On Jacquet modules of discrete series

Ivan Matić

April 17, 2014

Abstract

The purpose of this paper is to determine Jacquet modules of discrete series which are obtained by adding a pair of consecutive elements on which the ϵ-function equals one to a Jordan block of an irreducible strongly positive representation. Such representations present the first inductive step in the realization of discrete series starting from the strongly positive ones. We are interested in determining Jacquet modules with respect to the maximal parabolic subgroups, with an irreducible essentially square-integrable representation on the general linear part.

1 Introduction

Discrete series representations present one of the most extensively studied parts of the unitary duals of reductive p-adic groups, with numerous applications in harmonic analysis and theory of automorphic forms. In the case of p-adic classical groups, this prominent class of representations has been classified in the work of Mœglin and Tadić ([9, 11]), under a natural hypothesis which now follows from the work of Arthur ([1]), some further details on the completion of this classification can be seen in [10]. According to this classification, discrete series are in bijective correspondence with so-called admissible triples consisting of Jordan block, ϵ-function and partial cuspidal support. Furthermore, each discrete series can be obtained as a result of an

MSC2000: primary 22E35; secondary 22E50

Keywords: discrete series, classical p-adic groups, Jacquet modules

inductive procedure consisting of repeated adding of new consecutive pairs to Jordan block, starting from the strongly positive discrete series.

Thus, the strongly positive discrete series serve as a cornerstone in such construction of discrete series. An algebraic classification of such representations is given in [5] and, based on that classification, a complete description of Jacquet modules of strongly positive discrete series has been obtained in [7]. The results obtained there show that, if we denote by $\pi \otimes \sigma$ an irreducible representation contained in Jacquet module of strongly positive representation with respect to the maximal parabolic subgroup, then σ is also a strongly positive discrete series, while π is a ladder representation of particular type. We note that such representations of general linear groups have lately been studied in detail in [3] and [4].

On the other hand, it is evident that on the classical group parts of Jacquet modules of non-strongly positive discrete series appear representations belonging to different classes, ranging from discrete series to the non-tempered ones. It is natural to initially extend the investigation of Jacquet modules of discrete series started in [7] to discrete series obtained by adding a pair of consecutive elements on which the ϵ-function equals one to a Jordan block of strongly positive discrete series. This class of representations already shows substantial differences from the strongly positive case and has played a fundamental role in determination of the first occurrence indices of discrete series of metaplectic groups in [6]. In this paper, we are interested in deriving Jacquet modules of representations of mentioned type with respect to maximal parabolic subgroups whose general linear part consists of an irreducible essentially square-integrable representation. Even in this case, we obtain a variety of different representations appearing in Jacquet modules, which are then considered separately.

We note that recently the main properties defining the ϵ-function attached to a discrete series representation have been rewritten in terms of Jacquet modules in [17] and these results are mainly expressed using Jacquet modules of the type analogous to one which we study in the paper.

In our determination of Jacquet modules we use elementary but non-standard methods which are essentially different from the ones used in [7]. First, starting from certain embeddings of discrete series, we apply the structural formula of Tadić ([15]), combined with results of [7], to obtain all (not necessarily irreducible) elements appearing in Jacquet modules of induced representations containing our discrete series. Then, using a description of composition series of certain generalized principle series, obtained in [12] and enhanced by Proposition 3.2 of [8], we derive all possible candidates for Jacquet modules of investigated discrete series, together with their multiplicity. To deduce whether obtained irreducible constituent $\pi \otimes \sigma$ appears in Jacquet module of the observed discrete series or not, using a case-by-case consideration we derive certain element $\pi' \otimes \sigma'$ appearing in Jacquet module of σ and, by means of transitivity of Jacquet modules, turn our attention to representations of general linear groups having $\pi \otimes \pi'$ in their Jacquet modules. This puts us in position to deduce further information carried in obtained irreducible representation $\pi \otimes \sigma$ and, consequently, [17] can be used to determine whether such representation belongs to Jacquet module of the observed discrete series or not.

Our results, besides being interesting by themselves, might have an application in the theory of automorphic forms, where both discrete series and their Jacquet modules have an important role. Also, our results can be used to identify discrete series subquotients of generalized principal series (as has been done in [8]), which have an application in the determination of the unitary duals of classical p-adic groups.

We now describe the content of the paper in more detail. In the second section we recall required notations and preliminaries. In the third section we begin our study of Jacuet modules, in certain elementary cases which are used afterwards in the paper. In the fourth section we provide a description of Jacquet modules in the most complicated case, which we divide in several subcases. An exceptional case is handled in Section 5.

The author would like to thank Marko Tadić for useful conversations on Jacquet modules of representations of general linear groups and to Goran Muić for his active interest in the publication of this paper.

2 Notation and preliminaries

Let F denote a non-archimedean local field of characteristic different than two. We consider usual towers of symplectic and orthogonal groups $G_n = G(V_n)$, that are the groups of isometries of F-spaces $(V_n, (\ , \ ))$, $n \geq 0$. Here the form $(\ , \ )$ is non-degenerate and it is skew-symmetric if the tower is symplectic and symmetric otherwise. The set of standard parabolic subgroups will be fixed in a usual way, i.e., in the usual matrix realization of the classical group G_n we fix a minimal F-parabolic subgroup consisting of upper-triangular matrices. Then the Levi factors of standard parabolic subgroups have the form $M \simeq GL(n_1, F) \times \cdots \times GL(n_k, F) \times G_{n'}$, where $GL(m, F)$ denotes a general linear group of rank m over F. For representations δ_i of $GL(n_i, F)$, $i = 1, 2, \ldots, k$, and a representation τ of $G_{n'}$, the normalized parabolically induced representation $\mathrm{Ind}_M^{G_n}(\delta_1 \otimes \cdots \otimes \delta_k \otimes \tau)$ will be denoted by $\delta_1 \times \cdots \times \delta_k \rtimes \tau$.

The set of all irreducible admissible representations of G_n will be denoted by $Irr(G_n)$. Let $R(G_n)$ denote a Grothendieck group of admissible representations of finite length of G_n and set $R(G) = \oplus_{n \geq 0} R(G_n)$. In a similar way we define $R(GL) = \oplus_{n \geq 0} R(GL(n, F))$. For $\sigma \in Irr(G_n)$ and $1 \leq k \leq n$ we denote by $r_{(k)}(\sigma)$ the normalized Jacquet module of σ with respect to the parabolic subgroup $P_{(k)}$ having Levi factor equal to $GL(k, F) \times G_{n-k}$. Then $r_{(k)}(\sigma)$ can be interpreted as an element of $R(GL) \otimes R(G)$. For $\sigma \in Irr(G_n)$ we introduce $\mu^*(\sigma) \in R(GL) \otimes R(G)$ by

$$\mu^*(\sigma) = \sum_{k=0}^{n} \text{s.s.}(r_{(k)}(\sigma))$$

(s.s. denotes the semisimplification) and extend μ^* linearly to the whole of $R(G)$.

Using Jacquet modules for the maximal parabolic subgroups of $GL(n, F)$ we can also define $m^*(\pi) = \sum_{k=0}^{n} \text{s.s.}(r_k(\pi)) \in R(GL) \otimes R(GL)$, for an irreducible representation π of $GL(n, F)$, and then extend m^* linearly to the whole of $R(GL)$. Here $r_k(\pi)$ denotes the normalized Jacquet module of π with respect to the parabolic subgroup having Levi factor equal to $GL(k, F) \times GL(n-k, F)$.

The results of [18] show that an irreducible essentially square-integrable representation δ of $GL(n, F)$ is attached to the segment and we set $\delta = \delta([\nu^a\rho, \nu^b\rho])$, where $a, b \in \mathbb{R}$ such that $b-a$ is a nonnegative integer and ρ is an irreducible unitary representation of $GL(n_\rho, F)$ (this defines n_ρ). We recall that $\delta([\nu^a\rho, \nu^b\rho])$ is a unique irreducible subrepresentation of the induced representation $\nu^b\rho \times \nu^{b-1}\rho \times \cdots \times \nu^a\rho$.

We will also frequently use the following equation:

$$m^*(\delta([\nu^a\rho, \nu^b\rho])) = \sum_{i=a-1}^{b} \delta([\nu^{i+1}\rho, \nu^b\rho]) \otimes \delta([\nu^a\rho, \nu^i\rho]).$$

Note that multiplicativity of m^* implies

$$m^*(\prod_{j=1}^{n}\delta([\nu^{a_j}\rho_j,\nu^{b_j}\rho_j])) =$$
$$= \prod_{j=1}^{n}(\sum_{i_j=a_j-1}^{b_j}\delta([\nu^{i_j+1}\rho_j,\nu^{b_j}\rho_j])\otimes\delta([\nu^{a_j}\rho_j,\nu^{i_j}\rho_j])).$$

We take a moment to state the result, derived in [15], which presents a crucial structural formula for our calculations with Jacquet modules of representations of classical groups.

Lemma 2.1. *Let ρ be an irreducible cuspidal representation of $GL(m,F)$ and $k,l\in\mathbb{R}$ such that $k+l\in\mathbb{Z}_{\geq 0}$. Let σ be an admissible representation of finite length of G_n. Write $\mu^*(\sigma)=\sum_{\tau,\sigma'}\tau\otimes\sigma'$. Then the following holds:*

$$\begin{aligned}\mu^*(\delta([\nu^{-k}\rho,\nu^{l}\rho])\rtimes\sigma) \quad = \quad & \sum_{i=-k-1}^{l}\sum_{j=i}^{l}\sum_{\tau,\sigma'}\delta([\nu^{-i}\widetilde{\rho},\nu^{k}\widetilde{\rho}])\times\delta([\nu^{j+1}\rho,\nu^{l}\rho])\times\tau \\ \otimes \quad & \delta([\nu^{i+1}\rho,\nu^{j}\rho])\rtimes\sigma'.\end{aligned}$$

We omit $\delta([\nu^{x}\rho,\nu^{y}\rho])$ if $x>y$.

We briefly recall the Langlands classification for general linear groups. As in [2], we favor the subrepresentation version of this classification over the quotient one. Main advantage of this version is that it enables us to recover some useful representations from the certain members of their Jacquet modules.

For every irreducible essentially square-integrable representation δ of $GL(n,F)$, there exists an $e(\delta)\in\mathbb{R}$ such that $\nu^{-e(\delta)}\delta$ is unitarizable. Suppose that $\delta_1,\delta_2,\ldots,\delta_k$ are irreducible, essentially square-integrable representations of $GL(n_1,F),GL(n_2,F),\ldots,GL(n_k,F)$ with $e(\delta_1)\leq e(\delta_2)\leq\ldots\leq e(\delta_k)$. Then the induced representation $\delta_1\times\delta_2\times\cdots\times\delta_k$ has a unique irreducible subrepresentation, which we denote by $L(\delta_1\times\delta_2\times\cdots\times\delta_k)$. This irreducible subrepresentation is called the Langlands subrepresentation, and it appears with the multiplicity one in $\delta_1\times\delta_2\times\cdots\times\delta_k$. Every irreducible representation π of $GL(n,F)$ is isomorphic to some $L(\delta_1\times\delta_2\times\cdots\times\delta_k)$. For given π, the representations $\delta_1,\delta_2,\ldots,\delta_k$ are unique up to a permutation.

Similarly, throughout the paper we use a subrepresentation version of Langlands classification for classical groups, which also happens to be more appropriate for our Jacquet module considerations. Thus, we realize a non-tempered irreducible representation π of G_n as a unique irreducible (Langlands) subrepresentation of the induced representation of the form $\delta_1 \times \delta_2 \times \cdots \times \delta_k \rtimes \tau$, where τ is a tempered representation of G_t (this defines t), δ_i is an irreducible essentially square integrable representation of $GL(n_{\delta_i}, F)$ attached to the segment $[\nu^{a_i}\rho_i, \nu^{b_i}\rho_i]$ for $i = 1, 2, \ldots, k$, and $a_1 + b_1 \leq a_2 + b_2 \leq \cdots \leq a_k + b_k < 0$ (note that $e(\delta([\nu^{a_i}\rho_i, \nu^{b_i}\rho_i])) = a_i + b_i$). In this case, we write $\pi = L(\delta_1 \times \delta_2 \times \cdots \times \delta_k \rtimes \tau)$.

We will now recall the Mœglin-Tadić classification of discrete series for classical groups, which presents a framework for our study. We fix a certain tower of classical groups (symplectic or orthogonal). Every discrete series representation of such group is uniquely described by its three invariants: a partial cuspidal support, Jordan block and ϵ-function.

A partial cuspidal support of a discrete series $\sigma \in Irr(G_n)$ is an irreducible cuspidal representation σ_{cusp} of some G_m with the property that there is an irreducible admissible representation π of $GL(n_\pi, F)$ such that σ is a subrepresenation of $\pi \rtimes \sigma_{cusp}$.

Jordan block of σ, which we denote by $Jord(\sigma)$, is a set of all pairs (c, ρ) where $\rho \simeq \widetilde{\rho}$ is an irreducible cuspidal representation of some $GL(n_\rho, F)$ and $c > 0$ is an integer such that the following two conditions are satisfied:

1. c is even if and only if $L(s, \rho, r)$ has a pole at $s = 0$. The local L-function $L(s, \rho, r)$ is the one defined by Shahidi (see for instance [13], [14]), where $r = \bigwedge^2 \mathbb{C}^{n_\rho}$ is the exterior square representation of the standard representation on $\mathbb{C}^{n_\rho}$ of $GL(n_\rho, \mathbb{C})$ if G_n is a symplectic or even-orthogonal group and $r = \mathrm{Sym}^2\mathbb{C}^{n_\rho}$ is the symmetric-square representation of the standard representation on $\mathbb{C}^{n_\rho}$ of $GL(n_\rho, \mathbb{C})$ if G_n is an odd-orthogonal group.

2. The induced representation
$$\delta([\nu^{-(c-1)/2}\rho, \nu^{(c-1)/2}\rho]) \rtimes \sigma$$
is irreducible.

To explain the notion of the ϵ-function, we first define Jordan triples. This are the triples of the form $(Jord, \sigma', \epsilon)$ where

- σ' is an irreducible cuspidal representation of some G_n.

- $Jord$ is a finite set (possibly empty) of pairs (c,ρ) ($\rho \simeq \widetilde{\rho}$ is an irreducible cuspidal representation of $GL(n_\rho, F)$, $c > 0$ an integer) such that c is even if and only if $L(s,\rho,r)$ has a pole at $s = 0$ (as above). For an irreducible cuspidal representation $\rho \simeq \widetilde{\rho}$ of $GL(n_\rho, F)$ we write $Jord_\rho = \{c : (c,\rho) \in Jord\}$. If $Jord_\rho \neq \emptyset$ and $c \in Jord_\rho$, we put $c_- = \max\{d \in Jord_\rho : d < c\}$, if it exists.

- ϵ is a function defined on a subset of $Jord \cup (Jord \times Jord)$ and attains values 1 and -1. If $(c,\rho) \in Jord$, then $\epsilon(c,\rho)$ is not defined if and only if c is odd and $(c',\rho) \in Jord(\sigma')$ for some positive integer c'. Further, ϵ is defined on a pair $(c,\rho),(c',\rho') \in Jord$ if and only if $\rho \simeq \rho'$ and $c \neq c'$.

The following compatibility conditions must hold for different $c, c', c'' \in Jord_\rho$:

1. If $\epsilon(c,\rho)$ is defined (hence $\epsilon(c',\rho)$ is also defined), then $\epsilon((c,\rho),(c',\rho)) = \epsilon(c,\rho) \cdot \epsilon(c',\rho)^{-1}$.

2. $\epsilon((c,\rho),(c'',\rho)) = \epsilon((c,\rho),(c',\rho)) \cdot \epsilon((c',\rho),(c'',\rho))$.

3. $\epsilon((c,\rho),(c',\rho)) = \epsilon((c',\rho),(c,\rho))$.

Listed properties show that it is enough to know the value of ϵ on the consecutive pairs $(c_-,\rho),(c,\rho)$ to define ϵ on all pairs.

Suppose that, for Jordan triple $(Jord,\sigma',\epsilon)$, there is $(c,\rho) \in Jord$ such that $\epsilon((c_-,\rho),(c,\rho)) = 1$. If we put $Jord' = Jord \backslash \{(c_-,\rho),(c,\rho)\}$ and consider the restriction ϵ' of ϵ to $Jord' \cup (Jord' \times Jord')$, we obtain a new Jordan triple $(Jord',\sigma',\epsilon')$, and we say that such Jordan triple is subordinated to $(Jord,\sigma',\epsilon)$.

We say that Jordan triple $(Jord,\sigma',\epsilon)$ is a triple of alternated type if $\epsilon((c_-,\rho),(c,\rho)) = -1$ holds whenever c_- is defined and there is an increasing bijection $\phi_\rho : Jord_\rho \to Jord'_\rho(\sigma')$, where

$$Jord'_\rho(\sigma') = \begin{cases} Jord_\rho(\sigma') \cup \{0\} & \text{if } a \text{ is even and } \epsilon(\min Jord_\rho, \rho) = 1; \\ Jord'_\rho(\sigma') & \text{otherwise.} \end{cases}$$

Jordan triple $(Jord,\sigma',\epsilon)$ dominates the Jordan triple $(Jord',\sigma',\epsilon')$ is there is a sequence of Jordan triples $(Jord_i,\sigma',\epsilon_i)$, $0 \leq i \leq k$, such that $(Jord_0,\sigma',\epsilon_0) = (Jord,\sigma',\epsilon)$, $(Jord_k,\sigma',\epsilon_k) = (Jord',\sigma',\epsilon')$ and $(Jord_i,\sigma',\epsilon_i)$ is subordinated to $(Jord_{i-1},\sigma',\epsilon_{i-1})$ for $i \in \{1,2,\ldots,k\}$. Jordan triple

$(Jord, \sigma', \epsilon)$ is called the admissible triple if it dominates a triple of alternated type.

The classification given in [9] and [11] states that there is one-to-one correspondence between the set of all discrete series in $Irr(G)$ and the set of all admissible triples $(Jord, \sigma', \epsilon)$ given by $\sigma = \sigma_{(Jord,\sigma',\epsilon)}$, such that $\sigma_{cusp} = \sigma'$ and $Jord(\sigma) = Jord$. Further, if $(c, \rho) \in Jord$ is such that $\epsilon((c_{-}, \rho), (c, \rho)) = 1$, we set $Jord' = Jord \setminus \{(c_{-}, \rho), (c, \rho)\}$ and consider the restriction ϵ' of ϵ to $Jord' \cup (Jord' \times Jord')$. Then $(Jord', \sigma', \epsilon')$ is an admissible triple and σ is a subrepresentation of $\delta([\nu^{-(c_{-}-1)/2}\rho, \nu^{(c-1)/2}\rho]) \rtimes \sigma_{(Jord',\sigma',\epsilon')}$. Such induced representation has exactly two discrete series subrepresentations, which are mutually non-isomorphic. Moreover, the induced representation $\delta([\nu^{-(c_{-}-1)/2}\rho, \nu^{(c_{-}-1)/2}\rho]) \rtimes \sigma_{(Jord',\sigma',\epsilon')}$ is a direct sum of two non-isomorphic tempered representations τ_{+} and τ_{-} and there is the unique $\tau \in \{\tau_{+}, \tau_{-}\}$ such that σ is a subrepresentation of $\delta([\nu^{(c_{-}+1)/2}\rho, \nu^{(c-1)/2}\rho]) \rtimes \tau$.

We shall also say that discrete series σ and its corresponding admissible triple $(Jord, \sigma', \epsilon)$ are attached to each other.

Further, if $Jord_{\rho} \neq \emptyset$ consists of even numbers, then $\epsilon(c, \rho)$ is also defined for all $c \in Jord_{\rho}$. Let us denote by $c_{min,\rho}$ minimum of the set $Jord_{\rho}$. Then it is enough to define $\epsilon(c_{min,\rho}, \rho)$, and it equals 1 if and only if there exists an irreducible representation $\pi \in R(G)$ such that

$$\sigma \hookrightarrow \delta([\nu^{\frac{1}{2}}\rho, \nu^{(c_{min,\rho}-1)/2}\rho]) \rtimes \pi.$$

Also, if $Jord_{\rho} \neq \emptyset$ consists of odd numbers and $\rho \rtimes \sigma'$ reduces (equivalently, $Jord_{\rho}(\sigma') = \emptyset$), then $\epsilon(c, \rho)$ is also defined for all $c \in Jord_{\rho}$. The induced representation $\rho \rtimes \sigma'$ reduces into two non-isomorphic irreducible tempered representations, which we denote by $\tau_1^{(\sigma',\rho)}$ and $\tau_{-1}^{(\sigma',\rho)}$. We also denote the maximum of the set $Jord_{\rho}$ by $c_{max,\rho}$. It is enough to define $\epsilon(c_{max,\rho}, \rho)$, and it equals 1 if and only if there exists an irreducible representation $\pi' \in R(GL)$ such that

$$\sigma \hookrightarrow \pi' \times \delta([\nu\rho, \nu^{(c_{max,\rho}-1)/2}\rho]) \rtimes \tau_1^{(\sigma',\rho)}.$$

It is proved in [9, 11] that triples of alternated type correspond to strongly positive discrete series and definition of such triples shows that the strongly positive discrete series are completely determined by their partial cuspidal support and Jordan block. Since all strongly positive discrete series which appear in this paper share a common partial cuspidal support we will define only Jordan block when introducing them. This procedure is also summarized in Proposition 1.2 of [12].

