituent $\delta([\nu^{-b+1}\rho, \nu^b\rho]) \otimes \sigma_1$ of $\mu^*(\sigma)$ such that $\langle[\nu^{-a}\rho, \nu^{b-1}\rho]\rangle \rtimes \sigma_1$ contains σ'. Since $b \geq \frac{3}{2}$, [17, Proposition 7.2] implies that σ_1 is a discrete series, contradicting [8, Theorem 3.9].

- $-2a+1 \in \mathrm{Jord}_\rho(\sigma)$, $2b+1 \notin \mathrm{Jord}_\rho(\sigma)$.

In this case, there is a discrete series $\sigma' \in R(G)$ with the property $\mathrm{Jord}(\sigma') = \mathrm{Jord}(\sigma) \cup \{(2b+1,\rho)\} \setminus \{(-2a+1,\rho)\}$ such that τ' is an irreducible subrepresentation of $\delta([\nu^a\rho,\nu^{-a}\rho])^{l+1} \times \delta([\nu^{-b}\rho,\nu^b\rho])^{k-l} \rtimes \sigma'$. Note that $\delta([\nu^{-b}\rho,\nu^b\rho])^{k-l} \rtimes \sigma'$ is irreducible. Using the Frobenius reciprocity and the structural formula we deduce that there is an irreducible constituent $\delta([\nu^{a+1}\rho,\nu^{-a}\rho]) \otimes \sigma_1$ of $\mu^*(\sigma)$ such that $\langle[\nu^{-a+1}\rho,\nu^b\rho]\rangle \rtimes \pi_1$ contains $\delta([\nu^{-b}\rho,\nu^b\rho])^{k-l} \rtimes \sigma'$ for an irreducible subquotient π_1 of $\delta([\nu^{-b}\rho,\nu^b\rho])^{k-l} \rtimes \sigma_1$. If $a < -\frac{1}{2}$, it follows from [17, Proposition 7.2] that σ_1 is a discrete series. If $a = -\frac{1}{2}$, it can be easily obtained from [8, Lemma 3.6] that σ_1 is a discrete series such that $(2,\rho) \notin \mathrm{Jord}(\sigma_1)$. Consequently, π_1 is a tempered subrepresentation of $\delta([\nu^{-b}\rho,\nu^b\rho])^{k-l} \rtimes \sigma_1$. In the same way as in the previously considered cases we obtain that $\langle[\nu^{-a+1}\rho,\nu^b\rho]\rangle \rtimes \sigma_1$ contains σ'. If $a < -\frac{1}{2}$, this contradicts [8, Theorem 3.9], and if $a = -\frac{1}{2}$ this contradicts [8, Proposition 3.7].

- $-2a+1 \in \mathrm{Jord}_\rho(\sigma)$, $2b+1 \in \mathrm{Jord}_\rho(\sigma)$.

In this case, there is a discrete series $\sigma' \in R(G)$ with the property $\mathrm{Jord}(\sigma') = \mathrm{Jord}(\sigma) \setminus \{(-2a+1,\rho),(2b+1,\rho)\}$ and an irreducible tempered subrepresentation τ'' of $\delta([\nu^{-b}\rho,\nu^b\rho])^{k-l+1} \rtimes \sigma'$ such that τ' is an irreducible subrepresentation of $\delta([\nu^a\rho,\nu^{-a}\rho])^{l+1} \rtimes \tau''$. In the same way as in the previously considered cases, we first deduce that there is a discrete series $\sigma_1 \in R(G)$ such that $\mu^*(\sigma) \geq \delta([\nu^{a+1}\rho,\nu^{-a}\rho]) \otimes \sigma_1$ and $\langle[\nu^{-a+1}\rho,\nu^b\rho]\rangle \times \delta([\nu^{-b}\rho,\nu^b\rho])^{k-l} \rtimes \sigma_1$ contains τ''. Note that $2b+1 \in \mathrm{Jord}_\rho(\sigma)$ implies $2b+1 \in \mathrm{Jord}_\rho(\sigma_1)$, so $\delta([\nu^{-b}\rho,\nu^b\rho])^{k-l} \rtimes \sigma_1$ is irreducible. Also, $-2a+1 \notin \mathrm{Jord}_\rho(\sigma_1)$. Using the same reasoning again, we obtain that there is a discrete series $\sigma_2 \in R(G)$ such that $\mu^*(\sigma_1) \geq \delta([\nu^{-b+1}\rho,\nu^b\rho]) \otimes \sigma_2$ and $\langle[\nu^{-a+1}\rho,\nu^{b-1}\rho]\rangle \rtimes \sigma_2$ contains σ'. Again, this contradicts either [8, Theorem 3.9] or [8, Proposition 3.7]. □

The following lemma enables an inductive procedure for the construction of irreducible tempered subquotients.

Lemma 3.6. *Let $\tau_1,\tau_2 \in R(G)$ denote irreducible tempered representations and suppose that there is an irreducible square-integrable $\delta \in R(GL)$ and a positive integer m such that τ_1 is a subrepresentation of $\delta^m \rtimes \tau_2$ and that $\mu^*(\tau_2)$ does not contain an irreducible constituent of the form $\delta \otimes \pi$. Suppose that for an irreducible cuspidal self-contragredient representation $\rho_1 \in R(GL)$ and c,d such that $\frac{1}{2} \leq c \leq d$, $d-c \in \mathbb{Z}, 2c \in \mathbb{Z}$, the induced representation*

$\langle[\nu^c\rho_1,\nu^d\rho_1]\rangle \rtimes \tau_2$ has a unique irreducible tempered subquotient τ', which is a subrepresentation. Also, suppose that $\delta \rtimes \tau'$ reduces if and only if $\delta \rtimes \tau_2$ reduces. If $\delta \notin \{\delta([\nu^{-c+1}\rho_1,\nu^{c-1}\rho_1]), \delta([\nu^{-d}\rho_1,\nu^d\rho_1])\}$, then $\langle[\nu^c\rho_1,\nu^d\rho_1]\rangle \rtimes \tau_1$ has a unique irreducible tempered subquotient, which is a subrepresentation.

Proof. Let $\delta \cong \delta([\nu^{-z}\rho',\nu^z\rho'])$. Let us first prove that $\delta \times \langle[\nu^c\rho_1,\nu^d\rho_1]\rangle$ is irreducible if $(z,\rho') \neq (c-1,\rho_1)$. Denote by π an irreducible subquotient of $\delta \times \langle[\nu^c\rho_1,\nu^d\rho_1]\rangle$, and let $\pi \cong L(\delta_1,\delta_2,\dots,\delta_n)$ where $\delta_i \cong \delta([\nu^{x_i}\rho^{(i)},\nu^{y_i}\rho^{(i)}])$. It follows that there is a unique $j \in \{1,2,\dots,n\}$ such that $(x_j,\rho^{(j)}) = (-z,\rho')$ and for $i \in \{1,2,\dots,j-1\}$ we have $x_i > x_j$. Since $e(\delta_i) \leq e(\delta_j)$ for $i \in \{1,2,\dots,j-1\}$, we obtain $y_i < y_j$ for $i \in \{1,2,\dots,j-1\}$. Thus, for $i \in \{1,2,\dots,j-1\}$ we have $\delta_i \times \delta_j \cong \delta_j \times \delta_i$, so π is a subrepresentation of

$$\delta_j \times \delta_1 \times \cdots \times \delta_{j-1} \times \delta_{j+1} \times \cdots \times \delta_n.$$

Frobenius reciprocity and transitivity of the Jacquet modules imply that the Jacquet module of π with respect to the appropriate parabolic subgroup contains an irreducible representation of the form $\delta_j \otimes \pi'$, such that the Jacquet module of π' with respect to the appropriate parabolic subgroup contains $\delta_1 \otimes \cdots \otimes \delta_{j-1} \otimes \delta_{j+1} \otimes \cdots \otimes \delta_n$.

Note that if the Jacquet module of $\langle[\nu^c\rho_1,\nu^d\rho_1]\rangle$ with respect to the appropriate parabolic subgroup contains an irreducible representation of the form $\delta' \otimes \pi_1$, with δ' essentially square-integrable, then $\delta' \cong \nu^c\rho_1$. Since the Jacquet module of $\delta \times \langle[\nu^c\rho_1,\nu^d\rho_1]\rangle$ with respect to the appropriate parabolic subgroup contains $\delta_j \otimes \pi'$ and $(z,\rho') \neq (c-1,\rho_1)$, it follows that $\delta_j \cong \delta$ and $\pi' \cong \langle[\nu^c\rho_1,\nu^d\rho_1]\rangle$. Consequently, every irreducible subquotient of $\delta \times \langle[\nu^c\rho_1,\nu^d\rho_1]\rangle$ is isomorphic to the unique irreducible subrepresentation of $\delta \times \nu^c\rho_1 \times \nu^{c+1}\rho_1 \times \cdots \times \nu^d\rho_1$. It can be easily seen that the Jacquet module of $\delta \times \langle[\nu^c\rho_1,\nu^d\rho_1]\rangle$ contains $\delta \otimes \nu^c\rho_1 \otimes \nu^{c+1}\rho_1 \otimes \cdots \otimes \nu^d\rho_1$ with multiplicity one, so $\delta \times \langle[\nu^c\rho_1,\nu^d\rho_1]\rangle$ is irreducible. This leads to

$$\delta^m \rtimes \tau' \hookrightarrow \delta^m \times \langle[\nu^c\rho_1,\nu^d\rho_1]\rangle \rtimes \tau_2 \cong \langle[\nu^c\rho_1,\nu^d\rho_1]\rangle \times \delta^m \rtimes \tau_2.$$

If $\delta^m \rtimes \tau'$ is irreducible, then $\tau_1 \cong \delta^m \rtimes \tau_2$ and $\delta^m \rtimes \tau'$ is an irreducible tempered subrepresentation of $\langle[\nu^c\rho_1,\nu^d\rho_1]\rangle \rtimes \tau_1$. If we denote an irreducible tempered subquotient of $\langle[\nu^c\rho_1,\nu^d\rho_1]\rangle \rtimes \tau_1$ by τ'', using the cuspidal support considerations one can easily obtain that $\tau'' \cong \delta^m \rtimes \tau_3$, for an irreducible tempered representation τ_3 such that $\mu^*(\tau_3)$ does not contain an irreducible constituent of the form $\delta \otimes \pi$. The structural formula now implies that $\mu^*(\tau'')$

contains $\delta^m \otimes \tau_3$ with the multiplicity 2^m. On the other hand, since $(z, \rho') \notin \{(c-1, \rho_1), (d, \rho_1)\}$ it follows that if $\delta^m \otimes \tau_3$ appears in $\mu^*(\langle[\nu^c\rho_1, \nu^d\rho_1]\rangle \rtimes \tau_1)$, then τ_3 is an irreducible tempered subquotient of $\langle[\nu^c\rho_1, \nu^d\rho_1]\rangle \rtimes \tau_2$, so $\tau_3 \cong \tau'$. Since $\mu^*(\tau_1)$ contains $\delta^m \otimes \tau_2$ with the multiplicity 2^m and τ' is a unique irreducible tempered subquotient of $\langle[\nu^c\rho_1, \nu^d\rho_1]\rangle \rtimes \tau_2$, we deduce that $\mu^*(\langle[\nu^c\rho_1, \nu^d\rho_1]\rangle \rtimes \tau_1)$ contains $\delta^m \otimes \tau'$ with the multiplicity 2^m, so $\delta^m \rtimes \tau'$ is a unique irreducible tempered subquotient of $\langle[\nu^c\rho_1, \nu^d\rho_1]\rangle \rtimes \tau_1$.

Let us now assume that $\delta^m \rtimes \tau'$ reduces and let $\delta^m \rtimes \tau' = \tau^{(1)} + \tau^{(2)}$, for mutually non-isomorphic irreducible tempered representations $\tau^{(1)}$ and $\tau^{(2)}$. By the assumption of the lemma, the induced representation $\delta^m \rtimes \tau_2$ also reduces, and let $\delta^m \rtimes \tau_2 = \tau^{(3)} + \tau^{(4)}$, for mutually non-isomorphic irreducible tempered representations $\tau^{(3)}$ and $\tau^{(4)}$. By Lemma 2.3, for $i \in \{1, 2\}$ there is a $j \in \{3, 4\}$ such that $\tau^{(i)}$ is a subrepresentation of $\langle[\nu^c\rho_1, \nu^d\rho_1]\rangle \rtimes \tau^{(j)}$.

Note that for $i \in \{1, 2\}$ there is an irreducible tempered subrepresentation τ''_i of $\delta \rtimes \tau'$ such that $\tau^{(i)} \cong \delta^{m-1} \rtimes \tau''_i$. Since $\delta \rtimes \tau'$ reduces, $\mu^*(\tau''_i)$ contains $\delta \otimes \tau'$ with the multiplicity 1, so for $i \in \{1, 2\}$, $\mu^*(\tau^{(i)})$ contains $\delta^m \otimes \tau'$ with the multiplicity 2^{m-1}.

Applying the same argument, we deduce that $\mu^*(\tau^{(j)})$ contains $\delta^m \otimes \tau_2$ with the multiplicity 2^{m-1} for $j \in \{3, 4\}$. Following the same reasoning as in the previously considered case, we obtain that for each irreducible constituent of the form $\delta^m \otimes \pi$ appearing in $\mu^*(\langle[\nu^c\rho_1, \nu^d\rho_1]\rangle \rtimes \tau^{(j)})$, for $j \in \{3, 4\}$ and π tempered, we have $\pi \cong \tau'$. Also, for $j \in \{3, 4\}$, $\mu^*(\langle[\nu^c\rho_1, \nu^d\rho_1]\rangle \rtimes \tau^{(j)})$ contains $\delta^m \otimes \tau'$ with the multiplicity 2^{m-1}. Consequently, for every $j \in \{3, 4\}$ there is a unique $i \in \{1, 2\}$ such that $\tau^{(i)}$ is a subrepresentation of $\langle[\nu^c\rho_1, \nu^d\rho_1]\rangle \rtimes \tau^{(j)}$, and $\langle[\nu^c\rho_1, \nu^d\rho_1]\rangle \rtimes \tau_1$ contains a unique irreducible tempered subquotient, which is a subrepresentation. □

Proposition 3.7. *Suppose that $a \geq 1$, $\rho \cong \widetilde{\rho}$, and that $\mu^*(\tau)$ does not contain an irreducible constituent of the form $\delta([\nu^{-a+1}\rho, \nu^{a-1}\rho]) \otimes \pi$. Also, suppose that $2b+1 \notin Jord_\rho(\sigma)$. If $\langle[\nu^a\rho, \nu^b\rho]\rangle \rtimes \tau$ contains an irreducible tempered subquotient, then the following holds:*

(1) $x \in Jord_\rho(\sigma)$ for all $x \in \{2a-1, 2a+1, \ldots, 2b-1\}$, $\epsilon_\sigma((x_-, \rho), (x, \rho)) = -1$ for all $x \in \{2a+1, 2a+3, \ldots, 2b-1\}$,

(2) if $\mu^*(\tau)$ contains an irreducible constituent of the form $\delta([\nu^{-b}\rho, \nu^b\rho]) \otimes \pi_1$, and m stands for the largest positive integer such that $\mu^*(\tau)$ contains an irreducible constituent of the form $\delta([\nu^{-b}\rho, \nu^b\rho])^m \otimes \pi_2$, then $\mu^*(\tau)$ contains an irreducible constituent of the form $(\nu^b\rho)^{2m} \otimes \pi_3$.

Proof. Tempered representation τ can be written as a subrepresentation of an induced representation of the form $\delta_1 \times \cdots \times \delta_k \rtimes \tau'$, where δ_i is an irreducible square-integrable representation, $\delta_i \not\cong \delta([\nu^{-b}\rho, \nu^b\rho])$ for $i = 1, 2, \ldots, k$, and τ' is an irreducible tempered representation such that if $\mu^*(\tau')$ contains an irreducible constituent of the form $\delta \otimes \pi$ for δ irreducible and square-integrable, then $\delta \cong \delta([\nu^{-b}\rho, \nu^b\rho])$.

By Lemma 3.4, if $\langle[\nu^a\rho, \nu^b\rho]\rangle \rtimes \tau$ contains an irreducible tempered subquotient, then $\langle[\nu^a\rho, \nu^b\rho]\rangle \rtimes \tau'$ contains an irreducible tempered subquotient. If τ' is a discrete series representation, then an irreducible tempered subquotient of $\langle[\nu^a\rho, \nu^b\rho]\rangle \rtimes \tau'$ also has to be a discrete series since $2b+1 \notin \mathrm{Jord}_\rho(\sigma)$, and the claim of the proposition follows from [8, Theorem 3.4]. We note that nowhere in the proof of [8, Theorem 3.4] is used the fact that a is half-integral, so the proof also covers the case $a \in \mathbb{Z}$.

Suppose that τ' is not a discrete series. Since $2b + 1 \notin \mathrm{Jord}_\rho(\sigma)$, τ' is a subrepresentation of $\delta([\nu^{-b}\rho, \nu^b\rho])^m \rtimes \sigma$, we have $2a - 1 \in \mathrm{Jord}_\rho(\sigma)$ and an irreducible tempered subquotient τ_1 of $\langle[\nu^a\rho, \nu^b\rho]\rangle \rtimes \tau'$ is isomorphic to $\delta([\nu^{-b}\rho, \nu^b\rho])^m \rtimes \sigma'$, for a discrete series σ' such that $\mathrm{Jord}(\sigma') = \mathrm{Jord}(\sigma) \setminus \{(2a-1, \rho)\} \cup \{(2b+1, \rho)\}$.

Let us first consider the case $a < b$. The structural formula implies that $\mu^*(\langle[\nu^a\rho, \nu^b\rho]\rangle \rtimes \tau')$ does not contain an irreducible constituent of the form $(\nu^b\rho)^{2m+1} \otimes \pi$, so $\mu^*(\sigma')$ does not contain an irreducible constituent of the form $\nu^b\rho \otimes \pi$. It follows that $\mu^*(\tau_1)$ contains an irreducible constituent of the form $(\nu^b\rho)^{2m} \otimes \tau_2$, where τ_2 is an irreducible tempered subrepresentation of $\delta([\nu^{-b+1}\rho, \nu^{b-1}\rho])^m \rtimes \sigma'$. Thus, τ_2 is an irreducible subquotient of $\langle[\nu^a\rho, \nu^b\rho]\rangle \rtimes \tau''$, for an irreducible tempered subrepresentation τ'' of $\delta([\nu^{-b+1}\rho, \nu^{b-1}\rho])^m \rtimes \sigma$. Applying Lemma 3.4 again, we obtain that $\langle[\nu^a\rho, \nu^b\rho]\rangle \rtimes \sigma$ contains σ', and [8, Theorem 3.4] implies that $x \in \mathrm{Jord}_\rho(\sigma)$ for $x \in \{2a-1, 2a+1, \ldots, 2b-1\}$, $\epsilon_\sigma((x_-, \rho), (x, \rho)) = -1$ for all $x \in \{2a+1, 2a+3, \ldots, 2b-1\}$.

Note that τ can also be written as a subrepresentation of $\delta([\nu^{-b}\rho, \nu^b\rho])^m \rtimes \tau''$, for an irreducible tempered subrepresentation τ'' of $\delta_1 \times \cdots \times \delta_k \rtimes \sigma$. Since $2b + 1 \notin \mathrm{Jord}_\rho(\sigma)$ and $\delta_i \not\cong \delta([\nu^{-b}\rho, \nu^b\rho])$ for $i = 1, 2, \ldots, k$, the induced representation $\delta([\nu^{-b}\rho, \nu^b\rho])^m \rtimes \tau''$ reduces by Theorem 2.2 (3). Also, $a < b$ implies that $2b - 1 \in \mathrm{Jord}_\rho(\sigma)$, so $\delta([\nu^{-b+1}\rho, \nu^{b-1}\rho])^m \rtimes \tau''$ is irreducible. It is now a direct consequence of the structural formula that $(\nu^b\rho)^{2m} \otimes \delta([\nu^{-b+1}\rho, \nu^{b-1}\rho])^m \rtimes \tau''$ is a unique irreducible constituent of $\mu^*(\delta([\nu^{-b}\rho, \nu^b\rho])^m \rtimes \tau'')$ of the form $(\nu^b\rho)^{2m} \otimes \pi$, and appears there with multiplicity one. Thus, there is a unique irreducible subrepresentation of

$\delta([\nu^{-b}\rho, \nu^b\rho])^m \rtimes \tau''$ which contains an irreducible constituent of the form $(\nu^b\rho)^{2m} \otimes \pi$ in the Jacquet module with respect to an appropriate parabolic subgroup.

In the same way we conclude that an irreducible tempered subquotient $\tau^{(1)}$ of $\langle[\nu^a\rho, \nu^b\rho]\rangle \rtimes \tau$ is isomorphic to $\delta([\nu^{-b}\rho, \nu^b\rho])^m \rtimes \tau^{(2)}$, for an irreducible tempered subrepresentation $\tau^{(2)}$ of $\delta_1 \times \cdots \times \delta_k \rtimes \sigma'$. Thus, $\mu^*(\tau^{(1)})$ contains an irreducible constituent of the form $(\nu^b\rho)^{2m} \otimes \pi$, so $\mu^*(\langle[\nu^a\rho, \nu^b\rho]\rangle \rtimes \tau)$ also contains an irreducible constituent of such a form. Since $a < b$, we obtain that $\mu^*(\tau)$ contains an irreducible constituent of the form $(\nu^b\rho)^{2m} \otimes \pi$.

