Mathematics and Its Applications

D. S. Mitrinović, J. E. Pečarić and V. Volenec

Recent Advances in Geometric Inequalities

Springer-Science+Business Media, B.V.

Recent Advances in Geometric Inequalities

Mathematics and Its Applications (*East European Series*)

Recent Advances in Geometric Inequalities

D. S. Mitrinović
University of Belgrade, Yugoslavia

J. E. Pečarić
University of Zagreb, Yugoslavia

and

V. Volenec
University of Zagreb, Yugoslavia

SPRINGER-SCIENCE+BUSINESS MEDIA, B.V.

Mitrinović, Dragoslav S.
Recent advances in geometric inequalities.

(Mathematics and its applications (East European series))
Includes indexes.
1. Isoperimetric inequalities. I. Pečarić, J. E. II. Volenec, V. III. Title. IV. Title: Geometric inequalities. V. Series: Mathematics and its applications (Kluwer Academic Publishers). East European series.
QA447.M57 1989 516'.17 89-2450

ISBN 978-90-481-8442-2 ISBN 978-94-015-7842-4 (eBook)
DOI 10.1007/978-94-015-7842-4

printed on acid free paper

Contents

SERIES EDITOR'S PREFACE

Approach your problems from the right end and begin with the answers. Then one day, perhaps you will find the final question.

'The Hermit Clad in Crane Feathers' in R. van Gulik's *The Chinese Maze Murders*.

It isn't that they can't see the solution. It is that they can't see the problem.

G.K. Chesterton. *The Scandal of Father Brown* 'The point of a Pin'.

Growing specialization and diversification have brought a host of monographs and textbooks on increasingly specialized topics. However, the "tree" of knowledge of mathematics and related fields does not grow only by putting forth new branches. It also happens, quite often in fact, that branches which were thought to be completely disparate are suddenly seen to be related.

Further, the kind and level of sophistication of mathematics applied in various sciences has changed drastically in recent years: measure theory is used (non-trivially) in regional and theoretical economics; algebraic geometry interacts with physics; the Minkowsky lemma, coding theory and the structure of water meet one another in packing and covering theory; quantum fields, crystal defects and mathematical programming profit from homotopy theory; Lie algebras are relevant to filtering; and prediction and electrical engineering can use Stein spaces. And in addition to this there are such new emerging subdisciplines as "experimental mathematics", "CFD", "completely integrable systems", "chaos, synergetics and large-scale order", which are almost impossible to fit into the existing classification schemes. They draw upon widely different sections of mathematics. This programme, Mathematics and Its Applications, is devoted to new emerging (sub)disciplines and to such (new) interrelations as exempla gratia:

- a central concept which plays an important role in several different mathematical and/or scientific specialized areas;
- new applications of the results and ideas from one area of scientific endeavour into another;
- influences which the results, problems and concepts of one field of enquiry have and have had on the development of another.

The Mathematics and Its Applications programme tries to make available a careful selection of books which fit the philosophy outlined above. With such books, which are stimulating rather than definitive, intriguing rather than encyclopaedic, we hope to contribute something towards better communication among the practitioners in diversified fields.

Because of the wealth of scholarly research being undertaken in the Soviet Union, Eastern Europe, and Japan, it was decided to devote special attention to the work emanating from these particular regions. Thus it was decided to start three regional series under the umbrella of the main MIA programme.

Geometric inequalities have a wide range of applications, both within geometry itself as well beyond the traditional areas of geometry and geometric applications. For example, in the theory of complex functions, in the calculus of variations (broadly speaking), in the theory of embedding theorems for function spaces, and, more generally, in providing a priori estimates in several areas, e.g. differential equations, they are invaluable tools.

This book is not about these application areas; instead it is a unique and systematic encyclopaedic collection of geometric and related inequalities, which, I feel, will be of considerable value to mathematicians and scientists of widely varying signatures, not least because the authors have taken considerable pains to include also results published in the less accessible languages.

The unreasonable effectiveness of mathematics in science ...

Eugene Wigner

Well, if you know of a better 'ole, go to it.

Bruce Bairnsfather

What is now proved was once only imagined.

William Blake

As long as algebra and geometry proceeded along separate paths, their advance was slow and their applications limited.

But when these sciences joined company they drew from each other fresh vitality and thenceforward marched on at a rapid pace towards perfection.

Joseph Louis Lagrange.

Bussum, February 1988 Michiel Hazewinkel

PREFACE

In the following text we shall use the abbreviation AGI for this book, and the abbreviation GI for the book:

O. Bottema, R. Ž. Djordjević, R. R. Janić, D. S. Mitrinović, and P. M. Vasić. Geometric Inequalities, Groningen, 1969, 151 pp.

The book GI is very appreciated and has been much quoted in the mathematical literature. It contains about 400 varied geometric inequalities related to the elements of figures in the plane (triangles, quadrilaterals, n-gons, circles), and 225 authors are cited in it. The book AGI contains several thousands of inequalities, not only for elements of figures in the plane, as GI, but also for elements of figures in space and hyperspace (tetrahedra, polyhedra, simplices, polytopes, spheres). AGI cites over 750 names, and some of them are cited several times. The text has been updated and a lot of the most recent results up to the end of 1986 are included in AGI. AGI also contains, apart from numerous particular results, various methods for proof and for formation of geometric inequalities. This is the essential characteristic of AGI. It also contains many conjectures and unsolved problems and, consequently, will serve to provoke and inspire further research.

AGI is, at first glance, a synthesis of a large number of unconnected results, i.e., a unified and complete exposition of various geometric inequalities. We have insisted on finding the original source of each result, so that several historical priorities have been ascertained in AGI. AGI contains many new unpublished results addressed to the authors from several mathematicians and from the authors themselves.

Material on which data are difficult to obtain is presented in greater detail. It is important to note that results published in Chinese, Japanese, Serbo-Croatian, Bulgarian, Romanian, Hungarian and Dutch have also been considered. Such results are frequently quite unknown in the U.S.A., Canada and in Europe.

Chapters I-XVII were written on the basis of the very extensive literature published since 1968. However, many geometric inequalities, proved in the 19th and at the beginning of the 20th century, were forgotten and some of them later rediscovered. Such results, when they are not contained in GI from 1969, are incorporated in AGI. Special attention was paid to the existence of a triangle (Chapter I), and to the transformations (Chapters II, V, VII, XI). The same is true for some important geometric inequalities, i.e. the book contains complete reviews and unified treatments of recent results concerning the following inequalities: fundamental inequality - I.1; Gerretsen's inequalities - III.4; asymmetric trigonometric inequalities - VI.1 and XV.24; Finsler-Hadwiger's inequality - VII.1 and 2; polar moment of inertia inequality - XI.1, 3 and XVIII.2.22; Erdős-Mordell's inequality - XI.5 and XV.25; Neuberg-Pedoe's and Oppenheim's inequalities - XII.3 and XVIII.4.4; and Möbius-Neuberg's and Möbius-Pompeiu's theorems - XIII. Of course, many of these inequalities were considered in GI, but after the appearance of that book many new related results have appeared.

Chapters XVIII, XIX, and XX include inequalities about elementary figures in three-dimensional and n-dimensional Euclidean space. Here, by elementary figures we mean convex polyhedrons and polytopes, surfaces and hypersurfaces of the second order and finite sets of points. Thus, the inequalities about general convex sets and about regular infinite sets of points or other elements (such as tesselations, packings and coverings) are not taken into consideration, because there is a very abundant monographical literature about these objects.

The substance of Chapters XVIII-XX has not appeared in such a form until now, as opposed to the case of Chapters I-XVII, which were partially covered by GI. Therefore, during the preparation of these three chapters we took into consideration not only the recent advances but the older results, too. The older inequalities included in Chapter XX have been tossed about in a great number of books which will not be cited here. We take note only of the book F. G.-M., 'Exercices de géométrie, 4. éd., Paris-Tours 1912', which is probably the most complete of all books of this kind.

In the greatest part of the book we arranged inequalities with respect to the figures in which they appear, from the most general to the most special figures. The single inequalities are arranged with respect to the elements of figures which appear in them, and not with respect to the logical connections of their proofs. Only the shorter and simpler proofs are included in the book.

With the encyclopaedic content developed above, AGI will be a unique book in the existing mathematical literature. The expert will find some new material, since a state-of-the-art report has been given in the book about the results of geometric inequalities. However, the book is intended for a wide circle of readers - for the students attending high schools, colleges, universities, as well as for teachers and professors of colleges and universities. We had in mind that its content is extremely various - comprising material of very diverse levels of comprehension.

There is a great probability that in voluminous work, such as AGI, major or minor errors and various omissions occur. They can be made in all steps of the creation and production of a book. It is considered that there is no error-free book. In order that the errors in AGI be reduced at minimum, the authors turned for help to several mathematicians in the world, who offered their advice and criticism.

During the preparation of AGI the authors were in touch with many mathematicians dealing with geometric inequalities who read some of the preliminary versions of the chapters or sections of the manuscript and gave their comments. This enabled AGI to present a lot of unpublished inequalities obtained through private communications which have been cited. Thus, a great number of comments given in different stages of our work is included in AGI.

J. F. Rigby has taken part in writing Chapter III, and W. Janous in writing Chapter VI.

W. Janous and C. Tănăsescu have given considerable help as they have already read the whole manuscript. In many cases they improved the text by their suggestions and comments.

Without their assistance many misprints and even errors would probably have remained unnoticed.

The following mathematicians: A. Bager, K. Baron, S. J. Bilčev, O. Bottema, V. Čepulić, H. S. M. Coxeter, H. Demir, V. Devidé, L. Fejes-Tóth, J. Garfunkel, J. T. Groenman, S. Iwata, Dj. Kurepa, S. Kurepa, V. Mascioni, M. T. McGreggor, D. M. Milošević, A. Oppenheim, D. Pedoe, M. J. Pelling, K. Post, J. F. Rigby, D. Svrtan, G. Tsintsifas,

G. R. Veldkamp, D. Veljan kindly read different portions of AGI in various stages of preparations of the manuscript and made valuable suggestions, corrections, additions or comments.

R. R. Janić and B. Crstici assisted in collecting documentary material.

We have received invaluable help from R. R. Janić, D. Dj. Tošić and W. Janous, who have carefully read all the proofs of this book and have provided us with useful suggestions.

The authors feel indebted to all those mentioned above for the help which they gave, in one way or another.

We intend to keep a systematic check of the further development of geometric inequalities, and to make, from time to time, a research-expository paper, as well as perhaps a second edition of AGI, revised and updated, if AGI provokes a sufficient interest as a reference tool in research work.

We therefore invite the readers of AGI to send us any comment on the content and form of AGI, as well as on the methods used in the book so that later editions can be more complete and more accurate.

In particular, we invite the authors of papers from which some results were included in AGI to communicate us comments if their contributions are not presented correctly or completely.

After the appearance of GI (called "Bible of Bottema" in the Canadian journal Crux Mathematicorum) in 1969, during the period from 1969-1986 a large number of papers and problems concerning geometric inequalities were published in mathematical journals and this inspired us to compile encyclopaedic work AGI. We hope that AGI like GI will stimulate and motive new investigations in the development of geometric inequalities - branch of mathematics which permanently interested and attracted mathematicians from the 18th century to nowadays. We consider that this book is a good base for the various synthesis of apparently unconnected results about geometric inequalities, and also represents a rich source book for obtaining some deeper and essential generalizations.

The authors also wish to express their appreciation to D. Reidel Publishing Company, well known for their high-quality productions, and especially to the publisher Dr. D. J. Larner, assistant publisher O. A. Pols, as well as the series editor Professor M. Hazewinkel, for their most efficient handling of the publication of this book.

January 1, 1988
Belgrade/Zagreb
Yugoslavia

D. S. Mitrinović
Smiljanićeva 38
11000 Belgrade, Yugoslavia

J. E. Pečarić
Tehnološki fakultet
Ive Lole Ribara 126/3
41000 Zagreb, Yugoslavia

V. Volenec
Prirodoslovno-matematički fakultet
P. O. Box 187
41001 Zagreb, Yugoslavia

NOTATION AND ABBREVIATIONS

In this book the following notations are used (if not stated differently):

A, B, C	vertices or angles of a triangle
a, b, c	sides BC, CA, AB
h_a, h_b, h_c	altitudes
m_a, m_b, m_c	medians
w_a, w_b, w_c	angle-bisectors
R	radius of circumcircle
r	radius of incircle
s	semi-perimeter
r_a, r_b, r_c	radii of excircles
M_a, M_b, M_c	medians extended to the circumcircle
W_a, W_b, W_c	angle-bisectors extended to the circumcircle
H_a, H_b, H_c	altitudes extended to the circumcircle
O	circumcentre
I	incentre
H	orthocentre
G	centroid
I_a, I_b, I_c	excentres
K	Lhuilier-Lemoine point
Γ	Georgone point
N	Nagel point
Ω_1, Ω_2	Crelle-Brocard points
ω	Crelle-Brocard angle
S	center of Spieker's circle
g_a, g_b, g_c	Gergonne cevians
n_a, n_b, n_c	Nagel cevians
F or Δ	area of ABC
P	point in the interior of a triangle
R_1, R_2, R_3	distances from P to the vertices of ABC
r_1, r_2, r_3	distances from P to the sides of ABC
w_1, w_2, w_3	angle-bisectors of the angles BPC, CPA, APB
r_1', r_2', r_3'	Cevian segments PD, PE, PF

$x = s - a$, $y = s - b$, $z = s - c$

Σ	cyclic sum, for example: $\Sigma f(a)=f(a)+f(b)+f(c)$, $\Sigma f(b, c)=f(b, c)+ f(c, a)+f(a, b)$

Π	cyclic product, for example: $\Pi(bc) = \Pi bc = bc \cdot ca \cdot ab$, $\Pi GA = GA \cdot GB \cdot GC$.
$Q = \Sigma(b-c)^2$	
$MN^2 = \overline{MN}^2 = \|MN\|^2$	
$M_r(x, y, \ldots)$	mean of order r of the numbers x, y, ... For example for numbers x, y, z $M_r(x) = M_r(x, y, z) = (xyz)^{1/3}$ for $r = 0$ $= (\frac{1}{3} \Sigma x^r)^{1/r}$ for $r \neq 0$, $\|r\| < +\infty$ $= \min(x, y, z)$ for $r = -\infty$ $= \max(x, y, z)$ for $r = +\infty$
A, B, C, D	vertices or angles of a quadrilateral
a, b, c, d	sides AB, BC, CD, DA
p, q	diagonals AC, BD
L = 2s	perimeter of ABCD
F	area of ABCD
$A_1, \ldots, A_n$	vertices or angles of an n-gon
$a_1, \ldots, a_n$	its sides
L = 2s	its perimeter
F	its area
P	point in the interior of an n-gon
$R_k = PA_k$	
r_k	distance from P to the side $a_k = A_kA_{k+1}$
w_k	segment of the bisector of the angle $A_kPA_{k+1} = 2\delta_k$ from P to its intersection with the side a_k
{E}	equality is valid if and only if the triangle is equilateral
a/2b, 1/sin A sin B	a/(2b), 1/(sin A sin B)
{...}	equality is valid if and only if the conditions mentioned inside the accolade are satisfied
[n]*	in the paper [n] only the extremal case of the inequality was found but this inequality does not appear in the paper [n].
iff	if and only if
	D. S. Mitrinović (in cooperation with P. M. Vasić), <u>Analytic Inequalities</u>. Berlin-Heidelberg-New York, 1970.
GI	O. Bottema, R. Ž. Djordjević, R. R. Janić, D. S. Mitrinović, and P. M. Vasić, <u>Geometric Inequalities</u>, Groningen, 1969.
SM	V. P. Soltan and S. I. Mejdman, <u>Toždestva i neravenstva v treugol'nike</u>, Kišinev, 1982.
MO	A. W. Marshall and I. Olkin, <u>Inequalities: Theory of Majoration and Its Applications</u>, New York-London-Toronto-Sydney-San Francisco, 1979.

Other symbols are defined in the text.

ORGANIZATION OF THE BOOK

Besides the Preface, Notation and Abbreviations, and the Indexes, the book contains twenty chapters, each of which is divided into a number of sections, some of these into subsections some of which in smaller divisions. As a rule, the numeration of theorems/remarks is continuous throughout a subsection, or a section which does not contain subsections, or a chapter which does not contain sections.

After the end of the chapters I-VIII, XIII and XVIII-XX are quoted the bibliographical references. In the other chapters the references are mentioned after the sections, subsections or the smaller divisions of subsections.

The abbreviations of the cited journals are given according to Mathematical Reviews.

The book contains 16 figures, and graphs.

Chapter I

THE EXISTENCE OF A TRIANGLE*

0. Introduction

This Chapter is concerned with existence of a triangle satisfying prescribed conditions. Of course, it is well-known that a, b, c are sides of a triangle if and only if $a, b, c \geqslant 0$, $b+c \geqslant a$, $c+a \geqslant b$, $a+b \geqslant c$. If we wanted to rule out degenerate triangles, we would have to omit the equality signs.

Similarly, A, B, C are angles of a triangle if and only if

$$A, B, C \geqslant 0, \quad A + B + C = \pi,$$

with equality in the first conditions for degenerate triangles.

Note that apart from the sides a, b, c and the angles A, B, C the most important concepts in the geometry of the triangle are the variables R, r, s, where R is the radius of the circumcircle, r is the radius of the incircle and s is the semi-perimeter. The inequality which gives the necessary and sufficient condition for the existence of a triangle with given values of R, r, s is known in the literature as the 'fundamental inequality'. This inequality is considered in the first Section of this Chapter, which gives a history of the fundamental inequality and some critical analyses. It is interesting to note that this important inequality has been rediscovered a number of times in very different forms. Further, some authors called this inequality 'Blundon's fundamental inequality' although it was known more than a hundred years (1851) before the appearance of Blundon's papers. We also give several proofs of the fundamental inequality. Many of these proofs are new. At the end of this Section we give a geometric interpretation of the fundamental inequality [8].

The second Section gives a series of results concerning necessary and sufficient conditions for the existence of a triangle in terms of some other given elements of a triangle. Note that most of these results are due to G. Petrov [18], and in the book 'Geometric Inequalities' only a few of his results were stated. We hope that this Section, especially, will initiate some new contributions.

The third Section gives results about the existence of a triangle whose sides are equal to specified elements of another triangle. This part connects numerous isolated results, new and old, comments, and new proofs. Of course, we expect many similar contributions in the future.

1. The Fundamental Inequality

1.1. History

(a) In 1851, as an answer to Ramus' question, É. Rouché proved

* Chapter XIII also contains many criteria for the existence of a triangle and of some other figures in E^2 and E^3

the following result (see also GI 7.11):

THEOREM A

$$r\left(2R^2 + 10Rr - r^2 - 2\sqrt{R(R - 2r)^3}\right)^{1/2} \leqslant F \leqslant r\left(2R^2 + 10Rr - r^2 + 2\sqrt{R(R - 2r)^3}\right)^{1/2}. \tag{1}$$

If one of the signs $\leqslant$ in (1) means =, then the triangle is isosceles; and vice versa. If both signs $\leqslant$ in (1) stand for =, then the triangle is equilateral; and vice versa.

Note that using the formula $F = rs$, a simple transformation of (1) gives the following inequalities

$$2R^2 + 10Rr - r^2 - 2(R - 2r)\sqrt{R^2 - 2Rr} \leqslant s^2 \leqslant 2R^2 + 10Rr - r^2 + 2(R - 2r)\sqrt{R^2 - 2Rr}. \tag{2}$$

(b) E. Lemoine proved in 1891 the following result of R. Sondat from 1890 ([2], see also GI 13.8):

THEOREM B. A necessary and sufficient condition for the existence of a triangle with elements R, r and s, is

$$s^4 - 2(2R^2 + 10Rr - r^2)s^2 + r(4R + r)^3 \leqslant 0. \tag{3}$$

Equality in (3) holds only if the triangle is isosceles.

Note that two other results are also given in [2]. For the fundamental inequality the following result is important:

THEOREM B1. Let S be the area of the triangle OIH and let $a > b > c$; then

1° $\quad S = 2R^2 \sin\dfrac{B - C}{2} \sin\dfrac{A - C}{2} \sin\dfrac{A - B}{2};$

2° $\quad S = \dfrac{(b - c)(a - c)(a - b)}{8r};$

3° $\quad 16S^2 = -s^4 + 2(2R^2 + 10Rr - r^2)s^2 - r(4R + r)^3.$

Remark. The inequalities (1), (2), and (3) are equivalent, and Theorem B states that for example (2) is not only valid for every triangle, but conversely, if it is satisfied by R, r, and s, there exists a triangle with these data.

(c) Note that Theorem A is stated in the well-known book [3] dating from 1896.

(d) S. Nakajima in 1925 and 1926 ([4], [5]) proved the following result:

THEOREM D.

$$4R\left(R - \frac{2F}{s}\right)^3 \geqslant \left(s^2 + \frac{F^2}{s^2} - 2R^2 - \frac{10RF}{s}\right)^2 \tag{4}$$

with equality if and only if the triangle is isosceles.

In the proof he used the formula $F = rs$ and Theorem A, but as reference for Theorem A he gave [3]. Note that Theorem D is given in GI 7.8 but there we had to put $s = a + b + c$.

(e) In 1957, R. Frucht [6] proved the following result (GI 4.19):

THEOREM E. If $q = (\frac{1}{2}\Sigma(a - b)^2)^{1/2}$, then

$$\frac{1}{27}s(s + q)^2(s - 2q) \leqslant F^2 \leqslant \frac{1}{27}s(s - q)^2(s + 2q). \tag{5}$$

The first (second) equality sign in (5) holds for an isosceles triangle whose base is the largest (smallest) of the three sides; of course both equality signs apply when the triangle is equilateral, since then $q = 0$.

(f) In two important papers ([7], [8]) W. J. Blundon has drawn attention to these results. He proved the following theorem (see also GI 5.10):

THEOREM F. Let $(R,r) \to f(R,r)$ and $(R,r) \to F(R,r)$ be homogeneous real functions for $R, r > 0$. Then the strongest possible inequalities of the form

$$f(R, r) \leqslant s^2 \leqslant F(R, r) \tag{6}$$

are given by

$$f(R,r) = 2R^2 + 10Rr - r^2 - 2(R - 2r)\sqrt{R^2 - 2Rr}, \tag{7}$$

and

$$F(R,r) = 2R^2 + 10Rr - r^2 + 2(R - 2r)\sqrt{R^2 - 2Rr}, \tag{8}$$

with simultaneous equality only if the triangle is equilateral.

Blundon proved his theorem using the identity

$$\begin{aligned} -s^4 + 2(2R^2 + 10Rr - r^2)s^2 - r(4R + r)^3 &= \\ &= \frac{1}{4r^2}(a - b)^2(b - c)^2(c - a)^2 \\ &= \frac{1}{4r^2}(x - y)^2(y - z)^2(z - x)^2, \end{aligned} \tag{9}$$

where $x = s - a$, etc. Note that (9) is a simple consequence of 2° and 3° from Theorem B1.

Of course, (6) (with (7) and (8)) is the inequality (2). Blundon gave this form of the fundamental inequality and proved that these inequalities are the best possible.

(g) O. Bottema [9] proved the following theorem (see GI 14.27):

THEOREM G. If d denotes the distance between the circumcentre and the incentre of a triangle, then

$$(10)\qquad (R - d)(3R + d)^3 \leqslant 4R^2s^2 \leqslant (R + d)(3R - d)^3.$$

(h) In 1971, O. Bottema [10] considered Blundon's result. He noted that this is an old result and he called this result the 'fundamental inequality'. Furthermore, he gave the following form of the fundamental inequality

$$(11)\qquad I = (r^2 + s^2)^2 + 12Rr^3 - 20Rrs^2 + 48R^2r^2 - 4R^2s^2 + 64R^3r \leqslant 0,$$

and also a new geometric interpretation of it (see part 2.3).

(i) R. Frucht and M. S. Klamkin [11] considered the best possible inequalities in the form (6), but in the case when $f(R,r)$ and $F(R,r)$ are quadratic forms. Their results are generalizations of some inequalities of Gerretsen, Steinig and Blundon, but also a correction of a result of Blundon. In the proof they used the fundamental inequality.

(j) Some remarks on Bottema's geometric interpretation of fundamental inequality were given in [12].

(k) A new proof of the fundamental inequality was given in [13], and also a modification of Blundon's proof of Theorem F in [14] (see part 1.2 of this Section).

(l) Equality cases of Theorem E are also valid for inequalities (1), (2), and (9), because they are equivalent (see [15]).

Note that in GI 5.10, 7.11 and 14.27 it is specified that equality cases occur if and only if the triangle is equilateral, i.e. when both equalities occur.

A. Lupaş in [15] also proved the identity (9).

(m) V. N. Murty [16] gave some remarks about Bottema's geometric interpretation of the fundamental inequality. He considered this inequality in the forms (11) and

$$(12)\qquad 4R(R - 2r)^3 \geqslant (s^2 - 2R^2 - 10Rr + r^2)^2,$$

which could be deduced easily from (4).

(n) Two proofs of the fundamental inequality and several similar results are given in [17] (see part 3. of this Section).

(o) Of course, one can formulate several results similar to Theorem B. For example, since $F = rs$, the following two theorems are also valid:

THEOREM O1. A necessary and sufficient condition for the existence of a triangle, with elements R, r and F, is that

$$(13)\qquad F^4 - 2r^2(2R^2 + 10Rr - r^2)F^2 + r^5(4R + r)^3 \leqslant 0.$$

THEOREM O2. A necessary and sufficient condition for the existence of a triangle, with the elements R, F, and s, is (4) or

$$(14)\qquad s^4 - 2\left(2R^2 + 10R\,\frac{F}{s} - \frac{F^2}{s^2}\right)s^2 + \frac{F}{s}\left(4R + \frac{F}{s}\right)^2 \leqslant 0.$$

Since the set of variables (R, r, s), (R, r, F) and (R, F, s) are equivalent, in the following we shall use only the triple(R, r, s).

1.2. Proofs

(a) First, we shall give Blundon's proof of Theorem F with Rigby's modification (see [8] and [14]):

From the identity (9) it follows that (3) holds, which is equivalent to (2), since inequality (3) is quadratic in s^2.

Further, we know that $R - 2r \geqslant 0$ (Chapple-Euler's inequality, GI 5.1), and the distance d between its circumcentre and incentre is given by $d^2 = R(R - 2r)$; hence all triangles having given values of R and r can be regarded as having the same circumcircle and incircle.

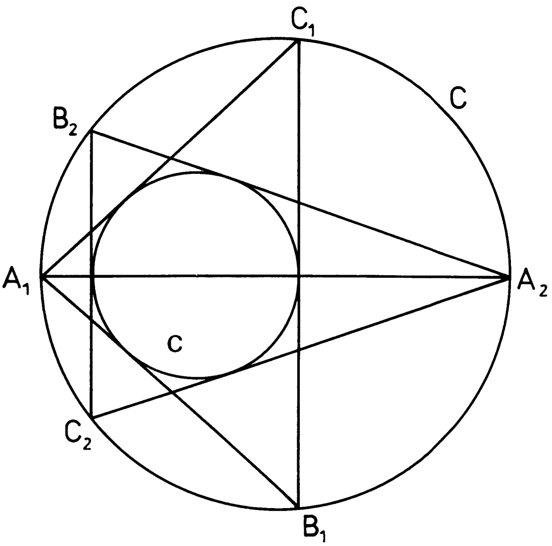


Fig. 1.

In the figure, circles C and c of radii R and r have their centres a distance d apart. B_1C_1 and B_2C_2 (both tangent to circle c) are perpendicular to the line joining the centres of the two circles.

It is not difficult to show that the triangles $A_1B_1C_1$ and $A_2B_2C_2$ have semiperimeters s_1 and s_2 given by

$$s_1^2 = f(R,r), \quad s_2^2 = F(R,r),$$

(since $f(R,r) \leqslant s^2 \leqslant F(R,r)$, it follows that, of all triangles inscribed in C and circumscribed to c, $A_1B_1C_1$ has the smallest perimeter and $A_2B_2C_2$ the largest). Now, let $s^2 \geqslant \phi(R,r)$ be any inequality valid for all triangles. For any given values of R, r (subject to the condition $R \geqslant 2r \geqslant 0$), we have just seen that there exists a triangle $A_1B_1C_1$ for which $s^2 = f(R,r)$. The inequality $s^2 \geqslant \phi(R,r)$ is valid for this triangle; hence $f(R,r) \geqslant \phi(R,r)$. This is true for all values of R, r such that $R \geqslant 2r \geqslant 0$, so $s^2 - f(R,r) \geqslant 0$ is a better inequality than $s^2 - \phi(R, r) \geqslant 0$. A similar argument holds for $F(R, r)$.

(b) Note that the major part of the proofs depends upon the following three theorems. Although the results are well-known (see for example [17]) we shall give their proofs for the sake of completeness.

THEOREM 1. (Sturm) The equation

$$t^3 + ut^2 + vt + w = 0, \tag{15}$$

with real coefficients u, v, w has real roots t_1, t_2, t_3 if and only if

$$(t_1 - t_2)^2(t_2 - t_3)^2(t_3 - t_1)^2 = -4u^3w + u^2v^2 + 18uvw - 4v^3 - 27w^2 \geqslant 0. \tag{16}$$

Proof. That the equality in (16) holds we can show using the Viète formulas. Further, if the roots are real, the inequality in (16) is obvious. But, if not, one root is real (say t_1) and two are complex conjugate (say $t_2 = A + Bi$ and $t_3 = A - Bi$, $B \neq 0$), and we have

$$(t_1 - t_2)(t_2 - t_3)(t_3 - t_1) = 2Bi((t_1 - A)^2 + B^2),$$

i.e. the reversed inequality in (16) is valid.

THEOREM 2. The equation (15) has positive roots if and only if (16) and

$$u < 0, \quad v > 0, \quad w < 0, \tag{17}$$

hold.

Proof. If the roots are positive, Viète's formulas give (17). Of course the roots are real, so (16) must be fulfilled.

Conversely, if u, v, w fulfill the conditions (16) and (17), then Theorem 1 implies that the roots of (15) must be real. Suppose that $t_1 \leqslant 0$, then (17) implies that the expression $t_1^3 + ut_1^2 + vt_1 + w$ is negative, which is a contradiction.

THEOREM 3. The roots of the equation (15) are the lengths of sides of a triangle if and only if (16), (17) and

$$u^3 - 4uv + 8w > 0 \tag{18}$$

hold.

Proof. Using Viète's formulas we have

$$(t_1 + t_2 - t_3)(t_1 - t_2 + t_3)(-t_1 + t_2 + t_3) = u^3 - 4uv + 8w.$$

So, (18) must also hold. Since the sides are real and positive, we infer that (16) and (17) must hold.

Now, we give several similar proofs of the fundamental inequality which depend on Theorems 1, 2, and 3.

The following results are known:

1) a, b, c are roots of the equation (GI, p. 72, [1], [2], [17, p. 17]):

$$t^3 - 2st^2 + (s^2 + r^2 + 4Rr)t - 4sRr = 0.$$

2) x, y, z (x = s - a, etc.) are roots of the equation (GI p. 72, [1], [2], [17, p. 19]):

$$t^3 - st^2 + r(4R + r)t - sr^2 = 0.$$

3) h_a, h_b, h_c are roots of the equation ([17, p. 21]):

$$2Rt^3 - (s^2 + r^2 + 4Rr)t^2 + 4s^2rt - 4s^2r^2 = 0.$$

4) r_a, r_b, r_c are roots of the equation (GI p. 49, [17, p. 23]):

$$t^3 - (4R + r)t^2 + s^2t - s^2r = 0.$$

5) sin A, sin B, sin C are roots of the equation

$$4R^2t^3 - 4Rst^2 + (s^2 + r^2 + 4Rr)t - 2sr = 0.$$

6) cos A, cos B, cos C are roots of the equation ([17, p. 26]):

$$4R^2t^3 - 4R(R + r)t^2 + (s^2 + r^2 - 4R^2)t + (2R + r)^2 - s^2 = 0.$$

7) cotan A, cotan B, cotan C are roots of the equation ([17, p. 28]):

$$2srt^3 - (s^2 - r^2 - 4Rr)t^2 + 2srt + (2R + r)^2 - s^2 = 0.$$

8) tan A, tan B, tan C are roots of the equation ([17, p. 29]):

$$(s^2 - (2R + r)^2)t^3 - 2srt^2 + (s^2 - 4Rr - r^2)t - 2sr = 0.$$

9) $\tan\frac{A}{2}$, $\tan\frac{B}{2}$, $\tan\frac{C}{2}$ are roots of the equation ([17, p. 30]):

$$st^3 - (4R + r)t^2 + st - r = 0.$$

10) $\cot\frac{A}{2}$, $\cot\frac{B}{2}$, $\cot\frac{C}{2}$ are roots of the equation ([17, p. 32]):

$$rt^3 - st^2 + (4R + r)t - s = 0.$$

11) $\sin^2\frac{A}{2}$, $\sin^2\frac{B}{2}$, $\sin^2\frac{C}{2}$ are roots of the equation ([17, p. 34]):

$$16R^2t^3 - 8R(2R - r)t^2 + (s^2 + r^2 - 8Rr)t - r^2 = 0.$$

12) $\cos^2\frac{A}{2}$, $\cos^2\frac{B}{2}$, $\cos^2\frac{C}{2}$ are roots of the equation ([17, p. 34]):

$$16R^2t^3 - 8R(4R + r)t^2 + (s^2 + (4R + r)^2)t - s^2 = 0.$$

Using the substitution $t \to 1/t$, we get the following similar results:

13) cosec A, cosec B, cosec C are roots of the equation:

$$2srt^3 - (s^2 + r^2 + 4Rr)t^2 + 4Rst - 4R^2 = 0.$$

14) sec A, sec B, sec C are roots of the equation:

$$(s^2 - (2R + r)^2)t^3 - (s^2 + r^2 - 4R^2)t^2 + 4R(R + r)t - 4R^2 = 0.$$

15) $\operatorname{cosec}^2\frac{A}{2}$, $\operatorname{cosec}^2\frac{B}{2}$, $\operatorname{cosec}^2\frac{C}{2}$ are roots of the equation:

$$r^2t^3 - (s^2 + r^2 - 8Rr)t^2 + 8R(2R - r)t - 16R^2 = 0.$$

16) $\sec^2\frac{A}{2}$, $\sec^2\frac{B}{2}$, $\sec^2\frac{C}{2}$ are roots of the equation

$$s^2t^3 - (s^2 + (4R + r)^2)t^2 + 8R(4R + r)t - 16R^2 = 0.$$

17) $1/a$, $1/b$, $1/c$ are roots of the equation

$$4sRrt^3 - (s^2 + r^2 + 4Rr)t^2 + 2st - 1 = 0.$$

18) $1/x$, $1/y$, $1/z$ are roots of the equation

$$sr^2t^3 - r(4R + r)t^2 + st - 1 = 0.$$

19) $1/r_a$, $1/r_b$, $1/r_c$ are roots of the equation

$$s^2rt^3 - s^2t^2 + (4R + r)t - 1 = 0.$$

20) $1/h_a$, $1/h_b$, $1/h_c$ are roots of the equation

$$4s^2r^2t^3 - 4s^2rt^2 + (s^2 + r^2 + 4Rr)t - 2R = 0.$$

Using the Theorem 1 and any of the above results we get (3) (i.e. (2)). For example, 1) and 2) were used in [1] (with $s \to F/r$) and [9]; 4) in [13]. Further, using Theorem 2 and any of the results: 2), 3), 4), 9), 10), 17), 18), 19), 20), we get Theorem B directly (i.e. Theorem O1 and O2). Of course, using Theorem 3 and 1) we can also get Theorem B ([17, pp. 54-56]).

1.3. A Geometric Interpretation

In order to investigate the set of triples (R, r, s) satisfying the fundamental inequality (3) i.e. (11), we remark that I is, of course, a homogeneous polynomial and hence only the ratios of R, r, and s are of interest (see [10] or [16]). Therefore, we introduce variables $x > 0$ and $y > 0$ defined by

(19) $$Rx = r \quad \text{and} \quad Ry = s.$$

This transforms (11) into

(20) $$(x^2 + y^2)^2 + 12x^3 - 20xy^2 + 48x^2 - 4y^2 + 64x \leq 0.$$

To each R-r-s triangle there corresponds to a point (x, y) of the graph of (20) in (the first quadrant of) the Cartesian plane; and, conversely, to each point (x, y) of the graph of (20) there corresponds infinitely many R-r-s triangles (one for each $R > 0$, for which r and s are then given by (19)). The graph of (20) is the shaded region in Figure 2.

Since the left member of (20) can be written (see [16]) as:

$$(y^2 - (2 + 10x - x^2))^2 - 4(1 - 2x)^3,$$

arc OA_1 in the figure is the graph of

$$y = \sqrt{(2 + 10x - x^2) - 2(1 - 2x)^{3/2}}, \quad 0 < x \leq 1/2,$$

and arc A_1D is the graph of

$$y = \sqrt{(2 + 10x - x^2) + 2(1 - 2x)^{3/2}}, \quad 0 < x \leq 1/2.$$

(Bottema [10] has shown that arc OA_1 and A_1D are parts of a hypocycloid of three cusps, or deltoid.) The points of arcs OA_1 and A_1D correspond to isosceles R-r-s triangles; the point A_1 corresponds to all equilateral R-r-s triangles; and the points on the segment OD (which are not part of the graph of (20)) correspond to degenerate triangles. Arc OA_1 is concave, arc A_1D is convex, the line $y = \sqrt{3}\,(1 + x)$ is tangent to both arcs at A_1, and the line $x = 0$ is tangent to arc OA_1 at O.

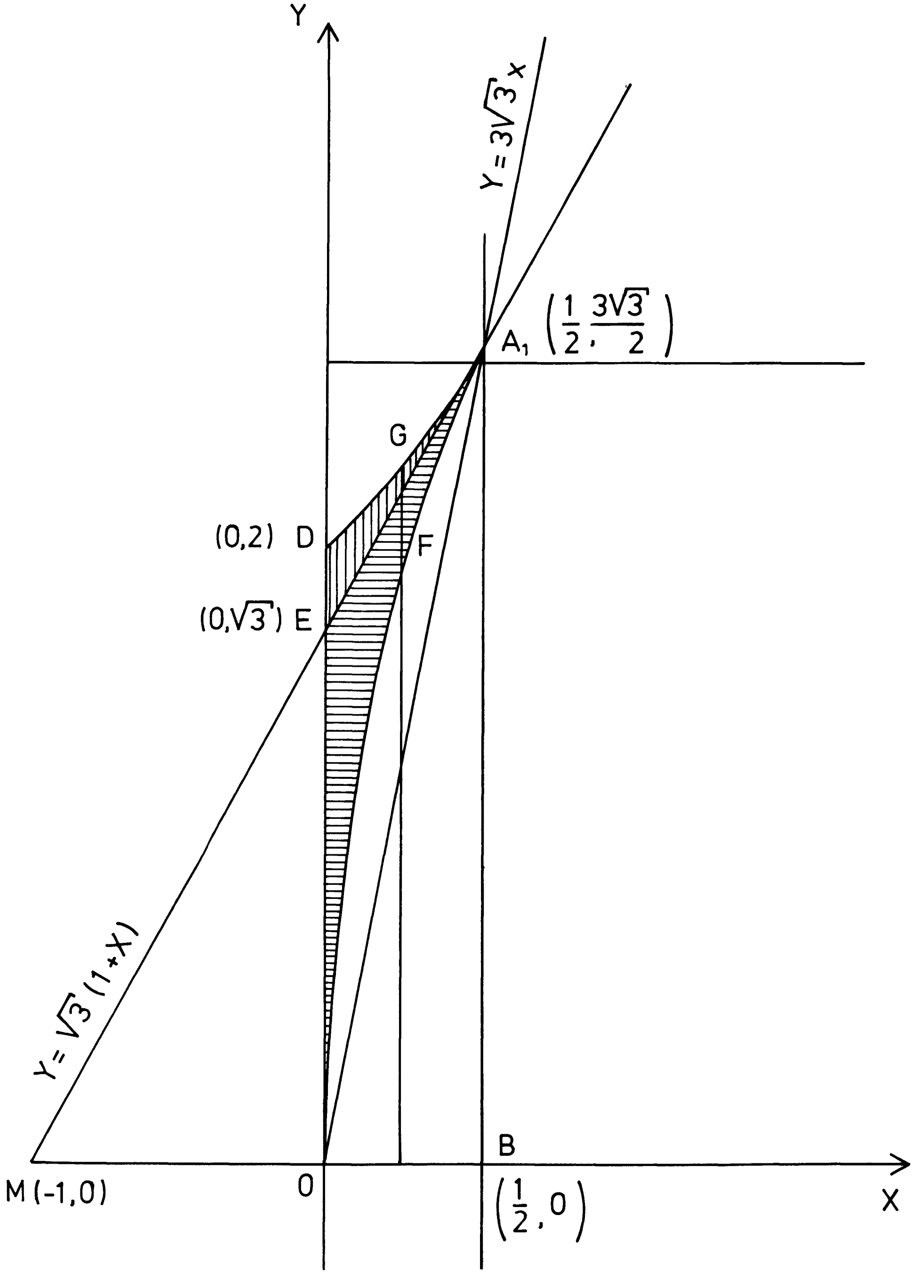

Y
$Y = 3\sqrt{3}\,x$
$A_1\left(\frac{1}{2}, \frac{3\sqrt{3}}{2}\right)$
G
(0,2) D
F
$(0,\sqrt{3})$ E
$Y = \sqrt{3}(1+X)$
B
M (-1,0)
O
$\left(\frac{1}{2}, 0\right)$
X

Fig. 2.

2. The Existence of a Triangle with Given Elements

As we saw, the theorem which states the existence of a triangle with elements R, r and s is the well-known 'fundamental inequality'. Here we shall give several similar results with other elements of the triangle.

1° A necessary and sufficient condition for the existence of a triangle with elements a, b and w_a, is

$$2b\frac{b+a}{2b+a} > w_a; \quad 2b\frac{b-a}{2b-a} < w_a \quad \text{for} \quad b-a > 0$$

and

$$w_a > 0 \quad \text{for} \quad b-a \leqslant 0.$$

If this condition is fulfilled, then there exists one and only one triangle with these given elements (in the following, for such an assertion, we shall write only (A)).

2° A necessary and sufficient condition for the existence of a triangle, with elements b, c and w_a, is

$$0 < w_a < \frac{2bc}{b+c}. \qquad \text{(A)}$$

3° A necessary and sufficient condition for the existence of a triangle, with elements a, b and m_a, is

$$a + 2b - 2m_a > 0, \quad a - 2b + 2m_a > 0, \quad -a + 2b + 2m_a > 0. \qquad \text{(A)}$$

4° A necessary and sufficient condition for the existence of a triangle, with elements b, c and m_a, is

$$b + c - 2m_a > 0, \quad b - c + 2m_a > 0, \quad -b + c + 2m_a > 0. \qquad \text{(A)}$$

5° A necessary and sufficient condition for the existence of a triangle, with elements a, b and r, is

$$r > 0, \quad 4r^6 + 4(2a^2 + 2b^2 + 7ab)r^4 + $$
$$+ 4(a^4 - a^3b - a^2b^2 - ab^3 + b^4)r^2 - a^2b^2(a-b)^2 \leqslant 0$$

If the condition for equality is fulfilled, then there exists only one triangle with these given elements; if not, there exist two triangles (in the following, for such an assertion, we shall write (B)).

6° A necessary and sufficient condition for the existence of a triangle, with elements a, b and h_a, is

$$b \geqslant h_a > 0. \qquad \text{(B)}$$

7° A necessary and sufficient condition for the existence of a triangle, with elements b, c and h_a, is

$$b \geqslant h_a, \quad c \geqslant h_a > 0.$$

If, in the case of two possible equalities, only one is fulfilled, then there exists one and only one triangle with given elements; if not, there exist two triangles (in the following, for such an assertion, we shall write (C)).

8° A necessary and sufficient condition for the existence of a triangle, with elements a, b and R, is

$$a \leqslant 2R, \quad b \leqslant 2R. \qquad \text{(C)}$$

9° A necessary and sufficient condition for the existence of a triangle, with elements a, m_a and w_b, is

$$2a\,\frac{a + 2m_a}{3a + 2m_a} > w_b; \quad 2a\,\frac{a - 2m_a}{3a - 2m_a} < w_b \quad \text{for } a - 2m_a > 0,$$

and

$$w_b > 0 \quad \text{for} \quad a - 2m_a \leqslant 0. \qquad \text{(A)}$$

10° A necessary and sufficient condition for the existence of a triangle, with elements a, m_a and m_b, is

$$3a + 2m_a - 4m_b > 0, \quad 3a - 2m_a + 4m_b > 0,$$

$$-3a + 2m_a + 4m_b > 0. \qquad \text{(A)}$$

11° A necessary and sufficient condition for the existence of a triangle, with elements a, m_b and m_c, is

$$3a + 2m_b - 2m_c > 0, \quad 3a - 2m_b + 2m_c > 0,$$

$$-3a + 2m_b + 2m_c > 0. \qquad \text{(A)}$$

12° A necessary and sufficient condition for the existence of a triangle, with elements a, h_a and h_b, is

$$a \geqslant h_b. \qquad \text{(B)}$$

13° A necessary and sufficient condition for the existence of a triangle, with elements a, h_b and h_c, is

$$a \geqslant h_b, \quad a \geqslant h_c. \qquad \text{(C)}$$

14° A necessary and sufficient condition for the existence of a triangle, with elements a, m_a and R, is either

$$2R \geqslant \frac{a^2 + 4m_a^2}{4m_a} \quad \text{for} \quad 2m_a \neq a, \tag{B}$$

or

$$a = 2R = 2m_a;$$

in the last case an infinity of solutions depending on a real parameter is possible.

15° A necessary and sufficient condition for the existence of a triangle, with elements a, m_a and h_a, is

$$m_a \geqslant h_a. \tag{B}$$

16° A necessary and sufficient condition for the existence of a triangle, with elements a, m_b and h_b, is

$$a \geqslant h_b, \quad m_b \geqslant h_b. \tag{C}$$

17° A necessary and sufficient condition for the existence of a triangle, with elements a, m_a and h_b, is

$$a \geqslant h_b, \quad 2m_a \geqslant h_b. \tag{C}$$

18° A necessary and sufficient condition for the existence of a triangle, with elements h, m_b and h_a, is

$$2m_b \geqslant h_a. \tag{B}$$

19° A necessary and sufficient condition for the existence of a triangle, with elements a, m_b and h_c, is

$$a \geqslant h_c, \quad 2m_b \geqslant h_c. \tag{C}$$

20° A necessary and sufficient condition for the existence of a triangle, with elements a, m_b and R, is

$$\left| \frac{a^2 - 2m_b^2}{m_b} \right| \leqslant 2R, \quad a \leqslant 2R. \tag{C}$$

21° A necessary and sufficient condition for the existence of a triangle, with elements a, h_b and w_b, is

$$a \geqslant h_b, \quad w_b \geqslant h_b. \tag{C}$$

22° A necessary and sufficient condition for the existence of a triangle, with elements a, h_b and w_c, is

$$a \geqslant h_b, \quad w_c^2 < 2a(a + \sqrt{a^2 - h_b^2}). \tag{B}$$

23° A necessary and sufficient condition for the existence of a triangle, with elements a, h_b and R, is

$$2R \geqslant a, \quad h_b \leqslant a, \tag{C}$$

but for $R = \frac{a^2}{2h_b}$ there exists only one solution, too.

24° A necessary and sufficient condition for the existence of a triangle, with elements a, h_b and r, is

$$a \geqslant h_b > 2r. \tag{B}$$

25° A necessary and sufficient condition for the existence of a triangle, with elements a, m_b and w_b, is

$$0 < w_b < a\,\frac{2m_b - a}{m_b} \quad \text{for} \quad m_b - a > 0, \tag{A}$$

and

$$w_b \leqslant m_b \quad \text{for} \quad m_b - a \leqslant 0. \tag{C}$$

26° A necessary and sufficient condition for the existence of a triangle, with elements a, $b + c$ and m_a, is

$$a < b + c, \quad 2m_a < b + c, \quad (b + c)^2 \leqslant a^2 + 4m_a^2. \tag{B}$$

27° A necessary and sufficient condition for the existence of a triangle, with elements a, $b + c$ and h_a, is

$$(b + c)^2 \geqslant a^2 + 4h_a^2. \tag{B}$$

28° A necessary and sufficient condition for the existence of a triangle, with elements a, $b + c$ and w_a, is

$$(b + c)^2 \geq a^2 + 4w_a^2, \quad 2(b + c)w_a > (b + c)^2 - a^2 > 0. \qquad \text{(B)}$$

29° A necessary and sufficient condition for the existence of a triangle, with elements a, w_a and m_a, is

$$m_a \geq w_a > \frac{4m_a^2 - a^2}{4m_a} > 0. \qquad \text{(B)}$$

30° A necessary and sufficient condition for the existence of a triangle, with elements a, b + c and R, is

$$4R \geq \frac{(b + c)^2}{\sqrt{(b + c)^2 - a^2}} \quad \text{for} \quad 2R \geq a. \qquad \text{(C)}$$

31° A necessary and sufficient condition for the existence of a triangle, with elements a, b + c and r, is

$$b + c \geq a\,\frac{a^2 + 4r^2}{a^2 - 4r^2} \quad \text{for} \quad a > 2r. \qquad \text{(B)}$$

32° A necessary and sufficient condition for the existence of a triangle, with elements a, m_a and r, is

$$r \leq \frac{a}{4m_a}(\sqrt{a^2 + 4m_a^2} - a). \qquad \text{(B)}$$

33° A necessary and sufficient condition for the existence of a triangle, with elements a, h_a, and w_a, is

$$h_a \geq w_a. \qquad \text{(B)}$$

34° A necessary and sufficient condition for the existence of a triangle, with elements a, h_a, and R, is

$$8h_aR \geq a^2 + 4h_a^2, \quad 2R \geq a. \qquad \text{(C)}$$

35° A necessary and sufficient condition for the existence of a triangle, with elements a, h_a, and r, is

$$h_a \geq \frac{2a^2r}{a^2 - 4r^2}, \quad a > 2r. \qquad \text{(B)}$$

36° A necessary and sufficient condition for the existence of a triangle, with elements a, w_a, and R, is

1. $8w_a R \geqslant a^2 + 4w_a^2$, for $a < 2w_a$; (A)

2. $2R \geqslant a$, for $a \geqslant 2w_a$,

where we have two solutions in the case

$$8w_a R \leqslant a^2 + 4w_a^2$$

and one solution in the case

$$8w_a R > a^2 + 4w_a^2.$$

37° A necessary and sufficient condition for the existence of a triangle, with elements a, w_a, and r, is

$$w_a \geqslant \frac{2a^2 r}{a^2 - 4r^2} \quad \text{for} \quad a > 2r. \qquad \text{(B)}$$

38° A necessary and sufficient condition for the existence of a triangle, with elements a, R, and r, is

$$\frac{4a^2 r}{a^2 - 4r^2} \geqslant 2R + \sqrt{4R^2 - a^2} \quad \text{for} \quad a > 2r; \quad 2R \geqslant a. \qquad \text{(A)}$$

39° A necessary and sufficient condition for the existence of a triangle, with elements a, m_c, and w_b, is

$$w_b > 4a \frac{a - m_c}{3a - 2m_c} \quad \text{for} \quad a > m_c, \quad w_b > 0 \quad \text{for} \quad a \leqslant m_c;$$

and

$$w_b < 4a \frac{a + m_c}{3a + 2m_c}. \qquad \text{(A)}$$

40° A necessary and sufficient condition for the existence of a triangle, with elements a, m_b, and w_a, is

$$w_a > 4(a - m_b) \frac{a - 2m_b}{3a - 4m_b} \quad \text{for} \quad a - 2m_b \geqslant 0, \quad \text{or}$$

$$w_a > 0 \quad \text{for} \quad a - 2m_b < 0, \quad a - m_b > 0, \quad \text{or}$$

$$w_a > 4(m_b - a)\frac{2m_b - a}{4m_b - 3a} \quad \text{for} \quad a - m_b \leqslant 0,$$

and

$$w_a < 4(a + m_b)\frac{a + 2m_b}{3a + 4m_b}. \tag{A}$$

Remark. The above results are given in the important paper of G. Petrov [18]. In GI 13.7 only 2°, 3°, 5°, 24°, 26°, 34°, and 36° are stated.

The following similar result is given in [19]:

41° A necessary and sufficient condition for the existence of a triangle, with elements a, $b + c$ and angle A ($0 < A < \pi$), is

$$a < b + c \leqslant a/\sin(A/2).$$

42° [43] A necessary and sufficient condition for the existence of a triangle with elements h_c, $a + b$, C, is

$$(a + b)\cos\frac{C}{2} \geqslant 2h_c. \tag{B}$$

43° (GI 13.4). Let p, q be real numbers such that $p + q = 1$. Then a triangle with the sides a, b, c exists if and only if

$$pa^2 + qb^2 \geqslant pqc^2 \quad \text{for all } p, q.$$

Remark. V. T. Janekoski [20], showed that Bohr's inequality for complex numbers (AI, p. 312):

$$a_1|z_1|^2 + a_2|z_2|^2 \geqslant |z_1 + z_2|^2, \quad a_1, a_2 > 0,\ 1/a_1 + 1/a_2 = 1;$$

and 43° are equivalent.

44° A necessary and sufficient condition that three positive numbers u, v, w are lenghts of the sides of a triangle is [21]:

$$\Sigma\frac{u^2 + v^2 - w^2}{2uv} > 1.$$

45° ([41]) Let r_a, r_b, r_c be arbitrary chosen positive numbers. Then there exists one and only one (real) triangle whose exradii are r_a, r_b, r_c; moreover necessarily

$$a = r_a(r_b + r_c)/(\Sigma r_b r_c)^{1/2}, \quad \text{etc.}$$

46° [42] If s, F, and C are given, then there exists a triangle ABC if and only if

$$s \geq \frac{\sqrt{2}(\sin C/2 + 1)\sqrt{F}}{\sqrt{\sin C}}.$$

Remarks. 1° [42] For $C = \pi/2$ we get that there exists a right triangle with semi-perimeter s and area F if and only if

$$s \geq (1 + \sqrt{2})\sqrt{F}.$$

2° Note that the following result is equivalent to the last one.

If s, r, and C are given, then there exists a triangle ABC if and only if

$$s \geq \frac{2(\sin C/2 + 1)^2 r}{\sin C}.$$

3° The result given in 46° is due to H. Ahlburg, and it is a generalization of a problem given by O. Bottema.

47° [29] A necessary and sufficient condition for the existence of a triangle with altitudes h_a, h_b, and h_c is

$$\left|\frac{h_a}{h_b} - \frac{h_a}{h_c}\right| \leq 1 \leq \frac{h_a}{h_b} + \frac{h_a}{h_c} \qquad \text{etc.}$$

Proof. Since $b = ah_a/h_b$, $c = ah_a/h_c$, this is equivalent to

$$|b - c| \leq a \leq b + c \quad (\text{etc.}).$$

3. Some Other Results

In this Section we shall give results concerning the existence of a triangle the sides of which are obtained as elements of any given triangle.

1. Let $x \to f(x)$ be any non-negative non-decreasing subadditive function on the domain $[0, 2s]$. If a, b, c form a triangle, the $f(a)$, $f(b)$, $f(c)$ form a triangle, too.

Remark. This result is given in GI 13.3. In [22] a special case of 1. is given, i.e. it is proved for a positive increasing function $x \to f(x)$ for which $f''(x) < 0$. Of course, the result from [22] is a simple consequence of the well-known Petrović inequality for convex functions (see AI, p. 22). The following special case of the above result is given in [24]:

$$a/(a + 1), \quad b/(b + 1), \quad c/(c + 1) \quad \text{form a triangle.}$$

2. If a triangle is acute, then a^2, b^2, c^2 form a triangle, too.

Remark. This is an old result (see for example GI 11.26).

3. A triangle whose sides are sin A, sin B, sin C exists.

Proof 1. This is a simple consequence of triangle inequalities $a < b + c$, etc. and $a = 2R \sin A$, etc.

Proof 2. ([17]) This follows from Theorem 3 and 5) of 1.2.

4. (GI 13.6) A triangle whose sides are $\cos\frac{A}{2}$, $\cos\frac{B}{2}$, $\cos\frac{C}{2}$ exists.

Proof. This is a consequence of 3. since $A' = (\pi - A)/2$, $B' = (\pi - B)/2$, $C' = (\pi - C)/2$ are angles of a triangle.

5. (GI 13.5) A triangle the sides of which are $\cos^2 \frac{A}{2}$, $\cos^2 \frac{B}{2}$, $\cos^2 \frac{C}{2}$ exists.

Proof. ([17]) This is a consequence of Theorem 3 and 12) of 1.2.

Remark. Using 5. and 1. for the function $f(x) = \sqrt{x}$, we get 4.

Further, we shall show that some results from [21], [25-27] are also consequences of 3.

6. If $0 < A, B, C < \pi/2$, then a triangle the sides of which are $\sin 2A$, $\sin 2B$, $\sin 2C$ exists.

Proof. This follows from 3., since $A' = \pi - 2A$, $B' = \pi - 2B$, $C' = \pi - 2C$ are angles of a triangle.

7. If $0 < A, B < \pi/2$, $0 < C < \pi$ then a triangle the sides of which are $\cos A$, $\cos B$, $\sin C$ exists.

Proof. This follows from 3., since $A' = \frac{\pi}{2} - A$, $B' = \frac{\pi}{2} - B$, $C' = \pi - C$ are angles of a triangle.

8. A triangle the sides of which are $\sin \frac{A}{2}$, $\sin \frac{B}{2}$, $\cos \frac{C}{2}$ exists.

Proof. This follows from 3., because $A' = \frac{A}{2}$, $B' = \frac{B}{2}$, $C' = \frac{\pi + C}{2}$ are angles of a triangle.

Remark. Of course, using other transformations for angles of a triangle (see [28-30] or Chapter V), we can get several similar results. Some results of this kind are given in [30].

9. If A, B, C are the angles of the triangle ABC, then there exists a triangle A'B'C' the sides of which are $\cos^2 \frac{A}{2}$, $\cos^2 \frac{B}{2}$, $\cos^2 \frac{C}{2}$. For every angle $A \leqslant \pi/3$ $(A \leqslant B \leqslant C)$ there exists exactly one triangle ABC similar to A'B'C'.

Remark. This extension of 5. is given in [31].

10. The minimum of $k(n)$ such that $k(n) + \sin \frac{A}{n}$, $k(n) + \sin \frac{B}{n}$, $k(n) + \sin \frac{C}{n}$ are sides of a triangle for $n \geqslant 1$ is $\sin \frac{\pi}{n}$.

Remark. This result of M. S. Klamkin [32] for $n = 1$ becomes 3.

11. If $F(\Theta) = u\left(\sin \frac{\Theta}{n} + \sin \frac{\pi}{n}\right) + v\left(\cos \frac{\Theta}{n} - \cos \frac{\pi}{n}\right)$, then for all u, $v \geqslant 0$ and $n \geqslant 1$ the triple $F(A)$, $F(B)$, $F(C)$ form a triangle.

Remark. This result is an answer of Hj. Stocker to the following problem posed by M. S. Klamkin ([33]):

Give a generalization of 3, 4, and 5 which includes all these results as special cases.

For another answer given by G. Bercea, see 36. and [33].

12. $\cos^\lambda (A/\lambda)$, $\cos^\lambda (B/\lambda)$, $\cos^\lambda (C/\lambda)$ form a triangle for all real $\lambda \geqslant 2$.

Proof. If $A \geqslant B \geqslant C$, it suffices to show that

$$\cos^\lambda (A/\lambda) + \cos^\lambda (B/\lambda) \geqslant \cos^\lambda (C/\lambda).$$

Since $\max \cos(C/\lambda) = 1$ and $\min(\cos^\lambda(A/\lambda) + \cos^\lambda(B/\lambda))$ occurs for $C = 0$, we can only prove that

$$\cos^{\lambda}(A/\lambda) + \cos^{\lambda}(B/\lambda) \geq 1 \quad \text{for} \quad A + B = \pi.$$

For $\lambda = 2$, the l.h.s. reduces to 1. For larger values of λ, the inequality immediately follows from the following:

LEMMA. $\cos^{\lambda}(A/\lambda)$ $(0 \leq A \leq \pi)$ is a non-decreasing function of λ for $\lambda \geq 2$.

For the proof of this Lemma, it is sufficient to prove that $dy/d\lambda \geq 0$, where $y = \cos^{\lambda}(A/\lambda)$. Here $y'/y = x \tan x + \log \cos x$, where $x = A/\lambda$. Then $D_x(y'/y) = x \sec^2 x \geq 0$. Also, log y is concave (in λ).

Finally, by means of 1.: $(\cos^{\mu}(A/\lambda)$, $\cos^{\mu}(B/\lambda)$, $\cos^{\mu}(C/\lambda)$ are sides of a triangle, where $\lambda \geq \mu \geq 0$, $\lambda \geq 2$.

Remark. The above results of P. Erdős and M.S. Klamkin [34] are generalizations of 4. and 5.

13. m_a, m_b, m_c are sides of a triangle (see [29] and [35] or [36]).

14. [37] Let c_a, c_b, c_c be three cevians of a triangle dividing the sides in the ratio v/u where $u + v = 1$. Then c_a, c_b, c_c form a triangle.

Proof. Let $\vec{A}$, $\vec{B}$, $\vec{C}$ denote three vectors from an origin O to the respective vertices of a triangle ABC. Now consider the three cevians c_a, c_b, c_c, whose endpoints are the respective endpoints of $\vec{A}$, $u\vec{B} + v\vec{C}$; $u\vec{C} + v\vec{A}$; and $\vec{C}$, $u\vec{A} + v\vec{B}$. Since $\Sigma(u\vec{B} + v\vec{C} - \vec{A}) = \vec{0}$, (c_a, c_b, c_c) form a triangle

15. A triangle whose sides are $1/h_a$, $1/h_b$, $1/h_c$ exists.

Proof 1. This is a simple consequence of $a + b \geq c$, etc. since $\frac{1}{h_a} = \frac{a}{2F}$, etc.

Proof 2. Note that 15 is a consequence of Theorems 3 and 20 of 1.2.

16. x, y, z form a triangle if and only if $s^2 < 4r(4R - r)$.

17. h_a, h_b, h_c form a triangle if and only if

$$(s^2 + r^2 + 4Rr)^3 < 32s^2Rr(s^2 + r^2).$$

18. r_a, r_b, r_c form a triangle if and only if

$$(4R + r)^3 < 4s^2(4R - r).$$

19. cos A, cos B, cos C form a triangle if and only if

$$2R + r < s \quad \text{and} \quad s^2(R - r) < R^2(3R + r).$$

20. cotan A, cotan B, cotan C form a triangle if and only if

$$2R + r < s \quad \text{and} \quad (s^2 - 4Rr - r^2)^3 < 16s^2r^2(8R^2 + 4Rr + r^2 - s^2).$$

21. tan A, tan B, tan C form a triangle if and only if

$$2R + r < s \quad \text{and} \quad s^2r^2 < (8R^2 + 4Rr + r^2 - s^2)(s^2 - (2R + r)^2).$$

22. $\tan \frac{A}{2}$, $\tan \frac{B}{2}$, $\tan \frac{C}{2}$ form a triangle if and only if

$$(4R + r)^3 < 4s^2(4R - r).$$

23. $\text{cotan} \frac{A}{2}$, $\text{cotan} \frac{B}{2}$, $\text{cotan} \frac{C}{2}$ form a triangle if and only if

$$s^2 < 4r(4R - r).$$

24. $\sin^2 \frac{A}{2}$, $\sin^2 \frac{B}{2}$, $\sin^2 \frac{C}{2}$ form a triangle if and only if

$$8R^3 + 4R^2r < s^2(2R - r).$$

Remarks. 1° The above results 16-24 were proved in [17] as consequences of Theorem 3 and 2-4), 6-11) of 1.2.

2° 24. is our correction of a result from [17].

3° 16. and 23. are equivalent since $x = r \,\text{cotan} \frac{A}{2}$, etc.

4° 18. and 22. are equivalent since $r_a = s \tan \frac{A}{2}$, etc.

Similarly, using Theorem 3 and 13-19) of 1.2 we can prove the following results:

25. cosec A, cosec B, cosec C form a triangle if and only if

$$(s^2 + r^2 + 4Rr)^3 < 32s^2Rr(s^2 + r^2).$$

26. sec A, sec B, sec C form a triangle if and only if

$$s > 2R + r$$

and

$$(s^2 + r^2 - 4R^2)^3 < 16R(s^2 - (2R + r)^2((R + r)(s^2 + r^2 - 4R^2) - 2R(s^2 - (2R + r)^2)).$$

27. $\text{cosec}^2 \frac{A}{2}$, $\text{cosec}^2 \frac{B}{2}$, $\text{cosec}^2 \frac{C}{2}$ form a triangle if and only if

$$(s^2 + r^2 - 8Rr)^3 < 32r^2R(s^2(2R - r) - r(r^2 + 6Rr + 16R^2)).$$

28. $\sec^2 \frac{A}{2}$, $\sec^2 \frac{B}{2}$, $\sec^2 \frac{C}{2}$ form a triangle if and only if

$$(s^2 + (4R + r)^2)^3 < 32Rs^2(rs^2 + (4R + r)^3).$$

29. $1/a$, $1/b$, $1/c$ form a triangle if and only if

$$(s^2 + r^2 + 4Rr)^3 < 32s^2Rr(s^2 + r^2).$$

30. $1/x$, $1/y$, $1/z$ form a triangle if and only if

$$(4R + r)^3 < 4sr(4R + r - 2s).$$

31. $1/r_a$, $1/r_b$, $1/r_c$ form a triangle if and only if

$$s^2 < 4r(4R - r).$$

Remark. 31 and 16 are equivalent since $1/r_a = x/(rs)$, etc.

32. If ABC is an acute triangle, then A, B, C form a triangle [30].

33. a^2h_a, b^2h_b, c^2h_c form a triangle.

Proof. (see [26]). This result is a simple consequence of inequalities $a < b + c$, etc., and formulas $a^2h_a = 2Fa$, etc.

34. $\frac{1}{a + c}$, $\frac{1}{b + c}$, $\frac{1}{a + b}$ form a triangle [39].

35. $a(s - a)$, $b(s - b)$, $c(s - c)$ form a triangle [23].

S. J. Bilčev and E. A. Velikova [45] gave the following generalization of this result:

If $k \geq 1$ is any arbitrary real number, then $a(2ks - (k + 1)a)$, $b(2ks - (k + 1)b)$, $c(2ks - (k + 1)c)$ are the sides of a triangle.

36. Now, we shall give a generalization of the well-known Möbius-Pompeiu's theorem. For this result see for example GI 15.5, but this is an old result (see [38]). More about Möbius-Pompeiu's theorem will be found in Chapter XIII.

Let D be an arbitrary point in the plane of a triangle ABC and let $BC = a$, $CA = b$, $AB = c$, $AD = p$, $BD = q$, $CD = r$. Then ap, bq, cr form a triangle.

Remark. As we said, this result is an answer to a problem of M. S. Klamkin (see 11.), i.e. G. Bercea [33] noted that 3., 4. and 5. follow from the above result in the cases when $D = O$ (circumcentre), $D = I$ (incentre) and $D = P$, where the point P is given by

$$PA/(s - a) = PB/(s - b) = PC/(s - c) = k.$$

37. Let a set T be defined by $T = \{(a, b, c): a, b, c \text{ are sides of a triangle}\}$.

1° The set $P(\subset R)$ defined by $P = \{p: (a, b, c) \in T \Rightarrow (a^p, b^p, c^p)\}$ is given by $P = [0, 1]$.

2° Sets U, V and W $(\subset R^+ \times R^+ \times R^+)$ defined by $U = \{(a, b, c): (a^p, b^p, c^p) \in T \text{ for every } p \in R^+\}$, $V = \{(a, b, c): (a^p, b^p, c^p) \in T \text{ for every } p < 0\}$, $W = \{(a, b, c): (a^p, b^p, c^p) \in T \text{ for every } p \in R\}$, are given by

$$U = \{(a, b, c): b = c \geq a > 0 \vee c = a > b > 0 \vee a = b > c > 0\},$$
$$V = \{(a, b, c): a \geq b = c > 0 \vee b > c = a > 0 \vee c > a = b > 0\},$$

$$W = \{(a, a, a): a \in R^+\}.$$

This result is due to D. D. Adamović [40].

38. [44] If ABC is an acute triangle, then a cos A, b cos B, c cos C and a sin A, b sin B, c sin C form triangles.

39. The following problem is given in [46]:

(i) Determine all real numbers λ such that, whenever a, b, c are the lenghts of three segments which can form a triangle, the same is true for $(b + c)^\lambda$, $(c + a)^\lambda$, $(a + b)^\lambda$.

(For $\lambda = -1$ we have 34.)

(ii) Determine all pairs of real numbers λ, μ such that, whenever a, b, c are the lengths of three segments which can form a triangle, the same is true for $(b + c + \mu a)^\lambda$, $(c + a + \mu b)^\lambda$, $(a + b + \mu c)^\lambda$.

40. [47] A triangle whose sides are $a + h_a$, $b + h_b$, $c + h_c$ exists.

REFERENCES

1. Ramus and É. Rouché, 'Question 233', Nouv. Ann. Math. 10 (1851), 353-355.
2. R. Sondat and E. Lemoine, 'Question 1593', Ibid. 9 (1890), 49 and 10 (1891), 43-47.
3. A. Laisant, Géométrie du Triangle, Paris 1896, p. 112.
4. S. Nakajima, 'Some Inequalities between the Fundamental Quantities of the Triangle', Tôhoku Math. J. 25 (1925), 115-121.
5. S. Nakajima, 'Some Relations among Fundamental Quantities of a Triangle Inscribed in a Circle', Tokyo But. Zassi 35 (1926), 75-86. (Japanese).
6. R. Frucht, 'Upper and Lower Bounds for the Area of a Triangle for whose Sides Two Symmetric Functions are Known', Can. J. Math. 9 (1957), 227-231.
7. W. J. Blundon, 'On Certain Polynomials Associated with the Triangle', Math. Mag. 36 (1963), 247-248.
8. W. J. Blundon, 'Inequalities Associated with the Triangle', Can. Math. Bull. 8 (1965), 615-626.
9. O. Bottema, Nieuw Arch. Wisk. 13 (1965), 246.
10. O. Bottema, 'Inequalities for R, r, and s', Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 338-352 (1971), 27-36.
11. R. Frucht and M. S. Klamkin, 'On Best Quadratic Triangle Inequalities'. Geometriae Dedicata 2 (1973), 341-348.
12. I. Paasche, 'Das Bottema-Deltoid als Enveloppe', Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 461-497 (1974), 121-125.
13. M. Marčev, 'Neravenstva meždu perimet'ra i radiusite na vpisanata i opisanata okr'žnost na tri'g'lnika i njakoi sledstvija ot tjah', Obučenieto po Mat. 6 (1976), 3-7.
14. J. F. Rigby, 'Quartic and Sextic Inequalities for the Sides of Triangles, and Best Possible Inequalities', Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 602-633 (1978), 195-202.
15. A. Lupaş, 'Problem 441', Mat. Vesnik 2 (15) (30) (1978), 293.
16. V. N. Murty, 'A New Inequality for R, r and s', Crux Math. 8 (1982), 62-68.
17. V. P. Soltan and S. I. Mejdman, Toždestva i neravenstva v

treugol'nike, Kišinev, 1982.
18. G. Petrov, 'O podmínkách konstrukce trojúhelníku', Časopis pro pěstováni matematiky 77 (1952), 77-92.
19. M. Berman and F. G. B. Maskell, 'Problem 94. The Two-Year Coll.', Math. J. 9 (1978), 300-301.
20. V. T. Janekoski, 'Neravenstvo na Bohr kako potreben i dovolen uslov za egzistencija na triagolnik', Ann. Fac. d'Électrotech. Méc. l'Univ. Skopje 4 (1970), 15-18.
21. V. T. Janekoski, 'Za eden potreben i dovolen uslov za egzistencija na triagolnik', Bull. Soc. Math. Phys. Macédoine 21 (1970), 85-87.
22. H. Carus and O. P. Lossers, 'Problem E 2823', Amer. Math. Monthly 87 (1980), 220 and 88 (1981), 540.
23. Ngo Tan and M. S. Klamkin, 'Problem 589', Crux Math. 7 (1981), 307.
24. Problem Gy 1489. Köz. Mat. Lap. 47 (1973), 77.
25. V. T. Janekoski, 'Egzistencija i nekoi osobini na triagolnici čiišto strani se trigonometriski funkcii na aglite od dadan triagolnik', Bull. Soc. Mat. Phys. Macédoine 21 (1970), 89-95.
26. V. T. Janekoski, 'Za nekoi neravenstva inducirani od dadeni triagolnici', Ibid. 23 (1972), 79-87.
27. Ž. Madevski, 'Četirite značajni točki vo triagolnikot kako težišta na pogodni izbrani masi', Ibid. 20 (1969), 31-46.
28. Ž. Madevski, 'A Procedure for Obtaining Relations between the Angles of a Triangle', Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 338-352 (1971), 87-88.
29. C. Pabst, 'Einige Beziehungen zwischen den drei Höhen und zwischen den drei seitenhalbirenden Ecktransversalen eines Dreiecks', Arch. Math. Phys. (2) 7 (1889), 10-26.
30. Ž. Madevski, 'Nekoi metodi za dobivanje relacii megu elementite od oden triagolnik', Dissertation, Skopje, 1973.
31. O. Bottema and K. Inkeri, 'Problem 353', Nieuw Arch. Wisk. 3 (21) (1973), 282 and 2 (22) (1974), 183-184.
32. M. S. Klamkin, Private communication.
33. M. S. Klamkin and Hj. Stocker, and G. Bercea, 'Aufgabe 729', Elem. Math. 29 (1974), 155 and 30 (1975), 134-135.
34. P. Erdös and M. S. Klamkin, 'A Triangle Inequality', Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 412-460 (1973), 117-118.
35. L. Toscano, 'Confronto degli Angoli di un Triangolo con quelli del Triangolo delle sue Mediane', Archimede 8 (1956), 278-279.
36. A. Bager, 'Some Inequalities for the Mediane of a Triangle', Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 338-352 (1971), 37-40.
37. A. Bager and M. S. Klamkin, 'Aufgabe 677', Elem. Math. 28 (1973), 129-130.
38. A. F. Möbius, 'Uber eine Methode, um von Relationen, welche der Longimetrie angehören, zu entsprechenden Sätzen der Planimetrie zu gelangen', Berichte über die Verhandlungen der Königlich Sächsischen Gesellschaft der Wissenschaften zu Leipzig. Math. Phys. Classe, Leipzig (1852), 41-54.
39. V. Linis and F. G. B. Maskell, 'Problem 14', Crux Math. 1/2 (1975/76), 7 and 28.
40. D. D. Adamović, 'Problem 151', Mat. Vesnik 6 (21) (1969), 470-471.
41. O. Bottema and H. Eves, 'Problem 786', Crux Math. 8 (1982), 277 and 10 (1984), 55-56.
42. O. Bottema and L. Somer, M. Kantrowitz, and H. Ahlburg, 'Problem

1004', Crux Math. 11 (1985), 15 and 22 (1986), 86-89.
43. A. Vrba and A. Toufara, 'Problem 11', Rozhledy Mat. Fyz. Praha 58 (1979/80), 84-86.
44. 'Problem 2795', Mat. v škole 1985, No. 5, 60.
45. S. J. Bilčev and E. A. Velikova, '$GT_e(k)$ - transformation for a triangle and some applications'.
46. D. S. Mitrinović and J. E. Pečarić, 'Problem 1171*', Crux Math. 12 (1986), 204.
47. V. Vâjâitu, Private communication.
48. K. Herterich, Die Konstruktion von Dreiecken, Stuttgart 1986.