3 Some elementary cases

In this section, we begin our determination of Jacquet modules of discrete series representation σ with respect to maximal parabolic subgroups by examining several elementary cases. Here and subsequently, we denote an admissible triple corresponding to σ by $(Jord, \sigma_{cusp}, \epsilon)$.

Throughout the paper we assume that σ is a discrete series representation of G_n and there are $d_-, d \in Jord_{\rho'}$ such that

$$\sigma \hookrightarrow \delta([\nu^{-(d_- -1)/2}\rho', \nu^{(d-1)/2}\rho']) \rtimes \sigma_{sp} \tag{1}$$

for strongly positive representation σ_{sp} such that $[d_-, d] \cap Jord_{\rho'}(\sigma_{sp}) = \emptyset$. Let us denote a discrete series subrepresentation of $\delta([\nu^{-(d_- -1)/2}\rho', \nu^{(d-1)/2}\rho']) \rtimes \sigma_{sp}$ different than σ by σ' and an admissible triple corresponding to σ' by $(Jord, \sigma_{cusp}, \epsilon')$.

We are interested in determining all irreducible constituents of $\mu^*(\sigma)$ of the form $\delta \otimes \pi$, where δ stands for an irreducible essentially square-integrable representation. We write δ in the form $\delta([\nu^a\rho, \nu^b\rho])$. It is well-know ([11]) that this forces $2b+1 \in Jord_\rho$. To keep the notation uniform, for $(c, \rho) \in Jord$ we denote by $\mu^*(\sigma)_{(c,\rho)}$ the sum of all irreducible constituents of $\mu^*(\sigma)$ of the form

$$\delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes \pi.$$

Let ρ denote an irreducible self-contragredient admissible representation of $GL(n_\rho, F)$ (this defines n_ρ) such that there is some $c \in \mathbb{R}$ such that $\nu^{(c-1)/2}\rho \otimes \pi$ is an irreducible constituent of $\mu^*(\sigma)$ for some irreducible representation π. Further, let us denote the minimal element of $Jord_\rho(\sigma)$ by $c_{min}(\rho)$.

In the following sequence of propositions we determine $\mu^*(\sigma)_{(c,\rho)}$ in some elementary cases. First we recall the following result (Theorem 8.2 of [17]).

Lemma 3.1. *If $c \neq c_{min}(\rho)$ and $a \geq (c_- + 3)/2$, there is a unique discrete series representation $\pi_{(a,\rho)}$ such that σ is a subrepresentation of the induced representation*

$$\delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \rtimes \pi_{(a,\rho)}.$$

Using this result, we obtain a description of certain Jacquet modules of σ.

Proposition 3.2. *If an irreducible representation $\delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes \pi$, for $c \neq c_{min}(\rho)$ and $a \geq (c_- + 3)/2$, appears in Jacquet module of σ with respect to an appropriate parabolic subgroup, then π is a unique discrete series representation such that σ is a subrepresentation of $\delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \rtimes \pi$, i.e., $\pi \simeq \pi_{(a,\rho)}$. Further, such irreducible constituent appears in $\mu^*(\sigma)$ with multiplicity one. In particular, for $c \neq c_{min}(\rho)$ and $\epsilon((c_-, \rho), (c, \rho)) = -1$, we have*

$$\mu^*(\sigma)_{(c,\rho)} = \sum_{a=(c_-+3)/2}^{(c-1)/2} \delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes \pi_{(a,\rho)}.$$

Proof. Using previous lemma we obtain $\mu^*(\sigma) \geq \delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes \pi_{(a,\rho)}$ for $a \geq (c_- + 3)/2$. Using the same result we get there is also a discrete series π' such that $\mu^*(\sigma') \geq \delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes \pi'$.

Let us prove that if an irreducible constituent of the form $\delta([\nu^a\rho, \nu^b\rho]) \otimes \pi''$ appears in $\mu^*(\sigma)$, then $\pi'' \simeq \pi_{(a,\rho)}$. Applying the structural formula for μ^* to the right hand side of (1) we obtain that there are $-(d_- + 1)/2 \leq i \leq j \leq (d-1)/2$ and an irreducible constituent $\delta \otimes \tau$ of $\mu^*(\sigma_{sp})$ such that

$$\delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \leq \delta([\nu^{-i}\rho', \nu^{(d_--1)/2}\rho']) \times \delta([\nu^{j+1}\rho', \nu^{(d-1)/2}\rho']) \times \delta$$

and

$$\pi'' \leq \delta([\nu^{i+1}\rho', \nu^j\rho']) \rtimes \tau.$$

If $((c-1)/2, \rho) = ((d-1)/2, \rho_1)$, since $a > (d_- - 1)/2$ we obtain $i = -(d_- + 1)/2$ and $\tau \simeq \sigma_{sp}$. Similarly, if $((c-1)/2, \rho) = ((d_- - 1)/2, \rho')$, we obtain $j = (d-1)/2$ and again $\tau \simeq \sigma_{sp}$. Finally, if $((c-1)/2, \rho) \notin \{((d-1)/2, \rho'), ((d_- - 1)/2, \rho')\}$ then, using Theorem 4.6 of [7] we deduce $i = -(d_- + 1)/2$, $j = (d-1)/2$ and τ is a strongly positive discrete series such that $Jord(\tau) = Jord(\sigma_{sp}) \setminus \{(c, \rho)\} \cup \{(2a-1, \rho)\}$. In any case, by [12], Theorem 2.1 we obtain that $\delta([\nu^{i+1}\rho', \nu^j\rho']) \rtimes \tau$ is a length three representation and in an appropriate Grothendieck group we have

$$\delta([\nu^{i+1}\rho', \nu^j\rho']) \rtimes \tau = \pi + \pi' + L(\delta([\nu^{-j}\rho', \nu^{-i-1}\rho']) \rtimes \tau).$$

It follows that both $\delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes \pi$ and $\delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes \pi'$ appear with multiplicity one in $\mu^*(\delta([\nu^{-(c_--1)/2}\rho', \nu^{(c-1)/2}\rho']) \rtimes \sigma_{sp})$. Thus, since $\delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes \pi'$ appears in $\mu^*(\sigma')$, it does not appear in $\mu^*(\sigma)$.

Let us now assume that $\delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-j}\rho', \nu^{-i-1}\rho']) \rtimes \tau)$ appears in $\mu^*(\sigma)$. Then the transitivity of Jacquet modules implies that

$\delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes \delta([\nu^{-j}\rho', \nu^{-i-1}\rho']) \otimes \tau$ is contained in Jacquet module of σ with respect to an appropriate parabolic subgroup and it is now easy to obtain a contradiction with the square-integrability of σ. Thus, we obtain $\pi'' \simeq \pi$ and $\delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes \pi$ appears in $\mu^*(\sigma)$ with multiplicity one.

If we assume that an irreducible constituent of the form $\delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-a+1}\rho, \nu^{(c_- -1)/2}\rho]) \rtimes \sigma_{sp})$ appears in $\mu^*(\sigma)$, transitivity of Jacquet modules immediately provides a contradiction with the square-integrability of σ. This ends the proof. $\square$

In a similar way we handle the case of the minimal element of $Jord_\rho$. In what follows, for a real number q we denote by $\lceil q \rceil$ the smallest integer which is not smaller than q. Similarly to Theorem 8.2 of [17], we obtain:

Lemma 3.3. *Let x' stand for $(c_{min}(\rho)-1)/2 - \lceil (c_{min}(\rho)-1)/2 \rceil + 1$. If $c_{min}(\rho)$ is even and $\epsilon(c_{min}(\rho), \rho) = -1$ set $x = x'+1$, otherwise set $x = x'$. If there is some irreducible representation π such that $\mu^*(\sigma) \geq \nu^{(c_{min}(\rho)-1)/2}\rho \otimes \pi$, then for $a \geq x$ there exists a unique discrete series representation $\pi_{(a,\rho)}$ such that σ is a subrepresentation of the induced representation*

$$\delta([\nu^a\rho, \nu^{(c_{min}(\rho)-1)/2}\rho]) \rtimes \pi_{(a,\rho)}.$$

Proof. Let us first assume that σ is a subrepresentation of the induced representation of the same form as in the right-hand side of (1) with $(d_-, \rho') \neq (c_{min}(\rho), \rho)$. Using [7], we see that there is an embedding

$$\sigma_{sp} \hookrightarrow \delta([\nu^a\rho, \nu^{(c_{min}(\rho)-1)/2}\rho]) \rtimes \sigma'_{sp},$$

for an appropriate strongly positive discrete series σ'_{sp}. Thus, we obtain

$$\begin{aligned} \sigma &\hookrightarrow \delta([\nu^{-(d_- -1)/2}\rho', \nu^{(d-1)/2}\rho']) \times \delta([\nu^a\rho, \nu^{(c_{min}(\rho)-1)/2}\rho]) \rtimes \sigma'_{sp} \\ &\simeq \delta([\nu^a\rho, \nu^{(c_{min}(\rho)-1)/2}\rho]) \times \delta([\nu^{-(d_- -1)/2}\rho', \nu^{(d-1)/2}\rho']) \rtimes \sigma'_{sp}. \end{aligned}$$

In consequence, there is some irreducible representation $\pi_{(a,\rho)}$ such that σ is a subrepresentation of $\delta([\nu^a\rho, \nu^{(c_{min}(\rho)-1)/2}\rho]) \rtimes \pi_{(a,\rho)}$. Frobenius reciprocity shows $\mu^*(\sigma) \geq \delta([\nu^a\rho, \nu^{(c_{min}(\rho)-1)/2}\rho]) \otimes \pi_{(a,\rho)}$. Lemma 2.1 implies that there are $-(d_- + 1)/2 \leq i \leq j \leq (d-1)/2$ and an irreducible constituent $\delta \otimes \tau$ of $\mu^*(\sigma_{sp})$ such that

$$\delta([\nu^a\rho, \nu^{(c_{min}(\rho)-1)/2}\rho]) \leq \delta([\nu^{-i}\rho', \nu^{(d_- -1)/2}\rho']) \times \delta([\nu^{j+1}\rho', \nu^{(d-1)/2}\rho']) \times \delta$$

and

$$\pi_{(a,\rho)} \leq \delta([\nu^{i+1}\rho', \nu^{j}\rho']) \rtimes \tau.$$

It directly follows that $\delta \simeq \delta([\nu^a\rho, \nu^{(c_{min}(\rho)-1)/2}\rho])$ and $\pi_{(a,\rho)}$ is an irreducible subquotient of $\delta([\nu^{-(d_{-}-1)/2}\rho', \nu^{(d-1)/2}\rho']) \rtimes \tau$, for strongly positive representation such that $[d_{-}, d] \cap Jord_{\rho'}(\tau) = \emptyset$. Using the square-integrability of σ we deduce $\pi_{(a,\rho)} \neq L(\delta([\nu^{-(d-1)/2}\rho', \nu^{(d_{-}-1)/2}\rho']) \rtimes \tau)$. Thus, Theorem 2.1 of [12] implies that $\pi_{(a,\rho)}$ is a discrete series representation and the observed irreducible constituent appears with multiplicity one. Since the same discussion can be made for σ', uniqueness of $\pi_{(a,\rho)}$ follows.

Let us now assume that σ is a subrepresentation of

$$\delta([\nu^{-(c_{min}(\rho)-1)/2}\rho, \nu^{(d-1)/2}\rho]) \rtimes \sigma_{sp},$$

where $[c_{min}(\rho), d] \cap Jord_{\rho}(\sigma_{sp}) = \emptyset$ and the ϵ-function of σ equals -1 on all pairs different than $((c_{min}(\rho), \rho), (d, \rho))$ and $((d, \rho), (c_{min}(\rho), \rho))$. For $a > \frac{1}{2}$, using Proposition 3.1 of [12], we get

$$\begin{aligned} \sigma &\hookrightarrow \delta([\nu^{-a+1}\rho, \nu^{(d-1)/2}\rho]) \times \delta([\nu^{-(c_{min}(\rho)-1)/2}\rho, \nu^{-a}\rho]) \rtimes \sigma_{sp} \\ &\simeq \delta([\nu^{-a+1}\rho, \nu^{(d-1)/2}\rho]) \times \delta([\nu^{a}\rho, \nu^{(c_{min}(\rho)-1)/2}\rho]) \rtimes \sigma_{sp} \\ &\simeq \delta([\nu^{a}\rho, \nu^{(c_{min}(\rho)-1)/2}\rho]) \times \delta([\nu^{-a+1}\rho, \nu^{(d-1)/2}\rho]) \rtimes \sigma_{sp}. \end{aligned}$$

In the same way as before we conclude that there is a unique discrete series $\pi_{(a,\rho)}$ such that σ is a subrepresentation of $\delta([\nu^a\rho, \nu^{(c_{min}(\rho)-1)/2}\rho]) \rtimes \pi_{(a,\rho)}$.

It remains to consider the case $a = \frac{1}{2}$ and $\epsilon(c_{min}(\rho), \rho) = 1$. By definition, there is some irreducible representation $\pi_{(1/2,\rho)}$ such that σ is a subrepresentation of $\delta([\nu^{1/2}\rho, \nu^{(c_{min}(\rho)-1)/2}\rho]) \rtimes \pi_{(1/2,\rho)}$. Again, Frobenius reciprocity implies $\mu^*(\sigma) \geq \delta([\nu^{1/2}\rho, \nu^{(c_{min}(\rho)-1)/2}\rho]) \otimes \pi_{(1/2,\rho)}$ and using Lemma 2.1 we deduce that $\pi_{(1/2,\rho)}$ is an irreducible subquotient of $\delta([\nu^{1/2}\rho, \nu^{(d-1)/2}\rho]) \rtimes \sigma_{sp}$. Square-integrability of σ shows $\pi_{(1/2,\rho)} \neq L(\delta([\nu^{-(d-1)/2}\rho, \nu^{-1/2}\rho]) \rtimes \sigma_{sp})$ and by Theorem 5.1 of [12] it is a uniquely defined discrete series. This proves the lemma. □

Analogously to Proposition 3.2 we have:

Proposition 3.4. *Let x' stand for $(c_{min}(\rho)-1)/2 - \lceil (c_{min}(\rho)-1)/2 \rceil + 1$. If $c_{min}(\rho)$ is even and $\epsilon(c_{min}(\rho), \rho) = -1$ set $x = x'+1$, otherwise set $x = x'$. If there is some irreducible representation π such that $\mu^*(\sigma) \geq \nu^{(c_{min}(\rho)-1)/2}\rho \otimes \pi$, then the following equality holds in $R(GL) \otimes R(G)$:*

$$\mu^*(\sigma)_{(c_{min}(\rho),\rho)} = \sum_{a=x}^{(c_{min}(\rho)-1)/2} \delta([\nu^a\rho, \nu^{(c_{min}(\rho)-1)/2}\rho]) \otimes \pi_{(a,\rho)}.$$

In the rest of the paper we deal with $\mu^*(\sigma)_{(c,\rho)}$ for $c \neq c_{min}(\rho)$ and $\epsilon((c_-,\rho),(c,\rho)) = 1$. Thus, to keep things simple, we assume $c = d$ and $\rho \simeq \rho'$, so σ can be given as an irreducible subrepresentation of the induced representation of the form

$$\sigma \hookrightarrow \delta([\nu^{-(c_--1)/2}\rho, \nu^{(c-1)/2}\rho]) \rtimes \sigma_{sp} \tag{2}$$

for strongly positive representation σ_{sp} such that $[c_-, c] \cap Jord_\rho(\sigma_{sp}) = \emptyset$. Let us denote by σ_{ind} the induced representation $\delta([\nu^{-(c_--1)/2}\rho, \nu^{(c-1)/2}\rho]) \rtimes \sigma_{sp}$ and by σ' a discrete series subrepresentation of σ_{ind} different than σ. We note that σ_{ind} is a length three representation, by [12], Theorem 2.1.

Also, applying the structural formula on the right hand side of (2) we see at once that if an irreducible constituent of the form $\delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes \pi$ appears in $\mu^*(\sigma)$, then $a \geq (c_- - 1)/2$.

Two possible cases shall be examined in separate sections.

4 Case $Jord_\rho(\sigma_{sp}) \neq \emptyset$ or c even.

Since in case that we consider in this section the classical-group part π of an irreducible constituent $\delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes \pi$ appearing in $\mu^*(\sigma)_{(c,\rho)}$ heavily depends on the left limit of the segment $[\nu^a\rho, \nu^{(c-1)/2}\rho]$, several cases will be treated separately. For simplicity of the notation, we write $\mu^*(\sigma)_{(a,c,\rho)}$ (resp., $\mu^*(\sigma_{ind})_{(a,c,\rho)}$) for the formal sum of all irreducible constituents of $\mu^*(\sigma)_{(c,\rho)}$ (resp., $\mu^*(\sigma_{ind})$) of the form $\delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes \pi$.

Results obtained in the previous section enable us to assume $a \leq (c_- + 1)/2$, since in the same way as in the proof of Proposition 3.2 it can be seen that $\mu^*(\sigma)_{(\rho,c,a)} = \delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes \pi_{(a,\rho)}$ for $a \geq (c_- + 3)/2$.

Also, it follows from [9, 11] that there is a unique irreducible tempered subrepresentation τ of $\delta([\nu^{-(c_--1)/2}\rho, \nu^{(c_--1)/2}\rho]) \rtimes \sigma_{sp}$ such that

$$\mu^*(\sigma)_{((c_-+1)/2,c,\rho)} = \delta([\nu^{(c_-+1)/2}\rho, \nu^{(c-1)/2}\rho]) \otimes \tau.$$

We begin our determination of $\mu^*(\sigma)_{(a,c,\rho)}$ with two elementary but useful technical result.

Lemma 4.1. *Suppose that $\delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes \pi$ is an irreducible constituent of $\mu^*(\sigma_{ind})$, with $a \leq (c_- - 1)/2$. Then $\delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes \pi$ does not appear in $\mu^*(L(\delta([\nu^{-(c-1)/2}\rho, \nu^{(c_--1)/2}\rho]) \rtimes \sigma_{sp}))$.*

Proof. Suppose, on the contrary, that $\delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes \pi$ is an irreducible constituent of $\mu^*(L(\delta([\nu^{-(c-1)/2}\rho, \nu^{(c_--1)/2}\rho]) \rtimes \sigma_{sp}))$, for $a \leq (c_- - 1)/2$. Then transitivity of Jacquet modules forces $\delta([\nu^{(c_-+1)/2}\rho, \nu^{(c-1)/2}\rho]) \otimes \pi' \leq \mu^*(L(\delta([\nu^{-(c-1)/2}\rho, \nu^{(c_--1)/2}\rho]) \rtimes \sigma_{sp}))$, for some irreducible representation π'. But it is well-known ([11]) that there are only two irreducible constituents of the form $\delta([\nu^{(c_-+1)/2}\rho, \nu^{(c-1)/2}\rho]) \otimes \pi'$ appearing in $\mu^*(\sigma_{ind})$ and each of them is contained either in $\mu^*(\sigma)$ or in $\mu^*(\sigma')$, a contradiction. □

Proposition 4.2. *For $-(c_- + 1)/2 \leq a \leq (c_- - 1)/2$, $\delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-(c_--1)/2}\rho, \nu^{a-1}\rho]) \rtimes \sigma_{sp})$ appears in $\mu^*(\sigma)_{(a,c,\rho)}$ with multiplicity one.*

Proof. From (2) we get

$$\sigma \hookrightarrow \delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \times \delta([\nu^{-(c_--1)/2}\rho, \nu^{a-1}\rho]) \rtimes \sigma_{sp}$$

and Frobenius reciprocity shows that the irreducible representation

$$\delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes \delta([\nu^{-(c_--1)/2}\rho, \nu^{a-1}\rho]) \otimes \sigma_{sp}$$

appears in Jacquet module of σ with respect to an appropriate parabolic subgroup. By transitivity of Jacquet modules, there is some irreducible representation π such that $\mu^*(\sigma)_{(a,c,\rho)}$ contains $\delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes \pi$ and $\mu^*(\pi) \geq \delta([\nu^{-(c_--1)/2}\rho, \nu^{a-1}\rho]) \otimes \sigma_{sp}$. We determine π calculating μ^* of the right-hand side of (2). By Lemma 2.1, there are $-(c_- + 1)/2 \leq i \leq j \leq (c-1)/2$ and an irreducible constituent $\delta \otimes \pi'$ of $\mu^*(\sigma_{sp})$ such that

$$\delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \leq \delta([\nu^{-i}\rho, \nu^{(c_--1)/2}\rho]) \times \delta([\nu^{j+1}\rho, \nu^{(c-1)/2}\rho]) \times \delta$$

and

$$\pi \leq \delta([\nu^{i+1}\rho, \nu^{j}\rho]) \rtimes \pi'.$$

Firstly, if $-i = a$ or $j + 1 = a$, using Theorem 4.6 of [7] and the fact $[c_-, c] \cap Jord_\rho(\sigma_{sp}) = \emptyset$, we obtain $\pi \leq \delta([\nu^{-(c_--1)/2}\rho, \nu^{a-1}\rho]) \rtimes \sigma_{sp}$.