Let us now consider the case $a = b$. Inspecting the cuspidal support of τ, we deduce at once that $2a - 1 \in \mathrm{Jord}_\rho(\sigma)$. In the same way as in the previously considered case, we get that an irreducible tempered subquotient $\tau^{(1)}$ of $\nu^b\rho \rtimes \tau$ is isomorphic to $\delta([\nu^{-b}\rho, \nu^b\rho])^m \rtimes \tau^{(2)}$, for an irreducible tempered subrepresentation $\tau^{(2)}$ of $\delta_1 \times \cdots \times \delta_k \rtimes \sigma'$, where σ' is a discrete series such that $\mathrm{Jord}(\sigma') = \mathrm{Jord}(\sigma) \setminus \{(2b-1, \rho)\} \cup \{(2b+1, \rho)\}$. From [17, Lemma 8.1] follows that there is an irreducible representation π_1 such that σ' is a subrepresentation of $\nu^b\rho \rtimes \pi_1$. By the assumption of the proposition, $\delta_i \times \nu^b\rho$ is irreducible for $i = 1, 2, \ldots, k$, and the standard commuting argument shows that $\tau^{(2)}$ is a subrepresentation of $\nu^b\rho \times \delta_1 \times \cdots \times \delta_k \rtimes \pi_1$. Consequently, $\mu^*(\tau^{(1)})$ contains an irreducible constituent of the form $(\nu^b\rho)^{2m+1} \otimes \pi$, so $\mu^*(\tau)$ contains an irreducible constituent of the form $(\nu^b\rho)^{2m} \otimes \pi$. □

The following result is contained in the proof of [6, Theorem 3.18], but for the sake of completeness we provide a proof here.

Lemma 3.8. *Let $\sigma' \in R(G)$ denote a discrete series, and let $\rho' \in R(GL)$ stand for an irreducible cuspidal self-contragredient representation. Suppose that c is such that $Jord_{\rho'}(\sigma')$ contains $\{2c-3, 2c-1, 2c+1\}$, and $\epsilon_{\sigma'}((2c-1, \rho'), (2c+1, \rho')) = 1$. We denote by τ' an irreducible tempered representation such that σ' is a subrepresentation of $\nu^c\rho' \rtimes \tau'$. Then $\epsilon_{\sigma'}((2c-3, \rho'), (2c-1, \rho')) = 1$ if and only if $\mu^*(\tau')$ contains an irreducible constituent of the form $\nu^{c-1}\rho' \times \nu^{c-1}\rho' \otimes \pi$.*

Proof. From $\epsilon_{\sigma'}((2c-1, \rho'), (2c+1, \rho')) = 1$ follows that τ' is an irreducible subrepresentation of $\delta([\nu^{-c+1}\rho', \nu^{c-1}\rho']) \rtimes \sigma''$, for a discrete series σ'' such that $\mathrm{Jord}(\sigma'') = \mathrm{Jord}(\sigma') \setminus \{(2c-1, \rho'), (2c+1, \rho')\}$.

Let us first assume that $\epsilon_{\sigma'}((2c-3, \rho'), (2c-1, \rho')) = 1$. Using Lemma 2.3, we deduce that there are irreducible representations π_1 and π_2 in $R(G)$

such that

$$\sigma' \hookrightarrow \delta([\nu^{c-1}\rho', \nu^c\rho']) \rtimes \pi_1$$

and

$$\sigma' \hookrightarrow \nu^{c-1}\rho' \rtimes \pi_2.$$

The structural formula implies that $\mu^*(\pi_1)$ contains an irreducible constituent of the form $\nu^{c-1}\rho' \otimes \pi_3$. Since $\nu^{c-1}\rho'$ is cuspidal, using Lemma 2.3 we get that there is also an irreducible representation π_4 such that π_1 is a subrepresentation of $\nu^{c-1}\rho' \rtimes \pi_4$.

Using $\sigma' \hookrightarrow \delta([\nu^{c-1}\rho', \nu^c\rho']) \times \nu^{c-1}\rho' \rtimes \pi_4$ and the Frobenius reciprocity, we conclude that $\mu^*(\nu^c\rho' \rtimes \tau')$ contains $\delta([\nu^{c-1}\rho', \nu^c\rho']) \times \nu^{c-1}\rho' \otimes \pi_4$. It directly follows that $\mu^*(\tau')$ does not contain an irreducible constituent of the form $\nu^c\rho' \otimes \pi_5$, so $\mu^*(\tau')$ contains an irreducible constituent of the form $\nu^{c-1}\rho' \times \nu^{c-1}\rho' \otimes \pi$.

Let us now assume that $\mu^*(\tau')$ contains an irreducible constituent of the form $\nu^{c-1}\rho' \times \nu^{c-1}\rho' \otimes \pi$. By [17, Corollary 4.2], τ' embeds into $\nu^{c-1}\rho' \times \nu^{c-1}\rho' \rtimes \pi$. Thus, σ' embeds into $\nu^c\rho' \times \nu^{c-1}\rho' \times \nu^{c-1}\rho' \rtimes \pi$, and by Lemma 2.3, there is an irreducible subquotient π' of $\nu^c\rho' \times \nu^{c-1}\rho' \times \nu^{c-1}\rho'$ such that σ' is a subrepresentation of $\pi' \rtimes \pi$. Since $\pi' \in \{L(\nu^{c-1}\rho', \nu^{c-1}\rho', \nu^c\rho'), \nu^{c-1}\rho' \times \delta([\nu^{c-1}\rho', \nu^c\rho'])\}$, it follows that σ' embeds into an induced representation of the form $\nu^{c-1}\rho' \rtimes \pi''$, and, by the definition of the ϵ-function, we have $\epsilon_{\sigma'}((2c-3, \rho'), (2c-1, \rho')) = 1$. □

Proposition 3.9. *Suppose that $a \geq 1$, $\rho \cong \widetilde{\rho}$, and that $\mu^*(\tau)$ does not contain an irreducible constituent of the form $\delta([\nu^{-a+1}\rho, \nu^{a-1}\rho]) \otimes \pi$. Also, suppose that $2b+1 \in Jord_\rho(\sigma)$. If $\langle[\nu^a\rho, \nu^b\rho]\rangle \rtimes \tau$ contains an irreducible tempered subquotient then $x \in Jord_\rho(\sigma)$ for all $x \in \{2a-1, 2a+1, \ldots, 2b-1\}$, $\epsilon_\sigma((x_-, \rho), (x, \rho)) = -1$ for all $x \in \{2a+1, 2a+3, \ldots, 2b-1\}$, and $\epsilon_\sigma((2b-1, \rho), (2b+1, \rho)) = 1$.*

Proof. In the same way as in the proof of Proposition 3.7, we write τ as a subrepresentation of an induced representation of the form $\delta_1 \times \cdots \times \delta_k \rtimes \tau'$, where δ_i is an irreducible square-integrable representation, $\delta_i \ncong \delta([\nu^{-b}\rho, \nu^b\rho])$ for $i = 1, 2, \ldots, k$, and τ' is an irreducible tempered representation such that if $\mu^*(\tau')$ contains an irreducible constituent of the form $\delta \otimes \pi$ for δ irreducible and square-integrable, then $\delta \cong \delta([\nu^{-b}\rho, \nu^b\rho])$.

If $\langle[\nu^a\rho,\nu^b\rho]\rangle \rtimes \tau$ contains an irreducible tempered subquotient, then Lemma 3.4 implies that $\langle[\nu^a\rho,\nu^b\rho]\rangle \rtimes \tau'$ also contains an irreducible tempered subquotient. If τ' is a discrete series representation, using $2b+1 \in \mathrm{Jord}_\rho(\sigma)$ we conclude that an irreducible tempered subquotient of $\langle[\nu^a\rho,\nu^b\rho]\rangle \rtimes \tau'$ is not a discrete series representation, and the claim of the proposition follows from [8, Theorem 3.12], the proof of which is also valid in the case $a \in \mathbb{Z}$.

In the rest of the proof we can assume that τ' is a subrepresentation of $\delta([\nu^{-b}\rho,\nu^b\rho])^m \rtimes \sigma$, for a positive integer m, and the cuspidal support considerations enable us to conclude that an irreducible tempered subquotient τ_1 of $\langle[\nu^a\rho,\nu^b\rho]\rangle \rtimes \tau'$ is a subrepresentation of $\delta([\nu^{-b}\rho,\nu^b\rho])^{m+1} \rtimes \sigma'$, for a discrete series σ' such that $\mathrm{Jord}(\sigma') = \mathrm{Jord}(\sigma) \setminus \{(2a-1,\rho),(2b+1,\rho)\}$. It follows that $\mu^*(\langle[\nu^a\rho,\nu^b\rho]\rangle \rtimes \tau')$ contains $\delta([\nu^{-b}\rho,\nu^b\rho])^{m+1} \otimes \sigma'$, and an easy application of the structural formula implies that $\mu^*(\sigma)$ contains an irreducible constituent of the form $\delta([\nu^{-b+1}\rho,\nu^b\rho]) \otimes \sigma_1$ such that σ' is an irreducible subquotient of $\langle[\nu^a\rho,\nu^{b-1}\rho]\rangle \rtimes \sigma_1$. Using [17, Proposition 7.2] we conclude that $2b-1 \in \mathrm{Jord}_\rho(\sigma)$, $\epsilon_\sigma((2b-1,\rho),(2b+1,\rho)) = 1$ and σ_1 is a discrete series. This completes the proof in the case $a = b$. If $a < b$, [8, Theorem 3.12] implies $x \in \mathrm{Jord}_\rho(\sigma)$ for $x \in \{2a-1,2a+1,\ldots,2b-3\}$, $\epsilon_\sigma((x_-,\rho),(x,\rho)) = -1$ for all $x \in \{2a+1,2a+3,\ldots,2b-3\}$.

It remains to prove $\epsilon_\sigma((2b-3,\rho),(2b-1,\rho)) = -1$ in the case $a < b$. There is an irreducible tempered subrepresentation τ_2 of $\delta([\nu^{-b}\rho,\nu^b\rho]) \rtimes \sigma'$ such that $\tau_1 \cong \delta([\nu^{-b}\rho,\nu^b\rho])^m \rtimes \tau_2$. Using $2b-1 \in \mathrm{Jord}_\rho(\sigma)$, in the same way as in the proof of [17, Lemma 4.1] we conclude that there is a unique irreducible subrepresentation of $\delta([\nu^{-b}\rho,\nu^b\rho]) \rtimes \sigma'$ which contains an irreducible representation of the form $\nu^b\rho \times \nu^b\rho \otimes \pi$ in its Jacquet module with respect to an appropriate parabolic subgroup. Since $a < b$, $\mu^*(\langle[\nu^a\rho,\nu^b\rho]\rangle \rtimes \tau')$ does not contain an irreducible constituent of the form $(\nu^b\rho)^{2m+2} \otimes \pi'$, so $\mu^*(\tau_2)$ does not contain an irreducible constituent of the form $\nu^b\rho \times \nu^b\rho \otimes \pi$.

From $\epsilon_\sigma((x_-,\rho),(x,\rho)) = -1$ for all $x \in \{2a+1,2a+3,\ldots,2b-3\}$, we deduce that $\mu^*(\sigma)$ does not contain an irreducible constituent of the form $\nu^y\rho \otimes \pi$ for $y \in \{a+1,a+2,\ldots,b-1\}$. This implies at once that $\mu^*(\langle[\nu^a\rho,\nu^b\rho]\rangle \rtimes \tau')$ also does not contain an irreducible constituent of the form $\nu^y\rho \otimes \pi$ for $y \in \{a+1,a+2,\ldots,b-1\}$. Using irreducibility of $\delta([\nu^{-b}\rho,\nu^b\rho]) \times \nu^y\rho$ for $y \in \{a+1,a+2,\ldots,b-1\}$, a simple commuting argument shows that $\mu^*(\sigma')$ does not contain an irreducible constituent of the form $\nu^y\rho \otimes \pi$ for $y \in \{a+1,a+2,\ldots,b-1\}$, so $\epsilon_{\sigma'}((x_-,\rho),(x,\rho)) = -1$ for all $x \in \{2a+1,2a+3,\ldots,2b-3\}$. This also implies that $\mu^*(\tau_2)$ does not contain an irreducible constituent of the form $\nu^y\rho \otimes \pi$ for $y \in \{a+1,a+2,\ldots,b-1\}$.

Using $2a-1 \notin \mathrm{Jord}_\rho(\sigma')$ and a repeated application of [17, Lemma 8.1], we obtain that there is a discrete series σ'' such that σ' is a subrepresentation of $\nu^a\rho \times \nu^{a+1}\rho \times \cdots \times \nu^{b-1}\rho \rtimes \sigma''$. By Lemma 2.3, there is an irreducible subquotient of $\nu^a\rho \times \nu^{a+1}\rho \times \cdots \times \nu^{b-1}\rho$ such that σ' is a subrepresentation of $\pi \rtimes \sigma''$. From $\epsilon_{\sigma'}((x_-,\rho),(x,\rho)) = -1$ for all $x \in \{2a+1, 2a+3, \ldots, 2b-3\}$ follows at once that $\pi \cong \langle[\nu^a\rho, \nu^{b-1}\rho]\rangle$.

In the same way as in the proof of Lemma 3.6 one can see that $\delta([\nu^{-b}\rho,\nu^b\rho]) \times \langle[\nu^a\rho,\nu^{b-1}\rho]\rangle$ is irreducible, so τ_2 is a subrepresentation of

$$\delta([\nu^{-b}\rho,\nu^b\rho]) \times \langle[\nu^a\rho,\nu^{b-1}\rho]\rangle \rtimes \sigma'' \cong \langle[\nu^a\rho,\nu^{b-1}\rho]\rangle \times \delta([\nu^{-b}\rho,\nu^b\rho]) \rtimes \sigma'',$$

and there is an irreducible tempered subrepresentation τ_3 of $\delta([\nu^{-b}\rho,\nu^b\rho]) \rtimes \sigma''$ such that τ_2 is a subrepresentation of $\langle[\nu^a\rho,\nu^{b-1}\rho]\rangle \rtimes \tau_3$. Note that $2b-1 \notin \mathrm{Jord}_\rho(\sigma'')$, so [14, Theorem 6.1] implies that $\nu^b\rho \rtimes \sigma''$ is irreducible. Thus, we have

$$\begin{aligned}\tau_3 &\hookrightarrow \delta([\nu^{-b}\rho,\nu^b\rho]) \rtimes \sigma'' \hookrightarrow \delta([\nu^{-b+1}\rho,\nu^b\rho]) \times \nu^{-b}\rho \rtimes \sigma'' \\ &\cong \delta([\nu^{-b+1}\rho,\nu^b\rho]) \times \nu^b\rho \rtimes \sigma'' \cong \nu^b\rho \times \delta([\nu^{-b+1}\rho,\nu^b\rho]) \rtimes \sigma'',\end{aligned}$$

so there is an irreducible subquotient σ_2 of $\delta([\nu^{-b+1}\rho,\nu^b\rho]) \rtimes \sigma''$ such that τ_3 is a subrepresentation of $\nu^b\rho \rtimes \sigma_2$. Embedding $\tau_3 \hookrightarrow \nu^b\rho \times \delta([\nu^{-b+1}\rho,\nu^b\rho]) \rtimes \sigma''$ implies that $\mu^*(\tau_3)$ contains an irreducible constituent of the form $\nu^b\rho \times \nu^b\rho \otimes \pi$. Thus, $\mu^*(\sigma_3)$ contains an irreducible constituent of the form $\nu^b\rho \otimes \pi$. Since $2b+1 \notin \mathrm{Jord}_\rho(\sigma'')$ and $\delta([\nu^{-b+1}\rho,\nu^{b-1}\rho]) \rtimes \sigma''$ is a length two representation, it follows that there are exactly two mutually non-isomorphic constituents of the form $\nu^b\rho \otimes \pi$ appearing in $\delta([\nu^{-b+1}\rho,\nu^b\rho]) \rtimes \sigma''$, and each of them appears there with multiplicity one. Now, using $2b-1, 2b+1 \notin \mathrm{Jord}_\rho(\sigma'')$ and [13, Theorem 2.1], we obtain that σ_2 is a discrete series subrepresentation of $\delta([\nu^{-b+1}\rho,\nu^b\rho]) \rtimes \sigma''$. Thus, there is an irreducible tempered subrepresentation τ_4 of $\delta([\nu^{-b+1}\rho,\nu^{b-1}\rho]) \rtimes \sigma''$ such that σ_2 is a subrepresentation of $\nu^b\rho \rtimes \tau_4$.

From embeddings

$$\tau_2 \hookrightarrow \langle[\nu^a\rho,\nu^{b-1}\rho]\rangle \rtimes \tau_3 \hookrightarrow \langle[\nu^a\rho,\nu^{b-1}\rho]\rangle \times \nu^b\rho \rtimes \sigma_2$$

and Lemma 2.3, we deduce that there is an irreducible subquotient π_1 of $\langle[\nu^a\rho,\nu^{b-1}\rho]\rangle \times \nu^b\rho$ such that τ_2 is a subrepresentation of $\pi_1 \rtimes \sigma_2$. It can be directly seen that every irreducible subquotient of $\langle[\nu^a\rho,\nu^{b-1}\rho]\rangle \times \nu^b\rho$ is isomorphic either to $L(\nu^a\rho, \nu^{a+1}\rho, \ldots, \nu^{b-2}\rho, \delta([\nu^{b-1}\rho,\nu^b\rho]))$ or to $\langle[\nu^a\rho,\nu^b\rho]\rangle$.

Suppose that $\pi_1 \cong L(\nu^a\rho, \nu^{a+1}\rho, \dots, \nu^{b-2}\rho, \delta([\nu^{b-1}\rho, \nu^b\rho]))$. Note that it can be seen in the same way as in the proof of Lemma 3.6 that the induced representation $L(\nu^a\rho, \nu^{a+1}\rho, \dots, \nu^{b-2}\rho) \times \nu^b\rho$ is irreducible. We have the following embeddings and isomorphisms:

$$\begin{aligned}
\tau_2 &\hookrightarrow L(\nu^a\rho, \nu^{a+1}\rho, \dots, \nu^{b-2}\rho, \delta([\nu^{b-1}\rho, \nu^b\rho])) \rtimes \sigma_2 \\
&\hookrightarrow L(\nu^a\rho, \nu^{a+1}\rho, \dots, \nu^{b-2}\rho) \times \delta([\nu^{b-1}\rho, \nu^b\rho]) \times \nu^b\rho \rtimes \tau_4 \\
&\cong L(\nu^a\rho, \nu^{a+1}\rho, \dots, \nu^{b-2}\rho) \times \nu^b\rho \times \delta([\nu^{b-1}\rho, \nu^b\rho]) \rtimes \tau_4 \\
&\hookrightarrow L(\nu^a\rho, \nu^{a+1}\rho, \dots, \nu^{b-2}\rho) \times \nu^b\rho \times \nu^b\rho \times \nu^{b-1}\rho \rtimes \tau_4 \\
&\cong \nu^b\rho \times \nu^b\rho \times L(\nu^a\rho, \nu^{a+1}\rho, \dots, \nu^{b-2}\rho) \times \nu^{b-1}\rho \rtimes \tau_4.
\end{aligned}$$

This implies that $\mu^*(\tau_2)$ contains an irreducible constituent of the form $\nu^b\rho \times \nu^b\rho \otimes \pi$, a contradiction. Thus, $\pi_1 \cong \langle[\nu^a\rho, \nu^b\rho]\rangle$.

Note that $\mathrm{Jord}_\rho(\sigma_2)$ contains $\{2b-3, 2b-1, 2b+1\}$ and $\epsilon_{\sigma_2}((2b-1, \rho), (2b+1, \rho)) = 1$. If $\epsilon_{\sigma_2}((2b-3, \rho), (2b-1, \rho)) = 1$, there is an irreducible representation π_2 such that σ_2 is a subrepresentation of $\nu^{b-1}\rho \rtimes \pi_2$. This yields

$$\tau_2 \hookrightarrow \langle[\nu^a\rho, \nu^b\rho]\rangle \rtimes \sigma_2 \hookrightarrow \langle[\nu^a\rho, \nu^b\rho]\rangle \times \nu^{b-1}\rho \rtimes \pi_2.$$

Since $a \leq b-1$, it can be seen in the same way as in the proof of Lemma 3.6 that $\langle[\nu^a\rho, \nu^b\rho]\rangle \times \nu^{b-1}\rho$ is irreducible, so τ_2 is a subrepresentation of

$$\nu^{b-1}\rho \times \langle[\nu^a\rho, \nu^b\rho]\rangle \rtimes \pi_2,$$

a contradiction. Thus, we have $\epsilon_{\sigma_2}((2b-3, \rho), (2b-1, \rho)) = -1$ and Lemma 3.8 implies that $\mu^*(\tau_4)$ does not contain an irreducible constituent of the form $\nu^{b-1}\rho \times \nu^{b-1}\rho \otimes \pi$.