Chapter II

DUALITY BETWEEN GEOMETRIC INEQUALITIES AND INEQUALITIES FOR POSITIVE NUMBERS

0. Introduction

As we noted in Chapter I, many of the geometric inequalities can be restricted to the three main sets of canonical variables, i.e.

(i) the sides a, b, c,

(j) the angles A, B, C,

(k) the circumradius R, inradius r and semi-perimeter s.

But, it is known that there exists a very simple transformation between the sides of a triangle and three non-negative numbers (see the next Section), so there exists a duality between all triangle inequalities and all inequalities involving three non-negative numbers.

1. Geometry of the Duality $(a, b, c) \Leftrightarrow (x, y, z)$

Let us consider an arbitrary triangle ABC and its inscribed circle (Figure 1).

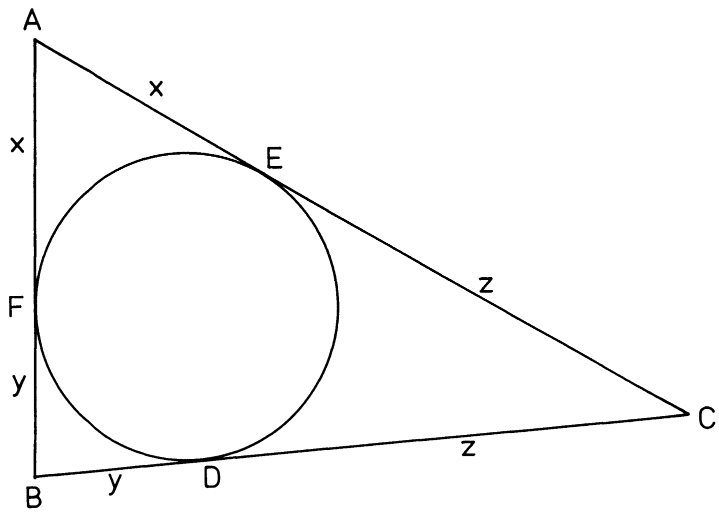


Fig. 1.

Since tangents from an external point to a circle are equal in length, we have

$$AF = AE = x, \quad BF = BD = y, \quad CD = CE = z.$$

Consequently,

(1) $a = y + z, \quad b = z + x, \quad c = x + y,$

(2) $x = s - a, \quad y = s - b, \quad z = s - c.$

The latter two equations imply that for any triangle ABC, the distances of the vertices to the closest points of tangency with the inscribed circle are three non-negative numbers x, y, z and, dually, that corresponding to any three non-negative numbers x, y, z, there exists a triangle whose sides are given by (1). In form (1), the basic triangle inequalities $a, b, c \geqslant 0$, $b + c \geqslant a$, $c + a \geqslant b$, $a + b \geqslant c$ are obviously satisfied.

One of the advantages of this duality is that we can use all the inequalities concerning any three non-negative numbers.

In [1] a table of equivalent forms in terms of (a, b, c), (A, B, C), (R, r, s) and (x, y, z) was given. Here we shall give only some simple examples. For some other examples see Chapter III.

If we use the notation

(3) $T_1 = \Sigma x, \quad T_2 = \Sigma yz, \quad T_3 = xyz,$

then

(4) $Q = \Sigma(b - c)^2 = \Sigma(y - z)^2 = 2(T_1^2 - 3T_2),$

(5) $\Sigma a^2 = 2T_1^2 - 2T_2,$

(6) $\Sigma bc = T_1^2 + T_2,$

(7) $abc = \Pi(y + z) = T_1T_2 - T_3,$

(8) $16F^2 = 2\Sigma a^2b^2 - \Sigma a^4 = 16r^2s^2 = 16T_1T_3,$

(9) $R = (T_1T_2 - T_3)/(4\sqrt{T_1T_3}),$

(10) $r = \sqrt{T_3/T_1}.$

2. Transformations

Another advantage of the (x, y, z)-representation is that it is very easy to make transformations preserving inequality. If $F(x, y, z) \geqslant 0$ is a valid inequality for all non-negative x, y, z, then so is $F(x', y', z') \geqslant 0$ where

(1) $x' = F_1(x, y, z), \quad y' = F_2(x, y, z), \quad z' = F_3(x, y, z)$

and where $F_1, F_2, F_3 \geqslant 0$. One particularly useful transformation is obtained by letting $F_1 = 1/x$, $F_2 = 1/y$, $F_3 = 1/z$, giving

$$(2) \qquad T_1' = \frac{T_2}{T_3}, \quad T_2' = \frac{T_1}{T_3}, \quad T_3' = \frac{1}{T_3} .$$

As an application to be used subsequently, the dual, via (2), of $T_1^2 \geqslant 3T_2$ is $T_2^2 \geqslant 3T_1T_3$.

In terms of the triangle representation, the above reads:
If $F(a, b, c) \geqslant 0$ is a triangle inequality, then so is $F(a', b', c') \geqslant 0$, where

$$(1)'^{*} \qquad a' = F_2(s - a, s - b, s - c) + F_3(s - a, s - b, s - c),$$

$$b' = F_3(s - a, s - b, s - c) + F_1(s - a, s - b, s - c),$$

$$c' = F_1(s - a, s - b, s - c) + F_2(s - a, s - b, s - c).$$

In particular,

$$(2) \qquad a' = \frac{a}{(s - b)(s - c)}, \quad b' = \frac{b}{(s - c)(s - a)},$$

$$c' = \frac{c}{(s - a)(s - b)} .$$

By multiplying through by $2(s - a)(s - b)(s - c)$ on the r.h.s., we get a similar triangle

$$(3) \qquad a' = 2a(s - a), \quad b' = 2b(s - b), \quad c' = 2c(s - c).$$

Since

$$s' - c' = \frac{1}{2}(c - a + b)(c + a - b) \quad \text{(and cyclically)},$$

and $\quad \Delta' = 8\Delta^2\Delta_1/s$**, we also get

$$(3)' \qquad R' = \frac{\Delta R}{\Delta_1}, \quad r' = \frac{\Delta^2}{s\Delta_1}, \quad s' = 8\Delta_1^2.$$

We now list some other simple transformations:

---

* The prime on (1)' is to indicate that the latter is a dual of (1).
** Δ_1 denotes the area of a triangle whose sides are $\sqrt{a}$, $\sqrt{b}$, $\sqrt{c}$, i.e.

$$4\Delta_1^2 = 4Rr + r^2 = \Sigma bc - s^2 = \Sigma xy.$$

Δ denotes the area of a triangle whose sides are a, b, c.

(4) $T_1' = T_1^2 - 2T_2, \quad T_2' = T_2^2 - 2T_1T_3, \quad T_3' = T_3^2$

(letting $F_1 = x^2, \quad F_2 = y^2, \quad F_3 = z^2$).

(4)' $T_1' = \dfrac{T_2^2 - 2T_1T_3}{T_3^2}, \quad T_2' = \dfrac{T_1^2 - 2T_2}{T_3^2}, \quad T_3' = \dfrac{1}{T_3^2}$

($F_1 = 1/x^2$, etc.).

(5) $T_1' = T_1^3 - 3T_1T_2 + 3T_3, \quad T_2' = T_2^3 + 3T_3^2 - 3T_1T_2T_3, \quad T_3' = T_3^3$

($F_1 = x^3$, etc.).

(5)' $T_1' = \dfrac{T_2^3 - 3T_1T_2T_3 + 3T_3^2}{T_3^3}, \quad T_2' = \dfrac{T_1^3 - 3T_1T_2 + 3T_3}{T_3^3}, \quad T_3' = \dfrac{1}{T_3^3}$

($F_1 = 1/x^3$, etc.).

(6) $T_1' = 2T_1, \quad T_2' = T_1^2 + T_2, \quad T_3' = T_1T_2 - T_3$

($F_1 = y + z$, etc.).

(6)' $T_1' = \dfrac{2T_1}{T_3}, \quad T_2' = \dfrac{T_2^2 + T_1T_3}{T_3^2}, \quad T_3' = \dfrac{T_1T_2 - T_3}{T_3^2}$

($F_1 = 1/y + 1/z$, etc.).

(7) $T_1' = 2T_2, \quad T_2' = T_2^2 + T_1T_3, \quad T_3' = T_1T_2T_3 - T_3^2$

($F_1 = x(y + z)$, etc.).

(7)' $T_1' = \dfrac{2T_1}{T_3}, \quad T_2' = \dfrac{T_1^2 + T_2}{T_3^2}, \quad T_3' = \dfrac{T_1T_2 - T_3}{T_3^3}$

($F_1 = (y + z)/xyz$, etc.).

(8) $$T_1' = 2(T_1^2 - 3T_2), \quad T_2' = T_1^4 - 6T_1^2T_2 + 9T_2^2,$$

$$T_3' = T_1^2T_2^2 + 18T_1T_2T_3 - 4T_2^3 - 4T_1^3T_3 - 27T_3^2$$

$$(F_1 = (y - z)^2, \text{ etc.}).$$

3. Examples

3.1. [1] Using inequality

(1) $$\Sigma(y - z)^2 = 2(T_1^2 - 3T_2) \geqslant 0$$

in the form $3(T_1^2 + T_2) \leqslant 4T_1^2 \leqslant 4(T_1^2 + T_2)$, we get GI 1.1.

3.2. [1] Using inequality

(2) $$(\Sigma x)^3 - 27xyz = T_1^3 - 27T_3 \geqslant 0$$

we get GI 5.11 and

(3) $$(4F)^2 \geqslant 27\,\Pi(b + c - a)^2.$$

3.3. [1] Using inequality

(4) $$\Sigma x^2(y + z) - 6xyz = T_1T_2 - 9T_3 \geqslant 0$$

(i.e. GI 1.4) we can get GI 1.3, 1.15, 5.1, 2.12, 5.30, 5.40, 5.41, 6.18, 6.21, 6.27.

3.4. [1] Using Schur's inequality (AI, p. 119) we get

(5) $$\Sigma x(x - y)(x - z) = T_1^3 + 9T_3 - 4T_1T_2 \geqslant 0$$

wherefrom it follows GI 1.6, 6.13 and the first inequality in 5.9.

3.5. [1] Using the inequality $2T_1^3 + 9T_3 \geqslant 4T_1T_2$ (weaker than (5)) we get GI 1.4, 8.14 and

(6) $$\Pi(\sin B + \sin C) \geqslant 8\,\Pi \sin A.$$

3.6. [1] Using the inequality $10T_1^3 \geqslant 27T_1T_3 + 27T_3$ (weaker than (5)) we get GI 1.5.

3.7. [1] Using the inequality $8T_1^3 + 27T_3 \geqslant 27T_1T_2$ (weaker than (5)) we get GI 5.42, 6.16, 5.12.

3.8. [1] Using inequality

(7) $$\Sigma x^2(y - z)^2 = 2(T_2^2 - 3T_1T_3) \geqslant 0,$$

we get GI 1.9, 5.5, 7.2, 10.3, 4.3, 4.6, 4.7, 4.9, 5.34, 5.46.

3.9. Of course, we can work in the opposite direction, i.e. using well known geometric inequalities we can obtain inequalities for positive numbers. For example the well known inequality

GI 2.1 $\Sigma \sin A \leqslant 3\sqrt{3}/2$

gives

$$27\Pi(y + z)^2 \geqslant 64xyz(x + y + z)^3, \quad \text{i.e.,}$$

$$27(T_1T_2 - T_3)^2 \geqslant 64T_1^3T_3$$

and these inequalities are equivalent to GI 1.12, 6.15, 5.3, 5.42, and 4.13.

3.10. A. Oppenheim [2] proposed the following problem:

Suppose that ABC is an acute-angled triangle, then

(8) $$16\Pi \cos^2 A + 4\Sigma \cos^2 B \cos^2 C \leqslant 1,$$

(9) $$4\Sigma \cos^2 B \cos^2 C \leqslant \Sigma \cos^2 A.$$

Equality occurs when ABC is equilateral or right-angled isosceles and in no other case.

Comment by M. S. Klamkin [3].

By virtue of the equality conditions, ABC should be restricted to non-obtuse triangles rather than acute triangles.

In a personal communication, A. W. Walker has pointed out that there is a flaw in the published solution (April, 1967, p. 441).

We now prove (9) first and then show how (8) follows from (9). By using the identity $2\cos^2 A = 1 + \cos 2A$ and then making the transformation $A' = \pi - 2A$, etc. (see Chapter VII), (9) becomes (after dropping primes)

(10) $$3\Sigma \cos A \geqslant 3 + 2\Sigma \cos B \cos C$$

where here ABC is a general triangle. In terms of the sides, (10) becomes

(11) $$3abc\Sigma a^2c + \Sigma b^5c \geqslant 4abc\Sigma c^3 + 2\Sigma b^3c^3 + 6(abc)^2,$$

which is equivalent to

(12) $$T_1^2T_2^2 + 4T_1T_2T_3 \geqslant 4T_1^3T_3 + 9T_3^2.$$

Since (7) and

(13) $$\Sigma y^2z^2(x - y)(x - z) = T_2^3 + 9T_3^2 - 4T_1T_2T_3 \geqslant 0,$$

are valid, it follows that (12) is equivalent to

(14) $$\Sigma x^4(y - z)^2 + (\Sigma xy)(\Sigma z^2(x - y)^2) \geqslant 0$$

which is obviously valid and consequently weaker than (13). There is equality in (14) only if $x = y = z$ or if two of x, y, z vanish which corresponds to the equilateral triangle or isosceles right triangle, respectively.

Now using (9), we will establish a stronger inequality then (8), i.e.

(15) $$16\Pi \cos^2 A + \Sigma\cos^2 A \leqslant 1.$$

Since (see part VII 2): $1 - \Sigma\cos^2 A = 2\Pi \cos A$, the latter is equivalent to

(16) $$(\Pi \cos A)(1 - 8\Pi\cos A) \geqslant 0.\ .$$

Since the triangle is non-obtuse, $\Pi\cos A \geqslant 0$. Also, it is known that $1 \geqslant 8\,\Pi\cos A$. There is equality in (16) only if the triangle is equilateral or if one of the angles is a right angle. This implies that there is equality in (8) only if the triangle is an equilateral or a right isosceles one.

Incidentally, (10) is also equivalent to the known inequality GI 6.12. A stronger related inequality is the second inequality in GI 6.13, or equivalently

(17) $$T_1^2T_2^2 + 2T_1T_2T_3 + 9T_3^2 \geqslant 4T_1^3T_3.$$

3.11. The following result was also proved by using the (x, y, z) representation ([4], [5]):

$$\Sigma\left(\frac{4s(s - a)}{3a^2}\right)^t \geqslant 3 \quad \text{for all} \quad t \leqslant 0 \quad \text{and}$$

$$\text{all } t \geqslant (\log \tfrac{3}{2})/(\log \tfrac{8}{3}).$$

3.12. [9] Using inequalities

$$2\Sigma \frac{yz}{y + z} \leqslant \Sigma x \leqslant \Sigma \frac{yz}{x}$$

we get

$$2\Sigma \frac{1}{a(s - a)} \leqslant \frac{1}{r^2} \leqslant \Sigma \frac{1}{(s - a)^2}.$$

3.13. [9] Using inequalities

$$2\Sigma xyz^2 \leqslant \Sigma yz(y^2 + z^2) \leqslant 2\Sigma x^4$$

we get

$$32\Delta^2 \leqslant \Sigma(a^2 + b^2)(b + c - a)(c + a - b) \leqslant 24\Delta^2 + \frac{1}{2}\Sigma(b + c - a)^4.$$

3.14. [9] Let x, y, z be non-negative numbers such that $\Sigma x = 1$. Then the following inequalities are valid

$$0 \leqslant \Sigma yz - 2xyz \leqslant 7/27.$$

This result was given in the XXV International Mathematical Olympiad, ČSSR, 1984. Now, Using substitutions $x = (s - a)/s$, etc., we get

$$\frac{40}{27} s^3 \leqslant (\Sigma a)(\Sigma bc) - 4abc < 2s^3.$$

Note that many other examples can be found in [1] and [9], and that table of non-negative forms is given in [1].

4. Some Important Non-Negative Quadratic Forms

M. S. Klamkin [8] proved the following result:
Let p, q, r, x, y, z be non-negative numbers. Then

(1) $$\Sigma \frac{p}{q + r} yz \geqslant \Sigma yz - \frac{1}{2}\Sigma x^2.$$

For $x = a^2$, etc., Klamkin obtained inequality

(2) $$\Sigma \frac{p}{q + r} b^2c^2 \geqslant 8F^2,$$

where he used (1.8).

Since the expression on the right-hand side of (1) is symmetric, we can use (1) and other identities for a triangle for generating many new inequalities. For example, for $x = a$, etc. we get

(3) $$\Sigma \frac{p}{q + r} bc \geqslant 2r(4R + r).$$

Comment by W. Janous. Since $F = (bc \sin A)/2$, etc., (2) becomes [8]:

(4) $$\sum \frac{p}{q + r} \operatorname{cosec}^2 A \geqslant 2,$$

which yields

for $p = q = r$ GI 2.50,
for $p = \sin^2 A$ etc.: $2 \leqslant \sum 1/(\sin^2 B + \sin^2 C)$,
for $p = \cos^2 A$ etc.: $2 \leqslant \sum \cot\text{an}^2 A/(\cos^2 B + \cos^2 C)$, etc.□

We shall note that analogously (3) becomes:

(5) $$\sum \frac{p}{q + r} \operatorname{cosec} A \geqslant \frac{4R + r}{s}.$$

D. S. Mitrinović and J. E. Pečarić [10] proved the following similar results:

Let p, q, r be real numbers such that $p + q > 0$, $q + r > 0$, $r + p > 0$. If x, y, z are non-negative numbers, then

(6) $$\sum \frac{p}{q + r} x^2 \geqslant \sum yz - \frac{1}{2}\sum x^2,$$

with equality if and only if

(7) $$p:q:r = (y + z - x):(x - y + z):(x + y - z).$$

Proof. Using Cauchy's inequality we get

$$\sum \frac{p}{q + r} x^2 = \frac{1}{2}(\sum(q + r))\left(\sum \frac{x^2}{q + r}\right) - \sum x^2 \geqslant \frac{1}{2}(\sum x)^2 - \sum x^2 =$$

$$= \sum yz - \frac{1}{2} \sum x^2.$$

From equality condition for Cauchy's inequality we get (7).

For $x = a^2$, etc. we get the following inequality [10] and [11]:

(8) $$\sum \frac{p}{q + r} a^4 \geqslant 8F^2.$$

Equality in (8) is valid if and only if

(9) $$p:q:r = (-a^2 + b^2 + c^2):(a^2 - b^2 + c^2):(a^2 + b^2 - c^2).$$

Bilčev and Velikova gave many special cases of (8). For example the following inequalities are valid [11]:

$$\sum a^3(s - a) \geqslant 8F^2, \quad \sum a^4/b \geqslant (\sum a^2)^2/(\sum a), \quad \sum a^5/(b + c) \geqslant 8F^2,$$

$$\sum \frac{a^3}{b(a+b)} \geqslant \frac{2F}{R}, \quad \sum a^2/(b+c) \geqslant 2F/R, \quad \sum (b^2+c^2)^{-1} \geqslant (2R^2)^{-1}.$$

They also gave the following inequality [11]:

(10) $$\sum \frac{p}{q+r} a^2 \geqslant 2r(4R+r).$$

Equality in (10) is valid if and only if

(11) $$p:q:r = (-a+b+c):(a-b+c):(a+b-c).$$

Of course, this follows from (6) for $x = a$, etc.

Remarks. 1° Proof of (6) is similar to Janous' proof of (8).

2° Tsintsifas [10] gave (8) for positive numbers p, q, r only.

3° Note that the left-hand sides of (6) and (1) are incomparable in general.

4° In fact, Mitrinović and Pečarić [10] proved a more general result.

The following result was also given in [8]:

Let a, b, c be the sides of a triangle and let x, y, z be real numbers. Then

(12) $$\sum\left(\frac{b}{c} + \frac{c}{b} - 1\right)x^2 - \sum\left(3 - \frac{b+c}{a}\right)yz \geqslant 0.$$

Remark. By replacing (x, y, z) by (x_1^2, x_2^2, x_3^2), i.e. if

$$a = x_2^2 + x_3^2, \quad b = x_3^2 + x_1^2, \quad c = x_1^2 + x_2^2$$

where x_1, x_2, x_3 are arbitrary real numbers, we can in a similar way consider inequalities for all real numbers. For such results see [1] and [6].

References

1. M. S. Klamkin, 'Duality in Triangle Inequalities', Ford Motor Company Preprint, July 1971. (also, see Notices of A.M.S., August 1971, p. 782).
2. A. Oppenheim, 'Problem E 1838', Amer. Math. Monthly 72 (1965), 1129.
3. M. S. Klamkin, Private communication.
4. M. T. McGregor, 'On Inequalities Related to Triangles', Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 544-576 (1976), 122-124.
5. J. F. Rigby, 'A Method of Obtaining Related Triangle Inequalities with Applications', Ibid. No. 412-460 (1973), 217-226.
6. M. S. Klamkin, 'Sums of Squares Representations for Triangle Inequalities', Ibid. No. 412-460 (1973), 113-116.
7. G. H. Hardy, J. E. Littlewood, and G. Pólya, Inequalities, Cambridge, 1952, pp. 57-60.
8. M. S. Klamkin, 'Two non-negative quadratic forms', Elem. Math. 28

(1973), 141-146.
9. S. J. Bilčev and G. S. Georgiev, 'V'rhu edin metod za dokazatelstvo na geometrični neravenstva črez algebrični neravenstva i obratno'. Matematika (Sofija), 1986, (to appear).
10. G. Tsintsifas and D. S. Mitrinović, J. E. Pečarić, M. S. Klamkin, and W. Janous, 'Problem 1051', Crux Math. **11** (1985), 187 and **12** (1986), 250-255.
11. S. J. Bilčev and E. A. Velikova, 'Ob odnoj probleme G. Tsintsifas'. Matematika i matem. obraz. BAN., Sofija, 1987, 586-573.
12. W. Janous, Private communication.

Chapter III

HOMOGENEOUS SYMMETRIC POLYNOMIAL GEOMETRIC INEQUALITIES

0. Introduction

P. J. van Albada and K. B. Stolarsky ([1], [2]) noted in 1971 that many inequalities in GI, Chapter I, for the sides of a triangle can be rewritten in the form $p(a, b, c) > 0$ or $p(a, b, c) \geqslant 0$ where $p(a, b, c)$ is a symmetric and homogeneous polynomial of degree n in the real variables a, b, c representing the sides of a triangle. They gave the general solution for such inequalities if $n \leqslant 3$.

Note that the substitutions

(1) $$x = \frac{1}{2}(b + c - a), \quad y = \frac{1}{2}(c + a - b), \quad z = \frac{1}{2}(a + b - c) \quad \text{or}$$

$$a = y + z, \quad b = z + x, \quad c = x + y$$

transform any inequality for the positive numbers x, y, z into an inequality for the sides a, b, c of a triangle, and conversely ([3], [4]), i.e. (x, y, z) is dual to (a, b, c) (see Chapter II). So, we can consider inequalities $P(x, y, z) \geqslant 0$ instead of $p(a, b, c) \geqslant 0$.

We shall also make use of the formulae

(2) $$\Sigma x = \frac{1}{2}\Sigma a = s, \quad \Sigma yz = r(4R + r), \quad xyz = r^2 s,$$

$$F = rs, \quad abc = 4Rrs$$

to derive inequalities for R, r, s, and F.

Further, we shall denote the elementary symmetric functions of x, y, z by T_1, T_2, T_3, i.e.

$$T_1 = x + y + z, \quad T_2 = yz + zx + xy, \quad T_3 = xyz.$$

In some proofs we shall use the following lemma:

LEMMA. The expression $X\lambda - Y\sqrt{\lambda\nu} + Z\nu$ (X, Y, Z real) will be non-negative for all non-negative values of λ and ν if $X \geqslant 0$, $Z \geqslant 0$ and $4XZ - Y^2 \geqslant 0$.

Proof. $X\lambda - Y\sqrt{\lambda\nu} + Z\nu = (\sqrt{X}\sqrt{\lambda} - \sqrt{Z}\sqrt{\nu})^2 + (2\sqrt{XZ} - Y)\sqrt{\lambda\nu}$.

1. General Results of P. J. van Albada and K. B. Stolarsky

As we have said, P. J. van Albada and K. B. Stolarsky gave general solutions for inequalities of the form $p(a, b, c) \geqslant 0$, i.e. $P(x, y, z) \geqslant 0$

for $n \leqslant 3$.

Degree 1.
It is obvious that such inequalities must be in the form

$$t(a + b + c) = 2t(x + y + z) \geqslant 0$$

which is true only if $t \geqslant 0$.

Degree 2.
Theorem 1. The inequality

$$P(x, y, z) = \lambda(\Sigma x^2 - \Sigma yz) + 4\mu\Sigma yz \geqslant 0$$

holds for all non-negative x, y, z only if $\lambda \geqslant 0$, $\mu \geqslant 0$.

Proof. $P(x, y, z) = \frac{1}{2}\lambda\Sigma(y - z)^2 + 4\mu\Sigma yz$, and $P(1, 0, 0) = \lambda$, $P(1, 1, 1) = 12\mu$.

We deduce using formula (0.1) that

$$\lambda(\Sigma a^2 - \Sigma bc) + \mu(-\Sigma a^2 + 2\Sigma bc) \geqslant 0$$

for all triangles only if $\lambda \geqslant 0$, $\mu \geqslant 0$ [1].

The polynomials $\Sigma x^2 - \Sigma yz$ and Σyz are the basic (symmetric) positive quadratics for positive numbers, whilst $\Sigma a^2 - \Sigma bc$ and $-\Sigma a^2 + 2\Sigma bc$ are the basic (symmetric) positive quadratics for the sides of a triangle.

Since $\Sigma x^2 - \Sigma yz = s^2 - 3r(4R + r)$, we have $s^2 \geqslant 3r(4R + r)$ i.e. GI 5.6.

Degree 3. ([1], [2], [4])
As in [4] we write

$$U = \Sigma x^3 - \Sigma x^2(y + z) + 3xyz = \Sigma x(x - y)(x - z)$$

$$= x(x - y)^2 + z(y - z)^2 + (x - y)(y - z)(x - y + z),$$

$$V = \Sigma x^2(y + z) - 6xyz = \Sigma x(y - z)^2,$$

$$W = xyz.$$

When $x, y, z \geqslant 0$ we have $U \geqslant 0$ (since without loss of generality $x \geqslant y \geqslant z$). This is Schur's inequality (AI, p. 119) with $n = 1$, and when we use (0.1) to express it in terms of a, b, and c we obtain Colins' inequality GI 1.6 (this inequality is from 1870). Also $V \geqslant 0$ and $W \geqslant 0$ when $x, y, z \geqslant 0$. Any symmetric cubic in x, y, z can be written in the form

$$P(x, y, z) = \lambda U + \mu V + \nu W.$$

THEOREM 2. [4] The inequality $\lambda U + \mu V + \nu W \geqslant 0$ holds for all x, y, z $\geqslant 0$ only if $\lambda, \mu, \nu \geqslant 0$.

Proof. $P(1, 0, 0) = \lambda$, $P(0, 1, 1) = 2\mu$, $P(1, 1, 1) = \nu$.

The polynomials U, V, W are the basic positive cubics for positive numbers. Using (0.1) we deduce that

$$\frac{1}{2}\lambda\left[-\Sigma a^3 + 2\Sigma a^2(b + c) - 9abc\right] + \mu\left[\Sigma a^3 - \Sigma a^2(b + c) + 3abc\right]$$
$$+ \frac{1}{8}\nu\left[-\Sigma a^3 + \Sigma a^2(b + c) - 2abc\right] \geqslant 0$$

for all triangles only if $\lambda, \mu, \nu \geqslant 0$ [1].

The three expressions in square brackets in the above inequality are the basic positive cubics for the sides of a triangle.

We have $U = s(s^2 - 16Rr + 5r)^2$, giving one of Gerretsen's inequalities $s^2 \geqslant 16Rr - 5r^2$ ([5], GI 5.14 and 5.25), and $V = 4rs(R - 2r)$, giving Chapple-Euler's inequality $R \geqslant 2r$ (GI 5.1).

Degree 4 ([2]).

A real symmetric form of degree 4 can be positive at (1, 1, 1), (1, 1, 0), (2, 1, 1) and (4, 3, 2) while negative at (1, 1, 1/2). (Note that in the previous cases, i.e. $n \leqslant 3$, from $P(1, 1, 1)$, $P(1, 1, 0)$, $P(2, 1, 1) \geqslant 0$ it follows that $P(a, b, c) \geqslant 0$ ([2]).

Indeed, define a symmetric form of degree 4 by

$$P(a, b, c) = A\Sigma a^4 + B\Sigma a^3(b + c) + C\Sigma a^2b^2 + D\Sigma a^2bc$$

where $A = 103/34$, $B = -4$, $C = 2$, $D = 3$. Then $P(1, 1, 0) = 1/17$, $P(1, 1, 1) = 3/34$, $P(2, 1, 1) = 275/34$, and $P(4, 3, 2) = 1815/34$, but $P(1, 1, 1/2) = -1/544$.

2. Special Inequalities

In [1], [3], [6], [8], and [9] 'special' inequalities were considered, i.e. inequalities for which equality occurs when $x = y = z$, or when $a = b = c$, i.e. when the triangle is equilateral.

Degree 2 [6].

THEOREM 3. The complete set of special quadratic inequalities is given by

$$\lambda(T_1^2 - 3T_2) \geqslant 0, \quad \lambda > 0,$$

i.e.

$$\lambda(\Sigma x^2 - \Sigma yz) \geqslant 0 \quad \text{(cf. Theorem 1)}.$$

Degree 3 [6].

THEOREM 4. The complete set of special cubic inequalities is given by

$$\lambda U + \mu V \geqslant 0, \quad \lambda, \mu \geqslant 0.$$

This result follows immediately from Theorem 2; in [6] the inequalities are expressed in the form

$$\alpha T_1^3 + \beta T_1 T_2 - 9(\beta + 3\alpha)T_3 \geqslant 0, \quad \alpha \geqslant 0, \quad \beta \geqslant -4\alpha.$$

<u>Degree 4</u> [1], [6], [9].
We shall give the formulation of the result from [9]. Write

$$A = \Sigma x^4 - \Sigma x^3(y + z) + \Sigma x^2 yz = \Sigma x^2(x - y)(x - z) =$$

$$= \frac{1}{2}\Sigma(y + z - x)^2(y - z)^2,$$

$$B = \Sigma x^3(y + z) - 2\Sigma y^2 z^2 = \Sigma yz(y - z)^2,$$

$$C = \Sigma y^2 z^2 - \Sigma x^2 yz = \frac{1}{2}\Sigma x^2(y - z)^2.$$

Then $A \geqslant 0$, $B \geqslant 0$, $C \geqslant 0$ are special quartic inequalities for non-negative x, y, z. In fact $A \geqslant 0$ and $C \geqslant 0$ for all real x, y, z; $A \geqslant 0$ is Schur's inequality (AI, p. 119) with $n = 2$. Any special symmetric quartic inequality for positive numbers has the form

$$\lambda A + \mu B + \nu C \geqslant 0.$$

THEOREM 5. The inequality $\lambda A + \mu B + \nu C \geqslant 0$, holds for all $x, y, z \geqslant 0$ only if $\lambda \geqslant 0$, $\nu \geqslant 0$, $\mu \geqslant -\sqrt{\lambda\nu}$.

<u>Proof</u>. Write $\lambda A + \mu B + \nu C = f(x, y, z)$, and suppose that $f(x, y, z) \geqslant 0$ for all positive x, y, z. Then $\lambda = f(1, 0, 0) \geqslant 0$ and $\nu = f(0, 1, 1) \geqslant 0$. Also

$$f(x, y, y) = (x - y)^2((x\sqrt{\lambda} - y\sqrt{\nu})^2 + 2(\mu + \sqrt{\lambda\nu})xy);$$

this expression is positive for all positive x, y only if $\mu + \sqrt{\lambda\nu} \geqslant 0$. Hence the given conditions for $f(x, y, z) \geqslant 0$ are necessary.

To show that the conditions are sufficient, we observe that

$$f(x, y, z) = \lambda A - \sqrt{\lambda\nu}B + \nu C + (\mu + \sqrt{\lambda\nu})B$$

and $\lambda A - \sqrt{\lambda\nu}B + \nu C \geqslant 0$ by the lemma since

$$4AC - B^2 = 3((x + y + z)(y - z)(z - x)(x - y))^2 \geqslant 0.$$

Alternatively,

$$\lambda A - \sqrt{\lambda\nu}B + \nu C = \frac{1}{2}\Sigma[(y + z - x)\sqrt{\lambda} - x\sqrt{\nu}]^2(y - z)^2 \geqslant 0.$$

The polynomials A, B and C are the <u>basic positive special quartics for positive numbers</u>. To obtain the <u>basic positive special quartics for</u>

triangles, we use (0.1):

$$A = \frac{1}{4}(\Sigma a^4 - 3\Sigma a^3(b + c) + 8\Sigma b^2c^2 - 3\Sigma a^2bc),$$

$$B = \frac{1}{2}(\Sigma a^4 - 2\Sigma a^3(b + c) + 4\Sigma b^2c^2 - \Sigma a^2bc),$$

$$C = \frac{1}{4}(\Sigma a^4 - \Sigma a^3(b + c) + \Sigma a^2bc).$$

Also

$$A = s^4 - r(20R - r)s^2 + 4r^2(4R + r)^2 \geqslant 0 \qquad [6, 6.5]$$

$$B = 4r((R + r)s^2 - r(4R + r)^2) \geqslant 0 \qquad [6, 6.7]$$

and

$$C = r^2((4R + r)^2 - 3s^2)$$

which gives $\sqrt{3}s \leqslant 4R + r$, i.e. GI 5.5.

Degree 6 [8], [9].
Inequalities of type $s^2 \gtreqless q(R, r)$ where q is a quadratic polynomial, and also various inequalities connecting the angles of a triangle, can be reduced to special sextic inequalities in x, y, z without any terms involving Σx^6 or $\Sigma x^5(y + z)$.

Write

$$J = \Sigma x^4(y^2 + z^2), \quad K = xyz\Sigma x^3, \quad L = \Sigma y^3z^3,$$

$$M = xyz\Sigma x^2(y + z), \quad N = x^2y^2z^2, \quad \text{and}$$

$$P = J - 2K - 2L + 2M - 6N = (y - z)^2(z - x)^2(x - y)^2$$

$$Q = K - M + 3N = xyz\Sigma x(x - y)(x - z) = xyzU$$

$$T = L - M + 3N = \Sigma yz(yz - zx)(yz - xy)$$

$$S = M - 6N = xyz\Sigma x(y - z)^2 = xyzV.$$

Then $P \geqslant 0$, $Q \geqslant 0$, $S \geqslant 0$, and we obtain $T \geqslant 0$ if we replace x, y, z by yz, zx, xy in the inequality $U \geqslant 0$. Also

$$P = 4F^2(4R^2 + 20Rr - 2r^2 - s^2) - 4r^3(4r^3 + r)^3$$

$$Q = F^2(s^2 - 16Rr + 5r^2)$$

$$T = r^3(4R + r)^3 - F^2(16Rr - 5r^2)$$

$$S = 4F^2r(R - 2r).$$

We can write

$$P = -4r^2((s^2 - 2R^2 - 10Rr + r^2)^2 - 4R(R - 2r)^3)$$

$$= -4r^2 I, \text{ say;}$$

$I \leqslant 0$ is the fundamental inequality for R, r and s (see Chapter I).

From $T \geqslant 0$ we deduce $s^2 \leqslant (4R + r)^3/(16R - 5r)$, but this is weaker than Gerretsen's inequality $s^2 \leqslant 4R^2 + 4Rr + 3r^2$.

Any special symmetric sextic inequality for positive numbers, with no terms involving Σx^6 or $\Sigma x^5(y + z)$, has the form

$$P(x, y, z) = \alpha P + \beta Q + \gamma T + \delta S \geqslant 0.$$

THEOREM 6. The inequality $\alpha P + \beta Q + \gamma T + \delta S \geqslant 0$ holds for all non-negative x, y, z only if $\alpha, \beta, \gamma \geqslant 0$ and $\delta \geqslant -\sqrt{\beta\gamma}$.

Proof. Suppose that $P(x, y, z) \geqslant 0$ for all positive x, y, z. Then $P(x, 1, 0) = x^2(\alpha(x - 1)^2 + \gamma x)$, so $x^{-4}P(x, 1, 0) \to \alpha$ as $x \to \infty$; hence $\alpha \geqslant 0$. Also $x^{-4}P(x, 1, 1) \to \beta$ as $x \to \infty$; hence $\beta \geqslant 0$, and $P(0, 1, 1) = \gamma$; hence $\gamma \geqslant 0$. Since $\beta, \gamma \geqslant 0$, we can write

$$P(x, y, z) = y^2(x - y)^2((x\sqrt{\beta} - y\sqrt{\gamma})^2 + 2(\delta + \sqrt{\beta\gamma})xy).$$

This expression is positive for all positive only if $\delta \geqslant -\sqrt{\beta\gamma}$ Hence the given conditions for $P(x, y, z) \geqslant 0$ are necessary.

To show that the conditions are sufficient, we observe that

$$P(x, y, z) = \alpha P + (\beta Q - \sqrt{\beta\gamma}S + \gamma T) + (\delta + \sqrt{\beta\gamma})S$$

and

$$\beta Q - \sqrt{\beta\gamma}S + \gamma T \geqslant 0 \quad \text{when} \quad \beta, \gamma \geqslant 0$$

by the lemma, since

$$4QT - S^2 = r^3s^2(16R - 5r)P \geqslant 0$$

because $R - 2r \geqslant 0$.

3. Best Possible Inequalities

Suppose that $X \geqslant 0$ and $Y \geqslant 0$ are two inequalities that hold for (let us say) all non-negative x, y, z, where X and Y are expressions in x, y, z and Y is not a constant multiple of X. If $X \geqslant Y$ for all non-negative x, y, z, with strict inequality for certain values of x, y, z, we say that $Y \geqslant 0$ is a better inequality than $X \geqslant 0$.

Two meanings can be given to the term "best possible inequality" in a set of inequalities (see [7], [8], and [9]). Let us say that an inequality is best possible in the weak sense if no inequality in the set is better, and best possible in the strong sense (we shall write only best) if it is better than every other inequality in the set.

These definitions can be extended to inequalities in (a, b, c) or in (R, r, s) etc. Our definition precludes us from saying that $kX \geqslant 0$ is better than $X \geqslant 0$ when $0 \leqslant k < 1$.

Degree 2 [3].
The only symmetric terms are linear combinations of T_1^2 and T_2. The following theorem is a consequence of Theorem 1.

THEOREM 7 (a) The best inequality of the type $T_1^2 \geqslant uT_2$ is

$$T_1^2 \geqslant 3T_2;$$

this is equivalent to

$$\Sigma x^2 - \Sigma yz \geqslant 0.$$

(b) Within the set of all symmetric quadratic inequalities in non-negative x, y, z, $\Sigma x^2 - \Sigma yz \geqslant 0$ and $\Sigma yz \geqslant 0$ are best possible only in the weak sense, since neither is better than the other.

Degree 3.
Frucht and Klamkin ([3], [7]) considered inequalities of types $T_1^3 \geqslant \alpha T_3$, $T_1T_2 \geqslant \beta T_3$ and $T_1^3 \geqslant vT_1T_2 + wT_3$; they showed that the best inequalities of these types are

$$T_1^3 \geqslant 27T_3 \qquad (U + 4V \geqslant 0 \quad \text{or} \quad s^2 \geqslant 27r^2),$$

$$T_1T_2 \geqslant 9T_3 \qquad (V \geqslant 0 \quad \text{or} \quad R \geqslant 2r)$$

and

$$T_1^3 + 9T_3 \geqslant 4T_1T_2 \qquad (U \geqslant 0).$$

We can also consider inequalities of other types; for instance, the best inequality of type

$$\Sigma x^3 \geqslant \lambda \Sigma x^2(y + z) \quad \text{is} \quad \Sigma x^3 \geqslant \tfrac{1}{2}\Sigma x^2(y + z).$$

These results, and the next theorem, are consequences of Theorem 2.

THEOREM 8. Within the set of all symmetric cubic inequalities in x, y, z,

$U \geqslant 0$, $V \geqslant 0$ and $W \geqslant 0$ are all best possible in the weak sense.

Degree 4.

THEOREM 9 [3]. The best inequality of the type

$$T_1^4 \geqslant uT_1^2T_2 + vT_2^2 \quad \text{is} \quad T_1^4 + 9T_2^2 \geqslant 6T_1^2T_2.$$

Note also that some other results are obtained in [3]. Klamkin also showed that there is no best possible inequality in the strong sense in the class

$$T_1^4 \geqslant uT_1^2T_2 + vT_2^2 + wT_1T_3,$$

but the following result is valid ([9]):

THEOREM 10. Within the set of all special symmetric inequalities, the inequalities $\lambda A - \sqrt{\lambda\nu}B + \nu C \geqslant 0$ ($\lambda, \nu \geqslant 0$; λ, ν not both zero) and $B \geqslant 0$ are all best possible in the weak sense.

Proof. This is a simple consequence of Theorem 5.

Degree 5 [3].

Any symmetric polynomial of degree 5 is a linear combination of T_1^5, $T_1^3T_2$, $T_1^2T_3$, $T_1T_2^2$, and T_2T_3. Klamkin uses the phrase "an inequality for (I, J, K)" to mean an inequality of the type $I \geqslant uJ + vK$. He has given a series of results for degree 5. For example, he has proved

THEOREM 11. The best possible inequality for $(T_1T_2^2, T_2T_3, T_1^2T_3)$ is

$$T_1T_2^2 + 3T_2T_3 - 4T_1^2T_3 \geqslant 0;$$

while, the best inequality for $(T_1^3T_2, T_2T_3, T_1T_2^2)$ reduces to the best inequality from Theorem 8.

Degree 6.

First, we shall give the following result:

THEOREM 12. Within the set of all special symmetric sextic inequalities in x, y, z with no terms involving Σx^6 or $\Sigma x^5(y + z)$, the inequalities $\beta Q - \sqrt{\beta\gamma}S + \gamma T \geqslant 0$ ($\beta, \gamma \geqslant 0$; β, γ not both zero), $P \geqslant 0$ and $S \geqslant 0$ are all best possible in the weak sense.

This is easily proved by the method of [9].

Some best possible inequalities in the strong sense, with some of the possible seven terms: T_1^6, $T_1^4T_2$, $T_1^2T_2^2$, T_2^3, $T_1^3T_3$, $T_1T_2T_3$ and T_3^2, are given in [3]. Here we shall give some of these results.

The best inequalities relating pairs of the above terms are well known and are consequences of the previous pairs. An exception is the pair $(T_2^3, T_1^3T_3)$ for which there is no inequality.

For $(T_2^3, T_3^2, T_1T_2T_3)$ the best inequality is

$$T_2^3 + 9T_3^2 - 4T_1T_2T_3 \geqslant 0.$$

4. Generalization of Gerretsen's Inequalities

Gerretsen's inequalities, ascribed to Steinig in GI 5.8 and 5.17 and in [9] and [10], but obtained earlier in a different form by Gerretsen [5] (GI 5.14 and 5.25) are

(1) $$G_1 = 5r^2 - 16Rr + s^2 \geqslant 0,$$

(2) $$G_2 = 4R^2 + 4Rr + 3r^2 - s^2 \geqslant 0.$$

The first has already been obtained as a consequence of $U \geqslant 0$, and the second follows from $P + 4T \geqslant 0$ (see Section 3 of this Chapter). Gerretsen proved these inequalities by considering expressions for the squares of the distances of the incentre of a triangle from the orthocentre and the centroid (these squares must be non-negative). Of course, these inequalities follow very simply from the fundamental inequality (see Chapter I):

$$2R^2 + 10Rr - r^2 + 2(R - 2r)\sqrt{R(R - 2r)}$$
$$= 4R^2 + 4Rr + 3r^2 - ((R - 2r) -$$
$$- \sqrt{R(R - 2r)})^2 \leqslant 4R^2 + 4Rr + 3r^2;$$

$$2R^2 + 10Rr - r^2 - 2(R - 2r)\sqrt{R(R - 2r)}$$
$$= \left((R - 2r) - \sqrt{R(R - 2r)}\right)^2 + 16Rr - 5r^2 \geqslant 16Rr - 5r^2.$$

W. J. Blundon [11] stated incorrectly that Gerretsen's inequalities are the best possible inequalities in the class $q(R, r) \leqslant s^2 \leqslant Q(R, r)$, where $q(R, r)$ and $Q(R, r)$ are quadratic forms with real coefficients. R. Frucht and M. S. Klamkin [7] gave correct results. Here we shall give the formulations and methods of J. F. Rigby ([8], [9]).

As we stated before, to investigate inequalities

(3) $$s^2 \leqslant \lambda R^2 + \mu Rr + (27 - 4\lambda - 2\mu)r^2,$$

where the coefficients of r^2 has been chosen to give equality when $x = y = z$, we multiply by F^2 and use (0.2) to obtain sextic inequalities in x, y, z of the type discussed in Section 3.

THEOREM 13 [8], [9]. An inequality of type (3) holds for all triangles only if it has the form

$$(4)\qquad s^2 \leqslant (1-\Theta^2)^{-1}(4R^2 + 4(1-\Theta-4\Theta^2)Rr + (3+8\Theta+5\Theta^2)r^2) + \varepsilon r(R-2r)$$

where $0 \leqslant \Theta \leqslant 1$ and $\varepsilon \geqslant 0$.

Within this set of special inequalities, those with $\varepsilon = 0$ are all best possible in the weak sense.

The next result is proved in a similar way.

THEOREM 14 [8], [9]. An inequality of type

$$s^2 \geqslant \lambda R^2 + \mu Rr + (27 - 4\lambda - 2\mu)r^2$$

holds for all triangles only if it has the form

$$(5)\qquad s^2 \geqslant (1-\omega^2)^{-1}(-4\omega^2R^2 + 4(4+\omega-\omega^2)Rr - (5+8\omega+3\omega^2)r^2) - \varepsilon r(R-2r)$$

where $0 \leqslant \omega < 1$ and $\varepsilon \geqslant 0$.

Within this set of special inequalities, those with $\varepsilon = 0$ are all best possible in the weak sense.

For $\Theta = \omega = \varepsilon = 0$, from (4) and (5) we get Gerretsen's inequalities.

The inequalities (4) and (5) have also been proved by Frucht and Klamkin [7]. They also showed that these inequalities are a consequence of the fundamental inequality.

Note that (4) and (5) can be expressed as a single set of inequalities:

THEOREM 15.

$$(5r^2 - 16Rr + s^2)\lambda - 4r(R-2r)\sqrt{\lambda\nu} + (4R^2 + 4Rr + 3r^2 - s^2)\nu + r(R-2r)\mu \geqslant 0$$

for all $\lambda, \mu, \nu \geqslant 0$.

If $\lambda < \nu$ we put $\lambda/\nu = \Theta^2$, $\mu/(\nu-\lambda) = \varepsilon$, to obtain (4); if $\lambda > \nu$ we put $\nu/\lambda = \omega^2$, $\mu/(\lambda-\nu) = \varepsilon$ to obtain (5); if $\lambda = \nu$ we obtain

$$4\lambda(R-2r)^2 + \mu r(R-2r) \geqslant 0.$$

When $\mu = 0$, the inequality in Theorem 15 can be written as

$$G_1\lambda - 4rE\sqrt{\lambda\nu} + G_2\nu \geqslant 0,$$

where $G_1 \geqslant 0$ and $G_2 \geqslant 0$ are Gerretsen's inequalities. Also

$$4G_1G_2 - (4rE)^2 = -4I \geqslant 0,$$

so Theorem 15 can be proved directly from the Lemma, and we have incidentally found another way of writing the fundamental inequality $I \leqslant 0$:

$$G_1G_2 \geqslant 4r^2E^2.$$

THEOREM 16. (a) An inequality of type $s \leqslant \lambda R + \mu r$ holds for all triangles only if it has the form

$$s \leqslant 2R + (3\sqrt{3} - 4)r + \alpha(R - 2r) + \beta r$$

where $\alpha, \beta \geqslant 0$, so that in this set of inequalities

$$s \leqslant 2R + (3\sqrt{3} - 4)r \tag{6}$$

is best possible in the strong sense.

(b) An inequality of type $s \geqslant \lambda R + \mu r$ holds for all triangles only if it has the form

$$s \geqslant 3\sqrt{3}r - \alpha(R - 2r) - \beta r$$

where $\alpha, \beta \geqslant 0$, so that in this set of inequalities

$$s \geqslant 3\sqrt{3}r \tag{7}$$

is the best possible in the strong sense.

These best-possible inequalities are derived in [10] and [11]. We obtain (6) in the form

$$s^2 \leqslant (2R + (3\sqrt{3} - 4)r)^2$$

by putting $\theta = 0$, $\varepsilon = 12\sqrt{3} - 20$ in (4); we obtain (7) in the form $s^2 \geqslant 27R^2$ by putting $\omega = 0$, $\varepsilon = 16$ in (5). Hence (6) and (7) are not best possible even in the weak sense when we square them; this is because we then consider them within a much wider set of inequalities.

References

1. P. J. van Albada, 'Geometric Inequalities and Their Geometry', Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 338-352 (1971), 41-45.
2. K. B. Stolarsky, 'Cubic Triangle Inequalities', Amer. Math. Monthly 78 (1971), 879-881.
3. M. S. Klamkin, 'Duality in Triangle Inequalities', Ford Motor Company Preprint, July 1971. (Also, see Notices Amer. Math. Soc., August 1971, p. 782).
4. J. F. Rigby, 'A Method of Obtaining Related Triangle Inequalities, with Applications', Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 412-460 (1973), 217-226.
5. J. C. Gerretsen, 'Ongelijkheden in de Driehoek', Nieuw Tijdschr. Wisk. 41 (1953), 1-7.
6. O. Bottema and J. T. Groenman, 'On Some Triangle Inequalities', Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 577-598 (1977), 11-20.

7. R. Frucht and M. S. Klamkin, 'On Best Quadratic Triangle Inequalities'. Geometriae Dedicata **2** (1973), 341-348.
8. J. F. Rigby, 'Sextic Inequalities for the Sides of a Triangle', Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. **498-541** (1975), 51-58.
9. J. F. Rigby, 'Quartic and Sextic Inequalities for the Sides of Triangles, and Best Possible Inequalities', Ibid. No. **602-633** (1978), 195-202.
10. O. Bottema, 'Inequalities for R, r and s', Ibid. No. **338-352** (1971), 27-36.
11. W. J. Blundon, 'Inequalities Associated with the Triangle', Canad. Math. Bull. **8** (1965), 615-626.

Chapter IV

DUALITY BETWEEN DIFFERENT TRIANGLE INEQUALITIES AND TRIANGLE INEQUALITIES WITH (R, r, s)

1. Some General Considerations and Some Applications

A very useful method in proving geometric inequalities is the transformation of any triangle inequality

(1) $$F(f_1(u_1, v_1, w_1), \ldots, f_n(u_n, v_n, w_n)) \geq 0,$$

where (u_i, v_i, w_i) $(i = 1, \ldots, n)$ are sets of triangle elements, into a triangle inequality with (R, r, s).

For example, if the following identities hold

(2) $$f_i(u_i, v_i, w_i) = g_i(R, r, s) \quad (i = 1, \ldots, n),$$

then (1) becomes

(3) $$F(g_1(R, r, s), \ldots, g_n(R, r, s)) \geq 0.$$

In many cases this inequality is equivalent to a known (R, r, s)-inequality (Chapple-Euler, Gerretsen or fundamental inequality, for example).

Thus, corresponding identities play a very important role for proving geometric inequalities. Some of these identities are given in II.3., and some others will be given in the next part of this chapter.

Of course, we can directly use any identity in order to generate geometric inequalities. For example, assume that we have an identity

(4) $$f(u, v, w) = g(G(R, r), H(s))$$

and an inequality

(5) $$H(s) \leq T(R, r).$$

If g is nondecreasing in the second variable we get

(6) $$f(u, v, w) \leq g(G(R, r), T(R, r))$$

and the reverse inequality if g is nonincreasing in the second variable.

Similarly, if we have an inequality

(7) $$G(R, r) \leq V(s)$$

and if g is nondecreasing in the first variable, then

(8) $$f(u, v, w) \leqslant g(V(s), R(s))$$

or the reverse inequality if g is nonincreasing in the first variable.

EXAMPLES: 1° (GI 5.14, Gerretsen):

$$12r(2R - r) \leqslant \Sigma a^2 \leqslant 8R^2 + 4r^2.$$

Proof. Since (II 3.10) i.e. $\Sigma a^2 = 2s^2 - 8Rr - 2r^2$, is valid, using the well known Gerretsen inequalities GI 5.8, i.e.

$$r(16R - 5r) \leqslant s^2 \leqslant 4R^2 + 4Rr + 3r^2$$

we get the above inequalities.

Remark. Of course, we can obtain very many such inequalities (see Chapter X of the book).

2° (R. R. Janić, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. **498-541** (1975), 183):

If a triangle is non-obtuse, then the following inequality is valid (A. W. Walker, 'Problem E 2388', Amer. Math. Monthly **79** (1972), 1135):

$$s^2 \geqslant 2R^2 + 8Rr + 3r^2.$$

Using this inequality and

$$\Sigma \frac{b + c}{a} = \frac{s^2 - 2Rr + r^2}{2Rr},$$

we obtain

$$\Sigma \frac{b + c}{a} \geqslant \frac{R^2 + 3Rr + 2r^2}{Rr}.$$

In the same way we obtain

$$\Sigma \frac{s - a}{b + c} \leqslant \frac{R^2 + 8Rr + 4r^2}{2R^2 + 10Rr + 4r^2},$$

from the identity

$$\Sigma \frac{s - a}{b + c} = \frac{1}{2}\left(1 + \frac{2r(3R + 2r)}{s^2 + r(2R + r)}\right).$$

Remark. For some other similar results for special triangles see Chapter X.

Further, we shall note that inequalities of the form (3) are homogeneous, so using the substitutions

(9) $$x = r/R \quad \text{and} \quad y = s/R,$$

this inequality becomes

(10) $F_1(f'_1(x, y), \ldots, f'_n(x, y)) \geqslant 0$

i.e.

(11) $F_2(x, y) \geqslant 0.$

For example, we showed in II 2.3. that the fundamental inequality can be written in the form

$$(x^2 + y^2)^2 + 12x^2 - 20xy^2 + 48x^2 - 4y^2 + 64x \leqslant 0.$$

Geometrically, this condition is described by the figure on page 10.

J. Garfunkel and G. Tsintsifas in an unpublished paper 'Inequalities through R-r-s triangles' used the same condition but they worked differently:

It is not difficult to prove the following proposition. Let C be a conic and $f(x, y) = 0$ its equation, let $M_1(x_1, y_1)$, $M_2(x_2, y_2)$ be two interior points of C (if C is an hyperbola M_1, M_2 belong to the interior of the same branch), then $f(x_1, y_1)f(x_2, y_2) > 0$.

The idea of the method will be shown by one of their examples: For every triangle ABC

(12) $\Sigma \cos A - \Sigma \cos B \cos C \geqslant 3/4.$

It is known that

$$\Sigma \cos A = \frac{R + r}{R} \quad \text{and} \quad \Sigma \cos A \cos B = \frac{x^2 + y^2}{4} - 1$$

(see the next part of this chapter). Therefore, (12) is equivalent to

(13) $x^2 + y^2 - 4x - 5 \leqslant 0.$

Inequality (13) is true for all nonexterior points of the circle $x^2 + y^2 - 4x - 5 = 0$. But, the points $O(0, 0)$, $D(0, 2)$, $A_1(\frac{1}{2}, \frac{3\sqrt{3}}{2})$ (see Figure 2 on page 10) are included in that circle. Hence, our conclusion follows immediately.

Remarks. 1° We shall note that the above inequality can be proved by using the well known Gerretsen and Chapple-Euler inequalities:

$$s^2 \leqslant 4R^2 + 4Rr + 3r^2 = 5R^2 + 4Rr - r^2 - (R - 2r)(R + 2r) \leqslant$$
$$\leqslant 5R^2 + 4Rr - r^2$$

which is equivalent to (13).

2° Garfunkel and Tsintsifas used their method for proving some inequalities for special triangles, too.

2. Some Equivalent Forms

(1) The sides of any triangle are the roots of the equation

$$t^3 - 2st^2 + (s^2 + r^2 + 4Rr)t - 4sRr = 0.$$

Proof. Using the identities

$$a = 2R \sin A = 4R \sin \frac{A}{2} \cos \frac{A}{2}, \quad s - a = r \cotan \frac{A}{2},$$

we get

$$\sin^2 \frac{A}{2} = \frac{ar}{4R(s-a)}, \quad \cos^2 \frac{A}{2} = \frac{a(s-a)}{4Rr}, \quad \text{i.e.}$$

$$\frac{ar}{4R(s-a)} + \frac{a(s-a)}{4Rr} = \sin^2 \frac{A}{2} + \cos^2 \frac{A}{2} = 1,$$

which is equivalent to

$$a^3 - 2sa^2 + (s^2 + r^2 + 4Rr)a - 4sRr = 0.$$

Similarly, we can prove that b and c satisfy the same condition. By using Viète's formulas, we directly obtain

(2) $\Sigma a = 2s,$

(3) $\Sigma bc = s^2 + r^2 + 4Rr,$

(4) $abc = 4sRr.$

Now, we shall give some other identities.

(5) $\Sigma a^2 = 2(s^2 - r^2 - 4Rr).$

Proof. Using the identity

$$\Sigma x^2 = T_1^2 - 2T_2,$$

and (2) and (3), we obtain (5).

(6) $\Sigma a^3 = 2s(s^2 - 3r^2 - 6Rr).$

Proof. Using the identity

$$\Sigma x^3 = T_1^3 - 3T_1T_2 + 3T_3$$

and (2-4) we get (6).

(7) $$\Pi(a + b) = 2s(s^2 + r^2 + 2Rr).$$

Proof. This is a similar consequence of the identity

$$\Pi(y + z) = T_1T_2 - T_3.$$

(8) $$\Sigma\frac{1}{a} = \frac{s^2 + r^2 + 4Rr}{4Rrs}.$$

Proof. This is a consequence of the identity

$$\Sigma\frac{1}{x} = T_2/T_3.$$

(9) $$\Sigma\frac{1}{bc} = \frac{1}{2Rr}.$$

Proof. This is a consequence of the identity

$$\Sigma\frac{1}{yz} = T_1/T_3.$$

(10) $$\Sigma\frac{1}{a^2} = \left(\frac{s^2 - 4Rr + r^2}{4Rrs}\right)^2 - \frac{1}{Rr}.$$

Proof. This is a consequence of the identity

$$\Sigma\frac{1}{x^2} = \frac{T_2^2 - 2T_1T_3}{T_3^2}.$$

(11) $$\Sigma\frac{a + b}{c} = \frac{s^2 + r^2 - 2Rr}{2Rr}.$$

Proof. This is a consequence of the identity

$$\Sigma\frac{y + z}{x} = \frac{T_1T_2 - 3T_3}{T_3}.$$

(12) a^{-1}, b^{-1}, c^{-1} are the roots of the equation

$$4srRt^3 - (s^2 + r^2 + 4Rr)t^2 + 2st - 1 = 0.$$

Proof. By the substitution $t \to 1/t$, we get (12) from (1).

Remarks. 1° Using Viète's formulas, we can get (8) and (9) from (12).

2° Similarly, we can get several similar results (see III 3.).

(13) x, y, z ($x = s - a$, etc.) are the roots of the equation

$$t^3 - st^2 + r(4R + r)t - sr^2 = 0.$$

Proof. By the substitution $t \to s - (s - t)$, we get (13) from (1).

(14) x^{-1}, y^{-1}, z^{-1} are the roots of the equation

$$sr^2t^3 - r(4R + r)t^2 + st - 1 = 0.$$

Proof. By the substitution $t \to 1/t$, we get (14) from (13).

Consequences of (13) and (14) are for example:

(15) $\Sigma xy = \Sigma ab - s^2 = 4Rr + r^2$;

(16) $xyz = r^2 s$,

which is the well known Heron formula

$$F = \sqrt{s(s - a)(s - b)(s - c)}.$$

(17) $\Sigma x^2 = s^2 - 2r(4R + r)$;

(18) $\Sigma x^3 = s(s^2 - 12Rr)$;

(19) $\Sigma \frac{1}{x} = \frac{4R + r}{sR}$;

(20) $\Sigma \frac{1}{xy} = \frac{1}{r^2}$;

(21) $\Sigma \frac{1}{x^2} = \frac{(4R + r)^2 - 2s^2}{s^2 r^2}$;

(22) $\Sigma \frac{a}{s - a} = \frac{4R - 2r}{r}$;

(23) $\Sigma \frac{a^2}{s - a} = \frac{4s(R - r)}{r}$;

(24) $\Sigma \frac{c}{(s - a)(s - b)} = \frac{2(4R + r)}{sr}$;

(25) $\Sigma \frac{c^2}{(s - a)(s - b)} = \frac{4(R + r)}{r}$;

(26) $\sin A$, $\sin B$, $\sin C$ are the roots of the equation

$$4R^2t^3 - 4Rst^2 + (s^2 + r^2 + 4Rr)t - 2sr = 0.$$

Proof. If we put $a = 2R \sin A$, etc., (1) becomes (26).

(27) $\sum \sin A = \frac{s}{R}$;

(28) $\sum \sin B \sin C = \frac{s^2 + 4Rr + r^2}{4R^2}$;

(29) $\prod \sin A = \frac{sr}{2R^2}$;

(30) $\sum \sin^2 A = \frac{s^2 - 4Rr - r^2}{2R^2}$;

(31) $\sum \sin^3 A = \frac{s(s^2 - 6Rr - 3r^2)}{4R^3}$;

(32) $\prod(\sin A + \sin B) = \frac{s(s^2 + r^2 + 2Rr)}{4R^3}$;

(33) $\sum \sin^4 A = \frac{s^4 - (8Rr + 6r^2)s^2 + r^2(4R + r)^2}{8R^4}$;

(34) cos A, cos B, cos C are the roots of the equation

$$4R^2t^3 - 4R(R + r)t^2 + (s^2 + r^2 - 4R^2)t + (2R + r)^2 - s^2 = 0.$$

Proof. By summing the equation $a = 2R \sin A$ and $s - a = r \operatorname{cotan} \frac{A}{2}$, and expressing sin A and cotan $\frac{A}{2}$ by cos A, we get

$$2R\sqrt{(1 - \cos A)(1 + \cos A)} + r\sqrt{\frac{1 + \cos A}{1 - \cos A}} = s$$

wherefrom by squaring we get

$$4R^2 \cos^3 A - 4R(R + r) \cos^2 A + (s^2 + r^2 - 4R^2) \cos A +$$
$$+ (2R + r)^2 - s^2 = 0.$$

Similarly we can prove that cos B and cos C satisfy the same condition, too.

(35) $\sum \cos A = \frac{R + r}{R}$;

(36) $\sum \cos A \cos B = \frac{r^2 + s^2 - 4R^2}{4R^2}$;

(37) $$\Pi \cos A = \frac{s^2 - (2R + r)^2}{4R^2};$$

(38) $$\Sigma \cos^2 A = \frac{6R^2 + 4Rr + r^2 - s^2}{2R^2};$$

(39) $$\Sigma \cos^3 A = \frac{(2R + r)^3 - 3s^2r}{4R^3} - 1;$$

(40) $$\Pi(\cos A + \cos B) = \frac{2Rr^2 + r^3 + s^2r}{4R^3};$$

(41) cosec A, cosec B, cosec C are the roots of the equation

$$2srt^3 - (s^2 + r^2 + 4Rr)t^2 + 4Rst - 4R^2 = 0.$$

Proof. By the substitution $t \to 1/t$ we get (41) from (34).

(42) $$\Sigma \operatorname{cosec} A = \frac{s^2 + r^2 + 4Rr}{2sr};$$

(43) $$\Sigma \operatorname{cosec} B \operatorname{cosec} C = \frac{2R}{r};$$

(44) $$\Sigma \operatorname{cosec}^2 A = \frac{(s^2 + r^2 + 4Rr)^2 - 16s^2Rr}{4s^2R^2};$$

(45) $$\Sigma \frac{\sin A + \sin B}{\sin C} = \frac{s^2 + r^2 - 2Rr}{2Rr};$$

(46) sec A, sec B, sec C are the roots of the equation

$$(s^2 - (2R + r)^2)t^3 - (s^2 + r^2 - 4R^2)t^2 + 4R(R + r)t - 4R^2 = 0;$$

(47) $$\Sigma \sec A = \frac{s^2 + r^2 - 4R^2}{s^2 - (2R - r)^2};$$

(48) $$\Sigma \sec B \sec C = \frac{4R(R + r)}{s^2 - (2R + r)^2};$$

(49) $$\Sigma \sec^2 A = \frac{(s^2 + r^2 - 4R^2)^2 - 8R(R + r)(s^2 - (2R + r)^2)}{(s^2 - (2R + r)^2)^2};$$

(50) $$\Sigma\frac{\cos A + \cos B}{\cos C} = \frac{(R + r)(s^2 + r^2 - 4R^2)}{R(s^2 - (2R + r)^2)} - 3;$$

(51) $$\Pi\frac{\sin A + \sin B}{\cos A + \cos B} = \frac{s}{r};$$

(52) $\sin^2 \frac{A}{2}$, $\sin^2 \frac{B}{2}$, $\sin^2 \frac{C}{2}$ are the roots of the equation

$$16R^2t^3 - 8R(2R - r)t^2 + (s^2 + r^2 - 8Rr)t - r^2 = 0.$$

Proof. Since $\cos A = 1 - 2\sin^2 \frac{A}{2}$, by the substitution $t \to 1 - 2t$ we get (52) from (34).

(53) $$\Sigma \sin^2 \frac{A}{2} = \frac{2R - r}{2R};$$

(54) $$\Sigma \sin^4 \frac{A}{2} = \frac{8R^2 + r^2 - s^2}{8R^2};$$

(55) $$\Sigma \sin^2 \frac{A}{2} \sin^2 \frac{B}{2} = \frac{s^2 + r^2 - 8Rr}{16R^2};$$

(56) $\cos^2 \frac{A}{2}$, $\cos^2 \frac{B}{2}$, $\cos^2 \frac{C}{2}$ are the roots of the equation

$$16R^2t^3 - 8R(4R + r)t^2 + (s^2 + (4R + r)^2)t - s^2 = 0;$$

(57) $$\Sigma \cos^2 \frac{A}{2} = \frac{4R + r}{2R};$$

(58) $$\Sigma \cos^2 \frac{B}{2} \cos^2 \frac{C}{2} = \frac{s^2 + (4R + r)^2}{16R^2};$$

(59) $\operatorname{cosec}^2 \frac{A}{2}$, $\operatorname{cosec}^2 \frac{B}{2}$, $\operatorname{cosec}^2 \frac{C}{2}$ are the roots of the equation

$$r^2t^3 - (s^2 + r^2 - 8Rr)t^2 + 8R(2R - r)t - 16R^2 = 0;$$

(60) $$\Sigma \operatorname{cosec}^2 \frac{A}{2} = \frac{s^2 + r^2 - 8Rr}{r^2};$$

(61) $$\Sigma \operatorname{cosec}^2 \frac{B}{2} \operatorname{cosec}^2 \frac{C}{2} = \frac{8R(2R - r)}{r^2};$$

(62) $\sec^2 \frac{A}{2}$, $\sec^2 \frac{B}{2}$, $\sec^2 \frac{C}{2}$ are the roots of the equation

$$s^2t^3 - (s^2 + (4R + r)^2)t^2 + 8R(4R + r)t - 16R^2 = 0;$$

(63) $$\Sigma \sec^2 \frac{A}{2} = \frac{s^2 + (4R + r)^2}{s^2};$$

(64) $$\Sigma \sec^2 \frac{A}{2} \sec^2 \frac{B}{2} = \frac{8R(4R + r)}{s^2};$$

(65) $$\Sigma \cos 2A = \frac{3R^2 + 4Rr + r^2 - s^2}{R^2}.$$

Proof. This follows from (38), since $\cos 2A = 2\cos^2 A - 1$.

(66) $$\Sigma \sin 2A = 4\Pi \sin A = \frac{2rs}{R^2};$$

(67) $$\Pi \sin 2A = 8\Pi \sin A \, \Pi \cos A = \frac{sr(s^2 - (2R + r^2))}{R^4};$$

(68) cotan A, cotan B, cotan C are the roots of the equation

$$2srt^3 - (s^2 - r^2 - 4Rr)t^2 + 2srt + (2R + r)^2 - s^2 = 0.$$

Proof. If sin A and cotan $\frac{A}{2}$ are expressed in terms of cotan A, then $2R \sin A + r \cot\frac{A}{2} = s$ becomes

$$\frac{2R}{\sqrt{1 + \cot^2 A}} + r\sqrt{1 + \cot^2 A} = s - r \cot A.$$

After squaring and simplifying we get that cotan A is a root of the above equation. The same is valid for cotan B and cotan C.

(69) $$\Sigma \cot A = \frac{s^2 - r^2 - 4Rr}{2sr};$$

(70) $$\Pi \cot A = \frac{s^2 - (2R + r)^2}{2sr};$$

(71) $$\Sigma \cot^2 A = \frac{(s^2 - r^2 - 4Rr)^2}{4s^2r^2} - 2;$$

(72) $$\Pi(\cot A + \cot B) = \frac{2R^2}{sr};$$

(73) $$\Sigma \cot^3 A = \frac{(s^2 - r^2 - 4Rr)^3 - 48s^2R^2r^2}{8s^3r^3};$$

(74) tan A, tan B, tan C are the roots of the equation

$$(s^2 - (2R + r)^2)t^3 - 2srt^2 + (s^2 - 4Rr - r^2)t - 2sr = 0.$$

Proof. By the substitution $t \to 1/t$ we get (74) from (68).

(75) $$\Sigma \tan A = \frac{2sr}{s^2 - (2R + r)^2};$$

(76) $$\Sigma \tan A \tan B = \frac{s^2 - r^2 - 4Rr}{s^2 - (2R + r)^2};$$

(77) $$\Sigma \tan^2 A = \frac{4s^2r^2 - 2(s^2 - r^2 - 4Rr)(s^2 - (2R + r)^2)}{(s^2 - (2R + r)^2)^2};$$

(78) $$\Sigma \tan^3 A = \frac{8sr(s^2r^2 - 3R^2(s^2 - (2R + r)^2))}{(s^2 - (2R + r)^2)^3};$$

(79) $$\Pi(\tan A + \tan B) = \frac{8sR^2r}{(s^2 - (2R + r)^2)^2};$$

(80) $$\Sigma \cot 2A = \frac{s^2 - r^2 - 4Rr}{4sr} + \frac{sr}{(2R + r)^2 - s^2};$$

(81) $$\Pi \cot 2A = \frac{(2s^2 - (2R + r)^2 - r^2 - 4Rr)^2 - 16s^2r^2}{16sr(s^2 - (2R + r)^2)}$$

(82) $\tan \frac{A}{2}$, $\tan \frac{B}{2}$, $\tan \frac{C}{2}$ are the roots of the equation

$$st^3 - (4R + r)t^2 + st - r = 0.$$

Proof. If in $2R \sin A + r \cot \frac{A}{2} = s$ the functions $A \to \sin A$ and $A \to \cot \frac{A}{2}$ are expressed in terms of $\tan \frac{A}{2}$, after simplifying, we find that $\tan \frac{A}{2}$ is the root of the above equation. The same is valid for $\tan \frac{B}{2}$ and $\tan \frac{C}{2}$.

(83) $$\Sigma \tan \frac{A}{2} = \frac{4R + r}{s};$$

(84) $$\Pi \tan \frac{A}{2} = \frac{r}{s} \;;$$

(85) $$\Pi(\tan \frac{A}{2} + \tan \frac{B}{2}) = \frac{4R}{s} \;;$$

(86) $$\Sigma \tan^2 \frac{A}{2} = \frac{(4R + r)^2 - 2s^2}{s^2} \;;$$

(87) $$\Sigma \tan^3 \frac{A}{2} = \frac{(4R + r)^3 - 12s^2R}{s^3} \;;$$

(88) $\cot an \frac{A}{2}$, $\cot an \frac{B}{2}$, $\cot an \frac{C}{2}$ are the roots of the equation

$$rt^3 - st^2 + (4R + r)t - s = 0.$$

Proof. By the substitution $t \to 1/t$ we get (88) from (82).

(89) $$\Sigma \cot an \frac{A}{2} = \Pi \cot an \frac{A}{2} = \frac{s}{r} \;;$$

(90) $$\Sigma \cot an \frac{B}{2} \cot an \frac{C}{2} = \frac{4R + r}{r} \;;$$

(91) $$\Sigma \cot an^2 \frac{A}{2} = \frac{s^2 - 2r(4R + r)}{r^2} \;;$$

(92) $$\Sigma \cot an^3 \frac{A}{2} = \frac{s(s^2 - 12Rr)}{r^3} \;;$$

(93) $$\Pi(\cot an \frac{A}{2} + \cot an \frac{B}{2}) = \frac{4sR}{r^2} \;;$$

(94) $$\Sigma a \sin A = \frac{\Sigma a^2}{2R} = \frac{s^2 - r^2 - 4Rr}{R} \;;$$

(95) $$\Sigma \frac{a}{s - a} = \Sigma \frac{\tan \frac{A}{2} + \tan \frac{B}{2}}{\tan \frac{C}{2}} = \Sigma \frac{\cot an \frac{A}{2} + \cot an \frac{B}{2}}{\cot an \frac{C}{2}} = \frac{4R - 2r}{r} \;;$$

(96) $\Sigma a \tan \frac{A}{2} = 2(2R - r)$ (V. Bobancu, Gaz. Mat. (Bucharest) B 20 (1969), 482);

(97) r_a, r_b, r_c are the roots of the equation

$$t^3 - (4R + r)t^2 + s^2t - s^2r = 0.$$

Proof. From elementary geometry it is well known that

$$\frac{r_a}{r} = \frac{s}{s - a}, \quad \text{i.e.} \quad x^{-1} = \frac{r_a}{rs}.$$

If we put this expression in (14), we get (97).