Now we calculate the multiplicity of $\delta([\nu^{-(c_--1)/2}\rho, \nu^{a-1}\rho]) \otimes \sigma_{sp}$ in $\mu^*(\delta([\nu^{-(c_--1)/2}\rho, \nu^{a-1}\rho]) \rtimes \sigma_{sp})$.

Again, there are $-(c_- + 1)/2 \leq i_1 \leq j_1 \leq a - 1$ and an irreducible constituent $\delta_1 \otimes \pi_1 \leq \mu^*(\sigma_{sp})$ such that

$$\delta([\nu^{-(c_--1)/2}\rho, \nu^{a-1}\rho]) \leq \delta([\nu^{-i_1}\rho, \nu^{(c_--1)/2}\rho]) \times \delta([\nu^{j_1+1}\rho, \nu^{a-1}\rho]) \times \delta_1$$

and

$$\sigma_{sp} \leq \delta([\nu^{i_1+1}\rho, \nu^{j_1}\rho]) \rtimes \pi_1.$$

Since $a-1 < (c_- - 1)/2$, we get $i_1 = -(c_- + 1)/2$. Further, the strong positivity of σ_{sp} forces $j_1 = -(c_- + 1)/2$, so $\pi_1 = \sigma_{sp}$. Thus, $\delta([\nu^{-(c_- -1)/2}\rho, \nu^{a-1}\rho]) \otimes \sigma_{sp}$ appears with the multiplicity one in $\mu^*(\delta([\nu^{-(c_- -1)/2}\rho, \nu^{a-1}\rho]) \rtimes \sigma_{sp})$. Since it, clearly, appears in $\mu^*(L(\delta([\nu^{-(c_- -1)/2}\rho, \nu^{a-1}\rho]) \rtimes \sigma_{sp}))$ it follows that $L(\delta([\nu^{-(c_- -1)/2}\rho, \nu^{a-1}\rho]) \rtimes \sigma_{sp})$ is a unique irreducible subquotient of $\delta([\nu^{-(c_- -1)/2}\rho, \nu^{a-1}\rho]) \rtimes \sigma_{sp}$ which contains $\delta([\nu^{-(c_- -1)/2}\rho, \nu^{a-1}\rho]) \otimes \sigma_{sp}$ in Jacquet module with respect to an appropriate parabolic subgroup.

Secondly, if $-i > a$ and $j+1 > a$, it follows that $\delta = \delta([\nu^a\rho, \nu^b\rho])$ for some $a \leq b \leq (c-1)/2$. This further gives $\delta([\nu^{-i}\rho, \nu^{(c_- -1)/2}\rho]) \times \delta([\nu^{j+1}\rho, \nu^{(c-1)/2}\rho]) \simeq \delta([\nu^{b+1}\rho, \nu^{(c-1)/2}\rho])$ and it directly follows $\pi \leq \delta([\nu^{b+1}\rho, \nu^{(c_- -1)/2}\rho]) \rtimes \pi'$, where, by Theorem 4.6 of [7], π' is a strongly positive discrete series such that $Jord(\pi') = Jord(\sigma_{sp}) \setminus \{(2b+1, \rho)\} \cup \{(2a-1, \rho)\}$.

Since $b \geq a$ and π' is strongly positive, it can be seen directly from the structural formula for μ^* that each irreducible constituent of the form $\delta([\nu^{-(c_- -1)/2}\rho, \nu^d\rho]) \otimes \pi''$ appearing in $\mu^*(\delta([\nu^{b+1}\rho, \nu^{(c_- -1)/2}\rho]) \rtimes \pi')$ satisfies $d > a$.

Consequently, $\mu^*(\sigma)_{(a,c,\rho)} \geq \delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-(c_- -1)/2}\rho, \nu^{a-1}\rho]) \rtimes \sigma_{sp})$. We have already seen that such irreducible constituent appears in $\mu^*(\sigma_{ind})$ with multiplicity two and it can be proved in completely analogous manner that it also appears in $\mu^*(\sigma')$. Thus, it appears in $\mu^*(\sigma)_{(a,c,\rho)}$ with multiplicity one. □

4.1 Case $a \leq 0$.

Let us first consider the case $a \leq 0$. Applying the structural formula for μ^* to the induced representation (2), in the following lemma we obtain all candidates for irreducible constituents of $\mu^*(\sigma)_{(a,c,\rho)}$.

Lemma 4.3. *Suppose $a \leq 0$. If $c_- > min(Jord_\rho(\sigma))$ and $a \leq ((c_-)_- - 1)/2$ then the following equality holds in $R(GL) \otimes R$:*

$$\begin{aligned} \mu^*(\sigma_{ind})_{(a,c,\rho)} &= 2\ \delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-(c_- -1)/2}\rho, \nu^{a-1}\rho]) \rtimes \sigma_{sp}) + \\ &\quad 2\ \delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-((c_-)_- -1)/2}\rho, \nu^{a-1}\rho]) \rtimes \sigma'_{sp}), \end{aligned}$$

where σ'_{sp} stands for strongly positive discrete series such that $Jord(\sigma'_{sp}) = Jord(\sigma_{sp}) \setminus \{((c_-)_-, \rho)\} \cup \{(c_-, \rho)\}$. Otherwise, in $R(GL) \otimes R$ we have

$$\mu^*(\sigma_{ind})_{(a,c,\rho)} = 2\ \delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-(c_- -1)/2}\rho, \nu^{a-1}\rho]) \rtimes \sigma_{sp}).$$

Proof. Let us comment only the $c_\_ > \min(Jord_\rho(\sigma))$ and $a \leq ((c_\_)_\_ - 1)/2$ case. In other cases $\mu^*(\sigma_{ind})_{(a,c,\rho)}$ can be obtained in the same way but more easily. Using Lemma 2.1, we deduce that there are $-(c_\_ + 1)/2 \leq i \leq j \leq (c-1)/2$ and an irreducible constituent $\delta \otimes \pi'$ of $\mu^*(\sigma_{sp})$ such that

$$\delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \leq \delta([\nu^{-i}\rho, \nu^{(c_\_-1)/2}\rho]) \times \delta([\nu^{j+1}\rho, \nu^{(c-1)/2}\rho]) \times \delta$$

and

$$\pi \leq \delta([\nu^{i+1}\rho, \nu^j\rho]) \rtimes \pi'.$$

Since $a \leq 0$, the strong positivity of σ_{sp} forces either $-i = a$ or $j+1 = a$. Also, $\pi' \simeq \sigma_{sp}$. If $-i = a$, it directly follows $j = (c_\_ - 1)/2$ and if $j+1 = a$ we have $i = -(c_\_ + 1)/2$. Now Proposition 3.1.(i) of [12] shows that in an appropriate Grothendieck group holds

$$\begin{aligned} \delta([\nu^{i+1}\rho, \nu^j\rho]) \rtimes \sigma_{sp} &= L(\delta([\nu^{-(c_\_-1)/2}\rho, \nu^{a-1}\rho]) \rtimes \sigma_{sp}) + \\ &\quad L(\delta([\nu^{-((c_\_)_\_-1)/2}\rho, \nu^{a-1}\rho]) \rtimes \sigma'_{sp}), \end{aligned}$$

and the lemma is proved. □

We are now ready to provide a description of $\mu^*(\sigma)_{(a,c,\rho)}$ for $a \leq 0$.

Theorem 4.4. *Suppose $a \leq 0$. If $c_\_ > min(Jord_\rho(\sigma))$, $a \leq ((c_\_)_\_ - 1)/2$ and $\epsilon(((c_\_)_\_, \rho), (c_\_, \rho)) = 1$ then in $R(GL) \otimes R$ we have:*

$$\begin{aligned} \mu^*(\sigma)_{(a,c,\rho)} &= \delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-(c_\_-1)/2}\rho, \nu^{a-1}\rho]) \rtimes \sigma_{sp}) + \\ &\quad 2\ \delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-((c_\_)_\_-1)/2}\rho, \nu^{a-1}\rho]) \rtimes \sigma'_{sp}), \end{aligned}$$

where σ'_{sp} stands for strongly positive discrete series such that $Jord(\sigma'_{sp}) = Jord(\sigma_{sp}) \setminus \{((c_\_)_\_, \rho)\} \cup \{(c_\_, \rho)\}$. Otherwise, in $R(GL) \otimes R$ we have

$$\mu^*(\sigma)_{(a,c,\rho)} = \delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-(c_\_-1)/2}\rho, \nu^{a-1}\rho]) \rtimes \sigma_{sp}).$$

Proof. We have already seen that $\delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-(c_\_-1)/2}\rho, \nu^{a-1}\rho]) \rtimes \sigma_{sp})$ appears in $\mu^*(\sigma)_{(a,c,\rho)}$ with multiplicity one. Previous lemma enables us to assume $c_\_ > \min(Jord_\rho(\sigma))$ and $a \leq ((c_\_)_\_ - 1)/2$. By Lemma 4.1, $\delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-((c_\_)_\_-1)/2}\rho, \nu^{a-1}\rho]) \rtimes \sigma'_{sp})$ can appear only in $\mu^*(\sigma)$ or in $\mu^*(\sigma')$.

It is not hard to see that there is a strongly positive representation π such that σ'_{sp} is a subrepresentation of the induced representation

$$\delta([\nu^{((c_\_)_\_+1)/2}\rho, \nu^{(c_\_-1)/2}\rho]) \rtimes \pi.$$

This provides an embedding

$$L(\delta([\nu^{-((c_-)_--1)/2}\rho, \nu^{a-1}\rho]) \rtimes \sigma'_{sp}) \hookrightarrow \delta([\nu^{-((c_-)_--1)/2}\rho, \nu^{a-1}\rho]) \times \delta([\nu^{((c_-)_-+1)/2}\rho, \nu^{(c_--1)/2}\rho]) \rtimes \pi.$$

Since $a \leq 0$, we have $\delta([\nu^{-((c_-)_--1)/2}\rho, \nu^{a-1}\rho]) \times \delta([\nu^{((c_-)_-+1)/2}\rho, \nu^{(c_--1)/2}\rho]) \simeq \delta([\nu^{((c_-)_-+1)/2}\rho, \nu^{(c_--1)/2}\rho]) \times \delta([\nu^{-((c_-)_--1)/2}\rho, \nu^{a-1}\rho])$ and by Lemma 3.2 of [11] there is an irreducible representation π' such that $L(\delta([\nu^{-((c_-)_--1)/2}\rho, \nu^{a-1}\rho]) \rtimes \sigma'_{sp})$ is a subrepresentation of $\delta([\nu^{((c_-)_-+1)/2}\rho, \nu^{(c_--1)/2}\rho]) \rtimes \pi'$. Frobenius reciprocity and transitivity of Jacquet modules show that if $\delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-((c_-)_--1)/2}\rho, \nu^{a-1}\rho]) \rtimes \sigma'_{sp})$ appears in $\mu^*(\sigma)$, then the Jacquet module of σ with respect to an appropriate parabolic subgroup contains

$$\delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes \delta([\nu^{((c_-)_-+1)/2}\rho, \nu^{(c_--1)/2}\rho]) \otimes \pi'.$$

Using transitivity of Jacquet modules again, we deduce that there is some irreducible constituent $\delta \otimes \pi'$ of $\mu^*(\sigma)$ such that Jacquet module of δ with respect to an appropriate parabolic subgroup contains $\delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes \delta([\nu^{((c_-)_-+1)/2}\rho, \nu^{(c_--1)/2}\rho])$. Since σ is an irreducible subrepresentation of σ_{ind}, we will determine δ from $\mu^*(\sigma_{ind})$.

By Lemma 2.1, there are $-(c_- + 1)/2 \leq i \leq j \leq (c-1)/2$ and an irreducible constituent $\delta' \otimes \pi''$ of $\mu^*(\sigma_{sp})$ such that

$$\delta \leq \delta([\nu^{-i}\rho, \nu^{(c_--1)/2}\rho]) \times \delta([\nu^{j+1}\rho, \nu^{(c-1)/2}\rho]) \times \delta'.$$

Since $a \leq 0$, from cuspidal support of δ we see that there are two possibilities to consider:

- $j + 1 = a$. Using Theorem 4.6 from [7] we deduce $-i = ((c_-)_- + 1)/2$. Consequently, δ is an irreducible subquotient of the induced representation $\delta([\nu^{((c_-)_-+1)/2}\rho, \nu^{(c_--1)/2}\rho]) \times \delta([\nu^a\rho, \nu^{(c-1)/2}\rho])$ which is irreducible ([18]). Thus, in this case

$$\delta \simeq \delta([\nu^{((c_-)_-+1)/2}\rho, \nu^{(c_--1)/2}\rho]) \times \delta([\nu^a\rho, \nu^{(c-1)/2}\rho]).$$

- $-i = a$. In the same way as in the previously considered case we get $j + 1 = ((c_-)_- + 1)/2$. Thus, δ is, in this case, an irreducible subquotient of the induced representation $\delta([\nu^a\rho, \nu^{(c_--1)/2}\rho]) \times \delta([\nu^{((c_-)_-+1)/2}\rho, \nu^{(c-1)/2}\rho])$, which is, by [18], length two representation which contains $\delta([\nu^{((c_-)_-+1)/2}\rho,$

$\nu^{(c_- -1)/2}\rho]) \times \delta([\nu^a\rho, \nu^{(c-1)/2}\rho])$ as an irreducible subquotient. Let us determine the multiplicity of $\delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes \delta([\nu^{((c_-)_-+1)/2}\rho, \nu^{(c_- -1)/2}\rho])$ in $m^*(\delta([\nu^a\rho, \nu^{(c_- -1)/2}\rho]) \times \delta([\nu^{((c_-)_-+1)/2}\rho, \nu^{(c-1)/2}\rho]))$.

There are $a-1 \leq i_1 \leq (c_- - 1)/2$ and $((c_-)_- - 1)/2 \leq j_1 \leq (c-1)/2$ such that

$$\delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \leq \delta([\nu^{i_1+1}\rho, \nu^{(c_- -1)/2}\rho]) \times \delta([\nu^{j_1+1}\rho, \nu^{(c-1)/2}\rho])$$

and

$$\delta([\nu^{((c_-)_-+1)/2}\rho, \nu^{(c_- -1)/2}\rho]) \leq \delta([\nu^a\rho, \nu^{i_1}\rho]) \times \delta([\nu^{((c_-)_-+1)/2}\rho, \nu^{j_1}\rho]).$$

Since $a < ((c_-)_-+1)/2$, from the first inequality we obtain $i_1 = a-1$ and $j_1 = (c_- -1)/2$. Therefore, $\delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes \delta([\nu^{((c_-)_-+1)/2}\rho, \nu^{(c_- -1)/2}\rho])$ appears with the multiplicity one in $m^*(\delta([\nu^a\rho, \nu^{(c_- -1)/2}\rho]) \times \delta([\nu^{((c_-)_-+1)/2}\rho, \nu^{(c-1)/2}\rho]))$ and it obviously appears in $m^*(\delta([\nu^{((c_-)_-+1)/2}\rho, \nu^{(c_- -1)/2}\rho]) \times \delta([\nu^a\rho, \nu^{(c-1)/2}\rho]))$. Again, we conclude $\delta \simeq \delta([\nu^{((c_-)_-+1)/2}\rho, \nu^{(c_- -1)/2}\rho]) \times \delta([\nu^a\rho, \nu^{(c-1)/2}\rho])$.

This enables us to conclude that Jacquet module of σ with respect to an appropriate parabolic subgroup contains

$$\delta([\nu^{((c_-)_-+1)/2}\rho, \nu^{(c_- -1)/2}\rho]) \otimes \delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes \pi'$$

and Proposition 7.2 of [17] shows $\epsilon(((c_-)_-, \rho), (c_-, \rho)) = 1$.

On the other hand, since discrete series σ and σ' are not isomorphic, definition of σ yields

$$\epsilon(((c_-)_-, \rho), (c_-, \rho)) \cdot \epsilon'(((c_-)_-, \rho), (c_-, \rho)) = -1.$$

Consequently, if $\delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-((c_-)_- -1)/2}\rho, \nu^{a-1}\rho]) \rtimes \sigma'_{sp})$ appears in $\mu^*(\sigma)$ then it is not an irreducible constituent of $\mu^*(\sigma')$ and the theorem is proved. □

4.2 Case $a \geq 1$.

We shall now consider the case $a \geq 1$. Let us begin with a technical lemma. From now on, x stands for the minimum of the set $\{d \in Jord_\rho : a \leq (d-1)/2\}$.

Lemma 4.5. *Suppose $a \geq 1$. If $x = c_-$ or $a = (x_- + 1)/2$ then for an irreducible constituent $\delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes \pi$ of $\mu^*(\sigma_{ind})$ we have*

$$\pi \leq \delta([\nu^{-a+1}\rho, \nu^{(c_--1)/2}\rho]) \rtimes \sigma_{sp},$$

Otherwise,

$$\pi \leq \delta([\nu^{-a+1}\rho, \nu^{(c_--1)/2}\rho]) \rtimes \sigma_{sp} \oplus \delta([\nu^{-(x-1)/2}\rho, \nu^{(c_--1)/2}\rho]) \rtimes \sigma'_{sp},$$

where σ'_{sp} denotes a strongly positive discrete series such that $Jord(\sigma'_{sp}) = Jord(\sigma_{sp}) \setminus \{(x, \rho)\} \cup \{(2a-1, \rho)\}$.

Proof. In a similar way as in the previously considered case, we see that there are $-(c_- + 1)/2 \leq i \leq j \leq (c-1)/2$ and an irreducible constituent $\delta \otimes \pi'$ of $\mu^*(\sigma_{sp})$ such that

$$\delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \leq \delta([\nu^{-i}\rho, \nu^{(c_--1)/2}\rho]) \times \delta([\nu^{j+1}\rho, \nu^{(c-1)/2}\rho]) \times \delta$$

and

$$\pi \leq \delta([\nu^{i+1}\rho, \nu^j\rho]) \rtimes \pi'.$$

If $-i = a$ or $j+1 = a$, we deduce $\pi \leq \delta([\nu^{-a+1}\rho, \nu^{(c_--1)/2}\rho]) \rtimes \sigma_{sp}$. Otherwise, Theorem 4.6 of [7] implies that $a \neq (x_- + 1)/2$ and $x < c$. This further gives $x < c_-$, $\delta \simeq \delta([\nu^a\rho, \nu^{(x-1)/2}\rho])$ and $\pi' \simeq \sigma'_{sp}$. Consequently,

$$\delta([\nu^{(x+1)/2}\rho, \nu^{(c-1)/2}\rho]) \leq \delta([\nu^{-i}\rho, \nu^{(c_--1)/2}\rho]) \times \delta([\nu^{j+1}\rho, \nu^{(c-1)/2}\rho])$$

and in the same way as before we conclude $\pi \leq \delta([\nu^{-(x-1)/2}\rho, \nu^{(c_--1)/2}\rho]) \rtimes \sigma'_{sp}$. □

In the following sequence of propositions we provide a complete description of $\mu^*(\sigma)_{(a,c,\rho)}$ for $a \geq 1$, using a case-by-case consideration.