Following the same lines as in the proof of Lemma 3.6, we obtain

$$\begin{aligned}
\tau_1 &\cong \delta([\nu^{-b}\rho, \nu^b\rho])^m \rtimes \tau_2 \hookrightarrow \delta([\nu^{-b}\rho, \nu^b\rho])^m \times \langle[\nu^a\rho, \nu^b\rho]\rangle \rtimes \sigma_2 \\
&\cong \langle[\nu^a\rho, \nu^b\rho]\rangle \times \delta([\nu^{-b}\rho, \nu^b\rho])^m \rtimes \sigma_2.
\end{aligned}$$

Since $2b+1 \in \mathrm{Jord}_\rho(\sigma_2)$, the induced representation $\delta([\nu^{-b}\rho, \nu^b\rho])^m \rtimes \sigma_2$ is irreducible. It follows that there is an irreducible representation π such that the Jacquet module of $\delta([\nu^{-b}\rho, \nu^b\rho])^m \rtimes \sigma_2$ with respect to an appropriate parabolic subgroup contains an irreducible quotient of the form $\nu^b\rho \otimes \cdots \otimes \nu^b\rho \otimes \pi$, where $\nu^b\rho$ appears $2m+1$ times. Using the Frobenius reciprocity, together with Lemma 2.3, we obtain that there is an irreducible representation π' such that $\delta([\nu^{-b}\rho, \nu^b\rho])^m \rtimes \sigma_2$ is a subrepresentation of $(\nu^b\rho)^{2m+1} \rtimes \pi'$.

Since $\mu^*(\delta([\nu^{-b}\rho, \nu^b\rho])^m \rtimes \sigma_2)$ contains $(\nu^b\rho)^{2m+1} \otimes \pi'$ and $\nu^b\rho \otimes \tau_4$ is a unique irreducible constituent of $\mu^*(\sigma_2)$ of the form $\nu^b\rho \otimes \pi$, an easy application of the structural formula implies $\pi' \cong \delta([\nu^{-b+1}\rho, \nu^{b-1}\rho])^m \rtimes \tau_4$.

Consequently, τ_1 is a subrepresentation of

$$\nu^a\rho \times \cdots \times \nu^{b-1}\rho \times \nu^b\rho \times (\nu^b\rho)^{2m+1} \times \delta([\nu^{-b+1}\rho, \nu^{b-1}\rho])^m \rtimes \tau_4,$$

and the Jacquet module of τ_1 with respect to the appropriate parabolic subgroup contains

$$\nu^a\rho \otimes \cdots \otimes \nu^{b-1}\rho \otimes (\nu^b\rho)^{2m+2} \otimes \delta([\nu^{-b+1}\rho, \nu^{b-1}\rho])^m \rtimes \tau_4. \qquad (1)$$

On the other hand, since τ_1 is an irreducible subquotient of $\langle[\nu^a\rho, \nu^b\rho]\rangle \rtimes \tau'$ and τ' is a subrepresentation of $\delta([\nu^{-b}\rho, \nu^b\rho])^m \rtimes \sigma$, the Jacquet module of $\langle[\nu^a\rho, \nu^b\rho]\rangle \times \delta([\nu^{-b}\rho, \nu^b\rho])^m \rtimes \sigma$ with respect to the appropriate parabolic subgroup also contains (1).

From $\epsilon_\sigma((2b-1, \rho), (2b+1, \rho)) = 1$ follows that there is an irreducible tempered subrepresentation τ_5 of $\delta([\nu^{-b+1}\rho, \nu^{b-1}\rho]) \rtimes \sigma_1$ such that $\mu^*(\sigma)$ contains $\nu^b\rho \otimes \tau_5$, and $\nu^b\rho \otimes \tau_5$ is a unique irreducible constituent of the form $\nu^b\rho \otimes \pi$ of $\mu^*(\sigma)$.

By the transitivity of Jacquet modules, the Jacquet module of $\langle[\nu^a\rho, \nu^b\rho]\rangle \times \delta([\nu^{-b}\rho, \nu^b\rho])^m \rtimes \sigma$ with respect to the appropriate parabolic subgroup contains an irreducible representation of the form $\nu^a\rho \otimes \cdots \otimes \nu^{b-1}\rho \otimes \pi_1$, with π_1 such that

$$\mu^*(\pi_1) \geq (\nu^b\rho)^{2m+2} \otimes \delta([\nu^{-b+1}\rho, \nu^{b-1}\rho])^m \rtimes \tau_4.$$

Since σ is a discrete series, using Lemma 3.1(3) we conclude

$$\pi_1 \leq \nu^b\rho \times \delta([\nu^{-b}\rho, \nu^b\rho])^m \rtimes \sigma.$$

The structural formula implies

$$\delta([\nu^{-b+1}\rho, \nu^{b-1}\rho])^m \rtimes \tau_4 \leq \delta([\nu^{-b+1}\rho, \nu^{b-1}\rho])^m \rtimes \tau_5.$$

It directly follows that both induced representations $\delta([\nu^{-b+1}\rho, \nu^{b-1}\rho])^m \rtimes \tau_4$ and $\delta([\nu^{-b+1}\rho, \nu^{b-1}\rho])^m \rtimes \tau_5$ are irreducible. If $\mu^*(\tau_5)$ contains an irreducible constituent of the form $\nu^{b-1}\rho \times \nu^{b-1}\rho \otimes \pi$, then $\mu^*(\delta([\nu^{-b+1}\rho, \nu^{b-1}\rho])^m \rtimes \tau_5)$ contains an irreducible constituent of the form $(\nu^{b-1}\rho)^{2m+2} \otimes \pi$, but using the structural formula, together with the description of $\mu^*(\tau_4)$, we obtain that $\mu^*(\delta([\nu^{-b+1}\rho, \nu^{b-1}\rho])^m \rtimes \tau_4)$ does not contain an irreducible constituent of such a form. Lemma 3.8 implies $\epsilon_\sigma((2b-3, \rho), (2b-1, \rho)) = -1$ and the proposition is proved. $\square$

Theorem 3.10. *Suppose that $a \geq 1$, $\rho \cong \widetilde{\rho}$, and that $\mu^*(\tau)$ does not contain an irreducible constituent of the form $\delta([\nu^{-a+1}\rho, \nu^{a-1}\rho]) \otimes \pi$. Then $\langle[\nu^a\rho, \nu^b\rho]\rangle \rtimes \tau$ contains an irreducible tempered subquotient if and only if $x \in Jord_\rho(\sigma)$ for all $x \in \{2a-1, 2a+1, \dots, 2b-1\}$, $\epsilon_\sigma((x_-, \rho), (x, \rho)) = -1$ for all $x \in \{2a+1, 2a+3, \dots, 2b-1\}$, and one of the following holds:*

(1) $2b+1 \notin Jord_\rho(\sigma)$ and $\mu^*(\tau)$ does not contain an irreducible constituent of the form $\delta([\nu^{-b}\rho, \nu^b\rho]) \otimes \pi$,

(2) $2b+1 \notin Jord_\rho(\sigma)$, $\mu^*(\tau)$ contains an irreducible constituent of the form $\delta([\nu^{-b}\rho, \nu^b\rho]) \otimes \pi_1$, and if m stands for the largest integer such that $\mu^*(\tau)$ contains an irreducible constituent of the form $\delta([\nu^{-b}\rho, \nu^b\rho])^m \otimes \pi_2$, then $\mu^*(\tau)$ contains an irreducible constituent of the form $(\nu^b\rho)^{2m} \otimes \pi_3$,

(3) $2b+1 \in Jord_\rho(\sigma)$ and $\epsilon_\sigma((2b-1, \rho), (2b+1, \rho)) = 1$.

Furthermore, if $\langle[\nu^a\rho, \nu^b\rho]\rangle \rtimes \tau$ contains an irreducible tempered subquotient then it contains an irreducible tempered subrepresentation.

Proof. The necessity part follows from Propositions 3.7 and 3.9. To prove the sufficiency part, we use an inductive procedure based on Lemma 3.6. By the classification of tempered representations, there is an ordered n-tuple $(\tau_1, \tau_2, \dots, \tau_n)$ of irreducible tempered representations $\tau_1, \tau_2, \dots, \tau_n \in R(G)$ such that $\tau \cong \tau_n$, τ_1 is a discrete series, and for $i = 2, 3, \dots, n$ there is an irreducible square-integrable representation $\delta_i \in R(GL)$ and a positive integer m_i such that τ_i is a subrepresentation of $\delta_i^{m_i} \rtimes \tau_{i-1}$, and $\delta_j \not\cong \delta_k$ for $j, k \in \{2, 3, \dots, n\}$, $j \neq k$. If there is an $i \in \{2, 3, \dots, n\}$ such that $\delta_i \cong \delta([\nu^{-b}\rho, \nu^b\rho])$, we can take $i = n$. Note that $\delta_i \not\cong \delta([\nu^{-a+1}\rho, \nu^{a-1}\rho])$ for all $i \in \{2, 3, \dots, n\}$, and $\tau_1 \cong \sigma$.

Suppose that $x \in \mathrm{Jord}_\rho(\sigma)$ for all $x \in \{2a-1, 2a+1, \dots, 2b-1\}$, and $\epsilon_\sigma((x_-, \rho), (x, \rho)) = -1$ for all $x \in \{2a+1, 2a+3, \dots, 2b-1\}$.

Let us first assume that $2b+1 \notin \mathrm{Jord}_\rho(\sigma)$. By [8, Theorem 3.4], there is a unique discrete series subrepresentation $\tau^{(1)}$ of $\langle[\nu^a\rho, \nu^b\rho]\rangle \rtimes \tau_1$, and $\mathrm{Jord}(\tau^{(1)}) = \mathrm{Jord}(\tau_1) \setminus \{(2a-1, \rho)\} \cup \{(2b+1, \rho)\}$. Let us assume that $\delta_n \not\cong \delta([\nu^{-b}\rho, \nu^b\rho])$. It follows that $\delta_2 \rtimes \tau^{(1)}$ reduces if and only if $\delta_2 \rtimes \tau_1$ reduces and Lemma 3.6 implies that $\langle[\nu^a\rho, \nu^b\rho]\rangle \rtimes \tau_2$ contains a unique irreducible tempered subquotient $\tau^{(2)}$, which is a subrepresentation. Also, if $n \geq 2$, since $\delta_2 \notin \{\delta([\nu^{-a+1}\rho, \nu^{a-1}\rho]), \delta([\nu^{-b}\rho, \nu^b\rho])\}$, $\delta_3 \rtimes \tau^{(2)}$ reduces if and only if $\delta_3 \rtimes \tau_2$ reduces. Repeating this procedure, we obtain that $\langle[\nu^a\rho, \nu^b\rho]\rangle \rtimes \tau$ contains an irreducible tempered subrepresentation.

If $\delta_n \cong \delta([\nu^{-b}\rho, \nu^b\rho])$, in the same way as in the previously considered case we conclude that $\langle[\nu^a\rho, \nu^b\rho]\rangle \rtimes \tau_{n-1}$ contains an irreducible tempered subrepresentation $\tau^{(n-1)}$. Suppose that $\mu^*(\tau)$ contains an irreducible constituent of the form $(\nu^b\rho)^{2m_n} \otimes \pi$. It can be seen in the same way as in the proof of [17, Lemma 4.1] that τ is a unique irreducible subquotient of $\delta([\nu^{-b}\rho, \nu^b\rho])^{m_n} \rtimes \tau_{n-1}$ which contains an irreducible constituent of the form $(\nu^b\rho)^{2m_n} \otimes \pi$ in the Jacquet module with respect to the appropriate parabolic subgroup.

The induced representation $\delta_n^{m_n} \rtimes \tau^{(n-1)}$ is an irreducible tempered subrepresentation of $\langle[\nu^a\rho, \nu^b\rho]\rangle \times \delta_n^{m_n} \rtimes \tau_{n-1}$. By Lemma 2.3 there is an irreducible subrepresentation τ' of $\delta_n^{m_n} \rtimes \tau_{n-1}$ such that $\delta_n^{m_n} \rtimes \tau^{(n-1)}$ embeds into $\langle[\nu^a\rho, \nu^b\rho]\rangle \rtimes \tau'$. Let $x = 0$ is $a < b$, and $x = 1$ is $a = b$. Then $\mu^*(\delta_n^{m_n} \rtimes \tau^{(n-1)})$ contains an irreducible constituent of the form $(\nu^b\rho)^{2m_n+x} \otimes \pi$, and it follows that $\mu^*(\tau')$ has to contain an irreducible constituent of the form $(\nu^b\rho)^{2m_n} \otimes \pi$, so $\tau' \cong \tau$.

Let us now assume that $2b+1 \in \mathrm{Jord}_\rho(\sigma)$. By [8, Theorem 3.12], there is a unique irreducible tempered subrepresentation $\tau^{(1)}$ of $\langle[\nu^a\rho, \nu^b\rho]\rangle \rtimes \tau_1$, and for $\delta \not\cong \delta([\nu^{-a+1}\rho, \nu^{a-1}\rho])$ the induced representation $\delta \rtimes \tau^{(1)}$ reduces if and only if $\delta \rtimes \tau_1$ reduces. If $\delta_n \not\cong \delta([\nu^{-b}\rho, \nu^b\rho])$, the claim follows in the same way as in the previously considered case.

It remains to consider the case $\delta_n \cong \delta([\nu^{-b}\rho, \nu^b\rho])$, and let $\tau^{(n-1)}$ stand for a unique irreducible tempered subrepresentation of $\langle[\nu^a\rho, \nu^b\rho]\rangle \rtimes \tau_{n-1}$. From $2b + 1 \in \mathrm{Jord}_\rho(\sigma)$ follows that $\tau \cong \delta([\nu^{-b}\rho, \nu^b\rho])^{m_n} \rtimes \tau_{n-1}$. The induced representation $\delta([\nu^{-b}\rho, \nu^b\rho])^{m_n} \rtimes \tau^{(n-1)}$ is an irreducible tempered subrepresentation of $\langle[\nu^a\rho, \nu^b\rho]\rangle \times \delta([\nu^{-b}\rho, \nu^b\rho])^{m_n} \rtimes \tau_{n-1}$, and the claim follows. $\square$

In the rest of this section we discuss irreducible tempered subquotients in the case $a = \frac{1}{2}$.

Proposition 3.11. *Suppose that $\rho \cong \widetilde{\rho}$ and $2b+1 \notin Jord_\rho(\sigma)$. If $\langle[\nu^{\frac{1}{2}}\rho, \nu^b\rho]\rangle \rtimes \tau$ contains an irreducible tempered subquotient, then the following holds:*

(1) $x \in Jord_\rho(\sigma)$ for all $x \in \{2, 4, \ldots, 2b-1\}$, $\epsilon_\sigma((x_-, \rho), (x, \rho)) = -1$ for all $x \in \{4, 6, \ldots, 2b-1\}$,

(2) if $b > \frac{1}{2}$, then $\epsilon_\sigma(2, \rho) = -1$,

(3) if $\mu^*(\tau)$ contains an irreducible constituent of the form $\delta([\nu^{-b}\rho, \nu^b\rho]) \otimes \pi_1$, and m stands for the largest integer such that $\mu^*(\tau)$ contains an irre-

ducible constituent of the form $\delta([\nu^{-b}\rho, \nu^b\rho])^m \otimes \pi_2$, then $\mu^*(\tau)$ contains an irreducible constituent of the form $(\nu^b\rho)^{2m} \otimes \pi_3$.

Proof. We write τ as an irreducible subrepresentation of $\delta_1 \times \cdots \times \delta_k \rtimes \tau'$, for irreducible square-integrable $\delta_1, \ldots, \delta_k \in R(GL)$, $\delta_i \not\cong \delta([\nu^{-b}\rho, \nu^b\rho])$ for $i = 1, \ldots, k$, and an irreducible tempered representation $\tau' \in R(G)$ such that if $\mu^*(\tau')$ contains an irreducible constituent of the form $\delta \otimes \pi$ for δ irreducible and square-integrable, then $\delta \cong \delta([\nu^{-b}\rho, \nu^b\rho])$.

In the same way as in the proof of Proposition 3.7 we deduce that if $\langle[\nu^{\frac{1}{2}}\rho, \nu^b\rho]\rangle \rtimes \tau$ contains an irreducible tempered subquotient, then $\langle[\nu^{\frac{1}{2}}\rho, \nu^b\rho]\rangle \rtimes \tau'$ also contains an irreducible tempered subquotient.

If τ' is a discrete series, then the claim of the proposition follows from [8, Proposition 3.7]. Let us assume that τ' is a subrepresentation of $\delta([\nu^{-b}\rho, \nu^b\rho])^m \rtimes \sigma$, for a positive integer m. If $b > \frac{1}{2}$, the rest of the proof follows the same lines as the one of Proposition 3.7.

Let us discuss the case $b = \frac{1}{2}$. Using the cuspidal support considerations we get that there is a discrete series $\sigma' \in R(G)$, $\mathrm{Jord}(\sigma') = \mathrm{Jord}(\sigma) \cup \{(2, \rho)\}$, such that an irreducible tempered subquotient τ_1 of $\nu^{\frac{1}{2}}\rho \rtimes \tau$ is isomorphic to $\delta([\nu^{-\frac{1}{2}}\rho, \nu^{\frac{1}{2}}\rho])^m \rtimes \tau''$, for an irreducible tempered subrepresentation τ'' of $\delta_1 \times \cdots \times \delta_k \rtimes \sigma'$.

It follows that $\mu^*(\nu^{\frac{1}{2}}\rho \times \delta([\nu^{-\frac{1}{2}}\rho, \nu^{\frac{1}{2}}\rho])^m \rtimes \sigma)$ contains $\delta([\nu^{-\frac{1}{2}}\rho, \nu^{\frac{1}{2}}\rho])^m \times \delta_1 \times \cdots \times \delta_k \otimes \sigma'$. Since $\mu^*(\sigma)$ does not contain an irreducible constituent of the form $\nu^x\rho \otimes \pi$ for $x \in \{\frac{1}{2}, -\frac{1}{2}\}$, we obtain that σ' is an irreducible subquotient of $\nu^{\frac{1}{2}}\rho \rtimes \sigma$. As in [8, Lemma 3.6(2)], we conclude that σ' is a subrepresentation of $\nu^{\frac{1}{2}}\rho \rtimes \sigma$. Consequently, $\mu^*(\tau'')$ contains an irreducible constituent of the form $\nu^{\frac{1}{2}}\rho \otimes \pi_1$, so $\mu^*(\tau_1)$ contains an irreducible constituent of the form $(\nu^{\frac{1}{2}}\rho)^{2m+1} \otimes \pi_2$. This implies that $\mu^*(\tau)$ contains an irreducible constituent of the form $(\nu^{\frac{1}{2}}\rho)^{2m} \otimes \pi_3$, and the proposition is proved. □

The following lemma can be proved in the same way as Lemma 3.8, details being left to the reader.

Lemma 3.12. *Let $\sigma' \in R(G)$ denote a discrete series, and let $\rho' \in R(GL)$ stand for an irreducible cuspidal self-contragredient representation. Suppose that $Jord_{\rho'}(\sigma')$ contains $\{2, 4\}$ and $\epsilon_{\sigma'}((2, \rho'), (4, \rho')) = 1$. Let τ' denote an irreducible tempered representation such that σ' is a subrepresentation of $\nu^{\frac{3}{2}}\rho' \rtimes \tau'$. Then $\epsilon_{\sigma'}(2, \rho') = 1$ if and only if $\mu^*(\tau')$ contains an irreducible constituent of the form $\nu^{\frac{1}{2}}\rho' \times \nu^{\frac{1}{2}}\rho' \otimes \pi$ or, equivalently, if and only if τ' embeds into an induced representation of the form $\nu^{\frac{1}{2}}\rho' \times \nu^{\frac{1}{2}}\rho' \rtimes \pi$.*

Proposition 3.13. *Suppose that $\rho \cong \widetilde{\rho}$ and $2b+1 \in Jord_\rho(\sigma)$. If $\langle[\nu^{\frac{1}{2}}\rho, \nu^b\rho]\rangle \rtimes \tau$ contains an irreducible tempered subquotient, then one of the following holds:*

(1) $b > \frac{1}{2}$, $x \in Jord_\rho(\sigma)$ for $x \in \{2, 4, \ldots, 2b-1\}$, $\epsilon_\sigma((x_-, \rho), (x, \rho)) = -1$ for all $x \in \{4, 6, \ldots, 2b-1\}$, $\epsilon_\sigma((2b-1, \rho), (2b+1, \rho)) = 1$, and $\epsilon_\sigma(2, \rho) = -1$,

(2) $b = \frac{1}{2}$ and $\epsilon_\sigma(2, \rho) = 1$.

Proof. Similarly as in the proof of Proposition 3.11, we suppose that $\langle[\nu^{\frac{1}{2}}\rho, \nu^b\rho]\rangle \rtimes \tau$ contains an irreducible tempered subquotient and write τ as an irreducible subrepresentation of $\delta_1 \times \cdots \times \delta_k \rtimes \tau'$, for irreducible square-integrable $\delta_1, \ldots, \delta_k \in R(GL)$, $\delta_i \not\cong \delta([\nu^{-b}\rho, \nu^b\rho])$ for $i = 1, \ldots, k$, and an irreducible tempered representation $\tau' \in R(G)$ such that if $\mu^*(\tau')$ contains an irreducible constituent of the form $\delta \otimes \pi$ for δ irreducible and square-integrable, then $\delta \cong \delta([\nu^{-b}\rho, \nu^b\rho])$.