(98) $r_a^{-1}, r_b^{-1}, r_c^{-1}$ are the roots of the equation

$$s^2rt^3 - s^2t^2 + (4R + r)t - 1 = 0;$$

(99) $\Sigma r_a = 4R + r;$

(100) $\Sigma r_b r_c = s^2;$

(101) $\Pi r_a = s^2 r,$ i.e.

(102) $F = \sqrt{r \Pi r_a};$

(103) $\Sigma r_a^2 = (4R + r)^2 - 2s^2;$

(104) $\Sigma r_a^3 = (4R + r)^3 - 12s^2R;$

(105) $\Pi(r_b + r_c) = 4s^2R;$

(106) $\Sigma \frac{1}{r_a} = \frac{1}{r};$

(107) $\Sigma \frac{1}{r_b r_c} = \frac{4R + r}{s^2 r};$

(108) $\Sigma \frac{1}{r_a^2} = \frac{s^2 - 2r(4R + r)}{s^2 r^2};$

(109) $\Sigma \frac{r_b + r_c}{r_a} = \frac{4R - 2r}{r}.$

Some interesting consequences are

(110) $r \Sigma r_b r_c = \Pi r_a;$

(111) $4R\Sigma r_b r_c = \Pi(r_b + r_c)$.

(112) $\Sigma \frac{a}{s - a} = \Sigma \frac{r_b + r_c}{r_a}$;

(113) h_a, h_b, h_c are the roots of the equation

$$2Rt^3 - (s^2 + r^2 + 4Rr)t^2 + 4s^2rt - 4s^2r^2 = 0.$$

Proof. From the equality $ah_a = 2F = 2sr$ we get $a = 2sr/h_a$, i.e. $h_a = 2sr/a$, so from (12) we get (113).

(114) h_a^{-1}, h_b^{-1}, h_c^{-1} are the roots of the equation

$$4s^2r^2t^3 - 4s^2rt^2 + (s^2 + r^2 + 4Rr)t - 2R = 0;$$

(115) $\Sigma h_a = \frac{1}{2R}(s^2 + r^2 + 4Rr)$;

(116) $\Sigma h_b h_c = 2s^2r/R$;

(117) $\Pi h_a = 2s^2r^2/R$,

i.e.

(118) $F = \sqrt{\frac{1}{2} R \Pi h_a}$;

(119) $\Sigma h_a^2 = \frac{1}{4R^2}((s^2 + r^2 + 4Rr)^2 - 16s^2Rr)$;

(120) $\Sigma \frac{1}{h_a} = \frac{1}{r}$;

(121) $\Sigma h_a^3 = \frac{1}{8R^3}((s^2 + r^2 + 4Rr)^3 - 24s^2Rr(s^2 + r^2 + 4Rr) + 48s^2R^2r^2)$;

(122) $\Pi(h_b + h_c) = \frac{s^2r}{R^2}(s^2 + r^2 + 2Rr)$;

(123) $\Sigma \frac{1}{h_b h_c} = \frac{s^2 + r^2 + 4Rr}{4s^2r^2}$;

(124) $$\Sigma\frac{1}{h_a^2} = \frac{s^2 - r^2 - 4Rr}{2s^2r^2} ;$$

(125) $$\Sigma\frac{h_b + h_c}{h_a} = \frac{s^2 + r^2 - 2Rr}{2Rr} .$$

Some further applications are

(126) $$\Sigma\frac{1}{r_a} = \Sigma\frac{1}{h_a} .$$

(127) $$\Pi\frac{h_b + h_c}{b + c} = \frac{sr}{2R^2} ;$$

(128) $$r\Sigma h_b h_c = \Pi h_a ;$$

(129) $$\Sigma\frac{b + c}{a} = \Sigma\frac{h_b + h_c}{h_a} ;$$

(130) $$\Pi w_a = \frac{16Rr^2s^2}{s^2 + 2Rr + r^2} .$$

Remark. For some other similar identities, see XI.1.

References

1. V. P. Soltan and S. I. Mejdman, Toždestva i neravenstva v treugol'nike. "Kišinev"-Stiinca, 1982.
2. A. Bager, 'Another Family of Goniometric Inequalities', Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 412-460 (1973), 207-216.

Chapter V

TRANSFORMATIONS FOR THE ANGLES OF A TRIANGLE

1. Some Applications of Pexider's Functional Equation in Geometry

In this chapter we shall give some results about transformations of the angles of a triangle or polygon. First, we shall give the following result:

THEOREM 1. Real continuous functions $A \to A_1(A)$, $B \to B_1(B)$, $C \to C_1(C)$ are the solutions of the equation

(1) $$A_1 + B_1 + C_1 = \pi$$

with condition

(2) $$A + B + C = \pi$$

if and only if they have the following form

(3) $$A_1 = kA + \lambda\pi, \quad B_1 = kB + \mu\pi, \quad C_1 = kC + \nu\pi$$

$$(k + \lambda + \mu + \nu = 1).$$

Proof. From (1) and (2) we get

(4) $$A_1(\pi - B - C) = \pi - B_1(B) - C_1(C).$$

If we put

(5) $$f(x) = A_1(\pi - x), \quad g(x) = \frac{\pi}{2} - B_1(x), \quad h(x) = \frac{\pi}{2} - C_1(x),$$

(4) becomes

(6) $$f(B + C) = g(B) + h(C),$$

which is the well known Pexider functional equation, the general continuous solution of which is

(7) $$f(x) = ax + c_1 + c_2, \quad g(x) = ax + c_1, \quad h(x) = ax + c_2.$$

So,

(8) $$A_1(A) = a(\pi - A) + c_1 + c_2, \quad B_1(B) = \frac{\pi}{2} - aB - c_1,$$

$$C_1(C) = \frac{\pi}{2} - aC - c_2,$$

and using the substitutions

$$a= -k, \quad c_1 + c_2 + a\pi = \lambda\pi, \quad \frac{\pi}{2} - c_1 = \mu\pi, \quad \frac{\pi}{2} - c_2 = \nu\pi,$$

from (8) we get (3).

The following result of Ž. Madevski ([1]) can be proved similarly.

THEOREM 2. Real continuous functions $A \to A_1(A)$, $B \to B_1(B)$, $C \to C_1(C)$ are solutions of the system

(9) $$A_1, B_1, C_1 \geqslant 0, \quad A_1 + B_1 + C_1 = \pi,$$

with condition

(10) $$A, B, C \geqslant 0, \quad A + B + C = \pi,$$

if and only if they have the form (3) with condition

(11) $$\lambda, \mu, k + \lambda, k + \mu \geqslant 0; \quad \lambda + \mu, k + \lambda + \mu \leqslant 1.$$

Remark. From (11) follows the condition $-1/2 \leqslant k \leqslant 1$.

In the above theorem we considered the class of all triangles. For this class we shall use the notation K, i.e. we can put

(12) $$K = \{(A, B, C) : A, B, C \geqslant 0, \quad A + B + C = \pi\}$$

or

(12') $$K = \{(A, B): A, B \geqslant 0, \quad A + B \leqslant \pi\}.$$

For the class of acute triangles we shall use the notation K_0, i.e.

(13) $$K_0 = \{(A, B, C): 0 \leqslant A, B, C \leqslant \pi/2, \quad A + B + C = \pi\},$$

or

(13') $$K_0 = \{(A, B): 0 \leqslant A, B \leqslant \pi/2, \quad A + B \geqslant \pi/2\},$$

and K_C for the classes of non-acute triangles with non-acute angle C:

(14) $$K_C = \{(A, B, C): 0 \leqslant A, B \leqslant \pi/2, \quad C \geqslant \pi/2, \quad A + B + C = \pi\}$$

or

(14') $$K_C = \{(A, B): A, B \geqslant 0, \quad A + B \leqslant \pi/2\}.$$

Similarly we have

$$K_A = \{(A, B, C): 0 \leqslant B, C \leqslant \pi/2, \quad A \geqslant \pi/2, \quad A + B + C = \pi\}$$

or

$$K_A = \{(A, B): A \geqslant \pi/2, \quad B \geqslant 0, \quad A + B \leqslant \pi\},$$

and

$$K_B = \{(A, B, C): 0 \leqslant A, C \leqslant \pi/2, \quad B \geqslant \pi/2, \quad A + B + C = \pi\}$$

or

$$K_B = \{(A, B): A \geqslant 0, \quad B \geqslant \pi/2, \quad A + B \leqslant \pi\}.$$

In OAB coordinate system the classes K, K_0, K_A, K_B and K_C are given by triangles OAB, PQR, APR, PBQ, ORQ, respectively, as it is shown in Figure 1.

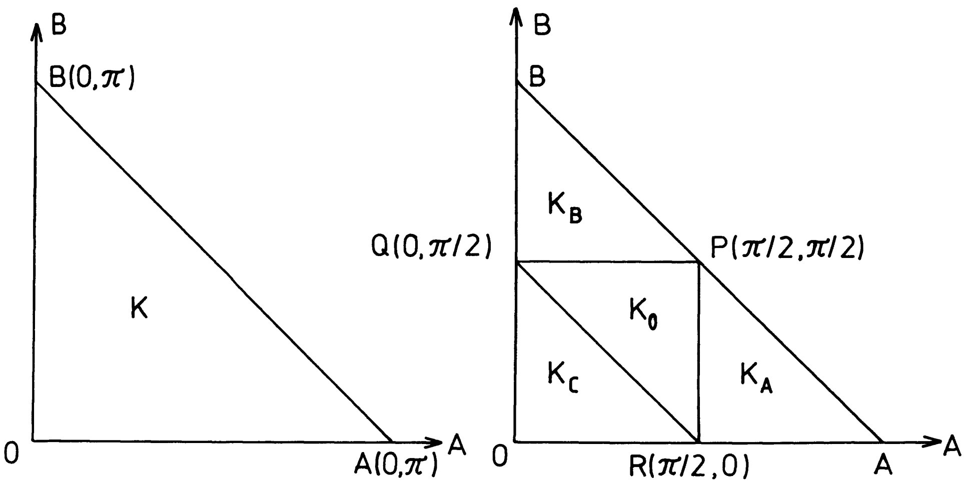


Fig. 1.

Similarly we can prove the following result:

LEMMA 1. (i) Functions A_1, B_1, C_1, defined by (3), transform K_0 in K if

$$(15) \qquad \lambda,\ \mu,\ \frac{k}{2} + \lambda,\ \frac{k}{2} + \mu \geqslant 0; \quad \frac{k}{2} + \lambda + \mu,\ k + \lambda + \mu \leqslant 1.$$

(ii) Functions A_1, B_1, C_1, defined by (3), transform K in K_0 if

$$(16) \qquad \lambda,\ \mu \geqslant 0; \quad 0 \leqslant k + \lambda,\ k + \mu \leqslant \frac{1}{2}; \quad \frac{1}{2} \leqslant \lambda + \mu,\ k + \lambda + \mu \leqslant 1.$$

THEOREM 3. Functions A_1, B_1, C_1, defined by (3), transform K_0 in K in one-to-one way if and only if $k = -2$, $\lambda = \mu = \nu = 1$, i.e. if and only if

$$A_1 = \pi - 2A, \quad B_1 = \pi - 2B, \quad C_1 = \pi - 2C. \tag{17}$$

Proof. We have from (3)

$$A = \frac{1}{k}A_1 - \frac{\lambda}{k}\pi, \quad B = \frac{1}{k}B_1 - \frac{\mu}{k}\pi. \tag{18}$$

So, using Lemma 1 we see that (15) must hold and

$$-\frac{\lambda}{k}, -\frac{\mu}{k} \geqslant 0; \quad 0 \leqslant \frac{1-\lambda}{k}, \frac{1-\mu}{k} \leqslant \frac{1}{2}; \tag{19}$$

$$\frac{1}{2} \leqslant -\frac{\lambda+\mu}{k}, \quad \frac{1-\lambda-\mu}{k} \leqslant 1.$$

If $\lambda, \mu > 0$, then from the first inequalities in (19) we get $k < 0$, and (19) becomes

$$\frac{k}{2} + \lambda, \frac{k}{2} + \mu \leqslant 1; \quad \frac{k}{2} + \lambda + \mu, \lambda, \mu \geqslant 1; \quad k + \lambda + \mu \leqslant 1. \tag{19'}$$

From (19') and (15) we get the condition $\frac{k}{2} + \lambda + \mu = 1$, i.e. $k = 2(1 - \lambda - \mu)$. From $\frac{k}{2} + \lambda \geqslant 0$ we get $\mu \leqslant 1$, what with (19') gives $\mu = 1$. Similarly we get $\lambda = 1$, and therefore $k = -2$.
If $\lambda = 0$ or $\mu = 0$ the systems (15) and (19) have no joint solution.

LEMMA 2. (i) Functions A_1, B_1, C_1, defined by (3), transform K in K_C if

$$\lambda, \mu, k + \lambda, k + \mu \geqslant 0; \quad \lambda + \mu, k + \lambda + \mu \leqslant \frac{1}{2}; \tag{20}$$

(ii) and K_C in K if

$$\lambda, \mu, \frac{k}{2} + \lambda, \frac{k}{2} + \mu \geqslant 0; \quad \lambda + \mu, \frac{k}{2} + \lambda + \mu \leqslant 1. \tag{21}$$

THEOREM 4. Functions A_1, B_1, C_1, defined by (3), transform one-to-one K in K_C if and only if $k = 1/2$, $\lambda = \mu = 0$, i.e. if and only if

$$A_1 = A/2, \quad B_1 = B/2, \quad C_1 = (C + \pi)/2. \tag{22}$$

Proof. Using Lemma 2 (ii) and (18) in the case $\lambda, \mu > 0$, we get $\lambda, \mu \geqslant 1/2$, what is in contradiction to (20). Similar results we get in the cases $\lambda = 0$, $\mu > 0$, and $\lambda > 0$, $\mu = 0$, so, we must put $\lambda = \mu = 0$. In this case from Lemma 2(ii) we get $k \geqslant 1/2$, what with (20) gives $k = 1/2$.

LEMMA 3. (i) Functions A_1, B_1, C_1, defined by (3), transform K_0 in K_C if

(23) $\lambda,\ \mu,\ \frac{k}{2} + \lambda,\ \frac{k}{2} + \mu \geqslant 0;\quad \frac{k}{2} + \lambda + \mu,\ k + \lambda + \mu \leqslant \frac{1}{2};$

(ii) and K_C in K_0 if

(24) $0 \leqslant \lambda,\ \mu,\ \frac{k}{2} + \lambda,\ \frac{k}{2} + \mu \leqslant \frac{1}{2};\quad \frac{1}{2} \leqslant \frac{k}{2} + \lambda + \mu,\ \lambda + \mu.$

THEOREM 5. Functions A_1, B_1, C_1, defined by (3), transform K_0 in K_c one-to-one if and only if $\lambda = \mu = 1/2$, $k = -1$, i.e. if and only if

(25) $A_1 = \frac{\pi}{2} - A,\quad B_1 = \frac{\pi}{2} - B,\quad C_1 = \pi - C.$

Using a generalization of Pexider's equation we can analogously extend Theorem 1:

THEOREM 6. Real continuous functions $A_i \to A_i'(A_i)$ $(i = 1, \ldots, n,\ n \geqslant 3)$ are solutions of equation

(26) $$\sum_{i=1}^{n} A_i' = (n - 2)\pi,$$

with condition

(27) $$\sum_{i=1}^{n} A_i = (n - 2)\pi,$$

if and only if they have the following form

(28) $$A_i'(A_i) = k_0 A_i + k_i(n - 2)\pi \quad (i = 1, \ldots, n) \quad \text{and}$$

$$\sum_{j=0}^{n} k_j = 1.$$

Now, we can prove the following theorem about transformations of the angles of polygons:

THEOREM 7. (i) Real continuous functions $A_i \to A_i'(A_i)$ $(i = 1, \ldots, n)$ are solutions of the system

(29) $$0 \leqslant A_i' \leqslant 2\pi \quad (i = 1, \ldots, n) \quad \sum_{i=1}^{n} A_i' = (n - 2)\pi,$$

with condition

(30) $$0 \leqslant A_i \leqslant 2\pi \quad (i = 1, \ldots, n), \quad \sum_{i=1}^{n} A_i = (n - 2)\pi,$$

if and only if they have the form (28) with condition

(31) $$0 \leqslant k_i(n-2) \leqslant 2, \quad 0 \leqslant 2k_0 + k_i(n-2) \leqslant 2 \quad (i = 1, \ldots, n).$$

(ii) Real continuous functions $A_i \to A_i'(A_i)$ $(i = 1, \ldots, n)$ are solutions of the system

(32) $$0 \leqslant A_i' \leqslant \pi \quad (i = 1, \ldots, n), \quad \sum_{i=1}^{n} A_i' = (n-2)\pi,$$

with condition

(33) $$0 \leqslant A_i \leqslant \pi \quad (i = 1, \ldots, n), \quad \sum_{i=1}^{n} A_i = (n-2)\pi,$$

if and only if they have the form (28) with condition

(34) $$0 \leqslant k_i(n-2) \leqslant 1, \quad 0 \leqslant k_0 + k_i(n-2) \leqslant 1 \quad (i = 1, \ldots, n).$$

Remarks. 1° Theorem 7 (ii) for $n = 3$ becomes Theorem 2.

2° Of course, we can also give the solution of system (29) with condition (33), or the solution of system (32) with condition (30).

3° Of course, it is possible to consider more general transformations. For example, Ž. Madevski [1] considered the affine transformations:

(35) $$A_1 = \xi_1 A + \xi_2 B + \xi_3 C, \quad B_1 = \eta_1 A + \eta_2 B + \eta_3 C,$$
$$C_1 = \zeta_1 A + \zeta_2 B + \zeta_3 C$$

with the following condition imposed

(36) $$\begin{vmatrix} \xi_1 & \xi_2 & \xi_3 \\ \eta_1 & \eta_2 & \eta_3 \\ \zeta_1 & \zeta_2 & \zeta_3 \end{vmatrix} \neq 0.$$

Note that from the conditions

(37) $$\Sigma A_1 = \pi \quad \text{and} \quad \Sigma A = \pi$$

we get

(38) $$\xi_i + \eta_i + \zeta_i = 1 \quad (i = 1, 2, 3),$$

so, (36) becomes

$$(39)\qquad \begin{vmatrix} \xi_1 & \xi_2 & \xi_3 \\ \eta_1 & \eta_2 & \eta_3 \\ 1 & 1 & 1 \end{vmatrix} \neq 0.$$

4° Note that the following result is valid (see [2]):

The angles $A_1, \ldots, A_n$ of simple closed polygons in E^2 satisfy $0 < A_i < 2\pi$ and $\Sigma A_i = (n-2)\pi$. If numbers A_i with these properties are given, then a polygon with the A_i as angles in the given order exists.

2. Some Applications

The above transformations play a very important role in geometry. Here we shall give some applications from [1, pp. 28-32]:

2.1. If A, B, C are angles of a triangle, then $A_1 = (\pi - A)/2$, etc. are also angles of a triangle. Using this transformation we have:

$$\Sigma \sin A \leqslant \frac{3\sqrt{3}}{2} \quad \text{(GI 2.1)} \Rightarrow \Sigma \cos \frac{A}{2} \leqslant \frac{3\sqrt{3}}{2} \quad \text{(GI 2.27)},$$

$$\text{GI 2.8} \Rightarrow \text{GI 2.28}$$

$$\text{GI 2.23} \Rightarrow \Pi \sin \frac{A}{2} \leqslant \frac{1}{8} \quad \text{(GI 2.12)}.$$

2.2. If A, B, C are angles of a triangle, then $A_1 = A/2$, $B_1 = B/2$, $C_1 = (\pi + C)/2$ are angles of a triangle from class K_C. Using this transformation we get:

$$\text{GI 2.2} \Rightarrow 0 < \sin \frac{A}{2} + \sin \frac{B}{2} + \cos \frac{C}{2} < 1 + \sqrt{2},$$

$$\text{GI 2.4} \Rightarrow \sin \frac{A}{2} + \sin \frac{B}{2} + \cos \frac{C}{2} \geqslant \sin A + \sin B - \sin C,$$

$$\text{GI 2.21} \Rightarrow 0 < \cos^2 \frac{A}{2} + \cos^2 \frac{B}{2} - \cos^2 \frac{C}{2} < 2.$$

2.3. If A, B, C are angles of an acute triangle, then $A_1 = \pi - 2A$, etc. are angles of a triangle. Using this transformation we get

$$\text{GI 2.27} \Rightarrow \text{GI 2.2 (2)},$$

$$\text{GI 2.1} \Rightarrow 0 < \Sigma \sin 2A \leqslant \frac{3\sqrt{3}}{2},$$

$$\text{GI 2.4} \Rightarrow \Sigma \sin 2A + \Sigma \sin 4A \geqslant 0 \quad (\Leftrightarrow \Sigma \sin 3A \cos A \geqslant 0),$$

$$\text{GI 2.56} \Rightarrow (\Sigma \cos A)^2 \leqslant \Sigma \sin^2 A.$$

2.4. If $(A, B, C) \in K_C$, then $A_1 = 2A$, $B_1 = 2B$, $C_1 = 2C - \pi$ are angles of a triangle, so we have

$$\text{GI 2.9} \Rightarrow 1 < \sin A + \sin B - \cos C \leqslant \frac{3}{2},$$

$$\text{GI 2.14} \Rightarrow \frac{3}{4} \leqslant \sin^2 A + \sin^2 B + \cos^2 C < 1,$$

$$\text{GI 2.27} \Rightarrow 2 < \cos A + \cos B + \sin C \leqslant \frac{3\sqrt{3}}{2},$$

$$\text{GI 2.24} \Rightarrow -\frac{1}{8} \leqslant \Pi \cos 2A < 1.$$

2.5. If A, B, C are angles of an acute triangle, then $A_1 = \frac{\pi}{2} - A$, $B_1 = \frac{\pi}{2} - B$, $C_1 = \pi - C$ are angles of a triangle from class K_C, so we have

$$\text{GI 2.8} \Rightarrow 0 < \cos A \cos B \sin C < \frac{3\sqrt{3}}{8},$$

$$\text{GI 2.2} \Rightarrow 0 < \cos A + \cos B + \sin C < 1 + \sqrt{2},$$

2.6. If $(A, B, C) \in K_C$, then $A_1 = \frac{\pi}{2} - A$, $B_1 = \frac{\pi}{2} - B$, $C_1 = \pi - C$ are angles of an acute triangle, so we have

$$\text{GI 2.2} \Rightarrow 2 < \cos A + \cos B + \sin C \leqslant \frac{3\sqrt{3}}{2},$$

$$\text{GI 2.3} \Rightarrow 2 < \cos^2 A + \cos^2 B + \sin^2 C \leqslant \frac{9}{4},$$

$$\text{GI 2.24} \Rightarrow -\frac{1}{8} \leqslant \sin A \sin B \cos C < 0,$$

$$\text{GI 2.30} \Rightarrow \cot A + \cot B - \tan C \geqslant 3\sqrt{3}.$$

Remark. For some other examples see for instance I.3. Of course, we gave only some applications of transformations of angles of a triangle. Very many other results can be found in [1].

3. On the Obtaining of Analytic Inequalities from Geometric Ones

3.1. If a triple (x, y, z) defines the angles of a triangle, we shall write $\Delta(x, y, z)$.

Let $x(t)$, $y(t)$, $z(t)$ be three real valued functions such that $\Delta(x(t), y(t), z(t))$ for every t in [a, b]. Then obviously, if R(A, B, C) is a relation between the angles of a triangle, then $R(x(t), y(t), z(t))$ will be a relation on $[a, b]$ (see [3]).

EXAMPLES. 1° $\Delta(\lambda_1 \cos t + \frac{1}{3}(\pi - \cos t), \lambda_2 \cos t + \frac{1}{3}(\pi - \cos t), \lambda_3 \cos t + \frac{1}{3}(\pi - \cos t))$, $\lambda_1, \lambda_2, \lambda_3 > 0$, $\Sigma\lambda_1 = 1$, $-\infty < t < +\infty$.

From GI 2.1 and GI 2.16 we get

$$\sin \frac{\pi - \cos t}{3} \Sigma \cos(\lambda_1 \cos t) + \cos \frac{\pi - \cos t}{3} \Sigma \sin(\lambda_1 \cos t) \leqslant \leqslant \frac{3\sqrt{3}}{2},$$

$$\cos \frac{\pi - \cos t}{3} \Sigma \cos(\lambda_1 \cos t) - \sin \frac{\pi - \cos t}{3} \Sigma \sin(\lambda_1 \cos t) \leqslant \frac{3}{2};$$

multiplying the first inequality by $\sin \frac{\pi - \cos t}{3}$ and the second by $\cos \frac{\pi - \cos t}{3}$ we get by addition

$$\Sigma \cos(\lambda_1 \cos t) \leqslant 3 \cos(\frac{1}{3} \cos t),$$

with equality for $\lambda_1 = \lambda_2 = \lambda_3 = 1/3$.

2° Replacing cost in 1° by 3 arccos t $(1/2 \leqslant t \leqslant 1)$, the same procedure yields

$$\Sigma \cos(3\lambda_1 \arccos t) \leqslant 3t \quad (1/2 \leqslant t \leqslant 1).$$

Remark. In [3] many other examples are given.

3.2. Let $f(z)$, $z = x + iy$, be a complex-valued function defined on a domain D of the plane Oxy; also let K be denoted by (12'), where the components A, B of the number A + iB and C, where $C = \pi - A - B$, are the numerical values of the angles of a triangle.

Then if $f(z)$ maps D into IntK, i.e. $f(D) \subseteq \mathrm{Int}K$, then the following correspondence

$$(A, B, C) = (\mathrm{Re}\, f(z), \mathrm{Im}\, f(z), \pi - \mathrm{Re}\, f(z) - \mathrm{Im}\, f(z)),$$

$$(x, y) \in D,$$

makes it possible to derive from every relation $R(A, B, C)$, valid for angles of a triangle, a new one $R^*(x, y)$ valid on D, i.e.

$$R(A, B, C) = R(\mathrm{Re}\, f(z), \mathrm{Im}\, f(z), \pi - \mathrm{Re}\, f(z) - \mathrm{Im}\, f(z)) \Leftrightarrow$$

$$\Leftrightarrow R^*(x, y).$$

EXAMPLE: 1° The function $1/\bar{z}$ maps

$$D = \{(x, y): x > 0,\ y > 0,\quad \pi(x^2 + y^2) > x + y\}$$

into IntK, i.e. $f(D) \subset \text{IntK}$. The correspondence in question is as follows

$$(A, B, C) = \left(\frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2}, \pi - \frac{x + y}{x^2 + y^2}\right), \qquad (x, y) \in D.$$

From the inequality of GI 2.16 we get

$$1 < \cos\frac{x}{x^2 + y^2} + \cos\frac{y}{x^2 + y^2} - \cos\frac{x + y}{x^2 + y^2} \leqslant \frac{3}{2},$$

or, for $y = x$

$$\cos(1/2x)\ \sin^2(1/4x) \leqslant 1/8 \quad (x \geqslant 1/\pi);$$

the equality holds for $x = 3/2\pi$.

Remark. For some other examples see [3].

References

1. Ž. Madevski, Nekoi metodi za dobivanje relacii megu elementite od eden triagolnik (Doktorska Disertacija), Skopje, 1973.
2. A. Rényi, 'Remark on the Angles of a Polygon', Časopis Pěst. Mat. 78 (1953), 305-306. (Czeck).
3. Ž. Madevski, 'A Note on the Obtaining of Analytic Inequalities from Geometric Ones'. God. zbor. Matem. fak. 29 (1978), 93-98.

Chapter VI

SOME TRIGONOMETRIC INEQUALITIES

0. Introduction

In the book [1] one finds that almost all triangles inequalities are symmetric in form when expressed in terms of the sides a, b, c or the angles A, B, C of a given triangle. No doubt that also assymmetric triangle inequalities play a very important role in geometric inequalities. It should be noted that many of these inequalities are still valid for real numbers A, B, C which satisfy the condition

$$A + B + C = p\pi,$$

where p is a natural number (which has to be odd in some cases). This also applies to the inequality of M. S. Klamkin [2] which can be specialized in many ways to obtain numerous well known inequalities.

1. Asymmetric Trigonometric Inequalities

1.1. Let us consider real numbers A, B, C and a positive integer p such that $A + B + C = p\pi$. Then, for every positive integer n and for any real x, y, z, the obvious inequality

$$\{x + (-1)^{pn}(y \cos nC + z \cos nB)\}^2 + (y \sin nC - z \sin nB)^2 \geqslant 0$$

may be translated to the more suggestive form

(1) $$\Sigma x^2 \geqslant 2(-1)^{np+1} xy \cos nA,$$

with equality if and only if

(2) $$x + (-1)^{pn}(y \cos nC + z \cos nB) = 0 \quad \text{and}$$

$$y \sin nC - z \sin nB = 0,$$

or, equivalently, if, and only if,

(3) $$2(-1)^{np+1} yz \cos nA = y^2 + z^2 - x^2$$

$$2(-1)^{np+1} xz \cos nB = z^2 + x^2 - y^2$$

$$2(-1)^{np+1} xy \cos nC = x^2 + y^2 - z^2.$$

The above equivalence becomes obvious by squaring both relations in (2) and adding the results, after noting that

$$\{x + (-1)^{np}(y \cos nC + z \cos nB)\}^2 + (y \sin nC - z \sin nB)^2$$

$$= \{z + (-1)^{np}(x \cos nB + y \cos nA)\}^2 + (x \sin nB - y \sin nA)^2$$

$$= \{y + (-1)^{np}(z \cos nA + x \cos nC)\}^2 + (z \sin nA - x \sin nC)^2.$$

(3) shows that x, y, z cannot be completely arbitrary real numbers if we wish to have the equality case in (1). Indeed, as noted by G. R. Veldkamp in a private communication, we get from (3)

$$|y^2 + z^2 - x^2| \leqslant 2|yz|$$

and two similar inequalities.

Hence

$$-2|yz| \leqslant y^2 + z^2 - x^2 \leqslant 2|yz|, \quad \text{or}$$

$$(|y| - |z|)^2 \leqslant x^2 \leqslant (|y| + |z|)^2.$$

But this is equivalent to

$$||y| - |z|| \leqslant |x| \leqslant |y| + |z|.$$

So, we have proved: If equality occurs in (1), then there exists a (possibly degenerate) triangle XYZ with $|x|$, $|y|$ and $|z|$ as its sides.

Let us now suppose that equality occurs in (1) and that moreover $p = 1$ and A, B, C are positive. Then we have, apart from ΔXYZ, a second ΔABC. We get from ΔXYZ (Δ the triangle)

(4) $$2|yz| \cos X = y^2 + z^2 - x^2$$

and two similar relations for cos Y and cos Z.

(3) and (4) lead to

(4') $$(-1)^{n+1} yz \cos nA = |yz| \operatorname{cox} X,$$

$$(-1)^{n+1} zx \cos nB = |zx| \cos Y,$$

$$(-1)^{n+1} xy \cos nC = |xy| \cos Z.$$

As the cases in which ΔXYZ is degenerate lead to no valuable insights, we may assume

$$||y| - |z|| < |x| < |y| + |z| \quad \text{etc.} \quad \text{and} \quad (xyz)^2 > 0.$$

In order to discuss (4') it now arises very naturally that we should distinguish the signs of xy, yz and zx and the parity of n. Let, e.g., yx, yz and zx be positive. Then ΔABC is unambiguously determined in its shape if and only if $n = 1$ or $n = 2$. For $n = 1$ we have $A = X$ etc., i.e. $\Delta ABC \sim \Delta XYZ$. If $n = 2$, we get $A = \frac{\pi}{2} - \frac{X}{2}$ etc., i.e. ΔABC is similar to the triangle having the excenters of ΔXYZ as its vertices.

Remarks. 1° M. S. Klamkin [2] extensively studied the various possibilities for the signs of x, y, z and the parity of n.

2° For $n = 1$ we obtain an extended version of the Barrow-Janić inequality (GI 2.20); i.e. if x, y, z are real numbers such that $xyz > 0$ and A, B, C are reals with $A + B + C = p\pi$ $(p \in N)$, then

$$(5) \qquad (-1)^{p+1}\Sigma x \cos A \leqslant \Sigma \frac{yz}{2x}$$

(for $xyz < 0$ the inequality has to be reversed).

For $p = 1$ and $A, B, C > 0$ inequality (5) becomes GI 2.20. But this case appeared much earlier as a problem in [3, p. 69].

3° If $\sin nA$, $\sin nB$, $\sin nC \neq 0$, (2) can be written in more suggestive form

$$(2') \qquad x/\sin nA = y/\sin nB = z/\sin nC.$$

If for example $\sin nA \neq 0$, we can also use (2'), but, in this case we shall understand equations of the form $A/B = C/D$ as $AD = BC$.

If $\sin nA = \sin nB = \sin nC = 0$, we get some trivial identities for $(x \pm y \pm z)^2$, and we shall eliminate these cases from our consideration. With these conventions we shall write (2') instead of (2) further on. The same conventions are valid for all related results.

Further, from the obvious inequalities similar to (3)

$$\{x + (-1)^{pn}(y \sin nC + z \sin nB)\}^2 + (y \cos nC - z \cos nB)^2 \geqslant 0$$

and

$$\{x + (-1)^{pn}(y \sin nC - z \cos nB)\}^2 + (y \cos nC - z \sin nB)^2 \geqslant 0$$

we obtain respectively

$$x^2 + y^2 + z^2 \geqslant 2(-1)^{pn}(yz \cos nA - zx \sin nB - xy \sin nC)$$

and

$$x^2 + y^2 + z^2 \geqslant 2(-1)^{pn}(yz \sin nA - zx \cos nB + xy \sin nC),$$

where $A, B, C \in R$, $A + B + C = p\pi$ $(p \in N)$. The cases $p = 1$ and $A, B, C > 0$ were given in [4].

Note also that the following equivalent form of (1) is valid:

Let $x, y, z \in R$, $p, n \in N$, $p_1, \ldots, p_n \in N$ with $\sum_{i=1}^{n} p_i = pn$, $A_i, B_i, C_i \in R$, $A_i + B_i + C_i = p_i\pi$, $i = 1, \ldots, n$. Then

(1') $$\Sigma x^2 \geqslant 2(-1)^{pn+1}\Sigma yz \cos(\sum_{i=1}^{n} A_i)$$

with equality if

(2'') $$x/\sin(\sum_{i=1}^{n} A_i) = y/\sin(\sum_{i=1}^{n} B_i) = z/\sin(\sum_{i=1}^{n} C_i).$$

This follows from (1) because

$$A = (\sum_{i=1}^{n} A_i)/n, \text{ etc.}$$

satisfy the conditions for this inequality.

Remark. O. Bottema and M. S. Klamkin [5] proved the well known Neuberg-Pedoe inequality (GI 10.8) by using (1') for $p = 1$ and $n = 2$.

1.2. $x, y, z \in R$, $n \in N_0$ $(:= \{0, 1, 2, \ldots\})$, $p \in N$, p odd, $A, B, C \in R$, $A + B + C = p\pi$

$$\Rightarrow \Sigma x^2 \geqslant 2(-1)^{n+[p/2]}\Sigma yz \sin \frac{2n+1}{2} A$$

with equality if

$$x/\cos \frac{2n+1}{2} A = y/\cos \frac{2n+1}{2} B = z/\cos \frac{2n+1}{2} C.$$

Proof. This follows from (1) with n replaced by $2n + 1$ and A by $(p\pi - A)/2$, etc.

Remark. The case $n = 0$, $p = 1$, $A, B, C > 0$ $x, y, z \geqslant 0$ is given in [7].

1.3. $x, y, z \in R$, $p, n \in N$, $A, B, C \in R$, $A + B + C = p\pi \Rightarrow$

(6) $$(\frac{1}{2}\Sigma x)^2 \geqslant \Sigma yz \sin^2 nA$$

with equality if

$$x/\sin 2nA = y/\sin 2nB = z/\sin 2nC.$$

Proof. This is also a consequence of (1) if we put $n \to 2n$ and use $\cos 2nA = 1 - 2\sin^2 nA$, etc.

Remark. This result in the case $p = 1$, $A, B, C > 0$ is given in [4]. The case $n = p = 1$ is also given in [5] and [8].

1.4. $x, y, z \in R$, $p, q \in N$, $q < 2p$, q odd, $n \in N_0$, $A, B, C, A_1, B_1, C_1 \in R$,

$$A + B + C = p\pi, \quad A_1 + B_1 + C_1 = q\pi \Rightarrow$$

(7) $$\sum x^2 \geq 2(-1)^{n+p-(q+1)/2} \sum yz \sin \frac{2n+1}{2}(2A - A_1).$$

Proof. Since $\Sigma(2A - A_1) = (2p - q)\pi$ and $2p - q$ is an odd natural number, this inequality follows from 1.2.

Remark. For $p = q = 1$, $A, B, C > 0$ and $A_1 = B_1 = C_1 = \pi/3$ we get from (7)

1° for $n = 0$ the result from [19],

2° for arbitrary n the result from Janous's unpublished solution of the problem from [19].

1.5. $p \in N$, p no multiple of 3, $A, B, C \in R$, $A + B + C = p\pi$, $x, y, z \in R$ with $xyz > 0 \Rightarrow$

(8) $$(-1)^{[p/3]} \sum x \sin A \leq \frac{\sqrt{3}}{2} \sum \frac{yz}{x}$$

with equality if and only if

$$\sin A : \sin B : \sin C = \cos A : \cos B : \cos C = \frac{1}{x} : \frac{1}{y} : \frac{1}{z}.$$

Proof. Putting $A_1 = \frac{2p\pi}{3} - A$, etc. (then $A_1 + B_1 + C_1 = p\pi$) and using (5) we get

$$(-1)^{p+1} \sum x \left(\cos \frac{2p\pi}{3} \cos A + \sin \frac{2p\pi}{3} \sin A\right) =$$

$$(-1)^{p+1} \sum x \left(-\frac{1}{2} \cos A + \frac{\sqrt{3}}{2}(-1)^{p-3[p/3]+1} \sin A\right) \leq \sum \frac{yz}{2x}$$

wherefrom we obtain (8).

The condition for equality follows from 1.1.

Remark. For $p = 1$, $A, B, C > 0$, (8) becomes

(9) $$\sum x \sin A \leq \frac{\sqrt{3}}{2} \sum \frac{yz}{x} \quad (xyz > 0).$$

This inequality was proved by P. M. Vasić [9] and generalizes GI 2.1. M. S. Klamkin [10] improved (9) in the case $x, y, z \geq 0$ to

(10) $$\sum x \sin A \leq \frac{1}{2}(\sum yz)\sqrt{\frac{x+y+z}{xyz}} \leq \frac{\sqrt{3}}{2} \sum \frac{yz}{x}.$$

He also gave another generalization of (9) in the case $x, y, z \geq 0$ (see [4])

(11) $$|\sum yz \sin nA| \leq \frac{\sqrt{3}}{2} \sum x^2$$

with equality if and only if $\sin nA = \sin nB = \sin nC = \pm \frac{\sqrt{3}}{2}$ and $x = y = z$.

Here we shall give the following generalization of the above results:
Let $x, y, z \geqslant 0$, $p, n \in N$, $A, B, C \in R$, $A + B + C = p\pi$ and $0 \leqslant r \leqslant 2$. Then

$$(12) \qquad \Sigma yz|\sin nA|^r \leqslant (\tfrac{1}{2}\Sigma x)^r (\Sigma yz)^{(2-r)/2}$$

or equivalently

$$(12') \qquad \Sigma x|\sin nA|^r \leqslant (\tfrac{1}{2}\Sigma yz)^r \frac{(x+y+z)^{(2-r)/2}}{(xyz)^{r/2}} .$$

For $r = 1$ we have also

$$(13) \qquad |\Sigma yz \sin nA| \leqslant \Sigma yz|\sin nA| \leqslant \tfrac{1}{2}\Sigma x(\Sigma yz)^{1/2} \leqslant \tfrac{\sqrt{3}}{2}\Sigma x^2$$

or equivalently

$$(13') \qquad |\Sigma x \sin nA| \leqslant \Sigma x|\sin nA| \leqslant \tfrac{1}{2}(\Sigma xy)\sqrt{\frac{x+y+z}{xyz}} \leqslant \frac{\sqrt{3}}{2}\Sigma \frac{yz}{x} .$$

Proof. That (12) and (12') and (13) and (13'), respectively, are equivalent we can see if we put $yz \to x$, $zx \to y$, $xy \to z$, and reverse. Further, using the inequality for weighted means, we get

$$(\Sigma yz|\sin nA|^r/\Sigma yz)^{1/r} \leqslant (\Sigma yz \sin^2 nA/\Sigma yz)^{1/2}.$$

Using now 1.3, we arrive at (12). Let now be $r = 1$.

The first inequalities in (13) and (13') are obvious, so we must only prove the third inequalities. Here, we shall give Klamkin's proof of the third inequality in (13') (i.e. the proof of the second inequality in (10)):

$$\sqrt{3}\Sigma \frac{yz}{x} \geqslant \Sigma yz\sqrt{\frac{x+y+z}{xyz}} ,$$

This inequality is equivalent with

$$3(\Sigma y^2z^2)^2 \geqslant xyz(\Sigma x)(\Sigma yz)^2.$$

Letting $x = 1/u$, $y = 1/v$, and $z = 1/w$ shows that this is equivalent to

$$3(\Sigma u^2)^2 \geqslant (\Sigma u)^2(\Sigma vw)$$

what is the third inequality in (13). Since

$$\left(\frac{1}{3}\Sigma u^2\right)^2 \geqslant \left(\frac{1}{3}\Sigma u\right)^4$$

by the power mean inequality, it suffices finally to show that

$$\left(\frac{1}{3}\Sigma u\right)^2 \geqslant \frac{1}{3}\Sigma vw,$$

and this is equivalent to $\Sigma(v - w)^2 \geqslant 0$.

Let us restate (12) in the equivalent form: for $x, y, z > 0$,

$$(12'') \qquad (\Sigma yz|\sin A|^r/\Sigma yz)^{1/r} \leqslant \frac{1}{2}\Sigma x(\Sigma yz)^{-1/2}$$

for any $A, B, C \in R$, $A + B + C \in \pi Z$, if $0 < r \leqslant 2$. (Z is the set of all integers.) The following problem is posed in [36]:

What is the number

$\mu(x, y, z) = \sup\{r > 0 | (12'')$ holds for any $A, B, C \in R$, $\Sigma A \in \pi Z\}$, (for fixed $x, y, z > 0$, of course)?

C. Tănăsescu communicated to us the following answer to this problem:

For $x, y, z > 0$, inequality (12'') holds for any $A, B, C \in R$ with $\Sigma A \in \pi Z$, if and only if

$$r \in I(x, y, z)$$

where

$$I(x, y, z) = \begin{cases} (0,2), \text{ if } x, y, z \text{ are the sides of a nonequilateral triangle,} \\ \left(0, 2\dfrac{\log \Sigma xy - \log x(y + z)}{\log \Sigma xy - \log(\frac{1}{2}\Sigma x)^2}\right), \quad \text{if} \quad x > y + z \\ \quad \text{or} \quad x = y = z \quad \text{and} \quad \dfrac{\sqrt{3}}{2} \leqslant \dfrac{(\Sigma x)/2}{(\Sigma yz)^{1/2}} < 1, \\ (0, +\infty), \text{ otherwise.} \end{cases}$$

Tănăsescu also noted the following consequence of the above result:

The 'universal' upper bound

$\mu^*: = \sup\{r > 0 | (12'')$ holds for any $x, y, z > 0$ and any $A, B, C \in R$, $\Sigma A \in \pi Z\}$

easily reveals to be

$$\mu^* = 2.$$

Remarks. 1° (W. Janous) Substitution of $\sin A = a/2R$ etc. yields (from Tănăsescu's result) for triangles

$$(\Sigma yza^r/\Sigma yz)^{1/r} \leqslant R\Sigma x(\Sigma yz)^{-1/2}$$

if and only if $r \in I(x, y, z)$. It is a generalization of GI 5.28.

2° Of course, we can give several similar results.

1.6. $x, y, z \geqslant 0$, $n \in N_0$, $p \in N$, p odd, $A, B, C \in R$, $A + B + C = p\pi$, $0 \leqslant r \leqslant 2$

$$\Rightarrow \Sigma yz\left|\cos\frac{2n + 1}{2}A\right|^r \leqslant (\frac{1}{2}\Sigma x)^r(\Sigma yz)^{/2-r)/2}.$$

Moreover, for $r = 1$,

$$|\Sigma yz \cos \frac{2n+1}{2} A| \leqslant \Sigma yz|\cos \frac{2n+1}{2} A| \leqslant \frac{1}{2} \Sigma x(\Sigma yz)^{1/2} \leqslant$$

$$\leqslant \frac{\sqrt{3}}{2} \Sigma x^2.$$

Proof. This is a simple consequence of (12) and (13), respectively, because for $n \to 2n + 1$ and $A \to (p\pi - A)/2$, etc. we have

$$|\sin(2n+1) \frac{p\pi - A}{2}| = |\cos \frac{2n+1}{2} A|, \text{ etc.}$$

1.7. $x, y, z, x_1, y_1, z_1 > 0$, $p, q, m, n \in N$, $A, B, C, A_1, B_1, C_1 \in R$, $A + B + C = p\pi$, $A_1 + B_1 + C_1 = q\pi$

$$\Rightarrow |\Sigma xx_1 \sin nA \sin mA_1| \leqslant \Sigma xx_1|\sin nA||\sin nA_1| \leqslant$$

$$\leqslant \frac{1}{4} \Sigma \frac{yz}{x} \Sigma \frac{y_1 z_1}{x_1}.$$

Proof. For $yz \to x^2$, $zx \to y^2$, $xy \to z^2$, 1.3 becomes

$$\Sigma x^2 \sin^2 nA \leqslant \frac{1}{4}(\Sigma \frac{yz}{x})^2.$$

It now follows from Cauchy's inequality that

$$\Sigma xx_1|\sin nA \sin mA_1| \leqslant (\Sigma x^2 \sin^2 nA \Sigma x_1^2 \sin^2 mA_1)^{1/2}$$

$$\leqslant \frac{1}{4} \Sigma \frac{yz}{x} \Sigma \frac{y_1 z_1}{x_1}.$$

Remark. For $n = m = p = q = 1$, $A, B, C, A_1, B_1, C_1 > 0$ we get Klamkin's two-triangle inequality from [11]. Klamkin also proved a similar m-triangle inequality. Of course, we could also give an analogous generalization of this result.

1.8. $x, y, z > 0$, $p, n \in N$, $A, B, C \in R$, $A + B + C = p\pi \Rightarrow$

$$(14) \qquad \Pi |\sin nA|^x \leqslant \left(\frac{(\Sigma yz)^2}{4xyz(x+y+z)}\right)^{(x+y+z)/2}$$

Proof.

$$(\Pi|\sin nA|^x)^{1/(x+y+z)} \leqslant (\Sigma x \sin^2 nA/\Sigma x)^{1/2} \leqslant$$

$$\leqslant ((\Sigma yz)^2/4xyz(\Sigma x))^{1/2},$$

where we used 1.3 and the power-mean-inequality with weights.

Remark. For $x = y = z = 1$, (14) becomes

(15) $$|\Pi \sin nA| \leqslant 3\sqrt{3}/8$$

what is given without proof in [12] (for $p = 1$).

The following similar result is given in [13]:

The maximum value of $P = \Pi \sin^x A$, where A, B, C are the angles of a triangle and x, y, z are given positive numbers is

$$P_{max} = \Pi\left\{\frac{x(x + y + z)}{(x + y)(x + z)}\right\}^{x/2}.$$

Proof. (W. Janous). The function $P(A, B, C)$ is continuous and non-negative over the compact set

(16) $$A + B + C = \pi, \quad 0 \leqslant A, B, C \leqslant \pi,$$

and vanishes just on its boundary. Consequently, P attains a maximum value at some interior point of the region (16), where $0 < A, B, C < \pi$. We use the method of Lagrange multipliers to find this maximum value.

Let

$$F(A, B, C, \lambda) = P(A, B, C) - \lambda(A + B + C - \pi).$$

Then, since $\partial F/\partial\lambda = 0$, the maximum value of F will occur when

(17) $$\frac{\partial F}{\partial A} = \frac{Px \cos A}{\sin A} - \lambda = 0, \quad \frac{\partial F}{\partial B} = \frac{Py \cos B}{\sin B} - \lambda = 0,$$

$$\frac{\partial F}{\partial C} = \frac{Pz \cos C}{\sin C} - \lambda = 0.$$

We show that $\cos A \cos B \cos C \neq 0$. If $\cos A = 0$, for example, then $\lambda = 0$, so also $\cos B = \cos C = 0$ and $A + B + C = 3\pi/2$, a contradiction. Now, from (17),

$$\lambda \neq 0 \quad \text{and} \quad \left(\frac{P}{\lambda}\right) = \frac{\tan A}{x} = \frac{\tan B}{y} = \frac{\tan C}{z} = \frac{\Sigma \tan A}{\Sigma x} = k, \quad \text{say.}$$

Since $\Sigma \tan A = \Pi \tan A$, we therefore have

$$k\Sigma x = k^3 xyz,$$

so $k^2 = (\Sigma x)/xyz$ and

(18) $$\tan^2 A = \frac{x\Sigma x}{yz}, \text{ etc.}$$

Finally, from $\sin A = \sqrt{\tan^2 A/(1 + \tan^2 A)}$, etc., we obtain

$$\sin A = \sqrt{\frac{x(x + y + z)}{(x + y)(x + z)}}, \text{ etc.}$$

and so

$$P_{max} = \Pi\left\{\frac{x(x + y + z)}{(x + y)(x + z)}\right\}^{x/2}.$$

Remark. 1° The proposer of this problem, M. S. Klamkin, noted that a closely related problem appears without solution in [14]. This problem asked the reader to show that (18) holds when P is a maximum.

2° For x, y, z > 0 there holds

$$P_{max} \leq \left(\frac{(\Sigma yz)^2}{4xyz(x + y + z)}\right)^{(x+y+z)/2}.$$

Proof. This inequality is equivalent to

$$\frac{\Pi(x(y + z))^{x/\Sigma x}}{\Pi(y + z)} \leq \frac{(\Sigma yz)^2}{4xyz(\Sigma x)^2}.$$

By the weighted A-G-inequality and the convexity of $f(t) = t^2$ we get

$$\Pi(x(y + z))^{x/\Sigma x} \leq \Sigma x^2(y + z)/\Sigma x = 2\Sigma x^2(y + z)/\Sigma(y + z) \leq$$

$$\leq 2(\Sigma x(y + z)/\Sigma(y + z))^2 = 2(\Sigma yz/\Sigma x)^2.$$

As also $\Pi(y + z) \geq 8xyz$, the claimed inequality follows.□

1.9. $x, y, z \geq 0$, $p \in N$, p odd, $n \in N_0$, $A, B, C \in R$, $A + B + C = p\pi$

$$\Rightarrow \Pi\left|\cos\frac{2n + 1}{2}A\right|^x \leq \left\{\frac{(\Sigma yz)^2}{4xyz(x + y + z)}\right\}^{\frac{x+y+z}{2}},$$

and for a triangle $\Pi\cos^x\frac{A}{2} \leq P_{max}$, where P_{max} is given as above.

1.10. $x, y, z \in R$, $yz \leq zx \leq xy$, $xy > 0$, A, B, C are angles of a triangle, ⇒

(19) $\quad yz + zx - xy < \Sigma yz\cos A.$

Proof. Using GI 2.16, i.e. $\Sigma\cos A > 1$, i.e.

$$1 + \cos C > 1 - \cos A + 1 - \cos B$$

we have

$$xy(1 + \cos C) > xy(1 - \cos A) + xy(1 - \cos B) \geq$$

$$\geq yz(1 - \cos A) + zx(1 - \cos B), \quad \text{i.e. (19).}$$

1.11. If the conditions of 1.10 are valid, then

$$yz + zx - xy < \Sigma yz \sin \frac{A}{2} .$$

1.12. If the conditions of 1.10 are valid, then

$$(20) \qquad yz + zx < \Sigma yz \cos \frac{A}{2} .$$

Proof. Using GI 2.27, i.e. $\Sigma \cos \frac{A}{2} > 2$, we have

$$xy \cos \frac{C}{2} > xy(1 - \cos \frac{A}{2}) + xy(1 - \cos \frac{B}{2}) \geq yz(1 - \cos \frac{A}{2}) +$$

$$+ zx(1 - \cos \frac{B}{2}), \quad \text{i.e. (20)}.$$

1.13. If the conditions of 1.8 are valid, then

$$xy > \Sigma yz \sin^2 \frac{A}{2} \quad \text{and} \quad yz + zx < \Sigma yz \cos^2 \frac{A}{2} .$$

1.14. $x, y, z \in R$, a, b, c are the sides of a triangle with area F, then ([6]):

$$(21) \qquad \left(\frac{\Sigma ax}{4F}\right)^2 \geq \Sigma \frac{yz}{bc} ,$$

with equality if and only if

$$\frac{x}{a(b^2 + c^2 - a^2)} = \frac{y}{b(c^2 + a^2 - b^2)} = \frac{z}{c(a^2 + b^2 - c^2)} .$$

Actually this result corresponds to the special case $n = 2$ of 1.1., of course in the case $A, B, C \geq 0$. To effect the conversion, just let $ax = x_1$, $by = y_1$, $cz = z_1$, and note that

$$4F^2 = a^2b^2 \sin^2 C = b^2c^2 \sin^2 A = c^2a^2 \sin^2 B,$$

$$\cos 2t = 1 - 2 \sin^2 t.$$

Another form equivalent to (21) is

$$(22) \qquad (\Sigma x)^2 R^2 \geq \Sigma yza^2$$

where R is the radius of the circumcircle of a triangle. The latter is due to Kooi (GI 14.1). It reduces to (21) by substituting $R = abc/4F$, etc. Another equivalent version is

$$(22') \qquad (\Sigma xa^2)^2 \geq 16F^2 \Sigma yz.$$

This form and the corresponding equality conditions are stated without proof by Oppenheim [15] who also remarked that it would be an interesting exercise to see how many triangle inequalities could be deduced from it.

Now we shall give an interesting example of the above results. Letting $yz = 1/a_1$, $zx = 1/b_1$, $xy = 1/c_1$ in (22), we obtain (see also 1) in XII.5.12:

$$\frac{(\Sigma a_1)^2 R^2}{a_1 b_1 c_1} \geqslant \Sigma \frac{a^2}{a_1}.$$

The latter inequality in which a_1, b_1, c_1 are restricted to be the sides of a triangle was proposed by Tomescu [16] as a problem. For other examples see GI 14.1, [2] and [17].

R. Ž. Djordjević [18] gave a refinement of a special case of Barrow-Janić', i.e. Wolstenholme's inequality (5). His result is contained in IX.4.6. (1).

1.15.

$$\Sigma bc \cos A > 0 \qquad ([26]).$$

2. Some Trigonometric Identities

In this Section we shall give some trigonometric identities for real numbers A, B, C which satisfy the condition $A + B + C = p\pi$ ($p \in N$).

2.1. $n \in N_0$, p odd, then

$$\Sigma \sin(2n+1)A = 4(-1)^{n+[p/2]} \Pi \cos \frac{2n+1}{2} A.$$

Proof. Denote $k = 2n + 1$. Then

$$\sin kA + \sin kB = 2 \sin k \frac{A+B}{2} \cos k \frac{A-B}{2} =$$

$$= 2 \sin k \frac{p\pi - C}{2} \cos k \frac{A-B}{2} =$$

$$= 2(-1)^{[kp/2]} \cos \frac{kC}{2} \cos k \frac{A-B}{2},$$

$$\sin kC = 2 \sin \frac{kC}{2} \cos \frac{kC}{2} = 2 \sin k \frac{p\pi - (A+B)}{2} \cos \frac{kC}{2} =$$

$$= 2(-1)^{[kp/2]} \cos k \frac{A+B}{2} \cos \frac{kC}{2}.$$

Hence

$$\Sigma \sin kA = 2(-1)^{[kp/2]} \cos \frac{kC}{2} \left(\cos k \frac{A-B}{2} + \cos k \frac{A+B}{2}\right) =$$

$$= 4(-1)^{[kp/2]}\Pi \cos \frac{kA}{2} .$$

Since $(-1)^{[kp/2]} = (-1)^{n+[p/2]}$, the identity follows.

2.2. $n \in N$, then

$$\Sigma \sin 2nA = 4(-1)^{np+1}\Pi \sin nA.$$

2.3. $n \in N_0$, p odd, then

$$\Sigma \cos(2n + 1)A = 1 + 4(-1)^{n+[p/2]}\Pi \sin \frac{2n + 1}{2} A.$$

Proof. We shall also use the notation $k = 2n + 1$. So we have

$$\cos kA + \cos kB = 2 \cos k \frac{A + B}{2} \cos k \frac{A - B}{2} ,$$

$$\cos kC = \cos(kp\pi - k(A + B)) = -\cos k(A + B) =$$

$$= 1 - 2 \cos^2 k \frac{A + B}{2} ,$$

$$\cos kC = 1 + 2 \cos k \frac{A + B}{2} (\cos k \frac{A - B}{2} - \cos k \frac{A + B}{2}) =$$

$$= 1 + 4 \cos k \frac{p\pi - C}{2} \sin \frac{kA}{2} \sin \frac{kB}{2} =$$

$$= 1 + 4(-1)^{n+[p/2]}\Pi \sin \frac{kA}{2} .$$

2.4. $n \in N$

$$\Rightarrow \Sigma \cos 2nA = 4(-1)^{np}\Pi \cos nA - 1.$$

2.5. $n \in N$,

$$\Rightarrow \Sigma \tan A = \Pi \tan A.$$

Proof. This follows from

$$\tan nA = \tan(pn\pi - n(B + C)) = -\tan n(B + C) =$$

$$= - \frac{\tan nB + \tan nC}{1 - \tan nB \tan nC} .$$

2.6. $n \in N_0$, p odd

$$\Rightarrow \Sigma \text{cotan} \frac{2n + 1}{2} A = \Pi \text{ cotan} \frac{2n + 1}{2} A.$$

Proof. This follows from 2.5. if we put $A \to \frac{p\pi - A}{2}$, etc.

2.7. $n \in N$,

$$\Rightarrow \Sigma \cos^2 nA = 1 + 2(-1)^{np}\Pi \cos nA.$$

Remark. This is equivalent with 2.4. because

$$\cos 2nA = 2\cos^2 nA - 1, \text{ etc.}$$

2.8. $n \in N$,

$$\Rightarrow \Sigma \sin^2 nA = 2(1 - (-1)^{np}\Pi \cos nA).$$

2.9. $n \in N_0$, p odd

$$\Rightarrow \Sigma \cos^2 \frac{2n+1}{2} A = 2\left(1 + (-1)^{n+[p/2]}\Pi \sin \frac{2n+1}{2} A\right).$$

2.10. $n \in N_0$, p odd

$$\Rightarrow \Sigma \sin^2 \frac{2n+1}{2} A = 1 - 2(-1)^{n+[p/2]}\Pi \sin \frac{2n+1}{2} A.$$

2.11. $n \in N$,

$$\Rightarrow \sin nA/\sin nB \sin nC = (-1)^{pn+1}(\cot nB + \cot nC).$$

Proof.

$$\sin nA/\sin nB \sin nC = \sin n(p\pi - (B + C))/\sin nB \sin nC =$$

$$= (-1)^{pn+1} \sin n(B + c)/\sin nB \sin nC =$$

$$= (-1)^{pn+1}(\cot nB + \cot nC).$$

2.12. $n \in N$,

$$\Rightarrow \sin nA/\cos nB \cos nC = (-1)^{pn+1}(\tan nB + \tan nC).$$

2.13. $n \in N$,

$$\Rightarrow \Sigma \cot nB \cot nC = 1.$$

2.14. $n \in N_0$, p odd

$$\Rightarrow \Sigma \cos \frac{2n+1}{2} A = 4\Pi \cos \frac{2n+1}{4}(A + B) =$$

$$= 4\Pi \cos \frac{2n+1}{4}(p\pi - A).$$

Proof. This follows from 2.1. if we put $A \to \frac{p\pi - A}{2}$, etc.

2.15. $n \in N_0$, p odd

$$\Rightarrow \Sigma \sin \frac{2n+1}{2} A = 4\Pi \sin \frac{2n+1}{4}(A+B) + (-1)^n.$$

2.16. $n \in N_0$, p odd

$$\Rightarrow \Sigma \tan \frac{2n+1}{2} B \tan \frac{2n+1}{2} C = 1.$$

Proof. This follows from 2.13. if we put $n \to 2n+1$, $A \to \frac{p\pi - A}{2}$, etc.

2.17. $n \in N$,

$$\Rightarrow \frac{\sin nA + \sin nB}{\cos nA + \cos nB} = \begin{cases} \cot \frac{nC}{2}, & \text{if } n \text{ and } p \text{ are odd,} \\ -\tan \frac{nC}{2}, & \text{if } n \text{ or } p \text{ is even.} \end{cases}$$

2.18. $n \in N$,

$$\Rightarrow \Sigma \tan nB \tan nC = 1 + (-1)^{pn+1} \Pi \sec nA.$$

2.19. $n \in N$,

$$\Rightarrow \Sigma \cot nA = \Pi \cot nA + (-1)^{pn+1} \Pi \operatorname{cosec} nA.$$

2.20. $n \in N$,

$$\Rightarrow (\sin^2 nA - \sin^2 nB)/\sin n(A-B) = (-1)^{pn+1} \sin nC.$$

2.21. $n \in N$,

$$\Rightarrow \sin^2 nA + (-1)^{pn+1} \sin nB \sin nC \cos nA =$$

$$= \sin^2 nB + (-1)^{pn+1} \sin nC \sin nA \cos nB =$$

$$= \sin^2 nC + (-1)^{pn+1} \sin nA \sin nB \cos nC.$$

2.22. $n \in N$,

$$\Rightarrow \Sigma \cos nA/\sin nB \sin nC = 2(-1)^{np+1}.$$

Proof.

$$\cos nA/\sin nB \sin nC = \cos(np\pi - n(B+C))/\sin nB \sin nC =$$

$$= (-1)^{pn} \cos n(B + C)/\sin nB \sin nC =$$

$$= (-1)^{pn}(\text{cotan } nB \text{ cotan } nC - 1),$$

so

$$\Sigma \frac{\cos nA}{\sin nB \sin nC} = (-1)^{pn}(\Sigma \text{ cotan } nB \text{ cotan } nC - 3) = 2(-1)^{np+1}$$

where we used 2.13.

2.23. $n \in N$, then

$$(-1)^{pn+1} \text{ cotan } nA + \sin nA/\sin nB \sin nC =$$

$$= (-1)^{pn+1} \text{ cotan } nB + \sin nB/\sin nA \sin nC$$

$$= (-1)^{pn+1} \text{ cotan } nC + \sin nC/\sin nA \sin nB.$$

Proof. This is a simple consequence of 2.11.

2.24. $n \in N$, then

$$\Pi(\text{cotan } nA + \text{cotan } nB) = (-1)^{n+1} \Pi \text{ cosec } nA.$$

2.25. $n \in N$, then

$$\Sigma(\text{cotan } nA + \text{cotan } nB)/(\tan nA + \tan nB) = 1.$$

Proof. Since $(\text{cotan } nA + \text{cotan } nB)/(\tan nA + \tan nB) = \text{cotan } nA \text{ cotan } nB$, the result follows from 2.13.

2.26. p odd, then

$$\Pi\left(1 + (-1)^{[p/2]} \tan \frac{A}{4}\right) = 2 + 2(-1)^{[p/2]}\Pi \tan \frac{A}{4} .$$

Proof. This is a simple consequence of $\tan (A + B + C)/4 = (-1)^{[p/2]}$.

2.27. $n \in N$, then

$$\Sigma \sin nA \cos nB \cos nC = \Pi \sin nA.$$

Proof. Using the identities

$$\sin A \cos B \cos C = \frac{1}{4}(\sin(A + B + C) + \sin(A + B - C) +$$

$$+ \sin(A - B + C) - \sin(B + C - A)),$$

$$\sin A \sin B \sin C = \frac{1}{4}(\sin(A + B - C) + \sin(B + C - A) +$$

$$+ \sin(C + A - B) - \sin(A + B + C))$$

we get for arbitrary real numbers A, B, C

$$\Sigma \sin A \cos B \cos C = \sin(A + B + C) + \Pi \sin A.$$

Now, using the substitutions $A \to nA$, $B \to nB$, $C \to nC$, where A, B, C are real numbers such that $A + B + C = p\pi$, we get 2.27.

Similarly using the substitutions $A \to \frac{2n + 1}{2} A$, etc. ($A + B + C = p\pi$), we get

2.28. $n \in N_0$, p odd, then

$$\Sigma \sin \frac{2n + 1}{2} A \cos \frac{2n + 1}{2} B \cos \frac{2n + 1}{2} C = (-1)^{n+[p/2]} +$$

$$+ \Pi \sin \frac{2n + 1}{2} A.$$

2.29. $n \in N$, then

$$\Sigma \cos nA \sin nB \sin nC = \Pi \cos nA - (-1)^{np}.$$

2.30. $n \in N_0$, p odd, then

$$\Sigma \cos \frac{2n + 1}{2} A \sin \frac{2n + 1}{2} B \sin \frac{2n + 1}{2} C = \Pi \cos \frac{2n + 1}{2} A.$$

Remarks. 1° The above results are generalizations of some identities from [20], [21] (see also [2] and [22]).

2° Some of the above identities were obtained using other identities and the transformation $A \to \frac{p\pi - A}{2}$, etc. for angles. Of course, using other transformations we can get series of similar results (for example, generalizations of some other identities from [20]).

3° Identities 2.27 and 2.30 are the same as 2.5 and 2.6, respectively.

3. Some Applications

3.1. $n, p \in N$, $A, B, C \in R$, $A + B + C = p\pi$, then

(1) $$-1 \leqslant (-1)^{np+1}\Pi \cos nA \leqslant 1/8.$$

Proof. For $x = y = z = 1$, 1.1 becomes

(2) $$(-1)^{np+1}\Sigma \cos nA \leqslant 3/2$$

and since $(-1)^{pn+1} \cos nA \geqslant -1$ etc., we also have a trivial companion inequality to (2)

$$(3) \qquad (-1)^{pn+1}\Sigma \cos nA \geqslant -3.$$

Now, using (2) and (3) with $n \to 2n$ and 2.4 we get (1).

Remark. The above statement for $p = 1$, $A, B, C > 0$ was proved in [2]. If in addition $n = 1$ we have GI 2.24.

3.2. $n \in N_0$, $p \in N$, p odd, $A, B, C \in R$, $A + B + C = p\pi$

$$\Rightarrow -1 \leqslant (-1)^{n+[p/2]}\Pi \sin \frac{2n+1}{2} A \leqslant 1/8.$$

In the case $p = 1$, $n = 0$ for a triangle the '-1' can be replaced by '0'.

Proof. Putting $n \to 2n + 1$ in (2) and (3) and using 2.3 we get the above result.
The modification for a triangle is just GI 2.12.

3.3. Note that the inequality

$$(4) \qquad |\Pi \sin nA| \leqslant 3\sqrt{3}/8$$

is a consequence of 2.2 and 1.5.(13), in the case $x = y = z$.
Similarly, the inequality

$$(5) \qquad \left|\Pi \cos \frac{2n+1}{2} A\right| \leqslant 3\sqrt{3}/8 \qquad (p \text{ odd})$$

is a consequence of 2.1 and 1.5.(13).

3.4. $n \in N_0$, $p \in N$, p odd, $A, B, C \in R$, $A + B + C = p\pi$, $a, b \in R$, $b > 2a$

$$(6) \qquad 3a - b \leqslant a\Sigma \sin^2 \frac{2n+1}{2} A + b(-1)^{n+[p/2]}\Pi \sin \frac{2n+1}{2} A \leqslant$$

$$\leqslant (6a + b)/8.$$

For $b < 2a$ the inequalities are reversed. If $n = 0$, $p = 1$, and A, B, C are the angles of a triangle, the constant '$3a - b$' in the first inequality can be replaced by 'a'.

Proof. Using 2.3 and $\sin^2 \frac{2n+1}{2} A = (1 - \cos(2n+1)A)/2$, etc. (6) becomes

$$-3(b - 2a) \leqslant (b - 2a)\Sigma \cos(2n+1)A \leqslant 3(b - 2a)/2$$

which is true for $b > 2a$ (for the second inequality see 2.3 and 3.1). Of course, for $b < 2a$, the reverse inequalities are valid. Similarly, we can prove the case $n = 0$, $p = 1$ for a triangle.

Remarks. 1° In the case $a = 1$, $b = 3$, $p = 1$ from the second inequality in (6) we obtain one result from [23].

2° For $b = 0$ the first inequality reads $3/4 \leqslant \Sigma \sin^2 \frac{2n+1}{2} A$. It generalizes GI 2.14.

3.5. $n, p \in N$, $A, B, C \in R$, $A + B + C = p\pi$. Then

$$|\Sigma \sin nA \cos nB \cos nC| \leqslant 3\sqrt{3}/8.$$

Proof. This is a simple consequence of 2.27 and (4).

Remark. For $n = p = 1$, $A, B, C > 0$, the following result is valid (cf. [21])

$$0 < \Sigma \sin A \cos B \cos C \leqslant 3\sqrt{3}/8.$$

3.6. $n \in N_0$, $p \in N$, p odd, $A, B, C \in R$, $A + B + C = p\pi$. Then

$$\frac{3}{2}(\Pi \cos^2 \frac{2n+1}{2} A)^{1/3} \leqslant$$

$$\leqslant (-1)^{n+[p/2]}\Sigma \sin \frac{2n+1}{2} A \cos \frac{2n+1}{2} B \cos \frac{2n+1}{2} C \leqslant \frac{9}{8}.$$

For $n = 0$, $p = 1$, $A, B, C > 0$ the left bound can be improved to $\max\{1, \frac{3}{2}(\Pi \cos^2 A/2)^{1/3}\}$ (cf. [2]).

Proof. These inequalities follow from 3.7 upon letting $A \to (p\pi - A)/2$ etc. and $n \to 2n + 1$.

Remark. For $n = 0$, $p = 1$, $A, B, C > 0$ this inequality reads

$$\max\{\Pi \sec \frac{A}{2}, \frac{3}{2}(\Pi \sec \frac{A}{2})^{1/3}\} \leqslant \Sigma \tan \frac{A}{2} \leqslant \frac{9}{8} \Pi \sec \frac{A}{2}.$$

3.7. $n, p \in N$, $A, B, C \in R$, $A + B + C = p\pi$. Then

$$\frac{3}{2}(\Pi \sin^2 nA)^{1/3} \leqslant (-1)^{pn+1}\Sigma \cos nA \sin nB \sin nC \leqslant 9/8.$$

Proof. The right inequality is a simple consequence of 2.29 and 3.1. Using 2.29 and 2.8 we obtain

$$(-1)^{np+1}\Sigma \cos nA \sin nB \sin nC = (\Sigma \sin^2 nA)/2$$

hence the left inequality follows by the A-G-inequality.

Remark. For $p = n = 1$ this inequality implies the result of [31].

3.8. $n \in N_0$, $p \in N$, p odd, $A, B, C \in R$, $A + B + C = p\pi$. Then

$$|\Sigma \cos \frac{2n+1}{2} A \sin \frac{2n+1}{2} B \sin \frac{2n+1}{2} C| \leqslant 3\sqrt{3}/8.$$

3.9. $n, p \in N$, $A, B, C \in R$, $A + B + C = p\pi$, $k \geqslant -1$. Then

(7) $$\Pi(1 + k \cos^2 nA) \geqslant k^2\Pi \sin^2 nA.$$

This inequality, due to C. Tănăsescu, generalizes results from [25] and [36].

3.10. $n, p \in N$, $A, B, C \in R$, $A + B + C = p\pi$, $k \geqslant -1$. Then

$$(8) \qquad \sum \frac{\sqrt{1 + k \cos^2 nB}}{|\sin nA|} \geqslant 3\sqrt[3]{|k|}.$$

This is a generalization of results from [24] and [36].

3.11. $n \in N_0$, $p \in N$, p odd, $A, B, C \in R$, $A + B + C = p\pi$, $\alpha \in R$. If $\tan \frac{2n+1}{2} B \tan \frac{2n+1}{2} C + u \geqslant 0$ etc., then

$$\sum \left(\tan \frac{2n+1}{2} B \tan \frac{2n+1}{2} C + u\right)^{\alpha} \leqslant 3^{1-\alpha}(1 + \sum u)^{\alpha},$$

where $0 < \alpha < 1$.

For $\alpha < 0$ or $\alpha > 1$ the inequality is reversed.

Proof. Let $0 < \alpha < 1$. Using Jensen's inequality for the concave function $f(x) = x^{\alpha}$ and 2.16, we get

$$\sum \left(\tan \frac{2n+1}{2} B \tan \frac{2n+1}{2} C + u\right)^{\alpha} \leqslant$$

$$\leqslant 3\left(\left(\sum \tan \frac{2n+1}{2} B \tan \frac{2n+1}{2} C + u\right)/3\right)^{\alpha} = 3^{1-\alpha}(1 + \sum u)^{\alpha}.$$

Similarly, the other cases are proved.

Remark. The above result generalizes GI 2.37.

3.12. $n, p \in N$, $A, B, C \in R$, $A + B + C = p\pi$. Then

$$(9) \qquad \sin nA \sin nB \leqslant \begin{cases} \sin^2 (nC/2), & \text{if } n \text{ or } p \text{ is even,} \\ \cos^2 (nC/2), & \text{if } n \text{ and } p \text{ are odd.} \end{cases}$$

Proof. $\sin nA \sin nB = (\cos n(A - B) - \cos n(A + B))/2 \leqslant (1 - \cos n(p\pi - C))/2 = (1 - (-1)^{np} \cos nC)/2$ which is equivalent to (9).

Remark. The case $p = 1$, $n = 2$, $A, B, C > 0$ is given in [27].