Proposition 4.6. *If $a \geq 1$, $x = c_-$ and $a \neq (x_- + 1)/2$, then there is a unique discrete series subrepresentation π of $\delta([\nu^{-a+1}\rho, \nu^{(c_--1)/2}\rho]) \rtimes \sigma_{sp}$ such that in $R(GL) \otimes R$ we have:*

$$\begin{aligned}\mu^*(\sigma)_{(a,c,\rho)} &= \delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-(c_--1)/2}\rho, \nu^{a-1}\rho]) \rtimes \sigma_{sp}) + \\ &\quad 2\ \delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes \pi.\end{aligned}$$

Proof. We have already seen that $\mu^*(\sigma)_{(a,c,\rho)}$ contains $\delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-(c_--1)/2}\rho, \nu^{a-1}\rho]) \rtimes \sigma_{sp})$. Further, the following equality holds in $R(GL) \otimes R$:

$$\begin{aligned}\mu^*(\sigma_{ind})_{(a,c,\rho)} &= 2\ \delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-(c_--1)/2}\rho, \nu^{a-1}\rho]) \rtimes \sigma_{sp}) + \\ &\quad 2\ \delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes \sigma_1 + 2\ \delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes \sigma_2,\end{aligned}$$

where σ_1 and σ_2 are mutually non-isomorphic discrete series subrepresentations of the induced representations $\delta([\nu^{-a+1}\rho, \nu^{(c_--1)/2}\rho]) \rtimes \sigma_{sp}$.

Since $Jord_\rho(\sigma_{sp})$ is non-empty, let us first assume that there is $y \in Jord_\rho$ such that $y_- = c$.

Let us denote by σ_1 a discrete series subrepresentation of the induced representation $\delta([\nu^{-a+1}\rho, \nu^{(c_--1)/2}\rho]) \rtimes \sigma_{sp}$ which is also a subrepresentation of $\delta([\nu^{-(c_--1)/2}\rho, \nu^{y}\rho]) \rtimes \sigma'_{sp}$, for strongly positive representation σ'_{sp} such that $Jord(\sigma'_{sp}) = Jord(\sigma_{sp}) \setminus \{(y,\rho)\} \cup \{(2a-1,\rho)\}$. If an irreducible subquotient π' of σ_{ind} contains $\delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes \sigma_1$ in $\mu^*(\pi')$, then transitivity of Jacquet modules shows that the Jacquet module of π' with respect to an appropriate parabolic subgroup contains $\delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes \delta([\nu^{-(c_--1)/2}\rho, \nu^{(y-1)/2}\rho]) \otimes \sigma'_{sp}$. Thus, there is some irreducible representation δ such that $\mu^*(\pi') \geq \delta \otimes \sigma'_{sp}$ and $m^*(\delta) \geq \delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes \delta([\nu^{-(c_--1)/2}\rho, \nu^{(y-1)/2}\rho])$. Since $\pi' \leq \sigma_{ind}$ there are $-(c_-+1)/2 \leq i \leq j \leq (c-1)/2$ and an irreducible constituent $\delta' \otimes \pi''$ of $\mu^*(\sigma_{sp})$ such that

$$\delta \leq \delta([\nu^{-i}\rho, \nu^{(c_--1)/2}\rho]) \times \delta([\nu^{j+1}\rho, \nu^{(c-1)/2}\rho]) \times \delta'.$$

It directly follows that either $-i = -(c_- - 1)/2$ or $j+1 = -(c_- - 1)/2$. If $j+1 = -(c_- - 1)/2$, then $i = -(c_- + 1)/2$ and $\delta' \simeq \delta([\nu^a\rho, \nu^{(y-1)/2}\rho])$, which implies

$$\delta \simeq \delta([\nu^{-(c_--1)/2}\rho, \nu^{(c-1)/2}\rho]) \times \delta([\nu^a\rho, \nu^{(y-1)/2}\rho]).$$

If $-i = -(c_- - 1)/2$, we get $j+1 = (c_- + 1)/2$ and again $\delta' \simeq \delta([\nu^a\rho, \nu^{(y-1)/2}\rho])$. Thus, in this case δ is an irreducible subquotient of

$$\delta([\nu^{-(c_--1)/2}\rho, \nu^{(c_--1)/2}\rho]) \times \delta([\nu^{(c_-+1)/2}\rho, \nu^{(c-1)/2}\rho]) \times \delta([\nu^a\rho, \nu^{(y-1)/2}\rho]). \quad (3)$$

Since $\delta([\nu^{-(c_--1)/2}\rho, \nu^{(c-1)/2}\rho]) \times \delta([\nu^a\rho, \nu^{(y-1)/2}\rho])$ is an irreducible subquotient of the induced representation (3) and it can be easily deduced that $\delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes \delta([\nu^{-(c_--1)/2}\rho, \nu^{(y-1)/2}\rho])$ appears with multiplicity one in both $m^*(\delta([\nu^{-(c_--1)/2}\rho, \nu^{(c-1)/2}\rho]) \times \delta([\nu^a\rho, \nu^{(y-1)/2}\rho]))$ and $m^*(\delta([\nu^{-(c_--1)/2}\rho,$

$\nu^{(c_{-}-1)/2}\rho]) \times \delta([\nu^{(c_{-}+1)/2}\rho, \nu^{(c-1)/2}\rho]) \times \delta([\nu^{a}\rho, \nu^{(y-1)/2}\rho]))$ we again get $\delta \simeq \delta([\nu^{-(c_{-}-1)/2}\rho, \nu^{(c-1)/2}\rho]) \times \delta([\nu^{a}\rho, \nu^{(y-1)/2}\rho])$.

Lemma 4.1 shows $\pi' \in \{\sigma, \sigma'\}$. Further, by Proposition 7.2 of [17] $\pi' \simeq \sigma$ forces $\epsilon((c,\rho),(y,\rho)) = 1$. If $\epsilon((c,\rho),(y,\rho)) = 1$ then the definition of σ' shows $\epsilon'((c,\rho),(y,\rho)) = -1$ and, consequently, $2\ \delta([\nu^{a}\rho, \nu^{(c-1)/2}\rho]) \otimes \sigma_1 \leq \mu^{*}(\sigma)_{(a,c,\rho)}$. Otherwise, in the same way we conclude $2\ \delta([\nu^{a}\rho, \nu^{(c-1)/2}\rho]) \otimes \sigma_1 \leq \mu^{*}(\sigma')_{(a,c,\rho)}$.

We also denote by σ_2 a discrete series subrepresentation of the induced representation $\delta([\nu^{-a+1}\rho, \nu^{(c_{-}-1)/2}\rho]) \rtimes \sigma_{sp}$ different than σ_1. Using Lemma 2.1 we see that $\delta([\nu^{a}\rho, \nu^{(c-1)/2}\rho]) \otimes \sigma_2$ does not appear as an irreducible constituent of $\mu^{*}(\delta([\nu^{-(c-1)/2}\rho, \nu^{(y-1)/2}\rho]) \rtimes \sigma''_{sp})$ for strongly positive representation σ''_{sp} such that $Jord(\sigma'_{sp}) = Jord(\sigma_{sp}) \setminus \{(y,\rho)\} \cup \{(c_{-},\rho)\}$. Thus, if $\epsilon((c,\rho),(y,\rho)) = 1$, $\mu^{*}(\sigma)_{(a,c,\rho)}$ does not contain $\delta([\nu^{a}\rho, \nu^{(c-1)/2}\rho]) \otimes \sigma_2$, since σ is a subrepresentation of $\delta([\nu^{-(c-1)/2}\rho, \nu^{(y-1)/2}\rho]) \rtimes \sigma''_{sp}$. Analogously, if $\epsilon'((c,\rho),(y,\rho)) = 1$ then $\mu^{*}(\sigma')_{(a,c,\rho)}$ does not contain $\delta([\nu^{a}\rho, \nu^{(c-1)/2}\rho]) \otimes \sigma_2$ so Lemma 4.1 implies $2\ \delta([\nu^{a}\rho, \nu^{(c-1)/2}\rho]) \otimes \sigma_2 \leq \mu^{*}(\sigma)_{(a,c,\rho)}$.

Now we assume $c = \max\ (Jord_{\rho})$ and denote by z an element in $Jord_{\rho}$ such that $(c_{-})_{-} = z$.

Similarly as in the previously considered case, let us denote by σ_3 a discrete series subrepresentation of $\delta([\nu^{-a+1}\rho, \nu^{(c_{-}-1)/2}\rho]) \rtimes \sigma_{sp}$ which is also a subrepresentation of $\delta([\nu^{-(z-1)/2}\rho, \nu^{a-1}\rho]) \rtimes \sigma'_{sp}$, for strongly positive representation σ'_{sp} such that $Jord(\sigma'_{sp}) = Jord(\sigma_{sp}) \setminus \{(z,\rho)\} \cup \{(2a-1,\rho)\}$. Also, we denote by σ_4 a discrete series subrepresentation of $\delta([\nu^{-a+1}\rho, \nu^{(c_{-}-1)/2}\rho]) \rtimes \sigma_{sp}$ different than σ_3. If we let π' stand for an irreducible subquotient of σ_{ind} such that $\delta([\nu^{a}\rho, \nu^{(c-1)/2}\rho]) \otimes \sigma_3$ appears in $\mu^{*}(\pi')$, then it follows that Jacquet module of π' with respect to an appropriate parabolic subgroup contains

$$\delta([\nu^{a}\rho, \nu^{(c-1)/2}\rho]) \otimes \delta([\nu^{-(z-1)/2}\rho, \nu^{a-1}\rho]) \otimes \delta([\nu^{(z+1)/2}\rho, \nu^{(c_{-}-1)/2}\rho]) \otimes \tau,$$

for some irreducible representation τ. Consequently, there is an irreducible constituent $\delta' \otimes \tau$ appearing in $\mu^{*}(\pi')$ such that Jacquet module of δ with respect to an appropriate parabolic subgroup contains $\delta([\nu^{a}\rho, \nu^{(c-1)/2}\rho]) \otimes \delta([\nu^{-(z-1)/2}\rho, \nu^{a-1}\rho]) \otimes \delta([\nu^{(z+1)/2}\rho, \nu^{(c_{-}-1)/2}\rho])$. Again, there are $-(c_{-}+1)/2 \leq i \leq j \leq (c-1)/2$ and an irreducible constituent $\delta' \otimes \pi''$ of $\mu^{*}(\sigma_{sp})$ such that

$$\delta \leq \delta([\nu^{-i}\rho, \nu^{(c_{-}-1)/2}\rho]) \times \delta([\nu^{j+1}\rho, \nu^{(c-1)/2}\rho]) \times \delta'.$$

From cuspidal support of δ we obtain $-(z-1)/2 \in \{-i, j+1\}$ and $\delta' \simeq \delta([\nu^{(z+1)/2}\rho, \nu^{(c_{-}-1)/2}\rho])$. In the same manner as in the previously considered case we get $\delta \simeq \delta([\nu^{-(z-1)/2}\rho, \nu^{(c-1)/2}\rho]) \times \delta([\nu^{(z+1)/2}\rho, \nu^{(c_{-}-1)/2}\rho])$.

This allows us to conclude that $\epsilon((z,\rho),(c_-,\rho)) = 1$ implies $\mu^*(\sigma)_{(a,c,\rho)} \geq 2\ \delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes \sigma_3$ and $\mu^*(\sigma')_{(a,c,\rho)} \geq 2\ \delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes \sigma_4$. On the other hand, $\epsilon((z,\rho),(c_-,\rho)) = -1$ leads to $\mu^*(\sigma)_{(a,c,\rho)} \geq 2\ \delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes \sigma_4$ and $\mu^*(\sigma')_{(a,c,\rho)} \geq 2\ \delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes \sigma_3$. This proves the proposition. □

Proposition 4.7. *Suppose $a \geq 1$, $x = c_-$ and $a = (x_-+1)/2$. If $\epsilon((x_-,\rho),(c_-,\rho)) = 1$ then in $R(GL) \otimes R$ we have:*

$$\begin{aligned}\mu^*(\sigma)_{(a,c,\rho)} &= \delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-(c_--1)/2}\rho, \nu^{a-1}\rho]) \rtimes \sigma_{sp}) + \\ &\quad 2\ \delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes \tau_{temp},\end{aligned}$$

where τ_{temp} stands for a unique irreducible tempered subquotient of $\delta([\nu^{-a+1}\rho, \nu^{(c_--1)/2}\rho]) \rtimes \sigma_{sp}$.
If $\epsilon((x_-,\rho),(c_-,\rho)) = -1$ then in $R(GL) \otimes R$ we have:

$$\mu^*(\sigma)_{(a,c,\rho)} = \delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-(c_--1)/2}\rho, \nu^{a-1}\rho]) \rtimes \sigma_{sp}).$$

Proof. In $R(GL) \otimes R$ we have

$$\begin{aligned}\mu^*(\sigma_{ind})_{(a,c,\rho)} &= 2\ \delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-(c_--1)/2}\rho, \nu^{a-1}\rho]) \rtimes \sigma_{sp}) + \\ &\quad 2\ \delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes \tau_{temp},\end{aligned}$$

so it is enough to consider $\delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes \tau_{temp}$. Since τ_{temp} is a subrepresentation of $\delta([\nu^{-a+1}\rho, \nu^{(c_--1)/2}\rho]) \rtimes \sigma_{sp}$, using Frobenius reciprocity we conclude that an irreducible subquotient π of σ_{ind} such that $\delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes \tau_{temp} \leq \mu^*(\pi)$ also contains $\delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes \delta([\nu^{-a+1}\rho, \nu^{(c_--1)/2}\rho]) \otimes \sigma_{sp}$ in the Jacquet module with respect to an appropriate parabolic subgroup. Therefore, there is an irreducible constituent $\delta \otimes \sigma_{sp}$ of $\mu^*(\pi)$ such that $m^*(\delta)$ contains $\delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes \delta([\nu^{-a+1}\rho, \nu^{(c_--1)/2}\rho])$.

Using the same procedure as in the proof of previous proposition, we get $\delta \leq \delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \times \delta([\nu^{-a+1}\rho, \nu^{(c_--1)/2}\rho])$.

Since $\delta([\nu^a\rho, \nu^{(c_--1)/2}\rho]) \times \delta([\nu^{-a+1}\rho, \nu^{(c-1)/2}\rho])$ is an irreducible subquotient of $\delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \times \delta([\nu^{-a+1}\rho, \nu^{(c_--1)/2}\rho])$ and multiplicity of $\delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes \delta([\nu^{-a+1}\rho, \nu^{(c_--1)/2}\rho])$ equals two in both $m^*(\delta([\nu^a\rho, \nu^{(c_--1)/2}\rho]) \times \delta([\nu^{-a+1}\rho, \nu^{(c-1)/2}\rho]))$ and $m^*(\delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \times \delta([\nu^{-a+1}\rho, \nu^{(c_--1)/2}\rho]))$, we conclude

$$\delta \simeq \delta([\nu^a\rho, \nu^{(c_--1)/2}\rho]) \times \delta([\nu^{-a+1}\rho, \nu^{(c-1)/2}\rho]).$$

Using Proposition 7.2 of [17] and $\epsilon((x_-,\rho),(c_-,\rho)) \cdot \epsilon'((x_-,\rho),(c_-,\rho)) = -1$, we deduce that $\mu^*(\sigma)$ contains $\delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes \tau_{temp}$ if and only if $\epsilon((x_-,\rho),(c_-,\rho)) = 1$ and it directly follows that $\mu^*(\sigma)$ contains either both or zero copies of $\delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes \tau_{temp}$. This ends the proof. □

Proposition 4.8. *Suppose $a \geq 1$, $x \neq c_-$ and $a = (x_- + 1)/2$. If $\epsilon(((c_-)_-, \rho), (c_-,\rho)) = 1$ then in $R(GL) \otimes R$ we have:*

$$\begin{aligned}\mu^*(\sigma)_{(a,c,\rho)} &= \delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-(c_- -1)/2}\rho, \nu^{a-1}\rho]) \rtimes \sigma_{sp}) + \\ & \quad 2\ \delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-((c_-)_- -1)/2}\rho, \nu^{a-1}\rho]) \rtimes \sigma'_{sp}),\end{aligned}$$

where σ'_{sp} denotes a strongly positive discrete series such that $Jord(\sigma'_{sp}) = Jord(\sigma_{sp}) \setminus \{((c_-)_-, \rho)\} \cup \{(c_-,\rho)\}$.
If $\epsilon(((c_-)_-, \rho), (c_-,\rho)) = -1$ then in $R(GL) \otimes R$ we have:

$$\mu^*(\sigma)_{(a,c,\rho)} = \delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-(c_- -1)/2}\rho, \nu^{a-1}\rho]) \rtimes \sigma_{sp}).$$

Proof. Since the following equality holds in $R(GL) \otimes R$:

$$\begin{aligned}\mu^*(\sigma_{ind})_{(a,c,\rho)} &= 2\ \delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-(c_- -1)/2}\rho, \nu^{a-1}\rho]) \rtimes \sigma_{sp}) + \\ & \quad 2\ \delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-((c_-)_- -1)/2}\rho, \nu^{a-1}\rho]) \rtimes \sigma'_{sp}),\end{aligned}$$

we discuss only the irreducible constituent $\delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-((c_-)_- -1)/2}\rho, \nu^{a-1}\rho]) \rtimes \sigma'_{sp})$. First, we note the following embeddings and isomorphism:

$$L(\delta([\nu^{-((c_-)_- -1)/2}\rho, \nu^{a-1}\rho]) \rtimes \sigma'_{sp}) \hookrightarrow \delta([\nu^{-((c_-)_- -1)/2}\rho, \nu^{a-1}\rho]) \rtimes \sigma'_{sp}$$

$$\begin{aligned}&\hookrightarrow \delta([\nu^{-((c_-)_- -1)/2}\rho, \nu^{a-1}\rho]) \times \delta([\nu^{((c_-)_- +1)/2}\rho, \nu^{(c_- -1)/2}\rho]) \rtimes \sigma_{sp} \\ &\simeq \delta([\nu^{((c_-)_- +1)/2}\rho, \nu^{(c_- -1)/2}\rho]) \times \delta([\nu^{-((c_-)_- -1)/2}\rho, \nu^{a-1}\rho]) \rtimes \sigma_{sp}.\end{aligned}$$

In consequence, if π is an irreducible subquotient of σ_{ind} such that $\mu^*(\pi)$ contains $\delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-((c_-)_- -1)/2}\rho, \nu^{a-1}\rho]) \rtimes \sigma'_{sp})$, then there is some irreducible representation π' such that $\delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes \delta([\nu^{((c_-)_- +1)/2}\rho, \nu^{(c_- -1)/2}\rho]) \otimes \pi'$ is contained in Jacquet module of π with respect to an appropriate parabolic subgroup. Using the structural formula for $\mu^*(\sigma_{ind})$ we conclude that such representation π contains the irreducible representation

$$\delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \times \delta([\nu^{((c_-)_- +1)/2}\rho, \nu^{(c_- -1)/2}\rho]) \otimes \pi'$$

in its Jacquet module with respect to an appropriate maximal parabolic subgroup. Now the rest of the proof runs as before. □

Proposition 4.9. *Suppose $a \geq 1$, $x \neq c_-$ and $a \neq (x_- + 1)/2$. We denote $(c_-)_-$ by y and suppose $x = y$. If $\epsilon((x,\rho),(c_-,\rho)) = 1$ then the following equality holds in $R(GL) \otimes R$:*

$$\begin{aligned} \mu^*(\sigma)_{(a,c,\rho)} &= \delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-(c_--1)/2}\rho, \nu^{a-1}\rho]) \rtimes \sigma_{sp}) + \\ &\quad 2\ \delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-(c_--1)/2}\rho, \nu^{(x-1)/2}\rho]) \rtimes \sigma_{sp}^{(1)}) + \\ &\quad 2\ \delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-(x-1)/2}\rho, \nu^{a-1}\rho]) \rtimes \sigma_{sp}^{(2)}) + \\ &\quad 4\ \delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes \sigma_1 + \delta([\nu^a\rho, \nu^{(c_--1)/2}\rho]) \otimes \sigma_2, \end{aligned}$$

where $\sigma_{sp}^{(1)}$ denotes a strongly positive discrete series such that $Jord(\sigma_{sp}^{(1)}) = Jord(\sigma_{sp}) \setminus \{(x,\rho)\} \cup \{(2a-1,\rho)\}$, $\sigma_{sp}^{(2)}$ denotes a strongly positive discrete series such that $Jord(\sigma_{sp}^{(2)}) = Jord(\sigma_{sp}) \setminus \{(x,\rho)\} \cup \{(c_-,\rho)\}$, while σ_1 and σ_2 are mutually non-isomorphic discrete series subrepresentations of $\delta([\nu^{-(x-1)/2}\rho, \nu^{(c_--1)/2}\rho]) \rtimes \sigma_{sp}^{(1)}$ and σ_1 is also a subrepresentation of $\delta([\nu^{-a+1}\rho, \nu^{(x-1)/2}\rho]) \rtimes \sigma_{sp}^{(2)}$.
If $\epsilon((x,\rho),(c_-,\rho)) = -1$ then the following equality holds in $R(GL) \otimes R$:

$$\begin{aligned} \mu^*(\sigma)_{(a,c,\rho)} &= \delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-(c_--1)/2}\rho, \nu^{a-1}\rho]) \rtimes \sigma_{sp}) + \\ &\quad 2\ \delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-(c_--1)/2}\rho, \nu^{(x-1)/2}\rho]) \rtimes \sigma_{sp}^{(1)}) + \\ &\quad \delta([\nu^a\rho, \nu^{(c_--1)/2}\rho]) \otimes \sigma_2, \end{aligned}$$

for $\sigma_{sp}^{(1)}$ and σ_2 as above.