In the same way as in the proof of Proposition 3.7 we obtain that $\langle[\nu^{\frac{1}{2}}\rho, \nu^b\rho]\rangle \rtimes \tau'$ contains an irreducible tempered subquotient. If τ' is a discrete series, the claim of the proposition follows from [8, Propositions 3.13, 3.14].

It remains to discuss the case $\tau' \cong \delta([\nu^{-b}\rho, \nu^b\rho])^m \rtimes \sigma$, for $m \geq 1$. If $b > \frac{3}{2}$, the claim of the proposition can be proved following the same lines as in the proof of Proposition 3.9, so we consider the case $b \leq \frac{3}{2}$. Let τ_1 denote an irreducible tempered subquotient of $\langle[\nu^{\frac{1}{2}}\rho, \nu^b\rho]\rangle \rtimes \tau'$.

Let us first assume that $b = \frac{1}{2}$. Then τ' is a subrepresentation of $\delta([\nu^{-\frac{1}{2}}\rho, \nu^{\frac{1}{2}}\rho])^{m+1} \rtimes \sigma'$, for a discrete series σ'. It follows that $\mu^*(\nu^{\frac{1}{2}}\rho \times \delta([\nu^{-\frac{1}{2}}\rho, \nu^{\frac{1}{2}}\rho])^m \rtimes \sigma)$ contains $\delta([\nu^{-\frac{1}{2}}\rho, \nu^{\frac{1}{2}}\rho])^{m+1} \otimes \sigma'$. Since σ is square-integrable, it follows that $\mu^*(\sigma)$ has to contain an irreducible constituent of the form $\nu^{\frac{1}{2}}\rho \otimes \pi$. By [17, Proposition 7.4] we have $\epsilon_\sigma(2, \rho) = 1$.

It remains to consider the case $b = \frac{3}{2}$. Using the cuspidal support considerations, we get that τ_1 is a subrepresentation of an induced representation of the form $\delta([\nu^{-\frac{3}{2}}\rho, \nu^{\frac{3}{2}}\rho])^{m+1} \rtimes \sigma_1$, for a discrete series σ_1 such that $\mathrm{Jord}(\sigma_1) = \mathrm{Jord}(\sigma) \setminus \{(4, \rho)\}$. Frobenius reciprocity implies

$$\mu^*(\langle[\nu^{\frac{1}{2}}\rho, \nu^{\frac{3}{2}}\rho]\rangle \times \delta([\nu^{-\frac{3}{2}}\rho, \nu^{\frac{3}{2}}\rho])^m \rtimes \sigma) \geq \delta([\nu^{-\frac{3}{2}}\rho, \nu^{\frac{3}{2}}\rho])^{m+1} \otimes \sigma_1,$$

and in the same way as in the proof of Proposition 3.9 we deduce that $\{2, 4\} \subseteq \mathrm{Jord}_\rho(\sigma)$, $\epsilon_\sigma((2, \rho), (4, \rho)) = 1$, and σ_1 is a subquotient of $\nu^{\frac{1}{2}}\rho \rtimes \sigma'$, for a discrete series σ' such that σ is a subrepresentation of $\delta([\nu^{-\frac{1}{2}}\rho, \nu^{\frac{3}{2}}\rho]) \rtimes \sigma'$.

Then $2, 4 \notin \mathrm{Jord}_\rho(\sigma')$, and by [8, Lemma 3.6], σ_1 is a subrepresentation of $\nu^{\frac{1}{2}}\rho \rtimes \sigma'$.

We denote by τ'' an irreducible tempered subrepresentation of $\delta([\nu^{-\frac{1}{2}}\rho, \nu^{\frac{1}{2}}\rho]) \rtimes \sigma'$ such that σ embeds into $\nu^{\frac{3}{2}}\rho \rtimes \tau''$.

There is an irreducible tempered subrepresentation τ_2 of $\delta([\nu^{-\frac{3}{2}}\rho, \nu^{\frac{3}{2}}\rho]) \rtimes \sigma_1$ such that τ_1 is isomorphic to $\delta([\nu^{-\frac{3}{2}}\rho, \nu^{\frac{3}{2}}\rho])^m \rtimes \tau_2$. Since $\mu^*(\langle[\nu^{\frac{1}{2}}\rho, \nu^{\frac{3}{2}}\rho]\rangle \rtimes \tau')$ does not contain an irreducible constituent of the form $(\nu^{\frac{3}{2}}\rho)^{2m+2} \otimes \pi$, it follows that $\mu^*(\tau_2)$ does not contain an irreducible constituent of the form $\nu^{\frac{3}{2}}\rho \times \nu^{\frac{3}{2}}\rho \otimes \pi$.

From

$$\tau_2 \hookrightarrow \delta([\nu^{-\frac{3}{2}}\rho, \nu^{\frac{3}{2}}\rho]) \times \nu^{\frac{1}{2}}\rho \rtimes \sigma' \cong \nu^{\frac{1}{2}}\rho \times \delta([\nu^{-\frac{3}{2}}\rho, \nu^{\frac{3}{2}}\rho]) \rtimes \sigma'$$

Lemma 2.3 and Theorem 2.2 (1), we get that there is an irreducible subrepresentation τ_3 of $\delta([\nu^{-\frac{3}{2}}\rho, \nu^{\frac{3}{2}}\rho]) \rtimes \sigma'$ such that τ_2 embeds into $\nu^{\frac{1}{2}}\rho \rtimes \tau_3$. Since $\mu^*(\tau_2)$ does not contain an irreducible constituent of the form $\nu^{\frac{3}{2}}\rho \times \nu^{\frac{3}{2}}\rho \otimes \pi$, we deduce that τ_3 is not a subrepresentation of an induced representation of the form $\delta([\nu^{\frac{1}{2}}\rho, \nu^{\frac{3}{2}}\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^{\frac{3}{2}}\rho]) \rtimes \pi$. Also, from $2 \notin \mathrm{Jord}_\rho(\sigma')$ we get that $\mu^*(\tau_3)$ does not contain an irreducible constituent of the form $\nu^{\frac{1}{2}}\rho \otimes \pi$.

By [14, Theorem 6.1], $\nu^{\frac{3}{2}}\rho \rtimes \sigma'$ is irreducible. This leads to:

$$\begin{aligned}\tau_3 &\hookrightarrow \delta([\nu^{-\frac{1}{2}}\rho, \nu^{\frac{3}{2}}\rho]) \times \nu^{-\frac{3}{2}}\rho \rtimes \sigma' \\ &\cong \delta([\nu^{-\frac{1}{2}}\rho, \nu^{\frac{3}{2}}\rho]) \times \nu^{\frac{3}{2}}\rho \rtimes \sigma' \cong \nu^{\frac{3}{2}}\rho \times \delta([\nu^{-\frac{1}{2}}\rho, \nu^{\frac{3}{2}}\rho]) \rtimes \sigma',\end{aligned}$$

and Lemma 2.3 implies that there is an irreducible subquotient σ_2 of $\delta([\nu^{-\frac{1}{2}}\rho, \nu^{\frac{3}{2}}\rho]) \rtimes \sigma'$ such that τ_3 embeds into $\nu^{\frac{3}{2}}\rho \rtimes \sigma_2$. Since $\mu^*(\tau_3)$ contains an irreducible constituent of the form $\nu^{\frac{3}{2}}\rho \times \nu^{\frac{3}{2}}\rho \otimes \pi$, it follows that $\mu^*(\sigma_2)$ contains an irreducible constituent of the form $\nu^{\frac{3}{2}}\rho \otimes \pi$. Using [13, Theorem 2.1], together with Theorem 2.2 (1), we get that σ_2 is a discrete series subrepresentation of $\delta([\nu^{-\frac{1}{2}}\rho, \nu^{\frac{3}{2}}\rho]) \rtimes \sigma'$, and let τ_4 denote an irreducible tempered subrepresentation of $\delta([\nu^{-\frac{1}{2}}\rho, \nu^{\frac{1}{2}}\rho]) \rtimes \sigma'$ such that σ_2 embeds into $\nu^{\frac{3}{2}}\rho \rtimes \tau_4$.

Let us now prove that $\mu^*(\tau_4)$ does not contain an irreducible constituent of the form $\nu^{\frac{1}{2}}\rho \times \nu^{\frac{1}{2}}\rho \otimes \pi$, which is, by [17, Corollary 4.5] equivalent to the fact that τ_4 does not embed into $\nu^{\frac{1}{2}}\rho \times \nu^{\frac{1}{2}}\rho \rtimes \sigma'$. Otherwise, σ_2 embeds in $\nu^{\frac{3}{2}}\rho \times \nu^{\frac{1}{2}}\rho \times \nu^{\frac{1}{2}}\rho \rtimes \sigma'$, and Lemma 2.3 implies that there is an irreducible subquotient π_1 of $\nu^{\frac{3}{2}}\rho \times \nu^{\frac{1}{2}}\rho \times \nu^{\frac{1}{2}}\rho$ such that σ_2 embeds into $\pi_1 \rtimes \sigma'$. It is easy to see, using Lemma 3.1(3), that $\pi_1 \cong \nu^{\frac{1}{2}}\rho \times \delta([\nu^{\frac{1}{2}}\rho, \nu^{\frac{3}{2}}\rho])$. Thus, τ_3 embeds into $\nu^{\frac{3}{2}}\rho \times \nu^{\frac{1}{2}}\rho \times \delta([\nu^{\frac{1}{2}}\rho, \nu^{\frac{3}{2}}\rho]) \rtimes \sigma'$, so there is an irreducible subquotient π_2 of $\nu^{\frac{3}{2}}\rho \times \nu^{\frac{1}{2}}\rho \times \delta([\nu^{\frac{1}{2}}\rho, \nu^{\frac{3}{2}}\rho])$ such that τ_3 embeds into $\pi_2 \rtimes \sigma'$. Since $\mu^*(\tau_3)$ does not contain an irreducible constituent of the form $\nu^{\frac{1}{2}}\rho \otimes \pi$, it follows that $\tau_3 \cong \delta([\nu^{\frac{1}{2}}\rho, \nu^{\frac{3}{2}}\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^{\frac{3}{2}}\rho])$, a contradiction.

From

$$\tau_2 \hookrightarrow \nu^{\frac{1}{2}}\rho \rtimes \tau_3 \hookrightarrow \nu^{\frac{1}{2}}\rho \times \nu^{\frac{3}{2}}\rho \rtimes \sigma_2,$$

we see that there is an irreducible subquotient π_3 of $\nu^{\frac{1}{2}}\rho \times \nu^{\frac{3}{2}}\rho$ such that τ_2 embeds into $\pi_3 \rtimes \sigma_2$. If $\pi_3 \cong \delta([\nu^{\frac{1}{2}}\rho, \nu^{\frac{3}{2}}\rho])$, we obtain

$$\tau_2 \hookrightarrow \delta([\nu^{\frac{1}{2}}\rho, \nu^{\frac{3}{2}}\rho]) \rtimes \sigma_2 \hookrightarrow \delta([\nu^{\frac{1}{2}}\rho, \nu^{\frac{3}{2}}\rho]) \times \nu^{\frac{3}{2}}\rho \rtimes \tau_4 \hookrightarrow \nu^{\frac{3}{2}}\rho \times \nu^{\frac{3}{2}}\rho \times \nu^{\frac{1}{2}}\rho \rtimes \tau_4,$$

a contradiction. Thus, τ_2 is a subrepresentation of $\langle[\nu^{\frac{1}{2}}\rho, \nu^{\frac{3}{2}}\rho]\rangle \rtimes \sigma_2$. This leads to an embedding

$$\tau_1 \hookrightarrow \langle[\nu^{\frac{1}{2}}\rho, \nu^{\frac{3}{2}}\rho]\rangle \times \delta([\nu^{-\frac{3}{2}}\rho, \nu^{\frac{3}{2}}\rho])^m \rtimes \sigma_2.$$

In the same way as in the proof of Proposition 3.9 we deduce that τ_1 is a subrepresentation of

$$\nu^{\frac{1}{2}}\rho \times (\nu^{\frac{3}{2}}\rho)^{2m+2} \times \delta([\nu^{-\frac{1}{2}}\rho, \nu^{\frac{1}{2}}\rho])^m \rtimes \tau_4,$$

so the Jacquet module of τ_1 with respect to the appropriate parabolic subgroup contains

$$\nu^{\frac{1}{2}}\rho \otimes (\nu^{\frac{3}{2}}\rho)^{2m+2} \otimes \delta([\nu^{-\frac{1}{2}}\rho, \nu^{\frac{1}{2}}\rho])^m \rtimes \tau_4.$$

Since τ_1 is an irreducible subquotient of $\langle[\nu^{\frac{1}{2}}\rho, \nu^{\frac{3}{2}}\rho]\rangle \times \delta([\nu^{-\frac{3}{2}}\rho, \nu^{\frac{3}{2}}\rho])^m \rtimes \sigma$, and $\nu^{\frac{3}{2}}\rho \otimes \tau''$ is a unique irreducible constituent of $\mu^*(\sigma)$ of the form $\nu^{\frac{3}{2}}\rho \otimes \pi$, following the same lines as in the proof of Proposition 3.9 we obtain that $\tau'' \cong \tau_4$. Thus, $\mu^*(\tau'')$ does not contain an irreducible constituent of the form $\nu^{\frac{1}{2}}\rho \times \nu^{\frac{1}{2}}\rho \otimes \pi$, and Lemma 3.12 implies $\epsilon_\sigma(2, \rho) = -1$. This completes the proof. □

Theorem 3.14. *Suppose that $b - \frac{1}{2}$ is a nonnegative integer and $\rho \cong \widetilde{\rho}$. Then $\langle[\nu^{\frac{1}{2}}\rho, \nu^b\rho]\rangle \rtimes \tau$ contains an irreducible tempered subquotient if and only if one of the following holds:*

(1) $b = \frac{1}{2}$, $2 \notin Jord_\rho(\sigma)$, and if $\mu^*(\tau)$ contains an irreducible constituent of the form $\delta([\nu^{-\frac{1}{2}}\rho, \nu^{\frac{1}{2}}\rho]) \otimes \pi_1$, and m stands for the largest integer such that $\mu^*(\tau)$ contains an irreducible constituent of the form $\delta([\nu^{-\frac{1}{2}}\rho, \nu^{\frac{1}{2}}\rho])^m \otimes \pi_2$, then $\mu^*(\tau)$ contains an irreducible constituent of the form $(\nu^{\frac{1}{2}}\rho)^{2m} \otimes \pi_3$,

(2) $b = \frac{1}{2}$, $2 \in Jord_\rho(\sigma)$, *and* $\epsilon_\sigma(2, \rho) = 1$,

(3) $b > \frac{1}{2}$, $2b + 1 \notin Jord_\rho(\sigma)$, $x \in Jord_\rho(\sigma)$ *for all* $x \in \{2, 4, \ldots, 2b - 1\}$, $\epsilon_\sigma((x_-, \rho), (x, \rho)) = -1$ *for all* $x \in \{4, 6, \ldots, 2b-1\}$, $\epsilon_\sigma(2, \rho) = -1$, *and if* $\mu^*(\tau)$ *contains an irreducible constituent of the form* $\delta([\nu^{-b}\rho, \nu^b\rho]) \otimes \pi_1$, *and* m *stands for the largest integer such that* $\mu^*(\tau)$ *contains an irreducible constituent of the form* $\delta([\nu^{-b}\rho, \nu^b\rho])^m \otimes \pi_2$, *then* $\mu^*(\tau)$ *contains an irreducible constituent of the form* $(\nu^b\rho)^{2m} \otimes \pi_3$,

(4) $b > \frac{1}{2}$, $2b + 1 \in Jord_\rho(\sigma)$, $x \in Jord_\rho(\sigma)$ *for all* $x \in \{2, 4, \ldots, 2b - 1\}$, $\epsilon_\sigma((x_-, \rho), (x, \rho)) = -1$ *for all* $x \in \{4, 6, \ldots, 2b - 1\}$, $\epsilon_\sigma((2b - 1, \rho), (2b + 1, \rho)) = 1$, *and* $\epsilon_\sigma(2, \rho) = -1$.

Furthermore, if $\langle [\nu^{\frac{1}{2}}\rho, \nu^b\rho] \rangle \rtimes \tau$ *contains an irreducible tempered subquotient then it contains an irreducible tempered subrepresentation.*

Proof. The proof can be obtained following the one of Theorem 3.10, using Propositions 3.11 and 3.13, together with [8, Theorems 3.8, 3.15, Proposition 3.13]. □

4 The case $a \geq 1$

Throughout this section, a and b denote real numbers such that $b - a$ is a non-negative integer, $2a \in \mathbb{Z}$, and $a \geq 1$.

We determine when the induced representation $\langle [\nu^a\rho, \nu^b\rho] \rangle \rtimes \tau$ reduces.

Lemma 4.1. *Suppose that* τ *is a subrepresentation of* $\delta([\nu^{-a+1}\widetilde{\rho}, \nu^{a-1}\widetilde{\rho}]) \rtimes \tau_1$, *for an irreducible tempered representation* τ_1. *Then the induced representation* $\nu^a\rho \rtimes \tau$ *contains* $L(\delta([\nu^{-a}\widetilde{\rho}, \nu^{a-1}\widetilde{\rho}]); \tau_1)$.

Proof. Let us first suppose that $\delta([\nu^{-a+1}\widetilde{\rho}, \nu^{a-1}\widetilde{\rho}]) \rtimes \tau_1$ is irreducible. Then in $R(G)$ we have

$$\begin{aligned} L(\delta([\nu^{-a}\widetilde{\rho}, \nu^{a-1}\widetilde{\rho}]); \tau_1) &\leq \delta([\nu^{-a+1}\widetilde{\rho}, \nu^{a-1}\widetilde{\rho}]) \times \nu^{-a}\widetilde{\rho} \rtimes \tau_1 \\ &= \nu^a\rho \times \delta([\nu^{-a+1}\widetilde{\rho}, \nu^{a-1}\widetilde{\rho}]) \rtimes \tau_1 \cong \nu^a\rho \rtimes \tau. \end{aligned}$$

Now we suppose that $\delta([\nu^{-a+1}\widetilde{\rho}, \nu^{a-1}\widetilde{\rho}]) \rtimes \tau_1$ reduces. It follows that $\rho \cong \widetilde{\rho}$ and $2a+1 \notin \mathrm{Jord}_\rho(\sigma)$. Using a repeated application of [15, Lemma 2.1] we obtain that there are irreducible square-integrable representations $\delta_1, \ldots, \delta_k \in R(GL)$ and an irreducible tempered representation $\tau_2 \in R(G)$ such that

$$\tau \cong \delta_1 \times \cdots \times \delta_k \rtimes \tau_2,$$

and τ_2 is a subrepresentation of an induced representation of the form $\delta_{k+1} \times \cdots \times \delta_l \rtimes \sigma$, for irreducible square-integrable representations $\delta_{k+1}, \ldots, \delta_l \in R(GL)$ such that $\delta_i \not\cong \delta_j$ for $i, j \in \{k+1, \ldots, l\}$, $i \neq j$, and $\delta_i \rtimes \sigma$ reduces for $i \in \{k+1, \ldots, l\}$. Obviously, we can take $\delta_{k+1} \cong \delta([\nu^{-a+1}\rho, \nu^{a-1}\rho])$, and there is an irreducible tempered subrepresentation τ_3 of $\delta_{k+2} \times \cdots \times \delta_l \rtimes \sigma$ such that τ_2 is a subrepresentation of $\delta([\nu^{-a+1}\rho, \nu^{a-1}\rho]) \rtimes \tau_3$. Also, it follows that τ_1 is a subrepresentation of $\delta_1 \times \cdots \times \delta_k \rtimes \tau_3$.