3.13. $n \in N_0$, $p \in N$, p odd, $A, B, C \in R$, $A + B + C = p\pi$. Then

$$(10) \qquad \left|\sum \tan \frac{2n+1}{2} A\right| \geqslant \sqrt{3}.$$

Proof. If we let $u = \tan \frac{2n+1}{2} A$ etc. then 2.16 reads $\sum uv = 1$.

From

$$\Sigma u^2 \geqslant \Sigma uv \quad \text{(which is equivalent to } \Sigma(u - v)^2 \geqslant 0)$$

we obtain

$$\Sigma u^2 \geqslant 1.$$

This and the identity $(\Sigma u)^2 = \Sigma u^2 + 2\Sigma uv$ finally yield (10).

Remarks. 1° This inequality generalizes GI 2.33.

2° By the general mean-inequality we obtain from (10) for $r \geqslant 1$

$$\Sigma \left|\tan \frac{2n+1}{2} A\right|^r \geqslant 3^{1-r/2}.$$

(Compare this with GI 2.35 ($r = 2$) and GI 2.36 ($r = 6$).)

3.14. $n, p \in N$, $A, B, C \in R$, $A + B + C = p\pi$. Then

(11) $$|\Sigma \cot nA| \geqslant \sqrt{3}.$$

Proof. Using the identity 2.13 and the idea of the proof of 3.13 the result follows.

Remarks. 1° This inequality generalizes GI 2.28.

2° (11) and the general mean-inequality for $r \geqslant 1$ imply

$$\Sigma |\cot nA|^r \geqslant 3^{1-r/2}.$$

(Compare this with GI 2.39 ($r = 2$) and GI 2.65 ($r \in N$).)

3.15. $n \in N_0$, $p \in N$, p odd, $A, B, C \in R$, $A + B + C = p\pi$. Then

(12) $$\Sigma \cot^2 \frac{2n+1}{2} A \geqslant \Sigma \cot \frac{2n+1}{2} A \, \Sigma \cot(2n+1)A.$$

Proof. We can use the idea of the proof of GI 2.44 and identity 2.5.

3.16. $n \in N_0$, $p \in N$, p odd, $A, B, C \in R$, $A + B + C = p\pi$, $r \in R$, $r > 0$. Then

(13) $$\Sigma \left|\sec \frac{2n+1}{2} A\right|^r \geqslant 2^r 3^{1-r/2}.$$

Proof. This follows from 3.3, inequality (5) and $M_r \geqslant M_0$ for $r > 0$.

Remark. For $n = 0$, $p = 1$, A, B, C angles of a triangle (13) is GI 2.48.

3.17. $n, p \in N$, $A, B, C \in R$, $A + B + C = p\pi$, $r \in R$, $r > 0$. Then

(14) $$\Sigma |\operatorname{cosec} nA|^r \geqslant 2^r 3^{1-r/2}.$$

Proof. This follows from 3.3, inequality (4) and $M_r \geqslant M_0$ for $r > 0$.

Remark. For $n = p = 1$, A, B, C angles of a triangle we get from

(14) GI 2.49 (for $r = 1$) and GI 2.50 (for $r = 2$).

3.18. $n \in N_0$, $p \in N$, p odd, $A, B, C \in R$, $A + B + C = p\pi$

(15) $$24\Pi \cos(2n + 1)A \leqslant \Sigma \cos^2(2n + 1)(B - C).$$

Proof. We can use the idea of the proof of GI 2.26.

Remarks. 1° This is a generalization of a result from [37].

2° Letting $A \to (p\pi - A)/2$ etc., we obtain from (15)

$$24(-1)^{n+[p/2]}\Pi \sin \frac{2n + 1}{2} A \leqslant \Sigma \cos^2 \frac{2n + 1}{2} (B - C).$$

This inequality generalizes the result from [32].

3° The following refinement of the last-stated inequality (similar to Klamkin's inequality [32]) also holds:

$$\Sigma \cos^2 \frac{2n + 1}{2} (B - C) \geqslant 1 + 16(-1)^{n+[p/2]}\Pi \sin \frac{2n + 1}{2} A \geqslant$$

$$\geqslant 24(-1)^{n+[p/2]}\Pi \sin \frac{2n + 1}{2} A.$$

Proof. Because of 3.2 the second inequality follows immediately. For the first one we start from Klamkin's triangle-inequality [32]

(16) $$\Sigma \cos^2 \frac{B - C}{2} \geqslant 1 + 16\Pi \sin \frac{A}{2} .$$

Using 2.1 and 2.3 we get

$$\Sigma \cos^2 \frac{B - C}{2} = \frac{1}{2}(3 + \Sigma \cos(B - C)) =$$

$$= \frac{1}{2}(3 + \Sigma \sin B \sin C + \Sigma \cos B \cos C) =$$

$$= \frac{1}{2}(3 + \frac{1}{2}\{(\Sigma \sin A)^2 - \Sigma \sin^2 A +$$

$$+ (\Sigma \cos A)^2 - \Sigma \cos^2 A\}) =$$

$$= \frac{3}{4} + \frac{1}{4}\{(4\Pi \cos \frac{A}{2})^2 + (1 + 4\Pi \sin \frac{A}{2})^2\}.$$

Therefore, (16) reads equivalently

(17) $$(\Pi \cos \frac{A}{2})^2 + (\Pi \sin \frac{A}{2})^2 \geqslant \frac{7}{2} \Pi \sin \frac{A}{2} .$$

If A, B, C are angles of an acute-angled triangle, $A \to \pi - 2A$ etc. transforms (17) to

$$(18) \qquad (\Pi \sin A)^2 + (\Pi \cos A)^2 \geqslant \frac{7}{2} \Pi \cos A.$$

As $\Pi \cos A \leqslant 0$ for non-acute triangles, (18) is valid for all triangles. Next we show that (18) holds for all $A, B, C \in R$ such that $A + B + C = p\pi$. Indeed, let $A = u\pi + A_1$, $B = v\pi + B_1$, $C = w\pi + C_1$, $u, v, w \in Z$, $0 \leqslant A_1, B_1, C_1 \leqslant \pi$.
Then there are the following three possibilities

(a) $A_1 = B_1 = C_1$. Then (18) is equivalent to $1 \geqslant \frac{7}{2}(-1)$.

(b) $\Sigma A_1 = \pi$. Then (18) is equivalent to

$$(\Pi \sin A_1)^2 + (\Pi \cos A_1)^2 \geqslant \frac{7}{2} \Pi \cos A_1$$

which holds true on grounds of (18) for triangles.

(c) $\Sigma A_1 = 2\pi$. Putting $A_1 = \pi - A_2$ etc. and noting that $\Sigma A_2 = \pi$, the validity of (18) follows as in case (b).
For the general case we only have to substitute in (18) $A \to (2n + 1)A$, $B \to (2n + 1)B$ and $C \to (2n + 1)C - (2pn + p - 1)\pi$.
Thus,

$$(19) \qquad (\Pi \sin(2n + 1)A)^2 + (\Pi \cos(2n + 1)A)^2 \geqslant \frac{7}{2} \Pi \cos(2n + 1)A.$$

Putting here $A \to (p\pi - A)/2$ etc., we finally obtain

$$\left(\Pi \cos \frac{2n + 1}{2} A\right)^2 + \left(\Pi \sin \frac{2n + 1}{2} A\right)^2 \geqslant$$

$$\geqslant \frac{7}{2}(-1)^{n+[p/2]} \Pi \sin \frac{2n + 1}{2} A$$

which is equivalent to the first inequality at the beginning of 3°.□

4° It should be noted that (19) is equivalent to

$$\Sigma \cos^2(2n + 1)(B - C) \geqslant 1 + 16\Pi \cos(2n + 1)A.$$

Because of 3.1, this inequality is better than (15).
For $p = 1$, $n = 0$ this inequality improves GI 2.26.□

5° Inequalities (17) and (18) are equivalent to the following triangle-inequalities (believed to be new)

$$\left(\Pi \cot \frac{A}{2}\right)^2 + 1 \geqslant \frac{7}{2} \Pi \operatorname{cosec} \frac{A}{2}$$

and

$$(\Pi \tan A)^2 + 1 \geq \frac{7}{2} \Pi \sec A.$$

6° Here we shall quote the following triangle-inequalities which are of a type similar to (16) (see [33], [34] and [35], respectively).

(i) $$3 + \Sigma \cos \frac{B - C}{2} \geq 4\Sigma \cos A,$$

(ii) $$\Sigma \sin^2 \frac{A}{2} \geq 1 - \frac{1}{4} \Pi \cos \frac{B - C}{2}$$ or equivalently

$$\Pi \cos \frac{B - C}{2} \geq 8 \Pi \sin \frac{A}{2},$$

(iii) $$\Sigma \cos \frac{B - C}{2} \leq \Sigma \cos A + \Sigma \sin \frac{A}{2} \leq 3.$$

3.19. It should be noted that very many inequalities of [28] and [29] are still valid if the assumptions that α, β, γ, (in these papers) be angles of a triangle are replaced by $A, B, C \in R$, $A + B + C = p\pi$ and $p \in N$ (where sometimes p has to be odd). We now give three examples from [29].

(30) now reads: $p, n \in N$, $A, B, C \in R$, $A + B + C = p\pi$. Then

$$\frac{16}{3} \Pi \sin^2 nA \leq \Sigma \sin^2 2nA.$$

(Its proof follows from 2.2 and the M_1-M_2-inequality.)

(54) now reads: $n \in N_0$, $p \in N$, p odd, $A, B, C \in R$, $A + B + C = p\pi$. Then

$$\frac{16}{3} \Pi \cos^2 \frac{2n + 1}{2} A \leq \Sigma \sin^2 (2n + 1)A.$$

(Its proof follows from 2.1 and the M_1-M_2-inequality.)

(80) now reads: $p, n \in N$, $A, B, C \in R$, $A + B + C = p\pi$. Then

$$4\Sigma \sin^4 nA \leq 9\Sigma \cos^2 2nA.$$

(Its proof - entirely different from the one in [29] - is as follows. Since $\sin^4 nA = (1 - \cos 2nA)^2/4$, the claimed inequality becomes

$$3 \leq 8\Sigma \cos^2 2nA + 2\Sigma \cos 2nA.$$

This inequality is valid on grounds of 2.4, 2.7, and 3.1.)

3.20. $n, p \in N$, $A, B, C \in R$, $A + B + C = p\pi$, $r \in N$, $r \geq 2$. Then

$$\max\left\{\frac{9}{4^r}\binom{2r}{r} - \frac{3}{2}, \frac{3}{2^{r-1}}\right\} \leq S_{2r} := \Sigma\left[\sin^{2r} nA + \cos^{2r} nA\right] \leq 3.$$

Proof. Starting from the formulae

$$\begin{Bmatrix}\sin x\\ \cos x\end{Bmatrix}^{2r} = \frac{1}{2^{2r-1}} \sum_{k=0}^{r-1} (\mp 1)^{r-k}\binom{2r}{k} \cos 2(r-k)x + \frac{1}{2^{2r}}\binom{2r}{r}$$

we get

$$S_{2r} = \frac{1}{2^{2r-1}} \sum_{\substack{k=0\\ k\equiv r \pmod 2}}^{r-1} \binom{2r}{k} \cos 2(r-k)nA + \frac{3}{2^{2r-1}}\binom{2r}{r}.$$

By 3.1, inequality (2) it follows that

$$\Sigma \cos 2(r-k)nA \geqslant -3/2.$$

Therefore,

$$S_{2r} \geqslant \frac{3}{2^{2r-1}} \left\{\binom{2r}{r} - \sum_{\substack{k=0\\ k\equiv r \pmod 2}}^{r-1} \binom{2r}{r}\right\}.$$

As the last sum equals $(2^{2r-1} - \binom{2r}{r})/2$, we have

$$S_{2r} \geqslant \frac{9}{2^{2r}} \left\{\binom{2r}{r} - \frac{3}{2}\right\}.$$

Since $M^r(\sin^2 nA, \cos^2 nA) \geqslant M_1(\sin^2 nA, \cos^2 nA) = 1/2$, etc., the inequality

$$S_{2r} \geqslant 3/2^{r-1}$$

follows.

Remark. This inequality is similar to a possible generalization of the problem from [30], which reads

$$5/2 < \Sigma[\sin^4 A/4 + \cos^4 A/4] \leqslant 21/8,$$

where A, B, C are angles of a triangle.

4. Open Questions

4.1. Let $0 < r < 1$ and $A, B, C \in R$ such that $A + B + C = p\pi$, $p \in N$. Give best lower bounds for

(a) $$\Sigma \left|\tan \frac{2n+1}{2} A\right|^r, \quad \text{where } n \in N_0 \text{ and } p \text{ odd.}$$

(b) $\sum |\cot \text{an}\ nA|^r$, where $n \in N$.

4.2. Improve the lower bound of S_{2r} from 3.20. The one given in 3.20 is not very sharp if r becomes great.

References

1. GI: O. Bottema, R. Ž. Djordjević, R. R. Janić, D. S. Mitrinović, and P.M. Vasić, Geometric Inequalities, Groningen, 1969.
2. M. S. Klamkin, 'Asymmetric Triangle Inequalities', Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 357-380 (1971), 33-44.
3. J. Wolstenholme, A Book of Mathematical Problems, London-Cambridge, 1867.
4. V. N. Murty, K. Satyanarayana, and M. S. Klamkin, 'Problem 715', Crux Math. 8 (1982), 48 and 9 (1983), 58-62.
5. O. Bottema and M. S. Klamkin, 'Joint Triangle Inequalities', Simon Stevin 48 (1974/75), 3-8.
6. M. S. Klamkin, 'Notes on Inequalities Involving Triangles or Tetrahedrons', Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 330-337 (1970), 1-15.
7. B. Milisavljević, 'Problem 457', Mat. Vesnik 2 (1978), 423.
8. J. Szikszai, 'Problem F.2302', Köz. Mat. Lap. 62 (1981), 126 and 63 (1981), 128-130.
9. P. M. Vasić, 'Some Inequalities for the Triangle', Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 274-301 (1969), 121-126.
10. M. S. Klamkin, 'On a Triangle Inequality', Crux Math. 10 (1984), 139-140.
11. M. S. Klamkin, 'An Identity for Simplexes and Related Inequalities', Simon Stevin 48 (1974/75), 57-64.
12. T. Chaundy, The Differential Calculus, Oxford, 1935, p. 292.
13. M. S. Klamkin and W. Janous, 'Problem 908', Crux Math. 10 (1984), 19 and 11 (1985), 93-94.
14. G. H. Hardy, A Course of Pure Mathematics, Cambridge, 1958, p. 311.
15. A. Oppenheim, 'Problem E 1724', Amer. Math. Monthly 72 (1965), 792.
16. I. Tomescu and A. W. Walker, 'Problem E 2221', Ibid. 78 (1971), 82-83.
17. V. Nicula, 'On an Inequality' (Romanian), Gaz. Mat. (Bucharest) 89 (1984), 273-276.
18. R. Ž. Djordjević, 'Some Inequalities for the Elements of a Triangle', Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 274-301 (1969), 124-126.
19. M. S. Klamkin and C. S. Karuppan Chetty, 'Problem E 2958', Amer. Math. Monthly 91 (1985), 289.
20. T. Lalesco, La Géométrie du Triangle, Paris, 1952, pp. 101-104.
21. R. R. Janić and Z. M. Stojaković, 'Problem 173', Mat. Vesnik 6 (1969), 351 and 7 (1970), 276.
22. S. Tache, Gaz. Mat. (Bucharest) 2 (86) (1981), 57-60.
23. V. O. Gordon, 'Problem 466', Mat. v škole 2 (1968), 89 and 6 (1968), 77.
24. J. Garfunkel and J. Gillis, 'Problem E 2119', Amer. Math. Monthly 75 (1968) and 76 (1969), 831.
25. M. S. Klamkin, 'Problem 313', Pi Mu Epsilon J. 6 (1974), 46 and 6

(1975), 109.
26. P. Szilaghi, 'Problem 11676', Gaz. Mat. (Bucharest) B22 (1971), 748.
27. Ju. Gerasimov, 'Problem 489', Mat. v. Škole 3 (1968), 65 and 1 (1969), 78.
28. A. Bager, 'A Family of Goniometric Inequalities', Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 338-352 (1971), 5-25.
29. A. Bager, 'Another Family of Goniometric Inequalities, Ibid. No. 412-460 (1973), 207-216.
30. F. Leuenberger and Hj. Stocker, 'Aufgabe 871', Elem. Math. 36 (1981), 175 and 37 (1982), 166-167.
31. O. P. Mavlo, M. S. Klamkin, J. C. Linders, and F. J. Ronday, 'Problem 639'. Nieuw Arch. Wisk. 30 (1982), 116 and 31 (1983), 87-89.
32. G. Tsintsifas, S. C. Chan, J. Garfunkel, B. Prielipp, and M. S. Klamkin, 'Problem 653', Crux Math. 7 (1981), 179 and 8 (1982), 190-193.
33. G. Tsintsifas and V. N. Murty, 'Problem 696', Crux Math. 7 (1981), 302 and 8 (1982), 316-318.
34. J. Garfunkel and W. Janous, 'Problem 896', Crux Math. 9 (1983), 313 and 11 (1985), 62.
35. A. Bager and L. Bankoff, 'Problem E 2298', Amer. Math. Monthly 78 (1971), 543, 792 and 79 (1972), 520.
36. D. S. Mitrinović, J.E. Pečarić and W. Janous, Some Trigonometric Inequalities, RAD JAZU (Zagreb), 428 (1987), 103-127.
37. R. Bán, 'Problem F 2406', Köz Mat. Lapok 1983, No. 2, 77 and 1983, No. 7, 57.

Chapter VII

SOME OTHER TRANSFORMATIONS

0. Introduction

Section I.3 contains results concerning the existence of a triangle the sides of which are obtained as elements of any given triangle. Therefore we can use these results for generating many other inequalities, i.e. using any known inequality for the sides of a triangle

(1) $I(a, b, c) \geqslant 0,$

and any result from I.3, we get the inequality

(2) $I(a_1, b_1, c_1) \geqslant 0,$

where a_1, b_1, c_1 are the sides of a new triangle given as in I.3.

Of course, we can give many such examples, but here we shall give only a few.

EXAMPLES. 1° For acute triangles [1]

(3) $(\Sigma a^4)^2 > 2\Sigma a^8.$

Proof. Using I.3.2, $u = a^2$, $v = b^2$, $w = c^2$ are sides of a triangle, so (3) is equivalent to

$$(\Sigma u)\Pi(v + w - u) > 0$$

which is obvious.

2° Using I.3.12 we have for $\lambda \geqslant 2$ [2]:

$$\text{GI } 1.1 \Rightarrow 3\Sigma(\cos\frac{B}{\lambda}\cos\frac{C}{\lambda})^{\lambda} \leqslant (\Sigma\cos^{\lambda}(\frac{A}{\lambda}))^2 < 4\Sigma(\cos\frac{B}{\lambda}\cos\frac{C}{\lambda})^{\lambda};$$

$$\text{GI } 1.19 \Rightarrow \frac{1}{3}(\Sigma\cos^{\lambda}(\frac{A}{\lambda}))^2 \leqslant \Sigma\cos^{2\lambda}(\frac{A}{\lambda}) < \frac{1}{2}(\Sigma\cos^{\lambda}(\frac{A}{\lambda}))^2, \text{ etc.}$$

3° Using I.3.32 we get for an acute triangle ([2]):

$$\text{GI } 1.1 \Rightarrow \frac{\pi^2}{4} < \Sigma BC \leqslant \frac{\pi^2}{3};$$

$$\text{GI } 1.19 \Rightarrow \frac{\pi^2}{3} \leqslant \Sigma A^2 < \frac{\pi^2}{2}; \quad \text{etc.}$$

4° Using 1.3.33, a^2h_a, b^2h_b, c^2h_c are sides of a triangle. For this triangle we have $F_1 = 4F^3$ (see [3]), so

$$\text{GI } 4.4 \Rightarrow \Sigma a^4 h_a^2 \geqslant 16F^3\sqrt{3}.$$

Of course, there exist many other transformations which include other elements of a triangle. Some of these transformations will be considered in this Chapter.

1. Square-Root Transformation

This transformation is a consequence of I.3.1 for the function $x \to f(x) = \sqrt{x}$, i.e. there exists a triangle with sides $a' = \sqrt{a}$, $b' = \sqrt{b}$, $c' = \sqrt{c}$. Note that [4]:

$$16F'^2 = 2\Sigma(\sqrt{b})^2(\sqrt{c})^2 - \Sigma(\sqrt{a})^4 = 2\Sigma bc - \Sigma a^2 = 4r(4R + r), \text{ i.e.}$$

$$F' = \frac{1}{2}\sqrt{r(4R + r)}, \quad R' = \frac{a'b'c'}{4F'} = \sqrt{\frac{sR}{4R + r}},$$

$$r' = \sqrt{r(4R + r)}/(\Sigma\sqrt{a}), \quad h_a' = \sqrt{\frac{r}{a}(4R + r)}, \quad m_a' = \frac{1}{2}\sqrt{2b + 2c - a},$$

so, if we have an inequality of the form

$$I(a, b, c, F, R, r, h_a, h_b, h_c, m_a, m_b, m_c) \geqslant 0,$$

then there exists its dual inequality

$$I(\sqrt{a}, \sqrt{b}, \sqrt{c}, F', R', r', h_a', h_b', h_c', m_a', m_b', m_c') \geqslant 0.$$

EXAMPLES. 1° [4] GI 5.13 ⇒ GI 5.1,

$$\text{GI } 4.7 \Rightarrow s + \sqrt{3r(4R + r)} \leqslant \Sigma\sqrt{bc} \leqslant \frac{1}{3}(5s + \sqrt{3r(4R + r)}).$$

2° [4] From GI 10.8 it follows the first inequality in

$$\Sigma a_1(-a + b + c) \geqslant 4\sqrt{rr_1(4R + r)(4R_1 + r_1)} \geqslant 4\sqrt{3FF_1}.$$

This is an interpolating inequality for a result from [5].

3° Using the square-root transformation we can show that inequalities (3) and (10) from Section II.4 are consequence of inequalities (2) and (8) from the same section, respectively.

4° A very important inequality is the following inequality of O. Kooi (GI 14.1):

If x, y, z are real numbers, then

(1) $(\Sigma x)^2R^2 \geqslant \Sigma yza^2.$

If we put $x \to yza^2$, etc., we get

$$(\Sigma yza^2)^2R^2 \geqslant (abc)^2xyz(\Sigma x).$$

If $\lambda = xyz(\Sigma x) \geqslant 0$, we get

(2) $|\Sigma yza^2| \geqslant 4F\sqrt{\lambda}.$

Remark. Note that the following equivalent forms of (1) and (2) are valid

(1') $(\Sigma yz)^2R^2 \geqslant xyz\Sigma xa^2,$

(2') $|\Sigma xa^2| \geqslant 4F\sqrt{\Sigma yz} \quad (\Sigma yz \geqslant 0).$

For inequalities (1) and (2') see Chapter VI (inequalities (22) and (22')).

The square root duals of (1) and (2) are

(3) $(\Sigma x)^2 \dfrac{sR}{4R + r} \geqslant \Sigma yza,$

(4) $|\Sigma yza| \geqslant 2\sqrt{\lambda r(4R + r)} \quad (\lambda \geqslant 0).$

Note that we can also work in the opposite direction. This method will be given in the following example:

EXAMPLE. 5° [6] If a, b, c denote the sides of a triangle, then

$$(\Sigma a)(\Sigma a^2) \geqslant 9\Pi(b^2 + c^2 - a^2)/\Pi(b + c - a)$$

with equality if and only if $a = b = c$.

Proof. If ABC is non-acute, the r.h.s. is $\leqslant 0$ and the inequality is trivial. For acute triangles, the inequality can be rewritten as $4F^2 \geqslant 9r_1^2$ where F denotes the area of ABC and r_1 the inradius of the triangle Δ_1 of sides a^2, b^2, c^2. Since $4F^2 = 4R_1r_1 + r_1^2$ (see [6], since $a = \sqrt{a^2}$, etc.), the inequality reduces to the well known one, $R_1 \geqslant 2r_1$, for triangle Δ_1. Thus the proof is complete.

Note that the following inequality for R' is given in [7]:

(5) $R'^2 \geqslant 2abc/(\Sigma a)^2,$

and that for F' the well known Finsler-Hadwiger inequality (GI 10.3)

(6) $4F'^2 \geqslant \sqrt{3}F,$

is valid.

Of course, we can use any transformation several times. For such a generalization of (6) see a next result. Some further generalizations and some applications are given in the next section.

Ž. Mitrović [8] considered a sequence of triangles T_n (n = 0, 1, 2, ...) where the sides of T_n are equal to the square roots of the sides of T_{n-1}, i.e.

$$a_n = a_0^{2^{-n}}, \quad b_n = b_0^{2^{-n}}, \quad c_n = c_0^{2^{-n}}, \quad \text{and}$$

$$a_\infty = b_\infty = c_\infty = 1.$$

We shall write R, r, s, A, F, h, d = F/R, k instead of R_∞, r_∞, s_∞, A_∞, F_∞, h_∞, d_∞, 2^n, respectively.

Mitrović proved the following results:

$$R_n^k \leqslant R^{k-1}R_0; \quad s_n^k \leqslant s^{k-1}s_0; \quad r_n^k \geqslant r^{k-1}r_0;$$

$$\sin^k A_n \geqslant \sin^{k-1} A \sin A_0; \quad F_n^k \geqslant F^{k-1}F_0;$$

$$h_{a_n}^k \geqslant h^{k-1}h_{a_0}; \quad r_0/r \leqslant (F_n/F)^{k/2} \leqslant R_0/R; \quad d_n^k \geqslant d^{k-1}d_0$$

with equalities if and only if T_0 is equilateral.

2. Generalizations of the Finsler-Hadwiger Inequality and Applications

A very important inequality is the Finsler-Hadwiger inequality of GI 10.3 (see the previous section). The following generalization of this inequality is given by A. Oppenheim [9]:

(a) If ABC is a triangle of sides a, b, c and area F, there exists a triangle of sides $a^{1/p}$, $b^{1/p}$, $c^{1/p}$, (p > 1) and area F_p such that

$$(4F_p/\sqrt{3})^p \geqslant 4F/\sqrt{3}. \tag{1}$$

Equality holds only if a = b = c.

In other words, $(4F_p/\sqrt{3})^p$ is an increasing function of p (bounded of course by $(abc)^{2/3}$).

The corresponding circumradii satisfy the inequality

$$(R_p\sqrt{3})^p \leqslant R\sqrt{3}. \tag{2}$$

$(R_p\sqrt{3})^p$ is a decreasing function of p, bounded below by 1.

Proof. To facilitate the writing, suppose that ABC has sides a^p, b^p, c^p so that the second triangle has sides a, b, c. It is then a question of proving that

$$(3) \qquad E = U^p - 3^{p-1}(2\Sigma b^{2p}c^{2p} - \Sigma a^{4p})$$

where $U = 2\Sigma b^2c^2 - \Sigma a^4$ has minimum 0, attained for $a = b = c$.

Since E is homogeneous it is enough to determine stationary values subject to abc = const. Partial differentiation and Euler's theorem on homogeneous functions yield the conditions

$$a\frac{\partial E}{\partial a} = b\frac{\partial E}{\partial b} = c\frac{\partial E}{\partial c} = \frac{4}{3}pE,$$

whence

$$(4) \qquad 3a^2(b^2 + c^2 - a^2)U^{p-1} - 3^p a^{2p}(b^{2p} + c^{2p} - a^{2p}) = E,$$

and two like equations (5), (6) by cyclic permutation of a, b, c.

One solution of these three equations is plainly $a = b = c$ for which $E = 0$. If a different solution exists we may suppose by symmetry that

$$a > c \geqslant b \quad \text{or} \quad a \geqslant c > b.$$

From (4) and (5) by subtraction,

$$(7) \qquad (a^2 - b^2)(a^2 + b^2 - c^2)U^{p-1} = 3^{p-1}(a^{2p} - b^{2p})(a^{2p} + b^{2p} - c^{2p}).$$

Eliminate U^{p-1} between (6) and (7), we find that

$$(a^2 - b^2)E = 3^p c^p (a^{2p} + b^{2p} - c^{2p})(a^2(a^{2p-2} - c^{2p-2}) + $$
$$+ b^2(c^{2p-2} - b^{2p-2}))$$

which shows that $E > 0$.

Thus (a) follows: equality holds only for equilateral triangles.

A. Oppenheim gave as conjecture a further generalization of the above results. C. E. Carroll proved this conjecture, i.e. he proved the following result [10]:

(b) Suppose a, b, c are the sides of an acute or right triangle, $f(x) > 0$, $\log f(x)$ is a convex function of $\log x$, and

$$0 < \log(f(x)/f(y))/\log(x/y) < 1,$$

where x and y are distinct positive numbers. Then

$$G(f(a), f(b), f(c)) \geqslant f(G(a, b, c))$$

where

$$G(a, b, c) = \left(\frac{4F}{\sqrt{3}}\right)^{1/2} =$$

$$= \left(\frac{(a + b + c)(-a + b + c)(a - b + c)(a + b - c)}{3}\right)^{1/4}$$

with equality if and only if $a = b = c$.

Here we shall give an application of Oppenheim's result. Similarly, one can use the result of Carroll.

Let an inequality of the form

$$I(F, R, a, b, c) \geqslant 0$$

be given. If I is non-increasing in the first variable and non-decreasing in the second, then the following inequality is also valid:

$$I\left(\frac{\sqrt{3}}{4}\left(\frac{4F}{\sqrt{3}}\right)^{1/2}, \frac{1}{\sqrt{3}}R\sqrt{3})^{1/p}, a^{1/p}, b^{1/p}, c^{1/p}\right) \geqslant 0 \quad (p \geqslant 1).$$

A similar result for two or n triangle inequalities is also valid.

EXAMPLES. 1° [11] If $0 < t \leqslant 2$, and x, y, z are real numbers, then

$$\Sigma yza^t \leqslant \frac{1}{3}(\Sigma x)^2 (R\sqrt{3})^t, \tag{8}$$

$$|\Sigma yza^t| \geqslant \sqrt{3}\lambda\left(\frac{4F}{\sqrt{3}}\right)^{t/2} \quad (\lambda = xyz(\Sigma x) \geqslant 0). \tag{9}$$

These inequalities are generalizations of (1) and (2) from the previous section.

2° [12],[30]. Let p, q, r be real numbers such that $p + q > 0$, $q + r > 0$, $r + p > 0$. If $0 < t \leqslant 4$, then

$$\Sigma \frac{p}{q + r} a^t \geqslant \frac{3}{2}\left(\frac{4F}{\sqrt{3}}\right)^{t/2}. \tag{10}$$

This is a generalization of Tsintsifas' inequality (8) from Section II.4.

3° Let p, q, r be non-negative numbers. If $0 < t \leqslant 2$, then

$$\Sigma \frac{p}{q + r} b^t c^t \geqslant \frac{3}{2}\left(\frac{4F}{\sqrt{3}}\right)^t. \tag{11}$$

This is a generalization of Klamkin's inequality (2) from Section II.4.

3. Some Trigonometric Transformations

We shall now give some further examples in the case when f, g, h are three real functions such that if A, B, C are angles of a triangle, then f(A), g(B), h(C) are sides of a triangle. The first such example was given in 0, but we shall now give examples involving other inequalities for a triangle (not only inequalities for the sides of a triangle). All examples from this section are given in [2] and [3].

3.1. The result from I.3.3 says that a triangle Δ_1 whose sides are $a_1 = \sin A$, $b_1 = \sin B$, $c_1 = \sin C$, exists. In this case we also have

$$A_1 = A, \text{ etc.}, \quad s_1 = \frac{1}{2} \Sigma \sin A = 2\Pi \cos \frac{A}{2}, \quad F_1 = \frac{1}{2} \Pi \sin A,$$

$$R_1 = \frac{1}{2}, \quad r_1 = 2\Pi \sin \frac{A}{2}, \ h_{a_1} = \sin B \sin C, \text{ etc.},$$

$$r_{a_1} = 2 \sin \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}, \text{ etc.}$$

so we can use all inequalities of the form

$$I(a, b, c, A, B, C, s, F, R, r, h_a, h_b, h_c, r_a, r_b, r_c) \geqslant 0$$

for generating new inequalities with angles of a triangle.

EXAMPLES.

GI 1.20 $\Rightarrow 1 < \Sigma \tan \frac{B}{2} \tan \frac{C}{2} < \sqrt{3}$;

GI 5.1 $\Rightarrow$ GI 2.12; GI 5.20 $\Rightarrow$ GI 2.29;

GI 6.28 $\Rightarrow \Sigma \sin^2 \frac{A}{2} \geqslant 6\Pi \sin \frac{A}{2}$; etc.

3.2. The result I.3.4 says that a triangle Δ_2 whose sides are $a_2 = \cos \frac{A}{2}$, $b_2 = \cos \frac{B}{2}$, $c_2 = \cos \frac{C}{2}$ exists. In this case we also have

$$A_2 = \frac{\pi}{2} - \frac{A}{2}, \text{ etc.} \quad F_2 = \frac{1}{2} \Pi \cos \frac{A}{2},$$

$$s_2 = \frac{1}{2} \Sigma \cos \frac{A}{2} = 2\Pi \cos \frac{- A}{4},$$

$$s_2 - a_2 = \frac{1}{2}(\cos \frac{B}{2} + \cos \frac{C}{2} - \cos\frac{A}{2}) =$$

$$= 2 \cos \frac{\pi - A}{4} \sin \frac{\pi - B}{4} \sin \frac{\pi - C}{4},$$

$$R_2 = \frac{1}{2}, \ r_2 = (\Pi \cos \frac{A}{2})/\Sigma \cos \frac{A}{2}.$$

EXAMPLES.

GI 5.3 $\Rightarrow$ GI 2.27;

$$\text{GI } 4.10 \Rightarrow \Sigma \cos^4 \frac{A}{2} \geqslant 4\Pi\cos^2 \frac{A}{2},$$

$$\text{GI } 5.1 \Rightarrow \Sigma \cos \frac{A}{2} \geqslant 4\Pi\cos \frac{A}{2}, \text{ etc.}$$

Comment by J. F. Rigby. Δ_2 is an acute triangle, so any inequality for acute triangles can be used here to yield an inequality for all triangles.

3.3. The result I.3.5 says that a triangle Δ_3 whose sides are $a_3 = \cos^2 \frac{A}{2}$, $b_3 = \cos^2 \frac{B}{2}$, $c_3 = \cos^2 \frac{C}{2}$ exists. In this case we also have

$$s_3 = u\Pi \cos \frac{A}{2}, \quad F_3 = (uv)^{1/2}\Pi \cos \frac{A}{2}, \quad R_3 = \frac{1}{4}(uv)^{-1/2},$$

$$r_3 = (v/u)^{1/2}\Pi \cos \frac{A}{2},$$

where

$$u = \Sigma \tan \frac{A}{2}, \quad v = \Pi \tan \frac{A}{2}.$$

EXAMPLES.

$$\text{GI } 1.3 \Rightarrow \text{GI } 2.12;$$

$$\text{GI } 1.15 \Rightarrow \Sigma \tan \frac{A}{2} \geqslant 9\Pi \tan \frac{A}{2};$$

$$\text{GI } 5.11 \Rightarrow (\Sigma \tan \frac{A}{2})^3 \geqslant 27\Pi\tan \frac{A}{2};$$

$$\text{GI } 5.12 \Rightarrow (\Sigma \tan \frac{A}{2})^3 \geqslant \frac{27}{8} \Pi \sec \frac{A}{2}; \text{ etc.}$$

3.4. If $0 < A, B, C < \pi/2$, then a triangle Δ_4 the sides of which are $a_4 = \sin 2A$, $b_4 = \sin 2B$, $c_4 = \sin 2C$ exists (see I.3.6). In this case we also have

$$A_4 = \pi - 2A, \text{ etc.} \quad F_4 = \frac{1}{2} \Pi \sin 2A, \quad R_4 = \frac{1}{2},$$

$$r_4 = 2\Pi\cos A, \text{ etc.}$$

Note that this result is a simple consequence of 3.1 if we use the substitution $A \to \pi - 2A$, etc. (see proof of I.3.6). Of course, we can get similar results from I.3.7 and I.3.8 (see proofs of these results).

Remark. For an interesting application of 3.4 see X.2.3.

Comment by J. F. Rigby. It should be emphasized that when a triangle inequality is applied to Δ_4, we obtain an inequality for <u>acute</u> triangles only. Also, this transformation is the opposite of 3.2.

4. The Median-Dual Transformation and Its Generalizations

4.1. We obtain an important transformation as a consequence of I.3.13. This result shows that the triangle Δ_m whose sides are m_a, m_b, m_c exists. In this case we have that if

$$(1) \qquad I(a, b, c, F, R, r, h_a, h_b, h_c, m_a, m_b, m_c, r_a, r_b, r_c, w_a, w_b, w_c) \geqslant 0$$

is any inequality involving the stated elements of triangle ABC, it is equivalent to the following inequality, called the median dual of (1) [13]:

$$I(m_a, m_b, m_c, F_m, R_m, \hat{h}_a, \hat{h}_b, \hat{h}_c, \hat{m}_a, \hat{m}_b, \hat{m}_c, \hat{r}_a, \hat{r}_b, \hat{r}_c, \hat{w}_a, \hat{w}_b, \hat{w}_c) \geqslant 0,$$

where

$$F_m = \frac{3}{4} F, \quad R_m = \frac{\Pi m_a}{3F}, \quad r_m = \frac{3F}{2\Sigma m_a}, \quad \hat{h}_a = \frac{3F}{2m_a}, \text{ etc.,}$$

$$\hat{m}_a = \frac{3}{4} a, \text{ etc.,} \quad \hat{r}_a = \frac{3F}{2(m_b + m_c - m_a)}, \text{ etc.,}$$

$$\hat{w}_a = \frac{1}{m_b + m_c} \sqrt{m_b m_c (m_b + m_c - m_a)\Sigma m_a}, \text{ etc.}$$

EXAMPLES. 1° [12] Let p, q, r be real numbers such that $p + q > 0$, $q + r > 0$, $r + p > 0$. Then

$$(2) \qquad \Sigma \frac{p}{q + r} m_a^4 \geqslant \frac{9}{2} F^2$$

with equality if and only if

$$p:q:r = (5a^2 - b^2 - c^2):(-a^2 + 5b^2 - c^2):(-a^2 - b^2 + 5c^2),$$

and if $0 < t \leqslant 4$,

$$(3) \qquad \Sigma \frac{p}{q + r} m_a^t \geqslant \frac{3}{2}(\sqrt{3}F)^{t/2}.$$

These inequalities are the median-duals to II.4.(8) and VII.2.(10).

2° Let p, q, r be non-negative numbers and $0 < t \leqslant 2$. Then

(4) $$\sum \frac{p}{q+r} m_b^t m_c^t \geqslant \frac{3}{2}(\sqrt{3}F)^t.$$

This is the median-dual to VII.2.(11).

3° [11] Let x, y, z be real numbers and let $0 < t \leqslant 2$. Then

(5) $$3(\sqrt{3}F)^t \Sigma yzm_a^t \leqslant (\Sigma x)^2 (\Pi m_a)^t,$$

(6) $$|\Sigma yzm_a^t| \geqslant \sqrt{3\lambda}(\sqrt{3}F)^{t/2} \qquad (\lambda = xyz(\Sigma x) \geqslant 0).$$

These inequalities are the median-duals to VII.2.(8) and (9).

4° [13], [14] GI 4.7 $\Rightarrow 12F\sqrt{3} + 3\Sigma a^2 \leqslant 8\Sigma m_b m_c \leqslant 4F\sqrt{3} + 5\Sigma a^2$.

5° For $t = 2$, $x = a$, $y = b$, $z = c$, (5) becomes

$$\Sigma bcm_a^2 \leqslant (2\Pi m_a/3r)^2,$$

which was obtained by E. A. Velikova and S. J. Bilčev [13]. Similarly we can get some other of their inequalities. Note that Velikova and Bilčev gave in their paper about a hundred geometric inequalities and their median-duals. We shall quote some of their examples:

1) GI 4.1 $\Rightarrow 12F \leqslant \min(b^2 + c^2 + 4a^2,\ c^2 + a^2 + 4b^2,\ a^2 + b^2 + 4c^2)$;

2) GI 4.14 $\Rightarrow (\sqrt{3}F)^3 \leqslant (\Pi m_a)^2$;

3) $8m_b m_c \leqslant 4a^2 + b^2 + c^2 \Rightarrow s(s - a) \leqslant m_a^2$;

4) $\Sigma 1/m_a \leqslant 1/r \quad \Sigma bc \leqslant 2R\Sigma m_a$;

5) $\frac{4}{3}\Sigma m_a \leqslant \Sigma a^2/m_a \Rightarrow \frac{3}{4}\Sigma a \leqslant \Sigma m_a^2/a$;

6) $\Sigma(m_a - m_b)(m_a - m_c)a^2 \geqslant 0 \Rightarrow \Sigma(a - b)(a - c)m_a^2 \geqslant 0$;

7) $\Sigma(m_b + m_c)/a \leqslant (\Pi m_a)^2/F^3 \Rightarrow \Sigma(b + c)/m_a \leqslant 9R^2/F$;

8) $\frac{3\sqrt{3}}{F} \leqslant \Sigma(b + c - a)^{-2} \Rightarrow \frac{\sqrt{3}}{F} \leqslant \Sigma(m_b + m_c - m_a)^{-2}$;

9) $(\Sigma m_a)^{2/3}\Sigma a/(m_b + m_c) \geqslant 3\sqrt[3]{3F} \Rightarrow \Sigma m_a/(b + c) \geqslant \frac{9}{4}\sqrt[3]{\frac{r}{s}}$;

which is the conversion of 10.9.

10) $(\Pi m_a)^2 \Sigma (4a^2 + b^2 + c^2)^{-1} \geq \frac{9}{8} F^2 \Rightarrow$ GI 6.7;

11) $9F\Sigma a m_a \leq 18F^2 + (4\Pi m_a)(\Sigma m_a)$ (for acute triangle)

$\Rightarrow \Sigma m_a/h_a \leq 1 + R/r$;

12) $(\Sigma m_a)(\Sigma a/m_a) \geq 6s \Rightarrow \Sigma m_a/a \geq 3(\Sigma m_a)/2s$;

13) GI 4.2: $s^2 \geq 3F\sqrt{3} \Rightarrow$ GI 8.4: $(\Sigma m_a)^2 \geq 9F\sqrt{3}$.

Note that the triangle Δ_m has the following two properties (see [15] and [16]):

14) $r_m \leq \dfrac{3abc}{4(a^2 + b^2 + c^2)}$;

15) $R_m \geq \dfrac{a^2 + b^2 + c^2}{2(a + b + c)}$.

Furthermore, if A_m, B_m, C_m are the angles of a triangle Δ_m and if $a > b > c$, then [28]:

$$m_a < m_b < m_c, \quad A_m < B_m < C_m, \quad A > A_m, \quad A > B_m,$$

$$B > A_m, \quad C < C_m, \quad C < B_m, \quad B < C_m.$$

For some other properties of Δ_m see [29].

4.2. Let c_a, c_b, c_c denote three cevians of a triangle dividing the sides in the ratio v/u where $u + v = 1$. Then (see I.3.14) c_a, c_b, c_c form a triangle Δ_c. If we now let F', R', r', h'_a, c'_a, respectively, denote the area, circumradius, inradius, altitude and cevian to side c_a (with the same u, v), of the triangle Δ_c, it can be shown (see Klamkin's solution of a problem from [17]) that:

$$F' = (1 - uv)F, \quad R' = \frac{\Pi c_a}{4(1 - uv)F}, \quad r' = \frac{2(1 - uv)F}{\Sigma c_a},$$

$$h'_a = 2(1 - uv)F/c_a, \text{ etc.,} \quad c'^2_a = uc_c^2 + vc_b^2 - uvc_a^2, \text{ etc.}$$

Then corresponding to any triangle inequality

$$I(a, b, c, F, R, r, h_a, h_b, h_c, c_a, c_b, c_c) \geqslant 0$$

for ABC, we also have the dual cevian triangle inequality

$$I(c_a, c_b, c_c, F', R', r', h'_a, h'_b, h'_c, c'_a, c'_b, c'_c) \geqslant 0.$$

EXAMPLES. 11° GI 6.1 $\Rightarrow \sqrt{3}\Sigma c_a \geqslant 4(1 - uv)F\Sigma c_a^{-1}$, which is Klamkin's generalization of a problem of A. Bager ([17]).

4.3. A generalization of the median-dual transformation has been given by I. Ginčev [18].

Let $D(x_1, x_2, x_3)$ be a point in the plane of a triangle where x_1, x_2, x_3 are its barycentric coordinates, i.e. $\vec{OD} = \Sigma x_1\vec{OA}$, $\Sigma x_1 = 1$, where O is an arbitrary point in the plane. If c_a, c_b, c_c are its cevians, then

$$a' = \left|\frac{x_2 + x_3}{x_2x_3}\right| c_a, \quad b' = \left|\frac{x_3 + x_1}{x_3x_1}\right| c_b, \quad c' = \left|\frac{x_1 + x_2}{x_1x_2}\right| c_c$$

are the sides of a triangle Δ'_c, and for this triangle we have

$$F' = F/|\Pi x_1|, \quad r' = 2F/(\Sigma|x_1x_2 + x_1x_3|c_a),$$

$$R' = |\Pi(x_2 + x_3)|(\Pi c_a)/4F|\Pi x_1|, \quad h'_a = 2F/c_a|x_1x_2 + x_1x_3|,$$

$$c'_a = a/|x_1x_2 + x_1x_3|, \text{ etc.}$$

5. Transformations T_e, T_{e^2} and T_e^{-1}

Here, we shall consider three transformations which are connected with I.3.55.

5.1. In I.3.35 we noted that $a_1 = a(s - a)$, $b_1 = b(s - b)$, $c_1 = c(s - c)$ are sides of a triangle (say $\Delta_{T_{e^2}}$). S. J. Bilčev and E. A. Velikova [19] noted that for this triangle we also have

$$s_1 = r(4R + r), \quad s_1 - a_1 = \frac{FR}{s - a}, \quad F_1 = Fr\sqrt{r(4R + r)},$$

$$R_1 = Rs\sqrt{\frac{r}{4R + r}}, \quad r_1 = rs\sqrt{\frac{r}{4R + r}},$$

$$r_{a_1} = (s - a)\sqrt{r(4R + r)}, \quad h_{a_1} = 2Fr\sqrt{r(4R + r)}/a(s - a).$$

and they called this transformation the T_{e^2}- transformation. They also noted that $a \xrightarrow{T_{e^2} * T_{e^2}}$ Fra, etc., i.e. using the transformation $T_{e^2} * T_{e^2}$ we get the initial result (symbolically $I_{T_{e^2} * T_{e^2}} = I$ or $I \xleftrightarrow{T_{e^2}} I_{T_{e^2}}$).

EXAMPLES. 1° [19] GI 4.7 $\Leftrightarrow$ GI 5.5_I,

$$\text{GI } 5.5_{II} \Leftrightarrow \Sigma a^2 - \Sigma(b - c)^2 \leqslant 4s\sqrt{\tfrac{r}{3}(4R + r)}.$$

2° [11] If x, y, z are real numbers and $0 < t \leqslant 2$, then

$$\Sigma yza^t(s - a)^t \leqslant \tfrac{1}{3}(\Sigma x)^2(FsR^2/(4R + r))^{t/2},$$

$$|\Sigma yza^t(s - a)^t| \geqslant \sqrt{3\lambda}(16F^2r^3(4R + r)/3)^{t/4} \quad (\lambda = xyz(\Sigma x) \geqslant 0).$$

These results are the T_{e^2}-duals to VII.2.(8) and (9).

3° Let p, q, r be real numbers such that $p + q > 0$, $q + r > 0$, $r + p > 0$. If $0 < t \leqslant 4$, then

$$\Sigma \frac{p}{q + r} a^t(s - a)^t \geqslant \tfrac{3}{2}(16F^2r^3(4R + r)/3)^{t/4}.$$

This is the T_{e^2}- dual to VII.2(10).

5.2. Of course, since $a(s - a)$, etc. are the sides of a triangle, $a_2 = (a(s - a))^{1/2}$, $b_2 = (b(s - b))^{1/2}$, $c_2 = (c(s - c))^{1/2}$ are also sides of a triangle (say Δ_{T_e}). Bilčev and Velikova [20] noted that for this triangle we have

$$R_2 = \sqrt{Rr}, \quad F_2 = F/2, \quad r_2 = F/(\Sigma\sqrt{a(s - a)}),$$

$$h_{a_2} = F/(a(s - a))^{1/2}, \quad A_2 = (\pi - A)/2, \quad B_2 = (\pi - B)/2,$$

$$C_2 = (\pi - C)/2,$$

and they called this transformation T_e- transformation.

It is obvious that this transformation is in connection with the transformation given in VII.3.2.

Also, if we apply these transformations n times, we get a new triangle with the angles

(1) $$A^{(n)} = \frac{2^n - (-1)^n}{3 \cdot 2^n}\pi + \frac{(-1)^n}{2^n}A, \text{ etc.}$$

This result is connected with XII.2.1. A generalization of the T_e-transformation is given in [22].

EXAMPLES. 4° [20] GI 4.5 ⇒ $\Sigma(bc/r_b r_c)^{1/2} \geq 2\sqrt{3}$,

GI 6.1 and 5.22 ⇒ $\sqrt{6\sqrt{3}} \leq \Sigma\sqrt{\frac{a}{r_a}} \leq 3\sqrt{\frac{3R}{s}}$,

GI 7.5 ⇒ $\Sigma a^2 - \Sigma(b - c)^2 \leq 16Rr + 4\sqrt{3}F/9$,

GI 10.8 ⇒ $\Sigma \frac{b + c - a}{b_1 + c_1 - a_1} a \geq \frac{2F}{r_1}$,

GI 4.4 ⇒ GI 4.7_I.

5° [11] If x, y, z are real numbers and $0 < t \leq 1$, then

$$\Sigma yza^t(s - a)^t \leq \frac{1}{3}(\Sigma x)^2(Rr\sqrt{3})^t,$$

$$|\Sigma yza^t(s - a)^t| \geq \sqrt{3}\lambda(2F/\sqrt{3})^t \quad (\lambda = xyz(\Sigma x) > 0).$$

These inequalities are the T_e- duals to VII.2.(8) and (9).

6° Let p, q, r be real numbers such that $p + q > 0$, $q + r > 0$, $r + p > 0$. If $0 < t \leq 2$, then

$$\Sigma \frac{p}{q + r} a^t(s - a)^t \geq \frac{3}{2}(2F/\sqrt{3})^t.$$

This is the T_e- dual with VII.2(10), and a generalization of some results from [30].

7° Using (1) for n = 3 and GI 5.22 we get [21]:

$$\Sigma((\sqrt{2} + 1)\cos\frac{A}{8} - \sin\frac{A}{8})^{-1} \geq \sqrt{6 - 3\sqrt{2}}.$$

5.3. The inverse transformation to the T_e-transformation was also considered in [20]. This is the T_e^{-1}-transformation, and it gives a triangle of sides a_3, b_3, c_3, such that $a^2 = a_3(s_3 - a_3)$, etc. The following relations are valid for this triangle [20]:

$$a_3 = \frac{2R^2}{d}\sin 2A = \frac{a^2(b^2 + c^2 - a^2)}{4Fd}, \quad s_3 = \frac{2F}{d},$$

$$s_3 - a_3 = d \tan A, \quad A_3 = \pi - 2A, \quad R_3 = \frac{R^2}{d}, \quad r_3 = d,$$

$$F_3 = 2F, \quad r_{a_3} = \frac{2F}{d} \cot A, \quad h_{a_3} = \frac{2Fd}{R^2 \sin 2A},$$

where $d^2 = s^2 - (2R + r)^2$. Of course, it is obvious that the triangle ABC of sides a, b, c must be an acute triangle.

EXAMPLES. 8° [20] GI 5.1, 5.11, 5.12 ⇒

$$\Pi \cos A \leqslant \frac{2F^2}{27R^4} \leqslant \frac{F}{6\sqrt{3}R^2} \leqslant \frac{1}{8}.$$

It is obvious that this is valid for all triangles.

9° [11] If x, y, z are real numbers and $0 < t \leqslant 2$, then for acute triangles the following T_e^{-1}-duals to VII.2.(8) and (9) are valid:

$$\Sigma yz \sin^t 2A \leqslant \frac{1}{3}(\Sigma x)^2 (\sqrt{3}/2)^t,$$

$$|\Sigma yz \sin^t 2A| \geqslant \sqrt{3\lambda}(2Fd^2/R^4\sqrt{3})^{t/2}.$$

10° [20] GI 2.56 ⇒ $2R^2 + 8Rr + 3r^2 \leqslant s^2$.
This inequality is due to A. W. Walker (see Section X.2.3);

$$\text{GI } 5.7 \Rightarrow \Sigma b^2c^2 \geqslant \frac{4F^2}{R^2}(13R^2 + 8Rr + 2r^2 - 2s^2).$$

Remark. Of course, we can combine the above transformations, i.e. if we first use the T_e and later the T_{e^2}- transformation we get the $T_e * T_{e^2}$ - transformation. Bilčev noted that using this transformation from the known inequality

$$\sqrt{3}\Sigma \cos A \geqslant \Sigma \sin 2A$$

we get

$$\Sigma(b + c - a)\sqrt{a} \geqslant 4r(4R + r)^{3/2}/(3Rs)^{1/2}.$$

6. Parallelogram Transformations

In the papers [23], [24] and [25] the parallelogram transformations were considered. For example, let ABC be a given triangle and let $CM = m_c$. If we extend this median an equal length, we get a triangle ACC_1 with sides

$AC_1 = a$, $AC = b$ and $CC_1 = 2m_c$, i.e. we get the parallelogram transformation with respect to side c: PT(c). This transformation implies that if we have an inequality

$$I(a, b, c, F, m_c, R, r, r_a, r_b, r_c, h_a, h_b, h_c) \geqslant 0$$

then, using PT(c) we get the inequality

$$I\Big(a, b, 2m_c, F, c/2, abm_c/2F, 2F/(a + b + 2m_c), 2F/(b + 2m_c - a), 2F/(2m_c + a - b), 2F/(a + b - 2m_c), h_a, h_b, F/m_c\Big) \geqslant 0.$$

EXAMPLE. 1° GI 4.4 $\Rightarrow 3a^2 + 3b^2 - c^2 \geqslant 4F\sqrt{3}$, (see [26]),

$$\text{GI } 4.10 \Rightarrow 5a^4 + 5b^4 + c^4 + 8a^2b^2 - 4b^2c^2 - 4c^2a^2 \geqslant 16F^2,$$

$$\text{GI } 4.12 \Rightarrow 2a^4 + 2b^4 + 5a^2b^2 - b^2c^2 - c^2a^2 \geqslant 16F^2.$$

In each example there is equality only if $a:b:c = 1:1:\sqrt{3}$.

Of course, we can combine several parallelogram transformations (see [23]), or parallelogram transformations with some other transformations (for example with the median-dual transformation: MDT as in [24]). So the following transformations are also valid:

(1) Using the PT(c) * PT(b)-transformation we have:

$$I(a, b, c, F) \geqslant 0\,\{E\} \Rightarrow I\Big(a, (6a^2 + 3b^2 - 2c^2)^{1/2}, 2m_c, F\Big) \geqslant 0$$

with equality if and only if $a:b:c = 1:\sqrt{3}:\sqrt{7}$.

EXAMPLE. 2° GI 4.12 $\Rightarrow 20a^4 + 6b^4 + 2c^4 + 23a^2b^2 - 7b^2c^2 - 13c^2a^2 \geqslant 16F^2$.

(2) Using the PT(c) * PT(b) * PT(a)-transformation we have

$$I(a, b, c, F) \geqslant 0 \quad \{E\} \Rightarrow I((15a^2 + 10b^2 - 6c^2)^{1/2}, (6a^2 + 3b^2 - 2c^2)^{1/2}, 2m_c, F) \geqslant 0$$

with equality if and only if $a:b:c = \sqrt{3}:\sqrt{7}:\sqrt{19}$.

EXAMPLE. 3° GI 4.12 $\Rightarrow 131a^4 + 56b^4 + 20c^4 + 173a^2b^2 - 67b^2c^2 - 103c^2a^2 \geqslant 16F^2$.

(3) Using the PT(c) * MDT (or MDT * PT(c))-transformation we have

$$I(a, b, c, F) \geq 0 \{E\} \Rightarrow I(m_a, m_b, 3c/2, 3F/4) \geq 0$$

with equality if and only if $a:b:c = \sqrt{7}:\sqrt{7}:1$.

EXAMPLE. 4° GI 4.4 $\Rightarrow a^2 + b^2 + 13c^2 \geq 12F\sqrt{3}$.

(4) Using the PT(c) * PT(b) * MDT-transformation we have

$$I(a, b, c, F) \geq 0 \{E\} \Rightarrow I\left(m_a, \frac{1}{2}(-4a^2 + 5b^2 + 20c^2)^{1/2}, 3c/2, 3F/4\right) \geq 0$$

with equality if and only if $a:b:c = \sqrt{19}:\sqrt{13}:1$.

EXAMPLE. 5° GI 4.4 $\Rightarrow -5a^2 + 7b^2 + 31c^2 \geq 12F\sqrt{3}$.

Some other combinations and further generalizations of the parallelogram transformations were given in [24]. We shall give the following results:

Let be $M \in AB$ such that $AM:MB = m:n$. We extend the cevian AM to the point C_1 such that $AC_1 \| BC$. Then we have

$$I(a, b, c, F) \geq 0 \Rightarrow I\left(\frac{m}{n} a, b, \frac{1}{n}((m + n)(ma^2 + nb^2) - mnc^2)^{1/2}, \frac{m}{n} F\right) \geq 0.$$

EXAMPLE. 6° GI 4.4 $\Rightarrow m(2m + n)a^2 + n(m + 2n)b^2 - mnc^2 \geq 4mnF\sqrt{3}$.

7. A Transformation for Acute Triangles

We now return to I.3.2, i.e. if f, g, h are the sides of an acute triangle, then $g^2 + h^2 \geq f^2$ etc.; if we write $f^2 = a$, $g^2 = b$, $h^2 = c$, then a, b, c are the sides of a triangle. If ξ, η, ζ are the angles of this acute triangle, then ([27]):

$$(1) \qquad \cos^2 \zeta = (g^2 + h^2 - f^2)^2/4g^2h^2 = (b + c - a)^2/4bc = x^2/(x + z)(x + y).$$

In this way we can derive inequalities for the sides and angles of an acute triangle from inequalities for positive numbers x, y, z (see Chapter III).

EXAMPLE. [27] For what values of λ, μ, ν is the inequality

(2) $$64\lambda\Pi \cos^2 \xi + 16\mu\Sigma \cos^2 \eta \cos^2 \zeta + 4\nu\Sigma \cos^2 \xi - (\lambda + 3\mu + 3\nu) \geqslant 0$$

satisfied for all acute triangles? Using (1) we reduce (2) to

(3) $$(\nu - \lambda - 3\mu)P + (4\nu - 4\lambda - 12\mu)Q + (4\nu - 4\lambda + 4\mu)T + (4\nu - 12\lambda - 4\mu)S \geqslant 0,$$

where P, Q, T, S are given as in Theorem III.6. Using this theorem, necessary and sufficient conditions for the validity of inequality (3) are

$$4\nu - 4\lambda - 12\mu = 4\Theta^2 \text{ (say)},$$

$$4\nu - 4\lambda + 4\mu = 4\omega^2 \text{ (say)},$$

$$4\nu - 12\lambda - 4\mu = -4\Theta\omega + 2\varepsilon^2 \text{ (say)},$$

where $\Theta, \omega \geqslant 0$. Thus (2) is a valid inequality if and only if it can be written in the form

$$\Theta^2(16\Pi \cos^2 \xi - 4\Sigma \cos^2 \eta \cos^2 \zeta + 2\Sigma \cos^2 \xi - 1) + 4\omega^2(4\Pi \cos^2 \xi + \Sigma \cos^2 \eta \cos^2 \zeta + \Sigma \cos^2 \xi - 1) + (-2\Theta\omega + \varepsilon^2)(-16\Pi \cos^2 \xi - \Sigma \cos^2 \xi + 1) \geqslant 0,$$

where $\Theta, \omega \geqslant 0$. Oppenheim's inequalities (GI 11.18) are given by $\Theta = 1$, $\omega = 0$, $\varepsilon^2 = 1$ and by $\Theta = 1$, $\omega = 0$, $\varepsilon^2 = 2$. Kooistra's inequality (GI 2.24) i.e. $\Sigma \sec^2 \xi \geqslant 12$, is given by $\Theta = 0$, $\omega = 1$, $\varepsilon^2 = 4$.

References

1. N. Schaumberger, B. Priellip, E. T. H. Wang, and W. Laurier, 'Problem 251', The Coll. Math. J. 6 (1983), 260, 8 (1985), 159-160.
2. Ž. Madevski, 'Nekoi metodi za dobivanje relacii megu elementite od eden triagolnik' (Doctoral dissertation), Skopje, 1973.
3. Ž. Madevski and A. Samardžiski, 'Edna postapka za formiranje neravenstva za aglite vo triagolnikot', Godišen Zbornik PMF Skopje 21 (1971), 13-25.
4. S. J. Bilčev and E. A. Velikova, Priloženie na njakoi obrazi na neravenstvoto na poljarnija inercionen moment, Naučni trudove na VTU "A. Kančev" - Russe, 1986 (to appear).
5. G. Ling and V. D. Mascioni, 'Problem 1156', Math. Mag. 57 (1984), 47-48.
6. M. S. Klamkin, 'Problem P. 227', Canad. Math. Bull. 17 (1) (1974), 151-152.

7. I. Tomescu, 'Problem 13018', Gaz. Mat. (Bucharest) B 24 (1973), 306.
8. Ž. M. Mitrović, 'Inequalities for a Sequence of Triangles', Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 381-409 (1972),
9. A. Oppenheim, 'Inequalities Involving Elements of Triangles, Quadrilaterals or Tetrahedra', Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 461-497 (1974), 257-263.
10. C. E. Carroll, 'Proof of Oppenheim's Area Inequalities for Triangles and Quadrilaterals', Aequat. Mathematicae 24 (1982), 97-109.
11. D. S. Mitrinović, J. E. Pečarić, S. J. Bilčev, and E. A. Velikova, 'On an Inequality of O. Kooi', Matematika i matematičesko obrazovanie. Sofija, 1988, 566-568.
12. W. Janous, Private communication.
13. E. A. Velikova and S. J. Bilčev, 'The Median Duality in a Triangle and Its Applications to Geometrical Inequalities' (Bulgarian), Matematika i matematičesko obrazovanie. Sofija, 1985, pp. 14-43
14. A. Bager, 'Some Inequalities for the Medians of a Triangle', Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 338-352 (1971), 37-40.
15. J. Garfunkel and G. Tsintsifas, M. S. Klamkin, L. Bankoff, and N. Murty, 'Problem 733', Crux. Math. 8 (1982), 106 and 9 (1983), 121, 149, 210.
16. J. Garfunkel, G. Tsintsifas, and L. Bankoff, 'Problem 835', Ibid. 9 (1983), 113 and 10 (1984), 227.
17. A. Bager, I. Paasche, and M. S. Klamkin, 'Aufgabe 677', Elem. Math. 28 (1973), 129-130.
18. I. Ginčev, 'Application of the Cevian Duality to Prove Identities and Inequalities for a Triangle' (Bulgarian), Matematika i matematičesko obrazovanie. Sofija, 1986, pp. 559-566.
19. S. J. Bilčev and E. A. Velikova, 'Vrhu T_{e^2}-dualnostta v tri'g'lnika', Obučenieto po matematika, Sofija, 1987 (to appear).
20. S. J. Bilčev and E. A. Velikova, Priloženie na njakoi preobrazuvanija za polučavane na novi geometrični neravenstva, Naučni trudove na VTU "A. Kančev" - Russe, 1986 (to appear).
21. S. J. Bilčev, 'Problem 1158', Crux. Math. 12 (1986), 140.
22. S. J. Bilčev and E. A. Velikova, '$GT_e(k)$-Transformation for a Triangle and Some Applications', (to appear).
23. S. J. Bilčev and E. A. Velikova, 'Njakoi asimetrični neravenstva za tri'g'lnika', Matematika (Sofija), 1986, No. 4, 13-17.
24. S. J. Bilčev and E. A. Velikova, 'On One Generalization of the Parallelogram's Transformation' (Bulgarian), Matematika i matemati matematičesko obrazovanie. Sofija, 1986, pp. 525-531.
25. S. J. Bilčev, 'Asimetrični geometrični neravenstva', in: 'Olimpiada po matematika za srednoškolci 1985 godina', MNP, CKPOM, Sofija, 1986.
26. Problem 2800. Mat. v. škole 1985, No. 5, 61.
27. J. F. Rigby, 'Sextic Inequalities for the Sides of a Triangle', Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 498-541 (1975), 51-58.
28. L. Toscano, 'Confronto degli angoli di un triangolo con quelli del triangolo delle sue mediane', Archimede VIII (1956), 278-279.
29. L. Toscano, 'Relazioni degli angoli di un triangolo con quelli del triangolo delle sue mediane', Bollettino della Soc. Mat. Calabrese

(5) 24 (1973), 106-108.
30. S. J. Bilčev and E. A. Velikova, 'Ob odnoj probleme Tsintsifasa' (Russian), Matematika i matematičesko obrazovanie, Sofija, 1987, 586-593.

Chapter VIII

CONVEX FUNCTIONS AND GEOMETRIC INEQUALITIES

1. Jensen's and Related Inequalities and Geometric Inequalities

1.1. That convex functions play an important role in generating inequalities for the triangles was apparently first noticed by M. Petrović in 1916 (see [1-3]) who obtained the first general inequalities which include a convex function and the sides of a triangle:

THEOREM 1. If $f:[0, +\infty) \to R$ is a convex function and a, b, c are the sides of a triangle, then

$$3f(2s/3) \leqslant \Sigma f(a) \leqslant f(0) + 2f(s). \tag{1}$$

M. Petrović proved (1) using

$$3f(s/3) \leqslant \Sigma f(x) \leqslant f(s) + 2f(0), \tag{2}$$

where $s = (a + b + c)/2$ and $x = s - a$, etc. The first inequality in (2) is a special case of Jensen's inequality, and the second one of the Petrović inequality for convex functions (see for example AI, pp. 12 and 23). As an example of (1), M. Petrović gave GI 1.16, 1.17, 1.19, 1.20, and several other results are also consequences of these inequalities. But it is clear that using Jensen's and Petrović's inequalities we can get several similar results for other elements of a triangle. On the other hand, these results are consequences of majorization (see part 2), so here we shall only give a simple method of generating inequalities from identities.

Suppose the following inequality holds:

$$\sum_{k=1}^{n} \Phi_k = \Phi,$$

where $\Phi, \Phi_1, \ldots, \Phi_n$ are real functions defined on $U \subseteq R^m$. Then by Jensen's inequality for the convex functions $F:[a, b] \to R$ ($\Phi(x)$, $\Phi_k(x) \in [a, b]$, $\forall x \in U$, $1 \leqslant k \leqslant n$),

$$\sum_{k=1}^{n} F(\Phi_k) \geqslant nF(\frac{1}{n}\Phi). \tag{3}$$

If F is concave, the reverse inequality is valid.

If F is convex for $x \geqslant 0$ and if $F(0) = 0$, then by Petrović's inequality,

$$(4)\qquad \sum_{k=1}^{n} F(\Phi_k) \leqslant F(\Phi).$$

Again if F is concave, the reverse inequality holds.

Now we give some applications of (3) and (4) to triangle inequalities.

1° From $\Sigma \frac{s-a}{s} = 1$, we get $\Sigma F\left(\frac{s-a}{s}\right) \geqslant 3F(\frac{1}{3})$ if $F \in C_x$ (the cone of convex functions), and the reverse inequality if $F \in C_v$ (concave functions). If additionally $F(0) = 0$, $x \geqslant 0$, then $\Sigma F(\frac{s-a}{s}) \leqslant F(1)$ if $F \in C_x$ and the reverse if $F \in C_v$.

For $F(x) = 1/x$ and $F(x) = \sqrt{x}$, respectively, we have GI 1.15 and 1.20.

To avoid constant repetition, it will be assumed in each of the following cases that we have the same conditions for F as in 1°, i.e., in the first inequality $F \in C_x$ and in the second inequality, we additionally have $F(0) = 0$, $x \geqslant 0$. Also if $F \in C_v$, then the two inequalities are reversed.

$$2^\circ \quad \Sigma \frac{r}{r_a} = 1 \quad [4] \Rightarrow \Sigma F\left(\frac{r}{r_a}\right) \geqslant 3F(\tfrac{1}{3}) \quad \text{and} \quad \Sigma F\left(\frac{r}{r_a}\right) \leqslant F(1).$$

For $F(x) = 1/x$ we have the left hand side of GI 5.41. Also, since $\frac{r}{r_a} = \frac{h_a - r}{h_a + r_a}$ [5], we get $\Sigma F\left(\frac{h_a - r}{h_a + r_a}\right) \geqslant 3F(\frac{1}{3})$ and $\Sigma F\left(\frac{h_a - r}{h_a + r_a}\right) \leqslant F(1)$.

<u>Remark</u>. Since $(s - a)/s = r/r_a$, examples 1° and 2° are equivalent.

$$3^\circ \quad \Sigma \frac{r}{h_a} = 1 \quad [4] \Rightarrow \Sigma F\left(\frac{r}{h_a}\right) \geqslant 3F(\tfrac{1}{3}) \quad \text{and} \quad \Sigma F\left(\frac{r}{h_a}\right) \leqslant F(1).$$

For $F(x) = 1/x$, we have GI 6.2.

$$4^\circ \quad \Sigma \frac{r_a}{4R + r} = 1 \quad [4] \Rightarrow \Sigma F\left(\frac{r_a}{4R + r}\right) \geqslant 3F(\tfrac{1}{3}) \quad \text{and}$$

$$\Sigma F\left(\frac{r_a}{4R + r}\right) \leqslant F(1).$$

For $F(x) = x^n$, $(n \geqslant 1,\ n \in R)$, we have $\Sigma r_a^n \geqslant 3\left(\frac{4R + r}{3}\right)^n$, which is better than the following result from [7]: $\Sigma r_a^n \geqslant 3(R + r)^n$.

$$5^\circ \quad \Sigma \sin^2 \frac{A}{2} = \frac{2R - r}{2R} \Rightarrow \Sigma F(\sin^2 \tfrac{A}{n}) \geqslant 3F\left(\frac{2R - r}{6R}\right) \quad \text{and}$$

$$\Sigma F(\sin^2 \tfrac{A}{2}) \leqslant F\left(\frac{2R - r}{2R}\right).$$

Since $\sin^2 \frac{A}{2} = \frac{1}{4R} \frac{a^2}{r_b + r_c}$ [6], we have $\sum \frac{a^2}{r_b + r_c} = 4R - 2r \Rightarrow$

$$\sum F\left(\frac{a^2}{r_b + r_c}\right) \geqslant 3F\left(\frac{4R - 2r}{3}\right) \quad \text{and} \quad \sum F\left(\frac{a^2}{r_b + r_c}\right) \leqslant F(4R - 2r).$$

These results are improvements and extensions of results from [6].

6° $\sum \sin A = \frac{s}{R} \Rightarrow \sum F(\sin A) \geqslant 3F(\frac{s}{3R})$ and $\sum F(\sin A) \leqslant F(\frac{s}{R})$.

For $F(x) = x^n$ $(n \geqslant 1,\ n \in R)$, we have: $\sum \sin^n A \geqslant 3^{1-n} \frac{s^n}{R^n}$. Since $s \geqslant 3r\sqrt{3}$ (GI 5.11) we get a result from [7]:

$$\sum \sin^n A \geqslant 3^{1+\frac{n}{2}} (\frac{r}{R})^n \quad (n \geqslant 1,\ n \in R).$$

7° $\sum \tan \frac{B}{2} \tan \frac{C}{2} = 1 \Rightarrow \sum F(\tan \frac{B}{2} \tan \frac{C}{2}) \geqslant 3F(\frac{1}{3})$ and

$$\sum F(\tan \frac{B}{2} \tan \frac{C}{2}) \leqslant F(1).$$

For $F(x) = x^n$ $(n \geqslant 1, n$ real), from the first inequality we get a result from [8].

1.2. In Section 1.1, we started out with a known identity. Here we start out with a known inequality

$$(5) \qquad \sum_{k=1}^{n} \Phi_k \geqslant \Phi.$$

Also, we write $F \in A$, $F \in B$, $F \in C$, $F \in D$ to denote that F is a function which is convex non-decreasing, convex non-increasing, concave non-increasing, or concave non-decreasing, respectively, in the domain involved.

It now follows immediately from Jensen's inequality that if $F \in A$ and if (5) is valid, then

$$(6) \qquad \sum_{k=1}^{n} F(\Phi_k) \geqslant nF(\frac{1}{n}\Phi).$$

Analogously, (6) is valid if the reverse inequality in (5) holds and $F \in B$; the reverse inequality in (6) is valid if (5) holds and $F \in C$; and finally the reverse inequality in (6) is valid if the reverse inequality in (5) is valid and $F \in D$.

We now apply these results to give extensions to a sample of known inequalities in GI. The reader can do the same with respect to other inequalities in GI and elsewhere.

8° GI 2.3 $\Rightarrow \Sigma F(\sin^2 A) \leqslant 3F(\frac{3}{4})$, $(F \in D,\ x > 0)$.

By specializing the function to $x \to F(x) = x^{k/2}$ $(0 < k \leqslant 2)$, and $x \to \log x$, we immediately obtain (for $k < 0$ we can use the inequality for means)

(7) $$M_k(\sin A,\ \sin B,\ \sin C) \leqslant \sqrt{3}/2 \quad (k \leqslant 2)$$

which is better than GI 2.6 (and also contains GI 2.5, 2.7 and 2.49).

Note that the same method and the weighted version of Jensen's inequality was used in Chapter VII, where a generalization of (7) was obtained.

9° GI 2.33 $\Rightarrow \Sigma F(\tan \frac{A}{2}) \geqslant 3F(\sqrt{3}/3)$, $(F \in A,\ x > 0)$.

For $F(x) = x^2,\ x^6$, we obtain GI 2.35 and 2.36, respectively.

Since $\tan \frac{A}{2} = \dfrac{r + r_a}{b + c} = \dfrac{a}{r_b + r_c}$, we also have

$$\Sigma F\left(\frac{r + r_a}{b + c}\right) = \Sigma F\left(\frac{a}{r_b + r_c}\right) \geqslant 3F(\sqrt{3}/3), \quad (F \in A,\ x > 0).$$

This generalizes an inequality in [6].