Proof. First, in $R(GL) \otimes R$ we have (we note that here is also used Proposition 3.2 of [8])

$$\begin{aligned} \mu^*(\sigma_{ind})_{(a,c,\rho)} &= 2\ \delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-(c_--1)/2}\rho, \nu^{a-1}\rho]) \rtimes \sigma_{sp}) + \\ &\quad 4\ \delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-(c_--1)/2}\rho, \nu^{(x-1)/2}\rho]) \rtimes \sigma_{sp}^{(1)}) + \\ &\quad 2\ \delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-(x-1)/2}\rho, \nu^{a-1}\rho]) \rtimes \sigma_{sp}^{(2)}) + \\ &\quad 4\ \delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes \sigma_1 + 2\ \delta([\nu^a\rho, \nu^{(c_--1)/2}\rho]) \otimes \sigma_2. \end{aligned}$$

Again, we have already seen that $\delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-(c_--1)/2}\rho, \nu^{a-1}\rho]) \rtimes \sigma_{sp})$ appears in both $\mu^*(\sigma)$ and $\mu^*(\sigma')$ with multiplicity one.

It directly follows that both σ and σ' are irreducible subrepresentations of the induced representation $\delta([\nu^{-(c_--1)/2}\rho, \nu^{(c-1)/2}\rho]) \times \delta([\nu^a\rho, \nu^{(x-1)/2}\rho]) \rtimes \sigma_{sp}^{(1)}$. It can be easily seen that $\delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes \delta([\nu^{-(c_--1)/2}\rho, \nu^{(x-1)/2}\rho])$

appears in $m^*(\delta([\nu^{-(c_--1)/2}\rho, \nu^{(c-1)/2}\rho]) \times \delta([\nu^a\rho, \nu^{(x-1)/2}\rho]))$ with multiplicity two. Now transitivity of Jacquet modules shows that there is some irreducible constituent $\delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes \pi'$ appearing in both $\mu^*(\sigma)$ and $\mu^*(\sigma')$ such that $\mu^*(\pi') \geq \delta([\nu^{-(c_--1)/2}\rho, \nu^{(x-1)/2}\rho]) \otimes \sigma_{sp}^{(1)}$.

Description of $\mu^*(\sigma_{ind})_{(a,c,\rho)}$ forces $\pi' \simeq L(\delta([\nu^{-(c_--1)/2}\rho, \nu^{(x-1)/2}\rho]) \rtimes \sigma_{sp}^{(1)})$ and, since $\mu^*(L(\delta([\nu^{-(c_--1)/2}\rho, \nu^{(x-1)/2}\rho]) \rtimes \sigma_{sp}^{(1)}))$ contains $\delta([\nu^{-(c_--1)/2}\rho, \nu^{(x-1)/2}\rho]) \otimes \sigma_{sp}^{(1)}$ with multiplicity one, $\delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-(c_--1)/2}\rho, \nu^{(x-1)/2}\rho]) \rtimes \sigma_{sp}^{(1)})$ appears in both $\mu^*(\sigma)$ and $\mu^*(\sigma')$ with multiplicity two.

On the other hand, $L(\delta([\nu^{-(x-1)/2}\rho, \nu^{a-1}\rho]) \rtimes \sigma_{sp}^{(2)})$ is a subrepresentation of

$$\delta([\nu^{-(x-1)/2}\rho, \nu^{a-1}\rho]) \times \delta([\nu^{(x+1)/2}\rho, \nu^{(c_--1)/2}\rho]) \rtimes \sigma_{sp}^{(1)}.$$

Since $a-1 < (x-1)/2$, Frobenius reciprocity implies that $\mu^*(L(\delta([\nu^{-(x-1)/2}\rho, \nu^{a-1}\rho]) \rtimes \sigma_{sp}^{(2)}))$ contains $\delta([\nu^{-(x-1)/2}\rho, \nu^{a-1}\rho]) \times \delta([\nu^{(x+1)/2}\rho, \nu^{(c_--1)/2}\rho]) \otimes \sigma_{sp}^{(1)}$. Consequently, if π is an irreducible subquotient of σ_{ind} such that $\delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-(x-1)/2}\rho, \nu^{a-1}\rho]) \rtimes \sigma_{sp}^{(2)})$ is contained in $\mu^*(\pi)$, then Jacquet module of π with respect to an appropriate parabolic subgroup contains $\delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes \delta([\nu^{-(x-1)/2}\rho, \nu^{a-1}\rho]) \times \delta([\nu^{(x+1)/2}\rho, \nu^{(c_--1)/2}\rho]) \otimes \sigma_{sp}^{(1)}$. Therefore, there is some irreducible constituent $\delta \otimes \sigma_{sp}^{(1)}$ of $\mu^*(\pi)$ such that $m^*(\delta)$ contains $\delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes \delta([\nu^{-(x-1)/2}\rho, \nu^{a-1}\rho]) \times \delta([\nu^{(x+1)/2}\rho, \nu^{(c_--1)/2}\rho])$. From $\mu^*(\sigma_{ind})$ we obtain that δ is an irreducible subquotient of $\delta([\nu^{-(x-1)/2}\rho, \nu^{(c_--1)/2}\rho]) \times \delta([\nu^{(x+1)/2}\rho, \nu^{(c-1)/2}\rho])$ and, in a standard way, we conclude

$$\delta \simeq \delta([\nu^{-(x-1)/2}\rho, \nu^{(c-1)/2}\rho]) \times \delta([\nu^{(x+1)/2}\rho, \nu^{(c_--1)/2}\rho]).$$

Proposition 7.2 of [17] and definition of representations σ and σ' show that $\mu^*(\sigma)$ contains $\delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-(x-1)/2}\rho, \nu^{a-1}\rho]) \rtimes \sigma_{sp}^{(2)})$ if and only if $\epsilon((x, \rho), (c_-, \rho)) = 1$.

Now we consider irreducible constituent $\delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes \sigma_1$. From Proposition 3.2 of [8] we obtain the following embeddings and isomorphism:

$$\begin{aligned} \sigma_1 &\hookrightarrow \delta([\nu^{-a+1}\rho, \nu^{(c_--1)/2}\rho]) \rtimes \sigma_{sp}^{(2)} \\ &\hookrightarrow \delta([\nu^{-a+1}\rho, \nu^{(c_--1)/2}\rho]) \times \delta([\nu^a\rho, \nu^{(x-1)/2}\rho]) \rtimes \sigma_{sp}^{(1)} \\ &\simeq \delta([\nu^a\rho, \nu^{(x-1)/2}\rho]) \times \delta([\nu^{-a+1}\rho, \nu^{(c_--1)/2}\rho]) \rtimes \sigma_{sp}^{(1)} \\ &\hookrightarrow \delta([\nu^a\rho, \nu^{(x-1)/2}\rho]) \times \delta([\nu^a\rho, \nu^{(c_--1)/2}\rho]) \times \delta([\nu^{-a+1}\rho, \nu^{a-1}\rho]) \rtimes \sigma_{sp}^{(1)}. \end{aligned}$$

Since the representation $\delta([\nu^a\rho, \nu^{(x-1)/2}\rho]) \times \delta([\nu^a\rho, \nu^{(c_--1)/2}\rho])$ is irreducible, by Lemma 3.2 of [11], there is an irreducible representation τ such that σ_1 is a subrepresentation of $\delta([\nu^a\rho, \nu^{(x-1)/2}\rho]) \times \delta([\nu^a\rho, \nu^{(c_--1)/2}\rho]) \rtimes \tau$ and Frobenius reciprocity yields $\mu^*(\sigma_1) \geq \delta([\nu^a\rho, \nu^{(x-1)/2}\rho]) \times \delta([\nu^a\rho, \nu^{(c_--1)/2}\rho]) \otimes \tau$. In consequence, for an irreducible subquotient π of σ_{ind} such that $\mu^*(\pi) \geq \delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes \sigma_1$ there is some irreducible constituent $\delta \otimes \tau$ of $\mu^*(\pi)$ such that Jacquet module of δ with respect to an appropriate parabolic subgroup contains $\delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes \delta([\nu^a\rho, \nu^{(x-1)/2}\rho]) \times \delta([\nu^a\rho, \nu^{(c_--1)/2}\rho])$.

It follows at once from the cuspidal support of δ and structural formula for $\mu^*(\sigma_{ind})$ that δ is isomorphic to $\delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \times \delta([\nu^a\rho, \nu^{(x-1)/2}\rho]) \times \delta([\nu^a\rho, \nu^{(c_--1)/2}\rho])$. In the same way as before we conclude that $\mu^*(\sigma)$ contains $\delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes \sigma_1$ if and only if $\epsilon((x, \rho), (c_-, \rho)) = 1$.

What is left is to show that $\delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes \sigma_2$ appears in both $\mu^*(\sigma)$ and $\mu^*(\sigma')$.

In the previous subsection we have seen that $\delta([\nu^{-(x-1)/2}\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-(c_--1)/2}\rho, \nu^{-(x+1)/2}\rho]) \rtimes \sigma_{sp})$ appears in both $\mu^*(\sigma)$ and $\mu^*(\sigma')$. Further, we have the following embeddings and isomorphisms (note that the representation $\delta([\nu^{-(c_--1)/2}\rho, \nu^{-(x+1)/2}\rho]) \rtimes \sigma_{sp}^{(1)}$ is irreducible by the results of [12]):

$$\begin{aligned} L(\delta([\nu^{-(c_--1)/2}\rho, \nu^{-(x+1)/2}\rho]) \rtimes \sigma_{sp}) &\hookrightarrow \delta([\nu^{-(c_--1)/2}\rho, \nu^{-(x+1)/2}\rho]) \rtimes \sigma_{sp} \\ &\hookrightarrow \delta([\nu^{-(c_--1)/2}\rho, \nu^{-(x+1)/2}\rho]) \times \delta([\nu^a\rho, \nu^{(x-1)/2}\rho]) \rtimes \sigma_{sp}^{(1)} \\ &\simeq \delta([\nu^a\rho, \nu^{(x-1)/2}\rho]) \times \delta([\nu^{-(c_--1)/2}\rho, \nu^{-(x+1)/2}\rho]) \rtimes \sigma_{sp}^{(1)} \\ &\simeq \delta([\nu^a\rho, \nu^{(x-1)/2}\rho]) \times \delta([\nu^{(x+1)/2}\rho, \nu^{(c_--1)/2}\rho]) \rtimes \sigma_{sp}^{(1)}. \end{aligned}$$

Further, from description of the composition series of induced representation $\delta([\nu^{-(c_--1)/2}\rho, \nu^{-(x+1)/2}\rho]) \rtimes \sigma_{sp}$ given in Proposition 3.1 of [12], we deduce that there is no irreducible constituent of the form $\delta([\nu^{(x+1)/2}\rho, \nu^{(c_--1)/2}\rho]) \otimes \pi$ appearing in $\mu^*(L(\delta([\nu^{-(c_--1)/2}\rho, \nu^{-(x+1)/2}\rho]) \rtimes \sigma_{sp}))$. Thus, $L(\delta([\nu^{-(c_--1)/2}\rho, \nu^{-(x+1)/2}\rho]) \rtimes \sigma_{sp})$ is contained in the kernel of an intertwining operator

$$\begin{aligned} \delta([\nu^a\rho, \nu^{(x-1)/2}\rho]) \times \delta([\nu^{(x+1)/2}\rho, \nu^{(c_--1)/2}\rho]) \rtimes \sigma_{sp}^{(1)} &\to \\ \delta([\nu^{(x+1)/2}\rho, \nu^{(c_--1)/2}\rho]) \times \delta([\nu^a\rho, \nu^{(x-1)/2}\rho]) \rtimes \sigma_{sp}^{(1)}&, \end{aligned}$$

which is, according to [18], isomorphic to

$$L(\delta([\nu^a\rho, \nu^{(x-1)/2}\rho]) \times \delta([\nu^{(x+1)/2}\rho, \nu^{(c_--1)/2}\rho])) \rtimes \sigma_{sp}^{(1)}.$$

In this way we conclude that both representations σ and σ' contain irreducible representation

$$\delta([\nu^{-(x-1)/2}\rho,\nu^{(c-1)/2}\rho])\otimes L(\delta([\nu^{a}\rho,\nu^{(x-1)/2}\rho])\times\delta([\nu^{(x+1)/2}\rho,\nu^{(c_{-}-1)/2}\rho]))\otimes\sigma_{sp}^{(1)}$$

in Jacquet module with respect to an appropriate parabolic subgroup.

If $\delta\otimes\sigma_{sp}^{(1)}$ is an irreducible constituent of $\mu^*(\sigma)$ or $\mu^*(\sigma')$ such that $m^*(\delta)$ contains $\delta([\nu^{-(x-1)/2}\rho,\nu^{(c-1)/2}\rho])\otimes L(\delta([\nu^{a}\rho,\nu^{(x-1)/2}\rho])\times\delta([\nu^{(x+1)/2}\rho,\nu^{(c_{-}-1)/2}\rho]))$, in the same fashion as before we get that δ is an irreducible subquotient of

$$\delta([\nu^{-(x-1)/2}\rho,\nu^{(c_{-}-1)/2}\rho])\times\delta([\nu^{(c_{-}+1)/2}\rho,\nu^{(c-1)/2}\rho])\times$$
$$\delta([\nu^{a}\rho,\nu^{(x-1)/2}\rho])\times\delta([\nu^{(x+1)/2}\rho,\nu^{(c_{-}-1)/2}\rho]).$$

It is not hard to see that the unique irreducible subquotient of this induced representation which contains $\delta([\nu^{-(x-1)/2}\rho,\nu^{(c-1)/2}\rho])\otimes L(\delta([\nu^{a}\rho,\nu^{(x-1)/2}\rho])\times\delta([\nu^{(x+1)/2}\rho,\nu^{(c_{-}-1)/2}\rho]))$ in Jacquet module with respect to an appropriate parabolic subgroup is isomorphic to

$$\delta([\nu^{-(x-1)/2}\rho,\nu^{(c-1)/2}\rho])\times L(\delta([\nu^{a}\rho,\nu^{(x-1)/2}\rho])\times\delta([\nu^{(x+1)/2}\rho,\nu^{(c_{-}-1)/2}\rho]))$$

(which is irreducible by Lemma 1.3.3 of [2]). Thus, $\delta([\nu^{-(x-1)/2}\rho,\nu^{(c-1)/2}\rho])\times L(\delta([\nu^{a}\rho,\nu^{(x-1)/2}\rho])\times\delta([\nu^{(x+1)/2}\rho,\nu^{(c_{-}-1)/2}\rho]))\otimes\sigma_{sp}^{(1)}$ appears in both $\mu^*(\sigma)$ and $\mu^*(\sigma')$. By the structural formula, such irreducible constituent appears with multiplicity two in $\mu^*(\sigma_{ind})$, so it appears with multiplicity one in both $\mu^*(\sigma)$ and $\mu^*(\sigma')$.

Using [18] and Lemma 1.3.3 of [2], we can assert that in an appropriate Grothendieck group holds

$$\delta([\nu^{-(x-1)/2}\rho,\nu^{(c-1)/2}\rho])\times\delta([\nu^{a}\rho,\nu^{(x-1)/2}\rho])\times\delta([\nu^{(x+1)/2}\rho,\nu^{(c_{-}-1)/2}\rho])=$$
$$\delta([\nu^{-(x-1)/2}\rho,\nu^{(c-1)/2}\rho])\times\delta([\nu^{a}\rho,\nu^{(c_{-}-1)/2}\rho])+$$
$$\delta([\nu^{-(x-1)/2}\rho,\nu^{(c-1)/2}\rho])\times L(\delta([\nu^{a}\rho,\nu^{(x-1)/2}\rho])\times\delta([\nu^{(x+1)/2}\rho,\nu^{(c_{-}-1)/2}\rho])).$$

Since $\delta([\nu^{a}\rho,\nu^{(c-1)/2}\rho])\otimes\delta([\nu^{-(x-1)/2}\rho,\nu^{(c_{-}-1)/2}\rho])$ appears with multiplicity three in $m^*(\delta([\nu^{-(x-1)/2}\rho,\nu^{(c-1)/2}\rho])\times\delta([\nu^{a}\rho,\nu^{(x-1)/2}\rho])\times\delta([\nu^{(x+1)/2}\rho,\nu^{(c_{-}-1)/2}\rho]))$ and with multiplicity two in $m^*(\delta([\nu^{-(x-1)/2}\rho,\nu^{(c-1)/2}\rho])\times\delta([\nu^{a}\rho,\nu^{(c_{-}-1)/2}\rho]))$ it follows that $\delta([\nu^{a}\rho,\nu^{(c-1)/2}\rho])\otimes\delta([\nu^{-(x-1)/2}\rho,\nu^{(c_{-}-1)/2}\rho])$ appears with multiplicity one in

$$m^*(\delta([\nu^{-(x-1)/2}\rho,\nu^{(c-1)/2}\rho])\times L(\delta([\nu^{a}\rho,\nu^{(x-1)/2}\rho])\times\delta([\nu^{(x+1)/2}\rho,\nu^{(c_{-}-1)/2}\rho]))).$$

Using the structural formula we also obtain that if $\delta \otimes \sigma_{sp}^{(1)}$ is an irreducible constituent of $\mu^*(\sigma_{ind})$ such that $m^*(\delta) \geq \delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes \delta([\nu^{-(x-1)/2}\rho, \nu^{(c_--1)/2}\rho])$, then δ is either isomorphic to $\delta([\nu^{-(x-1)/2}\rho, \nu^{(c-1)/2}\rho]) \times\delta([\nu^a\rho, \nu^{(c_--1)/2}\rho])$ or to $\delta([\nu^{-(x-1)/2}\rho, \nu^{(c-1)/2}\rho]) \times L(\delta([\nu^a\rho, \nu^{(x-1)/2}\rho])\times \delta([\nu^{(x+1)/2}\rho, \nu^{(c_--1)/2}\rho]))$.

Transitivity of Jacquet modules shows that for $\pi \in \{\sigma, \sigma'\}$ there is an irreducible constituent $\delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes \pi'$ of $\mu^*(\pi)$ such that $\mu^*(\pi')$ contains $\delta([\nu^{-(x-1)/2}\rho, \nu^{(c_--1)/2}\rho]) \otimes \sigma_{sp}^{(1)}$. Description of $\mu^*(\sigma_{ind})_{(a,c,\rho)}$ given in the beginning of the proof leads to $\pi' \in \{\sigma_1, \sigma_2\}$.

Let us denote by τ an element of the set $\{\sigma, \sigma'\}$ which is not an irreducible subrepresentation of $\delta([\nu^{-(x-1)/2}\rho, \nu^{(c_--1)/2}\rho]) \rtimes \sigma'_{sp}$, where σ'_{sp} denotes a strongly positive discrete series such that $Jord(\sigma'_{sp}) = Jord(\sigma_{sp}) \setminus \{(x,\rho)\} \cup \{(c,\rho)\}$. Then, as we have already proved, $\delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes \sigma_1$ does not appear in $\mu^*(\tau)$, and hence $\delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes \sigma_2 \leq \mu^*(\tau)$. Also, such irreducible constituent appears in $\mu^*(\tau)$ with multiplicity one, since otherwise $\delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes \delta([\nu^{-(x-1)/2}\rho, \nu^{(c_--1)/2}\rho]) \otimes \sigma_{sp}^{(1)}$ would appear in Jacquet module of τ with respect to an appropriate parabolic subgroup with multiplicity two and, consequently, $\delta([\nu^{-(x-1)/2}\rho, \nu^{(c-1)/2}\rho]) \times L(\delta([\nu^a\rho, \nu^{(x-1)/2}\rho]) \times \delta([\nu^{(x+1)/2}\rho, \nu^{(c_--1)/2}\rho])) \otimes \sigma_{sp}^{(1)}$ would appear in $\mu^*(\tau)$ with multiplicity two, which is impossible.