It is proved in [15, Lemmas 2.2, 2.3] that $L(\delta([\nu^{-a}\rho, \nu^{a-1}\rho]); \tau_3)$ is an irreducible subquotient of $\nu^a\rho \rtimes \tau_2$. In $R(G)$ we have

$$L(\delta([\nu^{-a}\widetilde{\rho}, \nu^{a-1}\widetilde{\rho}]); \tau_1) \leq \delta_1 \times \cdots \times \delta_k \times \delta([\nu^{-a}\widetilde{\rho}, \nu^{a-1}\widetilde{\rho}]) \rtimes \tau_3.$$

Thus, there is an irreducible subquotient π of $\delta([\nu^{-a}\widetilde{\rho}, \nu^{a-1}\widetilde{\rho}]) \rtimes \tau_3$ such that $L(\delta([\nu^{-a}\widetilde{\rho}, \nu^{a-1}\widetilde{\rho}]); \tau_1)$ is contained in $\delta_1 \times \cdots \times \delta_k \rtimes \pi$. An easy application of the structural formula implies that $\pi \cong L(\delta([\nu^{-a}\widetilde{\rho}, \nu^{a-1}\widetilde{\rho}]); \tau_3)$. This leads to

$$\begin{aligned} L(\delta([\nu^{-a}\widetilde{\rho}, \nu^{a-1}\widetilde{\rho}]); \tau_1) &\leq \delta_1 \times \cdots \times \delta_k \rtimes L(\delta([\nu^{-a}\widetilde{\rho}, \nu^{a-1}\widetilde{\rho}]); \tau_3) \\ &\leq \delta_1 \times \cdots \times \delta_k \times \nu^a\rho \rtimes \tau_2 \\ &= \nu^a\rho \times \delta_1 \times \cdots \times \delta_k \rtimes \tau_2 = \nu^a\rho \rtimes \tau. \end{aligned}$$

This ends the proof. □

Corollary 4.2. *Suppose that there is an irreducible tempered representation τ_1 such that τ is a subrepresentation of $\delta([\nu^{-a+1}\widetilde{\rho}, \nu^{a-1}\widetilde{\rho}]) \rtimes \tau_1$. Then the induced representation $\langle[\nu^a\rho, \nu^b\rho]\rangle \rtimes \tau$ reduces.*

Proof. Using the previous lemma, in the same way as in the proof of [7, Proposition 3.5] we deduce that $\langle[\nu^a\rho, \nu^b\rho]\rangle \rtimes \tau$ contains both irreducible representations

$$L(\nu^{-b}\widetilde{\rho}, \ldots, \nu^{-a}\widetilde{\rho}; \tau), L(\nu^{-b}\widetilde{\rho}, \ldots, \nu^{-a-1}\widetilde{\rho}, \delta([\nu^{-a}\widetilde{\rho}, \nu^{a-1}\widetilde{\rho}]); \tau_1)$$

so it reduces. □

Proposition 4.3. *Suppose that ρ is not a self-contragredient representation. Then the induced representation $\langle[\nu^a\rho, \nu^b\rho]\rangle \rtimes \tau$ reduces if and only if $\mu^*(\tau)$ contains an irreducible constituent of the form $\delta([\nu^{-a+1}\widetilde{\rho}, \nu^{a-1}\widetilde{\rho}]) \otimes \pi$.*

Proof. If $\mu^*(\tau)$ contains an irreducible constituent of the form $\delta([\nu^{-a+1}\widetilde{\rho}, \nu^{a-1}\widetilde{\rho}])\otimes \pi$, the claim follows from Lemma 2.4 and Corollary 4.2. Let us now suppose that $\mu^*(\tau)$ does not contain an irreducible constituent of the form $\delta([\nu^{-a+1}\widetilde{\rho}, \nu^{a-1}\widetilde{\rho}]) \otimes \pi$, and let $L(\delta_1, \dots, \delta_k; \tau')$ denote an irreducible non-tempered subquotient of $\langle[\nu^a\rho, \nu^b\rho]\rangle \rtimes \tau$, where $\delta_i \cong \delta([\nu^{x_i}\widetilde{\rho}, \nu^{y_i}\widetilde{\rho}])$ for $i = 1, \dots, k$.

If $x_i = y_i$ for all $i \in \{1, 2, \dots, k\}$ it follows from Lemmas 2.6(1) and 3.2 that $k = b - a + i$ and $\tau' \cong \tau$.

Suppose that $x_k = -y_k - 1$ and $x_k \neq y_k$. From Lemma 2.6(2) follows that $k \in \{1, 2, \dots, b - a + 1\}$, and there is an irreducible tempered representation $\tau_1 \in R(G)$, such that τ is a subrepresentation of

$$\delta([\nu^{-b+k}\widetilde{\rho}, \nu^{b-k}\widetilde{\rho}]) \rtimes \tau_1$$

and τ' is a subquotient of $\langle[\nu^a\rho, \nu^{b-k}\rho]\rangle \rtimes \tau_1$. Since $\mu^*(\tau)$ does not contain an irreducible constituent of the form $\delta([\nu^{-a+1}\widetilde{\rho}, \nu^{a-1}\widetilde{\rho}]) \otimes \pi$, we deduce $k < b - a + 1$, which contradicts Lemma 3.2.

Since by Lemma 3.2 every irreducible subquotient of $\langle[\nu^a\rho, \nu^b\rho]\rangle \rtimes \tau$ is non-tempered, we obtain that every irreducible subqotient of $\langle[\nu^a\rho, \nu^b\rho]\rangle \rtimes \tau$ has to be isomorphic to $L(\nu^{-b}\widetilde{\rho}, \nu^{-b+1}\widetilde{\rho}, \dots, \nu^{-a}\widetilde{\rho}; \tau)$. Using [7, Lemma 3.2] we get that $L(\nu^{-b}\widetilde{\rho}, \nu^{-b+1}\widetilde{\rho}, \dots, \nu^{-a}\widetilde{\rho}; \tau)$ appears in the composition series of $\langle[\nu^a\rho, \nu^b\rho]\rangle \rtimes \tau$ with multiplicity one, so $\langle[\nu^a\rho, \nu^b\rho]\rangle \rtimes \tau$ is irreducible. □

Lemma 4.4. *Suppose that c and d are such that $2c$, $2d$ are positive integers, $c \geq 1$, and $d - c$ is a positive integer. Let $\rho_1 \in R(GL)$ denote an irreducible self-contragredient representation. Let $\tau_1 \in R(G)$ denote an irreducible tempered subrepresentation of $\delta_1 \times \cdots \times \delta_k \rtimes \sigma_1$, for discrete series $\delta_1, \dots, \delta_k \in R(GL)$ and $\sigma_1 \in R(G)$. Suppose that $\delta_i \not\cong \delta([\nu^{-c+1}\rho_1, \nu^{c-1}\rho_1])$, for $i = 1, 2, \dots, k$, $x \in Jord_{\rho_1}(\sigma_1)$ for all $x \in \{2c-1, 2c+1, \dots, 2d-3\}$, $2d - 1 \notin Jord_{\rho_1}(\sigma_1)$, and $\epsilon_{\sigma_1}((x-2, \rho_1), (x, \rho_1)) = -1$ for all $x \in \{2c + 1, \dots, 2d-3\}$. Suppose that $\mu^*(\tau_1)$ contains an irreducible constituent of the form $\delta([\nu^{-d+1}\rho_1, \nu^{d-1}\rho_1]) \otimes \pi$, and let m denote the largest integer such that $\mu^*(\tau_1)$ contains an irreducible constituent of the form $\delta([\nu^{-d+1}\rho_1, \nu^{d-1}\rho_1])^m \otimes \pi$. Suppose that $\mu^*(\tau_1)$ does not contain an irreducible constituent of the form $(\nu^{d-1}\rho_1)^{2m} \otimes \pi$. Then there is an irreducible tempered representation $\tau_2 \in R(G)$ such that $\langle[\nu^c\rho_1, \nu^d\rho_1]\rangle \rtimes \tau_1$ contains $L(\delta([\nu^{-d}\rho_1, \nu^{d-1}\rho_1]); \tau_2)$.*

Proof. Let τ_1' denote an irreducible tempered representation such that τ_1 embeds into $\delta([\nu^{-d+1}\rho_1, \nu^{d-1}\rho_1])^m \rtimes \tau_1'$. By Theorem 3.10, there is an irreducible tempered subrepresentation of $\langle[\nu^c\rho_1, \nu^{d-1}\rho_1]\rangle \rtimes \tau_1'$, which we denote by τ_2'. Note that $\delta([\nu^{-d+1}\rho_1, \nu^{d-1}\rho_1])^l \rtimes \tau_2'$ is irreducible for a positive integer l, and let $\tau_2 \cong \delta([\nu^{-d+1}\rho_1, \nu^{d-1}\rho_1])^{m-1} \rtimes \tau_2'$ (we omit $\delta([\nu^{-d+1}\rho_1, \nu^{d-1}\rho_1])^{m-1}$ if $m = 1$).

We have the following embeddings and an isomorphism:

$$\begin{aligned} L(\delta([\nu^{-d}\rho_1, \nu^{d-1}\rho_1]); \tau_2) &\hookrightarrow \delta([\nu^{-d}\rho_1, \nu^{d-1}\rho_1]) \times \delta([\nu^{-d+1}\rho_1, \nu^{d-1}\rho_1])^{m-1} \rtimes \tau_2' \\ &\cong \delta([\nu^{-d+1}\rho_1, \nu^{d-1}\rho_1])^{m-1} \times \delta([\nu^{-d}\rho_1, \nu^{d-1}\rho_1]) \rtimes \tau_2' \\ &\hookrightarrow \delta([\nu^{-d+1}\rho_1, \nu^{d-1}\rho_1])^{m} \times \nu^{-d}\rho_1 \rtimes \tau_2'. \end{aligned}$$

Lemma 2.3 implies that there is an irreducible subquotient π_1 of $\nu^{-d}\rho_1 \rtimes \tau_2'$ such that $L(\delta([\nu^{-d}\rho_1, \nu^{d-1}\rho_1]); \tau_2)$ is a subrepresentation of $\delta([\nu^{-d+1}\rho_1, \nu^{d-1}\rho_1])^m \rtimes \pi_1$. Since $\mu^*(L(\delta([\nu^{-d}\rho_1, \nu^{d-1}\rho_1]); \tau_2))$ contains $\delta([\nu^{-d}\rho_1, \nu^{d-1}\rho_1]) \otimes \tau_2$, τ_2' is tempered and $\mu^*(\tau_2')$ does not contain an irreducible constituent of the form $\delta([\nu^{-d+1}\rho_1, \nu^{d-1}\rho_1]) \otimes \pi$, it follows that $\mu^*(\pi_1)$ contains an irreducible constituent of the form $\nu^{-d}\rho_1 \otimes \pi$, so $\pi_1 \cong L(\nu^{-d}\rho_1; \tau_2')$.

Since $\mu^*(\tau_1')$ does not contain an irreducible constituent of the form $\delta([\nu^{-d+1}\rho_1, \nu^{d-1}\rho_1]) \otimes \pi$, it directly follows that $L(\nu^{-d}\rho_1, \tau_1')$ is a unique non-tempered irreducible subquotient of $\nu^{d}\rho_1 \rtimes \tau_1'$. Using $2d-1 \notin \mathrm{Jord}_{\rho_1}(\sigma_1)$ and the cuspidal support considerations we conclude that there are no irreducible tempered subquotients of $\nu^{d}\rho_1 \rtimes \tau_1'$. Thus, $\nu^{d}\rho_1 \rtimes \tau_1'$ is irreducible. This leads to

$$\begin{aligned} L(\nu^{-d}\rho_1; \tau_2') &\hookrightarrow \nu^{-d}\rho_1 \rtimes \tau_2' \hookrightarrow \nu^{-d}\rho_1 \times \langle[\nu^{c}\rho_1, \nu^{d-1}\rho_1]\rangle \rtimes \tau_1' \\ &\cong \langle[\nu^{c}\rho_1, \nu^{d-1}\rho_1]\rangle \times \nu^{-d}\rho_1 \rtimes \tau_1' \cong \langle[\nu^{c}\rho_1, \nu^{d-1}\rho_1]\rangle \times \nu^{d}\rho_1 \rtimes \tau_1'. \end{aligned}$$

Using Lemma 2.3 again, we obtain that there is an irreducible subquotient π_2 of $\langle[\nu^{c}\rho_1, \nu^{d-1}\rho_1]\rangle \times \nu^{d}\rho_1$ such that $L(\nu^{-d}\rho_1; \tau_2')$ is a subrepresentation of $\pi_2 \rtimes \tau_1'$. Since $\mu^*(L(\nu^{-d}\rho_1; \tau_2')) \geq \nu^{-d}\rho_1 \otimes \tau_2'$ and τ_1' is tempered, we deduce $\pi_2 \cong \langle[\nu^{c}\rho_1, \nu^{d}\rho_1]\rangle$.

In the proof of Lemma 3.6 we have seen that the induced representation $\delta([\nu^{-d+1}\rho_1, \nu^{d-1}\rho_1]) \times \langle[\nu^{c}\rho_1, \nu^{d}\rho_1]$ is irreducible, so we have

$$\begin{aligned} L(\delta([\nu^{-d}\rho_1, \nu^{d-1}\rho_1]); \tau_2) &\hookrightarrow \delta([\nu^{-d+1}\rho_1, \nu^{d-1}\rho_1])^m \times \langle[\nu^{c}\rho_1, \nu^{d}\rho_1]\rangle \rtimes \tau_1' \\ &\cong \langle[\nu^{c}\rho_1, \nu^{d}\rho_1]\rangle \times \delta([\nu^{-d+1}\rho_1, \nu^{d-1}\rho_1])^m \rtimes \tau_1'. \end{aligned}$$

Now there is an irreducible subquotient π_3 of $\delta([\nu^{-d+1}\rho_1, \nu^{d-1}\rho_1])^m \rtimes \tau_1'$ such that $L(\delta([\nu^{-d}\rho_1, \nu^{d-1}\rho_1]); \tau_2)$ is a subrepresentation of $\langle[\nu^{c}\rho_1, \nu^{d}\rho_1]\rangle \rtimes \pi_3$.

In $R(G)$ we have $\delta([\nu^{-d+1}\rho_1, \nu^{d-1}\rho_1])^m \rtimes \tau_1' = \tau_1 + \tau_{-1}$, where τ_{-1} is an irreducible tempered representation which embeds into an induced representation of the form $(\nu^{d-1}\rho_1)^{2m} \rtimes \pi_4$. If $\pi_3 \cong \tau_{-1}$, we get

$$\begin{aligned} L(\delta([\nu^{-d}\rho_1, \nu^{d-1}\rho_1]); \tau_2) &\hookrightarrow \langle[\nu^c\rho_1, \nu^d\rho_1]\rangle \times (\nu^{d-1}\rho_1)^{2m} \rtimes \pi_4 \\ &\cong (\nu^{d-1}\rho_1)^{2m} \times \langle[\nu^c\rho_1, \nu^d\rho_1]\rangle \rtimes \pi_4. \end{aligned}$$

Let $x = 1$ if $c = d-1$, and let $x = 0$ otherwise. Frobenius reciprocity implies that $\mu^*(L(\delta([\nu^{-d}\rho_1, \nu^{d-1}\rho_1]); \tau_2))$ contains an irreducible constituent of the form $(\nu^{d-1}\rho_1)^{2m+x} \otimes \pi$.

Using $\tau_2 \cong \delta([\nu^{-d+1}\rho_1, \nu^{d-1}\rho_1])^{m-1} \rtimes \tau_2'$, together with the structural formula and the definition of τ_2', we deduce that if $\mu^*(\tau_2)$ contains an irreducible constituent of the form $(\nu^{d-1}\rho_1)^l \otimes \pi$ then $l \leq 2m + x - 2$. Since $L(\delta([\nu^{-d}\rho_1, \nu^{d-1}\rho_1]); \tau_2)$ is an irreducible subquotient of $\delta([\nu^{-d}\rho_1, \nu^{d-1}\rho_1]) \rtimes \tau_2$, using the structural formula we see at once that if $\mu^*(L(\delta([\nu^{-d}\rho_1, \nu^{d-1}\rho_1]); \tau_2))$ contains an irreducible constituent of the form $(\nu^{d-1}\rho_1)^l \otimes \pi$ then $l \leq 2m + x - 1$, a contradiction. Thus, $\pi_3 \cong \tau_1$ and $L(\delta([\nu^{-d}\rho_1, \nu^{d-1}\rho_1]); \tau_2)$ is a subrepresentation of $\langle[\nu^c\rho_1, \nu^d\rho_1]\rangle \rtimes \tau_1$. □

Theorem 4.5. *Suppose that $a \geq 1$ and $\rho \cong \widetilde{\rho}$. The induced representation $\langle[\nu^a\rho, \nu^b\rho]\rangle \rtimes \tau$ reduces if and only if one of the following holds:*

(1) $\mu^*(\tau)$ contains an irreducible constituent of the form $\delta([\nu^{-a+1}\rho, \nu^{a-1}\rho]) \otimes \pi$.

(2) We have $x \in Jord_\rho(\sigma)$ for all $x \in \{2a-1, 2a+1, \dots, 2b+1\}$, $\epsilon_\sigma((x_-, \rho), (x, \rho)) = -1$ for all $x \in \{2a-1, 2a+1, \dots, 2b-3\}$, and $\epsilon_\sigma((2b-1, \rho), (2b+1, \rho)) = 1$.

(3) We have $x \in Jord_\rho(\sigma)$ for all $x \in \{2a-1, 2a+1, \dots, 2b-1\}$, $\epsilon_\sigma((x_-, \rho), (x, \rho)) = -1$ for all $x \in \{2a-1, 2a+1, \dots, 2b-3\}$, $2b+1 \notin Jord_\rho(\sigma)$, and if $\mu^*(\tau)$ contains an irreducible constituent of the form $\delta([\nu^{-b}\rho, \nu^b\rho]) \otimes \pi_1$, and m stands for the largest integer such that $\mu^*(\tau)$ contains an irreducible constituent of the form $\delta([\nu^{-b}\rho, \nu^b\rho])^m \otimes \pi_2$, then $\mu^*(\tau)$ contains an irreducible constituent of the form $(\nu^b\rho)^{2m} \otimes \pi_3$,

(4) There is a $c \in \{a, a+1, \dots, b-1\}$ such that $x \in Jord_\rho(\sigma)$ for all $x \in \{2a-1, 2a+1, \dots, 2c-1\}$, $\epsilon_\sigma((x_-, \rho), (x, \rho)) = -1$ for all $x \in \{2a+1, 2a+3, \dots, 2c-1\}$, and if $2c+1 \in Jord_\rho(\sigma)$ then $\epsilon_\sigma((2c-1, \rho), (2c+1, \rho)) = 1$.

Proof. If (1) holds, $\langle[\nu^a\rho, \nu^b\rho]\rangle \rtimes \tau$ reduces by Corollary 4.2. If either (2) or (3) holds, it follows from Theorem 3.10 that $\langle[\nu^a\rho, \nu^b\rho]\rangle \rtimes \tau$ contains an irreducible tempered subquotient, so it reduces.

Let us now suppose that (4) holds. If $\langle[\nu^a\rho, \nu^c\rho]\rangle \rtimes \tau$ contains an irreducible tempered subquotient, we denote it by τ_1, and it can be seen in the same way as in the proof of [7, Proposition 3.5] that $\langle[\nu^a\rho, \nu^b\rho]\rangle \rtimes \tau$ contains $L(\nu^{-b}\rho, \ldots, \nu^{-c-1}\rho; \tau_1)$, so it reduces.

If $\langle[\nu^a\rho, \nu^c\rho]\rangle \rtimes \tau$ does not contain an irreducible tempered subquotient, using Theorem 3.10 we obtain that $2c+1 \notin \mathrm{Jord}_\rho(\sigma)$, $\mu^*(\tau)$ contains an irreducible constituent of the form $\delta([\nu^{-c}\rho, \nu^c\rho]) \otimes \pi_1$, and if m stands for the largest integer such that $\mu^*(\tau)$ contains an irreducible constituent of the form $\delta([\nu^{-c}\rho, \nu^c\rho])^m \otimes \pi_2$, then $\mu^*(\tau)$ does not contain an irreducible constituent of the form $(\nu^c\rho)^{2m} \otimes \pi_3$. Now Lemma 4.4 implies that there is an irreducible tempered representation $\tau_2 \in R(G)$ such that $\langle[\nu^a\rho, \nu^{c+1}\rho]\rangle \rtimes \tau$ contains $L(\delta([\nu^{-c-1}\rho, \nu^c\rho]); \tau_2)$. Using an inductive application of Lemma 2.7, we deduce that

$$L(\nu^{-b}\rho, \ldots, \nu^{-c-2}\rho, \delta([\nu^{-c-1}\rho, \nu^c\rho]); \tau_2)$$

is an irreducible subquotient of $\langle[\nu^a\rho, \nu^b\rho]\rangle \rtimes \tau$, so $\langle[\nu^a\rho, \nu^b\rho]\rangle \rtimes \tau$ reduces.

Let us now assume that neither of (1), (2), (3), (4) holds. Then $\langle[\nu^a\rho, \nu^b\rho]\rangle \rtimes \tau$ does not contain an irreducible tempered subquotient. Let $L(\delta_1, \ldots, \delta_k; \tau')$ denote an irreducible non-tempered subquotient of $\langle[\nu^a\rho, \nu^b\rho]\rangle \rtimes \tau$, and $\delta_i \cong \delta([\nu^{x_i}\rho, \nu^{y_i}\rho])$ for $i = 1, \ldots, k$.

If $x_i = y_i$ for all $i \in \{1, 2, \ldots, k\}$, using Lemma 2.6(1) we get that there is a $c \in \{a-1, a, \ldots, b-1\}$ such that τ' is an irreducible subquotient of $\langle[\nu^a\rho, \nu^c\rho]\rangle \rtimes \tau$. Theorem 3.10 implies that $c = a-1$ and $\tau' \cong \tau$.

Suppose that there is $i \in \{1, 2, \ldots, k\}$ such that $x_i \neq y_i$. Lemma 2.6 implies $i = k$ and $x_k = -y_k - 1$. Since $\mu^*(\tau)$ does not contain an irreducible constituent of the form $\delta([\nu^{-a+1}\rho, \nu^{a-1}\rho]) \otimes \pi$, in the same way as in the proof of Proposition 4.3 we deduce that $k \leq b-a$ and there is an irreducible tempered representation $\tau_1 \in R(G)$ such that τ is a subrepresentation of

$$\delta([\nu^{-b+k}\rho, \nu^{b-k}\rho]) \rtimes \tau_1$$

and τ' is an irreducible tempered subquotient of $\langle[\nu^a\rho, \nu^{b-k}\rho]\rangle \rtimes \tau_1$. It directly follows that there are irreducible square-integrable representations $\delta'_1, \ldots, \delta'_l \in R(GL)$ such that τ_1 is a subrepresentation of $\delta'_1 \times \cdots \times \delta'_l \rtimes \sigma$. Using $a \leq b-k$ and Theorem 3.10, we conclude that $x \in \mathrm{Jord}_\rho(\sigma)$ for all $x \in \{2a-1, 2a+1, \ldots, 2b-2k-1\}$, $\epsilon_\sigma((x_-, \rho), (x, \rho)) = -1$ for all $x \in \{2a+1, \ldots, 2b-2k-1\}$, and if $2b-2k+1 \in \mathrm{Jord}_\rho(\sigma)$ then $\epsilon_\sigma((2b-2k-1, \rho), (2b-2k+1, \rho)) = 1$. Since (4) does not hold, we obtain $b-k = b$, which is impossible.