10° GI 2.41 $\Rightarrow \Sigma F(\mathrm{cotan}\ \frac{A}{2}) \geqslant 3F(\sqrt{3})$, $(F \in A,\ x > 0)$.

For $x \to F(x) = x^2$, we obtain GI 2.43.

Since $\mathrm{cotan}\ \frac{A}{2} = \dfrac{a}{r_a - r} = \left(\dfrac{r_b + r_c}{r_a - r}\right)^{1/2}$, we also have

$$\Sigma F\left(\frac{a}{r_a - r}\right) = \Sigma F\left(\left(\frac{r_b + r_c}{r_a - r}\right)^{1/2}\right) \geqslant 3F(\sqrt{3}), \quad (F \in A,\ x > 0).$$

This generalizes some of the results in [6].

11° GI 4.5 $\Rightarrow \Sigma F(bc) \geqslant 3F(4\Delta/\sqrt{3})$, $(F \in A,\ x > 0,\ \Delta = \text{area})$.

For $F(x) = x^2$, we have GI 4.12.

GI 4.6 $\Rightarrow \Sigma F(bc) \geqslant 3F\left(\dfrac{4\Delta}{\sqrt{3}} + \dfrac{Q}{6}\right)$, $(F \in A,\ x > 0,\ Q = \Sigma(b - c)^2)$.

For $F(x) = x^2$, we get

$$\Sigma b^2c^2 \geqslant 16\Delta^2 + 4\Delta Q/\sqrt{3} + Q^2/12$$

which is an improvement on GI 4.12.

12° GI 5.29 $\Rightarrow \Sigma F(r_a) \geqslant 3F(s/\sqrt{3})$, $(F \in A,\ x > 0)$.

For $F(x) = x^2$, we have GI 5.34.

13° GI 6.28 $\Rightarrow \Sigma F(r_a/h_a) \geqslant 3F(1)$, $(F \in A,\ x > 0)$.

For $F(x) = x^n$ $(n \geqslant 1,\ n \in R)$, we have a result which was given in Remark 1 of GI 6.28.

14° A simple application of Jensen's inequality is GI 2.37, i.e. its generalization from VII 3.10. For some other applications, see Chapter VII.

Remark. The previous results were given by J. E. Pečarić, R. R. Janić and M. S. Klamkin. Now, we shall note that using weighted version of Jensen's inequality we can give the following generalization of (3):

$$(3') \qquad \sum_{k=1}^{n} p_k F(\Phi_k/p_k) \geqslant PF(\Phi/P),$$

where

$$P = \sum_{k=1}^{n} p_k, \quad \Phi_k/p_k \in [a, b] \quad (k = 1, \ldots, n).$$

For some applications of (3') see part XI.4.1. Here we shall only give an example of D. Milošević:

$$\Sigma \frac{1}{a} \cos^2 \frac{A}{2} = \frac{s}{4Rr} \Rightarrow \Sigma a^2 \sec \frac{A}{2} \geqslant 12Rr\sqrt{3}.$$

Comment by V. Mascioni. Note that

$$M_2(\sin A) \leqslant \sqrt{3}/2$$

(see Example 8°) is equivalent to

$$M_2(a) \leqslant \sqrt{3}R$$

and, since $R = \frac{1}{2}\Sigma a/\Sigma \sin A$, to

$$M_1(\sin A) \leqslant \frac{\sqrt{3}}{2}\,\frac{M_1(a)}{M_2(a)}\ .$$

Using cotan $\omega = (\Sigma a^2)/4F$ (ω - the Crelle-Brocard angle), one sees further that

$$M_2(\sin A) \leqslant \sqrt{3}/2$$

is also equivalent to

$$\frac{8}{9}\,\Pi \sin A \leqslant \tan \omega$$

which is due to D. P. Mavlo (Problem 639, Nieuw Arch. Wisk. (3) 30 (1982), 116 and 31 (1983), 87-89.

Inequality GI 4.12 which is used in Example 11°, is equivalent to

$$M_{-2}(a) \leqslant \sqrt{3}R.$$

The remark above implies then that inequality (7) also contains GI 4.5. Note also that inequalities GI 4.6 and 5.29 which are used in Examples 11° and 12°, respectively, are equivalent.□

Comment by W. Janous. Mitrinović, Pečarić and Janous (see VI.1.Remark 1°) have shown that

$$M_k(a) \leqslant R\sqrt{3} \quad (k = \log(9/4)/\log(4/3) > 2).$$

Hence

$$M_1(\sin A) \leqslant \frac{3}{2}\,\frac{M_1(a)}{M_k(a)}\ .\square$$

Comment by J. F. Rigby. Examples 10° and 13° for $F(x) = x^k$ $(k > 1)$ become $\Sigma \cos^k \frac{A}{2} \geqslant 3(\sqrt{3})^k$ and $\Sigma(r_a/h_a)^k \geqslant 3$. But $\frac{1}{3}\,\Sigma \operatorname{cotan}^k \frac{A}{2} \geqslant$ $(\Pi \operatorname{cotan}^k \frac{A}{2})^{k/3} \geqslant (3\sqrt{3})^{k/3} = (\sqrt{3})^k$ for $k \geqslant 0$ (using GI 2.42). See also GI 11.8 (put $\alpha = \frac{\pi}{2} - \frac{A}{2}$, $A = \pi - 2\alpha$). Also, $\frac{1}{3}\,\Sigma(r_a/h_a)^k \geqslant$ $(\Pi(r_a/h_a)^k)^{1/3} \geqslant 1$ (GI 6.27) for $k \geqslant 0$.

Note that (7) also contains GI 4.5 (it is equivalent to $M_{-1}(a) \leqslant \sqrt{3}R$).□

1.3. If $m \leqslant f''(x) \leqslant M$ and $S_n = \frac{1}{n^2}\sum_{\substack{i,j=1\\ i<j}}^{n} (x_i - x_j)^2$, then $x \to \frac{M}{2}x^2 - f(x)$ and $x \to f(x) - \frac{m}{2}x^2$ are convex functions and Jensen's inequality gives [9]:

$$\frac{m}{2}S_n \leqslant \frac{1}{n}\sum_{i=1}^{n} f(x_i) - f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \leqslant \frac{M}{2}S_n. \tag{8}$$

We shall now give some applications of (8) to geometric inequalities from [9].

15° $R \geqslant 2r \exp(\frac{1}{24} \Sigma(B - C)^2)$.

Proof. Consider the function $x \to f(x) = \log \sin x$ on $(0,\pi)$. We have $M = -1$. In (8) put $n = 3$, $x_1 = A/2$, $x_2 = B/2$, $x_3 = C/2$; we get

$$\Pi \sin \frac{A}{2} \leqslant \frac{1}{8} \exp(- \frac{1}{24} \Sigma(B - C)^2)$$

and since $\Pi \sin \frac{A}{2} = \frac{r}{4R}$ we get 15°.

The following examples were proved similarly in [9]:

16° $\frac{3\sqrt{3}}{2} \geqslant \Sigma \sin A \geqslant \frac{3\sqrt{3}}{2} - \frac{1}{2} \Sigma(B - C)^2$.

17° $\frac{r}{R} \geqslant \frac{1}{2} - \frac{1}{6} \Sigma(B - C)^2$.

Comment by V. Mascioni [24]. Since

$$\frac{27}{\pi^3} ABC \leqslant 1 - \frac{1}{24} \Sigma(B - C)^2 \leqslant \exp(- \frac{1}{24} \Sigma(B - C)^2)$$

(it is easy to prove this with the multiplier method), the following inequality of V. Mascioni (see IX.6.58):

$$\frac{2r}{R} \leqslant (\frac{3}{\pi})^3 ABC$$

is stronger than the one proved in Example 15°.

2. Majorization and Geometric Inequalities

That majorization can play a role in generating geometric inequalities was noted by Steinig [10, 11] and Oppenheim [12]. In the well known book [13] (denoted further in the text by MO), Chapter 8 considers these inequalities with respect to majorization.

We shall now give some definitions and main results on majorization:

1) A vector $y = (y_1, \ldots, y_n)$ is said to be majorized by a vector $x = (x_1, \ldots, x_n)$, in symbols $x \succ y$ or $y \prec x$, if after possible reordering of its components so that $x_1 \geqslant \ldots \geqslant x_n$ and $y_1 \geqslant \ldots \geqslant y_n$ we have

$$\sum_{r=1}^{k} x_r \geqslant \sum_{r=1}^{k} y_r, \quad \text{for } k = 1, \ldots, n - 1, \text{ and}$$

$$\sum_{r=1}^{n} x_r = \sum_{r=1}^{n} y_r.$$

But, if we have

$$\sum_{r=1}^{k} x_r \geqslant \sum_{r=1}^{k} y_r \quad \text{for } k = 1, \ldots, n$$

we write $y \prec_w x$ or $x \succ_w y$.

2) A real-valued function F defined on a set $A \subset R^n$ is said to be Schur-convex on A if

$$x \prec y \text{ on } A \Rightarrow F(x) \geqslant F(y).$$

If, in addition, $F(x) < F(y)$ whenever $x \prec y$ but x is not a permutation of y, then F is said to be strictly Schur-convex on A. Similarly, F is said to be Schur-concave on A if

$$x \prec y \text{ on } A \Rightarrow F(x) \leqslant F(y),$$

and F is strictly Schur-concave on A if strict inequality $F(x) > F(y)$ holds when x is not a permutation of y.

Of course, F is Schur-concave if and only if -F is Schur-convex.

3) Let $I \subset R$ be an open interval and let $F:I^n \to R$ not be continuously differentiable. Necessary and sufficient conditions for F to be Schur-convex on I^n are: F is symmetric on I^n and for all $i \neq j$

$$(x_i - x_j)\left(\frac{\partial F}{\partial x_i} - \frac{\partial F}{\partial x_j}\right) \geqslant 0.$$

4) A real-valued function F defined on a set $A \subset R^n$ satisfies

$$x \prec_w y \text{ on } A \Rightarrow F(x) \leqslant F(y)$$

if and only if F is increasing and Schur-convex on A.

We shall now give some important classes of Schur-convex functions:

(i) If $I \subset R$ is an interval and $g:I \to R$ is convex, then $F(x) = \sum_{i=1}^{n} g(x_i)$ is Schur-convex on I^n.

In this case the inequality $F(x) \leqslant F(y)$ from the definition of Schur-convex functions gives the well-known majorization theorem for convex functions.

(ii) A function $F:I^n \to R$ is convex if

$$F(\lambda x + (1 - \lambda)y) \leqslant \lambda F(x) + (1 - \lambda)F(y)$$

for all $\lambda \in [0, 1]$ and $x, y \in I^n$.

If F is symmetric and convex, then F is Schur-convex. Consequently, $x \prec y$ implies $F(x) \leqslant F(y)$.

(iii) A function $F: I^n \to R$ is said to be quasi-convex if

$$F(\lambda x + (1 - \lambda)y) \leqslant \max(F(x), F(y))$$

for all $\lambda \in [0, 1]$ and $x, y \in I^n$.

If F is symmetric and quasi-convex, then F is Schur-convex.

(iv) Denote by $x \to T_k(x)$ the kth elementary symmetric function of $x_1, \ldots, x_n$. That is

$$T_0(x) = 1, \quad T_1(x) = \sum_{i=1}^{n} x_i, \quad T_2(x) = \sum_{i<j} x_i x_j,$$

$$T_3(x) = \sum_{i<j<k} x_i x_j x_k, \ \ldots\ldots, \quad T_n(x) = \prod_{i=1}^{n} x_i.$$

The function T_k is increasing and Schur-concave on R_+^n ($R_+ = [0, +\infty)$). If $k > 1$, T_k is strictly Schur-concave on R_{++}^n ($R_{++} = (0, +\infty)$).

(v) The function $x \to F(x) = (T_k(x))^{1/k}$ is concave and increasing (in fact, strictly concave if $k > 1$) in $x \in R_+^n$; hence F is Schur-concave (strictly Schur-concave if $k > 1$) and increasing, $k = 1, \ldots, n$, for $x \in R_+^n$.

(vi) If $1 \leqslant p \leqslant k \leqslant n$, then $F_{k,p}(x) = (T_k(x)/T_{k-p}(x))^{1/p}$ is a concave function of x, for $x \in R_{++}^n$. Hence, $F_{k,p}$ is Schur-concave on R_{++}^n, $1 \leqslant p \leqslant k \leqslant n$.

Remark. All the above results in 2 are given in MO.

2.1. Inequalities for the Sides of a Triangle and Polygon

In what follows we shall use the following notation

(Δ) - for all triangles,

(Δ_a) - for acute triangles,

(Δ_o) - for obtuse triangles.

First, we shall give some results for the sides of a triangle:

1) $$\left(\frac{2s}{3}, \frac{2s}{3}, \frac{2s}{3}\right) \prec (a, b, c) \prec (s, s, 0), \quad (\Delta),$$

$$\left(\frac{s}{3}, \frac{s}{3}, \frac{s}{3}\right) \prec (x, y, z) \prec (s, 0, 0), \quad (\Delta).$$

These basic majorizations for the sides of a triangle yield generalizations of (1) and (2) to Schur-convex functions.

EXAMPLES. We shall give some examples from MO, pp. 199-201.

1° $1/3 \leqslant \Sigma a^2/(\Sigma a)^2 < 1/2$ (Δ) (GI 1.9)

2° $1/4 < \Sigma bc/(\Sigma a)^2 \leqslant 1/3$ (Δ) (GI 1.1)

3° For a real number $d \geqslant 0$,

$$\frac{4(9 - 2d)}{27} s^2 \leqslant \Sigma a^2 - \frac{d}{s} \Pi a < 2s^2 \quad (\Delta)$$

3° for $d = 36/35$ is an extension of GI 1.2.

4° For $d \geqslant 0$,

$$\frac{3(3d + 2)}{4} \leqslant \Sigma \frac{ds + a}{b + c} < \frac{5d + 4}{2} \quad (\Delta) \quad (d = 0 \text{ gives GI 1.16})$$

5° $\sqrt{s} < \Sigma\sqrt{s - a} < \sqrt{3s}$ (Δ) (GI 1.20)

6° $\Sigma\sqrt{a(s - a)} \leqslant \sqrt{2}s$ (Δ) (GI 5.47)

7° $1/4 < \Pi(b + c)/(\Sigma a)^3 \leqslant 8/27$ (Δ)

8° $9/s \leqslant \Sigma(s - a)^{-1}$ (Δ) (GI 1.15).

In MO, p. 199, the following results were also given:
$(a, b, c) \prec (s, s/2, s/2)$ and $(x, y, z) \prec (s/2, s/2, 0)$ (Δ_o);
but V. Mascioni has shown that these results are not valid (the same is true for several examples which are consequences of these results).

However, C. Tănăsescu communicated to us the following corrections of these results:

$$((2\sqrt{2} - 2)s, (2 - \sqrt{2})s, (2 - \sqrt{2})s) \prec (a, b, c) \quad (\Delta_o),$$

$$((\sqrt{2} - 1)s, (\sqrt{2} - 1)s, (\sqrt{2} - 1)^2 s) \prec (x, y, z) \quad (\Delta_o).$$

He also gave the following examples for these results:

$$\Sigma a^2/(\Sigma a)^2 \geqslant 2(\sqrt{2} - 1)^2; \quad \Sigma bc/(\Sigma a)^2 \leqslant (4\sqrt{2} - 5)/2;$$

$$\Pi(b + c)/(\Sigma a)^3 \leqslant (2 - \sqrt{2})/2; \quad \Sigma s/(s - a) \geqslant 5 + 4\sqrt{2};$$

$$\Sigma a^2 - \frac{d}{s} \Pi a \geqslant 4(\sqrt{2} - 1)^2(2 - (\sqrt{2} - 1)d)s^2;$$

$$\Sigma(ds + a)/(b + c) \geqslant ((2 + 5\sqrt{2})d + 2(5\sqrt{2} - 4))/4;$$

$$\Sigma\sqrt{s - a} \leqslant (\sqrt{2} - 1 + 2\sqrt{\sqrt{2} - 1})\sqrt{s};$$

$$\Sigma\sqrt{a(s - a)} \leqslant (\sqrt{2} - 1)(\sqrt{2\sqrt{2} - 2} + 2\sqrt[4]{2})s.$$

In fact, Tănăsescu proved the following result: If $a \geqslant b \geqslant c$, then for all obtuse triangle, if degeneracy (in any sense) is also allowed,

$$((2\sqrt{2} - 2)s,\ (2 - \sqrt{2})s,\ (2 - \sqrt{2})s) \prec (a,\ s - a/2,\ s - a/2) \prec$$
$$\prec (a,\ b,\ c) \prec (s,\ s - c,\ c) \prec$$
$$\prec (s,\ s,\ 0),$$

$$((\sqrt{2} - 1)s,\ (\sqrt{2} - 1)s,\ (\sqrt{2} - 1)^2 s) \prec (a/2,\ a/2,\ s - a) \prec$$
$$\prec (x,\ y,\ z) \prec$$
$$\prec (s - c,\ c,\ 0) \prec (s,\ 0,\ 0),$$

but it is evident that only the first majorizations are valid only for obtuse triangles, i.e. the other results are valid for all triangles.

2) $\left(\frac{a + b}{2},\ \frac{b + c}{2},\ \frac{c + a}{2}\right) \prec (a,\ b,\ c) \quad (\Delta)$.

EXAMPLE. 9° (MO, p. 201) $8\Pi a \leqslant \Pi(b + c)$ (GI 1.4).

3) $(a,\ b,\ c) \prec (2x,\ 2y,\ 2z) \quad (\Delta)$.

Here we shall give examples from MO, p. 202.

EXAMPLES. 10° $8xyz \leqslant abc$ (GI 1.3).

11° $2xyz/(\Sigma yz) \leqslant abc/(\Sigma bc)$,

12° $32xyz(\Sigma yz) \leqslant abc(\Sigma bc)$.

We shall now give some similar results for the sides of a polygon.

4) If $a_1, \ldots, a_n$ are the sides of a polygon with perimeter p and if $a_i \leqslant s$, $i = 1, \ldots, n$, where for some integer k, $p/(k + 1) \leqslant s \leqslant p/k$, then

$$\frac{1}{n}(p,\ \ldots,\ p) \prec (a_1,\ a_2,\ \ldots,\ a_n) \prec$$
$$\prec (\underbrace{s,\ \ldots,\ s}_{k},\ p - ks,\ \underbrace{0,\ \ldots,\ 0}_{n-k-1}).$$

This result is a generalization of Theorem 1 (in the case of Schur-convex functions, of course). A particular result in the case $p = sk$ for convex functions is given in [14], but as a consequence of Jensen's inequality and its conversion of Lah and Ribarič (see for example [14]). A simple consequence of this result is GI 16.5.

Note that for every polygon the following is valid

$$\frac{1}{n}(p,\ \ldots,\ p) \prec (a_1,\ \ldots,\ a_n) \prec (s,\ s,\ 0,\ \ldots,\ 0),$$

where s is semiperimeter of the polygon.

Remark. This result contains the following inequality of T. Popoviciu:

$$2^{m-1} \leqslant (\Sigma a_1)^m/(\Sigma a_1^m) \leqslant n^{m-1} \qquad (m \geqslant 1,\ m \in R).$$

The following two results are generalizations of (2) and (3) (see [14]).

5) If $a_1, \ldots, a_n$ are the sides of a polygon with perimeter $p = (n-1)s$ and if $a_i \leqslant s$ $(1 \leqslant i \leqslant n)$, then

$$(a_1, \ldots, a_n) \prec ((n-1)(s-a_1), \ldots, (n-1)(s-a_n)).$$

EXAMPLES. 13° $\Pi a_1 \geqslant (n-1)^n \Pi(s-a_1)$.

The latter result has been obtained previously by D. D. Adamović as answer to a problem of D. S. Mitrinović (AI, p. 209). For n = 3 we get 10°. Of course, this is a special case of the following result:

$$14° \quad T_k(a) \geqslant (n-1)^k T_k(s-a) \qquad (a = (a_1, \ldots, a_n)).$$

This inequality follows from the Schur-concavity of $T_k(x)$.

$$15° \quad T_k(a)/T_{k-1}(a) \geqslant (n-1)T_k(s-a)/T_{k-1}(s-a).$$

This inequality follows from the Schur-concavity of $T_k(x)/T_{k-1}(x)$.

$$16° \quad T_k(a)T_m(a) \geqslant (n-1)^{k+m} T_k(s-a)T_m(s-a).$$

6) Under the same condition as in 5) (see [14]):

$$(a_1, \ldots, a_n) \prec \left(s - \frac{a_1}{n-1}, \ldots, s - \frac{a_n}{n-1}\right).$$

EXAMPLE. 17° $(n-a)^n \Pi a_1 \leqslant \Pi(p-a_1)$.

For n = 3, this is 9°, i.e. GI 1.4.

7) Again under the same conditions as in (5), we can get the following results for a triangle and quadrilateral (see [14]):

$$\left(\frac{p+a}{2}, \frac{p+b}{2}, \frac{p+c}{2}\right) \prec (p-a, p-b, p-c),$$

$$\left(\frac{p+a+b}{2}, \frac{p+b+c}{2}, \frac{p+c+d}{2}, \frac{p+d+a}{2}\right) \prec$$

$$\prec (p-a, p-b, p-c, p-d).$$

2.2. Inequalities for the Angles of a Triangle and Polygon

We shall begin with some basic results for a triangle.

1) $(\pi/3,\ \pi/3,\ \pi/3) \prec (A, B, C) \prec (\pi, 0, 0) \quad (\Delta),$

$$(\pi/3,\ \pi/3,\ \pi/3) \prec (A, B, C) \prec (\pi/2,\ \pi/2,\ 0) \quad (\Delta_a),$$

$$(\pi/2,\ \pi/4,\ \pi/4) \prec (A, B, C) \prec (\pi, 0, 0) \quad (\Delta_o).$$

These results were given by A. Oppenheim in [12], but he used them only for convex functions. Of course, if we use them for Schur-convex functions as in MO, pp. 193-198 we get more general results.

We shall now give some examples from MO, pp. 194-198 with some extensions.

EXAMPLES. 1° The function $x \to \sin^k x$ is convex for $k < 0$, concave for $0 < k \leqslant 1$ on $[0, \pi]$, and convex for $k \geqslant 2$ on $[0, \pi/4]$, so the following results are valid:

(a) for $k < 0$, $\quad \sum \sin^k A \geqslant 3^{1+k/2}/2^k \quad (\Delta)$, and

$$\sum \sin^k A \geqslant 1 + 2^{1-k/2} \quad (\Delta_o);$$

(b) for $0 < k \leqslant 1$, $\quad 0 < \sum \sin^k A \leqslant 3^{1+k/2}/2^k \quad (\Delta)$,

$$2 < \sum \sin^k A \leqslant 3^{1+k/2}/2^k \quad (\Delta_a),$$

$$0 < \sum \sin^k A \leqslant 1 + 2^{1-k/2} \quad (\Delta_o);$$

(c) for $k < 0$, $\quad 3/2^k \leqslant \sum \sin^k (A/2) \quad (\Delta)$,

$$2^{-k/2} + 2\left(\frac{\sqrt{2 - \sqrt{2}}}{2}\right)^k \leqslant \sum \sin^k \frac{A}{2} \quad (\Delta_o);$$

(d) for $0 < k \leqslant 1$, $\quad 1 < \sum \sin^k (A/2) \leqslant 3/2^k \quad (\Delta)$,

$$2^{1-k/2} < \sum \sin^k (A/2) \leqslant 3/2^k \quad (\Delta_a),$$

$$1 < \sum \sin^k \frac{A}{2} \leqslant 2^{-k/2} + 2\left(\frac{\sqrt{2 - \sqrt{2}}}{2}\right)^k \quad (\Delta_o);$$

(e) for $k \geqslant 2$, $\quad 3/2^k \leqslant \sum \sin^k (A/2) < 2^{1-k/2} \quad (\Delta_a)$.

2° The function $x \to \log \sin x$ is concave on $(0, \pi)$ so the following inequalities are valid

$$0 < \Pi \sin A \leqslant 3\sqrt{3}/8 \quad (\Delta) \quad \text{(GI 2.7, 2.8)},$$

$$0 < \Pi \sin A \leqslant 1/2 \quad (\Delta_o),$$

$$0 < \Pi \sin A/2) \leqslant 1/8 \quad (\Delta_a) \quad \text{(GI 2.12)}.$$

3° Let $M_k(x, y, z)$ be the mean of order k of positive numbers x, y, z. Then from the above results in 1° and 2° we get for $k \leqslant 1$ and $k \neq -\infty$ (see MO, p. 196)

$$0 < M_k(\sin A, \sin B, \sin C) \leqslant \sqrt{3}/2 \quad (\Delta),$$

$$0 < M_k(\sin A, \sin B, \sin C) \leqslant ((1 + 2^{1-k/2})/3)^{1/k} \quad (\Delta_o).$$

For $0 < k \leqslant 1$ we have

$$(2/3)^{1/k} < M_k(\sin A, \sin B, \sin C) \leqslant \sqrt{3}/2 \quad (\Delta_a),$$

(not for $k \leqslant 1$ as in MO, p. 196).

For $k = -\infty$, the following results are valid (they are trivial)

$$0 < \min(\sin A, \sin B, \sin C) \leqslant \sqrt{3}/2 \quad (\Delta),$$

$$0 < \min(\sin A, \sin B, \sin C) \leqslant \sqrt{2}/2 \quad (\Delta_o).$$

4° The functions $x \to \cos^k x$ is convex for $k < 0$, concave for $0 < k \leqslant 1$ on $[0, \pi/2]$, and concave on $[0, \pi/4]$ for $0 < k \leqslant 2$. So the following results are valid:

(a) for $k < 0$, $\quad 3/2^k \leqslant \Sigma \cos^k A \quad (\Delta_a)$;

(b) for $0 < k \leqslant 1$, $\quad 1 < \Sigma \cos^k A \leqslant 3/2^k \quad (\Delta_a)$;

(c) for $k < 0$, $\quad 3^{1+k/2}/2^k \leqslant \Sigma \cos^k(A/2) \quad (\Delta)$, and

$$2^{-k/2} + 2\left(\frac{\sqrt{2 + \sqrt{2}}}{2}\right)^k \leqslant \Sigma \cos^k(A/2) \quad (\Delta_o);$$

(d) for $0 < k \leqslant 1$, $\quad 2 < \Sigma \cos^k(A/2) \leqslant 3^{1+k/2}/2^k \quad (\Delta)$, and

$$2 < \Sigma \cos^k(A/2) \leqslant 2^{-k/2} + 2\left(\frac{\sqrt{2 + \sqrt{2}}}{2}\right)^k \quad (\Delta_o);$$

(e) for $0 < k \leqslant 2$, $\quad 1 + 2^{1-k/2} < \Sigma \cos^k(A/2) \leqslant 3^{1+k/2}/2^k \quad (\Delta_a)$.

(f) Using the method of 1.2 we get $3/2^k \leqslant \Sigma \cos^k A$ for $k \geqslant 2$ (Δ_o) (V. Mascioni).

5° The function $x \to \log \cos x$ is concave on $(0, \pi/2)$, so the following inequalities are valid (MO, p. 197):

$$0 < \Pi \cos(A/2) \leqslant 3\sqrt{3}/8 \quad (\Delta) \quad \text{(GI 2.28)},$$

$$1/2 < \Pi \cos(A/2) \leqslant 3\sqrt{3}/8 \quad (\Delta_a) \quad \text{(GI 2.28)},$$

$$0 < \Pi \cos(A/2) \leqslant (1 + \sqrt{2})/4 \quad (\Delta_o).$$

6° The function $(A, B, C) \to T_2(\sin(A/2), \sin(B/2), \sin(C/2))$ is Schur-concave, so the following inequalities are valid (MO, p. 195):

$$0 < \Sigma \sin(B/2) \sin(C/2) \leqslant 3/4 \quad (\Delta),$$

$$1/2 < \Sigma \sin(B/2) \sin(C/2) \leqslant 3/4 \quad (\Delta_a), \quad \text{and}$$

$$0 < \Sigma \sin(B/2) \sin(C/2) \leqslant (2 - \sqrt{2})/4 + \sqrt{(2 - \sqrt{2})/2} \quad (\Delta_o).$$

7° The function $x \to \tan^k x$ $(k \geqslant 1)$ is convex on $(0, \pi/2)$, and the function $x \to \log \tan x$ is convex on $(0, \pi/2)$ so we have (MO, p. 197):

$$3^{1+k/2} \leqslant \Sigma \tan^k A \quad (\Delta_a), \qquad 3^{1-k/2} \leqslant \Sigma \tan^k(A/2) \quad (\Delta),$$

$$0 < \Pi \tan(A/2) \leqslant \sqrt{3}/9 \quad (\Delta_a).$$

8° In the previous examples we considered the trigonometric functions of the angles A, B, C and A/2, B/2, C/2. Of course, we can generalize the above results in a very simple way to the angles A/r, B/r, C/r. For such results see for example [15] and [16]. Here we shall only give results from [16]:

$$\sin(\pi/r) < \Sigma \sin(A/r) \leqslant 3 \sin(\pi/3r), \quad (1 \leqslant r),$$

$$2 + \cos(\pi/r) < \Sigma \cos(A/r) \leqslant 3 \cos(\pi/3r) \quad (2 \leqslant r),$$

$$\tan(\pi/r) > \Sigma \tan(A/r) \geqslant 3 \tan(\pi/3r) \quad (2 \leqslant r).$$

Note that for $r = 4$ we have $\sin(\pi/4) = \cos(\pi/4) = \sqrt{2}/2$, $\sin(\pi/12) = (\sqrt{6} - \sqrt{2})/4$, $\cos(\pi/12) = (\sqrt{6} + \sqrt{2})/4$, $\tan(\pi/12) = 2 - \sqrt{3}$.

9° (W. Janous) The function $x \to \log \dfrac{\sin tx}{tx}$, $0 < t \leqslant 1$, is concave on $(0, \pi)$. So we have

(i) for $t = 1$

$$0 < \Pi \sin A \leqslant (3\sqrt{3}/2\pi)^3 \Pi A \quad (\Delta)$$

$$(2/\pi)^2 \Pi A \leqslant \Pi \sin A \leqslant (3\sqrt{3}/2\pi)^3 \Pi A \quad (\Delta_a)$$

$$0 < \Pi \sin A < (16/\pi^3) \Pi A \quad (\Delta_o).$$

(ii) for $t = 1/2$

$$(1/4\pi)\Pi A < \Pi \sin(A/2) \leqslant (3/2\pi)^3 \Pi A \quad (\Delta)$$

$$(1/\pi^2)\Pi A < \Pi \sin(A/2) \leqslant (3/2\pi)^3 \Pi A \quad (\Delta_a)$$

$$(1/4\pi)\Pi A < \Pi \sin(A/2) < (8(\sqrt{2} - 1)/\pi^3)\Pi A \quad (\Delta_o).$$

Remarks. 1° We can get, from these inequalities, refinements concerning (Δ), (Δ_a), (Δ_o) of the results of IX.6.58 and IX.7.11 holding in any triangle.

2° We can get several interesting results using the following identities from [6]:

$$\Sigma\left(\frac{r + r_a}{b + c}\right)^k = \Sigma\left(\frac{a}{r_b + r_c}\right)^k = \Sigma\left(\frac{r_a - r}{a}\right)^k = \Sigma \tan^k(A/2),$$

$$\Sigma\left(\frac{r_a - r}{r_b + r_c}\right)^k = \Sigma \tan^{2k}(A/2),$$

$$\Sigma\left(\frac{a^2}{r_b + r_c}\right)^k = (4R)^k \Sigma \sin^{2k}(A/2),$$

$$\frac{1}{(4R)^k} \Sigma\left(\frac{a^2}{r_a - r}\right)^k = \frac{1}{2^k} \Sigma \left(\frac{h_b + h_c}{r + r_a}\right)^k =$$

$$= \Sigma \cos^{2k}(A/2).$$

2) Let ABC be any triangle with $A \leqslant B \leqslant C$, and let PQR be any triangle such that $P \geqslant C \geqslant Q \geqslant B$, then $A \geqslant R$ and (see [17]):

$$(P, Q, R) \succ (A, B, C).$$

In his answer to the problem from [17] Klamkin gave only the result for convex functions. It is obvious that for Schur-convex functions we have a more general result.

EXAMPLE. 10° Let ABC be an acute triangle of Bager's type II (see X.2.1), i.e. $\pi/6 \leqslant A \leqslant B \leqslant \pi/3 \leqslant C \leqslant \pi/2$. Then ([17]):

$$\frac{r}{R} \geqslant \frac{\sqrt{3} - 1}{2}.$$

Proof. Since $x \to \log \sin(x/2)$ is a concave function on $(0, \pi)$, we have

$$\Pi \sin(A/2) \geqslant \sin(\pi/4) \sin(\pi/6) \sin(\pi/12) = (\sqrt{3} - 1)/8,$$

so the above result follows from the identity $r/R = 4\prod \sin(A/2)$.

Remark. Using other Schur-convex (concave) functions, we can get several similar results:

$$\prod \sin A \geqslant \sqrt{3}/4, \quad \sum \cos A \geqslant (1+\sqrt{3})/2, \quad \text{etc.}$$

3) There exists a number s, $0 < s < 2$, such that [11]

$$(\cos^s(A/2),\ \cos^s(B/2),\ \cos^s(C/2)) \prec$$

$$\prec 3^{s/2}(\sin^s(A/2),\ \sin^s(B/2),\ \sin^s(C/2)).$$

Using this result for Schur-convex functions, we have a generalization of a result from [11], where a result is given only for convex functions (see the result (i) for Schur-convex functions). As a special case J. Steinig gives the following result:

(9) $$\sqrt{3}M_r(\sin(A/2)) \leqslant M_r(\cos(A/2)) \quad \text{for } r < s;$$

$$\sqrt{3}M_r(\sin(A/2)) \geqslant M_r(\cos(A/2)) \quad \text{for } r > s.$$

The inequalities are strict unless $A = B = C = \pi/3$, when equality holds for all r.

Remark. Steinig also proves several similar results. We shall give some of them:

1. There exists a number t, $0 < t < 1$, with the property that

$$3M_r(\tan(A/2)) \leqslant 2M_r(\cos(A/2)) \quad \text{for } r < t,$$

and the reverse inequality for $r > t$; the case of equality is the same as for (9).

2. There exists a number u, $-1 < u < 0$, such that

$$M_r(\operatorname{cotan}(A/2)) \leqslant 2M_r(\cos(A/2)) \quad \text{for } r < u,$$

and the reverse inequality for $r > u$, with equality as in (9).

3. There exists a number v, $-1 < v < 0$, with the following property

$$\sqrt{3}M_r(\tan(A/2)) \leqslant 2M_r(\sin(A/2)) \quad \text{for } r < v,$$

and the reverse inequality for $r > v$, with equality as in (9).

Comment by V. Mascioni. Some of Steinig's results may be improved using Čebyšev's inequality (AI, p. 36). For instance, if $k > 0$ we have

$$M_k\left(\operatorname{cotan}\frac{A}{2}\right) \geqslant \frac{M_k(\cos\frac{A}{2})}{M_{-k}(\sin\frac{A}{2})} \geqslant \left(\frac{4R}{r}\right)^{1/3} M_k\left(\cos\frac{A}{2}\right)$$

(for $k < 0$ these inequalities are reversed). For $k > 0$ this is better than Steinig's results since $(4R/r)^{1/3} \geqslant 2$.

We shall now give two similar results for polygons.

4) Let $A_1, \ldots, A_n$ be the angles of a convex polygon. Then

$$(\pi - 2\pi/n, \ldots, \pi - 2\pi/n) \prec (A_1, \ldots, A_n).$$

EXAMPLES. 11° [18] $\Sigma \sin^{\lambda} A_1 \leqslant n \sin^{\lambda}(2\pi/n)$ for $0 < \lambda \leqslant 1$, and the reverse inequality for $\lambda < 0$.

12° $\Pi \sin A_1 \leqslant \sin^n(2\pi/n)$.

13° $\Sigma \cos^{\lambda}(A_1/2) \leqslant n \sin^{\lambda}(\pi/n)$ for $0 < \lambda \leqslant 1$, and the reverse inequality for $\lambda < 0$.

14° $\Pi \cos(A_1/2) \leqslant \sin^n(\pi/n)$.

15° [19] $\Sigma \tan^{\lambda}(A_1/2) \geqslant n \cot an^{\lambda}(\pi/n)$ for $\lambda \geqslant 1$.

5) Let $A_1, \ldots, A_n$ be the angles of an arbitrary polygon. Then

$$(\pi/n, \ldots, \pi/n) \prec (A_1/(n - 2), \ldots, A_n/(n - 2)).$$

EXAMPLES. 16° $\Sigma \sin^{\lambda}(A_1/(n - 2)) \leqslant n \sin^{\lambda}(\pi/n)$ for $0 < \lambda \leqslant 1$, and the reverse inequality for $\lambda < 0$.

17° $\Pi \sin(A_1/(n - 2)) \leqslant \sin^n(\pi/n)$.

18° $\Sigma \cos^{\lambda}(A_1/(2n - 4)) \leqslant n \cos^{\lambda}(\pi/2n)$ for $0 < \lambda \leqslant 1$, and the reverse inequality for $\lambda < 0$.

19° $\Pi \cos(A_1/(2n - 4)) \leqslant \cos^n(\pi/2n)$.

20° $\Sigma \tan^{\lambda}(A_1/(2n - 4)) \geqslant n \tan^{\lambda}(\pi/2n)$ for $\lambda \geqslant 1$.

Equality in all examples holds only for regular n-gons. Note that 19° is given in [19] only for convex polygons.

2.3. Majorization for Other Elements of a Triangle

1) The following results are valid for exradii and altitudes:

$$\left(\frac{1}{3r}, \frac{1}{3r}, \frac{1}{3r}\right) \prec \left(\frac{1}{h_a}, \frac{1}{h_b}, \frac{1}{h_c}\right) \prec \left(\frac{1}{r_a}, \frac{1}{r_b}, \frac{1}{r_c}\right);$$

$$\left(\frac{r_a}{h_a}, \frac{r_b}{h_b}, \frac{r_c}{h_c}\right) \succ_w (1, 1, 1);$$

and as consequences of these results in MO, pp. 203-205, we obtain the following results: GI 6.8, 6.16, 6.18, 6.19, 6.20, 6.21, 6.22 and a result from Remark 1 from 6.28.

2) For each $t \geqslant 1$ (MO, pp. 206-207):

$$(\sqrt{3}/2)^t(a^t, b^t, c^t) \prec_w (r_a^t, r_b^t, r_c^t).$$

EXAMPLES. 1° $(\sqrt{3}/2)^t \Sigma a^t \leqslant \Sigma r_a^t, \quad t \geqslant 1$.
The case $t = 1$ is GI 5.29.

2° $(\sqrt{3}/2)M_k(a^t, b^t, c^t) \prec_w (r_a^t, r_b^t, r_c^t). \quad k \geqslant 1.$

3° From the Schur-concavity of Πx_i, $x_i > 0$, follows GI 5.35.

3) For $t \geqslant 1$ (MO, pp. 207-208)

$$(m_a^t, m_b^t, m_c^t) \prec_w (r_a^t, r_b^t, r_c^t).$$

EXAMPLE. 4° $M_k(m_a^t, m_b^t, m_c^t) \leqslant M_k(r_a^t, r_b^t, r_c^t)$, $k \geqslant 1$, $t \geqslant 1$.

2.4. Majorization and Isoperimetric-Type Inequalities

For a class C of plane figures, isoperimetric inequalities are often stated in one of these two forms:

(i) Of all figures in C with perimeter p, the figure G has the greatest area.

(ii) Of all figures in C with area F, the figure G has the least perimeter.

These are dual theorems; a particularly simple proof of the equivalence is given by Kazarinoff [20, p. 43].

In MO, pp. 208-214, it is shown that for plane figures possessing certain properties the area is a Schur-concave function of the parameters of the plane figure. Consequently, the area is maximized when these parameters are equal, from which the isoperimetric result follows, and therefore the corresponding inequality for the area of this figure follows.

Here we shall give without proof some of these results (MO, pp. 208-212):

1) The area of a triangle with one fixed side is a Schur-concave function of the other sides.

2) The area of a triangle is a Schur-concave function of the sides.
Remark. We note that a consequence of this result is GI 4.2.

3) The area of a quadrilateral inscribed in a circle is a Schur-concave function of the sides.
Remark. We note that a simple consequence of this result is GI 5.3.

4) Let H be a polygon of n sides $a_1, \ldots, a_n$ with vertices $A_1, \ldots,$

A_n, inscribed in a circle of radius r and containing the center O of the circle. The area of H is a Schur-concave function of the central angles $\Theta_1, \ldots, \Theta_n$ subtended by the arcs $A_1A_2, A_2A_3, \ldots, A_nA_1$, of the sides, and of the altitudes $h_1, \ldots, h_n$ from O to the corresponding sides.

Remark. As a consequence of (4), the area of a polygon containing O is maximized when the polygon is regular.

5) The area of a polygon of n sides with fixed perimeter inscribed in a circle is a Schur-concave function of the lengths of the sides.

3. Applications of Some Other Inequalities for Convex Functions

3.1. First, we note that the well-known Levinson inequality for 3-convex function (AI, p. 363) has the following simple consequence:

(1) If f is a 3-convex function on [0, 2s], then

$$\frac{1}{3}\,\Sigma f(a) - f\left(\frac{2s}{3}\right) \leqslant \frac{1}{3}\,\Sigma f(b+c) - f\left(\frac{4s}{3}\right).$$

Remark. If a function f is three times differentiable, then it is 3-convex if and only if $f'''(x) \geqslant 0$ for every $x \in (0, 2s)$.

EXAMPLE. 1° The function $x \to f(x) = x/(2s - x)$ is 3-convex, so the following inequality is valid

$$\Sigma\,\frac{b+c}{a} - \Sigma\,\frac{a}{b+c} \geqslant \frac{9}{2}\,.$$

(2) If $f:[0, \pi/2] \to R$ is a 3-convex function, then for every triangle

$$\Sigma f\left(\frac{\pi - A}{2}\right) - \Sigma f\left(\frac{A}{2}\right) \geqslant 3f\left(\frac{\pi}{3}\right) - 3f\left(\frac{\pi}{6}\right). \tag{10}$$

If f is strictly 3-convex, equality in (10) holds only for equilateral triangles.

Proof. In the proof we shall use the generalization of Levinson's inequality from [21, 22]; i.e. the following special case of this generalization:

Let $f:[0, 2a] \to R$ be a 3-convex function, $x_i \in [0, 2a]$ (i = 1, 2, 3) and

$$x_1 + x_3 \leqslant 2a, \quad x_2 \leqslant a, \tag{11}$$

then

$$\Sigma f(x_1) - 3f\left(\frac{1}{3}\,\Sigma x_1\right) \leqslant \Sigma f(2a - x_1) - 3f\left(2a - \frac{1}{3}\,\Sigma x_1\right). \tag{12}$$

If f is strictly 3-convex the equality in (12) holds if and only if $x_1 = x_2 = x_3$.

Let A, B, C be the angles of a triangle such that $A \geqslant B, C$. Using the substitutions $a = \pi/4$, $x_1 = A/2$, $x_2 = B/2$, $x_3 = C/2$, from (12) we get (10), since the conditions (11) are fulfilled.

EXAMPLES. 2° The function $x \to \tan^k x$ is 3-convex on $[0, \pi/2]$ for $k \in (0, 1) \cup (2, +\infty)$, so the following inequality is valid

$$\sum \cot^k(A/2) - \sum \tan^k(A/2) \geq 3^{1+k/2} - 3^{1-k/2}.$$

For $k = 1$, we get $\sum \cot A \geq \sqrt{3}$ (GI 2.38), since $\cot(A/2) - \tan(A/2) = 2 \cot A$. For $k = 2$, we get the first inequality in

$$\sum \cot^2(A/2) \geq 8 + \sum \tan^2(A/2) \geq 9.$$

The second inequality is GI 2.35. These inequalities are a refinement of GI 2.43.

3° The function $x \to \cos^k x$ is 3-convex on $[0, \pi/2]$ for $k \in (-\infty, 0) \cup [1, 2]$, so the following inequality is valid

$$\sum \cos^k(A/2) - \sum \sin^k(A/2) \leq (3^{1+k/2} - 3)/2^k.$$

For $k \in (0, 2/3]$ we have the reverse inequality.

For $k = 2$ we get $\sum \cos A \leq 3/2$, i.e. the second inequality from GI 2.16. For $k = -1$, we get the first inequality in

$$\sum \operatorname{cosec}(A/2) \geq 6 - 2\sqrt{3} + \sum \sec(A/2) \geq 6.$$

The second inequality follows from Example 4° (c) of (1) in 2.2. For $k = -2$, using GI 2.48, we get the following refinement of GI 2.52

$$\sum \operatorname{cosec}^2(A/2) \geq 8 + \sum \sec^2(A/2) \geq 12.$$

4° The function $x \to \log \sin x$ is 3-convex on $[0, \pi/2]$, so the following inequalities are valid:

$$\prod \tan(A/2) \leq \sqrt{3}/9 \text{ (GI 2.34)}, \quad \prod \cot(A/2) \geq 3\sqrt{3} \text{ (GI 2.42)}.$$

Remark. In all examples equality holds only for equilateral triangles.

3.2. The following result is a simple consequence of the well-known Popoviciu inequality for convex functions (AI, p. 174):

If $f:[0, \pi] \to R$ is a convex function, then

$$3f(\pi/3) - 2\sum f\left(\frac{\pi - A}{2}\right) + \sum f(A) \geq 0.$$

EXAMPLES.

5° $$3^{1+k/2}/2^k - 2\sum \cos^k(A/2) + \sum \sin^k A \begin{cases} \geq 0, & \text{for } k < 0, \\ \leq 0, & \text{for } 0 < k \leq 1, \end{cases} \quad (\Delta)$$

6° $$3/2^k - 2\sum \sin^k(A/2) + \sum \cos^k A \begin{cases} \geq 0, & \text{for } k < 0, \\ \leq 0, & \text{for } 0 < k \leq 1, \end{cases} \quad (\Delta_a)$$

7° $\frac{3\sqrt{3}}{8}\ \Pi \sin A \leqslant \Pi \cos^2(A/2) \quad (\Delta),$

8° $\frac{1}{8}\ \Pi \cos A \leqslant \Pi \sin^2(A/2) \quad (\Delta_a),$

9° $3^{1+k/2} - 2\Sigma \operatorname{cotan}^k(A/2) + \Sigma \tan^k A \geqslant 0 \quad \text{for } k \geqslant 1\ (\Delta_a).$

Comment by V. Mascioni. Example 7° may be extended: by Steinig's (9), there is an s, $0 < s < 2$, such that

$$\sqrt{3}M_r(\sin A/2) \leqslant M_r(\cos A/2) \quad \text{for } r < s.$$

Then, by Čebyšev's inequality (AI, p. 36), we have

$$M_k(\sin A) \leqslant 2M_k(\sin \tfrac{A}{2})M_k(\cos \tfrac{A}{2}) \leqslant \frac{2\sqrt{3}}{3} M_k^2(\cos \tfrac{A}{2}), \qquad (0 < k < s),$$

which directly implies 7°.

References

1. M. Petrović, 'Sur quelques fonctions des côtes et des angles d'un triangle', L'Enseignement Mathématique 18 (1916), 153-163.
2. M. Petrović, 'A propos d'un théorème de M. Pompeiu', Bulletin Mathématique de la Société Roumaine des Sciences 40 (1938), 205-208.
3. P. M. Vasić, 'Geometrijske nejednakosti u radovima Mihaila Petrovića', Mat. Biblioteka 38 (1968), 105-111.
4. A. Bager, 'A Family of Goniometric Inequalities', Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 338-352 (1971), 5-25.
5. R. Ž. Djordjević, 'Some Inequalities for the Triangle', Ibid. No. 247-273 (1969), 35-39.
6. M. S. Jovanović, 'Some Inequalities for Triangle Elements', Ibid. No. 498-541 (1975), 171-178.
7. D. Milošević, 'Jedna primena konveksnih funkcija', Matematika 1 (1979), 69-71.
8. V. Andrić, 'Jedna trigonometrijska nejednakost i njena primena', Ibid. 1 (1981), 78-79.
9. D. Andrica, The Improvement of Some Geometric Inequalities. The Seminar "Teodor Angheluţă", Cluj-Napoca 1983, 1-2.
10. J. Steinig, 'Sur quelques applications géométriques d'une inégalité relative aux fonctions convexes', Enseignement Math. (2) 11 (1965), 281-285.
11. J. Steinig, 'Some Trigonometric Inequalities', El. Math. 27 (1972), 121-129.
12. A. Oppenheim, 'Some Inequalities for Triangle', Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 357-380 (1971) 21-28.
13. A. W. Marshall and I. Olkin, Inequalities: Theory of Majorization and Its Applications, New York-London-Toronto-Sydney-San Francisco, 1979.
14. J. E. Pečarić, R. R. Janić, and M. S. Klamkin, 'Inequalities for

Polygons', Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 716-734 (1981), 62-64.
15. M. Mitrović, 'Problem 222', Mat. Vesnik 9 (24) (1972), 308.
16. M. Vowe and G. Bercea,'Aufgabe P 859', Praxis Math. 27 (1985), 309.
17. V. N. Murty and M. S. Klamkin, 'Problem 859', Crux Math. 11 (1985), 191.
18. M. R. Tasković, 'Généralisations de certains inégalités géométriques et quelques inégalités géométriques nouvelles', Mat. Vesnik 7 (22) (1970), 73-81.
19. H. Kappus,'Aufgabe P 521', Praxis Math. 7 (1973), 189, and 11 (1973), 305-306.
20. N. D. Kazarinoff, Geometric Inequalities, New York, 1961.
21. S. Lawrence and D. Segalman, 'A Generalization of Two Inequalities Involving Means', Proc. Amer. Math. Soc. 35 (1972), 96-100.
22. J. E. Pečarić, 'On an Inequality of N. Levinson', Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 678-715 (1980), 71-74.
23. M. S. Klamkin, 'On Yff's Inequality for the Brocard Angle of a Triangle', Elem. Math. 32 (1977), 118.
24. V. D. Mascioni and W. Janous, 'Aufgabe 899', Elem. Math. 38 (1983), 106 and 39 (1984), 102-103.
25. V. D. Mascioni, 'Problem E 3054', Amer. Math. Monthly 91 (1984).

Chapter IX

MISCELLANEOUS INEQUALITIES WITH ELEMENTS OF A TRIANGLE

1. Inequalities Involving only the Sides of a Triangle

1.1. $3(\Sigma b^2/c^2) - (\Sigma a^2)(\Sigma 1/a^2) \geqslant 0.$ {E}

A. W. Walker and L. Carlitz, 'Problem 774', Math. Mag. 43 (1970), 226 and 44 (1971), 172-173.

1.2. $\Sigma \frac{b^2}{c^2} \geqslant 3 \geqslant \Sigma \frac{b^2 + c^2 - a^2}{bc}.$ {E}

M. S. Klamkin, 'Asymmetric Triangle Inequalities', Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 357-380 (1971), 33-44.

1.3. $abc\Sigma \frac{b^2}{c^2} \geqslant \Sigma a^3 + \Sigma ab(b - a).$ {E}

M. S. Klamkin, The same reference as in 1.2.

1.4. $3\Sigma b/c \geqslant \Sigma a\Sigma 1/a.$ {E}

M. S. Klamkin, The same reference as in 1.2.

1.5. $\Sigma(2a - s)(b - c)^2 \geqslant 0.$ {E}

This inequality is equivalent to GI 1.6 and to the first inequality in GI 5.8 (the well-known Gerretsen inequality).

V. N. Murty and W. J. Blundon, 'Problem 708', Crux Math. 9 (1983), 49-50.

1.6. $\pm 2\sqrt{9 + 6\sqrt{3}}$ are the largest and smallest permissible values of k in inequality

$$\Sigma a^3 \geqslant 3abc + k\Pi(b - c).$$

M. S. Klamkin, D. J. Newman, C. C. Rousseau, and D. Shanks, 'Problem 71-78', SIAM Review 14 (1972), 656-657.

1.7. $3\Sigma a^2 b \geqslant 9abc + \Sigma a\Sigma a^2.$ {E}

1.8. $(\Sigma a)^3 \geqslant \Pi(2b + 2c - a).$ {E}

Remark. 1.7 and 1.8 are due to L. Goldstone.

1.9. $8(\Sigma a^3)^2 \geqslant 9\Pi(a^2 + bc)$. {E}

H. S. Hall and S. R. Knight, Higher Algebra, London (1940), p. 521.

1.10. $\Sigma \frac{a}{b+c} + \frac{3\Pi a}{\Pi(b+c)} < 2$.

This inequality is better than the second inequality of GI 1.16.

D. D. Adamović and I. Paasche, 'Problem 152', Mat. Vesnik 6 (21) (1969), 472.

1.11. Let $\lambda > 0$ be a real number. Then

$$\Pi \frac{s + \lambda a}{s - a} \geqslant (2\lambda + 3)^3. \quad \{E\}$$

This inequality is due to W. Janous. $\lambda = 3$ yields:

C. J. M. Swinkels, 'Problem 2799', Nieuw Tijds. Wisk. 1973, No. 4.

1.12. $|\Sigma \frac{a-b}{a+b}| < \frac{1}{3}(8\sqrt{2} - 5\sqrt{5})$.

D. S. Mitrinović and W. Janous, 'Problem 1080', Crux Math. 12 (1986), 11.

1.13. $\Sigma \frac{a-b}{s-b} \leqslant 0$.

S. G. Guba, 'Problem 674', Mat. v. Škole 1969, No. 5, 76, and 1970, No. 4, 78.

1.14. Let $a \leqslant b \leqslant c$ be sides of a triangle. Then

$$1 \leqslant \min(a/b, b/c, c/a) \cdot \max(a/b, b/c, c/a) < \frac{1}{2}(1 + \sqrt{5}),$$

with equality at the left-hand-side for isosceles triangle; the right-hand-estimate is the best possible one.

E. S. Langford, D. Singmaster, and G. Singmaster, 'Problem E 1705', Amer. Math. Monthly 71 (1964), 680.

1.15. $$\Sigma a^n \geqslant \frac{2^n \cdot 3^{\frac{n}{2}+1}}{2^{\frac{3n}{2}} + 3^{\frac{3n}{2}}} \left(s^n + \left(\frac{abc}{s}\right)^{\frac{n}{2}}\right), \quad n \geqslant 1. \quad \{E\}$$

This is a generalization of GI 1.2.

B. Milisavljević, 'Problem 448', Mat. Vesnik 2 (15) (30) (1978), 294-295.

1.16.(a) $\Sigma a^\lambda(a - b) \geqslant 0$, $\lambda > 0$. {E}

For $\lambda < 0$ the reverse inequality is valid.

(b) $\Sigma a^{\lambda} b(a - b) \geqslant 0, \quad \lambda \geqslant 2.$ {E}

(a) is due to M. S. Klamkin (the same reference as in 1.2), and (b) is given in

V. Cirtoaje, 'Problem 0:473', Gaz. Mat. (Bucharest) 91 (1986), 138.

1.17. Conjecture:

$$2s(\Sigma\sqrt{s - a}) \leqslant 3(\Sigma\sqrt{bc(s - a)}).$$

W. Janous, 'Problem E 3146', Amer. Math. Monthly 93 (1986), 299.

1.18. Let $\lambda \leqslant 1$ be a real number. Then

$$\Pi(a + b - \lambda c)^{a} \leqslant (2 - \lambda)^{2s}\Pi a^{a}. \quad \{E\}$$

This inequality due to W. Janous generalizes the results of

H. S. Hall and S. R. Knight, Higher Algebra, London, 1940, 523.

I. Stojanov, 'Problem 3', Matematika i Fizika 13 (1982), 334.

M. Selby and L.-W. Yip, 'Problem 233', The College Math. J. 15 (1984), 272-273.

B. Prielipp, E. M. Klein, and H. S. Lieberman, 'Problem 580', Pi Mu Epsilon J. 8 (1984), 43 and 8 (1985), 193.

1.19. $\quad abc\Sigma bc \geqslant 32xyz\Sigma yz \quad$ and $\quad abc\Sigma yz \geqslant 2xyz\Sigma bc.$

A. W. Walker, S. Reich, and M. Goldberg, 'Problem E 2284', Amer. Math. Monthly 78 (1971), 297 and 79 (1972), 183-184.

1.20. $$\Sigma \frac{c(a + b)}{a + b - c} \geqslant \frac{2(a + b + c)}{abc} \Pi(-a + b + c). \quad \{E\}$$

N. Pantazi, 'Problem 8969', Gaz. Mat. (Bucharest) B 19 (1968), 372 and B 20 (1969), 228.

1.21. $$\Pi(-a + b + c)^2 \geqslant \Pi(-a^2 + b^2 + c^2).$$

A. W. Walker, 'Problem 300', Nieuw Arch. Wisk. (3) 19 (1971), 224.

J. Wolstenholme, A Book of Mathematical Problems on Subjects Included in the Cambridge Course, London and Cambridge, 1867, 56.

Generalization. The functions appearing in 1.21 are special cases of the general function

$$D_{n,s}(x) = \Pi(-sa_1^x + a_2^x + \ldots + a_n^x),$$

where s and x are real. Although Walker gave 1.21 for real a, b and c, here we consider only results for positive values of $a_1, \ldots, a_n$. For example the following results are valid:

(1) $(D_{4,1}(1))^2 > 3D_{4,1}(2).$

We remark that (1) becomes an equality if we set $a_1 = a_2 = a_3$ and take the limit as $a_4 \to 0$. If we let $a_4 \to 0$ and then use the elementary inequality $3(a^2 + b^2 + c^2) \geq (a + b + c)^2$, we get 1.21.

(2) $D_{3,1}(x)D_{3,1}(y) \geq D_{3,1}(x + y) \quad (x, y > 0),$

with equality if and only if $a_1 = a_2 = a_3$.

(3) Suppose a_1^{x+y}, a_2^{x+y}, a_3^{x+y} are the sides of a triangle. If x and y are positive and $s \leq 1$, then

$$D_{3,s}(x)D_{3,s}(y) > 2(1 - s)^2 D_{3,s}(x + y).$$

C. E. Carroll, C. C. Yang, and S. Ahn, 'Some Triangle Inequalities and Generalizations', Canad. Math. Bull. 23 (3) (1980), 267-274.

1.22. $$\sum \frac{b}{c} > \frac{1}{2}(\sum a)\left(\sum \frac{1}{a^2}\right)^{1/2}.$$

Proof. If Ω denotes a Crelle-Brocard point of a triangle ABC, then $B\Omega = (2R \sin \omega)b/c$ etc. Since also

$$\sum(bc)^2 = 4F^2/\sin^2 \omega,$$

the given inequality can be rewritten as

$$2\sum A\Omega > 4(\sum a)FR/\prod a = \sum a.$$

The latter inequality is now an immediate consequence of the basic triangle inequality, since for any point P, $BP + CP > a$, $CP + AP > b$, $AP + BP > c$.

J. Brejcha and M. S. Klamkin, 'Aufgabe 771', Elem. Math. 31 (1976), 99, and 32 (1977), 97-98.

Comment by W. Janous. The right-hand-sides (RHS) of 1.4 and 1.22 are incomparable in general. Indeed, if $a = b = c$, then $\text{RHS}(1.4)/\text{RHS}(1.22) = 2/\sqrt{3} > 1$. On the other hand, if $a = 1$, $b = c \to \infty$, then $\text{RHS}(1.4)/\text{RHS}(1.22) \to 2/3 < 1$.

1.23. $$\sum \frac{(a + b)\sqrt{ab}}{\sqrt{a + b - c}} \geq 6\left(\frac{2s}{3}\right)^{3/2}. \quad \{E\}$$

This inequality is due to W. Janous.

1.24. If $x, y, z \in R$, then

$$\sum a^2(x - y)(x - z) \geq 0.$$

J. Wolstenholme, A Book of Mathematical Problems on Subjects Included in the Cambridge Course, London and Cambridge, 1867, 24.

M. S. Klamkin, The same reference as in 1.2.

1.25. $\sum x = 0 \Rightarrow \sum a^2 yz \leqslant 0.$

J. Wolstenholme, The same reference as in 1.24.

1.26. $\sum \frac{a^2}{x} = 0$ and $\sum x > 0 \Rightarrow xyz < 0.$

W. J. Greenstreet and H. W. Curjel, 'Problem 11433', Math. Quest. and Sol. from Educ. Times 59 (1893), 37.

1.27. $0 < a \leqslant b \leqslant c \Rightarrow$

$$4 < (\sum a)^2/(bc) \leqslant 9 \quad \{E\}, \quad 8 < (\sum a)^2/(ac) < +\infty,$$

$$9 \leqslant (\sum a)^2/(ab) < +\infty \quad \{E\}.$$

L. Ratz and G. Bach, 'Aufgabe 696', Elem. Math. 28 (1973), 76 and 29 (1974), 73-74.

1.28. $(b + c)^2(a + b - c)(a + c - b) \leqslant 4a^2bc.$

V. Tifui and B. Tudor, 'Problem 18368', Gaz. Mat. (Bucharest) 10 (86) (1981), 386.

1.29. $F = 1 \Rightarrow s \geqslant \sqrt{(s - b)(s - c)} + \sqrt{bc}$

with equality if $b = c$.

L. Pirsan, 'Asupra problemei E 2955', Gaz. Mat. (Bucharest) B 20 (1969), 139-140.

1.30. If $k \geqslant 1$, then

$$\sum \frac{a}{k(b + c) - a} \geqslant \frac{3}{2k - 1}. \quad \{E\}$$

M. S. Klamkin, 'Solution of Problem 689', Crux Math. 8 (1982), 308-309.

1.31. $\sum a^2(b + c) \geqslant 48\Pi(s - a).$

K. Čimev, 'Problem 1', Mat. i Fiz. (Sofija) 5, 4 (1961), 59.

1.32. $\sum a^4 + 5abc(\sum a) \geqslant 2(\sum bc)^2.$

S. Bilčev, 'Problem 3', Ob. po Matematika (Sofija), 1984, No. 3, 55 and 1985, No. 3, 59-60.

1.33. $(7(\sum a)^2 - 18(\sum bc))^{3/2} + 9(\sum a)^3 + 54abc \geqslant 36(\sum a)(\sum bc).$

S. Bilčev and D. Mihov, 'Problem 3', Matematika (Sofija) 1984, No. 5, 38, and 1984 No. 10, 31.

1.34. Let $g(a, b, c) = \Sigma \frac{a}{a + 2b} \frac{b - 4c}{b + 2c}$. Then

$$-5/3 < g(a, b, c) \leqslant -1.$$

W. Janous, 'Problem 1079', Crux. Math. 11 1985), 250.

1.35. $$(2^{n-2} - 2^{n-3} + 1)\Sigma a^n + 2(2^{n-3} + 1)abc\Sigma a^{n-3} \geqslant$$

$$\geqslant (2^{n-2}\Sigma bc - (2^{n-3} - 1)\Sigma a^2)\Sigma a^{n-2} \quad (n \in N).$$

This inequality is due to S. J. Bilčev.

2. Inequalities for the Angles of a Triangle

2.1. Bager's Graphs

Here we give Bager's two graphs of goniometric inequalities. One vertex in each graph is the constant function 1, the other letters represent certain normalized symmetric functions of (A, B, C) (a function of (A, B, C) is called normalized if it takes the value 1 at ($\pi/3$, $\pi/3$, $\pi/3$)). The arrows are numbered and represent the inequalities. For instance, from the first graph

$$a \xrightarrow{1} b$$

denotes the inequality

$$\frac{1}{9} \Sigma \text{ cotan } \frac{B}{2} \text{ cotan } \frac{C}{2} \leqslant \frac{1}{8} \Pi \text{ cosec } \frac{A}{2}. \tag{1}$$

In each inequality stated equality occurs if and only if A = B = C.

The first graph is from [1]. Of course, we give a modification of this graph because the conjectures from this paper were proved in [3] and [4]. The second graph is given in [2]. Of course, every inequality from the graphs can be found somewhere in the given references [1-7].

The letters representing normalized symmetric functions for Bager's first graph are:

$$a = \frac{1}{9} \Sigma \text{ cotan } \frac{B}{2} \text{ cotan } \frac{C}{2}, \quad b = \frac{1}{8} \Pi \text{ cosec } \frac{A}{2},$$

$$c = \frac{\sqrt{3}}{3} \Sigma \text{ cotan } A, \quad d = \frac{3\sqrt{3}}{8} \Pi \text{ cosec } A, \quad e = \frac{\sqrt{3}}{3} \Sigma \tan \frac{A}{2},$$

$$f = \frac{3\sqrt{3}}{8} \Pi \sec \frac{A}{2}, \quad g = \frac{2}{3} \Sigma \sin \frac{A}{2}, \quad h = \frac{1}{6} \Sigma \text{ cosec } \frac{A}{2},$$

$$i = \frac{1}{12} \Sigma \text{ cosec } \frac{B}{2} \text{ cosec } \frac{C}{2}, \quad j = 8\Pi \sin \frac{A}{2},$$

$$k = \frac{4}{3}\,\Sigma \sin\frac{B}{2}\sin\frac{C}{2}, \quad L = 8\Pi \cos A, \quad m = \frac{4}{3}\,\Sigma \cos B \cos C,$$

$$n = \frac{4}{9}\,\Sigma \sin B \sin C, \quad o = \frac{2}{3}\,\Sigma \cos A, \quad p = \frac{4}{9}\,\Sigma \cos\frac{B}{2}\cos\frac{C}{2},$$

$$q = \frac{\sqrt{3}}{3}\,\Sigma \cos\frac{A}{2}, \quad r = \frac{1}{4}\,\Sigma \sec\frac{B}{2}\sec\frac{C}{2}, \quad s = \frac{\sqrt{3}}{6}\,\Sigma \sec\frac{A}{2},$$

$$t = \frac{8\sqrt{3}}{9}\,\Pi \cos\frac{A}{2}, \quad u = \frac{\sqrt{3}}{9}\,\Pi \cot\!an\frac{A}{2}, \quad v = \frac{8\sqrt{3}}{9}\,\Pi \sin A,$$

$$w = 3\sqrt{3}\,\Pi \tan\frac{A}{2}, \; x = \frac{\sqrt{3}}{6}\,\Sigma\, \mathrm{cosec}\, A, \quad y = 3\sqrt{3}\,\Pi\, \mathrm{cotan}\, A.$$

The letters representing normalized symmetric functions for Bager's second graph are:

$$a = \frac{1}{3}\,\Sigma \cos 2(B - C), \quad b = 3\sqrt{3}\,\Pi\, \mathrm{cotan}\, A, \quad c = 8\Pi \cos A,$$

$$d = \frac{4}{3}\,\Sigma \cos B \cos C, \quad e = \frac{4}{9}\,\Sigma \sin 2B \sin 2C,$$

$$f = \frac{16}{9}(\Sigma \cos B \cos C)^2, \quad g = \frac{2\sqrt{3}}{9}\,\Sigma \sin 2A = \frac{8\sqrt{3}}{9}\,\Pi \sin A,$$

$$h = 64\Pi \sin^2\frac{A}{2}, \quad i = \frac{4}{27}(\Sigma \sin 2A)^2 = \frac{64}{27}\,\Pi \sin^2 A,$$

$$j = \frac{16}{3}\,\Sigma \cos^2 B \cos^2 C, \quad k = \frac{4}{9}\,\Sigma \sin^2 2A, \quad L = \frac{1}{3}\,\Sigma \cos(B - C),$$

$$m = 27\Pi \tan^2\frac{A}{2}, \quad n = \frac{16}{81}(\Sigma \sin B \sin C)^2, \quad o = 8\Pi \sin\frac{A}{2},$$

$$p = 3\sqrt{3}\,\Pi \tan\frac{A}{2}, \quad q = \frac{16}{9}\left(\Sigma \sin\frac{B}{2}\sin\frac{C}{2}\right)^2,$$

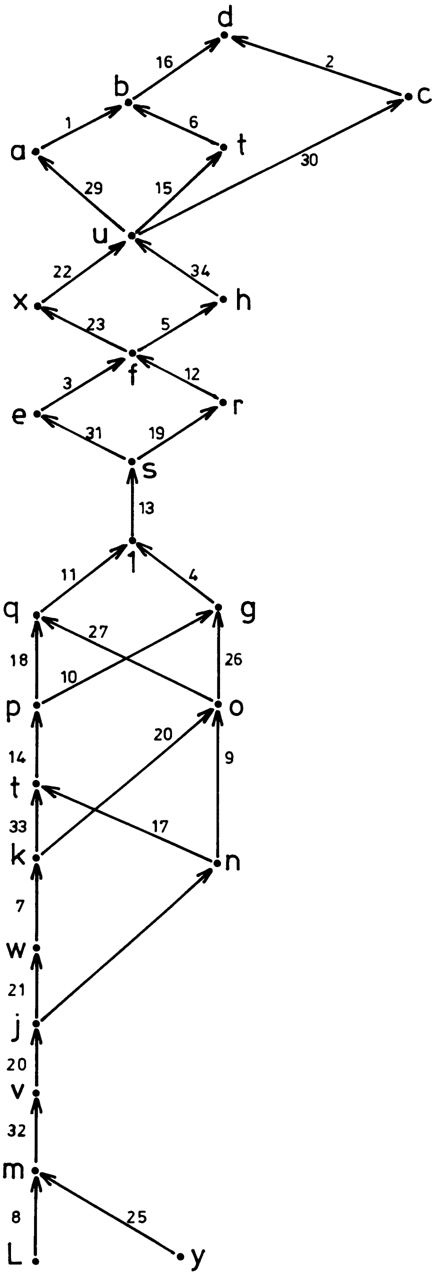
$$r = \frac{4}{9}\,\Sigma \sin B \sin C, \quad s = \frac{16}{27}\,\Sigma \sin^2 B \sin^2 C,$$

$$t = -\frac{2}{3}\,\Sigma \cos 2A, \quad u = \frac{4}{3}\,\Sigma \sin\frac{B}{2}\sin\frac{C}{2}, \quad v = \frac{2}{3}\,\Sigma \cos A,$$

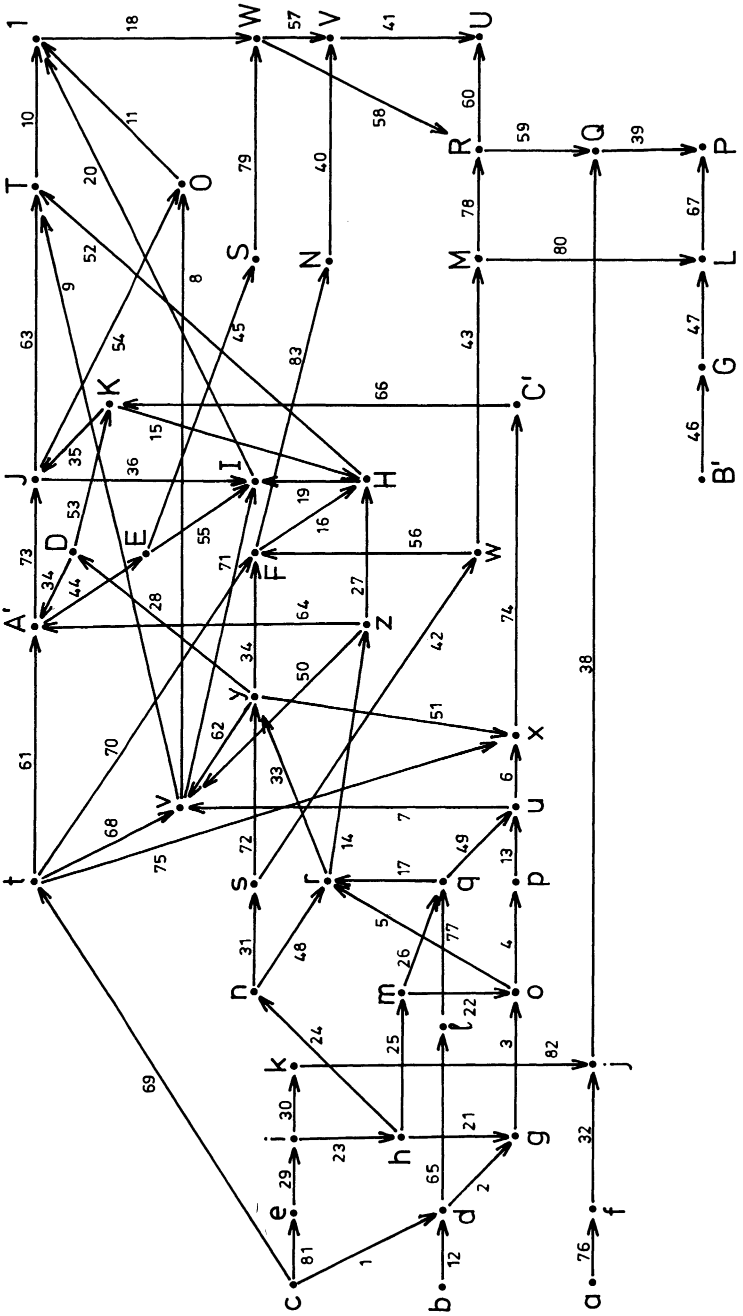
$$w = \frac{16}{81}(\Sigma \sin^2 A)^2, \quad x = \frac{2\sqrt{3}}{9}\,\Sigma \sin A = \frac{8\sqrt{3}}{9}\,\Pi \cos\frac{A}{2},$$

$$y = \frac{4}{27}(\Sigma \sin A)^2 = \frac{64}{27}\,\Pi \cos^2\frac{A}{2}, \quad z = \frac{4}{9}(\Sigma \cos A)^2,$$

$$A' = \frac{16}{27}\,\Sigma \cos^2\frac{B}{2}\cos^2\frac{C}{2}, \quad B' = \frac{4}{3}\,\Sigma \cos 2B \cos 2C,$$



Bager's First Graph



Bager's Second Graph

$$C' = \frac{1}{3}\sum \cos\frac{B-C}{2}, \quad D = \frac{16}{81}\left(\sum \cos\frac{B}{2}\cos\frac{C}{2}\right)^2,$$

$$E = \frac{16}{81}\left(\sum \cos^2\frac{A}{2}\right)^2, \quad F = \frac{4}{9}\sum \sin^2 A, \quad G = \frac{4}{9}\left(\sum \cos 2A\right)^2,$$

$$H = \frac{4}{9}\left(\sum \sin\frac{A}{2}\right)^2, \quad I = \frac{4}{9}\sum \cos^2\frac{A}{2}, \quad J = \frac{4}{27}\left(\sum \cos\frac{A}{2}\right)^2,$$

$$K = \frac{4}{9}\sum \cos\frac{B}{2}\cos\frac{C}{2}, \quad L = \frac{4}{3}\sum \cos^2 2A, \quad M = \frac{16}{27}\sum \sin^4 A,$$

$$N = \frac{16}{3}\sum \sin^2\frac{B}{2}\sin^2\frac{C}{2}, \quad O = \frac{2\sqrt{3}}{9}\sum \cos\frac{A}{2},$$

$$P = \frac{16}{3}\sum \cos^4 A, \quad Q = \frac{16}{9}\left(\sum \cos^2 A\right)^2, \quad R = \frac{4}{3}\sum \cos^2 A,$$

$$S = \frac{16}{27}\sum \cos^4\frac{A}{2}, \quad T = \frac{2}{3}\sum \sin\frac{A}{2}, \quad U = \frac{16}{3}\sum \sin^4\frac{A}{2},$$

$$V = \frac{16}{9}\left(\sum \sin^2\frac{A}{2}\right)^2, \quad W = \frac{4}{3}\sum \sin^2\frac{A}{2}.$$

Remark. Note that using Bager's graphs we can directly obtain inequalities from [8-15].