Therefore, $\delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes \sigma_2$ appears in both $\mu^*(\sigma)$ and $\mu^*(\sigma')$ with multiplicity one and the proposition is proved. $\square$

Proposition 4.10. *Suppose $a \geq 1$, $x \neq c_-$ and $a \neq (x_- + 1)/2$. We denote $(c_-)_-$ by y and suppose $x = y_-$. If $\epsilon((y,\rho),(c_-,\rho)) = 1$ then the following equality holds in $R(GL) \otimes R$:*

$$\begin{aligned}
\mu^*(\sigma)_{(a,c,\rho)} = \; & \delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-(c_--1)/2}\rho, \nu^{a-1}\rho]) \rtimes \sigma_{sp}) + \\
& 2\; \delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-(c_--1)/2}\rho, \nu^{(x-1)/2}\rho]) \rtimes \sigma_{sp}^{(1)}) + \\
& 2\; \delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-(y-1)/2}\rho, \nu^{a-1}\rho]) \rtimes \sigma_{sp}^{(2)}) + \\
& 4\; \delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-(y-1)/2}\rho, \nu^{(x-1)/2}\rho]) \rtimes \sigma_{sp}^{(3)}) + \\
& \delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-(c_--1)/2}\rho, \nu^{(y-1)/2}\rho]) \rtimes \sigma_{sp}^{(4)}) + \\
& 2\; \delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes \sigma_1,
\end{aligned}$$

where $\sigma_{sp}^{(1)}$ denotes a strongly positive discrete series such that $Jord(\sigma_{sp}^{(1)}) = Jord(\sigma_{sp}) \setminus \{(x,\rho)\} \cup \{(2a-1,\rho)\}$, $\sigma_{sp}^{(2)}$ denotes a strongly positive dis-

crete series such that $Jord(\sigma_{sp}^{(2)}) = Jord(\sigma_{sp}) \setminus \{(y,\rho)\} \cup \{(c_-,\rho)\}$*,* $\sigma_{sp}^{(3)}$ *denotes a strongly positive discrete series such that* $Jord(\sigma_{sp}^{(3)}) = Jord(\sigma_{sp}^{(2)}) \setminus \{(x,\rho)\} \cup \{(2a-1,\rho)\}$*,* $\sigma_{sp}^{(4)}$ *denotes a strongly positive discrete series such that* $Jord(\sigma_{sp}^{(4)}) = Jord(\sigma_{sp}) \setminus \{(y,\rho)\} \cup \{(2a-1,\rho)\}$*, while* σ_1 *denotes unique discrete series subrepresentation of* $\delta([\nu^{-(x-1)/2}\rho, \nu^{(c_--1)/2}\rho]) \rtimes \sigma_{sp}^{(1)}$*.*
If $\epsilon((y,\rho),(c_-,\rho)) = -1$ *then the following equality holds in* $R(GL) \otimes R$*:*

$$\begin{aligned}\mu^*(\sigma)_{(a,c,\rho)} \;=\;& \delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-(c_--1)/2}\rho, \nu^{a-1}\rho]) \rtimes \sigma_{sp}) + \\ & 2\ \delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-(c_--1)/2}\rho, \nu^{(x-1)/2}\rho]) \rtimes \sigma_{sp}^{(1)}) + \\ & \delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-(c_--1)/2}\rho, \nu^{(y-1)/2}\rho]) \rtimes \sigma_{sp}^{(4)}),\end{aligned}$$

for $\sigma_{sp}^{(1)}$ *and* $\sigma_{sp}^{(4)}$ *as above.*

Proof. We start with the following equality (again, Proposition 3.2 of [8] is used to obtained σ_1):

$$\begin{aligned}\mu^*(\sigma_{ind})_{(a,c,\rho)} \;=\;& 4\ \delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-(c_--1)/2}\rho, \nu^{a-1}\rho]) \rtimes \sigma_{sp}) + \\ & 4\ \delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-(c_--1)/2}\rho, \nu^{(x-1)/2}\rho]) \rtimes \sigma_{sp}^{(1)}) + \\ & 2\ \delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-(y-1)/2}\rho, \nu^{a-1}\rho]) \rtimes \sigma_{sp}^{(2)}) + \\ & 2\ \delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-(y-1)/2}\rho, \nu^{(x-1)/2}\rho]) \rtimes \sigma_{sp}^{(3)}) + \\ & 2\ \delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-(c_--1)/2}\rho, \nu^{(y-1)/2}\rho]) \rtimes \sigma_{sp}^{(4)}) + \\ & 2\ \delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes \sigma_1.\end{aligned}$$

In Proposition 4.2 we have seen that $\delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-(c_--1)/2}\rho, \nu^{a-1}\rho]) \rtimes \sigma_{sp})$ appears in both $\mu^*(\sigma)$ and $\mu^*(\sigma')$ with multiplicity one.

For $\pi \in \{L(\delta([\nu^{-(y-1)/2}\rho, \nu^{a-1}\rho]) \rtimes \sigma_{sp}^{(2)}), L(\delta([\nu^{-(y-1)/2}\rho, \nu^{(x-1)/2}\rho]) \rtimes \sigma_{sp}^{(3)}), \sigma_1\}$ it can be directly seen that $\mu^*(\pi)$ contains an irreducible constituent of the form $\delta([\nu^{(y+1)/2}\rho, \nu^{(c_--1)/2}\rho]) \otimes \pi'$, for some irreducible representation π'. Thus, if $\delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes \pi$ appears in $\mu^*(\tau)$ for $\pi \in \{L(\delta([\nu^{-(y-1)/2}\rho, \nu^{a-1}\rho]) \rtimes \sigma_{sp}^{(2)}), L(\delta([\nu^{-(y-1)/2}\rho, \nu^{(x-1)/2}\rho]) \rtimes \sigma_{sp}^{(3)}), \sigma_1\}$ and an irreducible subquotient τ of σ_{ind}, it follows that there is an irreducible constituent $\delta \otimes \pi'$ of $\mu^*(\tau)$ such that $m^*(\delta)$ contains $\delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes \delta([\nu^{(y+1)/2}\rho, \nu^{(c_--1)/2}\rho])$. From structural formula for $\mu^*(\sigma_{ind})$ we obtain that δ is an irreducible subquotient of some of the following representations:

$$\delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \times \delta([\nu^{(y+1)/2}\rho, \nu^{(c_--1)/2}\rho]),$$

$$\delta([\nu^a\rho,\nu^{(c_--1)/2}\rho])\times\delta([\nu^{(c_--1)/2}\rho,\nu^{(c-1)/2}\rho])\times\delta([\nu^{(y+1)/2}\rho,\nu^{(c_--1)/2}\rho]),$$
$$\delta([\nu^a\rho,\nu^{(x-1)/2}\rho])\times\delta([\nu^{(x+1)/2}\rho,\nu^{(c-1)/2}\rho])\times\delta([\nu^{(y+1)/2}\rho,\nu^{(c_--1)/2}\rho]),$$
$$\delta([\nu^a\rho,\nu^{(x-1)/2}\rho])\times\delta([\nu^{(x+1)/2}\rho,\nu^{(c_--1)/2}\rho])\times\delta([\nu^{(y+1)/2}\rho,\nu^{(c-1)/2}\rho]).$$

Since the first of these representation is an irreducible subquotient of other three and $\delta([\nu^a\rho,\nu^{(c-1)/2}\rho])\otimes\delta([\nu^{(y+1)/2}\rho,\nu^{(c_--1)/2}\rho])$ appears with multiplicity one in Jacquet module with respect to an appropriate parabolic subgroup of each of these four representations, we deduce

$$\delta\simeq\delta([\nu^a\rho,\nu^{(c-1)/2}\rho])\times\delta([\nu^{(y+1)/2}\rho,\nu^{(c_--1)/2}\rho]).$$

Thus, in the same way as before we may conclude that $\delta([\nu^a\rho,\nu^{(c-1)/2}\rho])\otimes\pi$ appears in $\mu^*(\sigma)$ for $\pi\in\{L(\delta([\nu^{-(y-1)/2}\rho,\nu^{a-1}\rho])\rtimes\sigma_{sp}^{(2)}),L(\delta([\nu^{-(y-1)/2}\rho,\nu^{(x-1)/2}\rho])\rtimes\sigma_{sp}^{(3)}),\sigma_1\}$ if and only if $\epsilon((y,\rho),(c_-,\rho))=1$.

One the other hand, both discrete series representations σ and σ' are subrepresentations of

$$\delta([\nu^{-(c_--1)/2}\rho,\nu^{(c-1)/2}\rho])\times\delta([\nu^a\rho,\nu^{(x-1)/2}\rho])\rtimes\sigma_{sp}^{(1)}.$$

Irreducibility of $\delta([\nu^{-(c_--1)/2}\rho,\nu^{(c-1)/2}\rho])\times\delta([\nu^a\rho,\nu^{(x-1)/2}\rho])$ and Frobenius reciprocity yield that Jacquet module of σ with respect to an appropriate parabolic subgroup contains $\delta([\nu^{-(c_--1)/2}\rho,\nu^{(c-1)/2}\rho])\times\delta([\nu^a\rho,\nu^{(x-1)/2}\rho])\otimes\sigma_{sp}^{(1)}$.

Since $m^*(\delta([\nu^{-(c_--1)/2}\rho,\nu^{(c-1)/2}\rho])\times\delta([\nu^a\rho,\nu^{(x-1)/2}\rho]))$ contains irreducible constituent $\delta([\nu^a\rho,\nu^{(c-1)/2}\rho])\otimes\delta([\nu^{-(c_--1)/2}\rho,\nu^{(x-1)/2}\rho])$ (with multiplicity two), transitivity of Jacquet modules shows that there is some irreducible constituent $\delta([\nu^a\rho,\nu^{(c-1)/2}\rho])\otimes\pi$ of $\mu^*(\sigma)$ such that $\mu^*(\pi)$ contains $\delta([\nu^{-(c_--1)/2}\rho,\nu^{(x-1)/2}\rho])\otimes\sigma_{sp}^{(1)}$.

Directly from description of $\mu^*(\sigma_{ind})$ given in the beginning of the proof, we conclude that π has to be isomorphic to $L(\delta([\nu^{-(c_--1)/2}\rho,\nu^{(x-1)/2}\rho])\rtimes\sigma_{sp}^{(1)})$. Further, it can be easily verified that $\mu^*(L(\delta([\nu^{-(c_--1)/2}\rho,\nu^{(x-1)/2}\rho])\rtimes\sigma_{sp}^{(1)}))$ contains $\delta([\nu^{-(c_--1)/2}\rho,\nu^{(x-1)/2}\rho])\otimes\sigma_{sp}^{(1)}$ with multiplicity one. In consequence, $\delta([\nu^a\rho,\nu^{(c-1)/2}\rho])\otimes L(\delta([\nu^{-(c_--1)/2}\rho,\nu^{(x-1)/2}\rho])\rtimes\sigma_{sp}^{(1)})$ appears in $\mu^*(\sigma)$ with multiplicity at least two.

In an analogous manner we deduce that $\delta([\nu^a\rho,\nu^{(c-1)/2}\rho])\otimes L(\delta([\nu^{-(c_--1)/2}\rho,\nu^{(x-1)/2}\rho])\rtimes\sigma_{sp}^{(1)})$ also appears in $\mu^*(\sigma')$ with multiplicity at least two, so $\delta([\nu^a\rho,\nu^{(c-1)/2}\rho])\otimes L(\delta([\nu^{-(c_--1)/2}\rho,\nu^{(x-1)/2}\rho])\rtimes\sigma_{sp}^{(1)})$ appears in both $\mu^*(\sigma)$ and $\mu^*(\sigma')$ with the multiplicity two.

In similar way it can be seen that $\delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-(c_--1)/2}\rho, \nu^{(y-1)/2}\rho]) \rtimes \sigma_{sp}^{(4)})$ appears in both $\mu^*(\sigma)$ and $\mu^*(\sigma')$ with the multiplicity one. $\square$

Proposition 4.11. *Suppose $a \geq 1$, $x \neq c_-$ and $a \neq (x_- + 1)/2$. We denote $(c_-)_-$ by y and suppose $x < y_-$. If $\epsilon((y,\rho),(c_-,\rho)) = 1$ then the following equality holds in $R(GL) \otimes R$:*

$$\begin{aligned}
\mu^*(\sigma)_{(a,c,\rho)} \quad = \quad & \delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-(c_--1)/2}\rho, \nu^{a-1}\rho]) \rtimes \sigma_{sp}) + \\
& 2\ \delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-(c_--1)/2}\rho, \nu^{(x-1)/2}\rho]) \rtimes \sigma_{sp}^{(1)}) + \\
& 4\ \delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-(y-1)/2}\rho, \nu^{x-1}\rho]) \rtimes \sigma_{sp}^{(2)}) + \\
& 2\ \delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-(y-1)/2}\rho, \nu^{a-1}\rho]) \rtimes \sigma_{sp}^{(3)}) + \\
& \delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-(c_--1)/2}\rho, \nu^{(z-1)/2}\rho]) \rtimes \sigma_{sp}^{(4)}) + \\
& 2\ \delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-(y-1)/2}\rho, \nu^{(z-1)/2}\rho]) \rtimes \sigma_{sp}^{(5)}),
\end{aligned}$$

where $\sigma_{sp}^{(1)}$ denotes a strongly positive discrete series such that $Jord(\sigma_{sp}^{(1)}) = Jord(\sigma_{sp}) \setminus \{(x,\rho)\} \cup \{(2a-1,\rho)\}$, $\sigma_{sp}^{(2)}$ denotes a strongly positive discrete series such that $Jord(\sigma_{sp}^{(2)}) = Jord(\sigma_{sp}^{(1)}) \setminus \{(y,\rho)\} \cup \{(c_-,\rho)\}$, $\sigma_{sp}^{(3)}$ denotes a strongly positive discrete series such that $Jord(\sigma_{sp}^{(3)}) = Jord(\sigma_{sp}) \setminus \{(y,\rho)\} \cup \{(c_-,\rho)\}$, $\sigma_{sp}^{(4)}$ denotes a strongly positive discrete series such that $Jord(\sigma_{sp}^{(4)}) = Jord(\sigma_{sp}) \setminus \{(z,\rho)\} \cup \{(2a-1,\rho)\}$, while $\sigma_{sp}^{(5)}$ denotes a strongly positive discrete series such that $Jord(\sigma_{sp}^{(5)}) = Jord(\sigma_{sp}^{(3)}) \setminus \{(z,\rho)\} \cup \{(2a-1,\rho)\}$.
If $\epsilon((y,\rho),(c_-,\rho)) = -1$ then the following equality holds in $R(GL) \otimes R$:

$$\begin{aligned}
\mu^*(\sigma)_{(a,c,\rho)} \quad = \quad & \delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-(c_--1)/2}\rho, \nu^{a-1}\rho]) \rtimes \sigma_{sp}) + \\
& 2\ \delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-(c_--1)/2}\rho, \nu^{(x-1)/2}\rho]) \rtimes \sigma_{sp}^{(1)}) + \\
& \delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-(c_--1)/2}\rho, \nu^{(z-1)/2}\rho]) \rtimes \sigma_{sp}^{(4)}),
\end{aligned}$$

for $\sigma_{sp}^{(1)}$ and $\sigma_{sp}^{(4)}$ as above.

Proof. We just note the equality

$$\begin{aligned}
\mu^*(\sigma_{ind})_{(a,c,\rho)} \;=\;& 2\,\delta([\nu^a\rho,\nu^{(c-1)/2}\rho])\otimes L(\delta([\nu^{-(c_- -1)/2}\rho,\nu^{a-1}\rho])\rtimes\sigma_{sp})+\\
& 4\,\delta([\nu^a\rho,\nu^{(c-1)/2}\rho])\otimes L(\delta([\nu^{-(c_- -1)/2}\rho,\nu^{(x-1)/2}\rho])\rtimes\sigma_{sp}^{(1)})+\\
& 4\,\delta([\nu^a\rho,\nu^{(c-1)/2}\rho])\otimes L(\delta([\nu^{-(y-1)/2}\rho,\nu^{x-1}\rho])\rtimes\sigma_{sp}^{(2)})+\\
& 2\,\delta([\nu^a\rho,\nu^{(c-1)/2}\rho])\otimes L(\delta([\nu^{-(y-1)/2}\rho,\nu^{a-1}\rho])\rtimes\sigma_{sp}^{(3)})+\\
& 2\,\delta([\nu^a\rho,\nu^{(c-1)/2}\rho])\otimes L(\delta([\nu^{-(c_- -1)/2}\rho,\nu^{(z-1)/2}\rho])\rtimes\sigma_{sp}^{(4)})+\\
& 2\,\delta([\nu^a\rho,\nu^{(c-1)/2}\rho])\otimes L(\delta([\nu^{-(y-1)/2}\rho,\nu^{(z-1)/2}\rho])\rtimes\sigma_{sp}^{(5)}).
\end{aligned}$$

The rest of the proof can be obtained in completely analogous manner as the proof of Proposition 4.10, details being left to the reader. □

4.3 Case $a = \frac{1}{2}$

In this subsection we discuss the remaining case $a = \frac{1}{2}$. Throughout this subsection we denote $\min(Jord_\rho)$ by c_{min}. Again, we start our determination with a technical lemma, the proof of which we omit, followed by some elementary situations.

Lemma 4.12. *If $c_- = c_{min}$ or $\epsilon(c_{min},\rho) = -1$ then for an irreducible constituent $\delta([\nu^{\frac{1}{2}}\rho,\nu^{(c-1)/2}\rho])\otimes\pi$ of $\mu^*(\sigma_{ind})$ we have*

$$\pi \le \delta([\nu^{\frac{1}{2}}\rho,\nu^{(c_- -1)/2}\rho])\rtimes\sigma_{sp},$$

Otherwise,

$$\pi \le \delta([\nu^{\frac{1}{2}}\rho,\nu^{(c_- -1)/2}\rho])\rtimes\sigma_{sp}\oplus\delta([\nu^{-(c_{min}-1)/2}\rho,\nu^{(c_- -1)/2}\rho])\rtimes\sigma'_{sp},$$

where σ'_{sp} denotes a strongly positive discrete series such that $Jord(\sigma'_{sp}) = Jord(\sigma_{sp})\setminus\{(c_{min},\rho)\}$, i.e., σ'_{sp} is a unique strongly positive discrete series such that σ_{sp} embeds in $\delta([\nu^{\frac{1}{2}}\rho,\nu^{(c_{min}-1)/2}\rho])\rtimes\sigma'_{sp}$. Also, $\epsilon_{\sigma'_{sp}}(min(Jord_\rho(\sigma'_{sp})),\rho) = -1$, where an admissible triple attached to σ'_{sp} is denoted by $(Jord(\sigma'_{sp}),\sigma_{cusp},\epsilon_{\sigma'_{sp}})$.

Proposition 4.13. *Suppose that $c_- = c_{min}$. If $\epsilon(c_-,\rho) = -1$, in $R(GL)\otimes R$ we have*

$$\mu^*(\sigma)_{(\frac{1}{2},c,\rho)} = \delta([\nu^{\frac{1}{2}}\rho,\nu^{(c-1)/2}\rho])\otimes L(\delta([\nu^{-(c_- -1)/2}\rho,\nu^{-\frac{1}{2}}\rho])\rtimes\sigma_{sp}).$$

If $\epsilon(c_-, \rho) = 1$, *in* $R(GL) \otimes R$ *we have*

$$\begin{aligned}\mu^*(\sigma)_{(\frac{1}{2},c,\rho)} &= \delta([\nu^{\frac{1}{2}}\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-(c_--1)/2}\rho, \nu^{-\frac{1}{2}}\rho]) \rtimes \sigma_{sp}) + \\ &\quad 2\ \delta([\nu^{\frac{1}{2}}\rho, \nu^{(c-1)/2}\rho]) \otimes \sigma_{ds},\end{aligned}$$

where σ_{ds} *denotes a unique discrete series subquotient of* $\delta([\nu^{-(c_--1)/2}\rho, \nu^{-\frac{1}{2}}\rho]) \rtimes \sigma_{sp}$.