Thus, every irreducible subquotient of $\langle[\nu^a\rho, \nu^b\rho]\rangle \rtimes \tau$ is isomorphic to $L(\nu^{-b}\rho, \nu^{-b+1}\rho, \ldots, \nu^{-a}\rho; \tau)$, and in the same way as in the proof of Proposition 4.3 we deduce that $\langle[\nu^a\rho, \nu^b\rho]\rangle \rtimes \tau$ is irreducible. □

We emphasize that if we write, as in the part (2) of Theorem 2.2, τ as a subrepresentation of $\delta^{(1)} \times \cdots \times \delta^{(r)} \rtimes \sigma$, for discrete series $\delta^{(1)}, \ldots, \delta^{(r)} \in R(G)$, then $\mu^*(\tau)$ contains an irreducible constituent of the form $\delta([\nu^{-a+1}\rho, \nu^{a-1}\rho]) \otimes \pi$ if and only if we have $\delta^{(i)} \cong \delta([\nu^{-a+1}\rho, \nu^{a-1}\rho])$ for some $i \in \{1, \ldots, r\}$.

Suppose that $2b+1 \notin \mathrm{Jord}_\rho(\sigma)$, $\mu^*(\tau)$ contains an irreducible constituent of the form $\delta([\nu^{-b}\rho, \nu^b\rho]) \otimes \pi_1$, and let m denote the largest integer such that $\mu^*(\tau)$ contains an irreducible constituent of the form $\delta([\nu^{-b}\rho, \nu^b\rho])^m \otimes \pi_2$. Then τ is a subrepresentation of $\delta([\nu^{-b}\rho, \nu^b\rho])^m \rtimes \tau'$, for an irreducible tempered representation τ' such that $\mu^*(\tau')$ does not contain an irreducible constituent of the form $\delta([\nu^{-b}\rho, \nu^b\rho]) \otimes \pi_3$. Since $2b+1 \notin \mathrm{Jord}_\rho(\sigma)$, the part (3) of Theorem 2.2 implies that the induced representation $\delta([\nu^{-b}\rho, \nu^b\rho])^m \rtimes \tau'$ reduces, and by the part (4) of Theorem 2.2, it is a direct sum of two mutually non-isomorphic tempered representations τ_1 and τ_2, and $\tau \in \{\tau_1, \tau_2\}$. By the part (1) of Lemma 2.5, there is a unique $i \in \{1, 2\}$ such that $\mu^*(\tau_i)$ contains an irreducible constituent of the form $(\nu^b\rho)^{2m} \otimes \pi$. Thus, if we additionally assume that $\mu^*(\tau)$ does not contain an irreducible constituent of the form $\delta([\nu^{-a+1}\rho, \nu^{a-1}\rho]) \otimes \pi$, $x \in \mathrm{Jord}_\rho(\sigma)$ for all $x \in \{2a-1, 2a+1, \ldots, 2b-1\}$, and $\epsilon_\sigma((x_-, \rho), (x, \rho)) = -1$ for all $x \in \{2a-1, 2a+1, \ldots, 2b-3\}$, then there is a unique $i \in \{1, 2\}$ such that $\langle[\nu^a\rho, \nu^b\rho]\rangle \rtimes \tau_i$ contains an irreducible tempered subquotient.

We also provide a rephrasing of the previous theorem:

Corollary 4.6. *Suppose that $\rho \cong \widetilde{\rho}$ and $a \geq 1$ such that $2a$ is an integer. The induced representation $\langle[\nu^a\rho, \nu^b\rho]\rangle \rtimes \tau$ reduces if and only if one of the following holds:*

(1) The Jacquet module of τ with respect to the appropriate parabolic subgroup contains an irreducible subquotient of the form $\delta([\nu^{-a+1}\rho, \nu^{a-1}\rho]) \otimes \pi$.

(2) There is a $c \in \{a, a+1, \ldots, b\}$ such that $\langle[\nu^a\rho, \nu^c\rho]\rangle \rtimes \tau$ contains an irreducible tempered subquotient.

(3) There is a $c \in \{a, a+1, \ldots, b\}$ and an irreducible tempered representation $\tau' \in R(G)$ such that $\langle[\nu^a\rho, \nu^c\rho]\rangle \rtimes \tau$ contains $L(\delta([\nu^{-c}\rho, \nu^{c-1}\rho]); \tau')$.

5 The case $a = \frac{1}{2}$

In this section, b denotes a real number such that $b - \frac{1}{2}$ is a non-negative integer. We determine when the induced representation $\langle[\nu^{\frac{1}{2}}\rho, \nu^b\rho]\rangle \rtimes \tau$ reduces.

Lemma 5.1. *If ρ is not a self-contragredient representation, then the induced representation $\langle[\nu^{\frac{1}{2}}\rho, \nu^b\rho]\rangle \rtimes \tau$ is irreducible.*

Proof. By Lemma 3.2, $\langle[\nu^{\frac{1}{2}}\rho, \nu^b\rho]\rangle \rtimes \tau$ does not contain an irreducible tempered subquotient. Let $L(\delta_1, \ldots, \delta_k; \tau')$ denote an irreducible non-tempered subquotient of $\langle[\nu^{\frac{1}{2}}\rho, \nu^b\rho]\rangle \rtimes \tau$, where $\delta_i \cong \delta([\nu^{x_i}\widetilde{\rho}, \nu^{y_i}\widetilde{\rho}])$ for $i = 1, \ldots, k$.

Suppose that there is an $i \in \{1, 2, \ldots, k\}$ such that $x_i = -y_i - 1$, $x_i \neq y_i$. From Lemma 2.6(2) follows that $x_k = -y_k - 1$, and from $x_k \neq y_k$ we obtain $x_k \neq -\frac{1}{2}$. Lemma 2.6(2) also implies that there is an irreducible tempered representation τ_1 and $c \in \{\frac{1}{2}, \frac{3}{2}, \ldots, b - 1\}$ such that τ' is an irreducible subquotient of $\langle[\nu^{\frac{1}{2}}\rho, \nu^c\rho]\rangle \rtimes \tau_1$, which is impossible by Lemma 3.2.

Consequently, we have $x_i = y_i$ for all $i \in \{1, 2, \ldots, k\}$. Now the rest of the proof follows in the same way as the one of Proposition 4.3. $\square$

The following lemma can be proved in the same way as Lemma 4.4, using Theorem 3.14.

Lemma 5.2. *Suppose that $c - \frac{1}{2}$ is a positive integer and let $\rho_1 \in R(GL)$ denote an irreducible self-contragredient representation. Let $\tau_1 \in R(G)$ denote an irreducible tempered representation, which is a subrepresentation of $\delta_1 \times \cdots \times \delta_k \rtimes \sigma_1$, for discrete series $\delta_1, \ldots, \delta_k \in R(GL)$ and $\sigma_1 \in R(G)$. Suppose that $x \in Jord_{\rho_1}(\sigma_1)$ for all $x \in \{2, 4, \ldots, 2c-3\}$, $2c - 1 \notin Jord_{\rho_1}(\sigma_1)$, $\epsilon_{\sigma_1}((x-2, \rho_1), (x, \rho_1)) = -1$ for all $x \in \{4, \ldots, 2c-3\}$ and $\epsilon_\sigma(2, \rho) = -1$ if $c \geq \frac{5}{2}$. Suppose that $\mu^*(\tau_1)$ contains an irreducible constituent of the form $\delta([\nu^{-c+1}\rho_1, \nu^{c-1}\rho_1]) \otimes \pi$, and let m denote the largest integer such that $\mu^*(\tau_1)$ contains an irreducible constituent of the form $\delta([\nu^{-c+1}\rho_1, \nu^{c-1}\rho_1])^m \otimes \pi$. Suppose that $\mu^*(\tau_1)$ does not contain an irreducible constituent of the form $(\nu^{c-1}\rho_1)^{2m} \otimes \pi$. Then there is an irreducible tempered representation $\tau_2 \in R(G)$ such that $\langle[\nu^{\frac{1}{2}}\rho_1, \nu^c\rho_1]\rangle \rtimes \tau_1$ contains $L(\delta([\nu^{-c}\rho_1, \nu^{c-1}\rho_1]); \tau_2)$.*

Theorem 5.3. *Suppose that $\rho \cong \widetilde{\rho}$ and let b be such that $b - \frac{1}{2}$ is a non-negative integer. The induced representation $\langle[\nu^{\frac{1}{2}}\rho, \nu^b\rho]\rangle \rtimes \tau$ reduces if and only if one of the following holds:*

(1) $2 \in Jord_\rho(\sigma)$ *and* $\epsilon_\sigma(2,\rho) = 1$.

(2) $2 \notin Jord_\rho(\sigma)$ *and if* $\mu^*(\tau)$ *contains an irreducible constituent of the form* $\delta([\nu^{-\frac{1}{2}}\rho, \nu^{\frac{1}{2}}\rho]) \otimes \pi_1$, *and* m *stands for the largest integer such that* $\mu^*(\tau)$ *contains an irreducible constituent of the form* $\delta([\nu^{-\frac{1}{2}}\rho, \nu^{\frac{1}{2}}\rho])^m \otimes \pi_2$, *then* $\mu^*(\tau)$ *contains an irreducible constituent of the form* $(\nu^{\frac{1}{2}}\rho)^{2m} \otimes \pi_3$.

(3) $b > \frac{1}{2}$, $x \in Jord_\rho(\sigma)$ *for all* $x \in \{2, 4, \ldots, 2b+1\}$, $\epsilon_\sigma((x_-,\rho),(x,\rho)) = -1$ *for all* $x \in \{4, 6, \ldots, 2b-3\}$, $\epsilon_\sigma(2,\rho) = -1$, *and* $\epsilon_\sigma((2b-1,\rho),(2b+1,\rho)) = 1$.

(4) $b > \frac{1}{2}$, $x \in Jord_\rho(\sigma)$ *for all* $x \in \{2, 4, \ldots, 2b-1\}$, $\epsilon_\sigma((x_-,\rho),(x,\rho)) = -1$ *for all* $x \in \{4, 6, \ldots, 2b-3\}$, $2b+1 \notin Jord_\rho(\sigma)$, $\epsilon_\sigma(2,\rho) = -1$, *and if* $\mu^*(\tau)$ *contains an irreducible constituent of the form* $\delta([\nu^{-b}\rho, \nu^b\rho]) \otimes \pi_1$, *and* m *stands for the largest integer such that* $\mu^*(\tau)$ *contains an irreducible constituent of the form* $\delta([\nu^{-b}\rho, \nu^b\rho])^m \otimes \pi_2$, *then* $\mu^*(\tau)$ *contains an irreducible constituent of the form* $(\nu^b\rho)^{2m} \otimes \pi_3$,

(5) $b > \frac{1}{2}$, *and there is a* $c \in \{\frac{3}{2}, \ldots, b-1\}$ *such that* $x \in Jord_\rho(\sigma)$ *for all* $x \in \{2, 4, \ldots, 2c-1\}$, $\epsilon_\sigma((x_-,\rho),(x,\rho)) = -1$ *for all* $x \in \{2, 4, \ldots, 2c-1\}$, $\epsilon_\sigma(2,\rho) = -1$, *and if* $2c+1 \in Jord_\rho(\sigma)$ *then* $\epsilon_\sigma((2c-1,\rho),(2c+1,\rho)) = 1$.

Proof. If $b > \frac{1}{2}$, the theorem can be proved following the same lines as in the proof of Theorem 4.5, just using Lemma 5.2 instead of Lemma 4.4.

Let us comment the case $b = \frac{1}{2}$. In the same way as in the proof of Lemma 2.6 we deduce that every irreducible non-tempered subquotient of $\nu^{\frac{1}{2}}\rho \rtimes \tau$ is isomorphic to $L(\nu^{-\frac{1}{2}}\rho; \tau)$, and it can be seen at once that $L(\nu^{-\frac{1}{2}}\rho; \tau)$ appears in the composition series of $\nu^{\frac{1}{2}}\rho \rtimes \tau$ with multiplicity one. Consequently, $\nu^{\frac{1}{2}}\rho \rtimes \tau$ reduces if and only if contains an irreducible tempered subquotient. Now Theorem 3.14 can be used to finish the proof. □

Suppose that $2 \notin \mathrm{Jord}_\rho(\sigma)$, $\mu^*(\tau)$ contains an irreducible constituent of the form $\delta([\nu^{-\frac{1}{2}}\rho, \nu^{\frac{1}{2}}\rho]) \otimes \pi_1$, and let m denote the largest integer such that $\mu^*(\tau)$ contains an irreducible constituent of the form $\delta([\nu^{-\frac{1}{2}}\rho, \nu^{\frac{1}{2}}\rho])^m \otimes \pi_2$. Then there is an irreducible tempered representation $\tau' \in R(G)$ such that τ is a subrepresentation of $\delta([\nu^{-\frac{1}{2}}\rho, \nu^{\frac{1}{2}}\rho])^m \rtimes \tau'$ and $\mu^*(\tau')$ does not contain an irreducible constituent of the form $\delta([\nu^{-\frac{1}{2}}\rho, \nu^{\frac{1}{2}}\rho]) \otimes \pi_3$. By the parts (2) and (3) of Theorem 2.2, $\delta([\nu^{-\frac{1}{2}}\rho, \nu^{\frac{1}{2}}\rho])^m \rtimes \tau'$ is a direct sum of two mutually non-isomorphic tempered representations, which we denote by τ_1 and τ_2. It follows from the part (2) of Lemma 2.5 that there is a unique $i \in \{1,2\}$ such that $\mu^*(\tau_i)$ contains an irreducible constituent of the form $(\nu^{\frac{1}{2}}\rho)^{2m} \otimes \pi_4$. Thus, there is a unique $i \in \{1,2\}$ such that $\langle[\nu^{\frac{1}{2}}\rho, \nu^b\rho]\rangle \rtimes \tau_i$ reduces.

We also note that Theorem 5.3 can be rephrased as follows:

Corollary 5.4. *Suppose that $\rho \cong \widetilde{\rho}$ and let b be such that $b - \frac{1}{2}$ is a non-negative integer. The induced representation $\langle[\nu^{\frac{1}{2}}\rho, \nu^b\rho]\rangle \rtimes \tau$ reduces if and only if one of the following holds:*

(1) There is a $c \in \{\frac{1}{2}, \frac{3}{2}, \ldots, b\}$ such that $\langle[\nu^{\frac{1}{2}}\rho, \nu^c\rho]\rangle \rtimes \tau$ contains an irreducible tempered subquotient.

(2) There is a $c \in \{\frac{3}{2}, \frac{5}{2}, \ldots, b\}$ and an irreducible tempered representation $\tau' \in R(G)$ such that $\langle[\nu^{\frac{1}{2}}\rho, \nu^c\rho]\rangle \rtimes \tau$ contains $L(\delta([\nu^{-c}\rho, \nu^{c-1}\rho]); \tau')$.

6 The case $a < 0$, a half-integral

In this section, until said otherwise, a and b denote half-integers such that $a < 0$ and $-a \leq b$. We first determine when the induced representation $\langle[\nu^a\rho, \nu^b\rho]\rangle \rtimes \tau$ reduces. At the end of the section, we provide a summary of our main results.

Let us first discuss the case of self-contragredient ρ.

Proposition 6.1. *Suppose that a is negative and half-integral, and $\rho \cong \widetilde{\rho}$. Let $L(\delta_1, \ldots, \delta_k; \tau')$ stand for an irreducible non-tempered subquotient of $\langle[\nu^a\rho, \nu^b\rho]\rangle \rtimes \tau$, and let $\delta_i \cong \delta([\nu^{x_i}\rho_i, \nu^{y_i}\rho_i])$ for $i = 1, \ldots, k$. Then for all $i = 1, \ldots, k$ we have $\rho_i \cong \rho$ and $x_i \in \{y_i, y_i - 1, -y_i - 1\}$. Also, there is at most one $i \in \{1, \ldots, k\}$ such that $x_i \notin \{y_i, y_i - 1\}$. If $j \in \{2, 3, \ldots, k\}$ is such that $x_i \in \{y_i, y_i - 1\}$ for all $i \in \{1, 2, \ldots, j-1\}$, then there are c and d, $a \leq c \leq \frac{1}{2}$, $\frac{1}{2} \leq d \leq b$, such that $L(\delta_j, \ldots, \delta_k; \tau')$ is a subquotient of $\langle[\nu^c\rho, \nu^d\rho]\rangle \rtimes \tau$.*

Proof. In the same way as in the proof of Lemma 2.6, we deduce that are i, j such that $a - 1 \leq i \leq j \leq b$, and an irreducible constituent $\delta \otimes \pi$ of $\mu^*(\tau)$ such that

$$\delta_1 \leq \langle[\nu^{-b}\rho, \nu^{-j-1}\rho]\rangle \times \langle[\nu^a\rho, \nu^i\rho]\rangle \times \delta$$

and

$$L(\delta_2,\ldots,\delta_k;\tau') \leq \langle[\nu^{i+1}\rho,\nu^j\rho]\rangle \rtimes \pi.$$

It follows at once that $\rho_i \cong \rho$, $i \in \{a-1,a\}$, and $j \in \{b-1,b\}$. Using the temperedness criterion, we obtain

$$\begin{aligned}\delta_1 \in &\{\nu^{-b}\rho,\nu^a\rho,\delta([\nu^{-b}\rho,\nu^{-b+1}\rho]),\delta([\nu^{-b}\rho,\nu^{b-1}\rho]),\\ &\delta([\nu^a\rho,\nu^{-a-1}\rho]),\delta([\nu^{-b}\rho,\nu^{b-2}\rho])\}.\end{aligned}$$

Furthermore, $\delta_1 \in \{\delta([\nu^{-b}\rho,\nu^{-b+1}\rho]),\delta([\nu^{-b}\rho,\nu^{b-2}\rho])\}$ implies $a=-b+1$. Also, if

$$\delta_1 \in \{\nu^{-b}\rho,\nu^a\rho,\delta([\nu^{-b}\rho,\nu^{-b+1}\rho])\},$$

then $\pi \cong \tau$.

Suppose that there is an $i \in \{1,2,\ldots,k\}$ such that $y_i \notin \{x_i,x_i+1\}$, and let us denote the minimal such i by $i_{\min}$. Then for $i < i_{\min}$ we have $y_i \in \{x_i,x_i+1\}$ and an inductive application of the first part of the proof shows that there are c and d such that $a \leq c \leq \frac{1}{2}$, $\frac{1}{2} \leq d \leq b$, and

$$L(\delta_{i_{\min}},\ldots,\delta_k;\tau') \leq \langle[\nu^c\rho,\nu^d\rho]\rangle \rtimes \tau.$$

Again, $\rho_{i_{\min}} \cong \rho$. We have already seen that $y_{i_{\min}} \in \{-x_{i_{\min}}-2,-x_{i_{\min}}-1\}$. Suppose that $y_{i_{\min}} = -x_{i_{\min}}-2$. Since $\mu^*(\langle[\nu^c\rho,\nu^d\rho]\rangle \rtimes \tau)$ contains $\delta_{i_{\min}} \otimes L(\delta_{i_{\min}+1},\ldots,\delta_k;\tau')$, it follows that $d \geq \frac{5}{2}$, $x_{i_{\min}} = -d$, $c=-d+1$, and there is an irreducible tempered representation τ_1 such that $\mu^*(\tau) \geq \delta([\nu^{-d+2}\rho,\nu^{d-2}\rho]) \otimes \tau_1$ and

$$L(\delta_{i_{\min}+1},\ldots,\delta_k;\tau') \leq \langle[\nu^{-d+2}\rho,\nu^{d-1}\rho]\rangle \rtimes \tau_1.$$

Since $-d+2<0$, Theorem 3.5 implies $i_{\min} < k$. We directly obtain $e(\delta_{i_{\min}+1}) \geq -1$ and $\rho_{i_{\min}+1} \cong \rho$. If $x_{i_{\min}+1} = y_{i_{\min}+1}$, from $e(\delta_{i_{\min}+1}) \geq -1$ follows that $\delta_{i_{\min}+1} \cong \nu^{-\frac{1}{2}}\rho$. Thus $\delta_{i_{\min}} \times \delta_{i_{\min}+1} \cong \delta_{i_{\min}+1} \times \delta_{i_{\min}}$, so we deduce that $\mu^*(\langle[\nu^{-d+1}\rho,\nu^d\rho]\rangle \rtimes \tau)$ contains an irreducible constituent of the form $\nu^{-\frac{1}{2}}\rho \otimes \pi$, which is impossible since $-d+1 \leq -\frac{3}{2}$ and τ is tempered. Consequently,

$$\delta_{i_{\min}+1} \in \{\delta([\nu^{-d+1}\rho,\nu^{d-2}\rho]),\delta([\nu^{-d+2}\rho,\nu^{d-3}\rho]),\delta([\nu^{-d+1}\rho,\nu^{d-3}\rho])\}$$

This again gives $\delta_{i_{\min}} \times \delta_{i_{\min}+1} \cong \delta_{i_{\min}+1} \times \delta_{i_{\min}}$, so $\mu^*(\langle[\nu^{-d+1}\rho,\nu^d\rho]\rangle \rtimes \tau)$ contains an irreducible constituent of the form $\delta_{i_{\min}+1} \otimes \pi$, and it follows directly from the structural formula that this is possible only if $\delta_{i_{\min}+1} \cong \delta([\nu^{-d+1}\rho, \nu^{d-2}\rho])$. Using

$$L(\delta_{i_{\min}}, \ldots, \delta_k; \tau') \hookrightarrow \delta_{i_{\min}+1} \times \delta_{i_{\min}} \rtimes L(\delta_{i_{\min}+2}, \ldots, \delta_k; \tau'),$$

together with the structural formula, we deduce that there is an irreducible tempered representation τ_1 such that $\mu^*(\tau)$ contains $\delta([\nu^{-d+2}\rho, \nu^{d-2}\rho]) \otimes \tau_1$ and $\mu^*(\langle[\nu^{-d+2}\rho, \nu^{d}\rho]\rangle \rtimes \tau_1)$ contains $\delta_{i_{\min}} \otimes L(\delta_{i_{\min}+2}, \ldots, \delta_k; \tau')$. It follows that $\mu^*(\tau_1)$ contains an irreducible constituent of the form $\delta([\nu^{-d+1}\rho, \nu^{x}\rho]) \otimes \pi$, for $x \leq d-2$, which contradicts the temperedness of τ_1. Thus, we obtain $y_{i_{\min}} = -x_{i_{\min}} - 1$.