References

1. A. Bager's, 'A Family of Goniometric Inequalities', Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 338-352 (1971), 5-25.
2. A. Bager, 'Another Family of Goniometric Inequalities', Ibid. No. 412-460 (1973), 207-216.
3. O. Bottema, 'Inequalities for R, r and s', Ibid. No. 338-352 (1971), 27-36.
4. E. Makai, Jr., 'A Goniometric Inequality', Ibid. No. 381-409 (1972), 139-142.
5. L. Bankoff, 'A Comparison of Two Trigonometric Inequalities', Ibid. No. 247-273 (1969), 51-52.
6. J. Steinig, 'Some Trigonometric Inequalities', Elem. Math. 27 (1972), 121-129.
7. O. Bottema, R. Ž. Djordjević, R. R. Janić, D. S. Mitrinović, and P. M. Vasić, Geometric Inequalities, Groningen, 1969.
8. W. J. Blundon, Math. Mag. 43 (1970), 48-50.
9. L. Bankoff, 'Problem 269', Pi Mu Epsilon J. 5 (1971), 242.
10. L. Carlitz and E. Neuman, 'Problem 270', Ibid. 5 (1972), 296; and 5 (1973), 437-438.
11. L. Bankoff, Math. Mag. 46 (1973), 103.
12. V. N. Murty, G. Tsintsifas, and M. S. Klamkin, 'Problem 544', Crux Math. 6 (1980), 153; 7 (1981), 150-153.
13. J. Garfunkel, 'Problem 974', Ibid. 10 (1984), 262.
14. M. S. Klamkin and R. L. Young, 'Problem 253', Coll. Math. J. 16 (1985), 224-225.

15. Ž. Mijalković and Š. Arslanagić, 'Problem 48', Matematika 1 (1978), 64-65.

2.2. Miscellaneous Inequalities for the Angles of a Triangle

2.2.1. $\Pi(1 - \cos A) \geqslant \Pi \cos A.$ {E}

Proof. (M. S. Klamkin) It is known that

$$IH^2 = 2r^2 - 4R^2\Pi \cos A = 4R^2(\Pi(1 - \cos A) - \Pi \cos A).$$

Since $IH^2 \geqslant 0$, the above inequality of Bager follows, with equality just when I and H coincide, that is just when the triangle is equilateral.

A. Bager, 'A Family of Goniometric Inequalities', Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 338-352 (1971), 5-25.

V. N. Murty, G. Tsintsifas, and M. S. Klamkin, 'Problem 544', Crux Math. 6 (1980), 153; 7 (1981), 150-153.

2.2.2. $\Pi(1 + \cos 2A) + \Pi \cos 2A \geqslant 0$

with equality for equilateral triangles or for isosceles right triangles.

This is an inequality of Bottema. Now, we shall give a proof from 'Problem 836', Crux Math., where it was shown that 2.2.2. could be proved by using 2.2.1.

Proof. (M. S. Klamkin) We assume without loss of generality that $A \geqslant B \geqslant C$. If $A = \pi/2$, then Bottema's inequality reduces to

$$\frac{1}{2}(1 - \cos 2(B - C)) \geqslant 0,$$

and this clearly holds, with equality just when $B = C = \pi/4$. If $A < \pi/2$, then the angles of the orthic triangle are $\pi - 2A$, $\pi - 2B$, $\pi - 2C$; and Bottema's inequality results if we apply inequality 2.2.1. to the orthic triangle. If $A > \pi/2$, then the angles of the orthic triangle are $2A - \pi$, $2B$, $2C$. If we now apply Bager's inquality 2.2.1. to the orthic triangle, the result is

$$(1 + \cos 2A)(1 - \cos 2B)(1 - \cos 2C) \geqslant -\cos 2A \cos 2B \cos 2C,$$

and Bottema's inequality will follow if

$$(1 + \cos 2B)(1 + \cos 2C) \geqslant (1 - \cos 2B)(1 - \cos 2C),$$

or, equivalently, if

$$\cos 2B + \cos 2C = 2 \cos(B + C) \cos(B - C) \geqslant 0.$$

This is clearly true, since both $B + C$ and $B - C$ are less than $\pi/2$.

Now, we shall show that Bager's inequality follows from Bottema's inequality, too. Let ABC be any triangle. Then $A' = \frac{(\pi - A)}{2}$, $B' = \frac{(\pi - B)}{2}$, $C' = \frac{(\pi - C)}{2}$ are also the angles of a triangle, and Bager's inequality results if we apply Bottema's inequality to triangle A'B'C'. Note however that, even though inequalities 2.2.1. and 2.2.2. are equivalent, the corresponding equalities are not, for equality holds in 2.2.2. but not in 2.2.1. when ABC is an isosceles right triangle.

Generalization. C. Cooper (see Editor's comments in Problem 836) gave the following generalization of Bottema's inequality:

$$\Pi(1 + \cos 2^n A) + \Pi \cos 2^n A \geqslant 0.$$

Proof. (N. S. Mendelson). Proceed inductively. If $A \geqslant B \geqslant C$ are the angles of a triangle, then either

$$\pi - 2A,\ \pi - 2B,\ \pi - 2C \quad \text{or} \quad 2A - \pi,\ 2B,\ 2C$$

are angles of a triangle. Replacing A, B, C in the inequality 2.2.2. by either of these sets increases the value of n by 1. The only thing to note is that

$$\cos 2^n(\pi - 2X) = \cos 2^n(2X - \pi) = \cos 2^{n+1}X, \quad n = 1, 2, 3, \ldots$$

O. Bottema, 'A New Inequality for the Angles of a Triangle', Crux Math. 8 (1982), 296-297.
V. N. Murty, M. S. Klamkin, and N. S. Mendelson, 'Problem 836', Crux Math. 9 (1983), 113; 10 (1984), 228-229 and 320.

2.2.3. $3\Sigma \sin^2 A - 2\Sigma \cos^3 A \leqslant 6.$

E. Just, B. Kabak, L. H. Cairoli, and M. S. Klamkin, 'Problem 394', Pi Mu Epsilon J. 6 (1977), 366; 6 (1978), 493-495.

2.2.4. $(\Sigma \sin A)(\Sigma \sin A + 8\Pi \sin A) \geqslant 4(\Sigma \sin B \sin C)^2.$

This inequality is due to S. J. Bilčev.

2.2.5. $\Sigma \sin^3 A \leqslant (\Sigma \sin A)(\Sigma \sin^2 \frac{A}{2}).$ {E}

A. Bager and H. Frischknecht, 'Aufgabe 672', Elem. Math. 27 (1972), 68, and 28 (1973), 75.

2.2.6. $\Sigma(\sin 3A - \sin 2A + \sin A) \geqslant 0.$ {E}

E. Braune and H. Frischknecht, 'Aufgabe 716', Elem. Math. 29 (1974), 52, and 30 (1975), 43.

2.2.7. $\Sigma \cos B \cos^2 C \leqslant \frac{1}{2} - \Pi \cos A,$

$$\Sigma \sin \frac{B}{2} \sin^2 \frac{C}{2} \leqslant \frac{1}{2} - \Pi \sin \frac{A}{2}.$$

I. Paasche, 'Aufgabe 250', Mat. Vesnik 10 (25) (1973), 209.

2.2.8. $\sum \tan^2 \frac{A}{2} \geqslant 2 - 8\prod \sin \frac{A}{2}$. {E}

J. Garfunkel and L. Bankoff, 'Problem 825', Crux Math. 9 (1983), 79 and 10 (1984), 168.

2.2.9. $2\sum \cos A + \sum \tan^2 \frac{A}{2} \geqslant 4$. {E}

A. Bager and L. Bankoff, 'Aufgabe 671', Elem. Math. 27 (1972), 68 and 28 (1973), 74.

2.2.10. $4\sum \tan \frac{A}{2} \leqslant \sqrt{3} + \sum \operatorname{cotan} \frac{A}{2} \leqslant 2\sum \operatorname{cosec} A$. {E}

A. Bager and P. Hohler, 'Aufgabe 675', Elem. Math. 27 (1973), 95 28 (1973), 100.

2.2.11. $2\sum \sin \frac{B}{2} \sin \frac{C}{2} \leqslant \sqrt{\sum \sin B \sin C} \leqslant \sum \cos A \leqslant \sum \sin \frac{A}{2}$.

These inequalities are due to M. S. Klamkin.
V. N. Murty, G. Tsintsifas, and M. S. Klamkin, 'Problem 544', Crux Math. 6 (1980), 153 and 7 (1981), 150-153.

2.2.12. $8\prod \sin A \leqslant \prod(\sin B + \sin C) \leqslant 2\sum \sin A = 8\prod \cos \frac{A}{2}$.

A. Viorel, 'On an Inequality' (Romanian), Gaz. Mat. (Bucharest) B 19 (1968), 336.
D. Milošević and S. Srećković, 'Problem 54', Matematika 1 (1979), 74-76.

2.2.13. $\frac{4}{9} \sum \sin B \sin C \leqslant \prod \cos \frac{B - C}{2} \leqslant \frac{2}{3} \sum \cos A$. {E}

J. Garfunkel, G. Tsintsifas, and V. N. Murty, 'Problem 768', Crux Math. 8 (1982), 210 and 9 (1983), 282-283.

2.2.14. $\sum \operatorname{cosec} \frac{A}{2} \cos \frac{B - C}{2} \geqslant 6$.

J. Garfunkel and M. S. Klamkin, 'Problem 585', Crux Math. 6 (1980), 284 and 7 (1981), 303-304.

2.2.15. $\frac{3\sqrt{3}}{2} \leqslant \sum \frac{\cos \frac{C}{2}}{\cos \frac{A - B}{2}} \leqslant \frac{1}{2} \prod \operatorname{cotan} \frac{A}{2}$. {E}

These inequalities are due to W. Janous.

2.2.16. $\sum \frac{\sin A}{(\sin B + \sin C)^2} < \frac{3}{4} \prod \operatorname{cosec} A$.

L. Constantinescu, 'Problem 17184', Gaz. Mat. (Bucharest) 83 (1978), 211.

2.2.17. $$\sum \frac{\tan \frac{B}{2} + \tan \frac{C}{2}}{\sin B + \sin C} \geqslant 2.$$

D. Miloševic and S. Srećković, The same reference as in 2.2.12.

2.2.18. $$\frac{3}{2} \leqslant \sum \frac{\cos \frac{B+C}{2}}{\cos \frac{B-C}{2}} < 2.$$

Remark. W. Janous noted that these inequalities are equivalent to GI 1.16.

C. Popa, 'Problem 18218', Gaz. Mat. (Bucharest) 85 (1980), 163.

2.2.19. $$\sum \frac{\cos A}{\cos(B - C)} \geqslant \frac{3}{2}.$$

V. Stoican, 'Problem 9720', Gaz. Mat. (Bucharest) B 20 (1969), 441 and B 21 (1970), 148.

2.2.20. $$(\sum \sin \frac{A}{2})(\sum \operatorname{cotan} \frac{A}{2}) \geqslant \frac{9\sqrt{3}}{2}.$$

D. Mavlo, Matematika (Sofija) 1985, No. 10, 48.

2.2.21. $$-3\sqrt{3}/8 < \sum \sin(B - C) \cos^3 A < 3\sqrt{3}/8.$$

M. S. Klamkin, 'Aufgabe 941', Elem. Math. 41 (1986), 78.

2.2.22. If, in a triangle, we have $a > b > c$ or $b > c > a$, or $c > a > b$, then

$$\prod \left(\frac{\sin B}{\sin C}\right)^{\cos A} < 1.$$

H. W. Segar, 'Problem 10615', Math. Questions 59 (1893), 93-94.

2.2.23. If $\lambda \in (-\infty, -1] \cup [0, +\infty))$, then

$$\sum \operatorname{cotan}^\lambda \frac{A}{2} \geqslant 3^{(\lambda+2)/2}. \quad \{E\}$$

Proof. Using the arithmetic-geometric mean inequality and GI 2.41 i.e. $\prod \operatorname{cotan} \frac{A}{2} \geqslant 3\sqrt{3}$, we get for $\lambda \in [0, +\infty)$

$$\sum \operatorname{cotan}^\lambda \frac{A}{2} \geqslant 3(\prod \operatorname{cotan} \frac{A}{2})^{\lambda/3} \geqslant 3^{(\lambda+2)/2}.$$

Using the inequality for means of order λ (< -1) and -1, and GI

2.33, i.e. $\Sigma \tan \frac{A}{2} \geqslant \sqrt{3}$, we get for $\lambda \in (-\infty, -1]$

$$\Sigma \cotan^{\lambda} \frac{A}{2} \geqslant 3(\frac{1}{3} \Sigma \cotan^{-1} \frac{A}{2})^{-\lambda} \geqslant 3(\sqrt{3})^{\lambda} = 3^{(\lambda+2)/2}.$$

In both cases equality holds if and only if the triangle is equilateral.

Remark. The above result is a generalization of GI 2.35, 2.36, 2.41 and 2.43. The case $\lambda = 0, \pm 1, \pm 2, \ldots,$ is given by V. F. Zvezdin.

V. F. Zvezdin, 'Problem 488', Mat. v Škole 1968, No. 3, 65 and 1969, No. 1, 77.

2.2.24. $\sin \frac{\pi}{n} < \Sigma \sin \frac{A}{n} < \frac{\pi}{n}$ $(n \in R,\ n \geqslant 1)$.

I. Bursuc, 'Problem 18378', Gaz. Mat. (Bucharest) 85 (1980), 365.

2.2.25. $-\frac{1}{2} < \Sigma \frac{\sin A}{(B - A)(C - A)} < 0$.

S. Găină, 'Metoda functiilor convexe', Gaz. Mat. (Bucharest) 85 (1980), 245.

2.2.26. $\sqrt{3} + 5\Sigma \cotan A \geqslant 3\Sigma \cosec A$. {E}

We were informed of this result, which is better than GI 2.62, by W. Janous.

2.2.27. $\sin^2 A + \sin B \sin C \leqslant \frac{25}{16}$.

S. Berkolajko, Kvant 1980, No. 1, 9.

2.2.28. Let m be a positive real number. Then

$$\sin \frac{A - B}{2} + \sin \frac{A - C}{2} + \frac{1}{m} \sin \frac{3A}{2} \leqslant \frac{m^2 + 2}{2m}.$$

This is a result of E. Mutu. For $m = 1$ we get a result of C. Ionescu-Ţiu.

Proof. (W. Janous) Let

$$t = \sin \frac{A - B}{2} + \sin \frac{A - C}{2} = 2 \sin \frac{2A - B - C}{4} \cos \frac{C - B}{4} =$$

$$= 2 \sin \frac{3A - \pi}{4} \cos \frac{C - B}{4}.$$

Since $m > 0$, the desired inequality becomes

$$y(m) = m^2 - 2mt + 2(1 - \sin \frac{3A}{2}) \geqslant 0.$$

The function $m \to y(m)$ has its minimum for $m_o = t$, i.e.

$$y(t) = -t^2 + 2\left(1 - \sin\frac{3A}{2}\right) =$$

$$= 2\left(1 - \sin\frac{3A}{2} - 2\sin^2\frac{3A - \pi}{4}\cos^2\frac{C - B}{4}\right) \geqslant$$

$$\geqslant 2\left(1 - \sin\frac{3A}{2} - 2\sin^2\frac{3A - \pi}{4}\right) =$$

$$= 2\left(\cos\left(\frac{3A}{2} - \frac{\pi}{2}\right) - \sin\frac{3A}{2}\right) = 0.$$

Remark. Of course, $y(t) = -\Delta/4$, where Δ is the corresponding discriminant, therefore $\Delta \leqslant 0$, and $y(m) \geqslant 0$.

E. Mutu, 'Problem 18008', Gaz. Mat. (Bucharest) 12 (1970), 728.

2.2.29. Let m be a real number. Then

$$m(m - 2)\sin\frac{A}{2} + \cos\frac{B - C}{2} \geqslant 0.$$

Proof. Let us consider the quadratic inequality

$$m^2\sin\frac{A}{2} - 2m\sin\frac{A}{2} + \cos\frac{B + C}{2} \geqslant 0.$$

Since the discriminant $\Delta = 4\sin\frac{A}{2}\left(-2\sin\frac{B}{2}\sin\frac{C}{2}\right) \leqslant 0$, the inequality is true for every $m \in R$.

2.2.30. $\sin^2 B + \sin^2 C \leqslant 1 + 2\sin B\sin C\cos A.$

D. Andrica, 'Problem 16048', Gaz. Math. (Bucharest) 81 (1976), 337.

2.2.31. $\operatorname{cosec} A + \operatorname{cosec} B \geqslant 8/(3 + 2\cos C).$

Ž. Mijalković, 'Jedna nejednakost o trouglu', Matematika 2 (1978), 67-68.

2.2.32. Let $0 < r, s, t \leqslant 1$. Then

$$0 < \sin rA\sin sB\sin tC \leqslant \sin rA_0\sin sB_0\sin tC_0,$$

where A_0, B_0, C_0 is the unique solution of

$$r\cot rA = s\cot sB = t\cot tC.$$

This result is due to W. Janous and it is a generalization of GI 2.10, 2.11 and 2.13.

2.2.33. The following results are valid:

$$0 < t \leqslant 1/2 \Rightarrow 3 \sin^2 \frac{t\pi}{3} \leqslant \Sigma \sin^2 tA < \sin^2 t\pi;$$

$$1/2 < t \leqslant \frac{3}{\pi} \arcsin \frac{1}{2} \sqrt{3 - \sqrt{3}} \Rightarrow 3 \sin^2 \frac{t\pi}{3} \leqslant \Sigma \sin^2 tA < < 2 \sin^2 \frac{t\pi}{2};$$

$$\frac{3}{\pi} \arcsin \frac{1}{2} \sqrt{3 - \sqrt{3}} < t < \frac{3}{\pi} \arccos \frac{\sqrt{33} - 1}{8} \Rightarrow$$

$$\sin^2 t\pi < \Sigma \sin^2 tA < 2 \sin^2 \frac{t\pi}{2};$$

$$\frac{3}{\pi} \arccos \frac{\sqrt{33} - 1}{8} \leqslant t \leqslant 1 \Rightarrow \sin^2 t\pi < \Sigma \sin^2 tA \leqslant 3 \sin^2 \frac{t\pi}{3}.$$

Remarks. 1° $\frac{3}{\pi} \arcsin \frac{1}{2} \sqrt{3 - \sqrt{3}} = 0.571077...$

$$\frac{3}{\pi} \arccos \frac{\sqrt{33} - 1}{8} = 0.8937467...$$

2° Using $\Sigma \cos^2 tA = 3 - \Sigma \sin^2 tA$ and $\Sigma \cos 2tA = 3 - 2(\Sigma \sin^2 tA)$ inequalities are easily obtained for $\Sigma \cos^2 tA$, $0 < t \leqslant 1$, and for $\Sigma \cos tA$, $0 < t \leqslant 2$.

These results are due to W. Janous, and they are generalizations of GI 2.3, 2.14, 2.21, 2.29 and 2.16.

2.2.34. $3/4 \leqslant \Sigma \cos A - \Sigma \cos B \cos C < 2$ in each triangle,

$3/4 \leqslant \Sigma \cos A - \Sigma \cos B \cos C < 1$ in acute triangle,

$(2\sqrt{2} - 1)/2 < \Sigma \cos A - \Sigma \cos B \cos C < 2$ in obtuse triangles.

These results are due to W. Janous, and they are improvements of (IV.1.12).

2.2.35. $3/4 \leqslant \Sigma \sin \frac{A}{2} - \Sigma \sin \frac{B}{2} \sin \frac{C}{2} < 1$ in each triangle,

$3/4 \leqslant \Sigma \sin \frac{A}{2} - \Sigma \sin \frac{B}{2} \sin \frac{C}{2} < (2\sqrt{2} - 1)/2$ in acute triangles,

$\sqrt{2 - \sqrt{2}}(1 - 1/\sqrt{2}) + (3\sqrt{2} - 2)/4 < \Sigma \sin \frac{A}{2} - \Sigma \sin \frac{B}{2} \sin \frac{C}{2} < 1$

in obtuse triangles.

W. Janous, 'Problem 1154', Crux Math. 12 (1986), 139.

2.2.36. $3 \sin(2\lambda\pi/3) \leqslant \Sigma(\tan \lambda B + \tan \lambda C)/(1 + \tan \lambda B \tan \lambda C) <$

$$< 2 \tan \lambda\pi,$$

where $0 < \lambda < 1/2$.

This is a Janous' generalization of a Stocker's problem.

Hj. Stocker and W. Janous, 'Aufgabe 923', Elem. Math. 41 (1986), 74-75.

2.2.37. Let $0 < \lambda \leq 1/2$. Then

$$\Sigma \cos \lambda A \geq 2 + \sqrt{2} \cos(\lambda\pi + \frac{\pi}{4}) + \Sigma \sin \lambda A.$$

This result is due to W. Janous. For $\lambda = 1/2$, we have a result from: M. S. Klamkin, 'Problem E3180', Amer. Math. Monthly 93 (1986), 812.

2.2.38. It is known that $k = (\log 9 - \log 4)/(\log 4 - \log 3)$ is maximal such that (see VI.1.5):

$$M_k : = M_k(\sin A, \sin B, \sin C) \leq \sqrt{3}/2.$$

(a)* Determine for $p > k$ the least value $m(p)$ such that $M_p \underset{(=)}{<} m(p)$.

(b) Let $0 < t \leq 1$ and $k \leq 1$. Then

(1) $$M_k(\sin tA, \sin tB, \sin tC) \leq \sin(t\pi/3).$$

(c)* Find the maximal $k = k(t)$ such that (1) is valid (From (b) it follows $k \geq 1$).

This result is due to W. Janous; (b) is a generalization of GI 2.6. and (a) and (c) are conjectures.

2.2.39. (i) If $k \leq 3$, then

(1) $$M_k(\sin A \sin B, \sin B \sin C, \sin C \sin A) \leq 3/4.$$

(ii) If $3 < k \leq 4$, then

(2) $$M_k(\sin A \sin B, \sin B \sin C, \sin C \sin A) \leq 3^{-1/4}.$$

(iii) If $A \leq B \leq C$ and $\cot^2 \frac{A}{2} \geq 7$, then for $k \leq 1 + \cot^2 \frac{A}{2}$ (1) is also valid.

Remark. (i) for $k = 1, 2, 3$ and (ii) for $k = 4$ were proved by V. Vâjâitu. J. E. Pečarić and B. Crstici noted that a simple extension of his proof gives the above result.

2.2.40. $$\Sigma BC \log A \leq \frac{\pi^2}{3} \log \frac{\pi}{3} \quad (A, B, C \text{ in radians}).$$

We were informed of this result by W. Janous.

2.2.41. If $A \leq B, C$, then

$$\frac{A}{2} \leqslant \text{arc cotan}(\sqrt{3} - \frac{9}{\pi} + \Sigma \frac{1}{A}).$$

This result is due to V. Mascioni.

2.2.42. $\frac{2}{\sqrt{3}} \Sigma \sin A \leqslant \Sigma \cos \frac{B - C}{2} \leqslant \frac{2}{\sqrt{3}} \Sigma \cos \frac{A}{2}$. {E}

J. Garfunkel, 'Problem 1083', Crux Math. 11 (1985), 288.

Proof and comments by C. Tănăsescu. We first perform some trivial transformations, after assuming, with no loss of generality, $A \geqslant B \geqslant C$, $\Sigma \cos \frac{B - C}{2} = \cos \frac{A - B}{2} + 2 \cos \frac{A - B}{4} \cos \frac{\pi - 3C}{4}$, $\Sigma \cos \frac{A}{2} = \cos \frac{C}{2} + 2 \cos \frac{A - B}{4} \cos \frac{\pi - C}{4}$, $\Sigma \sin A = \sin C + 2 \cos \frac{A - B}{2} \cos \frac{C}{2}$. So, by putting $t = \cos((A - B)/4) \in (1/\sqrt{2}, 1]$, the left and the right inequality, respectively, become

(1) $$h(t;C) = (\frac{8}{\sqrt{3}} \cos \frac{C}{2})t^2 - 2 \cos \frac{\pi - 3C}{4} t + (1 - \frac{4}{\sqrt{3}} \cos \frac{C}{2} + \frac{2}{\sqrt{3}} \sin C) \leqslant 0,$$

(2) $$g(t;C) = 2t^2 + 2(\cos \frac{\pi - 3C}{4} - \frac{2}{\sqrt{3}} \cos \frac{\pi - C}{4})t - 1 - \frac{2}{\sqrt{3}} \cos \frac{C}{2} \leqslant 0.$$

Since $C \in [0, \pi/3]$ and $h(0;C)$, $g(0;C) \leqslant 0$ $(\frac{8}{\sqrt{3}} \cos \frac{C}{2} > 2)$, it suffices to check that

$$h(1;C) \leqslant 0, \quad g(1;C) \leqslant 0,$$

moreover, since $h(1;\pi/3) = g(1;\pi/3) = 0$, it finally suffices to show that $\frac{dh(1;C)}{dC} > 0$, $\frac{dg(1;C)}{dC} > 0$, $C \in [0, \pi/3)$. And this is true. Indeed, we have:

$$\frac{dg(1;C)}{dC} = (6 - \frac{4}{\sqrt{3}}) \sin \frac{\pi - 3C}{4} + \frac{4}{\sqrt{3}}(\sin \frac{\pi - 3C}{4} + \sin \frac{2C}{4} - \sin \frac{\pi - C}{4}) > 0,$$

simply because $\sin(\alpha + \beta) \leqslant \sin \alpha + \sin \beta$ on $[0, \pi/4]$; further,

$$\frac{dh(1;C)}{dC} = \frac{2}{\sqrt{3}}(\cos C - \cos \frac{\pi - C}{2}) - \frac{3}{2} \sin \frac{\pi - 3C}{4} = \sin \frac{\pi - 3C}{4}(\frac{4}{\sqrt{3}} \sin \frac{\pi + C}{4} - \frac{3}{2}) > 0,$$

because $\sin \frac{\pi + C}{4} \geqslant \sin \frac{\pi}{4} > \frac{3\sqrt{3}}{8}$, on $[0, \pi/3)$.

Hence, both inequalities are strict for $C \in [0, \pi/3)$ and become equalities if and only if $C = \pi/3$, i.e. $A = B = C = \pi/3$, ABC is equilateral.

Remark 1. An almost similar device, i.e. the study of some appropriate second degree function, is used in the solution of the following, apparently insignificant problem: prove that for any non-degenerate triangle the following holds:

(3) $$1 - 8\Pi \sin\frac{A}{2} \geqslant \frac{2\sin B \sin C}{\cos\frac{B-C}{2}}\left(1 - \cos\frac{B-C}{2}\right). \quad \{E\}$$

Set $t = \cos\frac{B-C}{2} \in (0, 1]$, $x = \sin(A/2) \in (0, 1)$ and (3) reads:

$$f(x;t) = 2(1 + t)x^2 - 4t^2x + (t + 2t^2 - 2t^3) \geqslant 0,$$

where, for $t \in (0, 1]$,

$$\min_{x \in R} f(x;t) = f\left(\frac{t^2}{1 + t};t\right) = \frac{t(1 - t)(t + 3)}{t + 1} \geqslant 0.$$

So, the transformed inequality holds, with equality if and only if $t = 1$, $x = \frac{t^2}{1 + t} = \frac{1}{2}$, i.e. (3) holds, with equality if, and only if, $B = C$, $A = \pi/3$, that means only for equilateral triangles.

But (3) is not so insignificant as it appears because it is the goniometric transcription of a very interesting (and strong) linear inequality, namely

(4) $$R - 2r \geqslant w_a - h_a$$

(see also IX.11.19), due to L. Panaitopol. We are not able to give a viable geometric interpretation to (4): possibly that would involve the Euler nine points circle, or Feuerbach points, etc.

Remark 2. Although (1) and (2) are of great interest and beauty when written in goniometric form, they are also equivalent to some powerful and subtle metric inequalities, as one can see below.

First, since $\cos\frac{B-C}{2} = \frac{h_a}{w_a}$, (1) reads, succesively,

(5) $$\frac{2s}{R\sqrt{3}} \leqslant \Sigma\frac{h_a}{w_a} \quad \{E\} \quad \Rightarrow \frac{2s}{R\sqrt{3}} \leqslant \Sigma\frac{2F}{aw_a} \Rightarrow$$

(6) $$(Rr\sqrt{3})^{-1} \leqslant \Sigma(aw_a)^{-1} \quad \text{or} \quad \frac{4s}{abc\sqrt{3}} \leqslant \Sigma\frac{1}{aw_a} \text{ (see also IX.11.9)} \Rightarrow$$

$$\Rightarrow (Rr\sqrt{3})^{-1} \leqslant \Sigma(b + c)/(2abc\cos\frac{A}{2}) \Rightarrow$$

(7) $$\frac{8s}{\sqrt{3}} \leqslant \Sigma \frac{b + c}{\cos \frac{A}{2}} \quad \{E\},$$ (see also IX.4.10 for p = 1).

And now some trivialities: (5) also reads as

$$4s/\sqrt{3} \leqslant \Sigma bc/w_a \quad \{E\}; \qquad 2s/R\sqrt{3} \leqslant \Sigma h_1/w_a \quad \{E\},$$

where (h_1, h_2, h_3) is some permutation of (h_a, h_b, h_c).

From: $\Sigma \frac{h_a}{w_a} \leqslant \frac{2}{\sqrt{3}} \Sigma \cos \frac{A}{2}$ {E}, we get the following <u>conjecture</u>:

$$\Sigma \frac{h_1}{w_a} \leqslant \frac{2}{\sqrt{3}} \Sigma \cos \frac{A}{2}.$$

For isosceles triangles it is highly plausible and, in this problem, to be isosceles is not a drastic restriction.

Finally, we shall note that the following result, i.e. 2.2.43 is a simple consequence of the first inequality in 2.2.42 (put $A \to \frac{\pi}{2} - \frac{A}{2}$, etc.).

2.2.43. $$\Sigma \cos \frac{A}{2} \leqslant \frac{\sqrt{3}}{2} \Sigma \cos \frac{B - C}{4}.$$

J. T. Groenman, 'Problem 1152', Crux Math. 12 (1986), 138.

2.2.44. $$\Sigma((\sqrt{2} + 1) \cos \frac{A}{8} - \sin \frac{A}{8})^{-1} \geqslant \sqrt{6 - 3\sqrt{2}}.$$

S. Bilčev, 'Problem 1158', Crux Math. 12 (1986), 140.

2.2.45. $$\frac{\cos \frac{A}{4} + \sin \frac{A}{4}}{\cos \frac{B - C}{4}} < \frac{\cos \frac{A}{4} - \sin \frac{A}{4}}{\sin |\frac{B - C}{4}|} < \frac{2 \cos \frac{A}{4} - (2 \sin \frac{A}{2})^{1/2}}{\cos \frac{B - C}{4} + \sin |\frac{B - C}{4}| - (\cos \frac{B - C}{2})^{1/2}}.$$

<u>Proof</u>. (B. Crstici). Let $B > C$. Put $A/4 = x$, $(B - C)/4 = y$. The first inequality becomes $\sin(x + y) < \cos(x + y)$, what is true since $x + y < \pi/4$. The second inequality becomes

$$\frac{\cos x - \sin x}{\sin y} < \frac{(4 \cos^2 x - 2 \sin 2x)(\cos y + \sin y + \sqrt{\cos 2y})}{(1 + \sin 2y - \cos 2y)(2 \cos x + \sqrt{2} \sin 2x)}$$

i.e.

(1) $\sqrt{\sin x}(\sin y + \cos y) < \sqrt{\cos x}\cdot\sqrt{\cos 2y}.$

We noted that $x + y < \pi/4$. Also, we have $x + 2y < \pi/2$. Hence

$$\sin x < \sin(\frac{\pi}{2} - (x + 2y)),$$

i.e.

$$\sin x < \cos(x + 2y),$$

i.e.

$$\sin x(\sin y + \cos y)^2 < \sin x \cos 2y,$$

i.e. (1).

C. Ionescu-Ţiu, 'Problem 10465', Gaz. Mat. (Bucharest) B 21 (1970), 421.

2.2.46. $\Pi \frac{(\pi - A)^2}{A} \Pi \frac{\sin A}{\cos^2 \frac{A}{2}} \leqslant \left(\frac{8\pi}{3\sqrt{3}}\right)^3.$ {E}

2.2.47. $\Sigma \sin(\frac{A}{2} + B) > \Sigma \sin A.$

'Problem F 2600', Köz. Mat. Lapok 36 (1986), 319.

3. Inequalities with (R, r, s)

3.1. $\frac{R}{r} + \frac{r}{R} \geqslant \frac{5}{2}.$ {E}

L. Bankoff, 'Problem Q 417', Math. Mag. 40 (1967), 289.

3.2. $(\frac{r}{R})^t + (\frac{s}{R})^t \leqslant \frac{1 + 3^{3t/2}}{2^t}$ $(t > 0)$. {E}

This generalization of Onofras' inequality is given by I. Tomescu (see also paper of F. Fanaca, where many known inequalities are proved by using Jensen's inequality for convex functions).

E. Onofras, 'Problem 18300', Gaz. Mat. (Bucharest) 85 (1980), 266 and 86 (1981), 31-32.

I. Tomescu, 'Problem C 69', Gaz. Mat. (Bucharest) 85 (1980), 481.

F. Fanaca, 'Cîteva demontraţii ale inegalităţii lui Euler cu ajutorul inegalitaţilor algebrice, Aplicaţii'. Gaz. Mat. (Bucharest) 87 (1982), 16-20.

3.3. If $d = rs/R$, then

$$\frac{4d^2}{3\sqrt{3}} \leqslant 2dr \leqslant 3r^2\sqrt{3} \leqslant \frac{2ds}{3\sqrt{3}} \leqslant rs \leqslant \frac{3Rr\sqrt{3}}{2} \leqslant \frac{s^2}{3\sqrt{3}} \leqslant \frac{sR}{2} \leqslant \frac{3R^2\sqrt{3}}{4}.$$

K. Schuler, Praxis der Math. 9 (1967), 344.

3.4. If $d = rs/R$, then

$$9\sqrt{3}r - s \leqslant 4d \leqslant 6\sqrt{3}r \leqslant 2s \leqslant 3\sqrt{3}R.$$

I. Paasche, 'Problem 165', Mat. Vesnik 10 (25) (1973), 97.

3.5.
$$\frac{3r(4R + r)}{(7R - 5r)^2} \leqslant \frac{r}{2R - r} \leqslant \frac{3r}{4R + r} \leqslant \frac{r(4R + r)}{(2R - r)(2R + 5r)} \leqslant$$
$$\leqslant \frac{r(16R + 3r)}{(4R - r)(4R + 7r)} \leqslant \frac{r}{R + r} \leqslant \frac{r(16R - 5r)}{(4R + r)^2} \leqslant$$
$$\leqslant \frac{4r(12R^2 - 11Rr + r^2)}{(3R - 2r)(4R + r)^2} \leqslant \left(\frac{s}{4R + r}\right)^2 \leqslant \frac{R}{2(2R - r)} \leqslant$$
$$\leqslant \frac{4R^2 + 4Rr + 3r^2}{(4R + r)^2} \leqslant \frac{1}{3} \leqslant \frac{4R + r}{27r} \leqslant \frac{R^2}{4r(R + r)}.$$

This result, due to S. J. Bilčev, is an extension and interpolation of results from:

A. Bager and O. Reutter, 'Aufgabe 688', Elem. Math. 28 (1973), 20 and 29 (1974), 18-19.

A. Bager, Private communication.

Remark. In his solution of Aufgabe 688, L. Bankoff gave the following inequalities

$$2r(4R + r)^2/(2R - r) \leqslant 2r(4R + r)^2/(R + r) \leqslant$$
$$\leqslant 2r(16R - 5r) \leqslant 2s^2.$$

3.6. $$2s^2 \geqslant r(20R - r + \sqrt{3(12R + r)(4R - 5r)}) \geqslant 32Rr - 10r^2. \quad \{E\}$$

A. Bager and O. Reutter, 'Aufgabe 690', Elem. Math. 28 (1973), 48 and 29 (1974), 46-47.

3.7. If $x = r/R$ and $y = s/R$, then

$$y \geqslant \sqrt{x}(\sqrt{6} + \sqrt{2 - x}). \quad \{E\}$$

V. N. Murty and B. Prielipp, 'Problem 850', Crux Math. 9 (1983), 144, and 10 (1984), 241-242.

3.8. $$\sqrt{F} < R + (r/2).$$

This inequality is due to G. Mircea.

3.9. $$r^2 + rR(1 + \sqrt{27}) + s^2 - sR(5 + \sqrt{27}) + R^2(6 + 2\sqrt{27}) \geqslant 0. \quad \{E\}$$

I. Paasche, 'Problem 332', Mat. Vesnik 12 (27) (1975), 221.

3.10. Let u and v be constants such that the following inequality

$$(1) \qquad uRr - vr^2 \leqslant s^2$$

holds for every triangle. If in (1) equality occurs for equilateral triangles, then $2u - v = 27$, $u \in (-\infty, 16]$, $v \in (-\infty, 5]$. If we put $t_i = 16 - u = \frac{1}{2}(5 - v)$ for $i = 1, 2, 3$, then

$$\left.\begin{array}{l} t_1(Rr - 2r^2) \\ \frac{t_2}{16}(s^2 - 27r^2) \\ \frac{t_3}{5}(2s^2 - 27Rr) \end{array}\right\} + uRr - vr^2 \leqslant s^2$$

are the best possible improvements of (1), if $t_1 \geqslant 0$; $0 \leqslant t_2 \leqslant 16$, i.e. $u \geqslant 0$; and $0 \leqslant t_3 \leqslant 5$, i.e. $v \geqslant 0$.

I. Paasche, 'Problem 330', Mat. Vesnik 12 (27) (1975), 218-220.

3.11. The maximum values of the positive numbers A, B, C, D, E such that the inequalities

$$0 \leqslant \frac{Rr - 2r^2}{1} \leqslant \frac{s^2 - 27r^2}{A} \leqslant \frac{2s^2 - 27Rr}{B} \leqslant \frac{R^2 - 2Rr}{C} \leqslant$$

$$\leqslant \frac{R^2 - 4r^2}{D} \leqslant \frac{27R^2 - 4s^2}{E}$$

holds for any triangle are:

$$A = 16,\ B = 5,\ C = D = \frac{5}{8}\ ,\ E = \frac{20 + 5\sqrt{17}}{8}\ .$$

I. Paasche and A. Bager, 'Aufgabe 743', Elem. Math. 30 (1975), 63 and 31 (1976), 67-70.

O. Bottema, 'A Triangle Inequality', Elem. Math. 33 (1978), 36-38.

3.12. The best possible constants x and y in the inequalities

$$0 \leqslant \frac{s - r\sqrt{27}}{2} \leqslant \frac{R - 2r}{x} \leqslant \frac{R\sqrt{27} - 2s}{y}$$

are $x = 1$ and $y = \sqrt{27} - 4$.

I. Paasche, 'Problem 332', Mat. Vesnik 12 (27) (1975), 220-221.

3.13. If $M = \sup\{p : s \geqslant \sqrt{27}\ (\frac{R}{2})^p\ r^{1-p}\}$, then $M = \frac{1}{2}$.

This is Bottema's answer to a problem of A. Makowski.

A. Makowski, 'Problems and Remarks on Inequalities for a Triangle', Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 412-460 (1973), 127-130.

O. Bottema, 'Two Problems of A. Makowski', Ibid. No. 412-460 (1973), 131-133.

3.14. For $k \geqslant 1$, we have

$$(k + 1)r[2(k - 1)(7k - 9)R^2 - (k^2 + 17k - 16)Rr + 4r^2] \leqslant$$

$$\leqslant (k - 1)[k(k - 1)R - 4r]s^2.$$

For $k = 3$, we have the first inequality in

$$s^2 \geqslant \frac{4r}{3R - 2r}(12R^2 - 11Rr + r^2) \geqslant r(16R - 5r),$$

what is an interpolating inequality for the well-known Gerretsen inequality (GI 5.8).

S. J. Bilčev and E. A. Velikova, $GT_e(k)$-Transformation for a Triangle and Some Applications, (to appear).

4. Inequalities for the Sides and the Angles of a Triangle

4.1. If $A \geqslant B \geqslant C$, then $\Sigma a(C - B) \geqslant 0$.

M. Martin, 'Problem 11576', Gaz. Mat. (Bucharest) B 22 (1971), 680.

F. Cirjan, 'Problem 16354', Gaz. Mat. (Bucharest) 82 (1977), 20 and 179-180.

4.2. Let $p > 0$ and x, y, z and X, Y, Z be any permutations of a, b, c and A, B, C, respectively. Then

$$\Sigma\left(\frac{a - b + c}{xX}\right)^p > 3(2/\pi)^{p/3}.$$

This result is due to W. Janous, and it is an improvement of a result of C. T. Nedelcu ('Problem 18227', Gaz. Mat. (Bucharest) 85 (1980), 164).

4.3. $$8s^3\Pi(s - a) \geqslant (\Sigma a^2)^3\Pi \cos A.$$

A. W. Walker, 'Problem E 2245', Amer. Math. Monthly 77 (1970), 652; 78 (1971), 793-795; 79 (1972), 1034 and 80 (1973), 809-810.

4.4. $$\Sigma \cos^4 \frac{A}{2} \leqslant \frac{1}{2}\frac{s^3}{abc}. \quad \{E\}$$

Ngo Tan and H. Charles, 'Problem 608', Crux Math. 7 (1981), 49 and 8 (1982), 27.

4.5. $\Pi\left(\frac{c}{\cos B} + \frac{b}{\cos C} - a\right) \geqslant 27abc.$ {E}

R.R. Janić and I. Paasche, 'Problem 172', Mat. Vesnik 7 (22) (1970), 275-276.

4.6. The following inequalities are valid

(1) $\Sigma a^n \cos A \leqslant \frac{1}{3} \Sigma a^n \Sigma \cos A \leqslant \frac{1}{2} \Sigma a^n \leqslant (\frac{s}{2} \Sigma a^{2n-1})^{1/2}$;

(2) $\Sigma a^n \tan \frac{A}{2} \geqslant \frac{1}{3} \Sigma a^n \Sigma \tan \frac{A}{2} \geqslant \frac{\sqrt{3}}{3} \Sigma a^n \geqslant 3(\Sigma a^n)\Pi \tan \frac{A}{2}$;

(3) $\Sigma a^n \sin \frac{A}{2} \geqslant \frac{1}{3} \Sigma a^n \Sigma \sin \frac{A}{2} \geqslant 4\Sigma a^n \Pi \sin \frac{A}{2}$;

(4) $\Sigma a^n \cos \frac{A}{2} \leqslant \frac{1}{3} \Sigma a^n \Sigma \cos \frac{A}{2} \leqslant \frac{\sqrt{3}}{2} \Sigma a^n \leqslant (\frac{3s}{2} \Sigma a^{2n-1})^{1/2}$

with equalities if and only if the triangle is equilateral.

Proof. Here, we shall give only proof of (1). The proofs for the other results are similar. If $a \leqslant b \leqslant c$, then $\cos A \geqslant \cos B \geqslant \cos C$, and using the Čebyšev inequality for monotone sequences we get the first inequality in (1). The second inequality is equivalent to GI 2.16, and the third is a simple consequence of Cauchy's inequality.

Remark. (1) and (2) are generalizations and refinements of results of A. V. Nikulin and R. L. Sejncvit (Mat. v škole 1975, No. 6, 69-71). (2) is also a generalization and refinement of an inequality of W. Janous ('Problem O:88', Gaz. Mat. (Bucharest) 85 (1980), 392 and 86 (1981), 170). (4) is a refinement of an inequality of D. M. Milošević ('Aufgabe 921', Elem. Math. 40 (1985)).

Comment by W. Janous. The following generalizations of (2-4) are valid:

(2') $$\Sigma a^n \tan^p \frac{A}{2} \geqslant \frac{1}{3} \Sigma a^n \Sigma \tan^p \frac{A}{2} \geqslant \begin{cases} 3^{-p/2}\Sigma a^n \geqslant 3^p(\Sigma a^n)\Pi \tan^p \frac{A}{2}, & \text{where } p \geqslant 1 \\ 3^{(3-p)/2}(\Sigma a^n)\Pi \tan \frac{A}{2}, & \text{where } 0 \leqslant p \leqslant 3, \end{cases}$$

(3') $$\Sigma a^n \sin^p \frac{A}{2} \geqslant \frac{1}{3} \Sigma a^n \Sigma \sin^p \frac{A}{2} \geqslant 2^{3-p}(\Sigma a^n)\Pi \sin \frac{A}{2}, \quad \text{where } 0 \leqslant p \leqslant 3,$$

(4') $$\Sigma a^n \cos^p \frac{A}{2} \leqslant \frac{1}{3} \Sigma a^n \Sigma \cos^p \frac{A}{2} \leqslant (\tfrac{3}{4})^{p/2}\Sigma a^n \leqslant 3^{p/2}2^{1-p}(\tfrac{s}{2} \Sigma a^{2n-1})^{1/2}, \quad \text{where } 0 \leqslant p \leqslant 2.$$

4.7. Let $u, v, w \geqslant 0$, $u + v + w \leqslant 9$, $r, s, t \geqslant 0$ and $q > 0$. Then,

$$u\left(\frac{\sin A}{A}\right)^r + v\left(\frac{\sin B}{B}\right)^s + w\left(\frac{\sin C}{C}\right)^t < \Sigma a^q \Sigma a^{-q}.$$

Remark. For $r = s = t = 1$ we have a result of W. Janous. For $u = 4$, $v = 2$, $w = 3$, $r = s = 1$, $t = 0$, we have a result of G. F. Molea.

G. F. Molea, 'Problem 19833', Gaz. Mat. (Bucharest) 88 (1983), 334.

4.8. $$\frac{\sqrt{s}abc \sin C}{ab \sin^2 C + sc} \leq \frac{1}{2}\sqrt{abc} \leq \frac{ab \sin^2 C + sc}{4\sqrt{s} \sin C}.$$

Remark. This is a correction of a result of N. Plesu.

N. Plesu, 'Problem 19912', Gaz. Mat. (Bucharest) 88 (1983), 419 and 89 (1984), 372-373.

4.9. $$\Sigma a^2(b + c) \geq 6 \frac{a^2c^2}{b} \frac{\sin^3 B}{\Pi \cos \frac{A}{2}}.$$

M. Voicu, 'Problem 17746 and 18255', Gaz. Mat. (Bucharest) 84 (1979), 192 and 341-342, and 85 (1980), 214.

4.10. For $p \geq 1$

$$\Sigma(a + b) \sec^p \frac{C}{2} \geq 4(2/\sqrt{3})^p s.$$

W. Janous, 'Problem 1172', Crux Math. 12 (1986), 205.

4.11. $$3\Sigma a \leq \pi\Sigma a/A, \quad 3\Sigma a^2 \geq \pi\Sigma a^2/A.$$

A. Oppenheim, 'Problem E 2649', Amer. Math. Monthly 84 (1977), 294.

4.12. $$b/c > B/(A + B) \quad \text{and} \quad a/c > A/(A + B).$$

J. V. Uspensky, 'A Curious Case of the Use of Mathematical Induction in Geometry', Amer. Math. Monthly 34 (1927), 247-250.

4.13. $$\Sigma \frac{bc(2 \cos A - 1)}{(b - a)(c - a)} \sqrt{s - a} \geq 0.$$

This inequality is due to S. J. Bilčev and E. A. Velikova.

5. Inequalities with (a, b, c) and (R, r, s or F)

The main method for generating inequalities with (R, r, s or F) and other elements of a triangle is the use of identities and the main inequalities with (R, r, s or F) (i.e. inequalities GI 5.1, 5.3, 5.4, 5.8, 5.9, 5.10, 5.11, 7.9, 7.10, 7.11), as we said in chapter V. Of course, the best possible inequalities could be obtained by using GI 5.10 and 7.11 i.e. the well-known fundamental inequality. Here we shall give only three examples of applications of this inequality.

5.1. $$36r^2 \leq 18Rr \leq 12r(2R - r) \leq$$

$$\leq 4R^2 + 16Rr - 3r^2 - 4(R - 2r)\sqrt{R^2 - 2Rr} \leq \Sigma a^2 \leq$$

$$\leq 4R^2 + 16Rr - 3r^2 + 4(R - 2r)\sqrt{R^2 - 2Rr} \leq$$

$$\leq 8R^2 + 4r^2 \leq 9R^2.$$

Remarks. 1° Of course, the most interesting inequalities are

(1) $$12r(2R - r) \leq \Sigma a^2 \leq 8R^2 + 4r^2. \quad \{E\}$$

Note that using Theorems III.13 and 14 we can also consider the best possible inequalities of the form

$$\Sigma a^2 \leq uR^2 + vRr + wr^2 \quad (u, v, w \in R) \quad \text{and}$$

$$\Sigma a^2 \geq uR^2 + vRr + wr^2 \quad (u, v, w \in R).$$

These results are

$$\Sigma a^2 \leq 4(1 - \Theta^2)^{-1}(2R^2 + 2Rr\Theta(1 - 5\Theta) +$$

$$+ r^2(1 + 4\Theta + 2\Theta^2)) \quad (0 \leq \Theta < 1), \quad \{E\},$$

and

$$\Sigma a^2 \geq 4(1 - \omega^2)^{-1}(-2\omega^2 R^2 + 2Rr(3 + \omega) -$$

$$- r^2(3 + 4\omega + \omega^2)) \quad (0 \leq \omega < 1). \quad \{E\}$$

Similarly, we can get several results of the same type (for some results of this kind see paper of M. Marčev).

2° O. Muškarov considered the inequality

$$\Sigma a^2 \leq kR^2 + jr^2$$

and showed that there must be $k \geq 8$ and $j = 4(9 - k)$. The best inequality of this type holds for $k = 8$ and $j = 4$ (L. Panaitopol).

3° Of course, the above results are generalizations and extensions of results from SM, p. 39 and GI 5.13, 5.14 and 5.15.

L. Liviu, 'Problem 17055', Gaz. Mat. (Bucharest) 83 (1978), 82.

M. Marčev, 'Neravenstva meždu perimet'ra i radiusīte na vpisanata i opisanata okr'žnost na tri'g'lnika i njakoj sledstvija od tjah', Ob. Matematika (Sofija) 1 (1976), 3-7.

O. Muškarov, 'V'rhu edīn vid neravenstva v tri'g'lnik', Matematika (Sofija) 16 (1977), 32-35.

L. Panaītopol, 'A Geometric Inequality' (Romanian), Gaz. Mat. (Bucharest) 87 (1982), 113-114.

5.2. $36r^2 \leq 18Rr \leq 20Rr - 4r^2 \leq 2R^2 + 14Rr - 2(R - 2r)\sqrt{R^2 - 2Rr} \leq$

$$\leq \Sigma ab \leq 2R^2 + 14Rr + 2(R - 2r)\sqrt{R^2 - 2Rr} \leq 4(R + r)^2 \leq 9R^2.$$

Remark. This result is a refinement of GI 5.16, 5.17, 5.18, 5.19, 5.36, SM p. 39.

5.3. $8r(R - 2r) \leq 4(R - 2r)(R + r - \sqrt{R(R - 2r)}) \leq Q = \Sigma(b - c)^2 \leq$

$$\leq 4(R - 2r)(R + r + \sqrt{R(R - 2r)}) \leq 8R(R - 2r).$$

Remark. This is a refinement of GI 5.25.

A. Lupaş, 'Problem 441', Mat. Vesnik 2 (15) (30) (1978), 293.

5.4. $24\sqrt{3}r^3 \leq 12\sqrt{3}Rr^2 \leq abc \leq 4Rr(2R + (3\sqrt{3} - 4)r) \leq$

$$\leq 6\sqrt{3}R^2r \leq 3\sqrt{3}R^3, \quad \{E\}.$$

Remark. This result is a refinement and extension of results from SM, p. 39 and GI 5.27.

5.5. $72\sqrt{3}r^3 \leq 36\sqrt{3}Rr^2 \leq 12\sqrt{3}r^2(5R - 4r) \leq 4sr(5R - 4r) \leq \Sigma a^3 \leq$

$$\leq 4sR(2R - r) \leq 4R(2R + (3\sqrt{3} - 4)r)(2R - r) \leq$$

$$\leq 6\sqrt{3}R^2(2R - r). \quad \{E\}$$

Remarks. 1° The above result, an interpolation of a result from SM, p. 39, we obtained using results of Bottema-Veldkamp and Tsintsifas-Klamkin.

2° Since sr = F, we have the following refinement and extension of GI 7.7

$$\Sigma a^3 \geq 4F(5R - 4r) \geq 8F(2R - r) \geq \begin{Bmatrix} 4\sqrt{3}F(s - r\sqrt{3}) \\ 12FR \end{Bmatrix} \geq 24rF \geq$$

$$\geq 72\sqrt{3}r^3. \quad \{E\}$$

This is a result of A. Bager.

O. Bottema and G. R. Veldkamp, 'Problem 364', Nieuw Arch. Wisk. 22 (1974), 79 and 22 (1974), 266-267.

A. Bager (1972), Private communication.

G. Tsintsifas and M. S. Klamkin, 'Problem 816', Crux Math. 9 (1983), 46 and 10 (1984), 157.

5.6. $192\sqrt{3}r^3 \leq 96\sqrt{3}r^2R \leq 12\sqrt{3}r^2(9R - 2r) \leq 4sr(9R - 2r) \leq$

$$\leq \Pi(b + c) \leq 4s(2R^2 + 3Rr + 2r^2) \leq$$

$$4(2R + (3\sqrt{3} - 4)r)(2R^2 + 3Rr + 2r^2) \leqslant 6\sqrt{3}R(2R^2 + 3Rr + 2r^2) \leqslant$$
$$\leqslant 24\sqrt{3}R^3. \quad \{E\}$$

Remarks. 1° This is an interpolation of a result from SM, p. 39.
2° The fourth inequality is equivalent with the first inequality in

$$\Pi(b + c) \geqslant 4F(9R - 2r) \geqslant 2^6 Fr. \quad \{E\}$$

This result is a refinement of the following result of I. Dorobantu

$$\Pi(2 - \frac{a}{s}) \geqslant 2^6 r^4/F^2. \quad \{E\}$$

3° Note that M. S. Klamkin proved

$$\Pi(b + c) \leqslant 4s(2R^2 + 3Rr + 2r^2) \leqslant 8sR(R + 2r). \quad \{E\}$$

I. Dorobantu, 'Problem 9470', Gaz. Mat. (Bucharest) B 20 (1969), 162.
G. Tsintsifas and M. S. Klamkin, The same reference as in 5.5.

5.7. $$\frac{\sqrt{3}}{R} \leqslant \frac{2(5R - r)}{3\sqrt{3}R^2} \leqslant \frac{5R - r}{R(2R + (3\sqrt{3} - 4)r)} \leqslant \frac{5R - r}{Rs} \leqslant \Sigma \frac{1}{a} \leqslant$$
$$\leqslant \frac{(R + r)^2}{Rrs} \leqslant \frac{(R + r)^2}{3\sqrt{3}Rr^2} \leqslant \frac{\sqrt{3}R}{4r^2}. \quad \{E\}$$

Remarks. 1° This is an interpolation of a result from SM, p. 39.
2° Since $rs = F$, we have

$$\frac{r(5R - r)}{RF} \leqslant \Sigma \frac{1}{a} \leqslant \frac{(R + r)^2}{RF}, \quad \{E\}$$

what is better than GI 7.12. We also have

$$\frac{2(5R - r)}{R} \leqslant \Sigma a \Sigma \frac{1}{a} \leqslant \frac{2(R + r)^2}{Rr}. \quad \{E\}$$

5.8. $$\frac{1}{R^2} \leqslant \Sigma \frac{1}{a^2} \leqslant \frac{(R^2 + r^2)^2 + Rr^3}{R^2 r^3 (16R - 5r)} \leqslant \frac{1}{4r^2}. \quad \{E\}$$

Remark. This is a result from SM, p. 39. The first inequality was given by M. Erdman.
M. Erdman, 'Problem 9', Matematyka (Warszawa) 111 (1970), 367.

5.9. $$\frac{1}{2rR} \leqslant \frac{1}{3}(\Sigma \frac{1}{a})^2 \leqslant \Sigma \frac{1}{a^2} \leqslant \frac{1}{4r^2}. \quad \{E\}$$

Proof.

$$\frac{1}{2rR} = \frac{2s}{abc} = \frac{a + b + c}{abc} = \Sigma \frac{1}{bc} \leqslant \frac{1}{3}(\Sigma \frac{1}{a})^2 \leqslant \Sigma \frac{1}{a^2} \leqslant \frac{1}{4r^2} .$$

A. W. Walker and H. Meyer, 'Problem E 2248', Amer. Math. Monthly 77 (1970), 765 and 78 (1971), 678.

5.10. $$6 \leqslant \frac{7R - 2r}{R} \leqslant \Sigma \frac{b + c}{a} \leqslant \frac{2R^2 + Rr + 2r^2}{Rr} \leqslant \frac{3R}{r} . \quad \{E\}$$

B. M. Milisavljević, 'Some Inequalities Related to a Triangle', Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 498-541, (1978), 181-184.

SM, p. 40.

5.11. $$\frac{R^2 + 3Rr + 2r^2}{2R^2 + 3Rr + 2r^2} \leqslant \Sigma \frac{s - a}{b + c} \leqslant \frac{6R}{9R - 2r} . \quad \{E\}$$

B. M. Milisavljević, The same reference as in 5.10.

5.12. $$5 \min ((a - b)^2, (b - c)^2, (c - a)^2) \leqslant s^2 - 8Rr - 2r^2 .$$

Remark. This is a problem of M. S. Klamkin. He remarked that the problem was suggested by an inequality of D. S. Mitrinović (Amer. Math. Monthly 75 (1968), 1124).

M. S. Klamkin and A. A. Jagers, 'Problem 346', Nieuw Arch. Wisk. 21 (1973), 178 and 22 (1974), 90-91.

Comment by C. Tănăsescu. The following inequality is also valid:

(1) $$\min((a - b)^2, (b - c)^2, (c - a)^2) \leqslant 2r(R - 2r)$$

with equality if and only if $s^2 = 9r(2R - r)$.

(1) is best on the set of all (R, r)-triangles, taking s as free parameter.

From (1) we easily deduce

(2) $$\min ((a - b)^2, (b - c)^2, (c - a)^2) \leqslant R^2/4,$$

with equality if and only if $r = R/4$, $s = 3\sqrt{7}R/4$, or equivalently, after performing some trivial calculations, if and only if the sides are proportional to the numbers $\sqrt{7} - 1$, $\sqrt{7}$, $\sqrt{7} + 1$, respectively. This result, i.e. inequality (2), is due to L. Panaitopol.

Of course, (2) is also the best inequality on the set of all R-triangles with r and s as free parameters.

Another similar result is:

If $Q = \Sigma(a - b)^2$, $D = |\Pi(a - b)|$, then we have

(3) $$Q \min (|a - b|, |b - c|, |c - a|) \leqslant 3D,$$

with equality if and only if either the triangle is isosceles, or the sides form an arithmetic progression.

Remark 1. Of course (3) is equivalent to

(4) $$Q^2 \min((a-b)^2, (b-c)^2, (c-a)^2) \leqslant 9D^2.$$

Remark 2. Both (3) and (4) are still valid for any real numbers a, b, c, with the same equality conditions.

Remark 3. For the specialization n = 3 the Mitrinović-Prešić inequality (AI p. 341) reads

(5) $$\min((a-b)^2, (b-c)^2, (c-d)^2 \leqslant Q/6.$$

It is less sharp than (4).

L. Panaitopol, 'Problem C:2', Gaz. Mat. (Bucharest) 85 (1980), 4.

5.13. $$\Sigma a^2\sqrt{bc}/(\Sigma\sqrt{a})^2 \leqslant R^2. \quad \{E\}$$

Remark. This is a result of I. I. Tomescu. Using the idea of his proof and 5.1. we can get an interpolation of this inequality, i.e. the following result is valid:

$$\Sigma a^2\sqrt{bc}/(\Sigma\sqrt{a})^2 \leqslant \frac{1}{9}\Sigma a^2 \leqslant \frac{4}{9}(2R^2 + r^2) \leqslant R^2. \quad \{E\}$$

Comment by W. Janous. Let q be an arbitrary positive real number. Then

$$\Sigma a^p(bc)^q/(\Sigma a^q)^2 \leqslant \begin{Bmatrix} (2s)^p/3^{p+1} \\ 2^p(2R^2 + r^2)^{p/2}/3^{1+p/2} \end{Bmatrix} \leqslant$$

$$\leqslant R^p/3^{1-p/2}, \quad \text{if } \begin{cases} 0 < p \leqslant 1, \\ 1 < p \leqslant 2. \end{cases}$$

I. I. Tomescu, 'Problem 9374', Gaz. Mat. (Bucharest) 20, B7 (1969), 425-427.

5.14. $$9r^2 \leqslant \Sigma xy \leqslant \frac{9}{4}R^2. \quad \{E\}$$

SM, p. 40.

5.15. $$3\sqrt{3}r^3 \leqslant xyz \leqslant r^2(2R + (3\sqrt{3} - 4)r) \leqslant \frac{3\sqrt{3}}{2}Rr^2 \leqslant \frac{3\sqrt{3}}{8}R^3. \quad \{E\}$$

Remark. This is an interpolation of a result from SM, p. 40.

5.16. $$9r^2 \leqslant \frac{9}{2}Rr \leqslant r(8R - 7r) \leqslant \Sigma x^2 \leqslant (2R - r)^2. \quad \{E\}$$

Remark. We put the term $\frac{9}{2}Rr$ in the result from SM, p. 40.

5.17. $$9\sqrt{3}r^3 \leqslant \frac{9\sqrt{3}}{2} r^2R \leqslant 3\sqrt{3}r^2(4R - 5r) \leqslant sr(4R - 5r) \leqslant \Sigma x^3 \leqslant$$

$$\leqslant s(4R^2 - 8Rr + 3r^2) \leqslant (2R + (3\sqrt{3} - 4)r)(4R^2 - 8Rr + 3r^2) \leqslant$$

$$\leqslant \frac{3\sqrt{3}}{2} R(4R^2 - 8Rr + 3r^2) \leqslant \frac{3\sqrt{3}}{4} R^2(8R - 13r). \quad \{E\}$$

Remark. This is an interpolation of a result from SM, p. 40.

5.18. $$\frac{2\sqrt{3}}{R} \leqslant \frac{2(4R + r)}{3\sqrt{3}Rr} \leqslant \frac{4R + r}{r(2R + (3\sqrt{3} - 4)r)} \leqslant \Sigma \frac{1}{x} \leqslant$$

$$\leqslant \frac{4R + r}{3\sqrt{3}r^2} \leqslant \frac{3R}{2\sqrt{3}r^2}. \quad \{E\}$$

Remark. This is an interpolation of a result from SM, p. 40.

5.19. $$\frac{4}{R^2} \leqslant \frac{2}{Rr} \leqslant \Sigma \frac{1}{xy} = \frac{1}{r^2}. \quad \{E\}$$

SM, p. 41.

5.20. $$\frac{4}{R^2} \leqslant \frac{2}{Rr} \leqslant \frac{1}{r^2} \leqslant \frac{8R^2 - 5r^2}{r^2(4R^2 + 4Rr + 3r^2)} \leqslant \Sigma \frac{1}{x^2} \leqslant$$

$$\leqslant \frac{16R^2 - 24Rr + 11r^2}{r^3(16R - 5r)} \leqslant \frac{R^2}{4r^4}. \quad \{E\}$$

Remark. We put the term $2/Rr$ in the result from SM, p. 41.

5.21. $$\frac{12R^2 + 4Rr - 2r^2}{F^2} \leqslant \Sigma \frac{1}{x^2} \leqslant \frac{16R^2 - 8Rr + 6r^2}{F^2}. \quad \{E\}$$

5.22. $$\Sigma \frac{a^2}{x} \geqslant \frac{6sR}{r} \geqslant 12s. \quad \{E\}$$

5.23. $$1 \leqslant \Sigma a \Sigma a^3/(\Sigma bc)^2 \leqslant \frac{R}{2r}. \quad \{E\}$$

A. Bager and M. S. Klamkin, Elem. Math. 28 (1973), 102 and 29 (1974), 96-97.

5.24. $$2r\sqrt{3} \leqslant 2s/3 \leqslant \Sigma a^2/\Sigma a \leqslant s - r\sqrt{3}. \quad \{E\}$$

Remark. This result is due to A. Bager.

5.25. $$10 - \frac{8r}{R} \leqslant 2\Sigma a^3/abc \leqslant \frac{4R}{r} - 2 \leqslant \frac{9R}{r} - \frac{4r}{R} - 10. \quad \{E\}$$

Remark. This is a refinement and extension of a result of E. Braune and O. Bottema.

E. Braune and O. Bottema, 'Aufgabe 734', Elem. Math. 30 (1975), 18 and 31 (1976), 15-16.

5.26. $\Sigma a^4 \leqslant \frac{8}{3} R(R - r)(4R + r)^2 \leqslant 54R^3(R - r)$. {E}

Remark. This result is due to A. Bager.

5.27. $16F^2/3 \leqslant \Sigma a^2/\Sigma \frac{1}{a^2} \leqslant 9R^4$. {E}

A. Mirea, Gaz. Math. (Bucharest) 39 (1933), 411.

5.28. $16F^2/9R^2 \leqslant \Sigma a^2b^2/\Sigma a^2 \leqslant 3R^2$. {E}

A. Mirea, Gaz. Math. (Bucharest) 39 (1933), 411.

5.29. $16s^2r^2 \leqslant \Sigma a^2b^2 \leqslant 4s^2R^2$. {E}

Remark. This result is due to L. Goldstone.

5.30. $\frac{R}{r} \geqslant \frac{b}{c} + \frac{c}{b}$.

V. Băndilă, 'Problem C:474', Gaz. Mat. (Bucharest) 90 (1985), 65.

5.31. $a \geqslant b \geqslant c \Rightarrow \frac{b^2 + 2ac}{a + b + c} \leqslant \frac{2}{3}(2R + (3\sqrt{3} - 4)r) \leqslant R\sqrt{3}$. {E}

Proof. Using Čebyšev's inequality for monotone sequences we get

(1) $3(b^2 + 2ac) = 3(ac + bb + ca) \leqslant (a + b + c)^2$

i.e.

$$\frac{b^2 + 2ac}{a + b + c} \leqslant \frac{2s}{3} \leqslant \frac{2}{3}(2R + (3\sqrt{3} - 4)r) \leqslant R\sqrt{3}.$$

Remarks. 1° This is a refinement of a result of V. Boskoff and D. Mihet.

2° Analogously, (1) becomes

$$3(b^2 + 2ac) \leqslant 4s^2,$$

so, the following inequality is also valid

$$b^2 + 2ac \leqslant \frac{4}{3}(4R^2 + 4Rr + 3r^2).$$

V. Boskoff and D. Mihet, Gaz. Mat. (Bucharest) 90 (1985), 108.

5.32. $\Sigma\, 1/a^p \leqslant 3^{1-2p}(s/Rr)^p \quad (0 < p < 1)$. {E}

This result is due to W. Janous. For p = 1/2 we get a result from P. Flore, Gaz. Mat. (Bucharest) 83 (1978), 217.

5.33. $\Sigma a/x \geqslant 3\sqrt[3]{4R/r} \geqslant 6$. {E}

Remark. This result is due to A. Bager.

5.34.
$$\Sigma(b - c)^2 + k(rs\sqrt{3} + (4 - 2\sqrt{3})r(R - 2r)) \leqslant \begin{cases} \Sigma a^2, & \text{for } k = 4, \\ (\Sigma a)^2/2, & \text{for } k = 6. \end{cases} \quad \{E\}$$

I. Paasche and L. Carlitz, 'Aufgabe 642', Elem. Math. 26 (1971), 46 and 27 (1972), 39-40.

5.35. If $2S = \Sigma x^{-1}$, and $x = s - a$, etc. then, for $p \geqslant 1$,

$$\Sigma|S - x^{-1}|^p \geqslant 3^{1-p/2}(2r)^{-p}. \quad \{E\}$$

This result is due to W. Janous. For p = 2 we get a result from: P. Balev, Matematika (Sofia) 11 (1968), 33.

5.36. $(2r\sqrt{3})^{m+n+p} \leqslant a^m b^n c^p \leqslant (3\sqrt{3}R/2) \qquad (m, n, p \in R_+)$. {E}

Remark. This is Milisavljević's generalization of a problem of D. M. Milošević.

D. M. Milošević and B. Milisavljević, 'Problem 410', Mat. Vesnik 2 (15)(30)(1978), 425-426.

5.37. If $-1 \leqslant k \leqslant \dfrac{\log 9 - \log 4}{\log 4 - \log 3}$, then

$$2\sqrt{3}r \leqslant M_k(a, b, c) \leqslant \sqrt{3}R.$$

Remark. For the second inequality see GI 5.28.

J. Berkes, 'Certaines Inégalités Relatives au Triangle', Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 247-273 (1969), 151-152.

5.38. $\Sigma \dfrac{bc}{s - a} \geqslant (5 - 2r/R)s$.

S. J. Bilčev, Matematika (Sofija) 1985, No. 3, 52.

5.39. $4r(\Sigma a)^2 + 2\Sigma a^3 < 9abc + (\Sigma a)(\Sigma a^2)$.

S. J. Bilčev, Ob. po matematika (Sofija) 1984, No. 4, 62.

5.40. $4\Sigma 1/a - \Sigma a/bc \leqslant (5/R + 2/r)/\sqrt{3}$.

This inequality is due to S. J. Bilčev.

5.41. The following inequality is equivalent to 2.2.20:

$$\Sigma(a/(s-a))^{1/2} \geqslant 9(3Rr)^{1/2}/s.$$

This result is due to S. J. Bilčev. He also gave some other similar results.

5.42. The maximum value of the positive numbers A, B, C, D, E, F, G, H such that the inequalities

(1) $A(R-2r)r + 3\Sigma bc \leqslant (\Sigma a)^2,$

(2) $B(R-2r)r + \frac{36}{35}(s^2 + abc/s) \leqslant \Sigma a^2,$

(3) $C(R-2r)rs + 8xyz \leqslant abc,$

(4) $D(R-2r)rs + 8abc \leqslant \Pi(b+c),$

(5) $E(R-2r)rs + \frac{3}{8}\Pi(b+c) \leqslant \Sigma a^3,$

(6) $F(R-2r)rs + \frac{3}{2}(\Sigma a^3 + 3abc) \leqslant \Sigma a\Sigma a^2,$

(7) $G(R-2r)rs + \frac{2}{3}(\Sigma a^2 x) \leqslant abc,$

(8) $H(R-2r)rs + 48xyz \leqslant \Sigma bc(b+c),$

hold for any triangle are

$A = 4, \quad B = 24/7, \quad C = 4, \quad D = 4, \quad E = 1/4,$

$F = 0, \quad G = 3/4, \quad H = 28.$

I. Paasche and B. Milisavljević, 'Problem 325', Mat. Vesnik 12 (27) (1975), 315.

5.43 $$4F \leqslant 3^{\frac{1-2^n}{2}}\left(2\Sigma(bc)^{2^{1-n}} - \Sigma a^{2^{2-n}}\right)^{2^{n-1}} \quad (n = 1, 2, \ldots).$$ {E}

Remark. For n = 1 we get the following inequality of F. Finsler and H. Hadwiger.

$$4\sqrt{3}\,F \leqslant 2\,bc - \Sigma a^2.$$

D. D. Adamović, 'Problem 153', Mat. Vesnik 8 (23) (1971), 92.

5.44. Let F and s be fixed. The maximum and minimum values of one of the sides are roots of the equation

$$sx^2(x - s) + 4F^2 = 0.$$

G. H. Hardy, 'Problem 15689', Educational Times (2) 15 (1905), 8 and 74.

6. Inequalities Involving A, B, C and R, r, s or F

6.1. $$3\sqrt{3}\frac{r}{R} \leqslant \Sigma \sin A \leqslant 2 + (3\sqrt{3} - 4)\frac{r}{R} \leqslant \frac{3}{2}\sqrt{3}. \quad \{E\}$$

This is a refinement of a result from SM, p. 44, and an extension of GI 3.15 and of a result of M. Marčev.

M. Marčev, 'Neravenstva meždu perimet'ra i radiusite na vpisanata i opisanata okr'žnost na tri'g'lnika i njakoi sledsvija ot tjah', Ob. po matematika (Sofija) 1976, No. 6, 3-7.

6.2. $$\frac{9r}{2R} \leqslant \frac{r}{R^2}(5R - r) \leqslant \Sigma \sin A \sin B \leqslant \left(\frac{R + r}{R}\right)^2 \leqslant \frac{9}{4}. \quad \{E\}$$

This result from SM, p. 44, is an extension of a result of M. Marčev (see reference from 6.1).

6.3. $$\Sigma \sin A \sin B \leqslant \frac{s^2}{3R^2} = \frac{F^2}{3R^2r^2}. \quad \{E\}$$

Proof.

$$\Sigma \sin A \sin B = \frac{1}{4R^2}(s^2 + r^2 + 4Rr) \leqslant \frac{1}{4R^2}\left(s^2 + \frac{s^2}{3}\right),$$

where we used GI 5.5.

Remark. Since $x \to \sqrt{x}$ is a concave increasing function we have

$$\Sigma\sqrt{\sin A \sin B} \leqslant 3\left(\frac{1}{3}\Sigma \sin B \sin C\right)^{1/2} \leqslant 3\left(\frac{1}{3}\frac{F^2}{3R^2r^2}\right)^{1/2} = \frac{F}{Rr},$$

i.e.

$$\Sigma\sqrt{\sin A \sin B} \leqslant \frac{F}{Rr},$$

which is a result of I. Nanuti and V. Drulă.