Proof. It easily follows

$$\begin{aligned}\mu^*(\sigma_{ind})_{(\frac{1}{2},c,\rho)} &= 2\ \delta([\nu^{\frac{1}{2}}\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-(c_--1)/2}\rho, \nu^{-\frac{1}{2}}\rho]) \rtimes \sigma_{sp}) + \\ &\quad 2\ \delta([\nu^{\frac{1}{2}}\rho, \nu^{(c-1)/2}\rho]) \otimes \sigma_{ds}.\end{aligned}$$

It can be deduced from Theorem 5.1 of [12] that $\mu^*(\sigma_{ds}) \geq \delta([\nu^{\frac{1}{2}}\rho, \nu^{(c_--1)/2}\rho]) \otimes \sigma_{sp}$. Thus, if π is an irreducible subquotient of σ_{ind} such that $\mu^*(\pi) \geq \delta([\nu^{\frac{1}{2}}\rho, \nu^{(c-1)/2}\rho]) \otimes \sigma_{ds}$, then π is a discrete series representation (by Lemma 4.1) and there is an irreducible constituent $\delta \otimes \sigma_{sp}$ of $\mu^*(\pi)$ such that $m^*(\delta) \geq \delta([\nu^{\frac{1}{2}}\rho, \nu^{(c-1)/2}\rho]) \otimes \delta([\nu^{\frac{1}{2}}\rho, \nu^{(c_--1)/2}\rho])$. In standard way we obtain $\delta \simeq \delta([\nu^{\frac{1}{2}}\rho, \nu^{(c-1)/2}\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^{(c_--1)/2}\rho])$ and Proposition 7.4 of [17] shows that $\mu^*(\sigma) \geq \delta([\nu^{\frac{1}{2}}\rho, \nu^{(c-1)/2}\rho]) \otimes \sigma_{ds}$ if and only if $\epsilon(c_-, \rho) = 1$.

This ends the proof. □

Proposition 4.14. *Suppose* $c_- \neq c_{min}$ *and* $\epsilon(c_{min}, \rho) = -1$. *If* $\epsilon(((c_-)_-, \rho), (c_-, \rho)) = -1$ *in* $R(GL) \otimes R$ *we have*

$$\mu^*(\sigma)_{(\frac{1}{2},c,\rho)} = \delta([\nu^{\frac{1}{2}}\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-(c_--1)/2}\rho, \nu^{-\frac{1}{2}}\rho]) \rtimes \sigma_{sp}).$$

If $\epsilon(((c_-)_-, \rho), (c_-, \rho)) = 1$, *in* $R(GL) \otimes R$ *we have*

$$\begin{aligned}\mu^*(\sigma)_{(\frac{1}{2},c,\rho)} &= \delta([\nu^{\frac{1}{2}}\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-(c_--1)/2}\rho, \nu^{-\frac{1}{2}}\rho]) \rtimes \sigma_{sp}) + \\ &\quad 2\ \delta([\nu^{\frac{1}{2}}\rho, \nu^{(c-1)/2}\rho]) \otimes \tau,\end{aligned}$$

where τ *is a unique irreducible subquotient of* $\delta([\nu^{-(c_--1)/2}\rho, \nu^{-\frac{1}{2}}\rho]) \rtimes \sigma_{sp}$ *different than* $L(\delta([\nu^{-(c_--1)/2}\rho, \nu^{-\frac{1}{2}}\rho]) \rtimes \sigma_{sp})$.

Proof. The following equality holds in $R(GL) \otimes R$:

$$\begin{aligned}\mu^*(\sigma_{ind})_{(\frac{1}{2},c,\rho)} &= 2\ \delta([\nu^{\frac{1}{2}}\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-(c_--1)/2}\rho, \nu^{-\frac{1}{2}}\rho]) \rtimes \sigma_{sp}) + \\ &\quad 2\ \delta([\nu^{\frac{1}{2}}\rho, \nu^{(c-1)/2}\rho]) \otimes \tau,\end{aligned}$$

and the first part of Theorem 5.1 from [12] implies that there is some irreducible constituent of the form $\delta([\nu^{((c_-)_-+1)/2}\rho, \nu^{(c_--1)/2}\rho]) \otimes \pi$ appearing in $\mu^*(\tau)$. Now it can be obtained in the same fashion as in the proof of the previous proposition that $\mu^*(\sigma) \geq \delta([\nu^{\frac{1}{2}}\rho, \nu^{(c-1)/2}\rho]) \otimes \tau$ if and only if $\epsilon(((c_-)_-, \rho), (c_-, \rho)) = 1$. □

In the rest of this section we assume $c_- \neq c_{min}$ and $\epsilon(c_{min}, \rho) = 1$. Let us denote by x an element of $Jord_\rho$ such that $x_- = c_{min}$ and by y an element of $Jord_\rho$ such that $(c_-)_- = y$.

Proposition 4.15. *Suppose $x = c_-$, i.e. $y = c_{min}$. If $\epsilon((c_{min}, \rho), (c_-, \rho)) = 1$ then the following equality holds in $R(GL) \otimes R$:*

$$\begin{aligned}\mu^*(\sigma)_{(\frac{1}{2},c,\rho)} \;=\;& \delta([\nu^{\frac{1}{2}}\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-(c_--1)/2}\rho, \nu^{-\frac{1}{2}}\rho]) \rtimes \sigma_{sp}) + \\ & 2\ \delta([\nu^{\frac{1}{2}}\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-(c_{min}-1)/2}\rho, \nu^{-\frac{1}{2}}\rho]) \rtimes \sigma_{sp}^{(1)}) + \\ & 2\ \delta([\nu^{\frac{1}{2}}\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-(c_--1)/2}\rho, \nu^{(c_{min}-1)/2}\rho]) \rtimes \sigma_{sp}^{(2)}) + \\ & 4\ \delta([\nu^{\frac{1}{2}}\rho, \nu^{(c-1)/2}\rho]) \otimes \sigma_1 + \delta([\nu^{\frac{1}{2}}\rho, \nu^{(c-1)/2}\rho]) \otimes \sigma_2,\end{aligned}$$

where $\sigma_{sp}^{(1)}$ denotes a strongly positive discrete series such that $Jord(\sigma_{sp}^{(1)}) = Jord(\sigma_{sp}) \setminus \{(c_{min}, \rho)\} \cup \{(c_-, \rho)\}$, $\sigma_{sp}^{(2)}$ denotes a strongly positive discrete series such that $Jord(\sigma_{sp}^{(2)}) = Jord(\sigma_{sp}) \setminus \{(c_{min}, \rho)\}$, while σ_1 and σ_2 are mutually non-isomorphic discrete series subrepresentations of $\delta([\nu^{-(c_{min}-1)/2}\rho, \nu^{(c_--1)/2}\rho]) \rtimes \sigma_{sp}^{(2)}$ and $\mu^*(\sigma_1)$ contains an irreducible constituent of the form $\delta([\nu^{\frac{1}{2}}\rho, \nu^{(c_{min}-1)/2}\rho]) \otimes \tau$.

If $\epsilon((c_{min}, \rho), (c_-, \rho)) = -1$ then the following equality holds in $R(GL) \otimes R$:

$$\begin{aligned}\mu^*(\sigma)_{(\frac{1}{2},c,\rho)} \;=\;& \delta([\nu^{a}\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-(c_--1)/2}\rho, \nu^{-\frac{1}{2}}\rho]) \rtimes \sigma_{sp}) + \\ & 2\ \delta([\nu^{\frac{1}{2}}\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-(c_--1)/2}\rho, \nu^{(c_{min}-1)/2}\rho]) \rtimes \sigma_{sp}^{(2)}) + \\ & \delta([\nu^{\frac{1}{2}}\rho, \nu^{(c-1)/2}\rho]) \otimes \sigma_2,\end{aligned}$$

for $\sigma_{sp}^{(2)}$ and σ_2 as above.

Proof. In $R(GL) \otimes R$ holds

$$\begin{aligned}\mu^*(\sigma_{ind})_{(\frac{1}{2},c,\rho)} \;=\;& 2\ \delta([\nu^{\frac{1}{2}}\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-(c_--1)/2}\rho, \nu^{-\frac{1}{2}}\rho]) \rtimes \sigma_{sp}) + \\ & 2\ \delta([\nu^{\frac{1}{2}}\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-(c_{min}-1)/2}\rho, \nu^{-\frac{1}{2}}\rho]) \rtimes \sigma_{sp}^{(1)}) + \\ & 4\ \delta([\nu^{\frac{1}{2}}\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-(c_--1)/2}\rho, \nu^{(c_{min}-1)/2}\rho]) \rtimes \sigma_{sp}^{(2)}) + \\ & 4\ \delta([\nu^{\frac{1}{2}}\rho, \nu^{(c-1)/2}\rho]) \otimes \sigma_1 + 2\ \delta([\nu^{\frac{1}{2}}\rho, \nu^{(c-1)/2}\rho]) \otimes \sigma_2.\end{aligned}$$

We have already seen that $\delta([\nu^{a}\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-(c_{-}-1)/2}\rho, \nu^{-\frac{1}{2}}\rho]) \rtimes \sigma_{sp})$ appears in $\mu^*(\sigma)$ with multiplicity one.

Further, it follows directly that $\mu^*(L(\delta([\nu^{-(c_{min}-1)/2}\rho, \nu^{-\frac{1}{2}}\rho]) \rtimes \sigma_{sp}^{(1)}))$ contains some irreducible constituent of the form

$$\delta([\nu^{-(c_{min}-1)/2}\rho, \nu^{-\frac{1}{2}}\rho]) \times \delta([\nu^{(c_{min}+1)/2}\rho, \nu^{(c_{-}-1)/2}\rho]) \otimes \pi.$$

Thus, if π_1 is an irreducible subquotient of σ_{ind} with the property $\mu^*(\pi_1) \geq \delta([\nu^{\frac{1}{2}}\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-(c_{min}-1)/2}\rho, \nu^{-\frac{1}{2}}\rho]) \rtimes \sigma_{sp}^{(1)})$, then there is some irreducible constituent $\delta \otimes \pi$ of $\mu^*(\pi_1)$ such that

$$m^*(\delta) \geq \delta([\nu^{\frac{1}{2}}\rho, \nu^{(c-1)/2}\rho]) \otimes \delta([\nu^{-(c_{min}-1)/2}\rho, \nu^{-\frac{1}{2}}\rho]) \times \delta([\nu^{(c_{min}+1)/2}\rho, \nu^{(c_{-}-1)/2}\rho]).$$

From $\mu^*(\sigma_{ind})$ we deduce

$$\delta \simeq \delta([\nu^{(c_{min}+1)/2}\rho, \nu^{(c_{-}-1)/2}\rho]) \times \delta([\nu^{-(c_{min}-1)/2}\rho, \nu^{(c-1)/2}\rho])$$

and it follows that $\mu^*(\sigma) \geq \delta([\nu^{\frac{1}{2}}\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-(c_{min}-1)/2}\rho, \nu^{-\frac{1}{2}}\rho]) \rtimes \sigma_{sp}^{(1)})$ if and only if $\epsilon((c_{min}, \rho), (c_{-}, \rho)) = 1$.

Since, by Theorem 5.1 of [12], $\mu^*(\sigma_1) \geq \delta([\nu^{\frac{1}{2}}\rho, \nu^{(c_{min}-1)/2}\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^{(c_{-}-1)/2}\rho]) \otimes \tau'$, for some irreducible representation τ', in the same way we obtain that $\mu^*(\sigma) \geq \delta([\nu^{\frac{1}{2}}\rho, \nu^{(c-1)/2}\rho]) \otimes \sigma_1$ if and only if $\epsilon((c_{min}, \rho), (c_{-}, \rho)) = 1$.

From definition of σ we obtain

$$\sigma \hookrightarrow \delta([\nu^{-(c_{-}-1)/2}\rho, \nu^{(c-1)/2}\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^{(c_{min}-1)/2}\rho]) \rtimes \sigma_{sp}^{(2)}.$$

Since $m^*(\delta([\nu^{-(c_{-}-1)/2}\rho, \nu^{(c-1)/2}\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^{(c_{min}-1)/2}\rho])) \geq 2\, \delta([\nu^{\frac{1}{2}}\rho, \nu^{(c-1)/2}\rho]) \otimes \delta([\nu^{-(c_{-}-1)/2}\rho, \nu^{(c_{min}-1)/2}\rho])$, using Frobenius reciprocity we obtain that

$$\delta([\nu^{\frac{1}{2}}\rho, \nu^{(c-1)/2}\rho]) \otimes \delta([\nu^{-(c_{-}-1)/2}\rho, \nu^{(c_{min}-1)/2}\rho]) \otimes \sigma_{sp}^{(2)}$$

appears with multiplicity two in Jacquet module of σ with respect to an appropriate parabolic subgroup.

Transitivity of Jacquet modules implies that there is some irreducible constituent $\delta([\nu^{\frac{1}{2}}\rho, \nu^{(c-1)/2}\rho]) \otimes \pi$ of $\mu^*(\sigma)$ such that $\mu^*(\pi) \geq \delta([\nu^{-(c_{-}-1)/2}\rho, \nu^{(c_{min}-1)/2}\rho]) \otimes \sigma_{sp}^{(2)}$. Description of $\mu^*(\sigma_{ind})$ given in the beginning of the proof shows $\pi \simeq L(\delta([\nu^{-(c_{-}-1)/2}\rho, \nu^{(c_{min}-1)/2}\rho]) \rtimes \sigma_{sp}^{(2)})$. Also, since $\mu^*(L(\delta([\nu^{-(c_{-}-1)/2}\rho,$

$\nu^{(c_{min}-1)/2}\rho]) \rtimes \sigma_{sp}^{(2)})$ contains $\delta([\nu^{-(c_--1)/2}\rho, \nu^{(c_{min}-1)/2}\rho]) \otimes \sigma_{sp}^{(2)}$ with multiplicity one, $\delta([\nu^{\frac{1}{2}}\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-(c_--1)/2}\rho, \nu^{(c_{min}-1)/2}\rho]) \rtimes \sigma_{sp}^{(2)})$ is contained in $\mu^*(\sigma)$ with multiplicity at least two. Examining analogous properties of $\mu^*(\sigma')$ we get that such irreducible constituent is contained in $\mu^*(\sigma)$ with multiplicity exactly two.

It remains to consider $\delta([\nu^{\frac{1}{2}}\rho, \nu^{(c-1)/2}\rho]) \otimes \sigma_2$. First we note that in an appropriate Grothendieck group holds ([12], Proposition 3.1)

$$\delta([\nu^{(c_{min}+1)/2}\rho, \nu^{(c_--1)/2}\rho]) \rtimes \sigma_{sp} = L(\delta([\nu^{-(c_--1)/2}\rho, \nu^{-(c_{min}+1)/2}\rho]) \rtimes \sigma_{sp}) + \sigma_{sp}^{(1)}.$$

Since $\mu^*(\delta([\nu^{(c_{min}+1)/2}\rho, \nu^{(c_--1)/2}\rho]) \rtimes \sigma_{sp})$ contains $L(\delta([\nu^{\frac{1}{2}}\rho, \nu^{(c_{min}-1)/2}\rho]) \times \delta([\nu^{(c_{min}+1)/2}\rho, \nu^{(c_--1)/2}\rho])) \otimes \sigma_{sp}^{(2)}$ and such irreducible constituent does not appear in $\mu^*(\sigma_{sp}^{(1)})$ by Theorem 4.6 of [7], it has to appear in $\mu^*(L(\delta([\nu^{-(c_--1)/2}\rho, \nu^{-(c_{min}+1)/2}\rho]) \rtimes \sigma_{sp}))$.

Now from $\mu^*(\sigma) \geq \delta([\nu^{\frac{1}{2}}\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-(c_--1)/2}\rho, \nu^{-(c_{min}+1)/2}\rho]) \rtimes \sigma_{sp})$, we obtain that there is some irreducible constituent $\delta \otimes \sigma_{sp}^{(2)}$ of $\mu^*(\sigma)$ such that $m^*(\delta) \geq \delta([\nu^{\frac{1}{2}}\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{\frac{1}{2}}\rho, \nu^{(c_{min}-1)/2}\rho]) \times \delta([\nu^{(c_{min}+1)/2}\rho, \nu^{(c_--1)/2}\rho]))$. Using structural formula for $\mu^*(\sigma_{ind})$ it is not hard to see

$$\delta \simeq \delta([\nu^{\frac{1}{2}}\rho, \nu^{(c-1)/2}\rho]) \times L(\delta([\nu^{\frac{1}{2}}\rho, \nu^{(c_{min}-1)/2}\rho]) \times \delta([\nu^{(c_{min}+1)/2}\rho, \nu^{(c_--1)/2}\rho])).$$

This gives $m^*(\delta) \geq \delta([\nu^{\frac{1}{2}}\rho, \nu^{(c-1)/2}\rho]) \otimes \delta([\nu^{-(c_{min}+1)/2}\rho, \nu^{(c_--1)/2}\rho])$ and, by transitivity of Jacquet modules, there is some irreducible constituent $\delta([\nu^{\frac{1}{2}}\rho, \nu^{(c-1)/2}\rho]) \otimes \pi$ of $\mu^*(\sigma)$ such that $\mu^*(\pi) \geq \delta([\nu^{-(c_{min}+1)/2}\rho, \nu^{(c_--1)/2}\rho]) \otimes \sigma_{sp}^{(2)}$. It can be seen directly from the description of $\mu^*(\sigma_{ind})$ that $\pi \in \{\sigma_1, \sigma_2\}$.

The same conclusion can be made for $\mu^*(\sigma')$ and in the same way as in the proof of Proposition 4.9 we obtain that $\delta([\nu^{\frac{1}{2}}\rho, \nu^{(c-1)/2}\rho]) \otimes \sigma_2$ appears in $\mu^*(\sigma)$ with multiplicity one. □

Proposition 4.16. *Suppose $x = y$. If $\epsilon((x,\rho),(c_-,\rho)) = 1$ then the following equality holds in $R(GL) \otimes R$:*

$$\begin{aligned}
\mu^*(\sigma)_{(\frac{1}{2},c,\rho)} \; = \; & \delta([\nu^{\frac{1}{2}}\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-(c_--1)/2}\rho, \nu^{-\frac{1}{2}}\rho]) \rtimes \sigma_{sp}) + \\
& 2\ \delta([\nu^{\frac{1}{2}}\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-(c_--1)/2}\rho, \nu^{(c_{min}-1)/2}\rho]) \rtimes \sigma_{sp}^{(1)}) + \\
& 2\ \delta([\nu^{\frac{1}{2}}\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-(x-1)/2}\rho, \nu^{-\frac{1}{2}}\rho]) \rtimes \sigma_{sp}^{(2)}) + \\
& 4\ \delta([\nu^{\frac{1}{2}}\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-(x-1)/2}\rho, \nu^{(c_{min}-1)/2}\rho]) \rtimes \sigma_{sp}^{(3)}) + \\
& \delta([\nu^{\frac{1}{2}}\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-(c_--1)/2}\rho, \nu^{(x-1)/2}\rho]) \rtimes \sigma_{sp}^{(4)}) + \\
& 2\ \delta([\nu^{\frac{1}{2}}\rho, \nu^{(c-1)/2}\rho]) \otimes \sigma_1,
\end{aligned}$$

where $\sigma_{sp}^{(1)}$ *denotes a strongly positive discrete series such that* $Jord(\sigma_{sp}^{(1)}) = Jord(\sigma_{sp}) \setminus \{(c_{min}, \rho)\}$, $\sigma_{sp}^{(2)}$ *denotes a strongly positive discrete series such that* $Jord(\sigma_{sp}^{(2)}) = Jord(\sigma_{sp}) \setminus \{(x, \rho)\} \cup \{(c_{\_}, \rho)\}$, $\sigma_{sp}^{(3)}$ *denotes a strongly positive discrete series such that* $Jord(\sigma_{sp}^{(3)}) = Jord(\sigma_{sp}) \setminus \{(c_{min}, \rho), (x, \rho)\} \cup \{(c_{\_}, \rho)\}$, $\sigma_{sp}^{(4)}$ *denotes a strongly positive discrete series such that* $Jord(\sigma_{sp}^{(4)}) = Jord(\sigma_{sp}^{(1)}) \setminus \{(x, \rho)\} \cup \{(c_{min}, \rho)\}$, *while* σ_1 *is a discrete series subrepresentation of both induced representations* $\delta([\nu^{-(c_{min}-1)/2}\rho, \nu^{(x-1)/2}\rho]) \rtimes \sigma_{sp}^{(2)}$ *and* $\delta([\nu^{-(x-1)/2}\rho, \nu^{(c_{\_}-1)/2}\rho]) \rtimes \sigma_{sp}^{(4)}$.
If $\epsilon((x, \rho), (c_{\_}, \rho)) = -1$ *then the following equality holds in* $R(GL) \otimes R$:

$$\begin{aligned}\mu^*(\sigma)_{(\frac{1}{2},c,\rho)} \quad = \quad & \delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-(c_{\_}-1)/2}\rho, \nu^{-\frac{1}{2}}\rho]) \rtimes \sigma_{sp}) + \\ & 2\ \delta([\nu^{\frac{1}{2}}\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-(c_{\_}-1)/2}\rho, \nu^{(c_{min}-1)/2}\rho]) \rtimes \sigma_{sp}^{(1)}) + \\ & \delta([\nu^{\frac{1}{2}}\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-(c_{\_}-1)/2}\rho, \nu^{(x-1)/2}\rho]) \rtimes \sigma_{sp}^{(4)}),\end{aligned}$$

for $\sigma_{sp}^{(1)}$ *and* $\sigma_{sp}^{(4)}$ *as above.*

Proof. We provide only the main details of the proof since it mostly follows the same lines as in the proof of previous proposition. Using structural formula, Theorem 5.1 from [12] and Proposition 3.2 of [8] we get

$$\begin{aligned}\mu^*(\sigma_{ind})_{(\frac{1}{2},c,\rho)} \quad = \quad & 2\ \delta([\nu^{\frac{1}{2}}\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-(c_{\_}-1)/2}\rho, \nu^{-\frac{1}{2}}\rho]) \rtimes \sigma_{sp}) + \\ & 4\ \delta([\nu^{\frac{1}{2}}\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-(c_{\_}-1)/2}\rho, \nu^{(c_{min}-1)/2}\rho]) \rtimes \sigma_{sp}^{(1)}) + \\ & 2\ \delta([\nu^{\frac{1}{2}}\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-(x-1)/2}\rho, \nu^{-\frac{1}{2}}\rho]) \rtimes \sigma_{sp}^{(2)}) + \\ & 4\ \delta([\nu^{\frac{1}{2}}\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-(x-1)/2}\rho, \nu^{(c_{min}-1)/2}\rho]) \rtimes \sigma_{sp}^{(3)}) + \\ & 2\ \delta([\nu^{\frac{1}{2}}\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-(c_{\_}-1)/2}\rho, \nu^{(x-1)/2}\rho]) \rtimes \sigma_{sp}^{(4)}) + \\ & 2\ \delta([\nu^{\frac{1}{2}}\rho, \nu^{(c-1)/2}\rho]) \otimes \sigma_1.\end{aligned}$$