Suppose that there is an $i > i_{\min}$ such that $y_i = -x_i - 1$ and $y_i > 0$. Since $e(\delta_{i_{\min}}) = e(\delta_i)$, it follows that $e(\delta_i) = e(\delta_j)$ for $j \in \{i_{\min}, \ldots, i-1\}$, so $\delta_j \times \delta_i \cong \delta_i \times \delta_j$ for $j \in \{i_{\min}, \ldots, i-1\}$. This enables us to assume $i = i_{\min}+1$. It follows that $\mu^*(\langle[\nu^{c}\rho, \nu^{d}\rho]\rangle \rtimes \tau)$ contains $\delta_{i_{\min}} \times \delta_{i_{\min}+1} \otimes L(\delta_{i_{\min}+2}, \ldots, \delta_k; \tau')$. Also, $\mu^*(\langle[\nu^{c}\rho, \nu^{d}\rho]\rangle \rtimes \tau)$ contains irreducible constituents of the form $\delta_{i_{\min}} \otimes \pi$ and $\delta_{i_{\min}+1} \otimes \pi$. Now an easy application of the structural formula shows that $\{x_{i_{\min}}, x_{i_{\min}+1}\} = \{-d, c\}$, $c \leq -\frac{3}{2}$, and there is an irreducible tempered representation τ_1 such that $\mu^*(\tau) \geq \delta([\nu^{-d+1}\rho, \nu^{d-1}\rho]) \times \delta([\nu^{c+1}\rho, \nu^{-c-1}\rho]) \otimes \tau_1$ and $L(\delta_{i_{\min}+2}, \ldots, \delta_k; \tau')$ is a subquotient of $\langle[\nu^{c+1}\rho, \nu^{d-1}\rho]\rangle \rtimes \tau_1$.

Theorem 3.5 implies $i_{\min} + 2 \leq k$, so $x_{i_{\min}+2} \in \{c+1, -d+1\}$. Also, from $e(\delta_{x_{i_{\min}}}) \leq e(\delta_{x_{i_{\min}+2}})$ we obtain that either $y_{i_{\min}+2} = x_{i_{\min}+2} = -\frac{1}{2}$ or $y_{i_{\min}+2} = -x_{i_{\min}+2} - 1$. A standard commuting argument implies

$$L(\delta_{i_{\min}}, \ldots, \delta_k; \tau') \hookrightarrow \delta_{i_{\min}+2} \times \delta_{i_{\min}+1} \times \delta_{i_{\min}} \rtimes L(\delta_{i_{\min}+3}, \ldots, \delta_k; \tau'),$$

Thus, $\mu^*(\langle[\nu^{c}\rho, \nu^{d}\rho]\rangle \rtimes \tau)$ contains an irreducible constituent of the form $\delta_{i_{\min}+2} \otimes \pi$, and using $x_{i_{\min}+2} \in \{c+1, -d+1\}$ we get $x_{i_{\min}+2} = c = -d+1$. Now it can be seen at once that $\mu^*(\langle[\nu^{c}\rho, \nu^{d}\rho]\rangle \rtimes \tau)$ does not contain an irreducible constituent of the form $\delta_{i_{\min}+2} \times \delta_j \otimes \pi$, for $j \in \{i_{\min}, i_{\min}+1\}$ such that $x_j = c$. Consequently, there is a unique $i \in \{1, 2, \ldots, k\}$ such that $y_i \notin \{x_i, x_i + 1\}$. □

Lemma 6.2. *Suppose that c and d are half-integers such that $c \leq -\frac{1}{2}$, $-c \leq d$, and $\frac{5}{2} \leq d$. Let $\tau_1 \in R(G)$ denote an irreducible tempered representation, and let $\rho_1 \in R(GL)$ denote an irreducible cuspidal self-contragredient representation. Suppose that $\langle[\nu^{c}\rho_1, \nu^{d}\rho_1]\rangle \rtimes \tau_1$ contains $L(\delta_1, \ldots, \delta_k; \tau')$, where $\delta_i \cong \delta([\nu^{x_i}\rho_1, \nu^{y_i}\rho_1])$ for $i = 1, \ldots, k$, such that $x_1 = -y_1 - 1$ and $y_1 \geq \frac{1}{2}$. Then $c = -\frac{1}{2}$, $k = 2$, $\delta_2 \cong \nu^{-\frac{1}{2}}\rho_1$, and $\langle[\nu^{\frac{1}{2}}\rho_1, \nu^{d}\rho_1]\rangle \rtimes \tau_1$ reduces.*

Proof. We first show that $k \geq 2$. Suppose, on the contrary, that $k = 1$. It can be directly seen that $x_1 \in \{-d, c\}$. If $x_1 = c$, then $c \leq -\frac{3}{2}$, and an easy application of the structural formula implies that there is an irreducible tempered representation τ_2 such that $\mu^*(\tau) \geq \delta([\nu^{c+1}\rho_1, \nu^{-c-1}\rho_1]) \otimes \tau_2$ and τ' is an irreducible subquotient of $\langle[\nu^{c+1}\rho_1, \nu^d\rho_1]\rangle \rtimes \tau_2$, which contradicts Theorem 3.5. The case $x_1 = -d \neq c - 1$ can be handled in the same way. Let us consider the case $x_1 = -d = c - 1$, and write τ_1 as a subrepresentation of $\delta([\nu^{-d+2}\rho_1, \nu^{d-2}\rho_1])^{m_1} \rtimes \tau_2$, where τ_2 is a tempered representation such that $\mu^*(\tau_2)$ does not contain an irreducible constituent of the form $\delta([\nu^{-d+2}\rho_1, \nu^{d-2}\rho_1]) \otimes \pi$. Here we also allow $m_1 = 0$, in which case $\mu^*(\tau_1)$ does not contain an irreducible constituent of the form $\delta([\nu^{-d+2}\rho_1, \nu^{d-2}\rho_1]) \otimes \pi$. Using the cuspidal support considerations we deduce that τ' embeds into an induced representation of the form $\delta([\nu^{-d+2}\rho_1, \nu^{d-2}\rho_1])^{m_1} \rtimes \tau'_1$, where τ'_1 is a tempered representation such that $\mu^*(\tau'_1)$ does not contain an irreducible constituent of the form $\delta([\nu^{-d+2}\rho_1, \nu^{d-2}\rho_1]) \otimes \pi$. Consequently,

$$\begin{aligned} L(\delta([\nu^{-d}\rho_1, \nu^{d-1}\rho_1]); \tau') &\hookrightarrow \delta([\nu^{-d}\rho_1, \nu^{d-1}\rho_1]) \times \delta([\nu^{-d+2}\rho_1, \nu^{d-2}\rho_1])^{m_1} \rtimes \tau'_1 \\ &\cong \delta([\nu^{-d+2}\rho_1, \nu^{d-2}\rho_1])^{m_1} \times \delta([\nu^{-d}\rho_1, \nu^{d-1}\rho_1]) \rtimes \tau'_1, \end{aligned}$$

and there is an irreducible subquotient π_1 of $\delta([\nu^{-d}\rho_1, \nu^{d-1}\rho_1]) \rtimes \tau'_1$ such that $L(\delta([\nu^{-d}\rho_1, \nu^{d-1}\rho_1]); \tau')$ embeds into $\delta([\nu^{-d+2}\rho_1, \nu^{d-2}\rho_1])^{m_1} \rtimes \pi_1$. Using

$$\mu^*(L(\delta([\nu^{-d}\rho_1, \nu^{d-1}\rho_1]); \tau')) \geq \delta([\nu^{-d}\rho_1, \nu^{d-1}\rho_1]) \otimes \tau',$$

we obtain $\pi_1 \cong L(\delta([\nu^{-d}\rho_1, \nu^{d-1}\rho_1]); \tau'_1)$. Thus, $\mu^*(\langle[\nu^{-d+1}\rho_1, \nu^d\rho_1]\rangle \rtimes \tau_1)$ contains $\delta([\nu^{-d+2}\rho_1, \nu^{d-2}\rho_1])^{m_1} \otimes L(\delta([\nu^{-d}\rho_1, \nu^{d-1}\rho_1]); \tau'_1)$. This implies that $L(\delta([\nu^{-d}\rho_1, \nu^{d-1}\rho_1]); \tau'_1)$ is an irreducible subquotient of $\langle[\nu^{-d+1}\rho_1, \nu^d\rho_1]\rangle \rtimes \tau_2$.

Now we write τ_2 as a subrepresentation of $\delta([\nu^{-d+1}\rho_1, \nu^{d-1}\rho_1])^{m_2} \rtimes \tau_3$, where τ_3 is a tempered representation such that $\mu^*(\tau_3)$ does not contain an irreducible constituent of the form $\delta([\nu^{-d+1}\rho_1, \nu^{d-1}\rho_1]) \otimes \pi$. Here we also allow $m_2 = 0$. Similarly as before, τ'_1 can be written as a subrepresentation of $\delta([\nu^{-d+1}\rho_1, \nu^{d-1}\rho_1])^{m_2} \rtimes \tau'_2$, for an irreducible tempered representation τ'_2 such that $\mu^*(\tau'_2)$ does not contain an irreducible constituent of the form $\delta([\nu^{-d+1}\rho_1, \nu^{d-1}\rho_1]) \otimes \pi$. Let us denote by $\delta'_1, \ldots, \delta'_{m_3} \in R(GL)$, $\sigma_1 \in R(G)$ discrete series representations such that τ_3 embeds into $\delta'_1 \times \cdots \times \delta'_{m_3} \rtimes \sigma_1$.

We have

$$\begin{aligned} L(\delta([\nu^{-d}\rho_1, \nu^{d-1}\rho_1]); \tau'_1) &\hookrightarrow \delta([\nu^{-d}\rho_1, \nu^{d-1}\rho_1]) \times \delta([\nu^{-d+1}\rho_1, \nu^{d-1}\rho_1])^{m_2} \rtimes \tau'_2 \\ &\cong \delta([\nu^{-d+1}\rho_1, \nu^{d-1}\rho_1])^{m_2} \times \delta([\nu^{-d}\rho_1, \nu^{d-1}\rho_1]) \rtimes \tau'_2 \\ &\hookrightarrow \delta([\nu^{-d+1}\rho_1, \nu^{d-1}\rho_1])^{m_2+1} \times \nu^{-d}\rho_1 \rtimes \tau'_2, \end{aligned}$$

and in the same way as before we conclude that $L(\delta([\nu^{-d}\rho_1, \nu^{d-1}\rho_1]); \tau_1')$ is a subrepresentation of $\delta([\nu^{-d+1}\rho_1, \nu^{d-1}\rho_1])^{m_2+1} \rtimes L(\nu^{-d}\rho_1; \tau_2')$. This leads to $\mu^*(\langle[\nu^{-d+1}\rho_1, \nu^{d}\rho_1]\rangle \rtimes \tau_2) \geq \delta([\nu^{-d+1}\rho_1, \nu^{d-1}\rho_1])^{m_2+1} \otimes L(\nu^{-d}\rho_1; \tau_2')$.

From the structural formula and the description of τ_2, we get that $\mu^*(\sigma_1)$ contains an irreducible constituent of the form $\delta([\nu^{-d+2}\rho_1, \nu^{d-1}\rho_1]) \otimes \sigma_2$, and there is an irreducible subquotient π_1 of $\delta_1' \times \cdots \times \delta_{m_3}' \rtimes \sigma_2$ such that $L(\nu^{-d}\rho_1; \tau_2')$ is a subquotient of $\langle[\nu^{-d+2}\rho_1, \nu^{d}\rho_1]\rangle \rtimes \pi_1$. From [17, Proposition 7.2] we deduce that σ_2 is a discrete series, so π_1 is an irreducible tempered representation. Since $\mu^*(\langle[\nu^{-d+2}\rho_1, \nu^{d}\rho_1]\rangle \rtimes \pi_1) \geq \nu^{-d}\rho_1 \otimes \tau_2'$, it follows at once that τ_2' is an irreducible subquotient of $\langle[\nu^{-d+2}\rho_1, \nu^{d-1}\rho_1]\rangle \rtimes \pi_1$, which contradicts Theorem 3.5. Thus, $k \geq 2$.

Using Proposition 6.1 and $e(\delta_1) = -\frac{1}{2}$, we get that $\delta_j \cong \nu^{-\frac{1}{2}}\rho_1$ for all $j \in \{2, \ldots, k\}$. From $\delta_1 \times \nu^{-\frac{1}{2}}\rho_1 \cong \nu^{-\frac{1}{2}}\rho_1 \times \delta_1$ we obtain $k = 2$ and $c = -\frac{1}{2}$. Using the embedding

$$L(\delta_1, \delta_2; \tau') \hookrightarrow \nu^{-\frac{1}{2}}\rho_1 \times \delta_1 \rtimes \tau'$$

and Lemma 2.3, we easily deduce that $L(\delta_1, \delta_2; \tau')$ is a subrepresentation of $\nu^{-\frac{1}{2}}\rho_1 \rtimes L(\delta_1; \tau')$. Thus, $\mu^*(\langle[\nu^{-\frac{1}{2}}\rho_1, \nu^{d}\rho_1]\rangle \rtimes \tau_1)$ contains $\nu^{-\frac{1}{2}}\rho_1 \otimes L(\delta_1; \tau')$, so $L(\delta_1; \tau')$ is an irreducible subquotient of $\langle[\nu^{\frac{1}{2}}\rho_1, \nu^{d}\rho_1]\rangle \rtimes \tau_1$ and $\langle[\nu^{\frac{1}{2}}\rho_1, \nu^{d}\rho_1]\rangle \rtimes \tau_1$ reduces. This proves the lemma. □

Lemma 6.3. *Suppose that $\tau_1 \in R(G)$ is an irreducible tempered representation and let $\rho_1 \in R(GL)$ denote an irreducible cuspidal self-contragredient representation. If $\langle[\nu^{-\frac{1}{2}}\rho_1, \nu^{\frac{3}{2}}\rho_1]\rangle \rtimes \tau_1$ contains $L(\delta([\nu^{-\frac{3}{2}}\rho_1, \nu^{\frac{1}{2}}\rho_1]), \delta_2, \ldots, \delta_k; \tau')$ then there is a $c \in \{\frac{1}{2}, \frac{3}{2}\}$ such that $\langle[\nu^{\frac{1}{2}}\rho_1, \nu^{c}\rho_1]\rangle \rtimes \tau_1$ reduces.*

Proof. Let us first assume that $k \geq 2$. Since $e(\delta([\nu^{-\frac{3}{2}}\rho_1, \nu^{\frac{1}{2}}\rho_1])) = -\frac{1}{2}$, using Proposition 6.1 we obtain that $\delta_i \cong \nu^{-\frac{1}{2}}\rho_1$ for $i = 2, \ldots, k$. Since $\delta([\nu^{-\frac{3}{2}}\rho_1, \nu^{\frac{1}{2}}\rho_1]) \times \nu^{-\frac{1}{2}}\rho_1$ is irreducible, it easily follows that $k = 2$. Note that $L(\delta([\nu^{-\frac{3}{2}}\rho_1, \nu^{\frac{1}{2}}\rho_1]), \nu^{-\frac{1}{2}}\rho_1; \tau')$ is a subrepresentation of $\nu^{-\frac{1}{2}}\rho_1 \rtimes L(\delta([\nu^{-\frac{3}{2}}\rho_1, \nu^{\frac{1}{2}}\rho_1]); \tau')$, so $\mu^*(\langle[\nu^{-\frac{1}{2}}\rho_1, \nu^{\frac{3}{2}}\rho_1]\rangle \rtimes \tau_1)$ contains $\nu^{-\frac{1}{2}}\rho_1 \otimes L(\delta([\nu^{-\frac{3}{2}}\rho_1, \nu^{\frac{1}{2}}\rho_1]); \tau')$. Using the structural formula and temperedness of τ_1, we deduce that $\langle[\nu^{\frac{1}{2}}\rho_1, \nu^{\frac{3}{2}}\rho_1]\rangle \rtimes \tau_1$ contains $L(\delta([\nu^{-\frac{3}{2}}\rho_1, \nu^{\frac{1}{2}}\rho_1]); \tau')$, so $\langle[\nu^{\frac{1}{2}}\rho_1, \nu^{\frac{3}{2}}\rho_1]\rangle \rtimes \tau_1$ reduces.

Let us now assume that $k = 1$, and write τ_1 as a subrepresentation of $\delta([\nu^{-\frac{1}{2}}\rho_1, \nu^{\frac{1}{2}}\rho_1])^m \rtimes \tau_2$, where τ_2 is a tempered representation such that $\mu^*(\tau_2)$ does not contain an irreducible constituent of the form $\delta([\nu^{-\frac{1}{2}}\rho_1, \nu^{\frac{1}{2}}\rho_1]) \otimes \pi$. Here we also allow $m = 0$. Let $\sigma_1 \in R(G)$ denote a discrete series such that

τ_2 is a subrepresentation of $\delta_1' \times \cdots \times \delta_n' \rtimes \sigma_1$, for discrete series $\delta_1', \ldots, \delta_n' \in R(GL)$.

Using the cuspidal support considerations, we deduce that there is an irreducible tempered representation $\tau_1' \in R(G)$ such that τ' is a subrepresentation of $\delta([\nu^{-\frac{1}{2}}\rho_1, \nu^{\frac{1}{2}}\rho_1])^m \rtimes \tau_1'$. This yields

$$\begin{aligned} L(\delta([\nu^{-\frac{3}{2}}\rho_1, \nu^{\frac{1}{2}}\rho_1]); \tau') &\hookrightarrow \delta([\nu^{-\frac{3}{2}}\rho_1, \nu^{\frac{1}{2}}\rho_1]) \times \delta([\nu^{-\frac{1}{2}}\rho_1, \nu^{\frac{1}{2}}\rho_1])^m \rtimes \tau_1' \\ &\cong \delta([\nu^{-\frac{1}{2}}\rho_1, \nu^{\frac{1}{2}}\rho_1])^m \times \delta([\nu^{-\frac{3}{2}}\rho_1, \nu^{\frac{1}{2}}\rho_1]) \rtimes \tau_1' \\ &\hookrightarrow \delta([\nu^{-\frac{1}{2}}\rho_1, \nu^{\frac{1}{2}}\rho_1])^{m+1} \times \nu^{-\frac{3}{2}}\rho_1 \rtimes \tau_1', \end{aligned}$$

so $\mu^*(\langle[\nu^{-\frac{1}{2}}\rho_1, \nu^{\frac{3}{2}}\rho_1]\rangle \rtimes \tau_1)$ contains an irreducible constituent of the form $\delta([\nu^{-\frac{1}{2}}\rho_1, \nu^{\frac{1}{2}}\rho_1])^{m+1} \otimes \pi$. The structural formula implies that $\mu^*(\sigma_1)$ contains an irreducible constituent of the form $\nu^{\frac{1}{2}}\rho_1 \otimes \pi$, so $2 \in \mathrm{Jord}_{\rho_1}(\sigma)$ and, by [17, Proposition 7.4], $\epsilon_\sigma(2, \rho_1) = 1$. Now Theorem 5.3 implies that $\nu^{\frac{1}{2}}\rho_1 \rtimes \tau_1$ reduces. □

This leads us to the main result of this section.