I. Nanuti and V. Drulă, Gaz. Mat. (Bucharest) 83 (1978), 218.

6.4. $$\frac{3\sqrt{3}r^2}{2R^2} \leqslant \Pi \sin A \leqslant \frac{r}{2R^2}(2R + (3\sqrt{3} - 4)r) \leqslant \frac{3\sqrt{3}r}{4R} < \frac{3\sqrt{3}}{8} . \quad \{E\}$$

This is an interpolation of a result from SM, p. 44.

6.5. $$\frac{15r}{2R} \leqslant \frac{3r}{R^2}(2R - r) \leqslant \Sigma \sin^2 A \leqslant \frac{1}{R^2}(2R^2 + r^2) \leqslant \frac{9}{4} . \quad \{E\}$$

This result from SM, p. 44, is an extension of a result of M. Marčev (see reference from 6.1).

6.6. $$\frac{9\sqrt{3}r^2}{2R^2} \leqslant \frac{3\sqrt{3}r^2}{2R^3}(5R - 4r) \leqslant \frac{sr}{2R^3}(5R - 4r) \leqslant \Sigma \sin^3 A \leqslant \frac{s}{2R^2}(2R - r)$$
$$\leqslant \frac{2R + (3\sqrt{3} - 4)r}{2R^2}(2R - r) \leqslant \frac{3\sqrt{3}}{4R}(2R - r) < \frac{3\sqrt{3}}{2} . \quad \{E\}$$

This is an interpolation of a result from SM, p. 44.

6.7. $$12\sqrt{3}\,\frac{r^2}{R^2} \leqslant \frac{3\sqrt{3}r^2}{2R^3}(9R - 2r) \leqslant \frac{sr}{2R^3}(9R - 2r) = \Pi(\sin A + \sin B) \leqslant$$
$$\leqslant \frac{s}{2R^3}(2R^2 + 3Rr + 2r^2) \leqslant$$
$$\leqslant \frac{2R + (3\sqrt{3} - 4)r}{2R^3}(2R^2 + 3Rr + 2r^2) \leqslant$$
$$\leqslant \frac{3\sqrt{3}}{4R^2}(2R^2 + 3Rr + 2r^2) \leqslant 3\sqrt{3}. \quad \{E\}$$

This is an interpolation of a result from SM, p. 44.

6.8. $$6 \leqslant \frac{7R - 2r}{R} \leqslant \Sigma\,\frac{\sin A + \sin B}{\sin C} \leqslant \frac{2R^2 + Rr + 2r^2}{Rr} \leqslant \frac{3R}{r} . \quad \{E\}$$

SM, p. 45.

6.9. $$\frac{3r}{R} \leqslant \Sigma \cos A \leqslant \frac{3}{2} . \quad \{E\}$$

SM, p. 45.

6.10. $$\frac{7r - 2R}{2R} \leqslant \frac{4Rr - r^2 - R^2}{R^2} \leqslant \Sigma \cos A \cos B \leqslant \frac{r(R + r)}{R^2} \leqslant$$
$$\leqslant \frac{3r}{2R} \leqslant \frac{3}{4} . \quad \{E\}$$

This is an extension of GI 3.11 and of a result of M. Marčev (see reference from 6.1), and a refinement of a result from SM, p. 45.

6.11. $$\frac{9r}{4R} - 1 \leqslant \frac{6Rr - 2R^2 - 3r^2}{2R^2} \leqslant \Pi \cos A \leqslant \frac{r^2}{2R^2} \leqslant \frac{1}{8} . \quad \{E\}$$

Remark. Using GI 3.11 we gave a refinement of a result from SM, p. 45, and an extension of a result of M. Marčev (see reference from 6.1).

6.12. $$\frac{3}{4} \leqslant \frac{2Rr - r^2}{R^2} \leqslant \frac{R^2 - r^2}{R^2} \leqslant \Sigma \cos^2 A \leqslant \frac{3}{R^2}(R - r)^2 < 3. \quad \{E\}$$

Remark. We inserted the term $\frac{2Rr - r^2}{R^2}$ in the result from SM, p. 45, and in the result of M. Marčev (see reference in 6.1).

6.13. $$\frac{3}{8} \leqslant \frac{1}{2R^3}(2R^3 - 3Rr^2 - 4r^3) \leqslant \Sigma \cos^3 A \leqslant$$
$$\leqslant \frac{1}{4R^2}(4R^2 + 12Rr - 34r^2). \quad \{E\}$$

SM, p. 45.

6.14. $$\frac{4r^2}{R^2} \leqslant \frac{r^2}{4R^3}(9R - 2r) \leqslant \Pi(\cos A + \cos B) \leqslant$$
$$\leqslant \frac{1}{2R^3}(2R^2 r + 3Rr^2 + 2r^2) \leqslant 1. \quad \{E\}$$

SM, p. 46.

6.15. $$\sqrt{3} \leqslant \frac{2(2R - r)}{\sqrt{3}R} \leqslant \frac{3(2R - r)}{2R + (3\sqrt{3} - 4)r} \leqslant \frac{3}{s}(2R - r) \leqslant \Sigma \cot A \leqslant$$
$$\leqslant \frac{2R^2 + r^2}{sr} \leqslant \frac{2R^2 + r^2}{3\sqrt{3}r^2} \leqslant \frac{\sqrt{3}R^2}{4r^2} . \quad \{E\}$$

Remark. Using a result of D. M. Milošević and Š. Z. Arslanagić we gave an interpolation of a result from SM, p. 46.

D. M. Milošević and Š. Z. Arslanagić, Private communication.

6.16. $$\Sigma \cot A \geqslant \frac{s}{3r} . \quad \{E\}$$

V. Petkov, Matematika (Sofija) 1968, No. 7, 30.

6.17. $$\frac{12Rr - 4R^2 - 6r^2}{3\sqrt{3}Rr} \leqslant \frac{6Rr - 2R^2 - 3r^2}{r(2R + (3\sqrt{3} - 4)r)} \leqslant \frac{6Rr - 2R^2 - 3r^2}{rs} \leqslant$$
$$\leqslant \Pi \text{ cotan } A \leqslant \frac{r}{s} \leqslant \frac{1}{3\sqrt{3}} . \quad \{E\}$$

This is a refinement of a result from SM, p. 46.

6.18. $$1 \leqslant \frac{9(2R - r)^2}{4R^2 + 4Rr + 3r^2} - 2 \leqslant \Sigma \text{ cotan}^2 A \leqslant$$
$$\leqslant \frac{(2R^2 + r^2)^2}{r^3(16R - 5r)} - 2 \leqslant \frac{3R^3}{8r^3} - 2. \quad \{E\}$$

SM, p. 46.

6.19. $$\Sigma \text{ cotan}^2 A \geqslant \frac{s^2}{9r^2} - 2. \quad \{E\}$$

6.20. $$\frac{8}{3\sqrt{3}} \leqslant \frac{4R}{3\sqrt{3}r} \leqslant \Pi(\text{cotan } A + \text{cotan } B) \leqslant \frac{2R^2}{3\sqrt{3}r^2} . \quad \{E\}$$

SM, p. 47.

6.21. $$2\sqrt{3}\,\frac{r}{R} \leqslant \frac{2(4R + r)}{3\sqrt{3}R} \leqslant \frac{4R + r}{2R + (3\sqrt{3} - 4)r} \leqslant \Sigma \tan \frac{A}{2} \leqslant$$
$$\leqslant \frac{4R + r}{3\sqrt{3}r} \leqslant \frac{\sqrt{3}R}{2r} . \quad \{E\}$$

This is a refinement of a result from SM, p. 47.
Remark. The following result is also valid.

$$\Sigma \tan \frac{A}{2} \leqslant \frac{9Rr}{2F} \leqslant \frac{9R^2}{4F} .$$

This is a refinement of a result of T. Rajkov.
T. Rajkov, Kvant **1976**, No. 6, 56.

6.22. $$\Sigma \tan \frac{A}{2} \leqslant \frac{s}{3r} . \quad \{E\}$$

6.23. $$\frac{2r}{3\sqrt{3}R} \leqslant \frac{r}{2R + (3\sqrt{3} - 4)r} \leqslant \Pi \tan \frac{A}{2} \leqslant \frac{1}{3\sqrt{3}} . \quad \{E\}$$

This is a refinement of a result from SM, p. 47.

6.24. $$1 \leqslant \Sigma \tan^2 \frac{A}{2} \leqslant \frac{16R^2 - 24Rr + 11r^2}{r(16R - 5r)} . \quad \{E\}$$

SM, p. 47.

6.25. $\frac{1}{\sqrt{3}} \leqslant \frac{2(4R + r)^3 - 24R(4R^2 + 4Rr + 3r^2)}{3\sqrt{3}R(4R^2 + 4Rr + 3r^2)} \leqslant \Sigma \tan^3 \frac{A}{2} \leqslant$

$$\leqslant \frac{9R^3}{8\sqrt{3}r^3} - \frac{8}{\sqrt{3}} . \quad \{E\}$$

SM, p. 48.

6.26. $\Sigma \tan^3 \frac{A}{2} \geqslant \frac{3r}{s} . \quad \{E\}$

Proof.

$$\Sigma \tan^3 \frac{A}{2} = \frac{1}{s^3}(4R + r)^3 - 12s^2R) \geqslant \frac{1}{s^3}(3s^2(4R + r) - 12s^2R) = \frac{3r}{s},$$

where we used GI 5.5.

6.27. $\frac{8}{3\sqrt{3}} \leqslant \Pi(\tan \frac{A}{2} + \tan \frac{B}{2}) \leqslant \frac{4R}{3\sqrt{3}r} . \quad \{E\}$

SM, p. 48.

6.28. $3\sqrt{3} \leqslant \Sigma \operatorname{cotan} \frac{A}{2} \leqslant \frac{2R + (3\sqrt{3} - 4)r}{r} \leqslant \frac{3\sqrt{3}R}{2r} . \quad \{E\}$

This is a refinement of a result from SM, p. 48.

6.29. $9 \leqslant \Sigma \operatorname{cotan} \frac{A}{2} \operatorname{cotan} \frac{B}{2} = \frac{4R + r}{r} \leqslant \frac{9R}{2r} . \quad \{E\}$

SM, p. 48.

6.30. $\Sigma \operatorname{cotan} \frac{A}{2} \operatorname{cotan} \frac{B}{2} \leqslant \frac{s^2}{3r^2} . \quad \{E\}$

6.31. $9 \leqslant \frac{9R}{2r} \leqslant \frac{8R - 7r}{r} \leqslant \Sigma \operatorname{cotan}^2 \frac{A}{2} \leqslant \frac{(2R - r)^2}{r^2} . \quad \{E\}$

We put the term $\frac{9R}{2r}$ in the result from SM, p. 49.

6.32. $\Sigma \operatorname{cotan}^2 \frac{A}{2} \geqslant \frac{s^2}{3r^2} . \quad \{E\}$

6.33. $9\sqrt{3} \leqslant \frac{9\sqrt{3}R}{2r} \leqslant \frac{3\sqrt{3}(4R - 5r)}{r} \leqslant \Sigma \operatorname{cotan}^3 \frac{A}{2} \leqslant$

$$\leqslant \frac{3\sqrt{3}R(4(R - r)^2 - r^2)}{2r^3} . \quad \{E\}$$

Remark. We put the second term in the result from SM, p. 49.

6.34. $$24\sqrt{3} \leqslant 12\sqrt{3}\,\frac{R}{r} \leqslant \Pi\left(\cot\frac{A}{2} + \cot\frac{B}{2}\right) \leqslant \frac{6\sqrt{3}R^2}{r^2} . \quad \{E\}$$

SM, p. 49.

6.35. $$6 \leqslant \Sigma\,\frac{\tan\frac{A}{2} + \tan\frac{B}{2}}{\tan\frac{C}{2}} = \Sigma\,\frac{\cot\frac{A}{2} + \cot\frac{B}{2}}{\cot\frac{C}{2}} = \frac{4R - 2r}{r} . \quad \{E\}$$

SM, pp. 48-49.

6.36. $$\frac{3r}{8R} \leqslant \frac{r(2R - r)}{4R^2} \leqslant \Sigma\,\sin^2\frac{A}{2}\sin^2\frac{B}{2} \leqslant \frac{R^2 - Rr + r^2}{4R^2} \leqslant \frac{2R - r}{8R}. \quad \{E\}$$

6.37. $$\Sigma\,\cos^2\frac{A}{2} \geqslant \frac{9r}{2R} . \quad \{E\}$$

6.38. $$\Sigma\,\cos^2\frac{A}{2} \leqslant \frac{s^2}{6Rr} . \quad \{E\}$$

6.39. $$\frac{27r}{8R} \leqslant \frac{8R + 11r}{8R} \leqslant \frac{4R^2 + 6Rr - r^2}{4R^2} \leqslant \Sigma\,\cos^2\frac{A}{2}\cos^2\frac{B}{2} \leqslant$$

$$\leqslant \frac{5R^2 + 3Rr + r^2}{4R^2} \leqslant \frac{10R + 7r}{8R} \leqslant \frac{27}{16} . \quad \{E\}$$

6.40. $$\Sigma\,\cos^2\frac{A}{2}\cos^2\frac{B}{2} \geqslant \frac{s^2}{4R^2} . \quad \{E\}$$

6.41. $$2\sqrt{3} \leqslant \frac{4(5R - r)}{3\sqrt{3}R} \leqslant \frac{2(5R - r)}{2R + (3\sqrt{3} - 4)r} \leqslant \frac{2(5R - r)}{s} \leqslant \Sigma\,\operatorname{cosec} A \leqslant$$

$$\leqslant \frac{2(R + r)^2}{sr} \leqslant \frac{2(R + r)^2}{3\sqrt{3}r^2} \leqslant \frac{\sqrt{3}R^2}{2r^2} . \quad \{E\}$$

This is a refinement of a result from SM, p. 44.

6.42. $$\Sigma\,\operatorname{cosec} A \leqslant \frac{2s}{3r} . \quad \{E\}$$

6.43. $$\frac{4(5R - r)^2}{4R^2 + 4Rr + 3r^2} - \frac{4R}{r} \leq \Sigma \operatorname{cosec}^2 A \leq \frac{4(R + r)^4}{r^3(16R - 5r)} - \frac{4R}{r}. \quad \{E\}$$

This is a result from SM, p. 45. The following result is due to D. M. Milošević:

$$\frac{27R^2}{s^2} \leq \Sigma \operatorname{cosec}^2 A.$$

6.44. $$\Sigma \sin\frac{A}{2}\sqrt{\sin B \sin C} \leq \frac{s}{2R}. \quad \{E\}$$

L. Pîrsan, Gaz. Mat. (Bucharest) B 20 (1969), 663-664.

6.45. $$\Sigma \cos\frac{A}{2}\sqrt{\sin B \sin C} \leq \frac{s}{R}. \quad \{E\}$$

L. Pîrsan, Gaz. Mat. (Bucharest) B 20 (1969), 663-664.

6.46. $$5 - \frac{r}{R} \leq \Sigma \frac{\cos\frac{B}{2}\cos\frac{C}{2}}{\sin\frac{A}{2}} \leq \frac{(R + r)^2}{Rr}. \quad \{E\}$$

This result of D. M. Milošević is an extension of GI 2.57.

6.47. $$\frac{2sr}{R^4}(6Rr - 2R^2 - 3r^2) \leq \Pi \sin 2A \leq \frac{2r^3 s}{R^4} \leq \frac{2r^3(2R + (3\sqrt{3} - 4)r)}{R^4} \leq$$
$$\leq \frac{3r^3\sqrt{3}}{R^3} \leq \frac{3\sqrt{3}}{8}. \quad \{E\}$$

6.48. $$M_k(\sin A, \sin B, \sin C) \leq \frac{s}{3R} \leq \frac{2}{3} + (\sqrt{3} - \frac{4}{3})\frac{r}{R} \leq \frac{\sqrt{3}}{2} \quad (k \leq 1) \ \{E\}$$

Remark. This is a refinement of GI 2.6. For $k = 1/2$ we get the following refinement of GI 2.5

$$\Sigma\sqrt{\sin A} \leq \sqrt{\frac{3s}{R}} \leq \sqrt{3(2 + (3\sqrt{3} - 4)\frac{r}{R})} \leq 3\sqrt[4]{3/4}. \quad \{E\}$$

Of course, using identities from 3.2 and Jensen's inequality for convex function we can give series of similar results. Here, we shall give a selection of such results but only for the function $x \to \sqrt{x}$.

6.49. $$\Sigma\sqrt{\sin A \sin B} \geq 3\sqrt{\frac{3r}{2R}}. \quad \{E\}$$

6.50. $$\Sigma\sqrt{\tan\frac{A}{2}} \leq 3\sqrt{\frac{4R + r}{3s}} \leq \sqrt{\frac{4R + r}{\sqrt{3}r}} \leq 3\sqrt{\frac{R}{2\sqrt{3}r}}. \quad \{E\}$$

6.51. $\Sigma\sqrt{\cot\mathrm{an}\,\frac{A}{2}} \leqslant \sqrt{\frac{3s}{r}} \leqslant \sqrt{3(\frac{2R}{r} + 3\sqrt{3} - 4)} \leqslant 3\sqrt{\frac{\sqrt{3}R}{2r}}$. {E}

6.52. $\Sigma \text{ cotan } \frac{A}{4} \geqslant 3(2 - \sqrt{3}) + \frac{2s}{r} \geqslant 6 + \frac{s}{r} \geqslant 3(2 + \sqrt{3})$. {E}

This result is due to Š. Z. Arslanagić and D. M. Milošević.

6.53. $\Sigma \cos(B - C) \leqslant (1 + \frac{2r}{R})\Sigma \cos A$. {E}

This result is due to A. Bager.

6.54. $$\Sigma \sin^p A \geqslant 3(\Pi \sin A)^{p/3} \geqslant 3^{1+p/2}(r^2/2R^2)^{p/3} \geqslant$$

$$\geqslant 3^{1+p/2}(r/R)^p,$$ {E}

where p is a positive number.

This result of W. Janous is a generalization and refinement of a result from:

D. M. Milošević, 'Problem 391', Mat. Vesnik 2 (15)(30)(1978), 396.

6.55. $\Sigma \cos A - \Sigma \cos A \cos B \geqslant 1 - (\frac{r}{R})^2 \geqslant \frac{3}{4}$. {E}

This is a refinement of a result of J. Garfunkel and G. Tsintsifas (see inequality (12) from Chapter IV).

Comments by W. Janous. 1) As an upper bound we have

$$2 > 2 - 3\frac{r}{R} + (\frac{r}{R})^2 \geqslant \Sigma \cos A - \Sigma \cos A \cos B.$$

2) The following similar inequalities are also valid:

(i) $$0 < \frac{1}{3\sqrt{3}}\left((32 - 15\sqrt{3})\frac{r}{R} - (10 - 3\sqrt{3})(\frac{r}{R})^2\right) \leqslant$$

$$\leqslant \Sigma \sin A - \Sigma \sin A \sin B < 1.$$

Remark. It would be interesting to find also an upper bound depending on r and R which is less than 1.

(ii) $$\Sigma \text{ cotan } \frac{A}{2} \text{ cotan } \frac{B}{2} - \Sigma \text{ cotan } \frac{A}{2} \geqslant 5 - 3\sqrt{3} + \frac{2R}{r} \geqslant 9 - 3\sqrt{3}.$$

(iii) $$\frac{9}{16} \leqslant \frac{1}{4}(3 - \frac{r}{R} - (\frac{r}{R})^2) \leqslant \Sigma \cos^2\frac{A}{2} - \Sigma \cos^2\frac{A}{2}\cos^2\frac{B}{2} =$$

$$= \Sigma \sin^2\frac{A}{2} - \Sigma \sin^2\frac{A}{2}\sin^2\frac{B}{2} \leqslant 1 - \frac{r}{R} + \frac{1}{4}(\frac{r}{R})^2 \leqslant$$

$$\leqslant 1 - \frac{7r}{8R} < 1.$$

J. Garfunkel and G. Tsintsifas, Inequalities through R-r-s Triangles, Private communication.

6.56. $\frac{15}{8} \leqslant 2 - \frac{r^2}{2R^2} \leqslant \Sigma(\sin^4 \frac{A}{2} + \cos^4 \frac{B}{2}) \leqslant 3(1 - \frac{r}{R} + \frac{r^2}{2R^2}) < 3.$ {E}

This inequality is due to W. Janous.

6.57. $\Sigma \frac{\tan \frac{A}{2}}{\frac{1}{\sin B} + \frac{1}{\sin C}} \geqslant \frac{3r}{2R}.$

This inequality is due to S. J. Bilčev.

6.58. $\max (a^2, b^2, c^2) < 2F\Sigma \operatorname{cosec} A.$

Problem B-I-3. Rozhledy Mat. Fyz. Praha 59 (1980/81), 466.

6.59. $\Pi(\frac{3A}{\pi}) \geqslant \frac{2r}{R}.$ {E}

V. D. Mascioni and W. Janous, 'Aufgabe 899', Elem. Math. 38 (1983), 106 and 39 (1984), 102-103.

6.60. $3/(\Sigma 1/A) \leqslant \frac{\pi}{3} \sqrt{2r/R}.$

V. D. Mascioni, 'Aufgabe 930', Elem. Math. 40 (1985).

6.61. For a triangle ABC with circumradius R and inradius r, let $M = (R - 2r)/R$. An inequality $P \geqslant Q$ involving elements of a triangle ABC will be called strong or weak, respectively, according as

$$P - Q \leqslant M \quad \text{or} \quad P - Q \geqslant M.$$

(a) The following inequalities are strong:

(1) $\Sigma \sin \frac{A}{2} \geqslant \Sigma \cos A,$

(2) $\Sigma \sin^2 \frac{A}{2} \geqslant \frac{3}{4},$

(3) $\Sigma \cos A \geqslant \frac{3}{4} + 6\Pi \sin \frac{A}{2},$

(4) $\frac{3}{2} \geqslant \Sigma \sin \frac{A}{2},$

(5) $\Sigma \cos A \geqslant \frac{1}{2} + \Pi \cos \frac{B - C}{2},$

(6) $\sum \sin^2 \frac{A}{2} \geq 1 - \frac{1}{4} \prod \cos \frac{B - C}{2}$.

(b) The following inequalities are weak:

(7) $\sum \cos A \geq 12 \prod \sin \frac{A}{2}$,

(8) $1 \geq 8 \prod \sin \frac{A}{2}$.

(9) $\frac{9}{4} \geq \sum \sin B \sin C$,

(10) $\sum \cos^2 \frac{A}{2} \geq \sum \sin B \sin C$,

(11) $\sum \cos A \geq 2\sum \cos B \cos C$,

(12) $\sum \tan^2 \frac{A}{2} \geq 1$.

Further, we can also consider a third class, where

$$P - Q \leq M/2.$$

Such inequalities do exists, but they are quite rare. For example, inequality (2) belongs to this class (see Problem 856 from Crux Math.), which we may call 'super strong'.

J. Garfunkel, Private Communication.

J. Garfunkel and W. J. Blundon, 'Problem 856', Crux Math. 9 (1983), 179 and 10 (1984), 303-304.

6.62. $\sum \sin \frac{B}{2} \sin \frac{C}{2} \geq \frac{9\sqrt{3}r}{4s}$,

$$\sum \operatorname{cosec} \frac{A}{2} \geq 9\sqrt{3} \frac{R}{s} .$$

These inequalities are due to W. Janous.

6.63. $\frac{s}{R} + 3(2 - \sqrt{3}) \leq \sum \operatorname{cosec} \frac{A}{2} < 1 + \frac{s}{R}$.

These inequalities are due to D. M. Milošević.

6.64. $\sum \tan^2 \frac{B}{2} \tan^2 \frac{C}{2} \geq \frac{8R - 7r}{16R - 5r} \geq \frac{1}{3}$.

This result is due to D. M. Milošević.

6.65. 1° $\sum \sin^4 A \leq 2 - \frac{1}{2}(\frac{r}{R})^2 - 3(\frac{r}{R})^4 \leq 2 - 5(\frac{r}{R})^4$, {E}

2° $\sum \cos^2 2A \leqslant 3 - 6(\frac{r}{R})^2 - 12(\frac{r}{R})^4 \leqslant 3 - 36(\frac{r}{R})^4,$ {E}

3° $\sum \sin^2 2A \geqslant 6(\frac{r}{R})^2 + 12(\frac{r}{R})^4 \geqslant 36(\frac{r}{R})^4,$

4° $\sum \sin 2B \sin 2C \leqslant 5(\frac{r}{R})^2 + 8(\frac{r}{R})^3 \leqslant 9(\frac{r}{R})^2.$

6.66. If $x = d/R$, where $d = OI = (R^2 - 2Rr)^{1/2}$, then the following inequalities are valid

(1) $$1 + x + 2\sqrt{1 - x} \leqslant 2\sum \sin \frac{A}{2} \leqslant 1 - x + 2\sqrt{1 + x},$$

(2) $$\sqrt{3 + x}(2 + \sqrt{1 - x}) \leqslant 2\sum \cos \frac{A}{2} \leqslant \sqrt{3 - x}(2 + \sqrt{1 + x}),$$

(3) $$1 - x + 2(1 + x)\sqrt{1 - x} \leqslant 4\sum \sin \frac{A}{2} \sin \frac{B}{2} \leqslant$$
$$\leqslant 1 + x + 2(1 - x)\sqrt{1 + x},$$

(4) $$(3 + x)(1 + 2\sqrt{1 - x}) \leqslant 4\sum \cos \frac{A}{2} \cos \frac{B}{2} \leqslant (3 - x)(1 + 2\sqrt{1 + x});$$

(5) $$\frac{2}{1 + x} + \frac{4}{\sqrt{1 - x}} \leqslant \sum \operatorname{cosec} \frac{A}{2} \leqslant \frac{2}{1 - x} + \frac{4}{\sqrt{1 + x}},$$

(6) $$\frac{4}{1 - x} + \frac{8}{(1 + x)\sqrt{1 - x}} \leqslant \sum \operatorname{cosec} \frac{A}{2} \operatorname{cosec} \frac{B}{2} \leqslant$$
$$\leqslant \frac{4}{1 + x} + \frac{8}{(1 - x)\sqrt{1 + x}},$$

(7) $$\frac{2}{\sqrt{3 - x}}\left(2 + \frac{1}{\sqrt{1 + x}}\right) \leqslant \sum \sec \frac{A}{2} \leqslant \frac{2}{\sqrt{3 + x}}\left(2 + \frac{1}{\sqrt{1 - x}}\right),$$

(8) $$\frac{4}{3 - x}\left(1 + \frac{2}{\sqrt{1 + x}}\right) \leqslant \sum \sec \frac{A}{2} \sec \frac{B}{2} \leqslant \frac{4}{3 + x}\left(1 + \frac{2}{\sqrt{1 - x}}\right).$$

All inequalities are the best possible.

D. S. Mitrinović, J. E. Pečarić, C. Tănăsescu, and V. Volenec, Inequalities Involving R, r and s for Special Triangles, Rad JAZU (Zagreb), (to appear).

7. Inequalities with (a, b, c), (A, B, C) and (R, r, s or F)

7.1. $$9r \leqslant 12r\left(1 - \frac{r}{2R}\right) \leqslant \Sigma a \sin A \leqslant 4R + \frac{2r^2}{R} \leqslant 4R + r \leqslant \frac{9R}{2}. \quad \{E\}$$

Proof. Using the identity $\Sigma a \sin A = \frac{1}{2R} \Sigma a^2$, GI5.14 and GI5.1.

we get the above result.

Remark. This is a refinement of GI 3.14. Of course, using GI 5.10 we can give a stronger result, and using results from 5.1 we can give some other results.

7.2. $$3R \leqslant \Sigma a \tan \frac{A}{2} \leqslant 5R - 4r. \quad \{E\}$$

Proof. (D. M. Milošević and Š. Z. Arslanagić.) This is a simple consequence of the identity

$$\Sigma a \tan \frac{A}{2} = 2(2R - r).$$

D. M. Milošević, 'Aufgabe 919', Elem. Math. 40 (1985).

7.3. $$2^{n-1}3^{1+(n+1)/2}r^n \leqslant \Sigma a^n \cos \frac{A}{2} \leqslant (\frac{3}{2} s\Sigma a^{2n-1})^{1/2} \quad (n \geqslant 1). \quad \{E\}$$

Remarks. 1° For a refinement of the second inequality see inequalities (4) from 4.6. Here we state the original problem from Elem. Math.

2° Note that using this result we can give the following refinement of Problem 367 from Mat. Vesnik:

$$9r \leqslant \Sigma a \cos \frac{A}{2} \leqslant \sqrt{3}s \leqslant 2\sqrt{3}R + (9 - 4\sqrt{3})r \leqslant \frac{9R}{2}. \quad \{E\}$$

D. M. Milošević, 'Aufgabe 921', Elem. Math. 40 (1985).
B. Milisavljević and D. M. Milošević, 'Problem 367', Mat. Vesnik 12 (27) (1975), 418 and 1 (14)(29)(1977), 406-407.

7.4. $$\frac{3}{s} \leqslant \frac{9\sqrt{3}R}{2s^2} \leqslant \Sigma \frac{1}{(b + c)\cos^2 \frac{A}{2}} \leqslant \frac{1}{r\sqrt{3}}. \quad \{E\}$$

R. R. Janić and I. Paasche, 'Problem 124', Mat. Vesnik 5 (20) (1968), 555-557.

7.5. $$\frac{4}{27}(\Sigma a)^3 \leqslant \Sigma \frac{a^3}{\sin^2 A} \leqslant 12R^3\sqrt{3}. \quad \{E\}$$

D. M. Milošević, 'Problem 381', Mat. Vesnik 2 (15) (30) (1978), 99-100.

7.6. $$\Sigma \frac{1}{a} \cos^2 \frac{A}{2} \geqslant \frac{27r}{8F}. \quad \{E\}$$

T. Rajkov, Kvant 6 (1976), 56.

7.7. $$\frac{Q}{4F} + \sqrt{3} \leqslant \Sigma \cotan A \leqslant \sqrt{3} + \frac{3Q}{4F}. \quad \{E\}$$

Remark. This result is due to D. M. Milošević and Š. Z. Arslanagić.

7.8. $\frac{9\sqrt{3}}{2}\Sigma a^2 \geqslant (\Sigma a)^2(\Sigma \sin A) \geqslant 54F.$ {E}

N. Schaumberger and D. C. Fuller, 'Problem 225', The College Math. J. 15 (1984), 164-165.

7.9. $\Sigma\sqrt{a}\cos\frac{A}{2} \geqslant \frac{3}{2R}\sqrt{abc}.$ {E}

C. Ionescu-Ţiu, 'Problem 7847', Gaz. Mat. (Bucharest) B 17

7.10.(a) $\Pi(Aa) \geqslant (\frac{2\pi}{3})^2 rF,$ {E}

(b) $\Pi((\pi - A)a) \geqslant (\frac{4\pi\sqrt{3}}{9})^3 sF.$ {E}

Proof. (W. Janous)
(a) Since $F = abc/(4R)$, the given inequality is equivalent to 6.59.
(b) As before, the given inequality reads equivalently

(1) $$\Pi(\pi - A) \geqslant (\frac{4\pi\sqrt{3}}{9})^3 \frac{s}{4R} .$$

Since $\frac{s}{4R} = \Pi\cos\frac{A}{2}$, (1) becomes

(2) $$(\frac{9}{2\pi\sqrt{3}})^3 \geqslant \Pi \frac{\sin((\pi - A)/2)}{(\pi - A)/2} .$$

Because of $f(t) = \log\frac{\sin t}{t}$ being strictly concave, we obtain

$$\Sigma f((\pi - A)/2) \leqslant 3f((\Sigma(\pi - A)/2)/3) = 3f(\frac{\pi}{3}) = 3\log\frac{3\sqrt{3}}{2\pi}$$

with equality if and only if $A = B = C$. Thus, (2) is proved.
V. D. Mascioni, 'Problem E 3054', Amer. Math. Monthly 91 (1984), 515.

8. Inequalities for the Radii of Excircles and Other Elements of a Triangle*

8.1. $$27r^2 \leqslant \frac{27}{2}Rr \leqslant r(16R - 5r) \leqslant \Sigma r_b r_c \leqslant 4R^2 + 4Rr + 3r^2 \leqslant$$

$$\leqslant \frac{1}{3}(4R + r)^2 \leqslant \frac{27R^2}{4} .$$

* In all inequalities from this Section, equalities hold only if the triangle is equilateral.

Remark. This is a refinement of a result from SM, p. 43.

8.2. $$27r^3 \leqslant 3r^2(4R + r) \leqslant r^2(16R - 5r) \leqslant \Pi r_a \leqslant r(4R^2 + 4Rr + 3r^2) \leqslant$$
$$\leqslant \frac{r}{3}(4R + r)^2 \leqslant \frac{27}{4} rR^2 \leqslant \frac{27}{8} R^3.$$

Remark. This is a refinement of a result from SM, p. 43.

8.3. $$27r^2 \leqslant \frac{27}{2} Rr \leqslant \frac{27}{4} R^2 \leqslant \frac{R}{2}(16R - 5r) \leqslant 8R^2 - 5r^2 \leqslant \Sigma r_a^2 \leqslant$$
$$\leqslant 16R^2 - 24Rr + 11r^2 \leqslant \frac{4R - 5r}{3r} s^2.$$

Remark. This is a refinement and extension of GI 5.43 and of a result from SM, p. 43.

8.4. Let $u \geqslant 0$ and $v \leqslant 2u$ be given constants. The maximum value of the positive number $k = k(u, v)$ such that the inequality

$$s^2 + k(R - 2r)(uR - vr) \leqslant \Sigma r_a^2$$

holds for any triangle is

$$k = k_{max} = \min\left(\frac{4}{u}, \frac{12}{2u - v}\right).$$

I. Paasche, 'Problem 329', Mat. Vesnik 12 (27) (1975), 317-318.

8.5. $$81r^3 \leqslant 3rs^2 \leqslant 16R^3 + r^3 - 24Rr^2 \leqslant \Sigma r_a^3 \leqslant$$
$$\leqslant 64R^3 - 144R^2r + 72Rr^2 + r^3.$$

Remark. This is a refinement of a result from SM, p. 43.

8.6. $$216r^3 \leqslant 108Rr^2 \leqslant 12Rr(4R + r) \leqslant 4Rr(16R - 5r) \leqslant \Pi(r_b + r_c) \leqslant$$
$$\leqslant 4R(4R^2 + 4Rr + 3r^2) \leqslant \frac{4R}{3}(4R + r)^2 \leqslant 27R^3.$$

Remark. This is a refinement of a result from SM, p. 43. Note that the inequality: $\Pi(1 + r_b/r_c) \geqslant 8$ {E} is given in:

μ.σ.,'Su una relazione fra lati ed altezze di un triangolo rettineo', Boll. Soc. Mat. Calabrese 21 (1970), 87-89.

8.7. $$\frac{2}{R} \leqslant \Sigma \frac{1}{r_a} = \frac{1}{r}.$$

SM, p. 43.

8.8. $$\frac{4}{3R^2} \leqslant \frac{2}{3Rr} \leqslant \frac{4R + r}{r(4R^2 + 4Rr + 3r^2)} \leqslant \Sigma \frac{1}{r_b r_c} \leqslant \frac{4R + r}{r^2(16R - 5r)} \leqslant$$

$$\leqslant \frac{1}{3r^2} \leqslant \frac{R}{6r^3} .$$

SM, p. 43.

8.9. $$\frac{1}{3r^2} \leqslant \frac{8R - 7r}{r^2(16R - 5r)} \leqslant \Sigma \frac{1}{r_a^2} \leqslant \frac{(2R - r)^2}{r^2(4R^2 + 4Rr + 3r^2)} .$$

SM, p. 43.

8.10. $$6 \leqslant \Sigma \frac{r_b + r_c}{r_a} = \frac{4R - 2r}{r} .$$

SM, p. 43.

8.11. $$\frac{1}{r} \leqslant \Sigma \frac{r_a}{r_b r_c} \leqslant \frac{4R - 5r}{3r^2} .$$

Remark. The first inequality is given by M. Erdman.
M. Erdman, 'Problem 14', Matematyka (Warszawa) No. 3 (112)(1970), 190.

8.12. $$9r(\Sigma r_a)^2 + 9r^3 \geqslant 32\Pi r_a - 14r^2\Sigma r_a .$$

S. Reich and R. W. Frucht, 'Problem E 1930', Amer. Math. Monthly **73** (1966), 1017 and **75** (1968), 299.

8.13. $$\Sigma a r_a \geqslant 3Rs \geqslant 6F.$$

This is Milošević's interpolation of a result from:
Gh. Vernič, 'Problem 11934', Gaz. Mat. (Bucharest) **B 23** (1972), 169.

8.14. $$18 \frac{r}{s} \leqslant \Sigma \frac{a}{r_a} \leqslant 9 \frac{R}{s} .$$

Proof. Since $r_a = rs/x$ $(x = s - a)$ and $\Sigma ax = 2r(4R + r)$, we have

$$\Sigma a/r_a = \Sigma ax/rs = 2(4R + r)/s.$$

Using the inequality $2r \leqslant R$ we get the above inequalities.
Ž. M. Mitrović, 'Problem 215', Mat. Vesnik **9** (24) (1972), 181.

8.15. $$3\sqrt[6]{\frac{4R}{s}} \leqslant \Sigma\sqrt{\frac{a}{r_a}} \leqslant \sqrt{2}\sqrt{\frac{s}{r}} .$$

W. Janous communicated to us the following generalization:

For $p \in (-\infty, +\infty)$: $\Sigma(a/r_a)^p \geqslant 3(4R/s)^{p/3}$.

For $0 < p \leqslant 1/2$: $\Sigma(a/r_a)^p \leqslant 3^{-2p+1}2^p(s/r)^p$.

A. Makowski, 'Problems and Remarks on Inequalities for a Triangle', Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 412-460 (1973), 127-130.

8.16 $$\frac{25}{3\sqrt{3}} - \frac{8R}{s} \leqslant \frac{8R - 7r}{s} \leqslant \Sigma \frac{x}{r_a} \leqslant \begin{cases} \frac{(2R - r)^2}{F} \\ \frac{s}{r} - 2\sqrt{3} \end{cases}$$

Remark. This result is due to D. S. Mitrinović, J. E. Pečarić, and W. Janous. The terms of the right-hand-side are incomparable in general.

8.17. $$\frac{2R^2\sqrt{27}}{rs} - 4 \geqslant \Sigma \frac{a^2}{r_b r_c} \geqslant 4(\sqrt{2Rs}/(r\sqrt[4]{27}) - 1) \geqslant 4.$$

Remark. This result is due to I. Paasche, and it is a refinement of GI 5.30.

R. R. Janić, A. Lupaş, and I. Paasche, 'Problem 133', Mat. Vesnik 6 (21) (1969), 93-94.

8.18. $$\Sigma \frac{r_b r_c}{a^2} \geqslant \frac{(5R - r)(2R - r)}{3R^2} \geqslant \frac{9}{4}. \quad \{E\}$$

This inequality is due to D. M. Milošević.

8.19. $$16(4R^2 - 8Rr + 3r^2) \leqslant \Sigma \frac{a^4}{r_b r_c} \leqslant \frac{16}{r}(R^3 - R^2 r - r^3), \quad \{E\}$$

$$4\sqrt{3}(3R - 4r) \leqslant \Sigma \frac{a^3}{r_b r_c} \leqslant \frac{2R\sqrt{3}}{r}(3R - 4r). \quad \{E\}$$

These results are due to D. M. Milošević.

8.20. $$32Rr^2 \leqslant 2r^2(9R - 2r) \leqslant \Pi(r + r_a) \leqslant 4r(2R^2 + 3Rr + 2r^2) \leqslant 8R^3.$$

Proof. Since $F = sr = (s - c)r_c = abc/(4R)$, we have

$$\frac{r + r_c}{2R} = \frac{\frac{F}{s} + \frac{F}{s - c}}{\frac{abc}{2F}} = \frac{2F^2(a + b)}{s(s - c)abc} = \frac{2(s - a)(s - b)(a + b)}{abc} =$$

$$= \frac{(a + b)(c^2 - (a - b)^2)}{2abc} = \frac{b^2 + c^2 - a^2}{2bc} + \frac{a^2 + c^2 - b^2}{2ac} =$$

$$= \cos A + \cos B,$$

and therefore

$$\Pi(r + r_a) = 8R^3\Pi(\cos A + \cos B).$$

Now, using 6.14 we get the desired result.

Remark. This result is an extension and refinement of an inequality of M. Stanković.

M. Stanković, 'Problem 174', Mat. Vesnik 6 (21) (1969), 351.

8.21. There exists a number s, $0 < s < 1$, with the property that

$$\frac{\sqrt{3}}{2} M_r(a, b, c) \leqslant M_r(r_a, r_b, r_c) \quad \text{for } r > s,$$

and the reverse inequality for $r < s$.

J. Steinig, 'Sur quelques applications géométriques d'une inégalité relative aux fonctions convexes', Enseign. Math. (2) 11 (1965), 281-285.

8.22. $\Sigma a^{r_a}/r_a \geqslant (abc)^r/r.$

Proof. (W. Janous). Since $F = sr = (s - a)r_a$, the above inequality is equivalent to

$$\Sigma \frac{s - a}{s} a^{\frac{F}{s-a}} \geqslant (abc)^{\frac{F}{s}}.$$

This inequality is a simple consequence of the weighted arithmetic-geometric inequality $\Sigma ux \geqslant \Pi x^u$ ($\Sigma u = 1$), because $\Sigma \frac{s - a}{s} = 1$.

M. Burtea, 'Problem O 33', Gaz. Mat. (Bucharest) 84 (1979), 210.

8.23. $\Sigma \frac{r_b + r_c}{a} \geqslant 3\sqrt{3}.$

This result is due to D. M. Milošević.

8.24. $\frac{3\sqrt{3}}{2} \leqslant \Sigma \frac{r_b + r_c}{b + c} \leqslant \frac{s}{2r}$.

Remark. D. M. Milošević proved the following inequality

$$\sum \frac{r_b + r_c}{b + c} \leqslant \frac{s(3R + 2r)}{r(9R - 2r)}$$

which is stronger than the second inequality.

D. M. Milošević and M. Vowe, 'Aufgabe 916', Elem. Math. 39 (1984), 156 and 40 (1985), 152-153.

Š. Z. Arslanagić and D. M. Milošević, 'Some Inequalities for a Triangle', Radovi Matematički (Sarajevo) 2 (1986), 35-44.

8.25. $$\prod \frac{r_a - r}{r_b + r_c} \leqslant \frac{1}{27}.$$

M. S. Jovanović, 'Some Inequalities for Triangle Elements', Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 498-541 (1975), 171-178.

8.26. $$\sum a/r_a \leqslant 9R/s,$$

$$\sum a^2/(r_a - r) \leqslant 9R.$$

P. Balev, 'Problem 4', Mat. i Fiz. (Sofija) 4, 11 (1968), 56.

8.27. $$24\frac{r}{R} \leqslant \sum\left(\left(\frac{r_a}{r_b}\right)^{1/2} + \left(\frac{r_b}{r_a}\right)^{1/2}\right)^2 \leqslant 6\frac{R}{r}.$$

T. Zamfirescu and T. Albu, 'Problem 6621', Gaz. Mat. (Bucharest) B 15 (1964), 514 and B 16 (1965), 405.

8.28. If $a > b > c$, then

$$3R - 2r_a < 0, \quad 3R - 2r_b \gtrless 0, \quad 3R - 2r_c > 0.$$

L. Toscano, 'Les distances des centres des cercles tritangents d'un triangle à la droite des points isogones', Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 274-301 (1969), 115-119.

8.29. $$\frac{3}{2} \leqslant \frac{3(5R - 2r)}{9R - 2r} \leqslant \sum \frac{r_a - r}{r_a + r} \leqslant \frac{3}{2} + \frac{2R^2 - 3Rr - 2r^2}{2R^2 + 3Rr + 2r^2} < 2, \quad \{E\}$$

$$6 \leqslant 7 - \frac{2r}{R} \leqslant \sum \frac{r_a + r}{r_a - r} \leqslant 1 + \frac{2(R^2 + r^2)}{Rr} \leqslant 2 + \frac{2R}{r}. \quad \{E\}$$

These inequalities are due to D. M. Milošević.

8.30. (i) $$R \geqslant \frac{(r_a + r)^2}{4(r_a - r)},$$

(ii) $3r(\Sigma r_a) \leqslant \Sigma r_b r_c$.

S. Iwata, Encyclopedia of Geometry (Japanese), Tokyo, 1971, Vol. 5, p. 337.

8.31. Given the angle A and the side a of a triangle ABC. Then

(i) $0 < r < \frac{a}{2}(\sec \frac{A}{2} - \tan \frac{A}{2})$;

(ii) $a < r_a \leqslant \frac{a}{2}(\sec \frac{A}{2} + \tan \frac{A}{2})$;

(iii) $0 < r_b < \frac{a}{2}(\cotan \frac{A}{2} + \cosec \frac{A}{2})$.
(or r_c)

Proof. (i) Since $r = (s - a) \tan \frac{A}{2}$, and $b + c = 2R(\sin B + \sin C) = 4R \sin \frac{B + C}{2} \cos \frac{B - C}{2} = 4R \cos \frac{A}{2} \cos \frac{B - C}{2}$, we have

$$a < b + c \leqslant 4R \cos \frac{A}{2} = a/\sin \frac{A}{2} .$$

Thus,

$$0 < r = (s - a) \tan \frac{A}{2} = (\frac{b + c - a}{2}) \tan \frac{A}{2} \leqslant$$

$$\leqslant \left(\frac{a}{2 \sin \frac{A}{2}} - \frac{a}{2}\right) \tan \frac{A}{2} = \frac{a}{2} (\sec \frac{A}{2} - \tan \frac{A}{2}).$$

(ii) Since $\tan \frac{A}{2} = \frac{r_a}{s}$, we have

$$r_a = s \tan \frac{A}{2} = \frac{1}{2} (a + b + c) \tan \frac{A}{2} \leqslant$$

$$\leqslant \frac{1}{2} (a + \frac{a}{\sin \frac{A}{2}}) \tan \frac{A}{2} = \frac{a}{2} (\sec \frac{A}{2} + \tan \frac{A}{2}).$$

(iii) $r_b = (s - c) \cotan \frac{A}{2} = (\frac{a + b - c}{2}) \cotan \frac{A}{2}$.

Now,

$$b - c = 2R(\sin B - \sin C) = 4R \cos \frac{B + C}{2} \sin \frac{B - C}{2} =$$

$$= 4R \sin \frac{A}{2} \sin \frac{B - C}{2} ,$$

i.e.

$$-4R \sin \frac{A}{2} < b - c < 4R \sin \frac{A}{2} \quad (B - C \neq \pm\pi \text{ means non-equality}),$$

$$a + b - c < 4R \sin \frac{A}{2} + a$$

$$\frac{(a + b - c)}{2} \operatorname{cotan} \frac{A}{2} < (4R \sin \frac{A}{2} + a) \frac{1}{2} \operatorname{cotan} \frac{A}{2} =$$

$$= 2R \cos \frac{A}{2} + \frac{a}{2} \operatorname{cotan} \frac{A}{2} =$$

$$= \frac{a}{\sin A} \cos \frac{A}{2} + \frac{a}{2} \operatorname{cotan} \frac{A}{2} =$$

$$= \frac{a}{2} (\operatorname{cosec} \frac{A}{2} + \operatorname{cotan} \frac{A}{2}).$$

S. Iwata, Encyclopedia of Geometry (Japanese), Tokyo, 1971, Vol. 5, p. 345.

8.32. The following inequalities are valid:

(1) $\Sigma a/r_a^2 \geq 6/s$, (2) $\Sigma a^2/r_a \geq 12r$, (3) $\Sigma a(b + c)/r_a \geq 24r$,

(4) $\Sigma a^2/r_a^2 \geq 4$, (5) $\Sigma a^2/r_a^3 \leq 4(R - r)/3r^2$,

(6) $\frac{(a + c)^2}{r_c} + \frac{(a + b)^2}{r_b} - \frac{(b - c)^2}{r} \geq 8(r + r_a)$,

(7) $\frac{1}{r^3} - \Sigma \frac{1}{r_a^3} \geq \frac{24r}{F^2}$, (8) $\frac{1}{r^4} + \Sigma \frac{1}{r_a^4} \geq \frac{28}{F^2}$,

(9) $\frac{1}{r^5} - \Sigma \frac{1}{r_a^5} \geq \frac{40\sqrt{3}R}{F^3}$, (10) $\frac{1}{r^6} + \Sigma \frac{1}{r_a^6} \geq \frac{244\sqrt{3}}{3F^3}$,

(11) $\frac{1}{r^7} - \Sigma \frac{1}{r_a^7} \geq \frac{364R}{F^4}$.

Proof. These results are simple consequences of the following identities of L. Toscano:

(1) $\Sigma a/r_a^2 = 2(2R - r)/F$, (2) $\Sigma a^2/r_a = 4(R + r)$,

(3) $\Sigma a(b + c)/r_a = 12R$, (4) $\Sigma a^2/r_a^2 = \frac{2}{s^2} ((4R + r)^2 - s^2)$,

(5) $$\Sigma a^2/r_a^3 = 4(4R + r)(R - r)/Fr,$$

(6) $$\frac{(a + c)^2}{r_c} + \frac{(a + b)^2}{r_b} - \frac{(b - c)^2}{r} = 4(R + 2r_a),$$

(7) $$\frac{1}{r^3} - \Sigma \frac{1}{r_a^3} = \frac{12R}{F^2},$$ (8) $$\frac{1}{r^4} + \Sigma \frac{1}{r_a^4} = \frac{(\Sigma a^2)^2 + 8F^2}{2F^4},$$

(9) $$\frac{1}{r^5} - \Sigma \frac{1}{r_a^5} = \frac{10(\Sigma a^2)R}{F^4},$$ (10) $$\frac{1}{r^6} + \Sigma \frac{1}{r_a^6} = \frac{\Sigma a^6 + 15F^2\Sigma a^2}{F^6},$$

(11) $$\frac{1}{r^7} - \Sigma \frac{1}{r_a^7} = \frac{7((\Sigma a^2)^2 + 4F^2)R}{F^6}$$

and of inequalities GI 5.1, 5.5, 4.4 and 4.13. Of course, using some other inequalities we can obtain several new results.

L. Toscano, 'Sui raggi dei cerchi tritangenti e circoscritto di un triangolo', Bolletino della Soc. Mat. Calabrese (5) 21 (1970), 60-67.

9. Inequalities Involving Altitudes and Other Elements of a Triangle

9.1. Let be $A \geqslant B \geqslant C$. Then

$$\Sigma h_b/h_c \geqslant \Sigma h_c/h_b$$

with equality only for isosceles triangles.

M. Moisei and L. Olaru, 'Problem 16726', Gaz. Math. (Bucharest) 82 (1977), 291 and 83 (1978), 113-114.

9.2. $$\Sigma \frac{a^2}{h_b h_c} \geqslant 4. \quad \{E\}$$

Ž. Mitrović, A. Makowski, and I. Paasche, 'Problem 176', Mat. Vesnik 7 (22) (1970), 425-426.

μ.σ. 'Su una relazione fra lati ed altezze di un triangolo rettilineo', Bolletino della Soc. Mat. Calabrese 21 (1970), 87-89.

9.3. $$\Pi \frac{h_b + h_c}{b + c} \leqslant \frac{3}{8}\sqrt{3}. \quad \{E\}$$

Ž. M. Mitrović, A. Makowski, and Ž. Mijajlović, 'Problem 204', Mat. Vesnik 8 (23) (1971), 339-340.

Comment by W. Janous: We can strengthen it to its ultimate range: The inequality:

$$M_k\left(\frac{h_a + h_b}{a + b}, \frac{h_b + h_c}{b + c}, \frac{h_c + h_a}{c + a}\right) \leqslant \frac{\sqrt{3}}{2}$$

is valid for every triangle, if and only if $k \leqslant (\log 9 - \log 4)/(\log 4 - \log 3)$.

9.4. $\Sigma a^{h_a}/h_a \geqslant (abc)^r/r.$

M. Burtea, 'Problem O 33', Gaz. Mat. (Bucharest) 84 (1979), 210.

9.5. $9r \leqslant \frac{2r}{R}(5R - r) \leqslant \Sigma h_a \leqslant \frac{2}{R}(R + r)^2 \leqslant 2R + 5r \leqslant 3(R + r) \leqslant \leqslant \frac{9}{2} R.$ {E}

A weaker result is given by L. Bankoff, 'Problem 594', Math. Mag. 39 (1966), 130.

Remark. This is an extension of GI 6.11, 6.12 and 6.13, and a refinement of a result from SM, p. 41. The following result is also valid

$$9F/s \leqslant \Sigma h_a \leqslant 2s^2/(3R). \quad \{E\}$$

The first inequality (note that $F/s = r$) is given by N. Schaumberger.

9.6. $54r^3/R \leqslant \left\{\begin{matrix} 27r^2 \\ 6r^2(4R + r)/R \end{matrix}\right\} \leqslant 2r^2(16R - 5r)/R \leqslant \Sigma h_b h_c \leqslant \leqslant 2r(4R^2 + 4Rr + 3r^2)/R \leqslant 2r(4R + r)^2/3R \leqslant \leqslant \frac{27}{2} Rr \leqslant \frac{27}{4} R^2.$ {E}

Remark. This is a refinement of a result from SM, p. 41.

9.7. $27r^3 \leqslant \frac{2r^3}{R}(16R - 5r) \leqslant \Pi h_a \leqslant \frac{2r^2}{R}(4R^2 + 4Rr + 3r^2) \leqslant \leqslant \frac{2r^2}{3R}(4R + r)^2 \leqslant \frac{27}{2} r^2 R \leqslant \frac{27}{8} R^3.$ {E}

9.8. $27r^2 \leqslant \Sigma h_a^2 \leqslant s^2 \leqslant 27R^2/4.$ {E}

SM, p. 41.

Remark. V. Vâjâitu communicated to us that the best inequality of type $\Sigma h_a^2 \leqslant kR^2 + jr^2$ holds for $k = 4$ and $j = 11$. We note that the following interpolating inequality is valid

$$\Sigma h_a^2 \leqslant 4R^2 + 7r^2 + 8r^3/R \leqslant 4R^2 + 11r^2. \quad \{E\}$$

Indeed, the fundamental inequality can be written in the following form

$$s^4 - (8Rr - 2r^2)s^2 + r^2(4R + r)^2 \leqslant 4R((R + 3r)s^2 - r(4R + r)^2),$$

i.e.

$$(s^2 + r^2 + 4Rr)^2 - 16s^2Rr \leqslant 4R((R + 3r)s^2 - r(4R + r)^2),$$

i.e.

$$R\Sigma h_a^2 \leqslant (R + 3r)s^2 - r(4R + r)^2.$$

Now, using Gerretsen's inequality $s^2 \leqslant 4R^2 + 4Rr + 3r^2$, we get the first inequality. The second inequality is a simple consequence of the Chapple-Euler inequality.

9.9. $$216r^3 \leqslant \frac{2r^3}{R^2}(16R - 5r)(9R - 2r) \leqslant \Pi(h_b + h_c) \leqslant$$

$$\leqslant \frac{2r}{R^2}(4R^2 + 4Rr + 3r^2)(2R^2 + 3Rr + 2r^2) \leqslant 54R^2r \leqslant$$

$$\leqslant 27R^3. \quad \{E\}$$

SM, p. 42.

9.10. $$\frac{2}{R} \leqslant \Sigma \frac{1}{h_a} = \frac{1}{r}. \quad \{E\}$$

SM, p. 42.

9.11. $$\frac{4}{3R^2} \leqslant \frac{2}{3Rr} \leqslant \frac{5R - r}{r(4R^2 + 4Rr + 3r^2)} \leqslant \Sigma \frac{1}{h_bh_c} \leqslant$$

$$\leqslant \frac{(R + r)^2}{r^3(16R - 5r)} \leqslant \frac{R}{6r^3}. \quad \{E\}$$

SM, p. 42.

9.12. $$\frac{4}{3R^2} \leqslant \frac{2}{3Rr} \leqslant \frac{3(2R - r)}{r(4R^2 + 4Rr + 3r^2)} \leqslant \Sigma \frac{1}{h_a^2} \leqslant$$

$$\leqslant \frac{2R^2 + r^2}{r^3(16R - 5r)} \leqslant \frac{R}{6r^3}. \quad \{E\}$$

SM, p. 42.

9.13. $$6 \leq \frac{7R - 2r}{R} \leq \sum \frac{h_b + h_c}{h_a} \leq \frac{2R^2 + Rr + 2r^2}{Rr} \leq \frac{7R - 2r}{2r}. \quad \{E\}$$

SM, p. 42.

9.14. $$F \leq \frac{s}{3} \min (h_a, h_b, h_c).$$

E.C. Popa, 'On Some Geometric Inequalities (Romanian)', Gaz. Mat. (Bucharest) 87 (1982), 256-257.

9.15. $$\frac{3}{2} \leq \frac{115R - 38r}{75R - 22r} \leq \sum \frac{h_a - r}{h_a + r} < \frac{5}{3}, \quad \{E\}$$

$$6 \leq \frac{3(19R - 6r)}{9R - 2r} \leq \sum \frac{h_a + r}{h_a - r} < 7. \quad \{E\}$$

These results are due to D. M. Milošević. The second result is an extension and interpolation of GI 6.22, and the first one is an interpolation of a result from:

D. M. Milošević and O. P. Lossers, 'Aufgabe 900', Elem. Math. 38 (1983), 128 and 39 (1984), 130.

9.16. $$\sum \sqrt{b^2 + c^2 - h_a^2} \leq 6R.$$

This inequality is due to S. J. Bilčev.

9.17. If $k \leq K = \dfrac{\log 9 - \log 4}{\log 4 - \log 3} = 2.81...$, then

(1) $$M_k(h_a, h_b, h_c) \leq \frac{\sqrt{3}}{2} M_k(a, b, c).$$

If $-1 \leq k \leq K$, then

(2) $$0 \leq \frac{\sqrt{3}}{3} M_k(h_a, h_b, h_c) - \frac{2}{3} M_k(a, b, c) \leq R - 2r.$$

Remark. Inequality (1) was proved by F. Leuenberger, and (2) by J. Berkes. A similar result was proved by S. C. Dumitru:

$$\sum h_a^n < \frac{2^n - 1}{2^n} \sum a^n \quad (n > 2).$$

F. Leuenberger, 'Bemerkungen zu zwei Arbeiten von A. Makowski und J. Berkes', Elem. Math. 18 (1963), 33.

J. Berkes, 'Certaines Inégalités Relatives au Triangle', Univ.

Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 247-273 (1969), 151-152.
S. C. Dumitru, 'Problem 13337', Gaz. Mat. (Bucharest) B 24 (1973), 484.

9.18. Now we shall give some results of M. R. Tasković for the expressions:

$$I_\lambda(a, b, c) = \Sigma \frac{a^\lambda}{h_b^\lambda + h_c^\lambda}, \quad M_\lambda(a, b, c) = \Sigma \frac{a^\lambda + b^\lambda}{h_a^\lambda + h_b^\lambda},$$

$$N_\lambda(a, b, c) = \Sigma \frac{a^\lambda + b^\lambda}{h_c^\lambda} \quad (\lambda \neq 0 \text{ is a real number}).$$

The following results are valid (we shall write only I_λ instead of $I_\lambda(a, b, c)$, etc.):

1.1. $I_\lambda \geqslant 2^{\lambda-1}3^{1-\lambda/2} \quad (0 < \lambda \leqslant 1)$, {E}

and the reverse inequality if $-1 \leqslant \lambda \leqslant 0$.

1.2. $I_\lambda \geqslant \sqrt{3} \quad (\lambda \geqslant 1)$ and $I_\lambda \leqslant 4^{-1}3^{3/2} \quad (\lambda \leqslant -1)$, $\{E_1\}$

where here and further on $\{E_1\}$ means that equality holds if and only if $\lambda = 1$ or -1 and $a = b = c$.

1.3. $I_\lambda \leqslant 2^{-(1+\lambda/2)}3^{1-\lambda/2}(R/r)^{3\lambda/2} \quad (0 < \lambda \leqslant 2)$ and

$I_\lambda \geqslant 2^{-(1+\lambda)}3^{1-\lambda/2}(R/r)^{2\lambda} \quad (-1 \leqslant \lambda < 0)$. {E}

1.4. $I_\lambda \leqslant 2^{-1}3^{\lambda/2}(R/r)^{\lambda} \quad (0 < \lambda \leqslant 1)$,

$I_\lambda \leqslant 2^{\lambda-3}3^{2-3\lambda/2}(R/r)^{2\lambda} \quad (\lambda \geqslant 1)$ and

$I_\lambda \geqslant 2^{1+\lambda}3^{-3\lambda/2}(R/r)^{2\lambda} \quad (\lambda \leqslant -1)$. $\{E_1\}$

2.1. $M_\lambda \geqslant 2^{\lambda}3^{1-\lambda/2} \quad (\lambda > 0)$ and

$M_\lambda \geqslant 3^{1-\lambda/2}(R/r)^{\lambda} \quad (\lambda < 0)$. {E}

2.2. $M_\lambda \leqslant 2^{-\lambda}3^{1-\lambda/2}(R/r)^{2\lambda} \quad (0 < \lambda \leqslant 1)$, and

$M_\lambda \leqslant 2^{\lambda}3^{1-\lambda/2} \quad (-1 \leqslant \lambda < 0)$. {E}

2.3. $M_\lambda \leqslant 2^{\lambda-2}3^{2-3\lambda/2}(R/r)^{2\lambda} \quad (\lambda \geqslant 1)$, and

$M_\lambda \leqslant 3^{3/2}/2 \quad (\lambda \leqslant -1).$ {E_1}

3.1. $N_\lambda \geqslant 2^{1+\lambda}3^{1-\lambda/2} \quad (\lambda > 0)$, and

$N_\lambda \geqslant 2 \cdot 3^{1-\lambda/2}(R/r)^\lambda \quad (\lambda < 0).$ {E}

3.2. $N_\lambda \leqslant 2^{1-\lambda}3^{1-\lambda/2}(R/r)^{2\lambda} \quad (0 < \lambda \leqslant 1)$, and

$N_\lambda \leqslant 2^{1+\lambda}3^{1-\lambda/2} \quad (-1 \leqslant \lambda < 0).$ {E}

3.3. $N_\lambda \leqslant 2 \cdot 3^{\lambda/2}(R/r)^\lambda \quad (\lambda \geqslant 1)$,

$N_\lambda \leqslant 2^{\lambda-1}3^{2-3\lambda/2}(R/r)^\lambda \quad (\lambda > 0)$ and

$N_\lambda \leqslant 3^{3/2}/2 \quad (\lambda \leqslant -1).$ {E_1}

4.1. $I_\lambda \cdot \Sigma \sin^\lambda A \geqslant 9.$ {E}

4.2. $M_\lambda \geqslant 2I_\lambda.$ {E}

4.3. $6F^\lambda N_\lambda \leqslant N_{-\lambda}^2 \cdot 2^\lambda R^{2\lambda}.$ {E}

Remark. Some special cases of the above results were first given by R.R. Janić and M. R. Tasković.

M. R. Tasković, 'Généralisations de Certaines Inégalités Géometriques et Quelques Inégalités Géométriques Nouvelles', Mat. Vesnik 7 (22) (1970), 73-81.

R. R. Janić and M. R. Tasković, 'Quelques Inégalités Relatives à un Triangle', Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 247-273 (1969), 131-134.

9.19. $3R \leqslant \Sigma \dfrac{a^2}{h_b + h_c} < 4R.$

D. M. Milošević, 'Problem 107', Matematika 4 (1980), 70 and 2 (1981), 90.

9.20. $\dfrac{9r}{s} \leqslant \Sigma \dfrac{x}{h_a} \leqslant \dfrac{9R}{2s}.$ {E}

I. Paasche and Ž. Mitrović, 'Problem 166', Mat. Vesnik 7 (22) (1970), 275.

9.21. $\dfrac{2s}{R} \leqslant \dfrac{2(4R^2 + 6Rr - r^2)}{Rs} \leqslant \Sigma \dfrac{h_a}{x} \leqslant 2\dfrac{5R^2 + 3Rr + r^2}{Rs}.$ {E}

Proof. Since $h_a/x = 2r_a/a = 2s \tan\frac{A}{2}/(2R \sin A) = 2s(1 + \tan^2\frac{A}{2})/$ $/(4R) = (s^2 + r_a^2)/(2Rs)$, and $\Sigma r_a^2 = (4R + r)^2 - 2s^2$, we have

$$\Sigma h_a/x = (s^2 + (4R + r)^2)/(2Rs).$$

Now, using the well-known Gerretsen inequalities GI 5.8, i.e.

$$(1) \qquad r(16R - 5r) \leqslant s^2 \leqslant 4R^2 + 4Rr + 3r^2$$

we get the second and the third inequality. The first inequality is also a simple consequence of the second inequality in (1) and Chapple-Euler's inequality $R \geqslant 2r$.

Remark. The first two inequalities are Leuenberger's refinement of Mitrović's problem. Of course, the above proof also contains the following inequalities:

$$\frac{s}{R} \leqslant \frac{4R^2 + 6Rr - r^2}{Rs} \leqslant \Sigma \frac{r_a}{r} \leqslant \frac{5R^2 + 3Rr + r^2}{Rs}. \quad \{E\}$$

9.22. $\Sigma r_a^2 r_b^2/ab \geqslant 81r^2/4$. {E}

9.23. $\tan\frac{A}{2} \leqslant \frac{a}{2h_a}$.

E. G. Gotman, 'Teorema kosinusov i ee sledstvija', Kvant 7 (1972), 29-32.

9.24. $h_a \leqslant \sqrt{bc}\cos\frac{A}{2}$.

E. G. Gotman, 'Problem 2734', Mat. v Škole 1984, No. 3 and 1985, No. 2, 64.

9.25. $3 \leqslant 2(2 - \frac{r}{R}) \leqslant \Sigma h_a/r_a \leqslant 2(\frac{R}{r} + \frac{r}{R} - 1)$. {E}

Remark. The second inequality, due to D. M. Milošević, gives a refinement of the following inequality from Mat. Vesnik, Problem 232:

$$\Sigma h_a/r_a \geqslant 3. \quad \{E\}$$

Of course, using the method from VIII 1.2 we can give a generalization of an inequality from Problem 449 from Mat. Vesnik.

Ž. Mitrović and I. Paasche, 'Problem 232', Mat. Vesnik 9 (24) (1972), 311.

B. Milisavljević, 'Problem 449', Mat. Vesnik 2 (15)(30)(1978), 295.

9.26. $\frac{3}{2} \leqslant \frac{9\sqrt{3}}{4}\frac{R}{s} \leqslant \Sigma \frac{r_a}{h_b + h_c} \leqslant \frac{s}{2r\sqrt{3}}$. {E}

R. R. Janić and I. Paasche, 'Problem 124', Mat. Vesnik 5 (20) (1968), 555-557.

9.27. The following inequalities are valid:

(1) $$\left.\begin{matrix}\sqrt{3\sqrt{3}F}\\ \Sigma h_a/\sqrt{3}\end{matrix}\right\} \leqslant \Sigma x = s \leqslant \Sigma r_a/\sqrt{3},$$

(2) $$\Sigma h_b h_c \leqslant 3\sqrt{3}F \leqslant 3\Sigma yz \leqslant \frac{3}{4}\Sigma bc \leqslant \Sigma r_b r_c,$$

(3) $$\Pi h_a \leqslant 3\sqrt{3}xyz \leqslant \sqrt{3\sqrt{3}F^3} \leqslant \Pi r_a \leqslant 3\sqrt{3}abc/8.$$

I. Paasche and Ž. M. Mitrović, 'Problem 220', Mat. Vesnik 8 (23) (1971), 90 and 415.

9.28. $$a\sqrt{27} + 4h_a - r_a \leqslant 27R/2.$$

I. Paasche, 'Problem 327', Mat. Vesnik 12 (27)(1975), 316-317.

9.29. If $ax = b + c$, then

$$2 < h_a^x(xr)^{-x} < e.$$

Proof. Since $x = (b + c)/a$, $x + 1 = 2s/a$, $h_a a/2 = rs$, and $h_a = r(x + 1)$, we have $h_a^x(xr)^{-x} = (1 + 1/x)^x := f(x)$. The function $x \to f(x)$ is strictly increasing on $(1, +\infty)$, so $2 < f(x) < e$.

F. Leuenberger and R. Weissauer, 'Aufgabe 674', Elem. Math. 27 (1972), 95 and 28 (1973), 99.

9.30. $$(\Sigma r_a^n/\Sigma h_a^{-2n})^2 \leqslant ((\Sigma a)^6/1728)^n \quad (n \geqslant 1).$$

D. Milošević, 'Problem 381', Mat. Vesnik 2 (15)(30)(1978), 99-100.

9.31. The following inequalities are valid:

1° $$\Sigma\left(\frac{h_b + h_c}{r_b + r_c}\right)^\lambda \leqslant 3 \quad (0 < \lambda \leqslant 1) \quad \{E_1\}^*,$$

2° $$\Pi\frac{h_b + h_c}{r_a + h_a} \leqslant 1 \quad \{E\},$$ 3° $$\Pi\frac{r_a + h_a}{b + c} \leqslant \frac{3}{8}\sqrt{3} \quad \{E\},$$

4° $$\Sigma\frac{r_a + h_a}{r + r_a} \geqslant \frac{9}{2} \quad \{E\},$$

* $\{E_1\}$ means that equality holds if and only if $\lambda = 1$ and $a = b = c$.

5° $\sum\left(\frac{r_a + r}{h_a - r}\right)^{\lambda} \geq 3 \cdot 2^{\lambda} \quad (\lambda \geq 1) \quad \{E_1\},$

6° $\sum\left(\frac{r_a - r}{h_a - 2r}\right)^{\lambda} \geq 3 \cdot 6^{\lambda} \quad (\lambda \geq 1) \quad \{E_1\},$

7° $\prod \frac{h_a - r}{r_b + r_c} \leq \frac{1}{27} \quad \{E\},$

8° $\sum\left(\frac{b + c}{h_a - r}\right)^{\lambda} \geq 3 \cdot 12^{\lambda/2} \quad (\lambda \geq 1) \quad \{E_1\},$

9° $\sum\left(\frac{r_a}{h_a + 2r_a}\right)^{\lambda} \geq 3^{1-\lambda} \quad (\lambda \geq 1) \quad \{E_1\},$

10° $\sum\left(\frac{h_a - 2r}{h_a + 2r_a}\right)^{\lambda} \geq 3^{1-2\lambda} \quad (\lambda \geq 1/2)$

with equality if and only if the triangle is equilateral and $\lambda = 1/2$,

11° $\sum\left(\frac{r_b + r_c}{h_a + 2r_a}\right)^{\lambda} \geq 3(\frac{2}{3})^{\lambda} \quad (\lambda \geq 1) \quad \{E_1\},$

12° $\sum \frac{r + r_a}{h_a + 2r_a} \leq \frac{4}{3} \quad \{E\},$ 13° $\prod \frac{h_a - r}{b + c} \leq \frac{\sqrt{3}}{72} \quad \{E\},$

14° $\sum\left(\frac{h_a - r}{h_a + r_a}\right)^{\lambda} \geq 3^{1-\lambda} \quad (\lambda \geq 1) \quad \{E_1\},$

15° $\prod \frac{h_a - r}{h_a + r_a} \leq \frac{1}{27} \quad \{E\}.$

M. S. Jovanović, 'Some Inequalities for Triangle Elements', Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 498-541 (1975), 171-178.

9.32. $\sum \frac{r_b + r_c}{h_a + 2r_a} \geq \frac{4}{3}(2 - \frac{r}{R}). \quad \{E\}$

Proof. Since $r_a = F/(s - a)$, $h_a = 2F/a$, etc., we have

(1) $$\sum \frac{r_b + r_c}{h_a + 2r_a} = \frac{1}{F^2}(\sum a^2(s - a)^2).$$

Further, using the inequality $(\Sigma a(s - a))^2 \leqslant 3(\Sigma a^2(s - a)^2)$ and the identity $\Sigma a(s - a) = 2r(4R + r)$, we get from (1):

$$(2) \qquad \Sigma \frac{r_b + r_c}{h_a + 2r_a} \geqslant \frac{2}{3}\left(\frac{r + 4R}{s}\right)^2.$$

Now, using GI 5.7, i.e. $2s^2(2R - r) \leqslant R(4R + r)^2$, we get the desired inequality from (2).

Remark. This result is due to Š. Arslanagić and D. M. Milošević, and it is better than 9.31.11° for $\lambda = 1$.

9.33. $$4r\Sigma r_a \leqslant \Sigma a^2 \leqslant \frac{27}{4} R^4 \Sigma 1/h_a^2.$$

D. M. Milošević and S. Srećković, 'Problem 34', Matematika 1977, No. 1, 78-80.

9.34. $$\frac{2r}{R} \leqslant \frac{\Sigma h_a}{\Sigma r_a} \leqslant \frac{2}{3}\left(1 + \frac{r}{R}\right).$$

M. Marčev, 'Problem 2', Matematika (Sofija) 1976, No. 6, 32 and 1977, No. 3, 36-37.

9.35. $$\Sigma \frac{1}{h_a r_a} \geqslant \frac{1}{r(2R - r)}.$$

M. Grecu, 'Problem 20826', Gaz. Mat. (Bucharest) 91 (1986), 264.

10. Inequalities with Medians and Other Elements of a Triangle

10.1. $$\frac{9}{20} \Sigma bc < \Sigma m_b m_c < \frac{5}{4} \Sigma bc.$$

The constant 9/20 and 5/4 are the best possible.

Proof. We shall give Bottema's proof of the second inequality. Starting with $\Sigma m_a \leqslant \Sigma a$, which implies

$$\Sigma m_a^2 + 2\Sigma m_b m_c \leqslant \Sigma a^2 + 2\Sigma bc,$$

and using the identity $4\Sigma m_a^2 = 3\Sigma a^2$, we get

$$2\Sigma m_b m_c \leqslant \frac{1}{4} \Sigma a^2 + 2\Sigma bc \leqslant \frac{5}{2} \Sigma bc.$$

Therefore,

$$\Sigma m_b m_c \leqslant \frac{5}{4} \Sigma bc.$$

In fact, 5/4 is the best coefficient, as shown by the example a = b = 1, c = 0 and, therefore, $m_a = m_b = \frac{1}{2}$, $m_c = 1$, $\Sigma m_b m_c = \frac{5}{4}$, $\Sigma bc = 1$.