For $\pi \in \{L(\delta([\nu^{-(x-1)/2}\rho, \nu^{-\frac{1}{2}}\rho]) \rtimes \sigma_{sp}^{(2)}), L(\delta([\nu^{-(x-1)/2}\rho, \nu^{(c_{min}-1)/2}\rho]) \rtimes \sigma_{sp}^{(3)}), \sigma_1\}$ there is some irreducible constituent of the form $\delta([\nu^{(x+1)/2}\rho, \nu^{(c_{\_}-1)/2}\rho]) \otimes \pi'$ appearing in $\mu^*(\pi)$. If $\delta \otimes \pi'$ is an irreducible constituent of $\mu^*(\sigma_{ind})$ such that $m^*(\delta) \geq \delta([\nu^{\frac{1}{2}}\rho, \nu^{(c-1)/2}\rho]) \otimes \delta([\nu^{(x+1)/2}\rho, \nu^{(c_{\_}-1)/2}\rho])$, then it can be seen that $m^*(\delta)$ also contains $\delta([\nu^{(x+1)/2}\rho, \nu^{(c_{\_}-1)/2}\rho]) \otimes \delta([\nu^{\frac{1}{2}}\rho, \nu^{(c-1)/2}\rho])$. Consequently, $\mu^*(\sigma)$ contains $\delta([\nu^{\frac{1}{2}}\rho, \nu^{(c-1)/2}\rho]) \otimes \pi$ for $\pi \in \{L(\delta([\nu^{-(x-1)/2}\rho, \nu^{-\frac{1}{2}}\rho]) \rtimes \sigma_{sp}^{(2)}), L(\delta([\nu^{-(x-1)/2}\rho, \nu^{(c_{min}-1)/2}\rho]) \rtimes \sigma_{sp}^{(3)}), \sigma_1\}$ if and only if $\epsilon((x, \rho), (c_{\_}, \rho)) = 1$.

Since both $\mu^*(\sigma)$ and $\mu^*(\sigma')$ contain $\delta([\nu^{-(c_--1)/2}\rho, \nu^{(c-1)/2}\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^{(c_{min}-1)/2}\rho]) \otimes \sigma_{sp}^{(1)}$ and $\delta([\nu^{\frac{1}{2}}\rho, \nu^{(c-1)/2}\rho]) \otimes \delta([\nu^{-(c_--1)/2}\rho, \nu^{(c_{min}-1)/2}\rho])$ appears with multiplicity two in $m^*(\delta([\nu^{-(c_--1)/2}\rho, \nu^{(c-1)/2}\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^{(c_{min}-1)/2}\rho]))$, in the same way as in the proof of Proposition 4.15 we get that $\mu^*(\sigma)$ contains $\delta([\nu^{\frac{1}{2}}\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-(c_--1)/2}\rho, \nu^{(c_{min}-1)/2}\rho]) \rtimes \sigma_{sp}^{(1)})$ with multiplicity two.

Similarly, both σ and σ' are irreducible subrepresentations of

$$\delta([\nu^{-(c_--1)/2}\rho, \nu^{(c-1)/2}\rho]) \times L(\delta([\nu^{\frac{1}{2}}\rho, \nu^{(c_{min}-1)/2}\rho]) \times \delta([\nu^{(c_{min}+1)/2}\rho, \nu^{(x-1)/2}\rho])) \rtimes \sigma_{sp}^{(4)}.$$

Lemma 1.3.3 of [2] shows that the induced representation

$$\delta([\nu^{-(c_--1)/2}\rho, \nu^{(c-1)/2}\rho]) \times L(\delta([\nu^{\frac{1}{2}}\rho, \nu^{(c_{min}-1)/2}\rho]) \times \delta([\nu^{(c_{min}+1)/2}\rho, \nu^{(x-1)/2}\rho]))$$

is irreducible and it is not hard to see, using Frobenius reciprocity and transitivity of Jacquet modules, that Jacquet modules of both σ and σ' with respect to an appropriate parabolic subgroup contain

$$\delta([\nu^{\frac{1}{2}}\rho, \nu^{(c-1)/2}\rho]) \otimes \delta([\nu^{-(c_--1)/2}\rho, \nu^{(x-1)/2}\rho]) \otimes \sigma_{sp}^{(4)}.$$

Now it can be seen in the same way as before that $\delta([\nu^{\frac{1}{2}}\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-(c_--1)/2}\rho, \nu^{(x-1)/2}\rho]) \rtimes \sigma_{sp}^{(4)})$ appears in $\mu^*(\sigma)$ with multiplicity one. This ends the proof. □

The remaining case is settled in the following proposition. We omit the proof, since it can be obtained applying the same arguments as in the proof of the previous proposition.

Proposition 4.17. *Suppose $x < y$. If $\epsilon((y, \rho), (c_-, \rho)) = 1$ then the following equality holds in $R(GL) \otimes R$:*

$$\begin{aligned}
\mu^*(\sigma)_{(\frac{1}{2},c,\rho)} = \; & \delta([\nu^{\frac{1}{2}}\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-(c_--1)/2}\rho, \nu^{-\frac{1}{2}}\rho]) \rtimes \sigma_{sp}) + \\
& 2\ \delta([\nu^{\frac{1}{2}}\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-(c_--1)/2}\rho, \nu^{(c_{min}-1)/2}\rho]) \rtimes \sigma_{sp}^{(1)}) + \\
& 2\ \delta([\nu^{\frac{1}{2}}\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-(y-1)/2}\rho, \nu^{-\frac{1}{2}}\rho]) \rtimes \sigma_{sp}^{(2)}) + \\
& 4\ \delta([\nu^{\frac{1}{2}}\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-(y-1)/2}\rho, \nu^{(c_{min}-1)/2}\rho]) \rtimes \sigma_{sp}^{(3)}) + \\
& \delta([\nu^{\frac{1}{2}}\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-(c_--1)/2}\rho, \nu^{(x-1)/2}\rho]) \rtimes \sigma_{sp}^{(4)}) + \\
& 2\ \delta([\nu^{\frac{1}{2}}\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-(y-1)/2}\rho, \nu^{(x-1)/2}\rho]) \rtimes \sigma_{sp}^{(5)}).
\end{aligned}$$

where $\sigma_{sp}^{(1)}$ *denotes a strongly positive discrete series such that* $Jord(\sigma_{sp}^{(1)}) = Jord(\sigma_{sp}) \setminus \{(c_{min}, \rho)\}$, $\sigma_{sp}^{(2)}$ *denotes a strongly positive discrete series such that* $Jord(\sigma_{sp}^{(2)}) = Jord(\sigma_{sp}) \setminus \{(y,\rho)\} \cup \{(c_-, \rho)\}$, $\sigma_{sp}^{(3)}$ *denotes a strongly positive discrete series such that* $Jord(\sigma_{sp}^{(3)}) = Jord(\sigma_{sp}) \setminus \{(c_{min}, \rho), (y,\rho)\} \cup \{(c_-, \rho)\}$, $\sigma_{sp}^{(4)}$ *denotes a strongly positive discrete series such that* $Jord(\sigma_{sp}^{(4)}) = Jord(\sigma_{sp}^{(1)}) \setminus \{(x,\rho)\} \cup \{(c_{min}, \rho)\}$ *and* $\sigma_{sp}^{(5)}$ *denotes a strongly positive discrete series such that* $Jord(\sigma_{sp}^{(5)}) = Jord(\sigma_{sp}^{(4)}) \setminus \{(y,\rho)\} \cup \{(c_-, \rho)\}$.
If $\epsilon((y,\rho),(c_-,\rho)) = -1$ *then the following equality holds in* $R(GL) \otimes R$:

$$\begin{aligned} \mu^*(\sigma)_{(\frac{1}{2},c,\rho)} &= \delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-(c_--1)/2}\rho, \nu^{-\frac{1}{2}}\rho]) \rtimes \sigma_{sp}) + \\ & 2\ \delta([\nu^{\frac{1}{2}}\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-(c_--1)/2}\rho, \nu^{(c_{min}-1)/2}\rho]) \rtimes \sigma_{sp}^{(1)}) + \\ & \delta([\nu^{\frac{1}{2}}\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-(c_--1)/2}\rho, \nu^{(x-1)/2}\rho]) \rtimes \sigma_{sp}^{(4)}), \end{aligned}$$

for $\sigma_{sp}^{(1)}$ *and* $\sigma_{sp}^{(4)}$ *as before.*

5 Case $Jord_\rho(\sigma_{sp}) = \emptyset$ and c odd.

The purpose of this section is to provide a description of $\mu^*(\sigma)_{(c,\rho)}$ in an exceptional case. Throughout this section we assume that c is odd and $Jord_\rho(\sigma_{sp}) = \emptyset$. Consequently, $c = \max(Jord_\rho(\sigma_{sp}))$ and $\epsilon((c,\rho))$ is defined. Further, the induced representation $\rho \rtimes \sigma_{cusp}$ reduces in a direct sum of two nonisomorphic tempered representations which we denote by τ_1 and τ_{-1}. Also, $\epsilon((c,\rho)) = i$ if and only if there is some irreducible representation π such that σ is a subrepresentation of $\pi \times \delta([\nu\rho, \nu^{(c-1)/2}\rho]) \rtimes \tau_i$. Also, $\epsilon((c,\rho)) \neq \epsilon'((c,\rho))$.

Results obtained in the previous sections show that we only need to consider $\mu^*(\sigma)_{(a,c,\rho)}$ for $a \leq (c_- - 1)/2$. In the following theorem we provide a description of corresponding Jacquet modules.

Theorem 5.1. *For* $-(c_- - 1)/2 \leq a \leq 0$, *in* $R(GL) \otimes R$ *we have*

$$\mu^*(\sigma)_{(a,c,\rho)} = \delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-(c_--1)/2}\rho, \nu^{a-1}\rho]) \rtimes \sigma_{sp}),$$

while for $1 \leq a \leq (c_- - 1)/2$ *we have:*

$$\delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-(c_--1)/2}\rho, \nu^{a-1}\rho]) \rtimes \sigma_{sp}) + 2\ \delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes \sigma_{ds}^{(a)},$$

where $\sigma_{ds}^{(a)}$ *is a discrete series subrepresentation of* $\delta([\nu^{-a+1}\rho, \nu^{(c_{-}-1)/2}\rho]) \rtimes \sigma_{sp}$ *such that for the corresponding admissible triple* $(Jord^{(a)}, \sigma_{cusp}, \epsilon^{(a)})$ *holds* $\epsilon^{(a)}((c_{-}, \rho)) = \epsilon((c, \rho))$.

Proof. We only comment the case $a \geq 1$. In this case it is easy to obtain

$$\begin{aligned}\mu^*(\sigma_{ind})_{a,c,\rho} &= 2\ \delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes L(\delta([\nu^{-(c_{-}-1)/2}\rho, \nu^{a-1}\rho]) \rtimes \sigma_{sp}) + \\ & \quad 2\ \delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes \sigma_1 + 2\ \delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes \sigma_{-1},\end{aligned}$$

where σ_1 and σ_{-1} denote mutually non-isomorphic discrete series subrepresentations of $\delta([\nu^{-a+1}\rho, \nu^{(c_{-}-1)/2}\rho]) \rtimes \sigma_{sp}$. Further, we denote by $(Jord^{(i)}, \sigma_{cusp}, \epsilon^{(i)})$ an admissible triple corresponding to σ_i, for $i \in \{1, -1\}$, and assume $\epsilon^{(i)}((c_{-}, \rho)) = i$.

Proposition 4.2 shows that it is enough to consider $\delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes \sigma_1$ and $\delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes \sigma_{-1}$. Thus, suppose that $\delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes \sigma_i$ is an irreducible constituent of $\mu^*(\sigma)$, for some $i \in \{1, -1\}$. By [16], there is an irreducible representation π such that σ_i contains

$$\pi \otimes \delta([\nu\rho, \nu^{(c_{-}-1)/2}\rho]) \otimes \tau_i$$

in Jacquet module with respect to an appropriate parabolic subgroup. In consequence, there is some irreducible constituent $\delta \otimes \tau_i$ of $\mu^*(\sigma_i)$ such that $m^*(\delta)$ contains $\pi \otimes \delta([\nu\rho, \nu^{(c_{-}-1)/2}\rho])$. Calculating $\mu^*(\delta([\nu^{-a+1}\rho, \nu^{(c_{-}-1)/2}\rho]) \rtimes \sigma_{sp})$ we deduce

$$\delta \simeq \pi' \times \delta([\nu\rho, \nu^{(c_{-}-1)/2}\rho]) \times \delta([\nu\rho, \nu^{a-1}\rho])$$

where π' stands for an irreducible representation such that $\pi' \otimes \sigma_{cusp} \leq \mu^*(\sigma_{sp})$. Since there are no twists or ρ appearing in the cuspidal support of π', it easily follows $\pi \simeq \pi' \times \delta([\nu\rho, \nu^{a-1}\rho]) \simeq \delta([\nu\rho, \nu^{a-1}\rho]) \times \pi'$.

Transitivity of Jacquet modules shows that

$$\delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes \delta([\nu\rho, \nu^{a-1}\rho]) \times \pi' \otimes \delta([\nu\rho, \nu^{(c_{-}-1)/2}\rho]) \otimes \tau_i$$

appears in Jacquet module of σ with respect to an appropriate parabolic subgroup. Hence, there is some irreducible constituent $\delta' \otimes \tau_i$ of $\mu^*(\sigma)$ such that Jacquet module of δ' with respect to an appropriate parabolic subgroup contains $\delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes \delta([\nu\rho, \nu^{a-1}\rho]) \times \pi' \otimes \delta([\nu\rho, \nu^{(c_{-}-1)/2}\rho])$. In the same way as before we conclude

$$\delta' \simeq \delta([\nu\rho, \nu^{(c-1)/2}\rho]) \times \pi' \times \delta([\nu\rho, \nu^{(c_{-}-1)/2}\rho])$$

so Jacquet module of σ with respect to an appropriate parabolic subgroup contains

$$\pi' \times \delta([\nu\rho, \nu^{(c_- -1)/2}\rho]) \otimes \delta([\nu\rho, \nu^{(c-1)/2}\rho]) \otimes \tau_i.$$

It follows that $\mu^*(\sigma)$ contains some irreducible constituent $\pi' \times \delta([\nu\rho, \nu^{(c_- -1)/2}\rho]) \otimes \tau$ such that $\mu^*(\tau) \geq \delta([\nu\rho, \nu^{(c-1)/2}\rho]) \otimes \tau_i$. From $\mu^*(\sigma_{ind})$ we directly obtain $\tau \leq \delta([\rho, \nu^{(c-1)/2}\rho]) \rtimes \sigma_{cusp}$. In an appropriate Grothendieck group we have

$$\delta([\rho, \nu^{(c-1)/2}\rho]) \rtimes \sigma_{cusp} \leq \delta([\nu\rho, \nu^{(c-1)/2}\rho]) \rtimes \tau_1 \oplus \delta([\nu\rho, \nu^{(c-1)/2}\rho]) \rtimes \tau_{-1}$$

and it follows immediately that $\delta([\nu\rho, \nu^{(c-1)/2}\rho]) \otimes \tau_i$ appears with multiplicity one in Jacquet module of the right-hand side of the previous inequality and, by Frobenius reciprocity, it also appears in Jacquet module of unique irreducible subrepresentation of $\delta([\nu\rho, \nu^{(c-1)/2}\rho]) \rtimes \tau_i$. Thus, τ is a unique irreducible subrepresentation of $\delta([\nu\rho, \nu^{(c-1)/2}\rho]) \rtimes \tau_i$ and Proposition 7.5 of [17] shows $\epsilon((c, \rho)) = i$. Consequently, $\mu^*(\sigma)$ contains $\delta([\nu^a\rho, \nu^{(c-1)/2}\rho]) \otimes \sigma_i$ if and only if $\epsilon((c, \rho)) = \epsilon^{(i)}((c_-, \rho))$. This ends the proof. □

References

[1] J. Arthur, *The endoscopic classification of representations*, vol. 61 of American Mathematical Society Colloquium Publications, American Mathematical Society, Providence, RI, 2013. Orthogonal and symplectic groups.

[2] C. Jantzen, *Jacquet modules of p-adic general linear groups*, Represent. Theory, 11 (2007), pp. 45–83.

[3] A. Kret and E. Lapid, *Jacquet modules of ladder representations*, C. R. Math. Acad. Sci. Paris, 350 (2012), pp. 937–940.

[4] E. Lapid and A. Minguez, *On a determinantal formula of Tadić*, Amer. J. Math., to appear, (2014).

[5] I. Matić, *Strongly positive representations of metaplectic groups*, J. Algebra, 334 (2011), pp. 255–274.

[6] ——, *The conservation relation for discrete series representations of metaplectic groups*, Int. Math. Res. Not. IMRN, (2013), pp. 5227–5269.

[7] ——, *Jacquet modules of strongly positive representations of the metaplectic group* $\widetilde{Sp(n)}$, Trans. Amer. Math. Soc., 365 (2013), pp. 2755–2778.

[8] ——, *On discrete series subrepresentations of generalized principal series*, preprint, (2014).

[9] C. Mœglin, *Sur la classification des séries discrètes des groupes classiques p-adiques: paramètres de Langlands et exhaustivité*, J. Eur. Math. Soc. (JEMS), 4 (2002), pp. 143–200.

[10] ——, *Paquets stables des séries discrètes accessibles par endoscopie tordue; leur paramètre de Langlands*, preprint, (2013).

[11] C. Mœglin and M. Tadić, *Construction of discrete series for classical p-adic groups*, J. Amer. Math. Soc., 15 (2002), pp. 715–786.

[12] G. Muić, *Composition series of generalized principal series; the case of strongly positive discrete series*, Israel J. Math., 140 (2004), pp. 157–202.

[13] F. Shahidi, *A proof of Langlands' conjecture on Plancherel measures; complementary series for p-adic groups*, Ann. of Math. (2), 132 (1990), pp. 273–330.

[14] ——, *Twisted endoscopy and reducibility of induced representations for p-adic groups*, Duke Math. J., 66 (1992), pp. 1–41.

[15] M. Tadić, *Structure arising from induction and Jacquet modules of representations of classical p-adic groups*, J. Algebra, 177 (1995), pp. 1–33.

[16] ——, *On invariants of discrete series representations of classical p-adic groups*, Manuscripta Math., 135 (2011), pp. 417–435.

[17] ——, *On tempered and square integrable representations of classical p-adic groups*, Sci. China Math., 56 (2013), pp. 2273–2313.

[18] A. V. Zelevinsky, *Induced representations of reductive p-adic groups. II. On irreducible representations of* $GL(n)$, Ann. Sci. École Norm. Sup. (4), 13 (1980), pp. 165–210.

Ivan Matić
Department of Mathematics, University of Osijek
Trg Ljudevita Gaja 6, Osijek, Croatia
E-mail: imatic@mathos.hr