Theorem 6.4. *Suppose that $\rho \cong \widetilde{\rho}$ and let a, b denote half-integers such that $a \leq -\frac{1}{2}$ and $-a \leq b$. The induced representation $\langle[\nu^a\rho, \nu^b\rho]\rangle \rtimes \tau$ reduces if and only if there is a $c \in \{\frac{1}{2}, \frac{3}{2}, \ldots, b\}$ such that $\langle[\nu^{\frac{1}{2}}\rho, \nu^c\rho]\rangle \rtimes \tau$ reduces.*

Proof. Let us first assume that there is a $c \in \{\frac{1}{2}, \frac{3}{2}, \ldots, b\}$ such that $\langle[\nu^{\frac{1}{2}}\rho, \nu^c\rho]\rangle \rtimes \tau$ reduces. By Corollary 5.4, either there is a $d \in \{\frac{1}{2}, \frac{3}{2}, \ldots, c\}$ such that $\langle[\nu^{\frac{1}{2}}\rho, \nu^d\rho]\rangle \rtimes \tau$ contains an irreducible tempered subquotient, or there is a $d \in \{\frac{3}{2}, \frac{5}{2}, \ldots, c\}$ and an irreducible tempered representation $\tau' \in R(G)$ such that $\langle[\nu^{\frac{1}{2}}\rho, \nu^d\rho]\rangle \rtimes \tau$ contains $L(\delta([\nu^{-d}\rho, \nu^{d-1}\rho]); \tau')$.

Suppose that there is a $d \in \{\frac{1}{2}, \frac{3}{2}, \ldots, c\}$ such that $\langle[\nu^{\frac{1}{2}}\rho, \nu^d\rho]\rangle \rtimes \tau$ contains an irreducible tempered subquotient, which we denote by τ'. If $d \geq -a$, a repeated application of Lemma 2.7 implies that $\langle[\nu^a\rho, \nu^b\rho]\rangle \rtimes \tau$ contains

$$L(\nu^{-b}\rho, \nu^{-b+1}\rho, \ldots, \nu^{-d-1}\rho, \nu^a\rho, \nu^{a+1}\rho, \ldots, \nu^{-\frac{1}{2}}\rho; \tau'),$$

so it reduces. If $d < -a$, using Lemma 2.7 and Lemma 2.8, we obtain that $\langle[\nu^a\rho, \nu^b\rho]\rangle \rtimes \tau$ contains

$$L(\nu^{-b}\rho, \ldots, \nu^{a-1}\rho, \nu^a\rho, \nu^a\rho, \ldots, \nu^{-d-1}\rho, \nu^{-d-1}\rho, \nu^{-d}\rho, \ldots, \nu^{-\frac{1}{2}}\rho; \tau'),$$

so it reduces.

Suppose that there is a $d \in \{\frac{3}{2}, \frac{5}{2}, \dots, c\}$ and an irreducible tempered representation $\tau' \in R(G)$ such that $\langle [\nu^{\frac{1}{2}}\rho, \nu^d\rho] \rangle \rtimes \tau$ contains $L(\delta([\nu^{-d}\rho, \nu^{d-1}\rho]); \tau')$. If $d \geq -a$, using Lemma 2.7 we get that $\langle [\nu^a\rho, \nu^b\rho] \rangle \rtimes \tau$ contains

$$L(\nu^{-b}\rho, \nu^{-b+1}\rho, \dots, \nu^{-d-1}\rho, \nu^a\rho, \nu^{a+1}\rho, \dots, \nu^{-\frac{1}{2}}\rho, \delta([\nu^{-d}\rho, \nu^{d-1}\rho]); \tau'),$$

so it reduces. If $d < -a$, using Lemma 2.7 and Lemma 2.8, we obtain that $\langle [\nu^a\rho, \nu^b\rho] \rangle \rtimes \tau$ contains

$$L(\nu^{-b}\rho, \dots, \nu^{a-1}\rho, \nu^a\rho, \nu^a\rho, \dots, \nu^{-d-1}\rho, \nu^{-d-1}\rho, \nu^{-d}\rho, \dots, \nu^{-\frac{1}{2}}\rho, \delta([\nu^{-d}\rho, \nu^{d-1}\rho]); \tau'),$$

so it reduces.

Let us now assume that $\langle [\nu^{\frac{1}{2}}\rho, \nu^c\rho] \rangle \rtimes \tau$ is irreducible for every $c \in \{\frac{1}{2}, \frac{3}{2}, \dots, b\}$. By Theorem 3.5, $\langle [\nu^a\rho, \nu^b\rho] \rangle \rtimes \tau$ does not contain an irreducible tempered subquotient. Let us denote by $L(\delta_1, \dots, \delta_k; \tau')$ an irreducible non-tempered subquotient of $\langle [\nu^a\rho, \nu^b\rho] \rangle \rtimes \tau$, where $\delta_i \cong \delta([\nu^{x_i}\rho, \nu^{y_i}\rho])$ for $i = 1, \dots, k$. If $x_i = y_i$ for all $i \in \{1, 2, \dots, k\}$, it follows from Proposition 6.1 that there are c and d, $a \leq c \leq \frac{1}{2}$, $\frac{1}{2} \leq d \leq b$ such that $L(\delta_k; \tau')$ is a subquotient of $\langle [\nu^c\rho, \nu^d\rho] \rangle \rtimes \tau$. Thus, $x_k \in \{c, -d\}$ and if $c \neq d$ then τ' is an irreducible subquotient of $\langle [\nu^{c'}\rho, \nu^{d'}\rho] \rangle \rtimes \tau$, for $c' \leq d'$, which is impossible since $c' \leq \frac{1}{2}$. Thus, if $x_i = y_i$ for all $i \in \{1, 2, \dots, k\}$, we have

$$L(\delta_1, \dots, \delta_k; \tau') \cong L(\nu^{-b}\rho, \dots, \nu^{a-1}\rho, \nu^a\rho, \nu^a\rho, \dots, \nu^{-\frac{1}{2}}\rho, \nu^{-\frac{1}{2}}\rho; \tau).$$

Suppose that for all $i \in \{1, 2, \dots, k\}$ we have $x_i \in \{y_i, y_i - 1\}$ and there is a $j \in \{1, 2, \dots, k\}$ such that $x_j = y_j - 1$. We denote the largest such j by $j_{\max}$. From Proposition 6.1 we deduce that there is a d, $\frac{3}{2} \leq d \leq b$, such that $L(\delta_{j_{\max}}, \dots, \delta_k; \tau')$ is an irreducible subquotient of $\langle [\nu^{-d+1}\rho, \nu^d\rho] \rangle \rtimes \tau$, and $\delta_{j_{\max}} \cong \delta([\nu^{-d}\rho, \nu^{-d+1}\rho])$. If $j_{\max} = k$, it follows that τ' is an irreducible subquotient of $\langle [\nu^{-d+2}\rho, \nu^{d-1}\rho] \rangle \rtimes \tau$. If $-d+2 \leq -\frac{1}{2}$, this contradicts Theorem 3.5. If $-d+2 = \frac{1}{2}$, it follows that $\langle [\nu^{\frac{1}{2}}\rho, \nu^{d-1}\rho] \rangle \rtimes \tau$ reduces, a contradiction. Thus, $j_{\max} < k$ and $L(\delta_{j_{\max}+1}, \dots, \delta_k; \tau')$ is an irreducible subquotient of $\langle [\nu^{-d+2}\rho, \nu^{d-1}\rho] \rangle \rtimes \tau$. From $x_{j_{\max}+1} = y_{j_{\max}+1}$ we deduce that $\delta_{j_{\max}+1} \in \{\nu^{-d+2}\rho, \nu^{-d+1}\rho\}$. If $\delta_{j_{\max}+1} \cong \nu^{-d+1}\rho$, we have

$$\begin{aligned} L(\delta_{j_{\max}}, \dots, \delta_k; \tau') &\hookrightarrow \delta([\nu^{-d}\rho, \nu^{-d+1}\rho]) \times \nu^{-d+1}\rho \rtimes L(\delta_{j_{\max}+2}, \dots, \delta_k; \tau') \\ &\hookrightarrow \nu^{-d+1}\rho \times \nu^{-d+1}\rho \times \nu^{-d}\rho \rtimes L(\delta_{j_{\max}+2}, \dots, \delta_k; \tau'), \end{aligned}$$

so $\mu^*(L(\delta_{j_{\max}},\dots,\delta_k;\tau'))$ contains an irreducible constituent of the form $\nu^{-d+1}\rho\times\nu^{-d+1}\rho\otimes\pi$, which is impossible since $\mu^*(\langle[\nu^{-d+1}\rho,\nu^d\rho]\rangle\rtimes\tau)$ does not contain an irreducible constituent of such a form. Thus, $\delta_{j_{\max}+1}\cong\nu^{-d+2}\rho$ and $L(\delta_{j_{\max}+2},\dots,\delta_k;\tau')$ is an irreducible subquotient of $\langle[\nu^{-d+3}\rho,\nu^{d-1}\rho]\rangle\rtimes\tau$. Repeating this procedure, we get that there is an x, $-d+3\le x\le\frac{1}{2}$, such that τ' is an irreducible subquotient of $\langle[\nu^x\rho,\nu^{d-1}\rho]\rangle\rtimes\tau$, and we have already seen that this is impossible.

It remains to consider the case $x_i=-y_i-1$, with $y_i\ge\frac{1}{2}$, for some $i\in\{1,2,\dots,k\}$. It follows from Proposition 6.1 that there are c and d, $a\le c\le\frac{1}{2}$, $\frac{1}{2}\le d\le b$ such that $L(\delta_i,\dots,\delta_k;\tau')$ is a subquotient of $\langle[\nu^c\rho,\nu^d\rho]\rangle\rtimes\tau$. Since $\langle[\nu^{\frac{1}{2}}\rho,\nu^d\rho]\rangle\rtimes\tau$ is irreducible by our assumption, we obtain $c\le-\frac{1}{2}$. If $(c,d)\ne(-\frac{3}{2},\frac{3}{2})$, this contradicts either Lemma 6.2 or Lemma 6.3. Suppose that $(c,d)=(-\frac{3}{2},\frac{3}{2})$. Then $\delta_i\cong\delta([\nu^{-\frac{3}{2}}\rho,\nu^{\frac{1}{2}}\rho])$ and $L(\delta_{i+1},\dots,\delta_k;\tau')$ is an irreducible subquotient of $\langle[\nu^{-\frac{1}{2}}\rho,\nu^{\frac{3}{2}}\rho]\rangle\rtimes\tau_1$, for an irreducible tempered representation τ_1 such that $\mu^*(\tau)$ contains $\delta([\nu^{-\frac{1}{2}}\rho,\nu^{\frac{1}{2}}\rho])\otimes\tau_1$. From Theorem 3.5 we get $i+1\le k$. Using $e(\delta_i)\le e(\delta_{i+1})$ we easily obtain $\delta_{i+1}\cong\nu^{-\frac{1}{2}}\rho$, so $\delta_i\times\delta_{i+1}\cong\delta_{i+1}\times\delta_i$, and $\mu^*(\langle[\nu^{-\frac{3}{2}}\rho,\nu^{\frac{3}{2}}\rho]\rangle\rtimes\tau)$ contains an irreducible constituent of the form $\nu^{-\frac{1}{2}}\rho\otimes\pi$, which is impossible.

Consequently, every irreducible subquotient of $\langle[\nu^a\rho,\nu^b\rho]\rangle\rtimes\tau$ is isomorphic to

$$L(\nu^{-b}\rho,\dots,\nu^{a-1}\rho,\nu^a\rho,\nu^a\rho,\dots,\nu^{-\frac{1}{2}}\rho,\nu^{-\frac{1}{2}}\rho;\tau).$$

It is an easy consequence of the structural formula that

$$\nu^{-b}\rho\otimes\cdots\otimes\nu^{a-1}\rho\otimes\nu^a\rho\times\nu^a\rho\otimes\cdots\otimes\nu^{-\frac{1}{2}}\rho\times\nu^{-\frac{1}{2}}\rho\otimes\tau$$

appears with the multiplicity one in the Jacquet module of $\langle[\nu^a\rho,\nu^b\rho]\rangle\rtimes\tau$ with respect to the appropriate parabolic subgroup. Thus, $\langle[\nu^a\rho,\nu^b\rho]\rangle\rtimes\tau$ is irreducible. □

Let us now discuss the remaining case.

Proposition 6.5. *Suppose that $\rho\not\cong\widetilde{\rho}$ and let a, b denote half-integers such that $a\le-\frac{1}{2}$ and $-a\le b$. Then the induced representation $\langle[\nu^a\rho,\nu^b\rho]\rangle\rtimes\tau$ is irreducible.*

Proof. Again, we denote by $L(\delta_1,\dots,\delta_k;\tau')$ an irreducible non-tempered subquotient of $\langle[\nu^a\rho,\nu^b\rho]\rangle\rtimes\tau$, where $\delta_i\cong\delta([\nu^{x_i}\rho_i,\nu^{y_i}\rho_i])$ for $i=1,\dots,k$. It is enough to prove that $x_i=y_i$ for all $i\in\{1,\dots,k\}$, the rest of the proof follows in the same way as the one of Theorem 6.4.

In the same way as in the proof of Proposition 6.1 we deduce that for $i = 1, \dots, k$ we have $\rho_i \in \{\rho, \widetilde{\rho}\}$ and $x_i \in \{y_i, -y_i - 1\}$. Suppose that there is an $i \in \{1, \dots, k\}$ such that $x_i = -y_i - 1$, $y_i \geq \frac{1}{2}$ and that for $j < i$ we have $x_j = y_j$. In the same way as in the proof of Proposition 6.1 we see that there are c and d, $a \leq c \leq \frac{1}{2}$, $\frac{1}{2} \leq d \leq b$, such that $L(\delta_i, \dots, \delta_k; \tau')$ is a subquotient of $\langle[\nu^c\rho, \nu^d\rho]\rangle \rtimes \tau$. If $c = \frac{1}{2}$, it follows that $\langle[\nu^{\frac{1}{2}}\rho, \nu^d\rho]\rangle \rtimes \tau$ reduces, which contradicts Lemma 5.1. Thus, $c \leq -\frac{1}{2}$. We get $(x_i, \rho_i) \in \{(c, \rho), (-d, \widetilde{\rho})\}$, and there is an irreducible tempered representation τ_1 such that $\mu^*(\tau) \geq \delta([\nu^{-y_i}\rho_i, \nu^{y_i}\rho_i]) \otimes \tau_1$, and $L(\delta_{i+1}, \dots, \delta_k; \tau')$ is an irreducible subquotient of $\langle[\nu^{c'}\rho, \nu^{d'}\rho]\rangle \rtimes \tau_1$, for some $c' \in \{c, c+1\}$ and $d' \in \{d, d-1\}$. It follows that $i + 1 \leq k$. Now, following the same lines as in the proof of Lemma 6.2, we obtain that this is possible only if $c = -\frac{1}{2}$ and $\langle[\nu^{\frac{1}{2}}\rho, \nu^d\rho]\rangle \rtimes \tau$ reduces, which contradicts Lemma 5.1. □

We close this section with a detailed summary of our main results.

Theorem 6.6. *Let $\rho \in R(GL)$ stand for an irreducible cuspidal representation, and let $\tau \in R(G)$ denote an irreducible tempered representation. We denote by $\sigma \in R(G)$ a discrete series such that τ is a subrepresentation of $\delta^{(1)} \times \cdots \times \delta^{(r)} \rtimes \sigma$, for discrete series $\delta^{(1)}, \dots, \delta^{(r)} \in R(GL)$. Let $(Jord(\sigma), \sigma_{cusp}, \epsilon_\sigma)$ stand for the admissible triple corresponding to σ by the Mœglin-Tadić classification. Let a, b denote real numbers such that $b - a$ is a non-negative integer and $a + b \geq 0$.*

If $2a \notin \mathbb{Z}$, then the induced representation $\langle[\nu^a\rho, \nu^b\rho]\rangle \rtimes \tau$ is irreducible.

If $2a$ is a positive integer and $\rho \not\cong \widetilde{\rho}$, then $\langle[\nu^a\rho, \nu^b\rho]\rangle \rtimes \tau$ reduces if and only if $a \geq 1$ and $\mu^*(\tau)$ contains an irreducible constituent of the form $\delta([\nu^{-a+1}\widetilde{\rho}, \nu^{a-1}\widetilde{\rho}]) \otimes \pi$.

If $2a$ is a positive integer, $a \geq 1$, and $\rho \cong \widetilde{\rho}$, then $\langle[\nu^a\rho, \nu^b\rho]\rangle \rtimes \tau$ reduces if and only if one of the following holds:

(1) $\mu^*(\tau)$ contains an irreducible constituent of the form $\delta([\nu^{-a+1}\rho, \nu^{a-1}\rho]) \otimes \pi$.

(2) We have $x \in Jord_\rho(\sigma)$ for all $x \in \{2a-1, 2a+1, \dots, 2b+1\}$, $\epsilon_\sigma((x_-, \rho), (x, \rho)) = -1$ for all $x \in \{2a-1, 2a+1, \dots, 2b-3\}$, and $\epsilon_\sigma((2b-1, \rho), (2b+1, \rho)) = 1$.

(3) We have $x \in Jord_\rho(\sigma)$ for all $x \in \{2a-1, 2a+1, \dots, 2b-1\}$, $\epsilon_\sigma((x_-, \rho), (x, \rho)) = -1$ for all $x \in \{2a-1, 2a+1, \dots, 2b-3\}$, $2b+1 \notin Jord_\rho(\sigma)$, and

if $\mu^*(\tau)$ contains an irreducible constituent of the form $\delta([\nu^{-b}\rho, \nu^b\rho]) \otimes \pi_1$, and m stands for the largest integer such that $\mu^*(\tau)$ contains an irreducible constituent of the form $\delta([\nu^{-b}\rho, \nu^b\rho])^m \otimes \pi_2$, then $\mu^*(\tau)$ contains an irreducible constituent of the form $(\nu^b\rho)^{2m} \otimes \pi_3$,

(4) There is a $c \in \{a, a+1, \ldots, b-1\}$ such that $x \in Jord_\rho(\sigma)$ for all $x \in \{2a-1, 2a+1, \ldots, 2c-1\}$, $\epsilon_\sigma((x_-, \rho), (x, \rho)) = -1$ for all $x \in \{2a+1, 2a+3, \ldots, 2c-1\}$, and if $2c+1 \in Jord_\rho(\sigma)$ then $\epsilon_\sigma((2c-1, \rho), (2c+1, \rho)) = 1$.

If $b - \frac{1}{2}$ is a non-negative integer and $\rho \cong \widetilde{\rho}$, then $\langle[\nu^{\frac{1}{2}}\rho, \nu^b\rho]\rangle \rtimes \tau$ reduces if and only if one of the following holds:

(1) $2 \in Jord_\rho(\sigma)$ and $\epsilon_\sigma(2, \rho) = 1$.

(2) $2 \notin Jord_\rho(\sigma)$ and if $\mu^*(\tau)$ contains an irreducible constituent of the form $\delta([\nu^{-\frac{1}{2}}\rho, \nu^{\frac{1}{2}}\rho]) \otimes \pi_1$, and m stands for the largest integer such that $\mu^*(\tau)$ contains an irreducible constituent of the form $\delta([\nu^{-\frac{1}{2}}\rho, \nu^{\frac{1}{2}}\rho])^m \otimes \pi_2$, then $\mu^*(\tau)$ contains an irreducible constituent of the form $(\nu^{\frac{1}{2}}\rho)^{2m} \otimes \pi_3$.

(3) $b > \frac{1}{2}$, $x \in Jord_\rho(\sigma)$ for all $x \in \{2, 4, \ldots, 2b+1\}$, $\epsilon_\sigma((x_-, \rho), (x, \rho)) = -1$ for all $x \in \{4, 6, \ldots, 2b-3\}$, $\epsilon_\sigma(2, \rho) = -1$, and $\epsilon_\sigma((2b-1, \rho), (2b+1, \rho)) = 1$.

(4) $b > \frac{1}{2}$, $x \in Jord_\rho(\sigma)$ for all $x \in \{2, 4, \ldots, 2b-1\}$, $\epsilon_\sigma((x_-, \rho), (x, \rho)) = -1$ for all $x \in \{4, 6, \ldots, 2b-3\}$, $2b+1 \notin Jord_\rho(\sigma)$, $\epsilon_\sigma(2, \rho) = -1$, and if $\mu^*(\tau)$ contains an irreducible constituent of the form $\delta([\nu^{-b}\rho, \nu^b\rho]) \otimes \pi_1$, and m stands for the largest integer such that $\mu^*(\tau)$ contains an irreducible constituent of the form $\delta([\nu^{-b}\rho, \nu^b\rho])^m \otimes \pi_2$, then $\mu^*(\tau)$ contains an irreducible constituent of the form $(\nu^b\rho)^{2m} \otimes \pi_3$,

(5) $b > \frac{1}{2}$, and there is a $c \in \{\frac{3}{2}, \ldots, b-1\}$ such that $x \in Jord_\rho(\sigma)$ for all $x \in \{2, 4, \ldots, 2c-1\}$, $\epsilon_\sigma((x_-, \rho), (x, \rho)) = -1$ for all $x \in \{2, 4, \ldots, 2c-1\}$, $\epsilon_\sigma(2, \rho) = -1$, and if $2c+1 \in Jord_\rho(\sigma)$ then $\epsilon_\sigma((2c-1, \rho), (2c+1, \rho)) = 1$.

If $a < 0$, $2a \in \mathbb{Z}$, $a \notin \mathbb{Z}$, and $\rho \cong \widetilde{\rho}$, then $\langle[\nu^a\rho, \nu^b\rho]\rangle \rtimes \tau$ reduces if and only if there is a $c \in \{\frac{1}{2}, \frac{3}{2}, \ldots, b\}$ such that $\langle[\nu^{\frac{1}{2}}\rho, \nu^c\rho]\rangle \rtimes \tau$ reduces.

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