Bottema gave the first inequality as a conjecture, and he obtained the coefficient 9/20 using the example a = b = 1, c = 2, which implies $m_a = m_b = \frac{3}{2}$, $m_c = 0$. The proof of this inequality is given by A. Reznikov. Here we shall note that the first inequality is a simple consequence of the second, i.e. it is the median-dual of the second inequality (see VIII, 4).

O. Bottema, Comment. In the paper: M. Jovanović, 'Some Inequalities Involving Elements of a Triangle and a Polygon', Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 357-380 (1971), 81-85.

A. Reznikov, Odno geometričeskoe neravenstvo, Kvant 1975, No. 12, 56.

10.2. $$\frac{1}{4}\max((a-b)^2, (a-c)^2, (b-c)^2) < \Sigma m_b m_c <$$
$$< \frac{3}{4}\max((a+b)^2, (a+c)^2, (b+c)^2).$$

C. Boangiu, 'Problem 19809', Gaz. Mat. (Bucharest) 88 (1983), 331.

Comment by W. Janous. This inequality can be significantly improved to

$$\frac{21}{4}\min((a-b)^2, \ldots) \leqslant \frac{s^2}{2} + \frac{3}{4}\min((a-b)^2, \ldots) < \Sigma m_b m_c \leqslant$$
$$\leqslant \frac{3}{4}\max((a^2+ab+b^2, b^2+bc+c^2, c^2+ca+a^2) <$$
$$< \frac{3}{4}\max((a+b)^2, \ldots).$$

10.3. $$9r(R+r) \leqslant \Sigma m_b m_c \leqslant 5R^2 + \frac{9}{4}Rr + \frac{5}{2}r^2.$$

A. Bager, 'Some Inequalities for the Medians of a Triangle', Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 338-352 (1971), 37-40.

10.4. $$27r^2 \leqslant \frac{27}{2}Rr \leqslant 9r(2R-r) \leqslant \Sigma m_a^2 \leqslant 3(2R^2+r^2) \leqslant \frac{27}{4}R^2.$$

Remark. This is an extension of a result of A. Bager, and a refinement and extension of a result of V. Gh. Vodă.

A. Bager, The same reference as in 10.3.

V. Gh. Vodă, Gaz. Mat. (Bucharest) 83 (1978), 58-60.

10.5. $$\frac{9}{4} \leqslant \Sigma \frac{m_a^2}{bc} \leqslant \frac{1}{8Rr}(4R^2 + 6Rr + 8r^2) \leqslant \frac{1}{4r}(2R+5r) \leqslant \frac{9R}{8r}. \quad \{E\}$$

Proof. Since,

$$\Sigma a m_a^2 = \Sigma a(\frac{1}{4} b^2 + \frac{1}{4} c^2 + \frac{1}{2} bc \cos A) =$$

$$= \frac{1}{4} \Sigma(ab^2 + a^2 b) + \frac{1}{2} abc\Sigma \cos A$$

and

$$\Sigma(a^2 b + ab^2) = (\Sigma a)(\Sigma bc) - 3abc = 2s(s^2 - 2Rr + r^2),$$

$$abc = 4RF = 4Rrs, \quad \Sigma \cos A = (R + r)/R,$$

we get

$$\Sigma a m_a^2 = \frac{s}{2}(s^2 + 2Rr + 5r^2), \quad \text{i.e.} \quad \Sigma \frac{m_a^2}{bc} = \frac{s^2 + 2Rr + 5r^2}{8Rr} .$$

Now, using Gerretsen's inequalities (GI 5.8) and Chapple-Euler's inequality $R \geqslant 2r$, we get the desired inequalities.

Remark. In the above proof we used the idea from Bager's proof of the first inequality (see reference from 10.3). Bager also gave the median dual to his inequality:

$$\Sigma \frac{a^2}{m_b m_c} \geqslant 4,$$

wherefrom

$$\Sigma \frac{\sin^2 A}{m_b m_c} \geqslant \frac{1}{R^2} .$$

He also gave the following inequality equivalent to the first inequality from 10.5

$$\Sigma a m_a^2 \geqslant 9RF.$$

10.6. (a) $\Sigma \frac{m_a^2}{a^2} \geqslant \frac{9}{4}$ {E},

(b) $\Sigma \frac{m_a}{a} \geqslant \frac{3\sqrt{3}}{2}$. {E}

Remark. The above inequalities are due to V. Matizen. The first inequality is again given as problem in Mat. v. Škole and in Crux Math.

V. Matizen, Kvant 1976, No. 8, 16.

Ngo Tan and M. S. Klamkin, 'Problem 589', Crux Math. 6 (1980), 317 and 7 (1981), 307-308.

S. I. Majzus, 'Problem 2953', Mat. v Škole 1983, No. 2, 61 and 1984,

No. 1, 70.

10.7. $\Sigma m_b^2 m_c^2 \geqslant rs^2(4R + r) \geqslant 9F^2$. {E}

Remark. Using an inequality of M. Jovanović we have a refinement of an inequality of A. Bager.

M. Jovanović, 'Inequalities for the Triangle', Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 412-460 (1973), 155-157.

A. Bager, The same reference as in 10.3.

10.8. $\Sigma \frac{1}{m_b m_c} \leqslant \frac{\sqrt{3}}{F}$. {E}

Remark. This inequality of A. Bager (see reference from 10.3) was again obtained by M. Jovanović (see reference from 10.7).

10.9. $\Sigma \frac{m_a}{b + c} \geqslant \frac{3\sqrt{3}}{2} \frac{r}{R}$. {E}

M. Jovanović, The same reference as in 10.7.

10.10. $\Sigma \frac{a^2}{m_b^2 + m_c^2} \leqslant 2$. {E}

Remark. This inequality is due to M. Jovanović (see reference from 10.7). Its median-dual is

$$\Sigma \frac{m_a^2}{b^2 + c^2} \leqslant \frac{9}{8},$$

and this inequality is given in:

V. A. Visenskij, i dr., Sbornik zadač Kievskih matematičeskih olimpiad, Kiev 1984, p. 54.

10.11. $\Sigma \frac{m_b^2 m_c^2}{bc} \geqslant \frac{81}{4} r^2$. {E}

A. Bager and A. Makowski, 'Aufgabe 702', Elem. Math. 28 (1973), 133 and 29 (1974), 121.

10.12.(a) $\Pi m_a \geqslant r\Sigma m_a^2$, (b) $12R\Pi m_a \geqslant \Sigma a(b + c)m_a^2$,

(c) $4R\Sigma am_a \geqslant \Sigma bc(b + c)$, (d) $2R\Sigma \frac{1}{bc} \geqslant \Sigma \frac{m_a}{m_b m_c}$.

J. Garfunkel, G. Tsintsifas, and M. S. Klamkin, 'Problem 846', Crux Math. 9 (1983), 143 and 10 (1984), 267-269.

Comment by W. Janous. (a) also reads: $2s(\Pi m_a) \geqslant 2F\Sigma m_a^2$, yielding the median-dual: $\Sigma m_a \geqslant 2F(\Sigma a^2)/\Pi a = (\Sigma a^2)/2R$, i.e.

$$\Sigma a^2 \leqslant 2R\Sigma m_a.$$

(c) reads also $\Sigma am_a \geqslant F\Sigma(b + c)/a$, with its median-dual $\Sigma am_a \geqslant F\Sigma(m_b + m_c)/m_a$ yielding

$$\Sigma am_a \geqslant F \max (\Sigma(b + c)/a, \Sigma(m_b + m_c)/m_a) \geqslant 6F.$$

10.13. $(\Sigma a^2)(\Sigma am_a) \geqslant 4(\Pi m_a)(\Sigma a)$, $(\Sigma a^2)(\Sigma am_a) \geqslant 3(\Pi a)(\Sigma m_a)$.

M. S. Klamkin, 'Problem 79-19', SIAM Review 22 (1980), 509-511.

10.14. $$\max^2 (a, b, c) - \min^2 (a, b, c) \leqslant \Sigma am_a \leqslant s\left(4 \max^2 (a, b, c) - \min^2 (a, b, c)\right)^{1/2}$$

Proof (W. Janous). Suppose that $a \leqslant b \leqslant c$, then the first inequality becomes

$$2c^2 - 2a^2 < \Sigma a\sqrt{2b^2 + 2c^2 - a^2}.$$

Dividing by a^2 and putting $b/a = x$, $c/a = y$ we require the following inequality to be proved:

(1) $$(2x^2 + 2y^2 - 1)^{1/2} + x(2 + 2y^2 - x^2)^{1/2} + y(2 + 2x^2 - y^2)^{1/2} > 2y^2 - 2$$

where $1 \leqslant x \leqslant y$ and $y < x + 1$ (triangle inequality!).

It is easily checked that

$$x(2 + 2y^2 - x^2)^{1/2} > y^2 - 1, \quad (2x^2 + 2y^2 - 1)^{1/2} > 2y - 1$$

and $$y(2 + 2x^2 - y^2)^{1/2} > y^2 - 2y$$

whence the required inequality (1) follows.

Moreover, triangles like $a = b = 2$, $c \to 4$ (i.e. $m_a = m_b \to 3$ and $m_c \to 0$) show what the left-hand-estimation is quite sharp (in non-equilateral extremal cases).

Analogously, the second inequality becomes

$$\Sigma am_a = \frac{1}{2} \Sigma a(2b^2 + 2c^2 - a^2)^{1/2} \leqslant \frac{1}{2} \sqrt{4c^2 - a^2}(\Sigma a),$$

since $a \leqslant b \leqslant c$. This is obviously true.

Note that the first inequality is given in

A. Bumbăcea, 'Problem 19457', Gaz. Mat. (Bucharest) 87 (1982), 422.

10.15. P. Erdös (in a letter to L. Bankoff) posed two questions:

Let $\Sigma m_a = a + b + \alpha c$. Is it true that $\alpha \geqslant 0$? Is it true that max α occurs for the equilateral triangle?

A. Oppenheim showed that both conjectures are true, i.e. the following results are valid:

1° $\Sigma m_a \geqslant a + b$

with equality only for degenerate triangles of sides $b + 1$, b, 1 and medians $\frac{1}{2}(b - 1)$, $\frac{1}{2} b + 1$, $b + \frac{1}{2}$.

2° $2\Sigma m_a \leqslant 2a + 2b + (3\sqrt{3} - 4)c \quad (a, b \geqslant c)$. {E}

The median-duals to the above results are:

3° $\frac{3}{4} \Sigma a \geqslant m_c + m_b$

with equality only for degenerate triangles of sides $b + 1$, b, 1.

$$\frac{3}{2} \Sigma a \leqslant 2m_b + 2m_c + (3\sqrt{3} - 4)m_a \quad (b, c \leqslant a).$$

A. Oppenheim, Private communication.

10.16. $m_a \geqslant \frac{1}{2} \sqrt{a(8s - 9a)}$.

Ja. N. Sukonnik, 'Problem 2090', Mat. v škole 1979, No. 1, 68 and 1979, No. 6, 60.

Comment by W. Janous. In Jovanović, reference of 10.7, the inequality $m_a^2 \geqslant s(s - a)$ can be found. It is indeed stronger than one stated in 10.16.

10.17. $\Sigma m_a^6 \geqslant s^4(s^2 - 12Rr)$.

G. Bojčev, 'Geometrični neravenstva, sidiržašči medianite i njakoi drugi elementi na triigilnika', Ob. po matematika (Sofija) 1981, No. 1, 41-49.

10.18. $$\frac{c^2 - (a - b)^2}{2(a + b)} \leqslant a + b - 2m_c < \frac{c^2 + (a - b)^2}{4m_c}.$$

J. Vincze, 'Problem F 2294', Köz. Mat. Lap. 62 (1981), 31 and 63 (1981), 63-64.

10.19. $9r \leqslant \Sigma h_a \leqslant \frac{2}{3R} \Sigma m_a^2$.

M. Ibragimov, Kvant 1980, No. 7, 9.

10.20. $\Sigma m_a h_a \leqslant s^2$.

L. Panaitopol, 'Problem 0:324', Gaz. Math. (Bucharest) 87 (1982), 336.

10.21. $\Sigma h_b/m_c \leqslant 3$. {E}

Proof. (M. S. Klamkin and A. Meir). We establish here the stronger result

(1) $\Sigma h_1/m_a \leqslant 3$. {E}

where (h_1, h_2, h_3) is any permutation of (h_a, h_b, h_c). It is clear from (1) that there is no loss of generality in assuming that $a \leqslant b \leqslant c$, with the consequent

(2) $h_a \geqslant h_b \geqslant h_c$ and $m_a \geqslant m_b \geqslant m_c$.

With the assumptions (2), it follows by an elementary rearrangement inequality (see for example Hardy, Littlewood, Pólya, Inequalities. Cambridge, 1952, p. 261) that

(3) $\Sigma h_a/m_a \leqslant \Sigma h_1/m_a \leqslant \Sigma h_a/m_c$.

Apart from telling us which of the six permutations (h_1, h_2, h_3) will minimize and maximize the left side of (1), (3) shows that (1) becomes

(4) $h_a/m_c + h_b/m_b + h_c/m_a \leqslant 3$,

which we proceed to establish.

By Cauchy's inequality

$$(\Sigma h_a/m_c)^2 \leqslant (\Sigma h_a^2)(\Sigma 1/m_c^2),$$

and we need only show that

(5) $(\Sigma h_a^2)(\Sigma 1/m_a^2) \leqslant 9$.

Since $h_a = 2F/a$, etc., $4m_a^2 = 2b^2 + 2c^2 - a^2$, etc., (5) can be written in the form

(6) $$16F^2(\Sigma a^{-2})(\Sigma(2b^2 + 2c^2 - a^2)^{-1}) \leqslant 9.$$

Using $16F^2 = \Sigma(2b^2c^2 - a^4)$, (6) can be manipulated into the equivalent form

$$F(a^2, b^2, c^2) = a^2b^2c^2\Pi(2b^2 + 2c^2 - a^2) -$$

$$- (\Sigma(2b^2c^2 - a^4))(\Sigma b^2c^2)^2 \geqslant 0.$$

An easy but tedious calculation shows that $F(x, y, z)$ and F/z both vanish for $x = y$. Thus, by symmetry, we must have $F(x, y, z) = (y - z)^2(z - x)^2(x - y)^2$, and (5) follows from

(7) $$F(a^2, b^2, c^2) = (b^2 - c^2)^2(c^2 - a^2)^2(a^2 - b^2)^2 \geqslant 0.$$

Thus (4) and hence (1) are established.

Suppose equality holds in (1); then it also holds in (4). Now we have

$$a = b \quad \text{or} \quad b = c \quad \text{or} \quad c = a$$

from (7) and

$$h_am_c = h_bm_b = h_cm_a$$

from the equality in Cauchy's inequality. Since

$$a = b \Rightarrow h_a = h_b \Rightarrow m_c = m_b \Rightarrow c = b,$$

$$b = c \Rightarrow h_b = h_c \Rightarrow m_b = m_a \Rightarrow b = a,$$

and $c = a \Rightarrow a = b = c$ follows from $a \leqslant b \leqslant c$, we conclude that the triangle is equilateral. The converse is trivial.

J. Garfunkel, M. S. Klamkin, and A. Meir, 'Problem 517', Crux Math. 6 (1980), 44 and 7 (1981), 61-63.

J. Garfunkel, B. J. Venkatachala, C. R. Pranesachar, and C. S. Gardner, 'Problem E 2715', Amer. Math. Monthly 85 (1978), 384; 86 (1979), 705-706 and 87 (1980), 304.

10.22. (1) $m_a/h_a \leqslant R/2r$;

(2) $3 \leqslant \Sigma\sqrt{m_a/h_a} \leqslant 3\sqrt{R/2r}$.

The first result is due to Panaitopol, and the second to Milošević.

L. Panaitopol, Gaz. Mat. (Bucharest) 87 (1982), 428.

D. M. Milošević, 'Problem 107', Matematika 4 (1980), 70.

10.23. $\Sigma 1/m_a > 5/s$.

This is a problem of W. Janous (to appear in Crux Math.).

11. Inequalities with Angle-Bisectors and Other Elements of a Triangle

11.1. $\Sigma 1/w_a^2 \geqslant 9/s^2$.

D. Buşneag, 'Problem 10245', Gaz. Mat. (Bucharest) B 22 (1971), 615-616.

11.2.
$$\left.\begin{array}{c}\dfrac{8Rr^2s^2}{2R^2+3Rr+2r^2}\\[2ex] \dfrac{8r^2(16R-5r)}{9R-2r}\end{array}\right\} \leqslant \Pi w_a \leqslant \left\{\begin{array}{l}\dfrac{8Rrs^2}{9R-2r} \leqslant rs^2\\[2ex] \dfrac{8Rr^2(4R^2+4Rr+3r^2)}{2R^2+3Rr+2r^2}.\end{array}\right.$$

Remark. This is an extension and refinement of GI 8.14.

11.3. (i) Let $p > 0$ be a real number. Then

$$3^{p+1}r^p \leqslant 3^{p+1}(Rr^2/2)^{p/3} \leqslant \Sigma w_a^p . \quad \{E\}$$

(ii) Let $0 < p \leqslant 1$ be a real number. Then

$$\Sigma w_a^p \leqslant 3(s/\sqrt{3})^p \leqslant 3(3R/2)^p. \quad \{E\}$$

These inequalities are due to W. Janous. For $p = 1/2$ we get the results of

D. M. Milošević and B. Milisavljević, 'Problem 405', Mat. Vesnik 2 (15)(30)(1978), 424-425.

11.4. $$4\Sigma w_b w_c \leqslant (\Sigma bc)\left(2 + \frac{8abc}{\Pi(b+c)}\right) \leqslant 3(\Sigma bc). \quad \{E\}$$

Remark. The above result of G. Bercea gives a refinement of an inequality of A. Bager.

A. Bager and G. Bercea, 'Aufgabe 706', Elem. Math. 28 (1973), 156 and 29 (1974), 151-153.

11.5. $\Sigma a w_a \leqslant 2s^2/\sqrt{3}$. {E}

Proof. Using GI 8.12 for $\lambda = 1$ and GI 5.12 we get

$$(\Sigma a w_a)^2 \leqslant \frac{9}{2}\,abcs = 18Rrs^2 \leqslant 4s^4/3 \quad \{E\},$$

from which the required inequality follows.

Remark. This proof is a modification of Arslanagić's proof.

11.6.(a) $6rs \leqslant \Sigma aw_a \leqslant \Sigma am_a \leqslant \frac{\sqrt{3}}{2} \Sigma a^2$, {E}

(b) $6rs \leqslant \Sigma aw_a \leqslant \Sigma am_a \leqslant 3Rs$. {E}

Remark. These results are due to B. M. Milisavljević and R. R. Janić.

R. R. Janić, Comments. In: B. M. Milisavljević, 'Some Inequalities Related to a Triangle', Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 498-541 (1975), 181-184.

11.7. $\frac{2}{3}\sqrt{2}s < \sqrt{\Sigma w_a^2} \leqslant s$.

The constants $2\sqrt{2}/3$ and 1 are the best possible.

A. Mameev, I. Bajčev, and M. Petkov, 'Problem 3', Mat. Fiz. (Sofija) 2, No. 3 (1960), 60 and 3, No. 4 (1961), 46-47.

11.8. $\Sigma 1/w_a < 3R/F$.

V. Gridasov, 'Problem 4', Mat. Fiz. (Sofija) 3, No. 11 (1968), 57.

11.9.(a) $\sqrt{3}\Sigma \frac{1}{aw_a} \geqslant \frac{4s}{abc}$ {E},

(b) $3\sqrt{3}(\Sigma \frac{1}{aw_a})/(\Sigma aw_a) \geqslant 4\sqrt{\frac{2s}{(abc)^3}}$. {E}

J. T. Groenman and B. Priellip, 'Problem 815', Crux Math. 9 (1983), 46 and 10 (1984), 132.

11.10. $\Sigma aw_b w_c \leqslant Rs^3/(3r)$. {E}

This inequality is due to W. Janous.

11.11. $\frac{3}{2} \leqslant \Sigma\sqrt{1 - w_a^2/bc} < 3$.

M. Chirita, 'Problem 19456', Gaz. Mat. (Bucharest) 87 (1982), 422.

11.12. Let $a \geqslant b \geqslant c$ be the sides of a triangle and let x, y, z be real numbers such that $x \geqslant 0$ and $z \geqslant 0$.

(a) If $y \leqslant (x^{1/2} + z^{1/2})^2$, then

(1) $\Sigma xaw_a^2 \leqslant \frac{1}{2} abc\Sigma x/a$.

(b) If $y^2 \leqslant (xa^{1/2} + zc^{1/2})^2$, then

(2) $\Sigma x a w_a \leq (\frac{3}{2}\, abcs(\Sigma x^2))^{1/2}$.

Remark. For $x = a^{t-1}$, $y = b^{t-1}$, $z = c^{t-1}$ we get GI 8.12 and 8.13.

11.13. $\Pi(\sin B + \sin C) \leq \Pi \frac{s}{w_a}$. {E}

11.14. $\Sigma w_a \leq s\sqrt{3} \leq \Sigma r_a$, {E}

$\Sigma h_a^2 \leq \Sigma w_a^2 \leq s^2 \leq \Sigma m_a^2 \leq \Sigma r_a^2$. {E}

A. Bager, 'Some Inequalities for the Medians of a Triangle', Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 338-352 (1971), 37-40.

11.15. $9r \leq \Sigma h_a \leq \Sigma w_a \leq \Sigma m_a \leq \Sigma r_a \leq 9R/2$. {E}

Ž. M. Mitrović, R. R. Janić, and I. Paasche, 'Problem 196', Mat. Vesnik 8 (23)(1971), 243-244.

11.16. $\Pi(h_b + h_c) \leq 24\sqrt{3}\Pi \frac{F}{w_a}$. {E}

11.17. $1 < \Sigma h_a/w_a \leq 3$.

M. Žižovic, S. Srećković, and Š. Arslanagić, 'Problem 23', Matematika 1976, No. 1, 110-111.

11.18.(a) $6r/R \leq \Sigma w_a/m_a \leq 3$ {E},

(b) $3 \leq \Sigma m_a/w_a \leq \frac{3R}{2r}$ {E},

(c) $3 \leq \Sigma w_a/h_a \leq \frac{3R}{2r}$. {E}

B. Milisavljević, 'Problem 456', Mat. Vesnik 2 (15)(30)(1978), 422-423.

Remark. D. M. Milošević communicated to us the following similar inequalities

$\Sigma w_a/h_a \leq \sqrt{3(1 + R/r)}$ and $\Sigma m_a/w_a \geq (13/4) - (r/2R)$. {E}

11.19. $R - 2r \geq w_a - h_a$.

L. Panaitopol, 'Problem 0: 374', Gaz. Mat. (Bucharest) 88 (1983), 300.

11.20. $s^2 \leq \Sigma m_a w_a \leq 3(2R^2 + r^2) \leq 27R^2/4.$ {E}

This is Janous' extension of a result from
L. Panaitopol and E. Vasile, 'Problem 0: 139', Gaz. Mat. (Bucharest) 85 (1980), 223 and 86 (1981), 223.

11.21. $\Sigma w_a^6 \leq s^4(s^2 - 12rR) \leq \Sigma m_a^6.$ {E}

Proof. First note that

$$s^4(s^2 - 12rR) = s^4(s^2 - 12\frac{F}{s}\frac{abc}{4F}) = s^4(s^2 - \frac{3abc}{s}) =$$

$$= s^3(s^3 - 3abc) = s^3\Sigma(s - a)^3.$$

Now

$$w_a = \frac{2\sqrt{bc}}{b + c}\sqrt{s(s - a)} \leq \sqrt{s(s - a)}$$

with equality if and only if $b = c$. Similar expressions hold for w_b and w_c. Thus, $\Sigma w_a^6 \leq s^3(\Sigma(s - a)^3)$ with equality if and only if the triangle is equilateral. Also

$$m_a^2 = \frac{1}{4}(2b^2 + 2c^2 - a^2) = \frac{1}{4}((b + c - a)(b + c + a) +$$

$$+ b^2 + c^2 - 2bc) = (s - a)(s) + \frac{1}{4}(b - c)^2.$$

That is, $m_a^2 \geq s(s - a)$ with equality if and only if $b = c$. Similarly for m_b and m_c. It follows that

$$\Sigma m_a^6 \geq s^3(\Sigma(s - a)^3)$$

with equality if and only if the triangle is equilateral.
F. Leuenberger and S. Reich, 'Problem E 2060', Amer. Math. Monthly 75 (1968), 189 and 76 (1969), 197-198.

11.22.(a) $\Sigma \frac{w_a}{h_a + 2r_a} \geq 1$ {E},

(b) $\Sigma\left(\frac{w_a}{h_b + h_c}\right)^\lambda \geq (\frac{r}{4R})^{\lambda/3}$ $(\lambda > 0)$,

(c) $\Sigma\left(\frac{w_a}{h_a - 2r}\right)^\lambda \geq 3^{\lambda+1}$ $(\lambda \geq 1)$.

Equality in (b) and (c) holds if and only if the triangle is equilateral and $\lambda = 1$.

M. S. Jovanović, 'Some Inequalities for Triangle Elements', Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 541 (1975), 171-178.

11.23. $\sum \frac{b^2 + c^2}{w_a^2} \geq 8.$ {E}

M. Jovanović, 'Inequalities for the Triangle', Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 412-460 (1973), 155-157.

11.24. If $F(x, y, z)$ is a symmetric increasing function of x, y, z, then

$$F(w_a, w_b, w_c) \leq F(m_a, m_b, m_c). \quad \{E\}$$

Proof. This is a simple consequence of the following inequalities: $w_a \leq m_a$, $w_b \leq m_b$, and $w_c \leq m_c$.

Remark. The following inequalities are consequences of the above result:

$$\sum w_a^\lambda \leq \sum m_a^\lambda \quad (\lambda > 0), \qquad \prod w_a \leq \prod m_a, \qquad \sum w_a^\lambda \geq \sum m_a^\lambda \quad (\lambda < 0). \quad \{E\}$$

M. S. Klamkin, 'Problem 421', Pi Mu Epsilon J. 6 (1974/79), 483-484, 631.

11.25. $\sum m_a + \min(a, b, c) \leq \sum w_a + \max(a, b, c)$.

11.26. (a) $\frac{1}{2} \leq \frac{w_a + w_b + m_c}{a + b + c} \leq \frac{\sqrt{3}}{2}$,

(b) $\frac{1}{4} \leq \frac{h_a + m_b + w_c}{a + b + c} \leq \frac{\sqrt{3}}{2}$,

(c) $\frac{3}{8} \leq \frac{h_a + m_b + m_c}{a + b + c} \leq 1$,

where all the bounds are the best possible.

Remark. The second inequality from (b) is given as conjecture E 2504 in Amer. Math. Monthly 81 (1974), 1111. The above results are from the solution of L. Ting and R. Lo (see Editor's comment). The second inequality from (a), i.e.

(1) $$w_a + w_b + m_c \leq s\sqrt{3} \quad \{E\},$$

is again obtained by G. S. Lessells and M. J. Pelling. Later, B. E. Patuwo, R. S. D. Thomas, and C.-L. Wang proved the following better inequality

$$(2)\qquad \sqrt{s(s-a)} + \sqrt{s(s-b)} + m_c \leqslant s\sqrt{3}. \qquad \{E\}$$

Indeed, taking into account

$$w_a = \frac{2\sqrt{bc}}{b+c}\sqrt{s(s-a)} \leqslant \sqrt{s(s-a)}$$

we observe that (2) is much stronger than (1). A very simple proof of (2) was given by C. Tănăsescu. Here, we shall give his proof.

By the arithmetic-geometric means inequality we have

$$2\sqrt{(s-a)(s-b)} \leqslant (s-a) + (s-b) = c$$

with equality if and only if a = b. Consequently

$$4m_c^2 = 2a^2 + 2b^2 - c^2 = (a+b)^2 + (b-a)^2 - c^2 =$$

$$= (a+b)^2 - (c^2 - (b-a)^2) =$$

$$= (a+b)^2 - (b+c-a)(a+c-b) =$$

$$= (a+b)^2 - 4(s-a)(s-b) =$$

$$= (a+b+2\sqrt{(s-a)(s-b)})(a+b-2\sqrt{(s-a)(s-b)})$$

implies

$$4m_c^2 \leqslant 2s(2s - (\sqrt{s-a} + \sqrt{s-b})^2)$$

since $a + b - 2\sqrt{(s-a)(s-b)} = 2s - (\sqrt{s-a} + \sqrt{s-b})^2$, or the following important inequality

$$(3)\qquad \sqrt{s(s-a)} + \sqrt{s(s-b)} = \sqrt{2}\sqrt{s^2 - m_c^2},$$

with equality only for a = b.

So (2) follows from

$$(4)\qquad \sqrt{2}\cdot\sqrt{s^2 - m_c^2} + m_c \leqslant s\sqrt{3},$$

which is true by the Cauchy-Schwarz inequality:

$$(\sqrt{2}\cdot\sqrt{s^2 - m_c^2} + 1\cdot m_c)^2 \leqslant 3((s^2 - m_c^2) + m_c^2).$$

For the equality case in (2) it is then necessary and sufficient that a = b and $\sqrt{s^2 - m_c^2} = m_c\sqrt{2}$, i.e. a = b = c.

The last inequality (4) shows that (3) is even stronger than (2).

J. Garfunkel and C. S. Gardner, 'Problem E 2504', Amer. Math. Monthly 81 (1974), 1111 and 83 (1976), 289-290.
G. S. Lessells and M. J. Pelling, 'An Inequality for the Sum of Two Angle Bisectors and a Median', Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 577-598 (1977), 59-62.
B. E. Patuwo, R. S. D. Thomas, and C.-L. Wang, 'The Triangle Inequality of Lessells and Pelling', Ibid. No. 678-715 (1980), 45-47.
C. Tănăsescu, A Short Proof of a Triangle Inequality (unpublished).
Comments on 'Problem 936', Math. Mag. 59 (1986), 179.

11.27. Let w'_a, w'_b, w'_c be the external angle-bisectors of a triangle. Then for every non-isosceles triangle

$$\Sigma w'_a > 12F/(\Sigma a^2)^{1/2}.$$

V. Gridasov, 'Problem 3', Mat. Fiz. (Sofija), 1968, No. 2, 41-42.

11.28. Conjecture: $\Sigma w_a^{-1} > \Sigma a^{-1}$.

'Problem Gy 2188', Köz. Mat. Lapok 1984, No. 3, 126.

11.29. $$\Sigma \frac{1}{a} \leqslant \frac{3}{2} \Sigma \frac{1}{w_a}.$$

'Problem 2', Köz. Mat. Lapok 1984, No. 10, 458.

11.30. If $R = 1$, then

$$w_a \leqslant \cos^2 \frac{A}{2} \cos \frac{B - C}{2} \leqslant m_a.$$

J. Garfunkel and J. Dou, 'Problem 423', Crux Math. 5 (1979), 76 and 6 (1980), 26-27.

11.31.(1) $$\frac{(b + c)^2}{4bc} \leqslant \frac{m_a}{w_a},$$

with equality if and only if $b = c$.

(2) $$\frac{b^2 + c^2}{2bc} \leqslant \frac{m_a}{h_a},$$

with equality if and only if $b = c$ or A is a right angle.

Proof. If k_a denotes the symmedian to side a, then

$$m_a = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}, \quad w_a = \frac{\sqrt{bc}}{b + c}\sqrt{(b + c)^2 - a^2},$$

$$k_a = \frac{bc}{b^2 + c^2}\sqrt{2b^2 + 2c^2 - a^2}.$$

By the triangle inequality, $a^2 > (b - c)^2$, so $(b + c)^2 - a^2 < (b + c)^2 - (b - c)^2 = 4bc$, and therefore

$$(3) \qquad \frac{(b - c)^2}{4bc} \leqslant \frac{(b - c)^2}{(b + c)^2 - a^2},$$

with equality if and only if b = c. By adding unity to each side of (3), we obtain

$$\frac{(b + c)^2}{4bc} \leqslant \frac{2b^2 + 2c^2 - a^2}{(b + c)^2 - a^2},$$

which is equivalent to

$$\frac{b + c}{2\sqrt{bc}} \leqslant \sqrt{\frac{2b^2 + 2c^2 - a^2}{(b + c)^2 - a^2}},$$

and therefore

$$\frac{(b + c)^2}{4bc} \leqslant \frac{b + c}{2\sqrt{bc}} \sqrt{\frac{2b^2 + 2c^2 - a^2}{(b + c)^2 - a^2}} = \frac{m_a}{w_a},$$

which is (1). Equality holds if and only if b = c.

For (2), it is evident that

$$\frac{m_a}{h_a} \geqslant \frac{m_a}{k_a} = \frac{b^2 + c^2}{2bc}$$

since $h_a \leqslant k_a$. Because h_a and k_a divide BC in the ratios $c^2:b^2$ and $(a^2 + c^2 - b^2):(a^2 + b^2 - c^2)$, respectively, equality holds in (2) if and only if b = c or A is a right angle.

G. Tsintsifas, L. Bankoff, and L. Goldstone, ' Problem E 2471', Amer. Math. Monthly 81 (1974), 406 and 82 (1975), 523-524.

11.32. If $a \leqslant b \leqslant c$, i.e. $h_a \geqslant h_b \geqslant h_c$, $w_a \geqslant w_b \geqslant w_c$ or $m_a \geqslant m_b \geqslant m_c$, then

(1) $\qquad w_c \leqslant M_{-2}(h_a, h_b) \qquad \{E\},$

(2) $\qquad w_a \geqslant M_4(m_b, m_c) \qquad \{E\},$

where $M_n(x, y)$ is the power mean of order n for two positive numbers x and y.

The following results are also valid:

(3) If $w_c \leqslant M_n(h_a, h_b)$ for all triangles, then $n \geqslant -2$;

(4) If $w_a \geqslant M_n(m_b, m_c)$ for all triangles, then $n \leqslant \log 2/(\log 9 - \log 8)$.

Remark. The above results are an extension and generalization of a problem of Erdős and Leuenberger.

P. Erdős, F. Leuenberger, P. Bundschuh, and O. Reutter, 'Aufgabe 610', Elem. Math. 24 (1969), 139 and 25 (1970), 138-139.

11.33. Some Bager's graphs.

We use the notation from 2.1. The letters representing functions for Bager's third graph are:

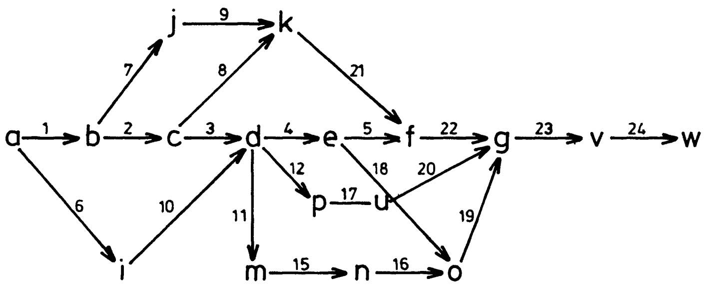
$$a = 18r^2/R, \quad b = 9r, \quad c = 2r(5R - r)/R, \quad d = \Sigma h_a,$$

$$e = \Sigma w_a, \quad f = \Sigma m_a, \quad g = \Sigma r_a = 4R + r, \quad i = (s^2 + F\sqrt{3})/2R,$$

$$j = \sqrt{9F\sqrt{3}}, \quad k = 3\sqrt{r(4R + r)}, \quad m = 2(R + r)^2/R, \quad n = 2R + 5r,$$

$$o = 3(R + r), \quad p = 2s^2/3R, \quad u = s\sqrt{3}, \quad v = 9R/2,$$

$$w = 9(3R - 2r)/4.$$



BAGER'S THIRD GRAPH

The letters representing functions for Bager's fourth graph are

$$a = 2r\sqrt{3}/R^2, \quad b = 9r/Rs, \quad c = 2s/3R^2, \quad d = \sqrt{3}/R,$$

$$e = (4R + r)/Rs, \quad f = 9/2s, \quad g = (5R - r)/Rs,$$

$$h = 9\sqrt{3}/2(4R + r), \quad i = 3\sqrt{3}/2(R + r), \quad A = \Sigma 1/a,$$

$$B = \frac{\sqrt{3}}{2}\Sigma 1/m_a, \quad C = \frac{\sqrt{3}}{2}\Sigma 1/w_a, \quad D = \frac{\sqrt{3}}{2}\Sigma 1/r_a = \sqrt{3}/2r = \frac{\sqrt{3}}{2}\Sigma 1/h_a,$$

$$j = s/3Rr, \quad E = \frac{1}{2}\Sigma\frac{1}{x} = (4R + r)/2F,$$

$k = 3(R + r)/Rs, \quad m = (16R - 5r)/3Rs, \quad n = 3(2R - r)/Rs,$

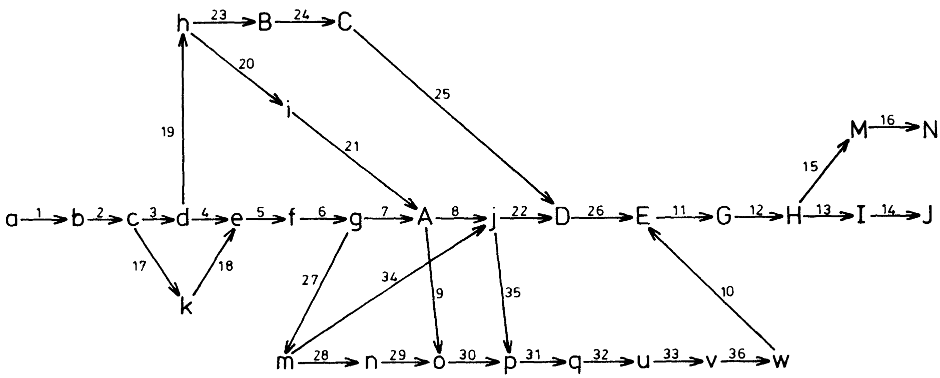
$o = (R + r)^2/RF, \quad p = (4R^2 + 4Rr + 3r^2)/3RF,$

$q = 3(R + r)/2F, \quad u = (5R - r)(R + r)/3RF, \quad v = (4R + r)^2/9RF,$

$w = (2R^2 + r^2)/RF, \quad G = 9R/4F, \quad H = (5R - r)/2F,$

$I = (8R^2 - 5r^2)/3RF, \quad J = (2R - r)^2/RF, \quad M = s/6r^2,$

$N = \sqrt{3}R/4r^2.$



BAGER'S FOURTH GRAPH

The letters representing functions for Bager's fifth graph are:

$a = 27r^2, \quad b = \Sigma h_b h_c = 2sF/R, \quad c = 3F\sqrt{3},$

$d = \Sigma yz = 3r(4R + r), \quad e = 27Rr/2, \quad f = 3r(5R - r),$

$g = \frac{3}{4}\Sigma bc, \quad h = s^2 = \Sigma r_b r_c, \quad i = \frac{3}{4}\Sigma a^2 = \Sigma m_a^2,$

$m = 3R(4R + r)/2, \quad n = 27R^2/4, \quad o = 3R(5R - r)/2,$

$p = 8R^2 - 5r^2, \quad t = 3(2R - r)^2, \quad u = 16R^2 - 24Rr + 11r^2,$

$$v = \Sigma r_a^2, \quad w = 3\Sigma x^2, \quad A = \Sigma h_a^2, \quad B = \Sigma m_b m_c,$$

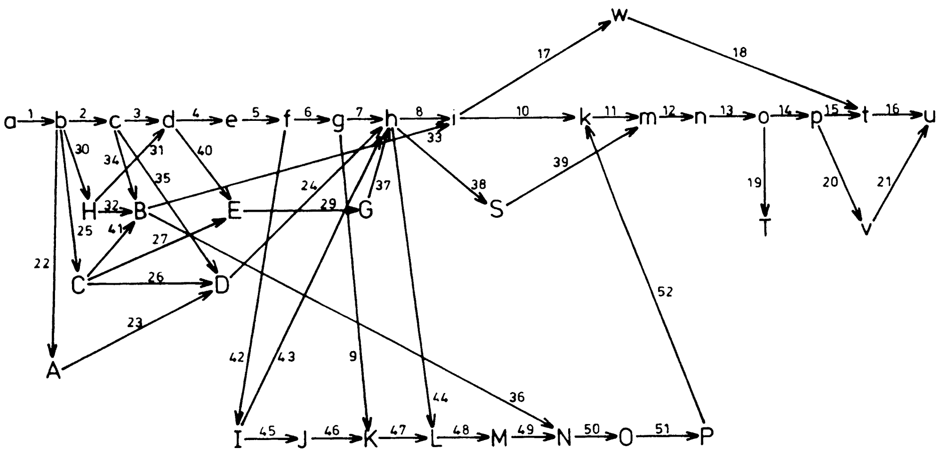
$$C = \Sigma w_b w_c, \quad D = \Sigma w_a^2, \quad E = s\sqrt{3r(4R + r)}, \quad G = \frac{3}{2} s\sqrt{6Rr},$$

$$H = 9r(R + r), \quad I = r(16R - 5r), \quad J = 9r(2R - r),$$

$$K = 3(R + r)^2, \quad L = 4R^2 + 4Rr + 3r^2, \quad M = 9r(R + r)/2,$$

$$N = 5R^2 + 2Rr + 3r^2, \quad O = (5R - r)(R + r),$$

$$P = (4R + r)^2/3, \quad S = 3Rs\sqrt{3}/2, \quad T = Rs^2/2r.$$



BAGER'S FIFTH GRAPH

The letters representing functions for Bager's sixth graph are:

$$a = 4/3R^2, \quad b = 9/(5R^2 + 2Rr + 3r^2), \quad c = 9/s^2,$$

$$d = \frac{1}{3} \Sigma \frac{1}{bc} = 2/3Rr, \quad e = \sqrt{3}/F, \quad f = \Sigma \frac{1}{r_b r_c} = \frac{4R + r}{sF},$$

$$g = 9R/2sF, \quad h = (5R - r)/sF, \quad i = \Sigma \frac{1}{h_b h_c},$$

$$j = 1/3r^2 = \frac{1}{3} \Sigma \frac{1}{yz}, \quad k = \Sigma 1/h_a^2, \quad m = (2R^2 + r^2)/F^2,$$

$$n = R(4R + r)/2F^2, \quad o = 9R^2/4F^2, \quad p = R(5R - r)/2F^2,$$

$$t = (8R^2 - 5r^2)/3F^2, \quad u = (2R - r)^2/F^2,$$

$$v = (16R^2 - 24Rr + 11r^2)/3F^2, \quad w = \frac{1}{3} \Sigma 1/x^2, \quad A = \Sigma \frac{1}{m_b m_c},$$

$$B = \Sigma \frac{1}{w_b w_c}, \quad C = \Sigma 1/w_a^2, \quad D = \Sigma 1/m_a^2,$$

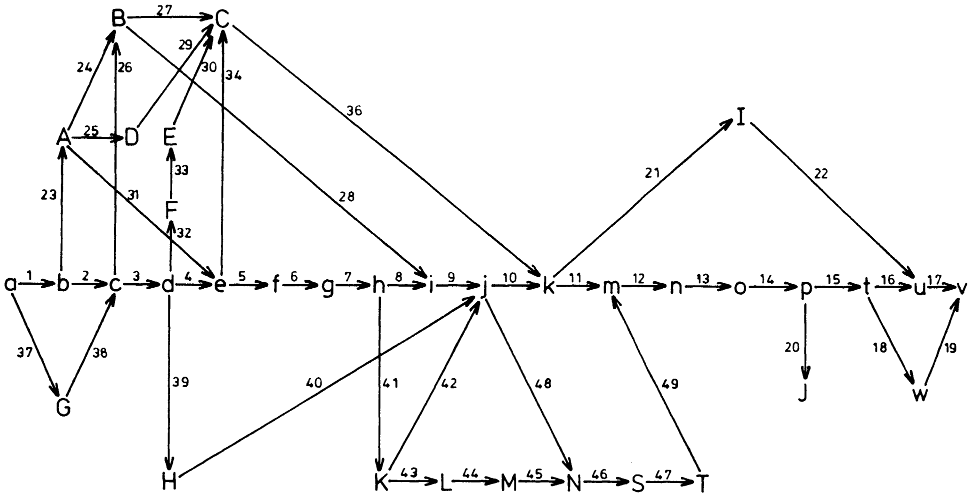
$$E = (4R^2 + 2r^2)/sRF, \quad F = 3(R + r)/sF, \quad G = \frac{2\sqrt{3}}{Rs},$$

$$H = \frac{4}{3} \Sigma 1/a^2, \quad I = \Sigma 1/r_a^2, \quad J = R/6r^3, \quad K = (16R - 5r)/3sF,$$

$$L = 3(2R - r)/sF, \quad M = (R + r)^2/F^2,$$

$$N = (4R^2 + 4Rr + 3r^2)/3F^2, \quad S = 3R(R + r)/2F^2,$$

$$T = (4R + r)^2/9F^2.$$



BAGER'S SIXTH GRAPH

A. Bager, Private communication.

12. <u>Inequalities with Some Elements of a Triangle Extended to the Circumcircle</u>

12.1. $\Sigma M_a/m_a \geq 4.$ {E}

<u>Proof</u>. (M.S. Klamkin) Since $m_a(M_a - m_a) = a^2/4$, etc., this inequality is equivalent to

$$(1) \qquad \Sigma \frac{a^2}{4m_a^2} = \Sigma \frac{a^2}{2b^2 + 2c^2 - a^2} \geq 1.$$

The median-dual of (1) is

$$\Sigma \frac{4m_a^2}{9a^2} = \Sigma \frac{2b^2 + 2c^2 - a^2}{9a^2} \geq 1,$$

and this is equivalent to

$$\Sigma\left(\frac{b^2}{c^2} + \frac{c^2}{b^2}\right) \geqslant 6,$$

which follows immediately from $x^2 + 1/x^2 \geqslant 2$. There is equality if and only if the triangle is equilateral.

J. Garfunkel, S. Rabinowitz, W. J. Blundon, and M. S. Klamkin, 'Problem 689', Crux Math. 7 (1981), 276 and 8 (1982), 307-309.

12.2. $\Pi \dfrac{M_a}{M_a - m_a} \geqslant 64.$

12.3. $(\Sigma m_a)(\Sigma M_a) \geqslant s^2.$ {E}

G. Kirov, Matematika (Sofija) 7 (1968), No. 3, p. 37.

12.4.(a) $\Sigma M_a \geqslant \frac{4}{3}\Sigma m_a,$

(b) Conjecture:

$$\Sigma M_a \geqslant \frac{2}{3}\sqrt{3}\Sigma a.$$

J. Garfunkel, 'Problem E 2505', Amer. Math. Monthly 81 (1974), 1111.

12.5. $\frac{4s}{\sqrt{3}} \leqslant \Sigma W_a \leqslant 5R + 2r.$ {E}

Remark. The second inequality is from Problem 628, and the first from Groenman's comment on this problem.

R. H. Eddy, S. C. Chan, J. Garfunkel, and J. T. Groenman, 'Problem 628', Crux Math. 7 (1981), 117 and 8 (1982), 115-116.

12.6. $\Pi W_a \geqslant \frac{8}{9}\sqrt{3}abc.$ {E}

Proof. (H. Eves) Let D and E denote the other ends of w_a and W_a, respectively. From similar triangles ABD and AEC it follows that $w_a = bc/W_a$ with similar expressions for w_b and w_c. Hence

$$\Pi w_a = (\Pi a^2)/(\Pi W_a)$$

and it suffices to show that

$$(1) \qquad \Pi w_a \leqslant \frac{3}{8}\sqrt{3}abc.$$

This is easily accomplished by means of inequalities GI 8.8 and 1.12:

(2) $w_a \leqslant \sqrt{s(s-a)}$ and $64s^3\Pi(s-a) \leqslant 27\Pi a^2$.

For we have, using those inequalities in turn,

$$\Pi w_a^2 \leqslant s^3\Pi(s-a) \leqslant \frac{27}{64}\Pi a^2$$

and (1) follows. As in (2), equality holds if and only if the triangle is equilateral.

J. Garfunkel, H. Eves, and R. H. Eddy, 'Problem 535', Crux Math. 6 (1980), 113 and 7 (1981), 120-121.

12.7. $\Sigma W_a \geqslant \frac{4s}{\sqrt{3}} \geqslant \frac{4}{3}\sqrt{s}(\Sigma\sqrt{s-a}) \geqslant \frac{4}{3}\Sigma w_a$.

Remark. This is a result from Prielipp's solution of Problem 795.

J. Garfunkel and B. Prielipp, 'Problem 795', Crux Math. 8 (1982), 303, and 9 (1983), 92.

12.8. $\Sigma w_a/W_a \leqslant 9/4$.

Ju. I. Gerasimov, 'Problem 718', Mat. v škole, 1970, No. 1, 82 and 1970, No. 5, 75-76.

12.9. $\Sigma(w_a W_a)^{1/2} \leqslant 2s \leqslant \Sigma\sqrt{m_a M_a}$.

Ja. I. Sukonnik, 'Problem 899', Mat. v škole, 1971, No. 2, 74 and 1971, No. 6, 76-77.

Chapter X

SPECIAL TRIANGLES

1. On the Triangles Satisfying $a^2 + b^2 + c^2 \gtreqless kR^2$

If for a triangle ABC with sides a, b, c and circumradius R the equation $a^2 + b^2 + c^2 = 9R^2$ holds, the triangle is equilateral. Each triangle satisfying $a^2 + b^2 + c^2 = 8R^2$ is a right triangle. Starting from these well-known properties V. Devidé [1] has investigated at length the special class of triangles defined by $a^2 + b^2 + c^2 = 6R^2$. O. Bottema [2] considered the general class of triangles (k-triangles) defined by $a^2 + b^2 + c^2 = kR^2$. In [12] it has been shown that $a^2 + b^2 + c^2 = 5R^2$ characterizes all the triangles for which the nine-point centre lies on the circumcircle.

On the other hand the following result is also known (see [3] or GI 11.27.1° and 3°).

Depending upon whether a triangle is acute, right, or obtuse, the following statements hold

(1) $$a^2 + b^2 + c^2 \gtreqless 8R^2$$

(2) $$s \gtreqless 2R + r.$$

Some other results were also given in GI 11.26, 11.27 and 11.28. A new proof of (2) is given in [4], and (1) as problem is again stated in [5]. A. Lupaş [6] gave six analogous results. Note that Lupaş had given these results to us in 1974. In 1976, W. J. Blundon [7] gave eleven inequalities of this kind, and in 1977 J. T. Groenman [7, 8] gave three new results. J. Garfunkel and G. Tsintsifas [9] gave a new proof of GI 11.28.

But the oldest result of this kind was given in 1889 by C. Pabst [13] who proved

$$\Pi(5m_a^2 - m_b^2 - m_c^2) \gtreqless 0,$$

depending upon whether a triangle is acute, right, or obtuse.

It is obvious that we can unify all these results if we consider the class of triangles defined by

(3) $$a^2 + b^2 + c^2 \gtreqless kR^2, \quad 0 < k \leqslant 9$$

(for k = 9 the sign $\gtreqless$ should be replaced by $\overline{\overline{<}}$).

For triangles defined in such a way we shall give several equivalent results.

Note that the following identity is well-known (see for example [2] or SM):

$$\Sigma a^2 = 2(s^2 - r^2 - 4Rr).$$

So (3) is equivalent to

$$2(s^2 - r^2 - 4Rr) \gtreqless kR^2,$$

i.e.

(4) $$s^2 \gtreqless r^2 + 4Rr + \frac{1}{2} kR^2,$$

i.e.

(5) $$s^2 \gtreqless (2R + r)^2 + \frac{1}{2} pR^2,$$

where $p = k - 8$. Note that the equality case is proved in [2].

Now, we shall give several results equivalent to (3):

1) $$\Sigma bc \gtreqless 2r^2 + 8Rr + \frac{1}{2} kR^2.$$

Proof. This is a simple consequence of (4) and of the identity

$$\Sigma bc = s^2 + 4Rr + r^2.$$

2) $$k\Pi a^2 \lesseqgtr 16F^2\Sigma a^2.$$

Proof. Using the identity $abc = 4sRr$, i.e. $\Pi a^2 = 16F^2R^2$, and (3) we get 2).

3) $$Q = \Sigma(a - b)^2 \gtreqless 2(\frac{1}{2} kR^2 - 8Rr - 2r^2).$$

Proof. Using the known identity $Q = 2(s^2 - 12Rr - 3r^2)$ and (4) we get the above result.

Remarks. 1° For $k = 8$ we get Blundon's result from [7].

2° Similarly we can prove the remaining results from this part of the Section, so we shall give only a few proofs.

4) $$\Sigma a^3 \gtreqless s(kR^2 - 4Rr - 4r^2).$$

Remark. For $k = 8$ $(p = 0)$ we get (1) and (2) from (3) and (4). Similarly, we can get

$$\Sigma a^3 \gtreqless 4s(R - r)(2R + r) \gtreqless 4(R - r)(2R + r)^2,$$

which is a refinement of Lupaş' result [6].

5) $\Pi(b + c) \gtrless s(kR^2 + 12Rr + 4r^2)$.

Remark. For k = 8 we get

$$\Pi(b + c) \gtrless 4s(R + r)(2R + r) \gtrless 4(R + r)(2R + r)^2.$$

6) $\Sigma \frac{1}{a} \gtrless \frac{kR^2 + 16Rr + 4r^2}{8Rrs}$.

7) $\Sigma \frac{b + c}{a} \gtrless \frac{kR^2 + 4Rr + 4r^2}{4Rr}$.

8) $\Sigma x^2 \gtrless \frac{1}{2} kR^2 - 4Rr - r^2 \quad (x = s - a, \text{ etc.})$.

9) $\Sigma x^3 \gtrless s(\frac{1}{2} kR^2 - 8Rr + r^2)$.

Remark. For k = 8 we get

$$\Sigma x^3 \gtrless s(4R^2 - 8Rr + r^2) \gtrless (2R + r)(4R^2 - 8Rr + r^2).$$

10) $\Sigma \frac{1}{x^2} \lessgtr 2 \frac{(16 - k)R^2 - r^2}{r^2(kR^2 + 2r^2 + 8Rr)}$.

11) $\Sigma h_a \gtrless \frac{1}{4R}(kR^2 + 4r^2 + 16Rr)$.

Remark. For k = 8 we get GI 11.28.

12) $\Sigma h_b h_c \gtrless \frac{r}{R}(kR^2 + 8Rr + 2r^2)$.

Remark. For k = 8 this is Lupaş' result from [6].

13) $\Pi h_a \gtrless \frac{r^2}{R}(kR^2 + 8Rr + 2r^2)$.

Remark. For k = 8 this is Lupaş' result from [6].

14) $\Pi(h_b + h_c) \gtrless \frac{s^2 r}{2R^2}(kR^2 + 4r^2 + 12Rr) \gtrless$

$$\gtrless \frac{r}{4R^2}(kR^2 + 4r^2 + 12Rr)(kR^2 + 2r^2 + 8Rr).$$

15) $\sum \frac{1}{h_a h_b} \lesseqgtr \frac{kR^2 + 4r^2 + 16Rr}{4r^2(kR^2 + 2r^2 + 8Rr)}$.

16) $\sum \frac{1}{h_a^2} \gtrless \frac{kR^2}{4F^2}$ and $\sum \frac{1}{h_a^2} \gtrless \frac{kR^2}{2r^2(kR^2 + 8Rr + 2r^2)}$.

17) $\sum \frac{h_b + h_c}{h_a} \gtrless \frac{kR^2 + 4Rr + 4r^2}{4Rr}$. 18) $\sum r_b r_c \gtrless (2R + r)^2 + \frac{p}{2} R^2$.

19) $\Pi r_a \gtrless r((2R + r)^2 + \frac{p}{2} R^2)$. 20) $\sum r_a^2 \lesseqgtr (16 - k)R^2 - r^2$.

21) $\sum r_a^3 \lesseqgtr (16 - 6p)R^3 + r^3$.

22) $\Pi(r_b + r_c) \gtrless 4R((2R + r)^2 + \frac{p}{2} R^2)$.

23) $\sum \frac{1}{r_b r_c} \lesseqgtr \frac{2(4R + r)}{2r(2R + r)^2 + prR^2}$. 24) $\sum \sin^2 A \gtrless \frac{k}{4}$.

25) $\sum \frac{1}{r_a^2} \gtrless \frac{1}{2F^2}(kR^2 - 8Rr - 2r^2)$ and

$$\sum \frac{1}{r_a^2} \gtrless \frac{kR^2 - 8Rr - 2r^2}{r^2(kR^2 + 8Rr + 2r^2)} .$$

Remark. The equality case is given in [2], as a generalization of a similar result of Devidé. For k = 8, 25) becomes GI 11.27.2°.

26) $\sum \sin A \sin B \gtrless \frac{1}{4R^2}(\frac{k}{2} R^2 + 8Rr + 2r^2)$.

27) $\sum \sin^3 A \gtrless \frac{s}{8R^3}(kR^2 - 4Rr - 4r^2)$.

Remark. For k = 8 we get

$$\sum \sin^3 A \gtrless \frac{s}{2R^3}(R - r)(2R + r) \gtrless \frac{1}{2R^3}(R - r)(2R + r)^2.$$

28) $\Pi(\sin A + \sin B) \gtrless \frac{s}{8R^3}(kR^2 + 12Rr + 4r^2)$.

Remark. For k = 8 we get

$$\Pi(\sin A + \sin B) \gtreqless \frac{s}{2R^3}(R + r)(2R + r) \gtreqless \frac{1}{2R^3}(R + r)(2R + r)^2.$$

29) $\sum \frac{\sin A + \sin B}{\sin C} \gtreqless \frac{1}{4Rr}(kR^2 + 4r^2 + 4Rr)$. 30) $\Pi \cos A \gtreqless \frac{1}{8} p$.

Remark. Another equivalent form of 30) is

30') $\Pi(b^2 + c^2 - a^2) \gtreqless p\Pi a^2$.

For $p = 0$ this is a result of C. Ciamberlini (GI 11.26). The equality cases in 30) and 30') are given in [2].

31) $\sum \cos^2 A \lesseqgtr \frac{1}{4}(12 - k)$.

Remark. For $k = 8$ we get Blundon's result from [7].

32) $\sum \cos A \cos B \gtreqless \frac{1}{8R^2}(8Rr + 4r^2 + (k - 8)R^2)$.

Remark. For $k = 8$ we get Blundon's result from [7].

33) $\sum \cos^3 A \lesseqgtr \frac{1}{8R^3}(4(2R + r)^2(R - r) - 3prR^2) - 1$.

34) $\Pi(\cos A + \cos B) \gtreqless \frac{r}{8R^3}(kR^2 + 12Rr + 4r^2)$.

35) $\sum \operatorname{cosec} A \gtreqless \frac{1}{4F}(kR^2 + 4r^2 + 16Rr)$. 36) $\sum \cos 2A \gtreqless 3 - \frac{k}{2}$.

Proof of 36). Using the identity $\sum \cos 2A = \frac{1}{R^2}(3R^2 + 4Rr + r^2 - s^2)$ and (4) the above result follows.

Remark. The equality case for $k = 6$ is given in [1]. Of course, we can prove 36) by using 31) and vice versa.

37) $\sum \cot A \gtreqless \frac{kR^2}{4F}$. 38) $\Pi \cot A \gtreqless \frac{pR^2}{4F}$.

39) $\sum \cot^2 A \gtreqless \frac{k^2R^2}{16F^2} - 2$.

40) $\sum \cot^3 A \gtreqless \frac{R^2}{8F^3}\left(\frac{k^3}{8} R^4 - 48F^2\right)$.

41) $\sum \tan^2 \frac{A}{2} \lesseqgtr 2\, \frac{(16 - k)R^2 - r^2}{2(2R + r)^2 + pR^2}$.

42) $\Sigma \tan^3 \frac{A}{2} \lesseqgtr \frac{1}{s^3}((64 - 6p)R^3 + r^3)$.

Proof. Using the identity $\Sigma \tan^3 \frac{A}{2} = \frac{1}{s^3}((4R + r)^3 - 12s^2R)$ and (5) the above result follows.

43) $\Sigma \cot^2 \frac{A}{2} \gtreqless \frac{1}{2r^2}(kR^2 - 8Rr - 2r^2)$.

44) $\Sigma \cot^3 \frac{A}{2} \gtreqless \frac{s}{2r^3}(kR^2 - 16Rr + 2r^2)$.

Remark. For k = 8 we get

$$\Sigma \cot^3 \frac{A}{2} \gtreqless \frac{s}{r^3}(4R^2 - 8Rr + r^2) \gtreqless \frac{1}{r^3}(2R + r)(4R^2 - 8Rr + r^2).$$

45) $\Sigma \sin^2 \frac{A}{2} \sin^2 \frac{B}{2} \gtreqless \frac{1}{16R^2}(\frac{k}{2} R^2 - 4Rr + 2r^2)$.

46) $\Sigma \sin^4 \frac{A}{2} \lesseqgtr \frac{1}{16R}\left((16 - k)R - 8r\right)$.

47) $\Pi \cos^2 \frac{A}{2} \gtreqless \frac{1}{32R^2}(kR^2 + 2r^2 + 8Rr)$.

48) $\Sigma \cos^2 \frac{A}{2} \cos^2 \frac{B}{2} \gtreqless \frac{1}{32R^2}((32 + k)R^2 + 24Rr + 4r^2)$.

49) $\Sigma a \sin A \gtreqless kR/2$.

50) $\Pi w_a \gtreqless 16Rr^2 \frac{kR^2 + 8Rr + 2r^2}{kR^2 + 12Rr + 4r^2}$ and $\Pi w_a \lesseqgtr \frac{32RF^2}{kR^2 + 12Rr + 4r^2}$.

Remark. For k = 8, we get Lupaş' result from [6] from the first result in 50).

51) $F^2 \gtreqless r^2(2R + r)^2 + \frac{1}{2} pR^2r^2$.

Remark. For k = 8 (p = 0) we get a result of Blundon [7].

52) $s^2 \gtreqless \frac{3k}{4} R^2 - \frac{1}{4} Q$.   53) $\Sigma bc \gtreqless kR^2 - \frac{Q}{2}$.

54) $\Sigma h_a \gtreqless \frac{k}{2} R - \frac{Q}{4R}$.

All equality cases from the previous results give properties of k-triangles (see [1] and [2]). Here we give some other results for these triangles.

1° $\Sigma \sec A = \dfrac{1}{pR^2}(pR^2 + 4r^2 + 8Rr)$.

2° $\Sigma \sec A \sec B = \dfrac{8(R + r)}{pR}$. 3° $\Sigma \tan A = \dfrac{4F}{pR^2}$.

4° $\Pi \tan A = \Sigma \tan A \tan B = \dfrac{1}{p}(p + 8)$.

5° $\Sigma \tan^2 A = \dfrac{16r^2}{p^2R^4}((2R + r)^2 + \dfrac{1}{2}\, pR^2) - 2\,\dfrac{p + 8}{p}$.

6° $\Pi(\tan A + \tan B) = 32F/(pR)^2$.

7° $\Sigma \tan^3 A = \dfrac{32F}{p^3R^6}(2F^2 - 3pR^4)$.

Now, we shall give several other results equivalent to (1) and (2), i.e. results for the case k = 8 (p = 0).

1° $\Sigma a^2(a + b) \gtreqless 16R^2 s$ (J. T. Groenman)

$\gtreqless 16R^2(2R + r)$.

2° $\Sigma \dfrac{1}{x} \gtreqless \dfrac{4R + r}{r(2R + r)}$. 3° $\Sigma \dfrac{a^2}{x} \gtreqless \dfrac{1}{r}(R - r)(2R + r)$.

4° $\Sigma \dfrac{c}{xy} \lesseqgtr \dfrac{2(4R + r)}{r(2R + r)}$. 5° $\Pi \dfrac{h_b + h_c}{b + c} \gtreqless (2 + \dfrac{r}{R})\dfrac{r}{R}$.

6° $\Sigma \sin A \gtreqless 2 + \dfrac{r}{R}$ (W. J. Blundon).

7° $\Pi \sin A \gtreqless \dfrac{(2R + r)r}{2R^2}$.

8° $\Pi(\cot A + \cot B) \lesseqgtr \dfrac{2R^2}{r(2R + r)}$.

9° $\Sigma \tan \dfrac{A}{2} \lesseqgtr \dfrac{4R + r}{2R + r}$. 10° $\Pi \tan \dfrac{A}{2} \leqslant \dfrac{r}{2R + r}$.

11° $\Pi(\tan \dfrac{A}{2} + \tan \dfrac{B}{2}) \lesseqgtr \dfrac{r}{2R + r}$.

12° $\Pi \cot \dfrac{A}{2} = \Sigma \cot \dfrac{A}{2} \gtreqless \dfrac{2R + r}{r}$.

13° $\Pi(\cotan \frac{A}{2} + \cotan \frac{B}{2}) \gtrless \frac{4R(2R + r)}{r^2}$. 14° $\Sigma a \sin A \gtrless 4R$.

15° $\Sigma \sin A - \Sigma \cos A \gtrless 1$ (J. T. Groenman).

16° $(r_a r_b r_c/r)^{1/2} \gtrless \frac{1}{2}(r + r_a + r_b + r_c)$ (J. T. Groenman [8]).

Remarks. 1° Of course, we can give generalizations of the above results in the case when the triangles satisfy (3). For example, the following generalization of 7° is valid.

$$\Pi \sin^2 A \gtrless \frac{r^2}{4R^4}((2R + r)^2 + \frac{p}{2} R^2),$$

but we gave only the much simpler results for the case k = 8.

2° Note that by simple modifications of the previous results we get new results. For example

$$\cotan \omega \gtrless \frac{kR^2}{4F} \quad (\omega \text{ is the Crelle-Brocard angle})$$

is a simple consequence of the identity $\cotan \omega = \Sigma \cotan A$. For k = 8 we get a result of W. J. Blundon. Similarly, using (3) and the identity $\Sigma m_a^2 = \frac{3}{4} \Sigma a^2$, we get

$$\Sigma m_a^2 \gtrless \frac{3k}{4} R^2.$$

For k = 8 we get a result of W. J. Blundon [7].

Furthermore, using the identity $\Sigma AG^2 = \frac{1}{3} \Sigma a^2$, we get

$$\Sigma AG^2 \gtrless \frac{k}{3} R^2.$$

For k = 8 we get a better result than the result from [10].

3° Comment by G. Tsintsifas. From $\Sigma bc = s^2 + r\Sigma r_a$ we have

$$2s^2 \gtrless 2r\Sigma r_a + kr^2.$$

4° The following similar result is also valid [11]:

Let a < b < c. Depending upon whether a triangle is acute, right, or obtuse, the following statements hold

$$\frac{a}{b} \gtrless \frac{w_b}{w_a} \left(\frac{a + c}{b + c}\right)^{1/2}.$$

5° For some other similar results see XI.1.3.

References

1. V. Devidé, 'Über eine Klasse von Dreiecken', "Rad" JAZU (Zagreb) 382 (1978), 55-64.
2. O. Bottema, 'On Triangles ABC Satisfying $a^2 + b^2 + c^2 = kR^2$', Ibid. 396 (1982), 23-28.
3. C. Ciamberlini, 'Sulla condizione necessaria e sufficiente affinchè un triangolo sia acutangolo, rettangolo o ottusangolo', Boll. Un. Mat. Ital. (2) 5 (1943), 37-41.
4. A. Tryboń, 'Zamkniety uklad twierdzeń o trójkatach', Matematyka 3 (47) (1957), 8-9.
5. Problem Gy. 1986. Köz. Mat. Lap. 62 (1981), 222.
6. A. Lupaş, 'Problem 427', Mat. Vesnik 2 (15)(30)(1978), 193-194.
7. J. T. Groenman, Private communication.
8. J. T. Groenman, P. J. van Albada, and L. Kuipers, 'Problem 472', Nieuw Arch. Wisk. 25 (1977), 187 and 26 (1978), 242-243.
9. J. Garfunkel and G. Tsintsifas, Inequalities through R-r-s Triangles. (unpublished).
10. 'Problem 1', Math. Spectrum (Oxford) 1 (1968/69), 59-60.
11. A. Radev, Mat. i Fiz. (Sofija) 6, 3 (1960), 55.
12. A. P. Guinand, D. J. Smeenk, J. T. Groenman, and W. Janous, 'Problem 1016', Crux Math. 11 (1985), 50 and 12 (1986), 128-129.
13. C. Pabst, 'Einige Beziehungen zwischen den drei Höhen und zwischen den drei seitenhalbierenden Ecktransversalen eines Dreiecks', Arch. Math. Phys. (2) 7 (1889), 10-26.

2. On Some Inequalities for Acute (Non-obtuse) Triangles

2.0. Introduction

In this Section we shall consider three interesting classes of inequalities for acute (non-obtuse) triangles.

In the first part of the Section we consider a question of Ono. A result which follows from this question is not correctly given in GI. We shall show that this result is analogous with two well-known inequalities for acute triangles (GI 11.5 and 11.6).

In the second part of the Section we consider an inequality for acute triangles of W. I. Gridasow. We shall show that this inequality is better than GI 11.5 and 11.6, i.e. it is a refinement of these inequalities, and therefore the same is valid for results from part 1 of the Section. Using these results we give several analogous results for other elements of a triangle.

In the third part of the Section we consider an inequality for non-obtuse triangles of A. W. Walker and give several equivalent inequalities.

2.1. A Question of Ono

T. Ono in 1914 [1] posed the following:

Prove or disprove that if a triangle has sides a, b, c and area F, then

$$(1)\qquad 27\Pi(b^2 + c^2 - a^2)^2 \leqslant (4F)^6.$$

Equality holds if and only if $a = b = c$.

G. Quijano [1] showed that (1) is not valid for all triangles. He noted that for all triangles the following inequality holds

$$(2) \qquad \sqrt{27}\,\Pi(b^2 + c^2 - a^2) \leqslant (4F)^3.$$

F. Balitrand [1] showed that (1) is valid for all acute triangles (not for all triangles as in GI 4.20). Of course, inequality (2) is trivial for non-acute triangles.

Since $\Pi a = 4FR$ and $\Pi \cos A = \frac{1}{4R^2}(s^2 - (2R + r)^2)$, we have

$$\Pi(b^2 + c^2 - a^2)^2 = 64(\Pi a)^4(\Pi \cos A)^2 = 4(s^2 - (2R + r)^2)^2(4F)^4,$$

and we get that (1) is equivalent to

$$(1') \qquad |s^2 - (2R + r)^2| \leqslant \frac{2\sqrt{3}}{9} F$$

and (2) to

$$(2') \qquad s^2 - (2R + r)^2 \leqslant \frac{2\sqrt{3}}{9} F.$$

Now we shall give a simple proof of (1'), i.e. of (2'), for acute triangles.

It is well-known that the following result is valid (GI 11.5 and 11.6, where a simple proof is given):

$$(3) \qquad \Pi \tan A = \Sigma \tan A \geqslant 3\sqrt{3}.$$

Equality holds if and only if the triangle is equilateral.

Since

$$(4) \qquad \Pi \tan A = \Sigma \tan A = \frac{2sr}{s^2 - (2R + r)^2},$$

we have $\frac{2F}{s^2 - (2R + r)^2} \geqslant 3\sqrt{3}$, whence we get (2').

Inequality (2') is a conversion of the well-known inequality

$$s \geqslant 2R + r \qquad \text{(GI 11.27.1°)}$$

which holds for non-obtuse triangles.

Note that (2') is equivalent to

$$s^2 - \frac{2\sqrt{3}}{9} rs - (2R + r)^2 \leqslant 0, \quad \text{i.e.}$$

$$s \leqslant \frac{\sqrt{3}}{9} r + \sqrt{\frac{1}{27} r^2 + (2R + r)^2}.$$

For some applications of (2'), see the next part: 2.2.

2.2. An Inequality of Gridasow

W. I. Gridasow [2] gave the following inequality for acute triangles

(5) $$(\Pi \tan A)(\Pi \sin A) \geq \frac{27}{8}.$$

Equality holds if and only if the triangle is equilateral.

Proof. (W. Mnich) Put $W = (\Pi \tan A \sin A)$. Using the identity $\Sigma \sin 2A = 4\Pi \sin A$, we get

$$W = \frac{1}{4}\Pi \tan A\, \Sigma \sin 2A = \frac{1}{2}(\Pi \frac{\sin A}{\cos A})(\Sigma \sin A \cos A) =$$

$$= \frac{1}{2}\Sigma \sin^2 A \tan B \tan C.$$

Now, using AG-inequality we get

$$\frac{W}{3} \geq \frac{1}{2}\sqrt[3]{\Pi \sin^2 A \tan A} = \frac{1}{2}\sqrt[3]{W^2},$$

wherefrom we get $W \geq 27/8$.

Further, using (4) and $\Pi \sin A = \dfrac{sr}{2R^2}$, (5) becomes

(6) $$s^2 - (2R + r)^2 \leq \frac{8F^2}{27R^2}.$$

Using GI 7.9, we get $\dfrac{8F^2}{27R^2} \leq \dfrac{2\sqrt{3}F}{9}$, so (6) is a better inequality than (2), i.e.

(7) $$s^2 - (2R + r)^2 \leq \frac{8F^2}{27R^2} \leq \frac{2\sqrt{3}F}{9}.$$

Note also that (6) is equivalent to [6]:

(8) $$s^2 \leq \frac{27R^2}{27R^2 - 8r^2}(2R + r)^2.$$

Of course, using (7) we can give a refinement of GI 11.5 and 11.6, i.e. the following results are valid for acute triangles

$$\Pi \tan A = \Sigma \tan A \geq \frac{27R^2}{4F} \geq 3\sqrt{3}, \quad \text{i.e.}$$

$$\Pi \tan A = \Sigma \tan A \geq \frac{27}{8}\Pi \operatorname{cosec} A \geq 3\sqrt{3}.$$

In what follows we shall give several similar, i.e. equivalent results for acute triangles. In all results, equality holds if and only if the triangle is equilateral. All results are simple consequences of (7) and well-known triangle identities.

1) $$\Sigma bc \leqslant 4R^2 + 8Rr + 2r^2 + \frac{8F^2}{27R^2} \leqslant 4R^2 + 8Rr + 2r^2 + \frac{2\sqrt{3}F}{9}.$$

2) $$\Sigma a^2 \leqslant 8R^2 + \frac{8F^2}{27R^2} \leqslant 8R^2 + \frac{2\sqrt{3}F}{9}.$$

3) $$Q = \Sigma(a - b)^2 \leqslant 4(2R^2 - 4Rr - r^2 - \frac{4F^2}{27R^2}) \leqslant$$
$$\leqslant 4(2R^2 - 4Rr - r^2 - \frac{\sqrt{3}F}{9}).$$

4) $$\Sigma a^3 \leqslant \frac{4F}{r}\left((R - r)(2R + r) + \frac{4F^2}{27R^2}\right) \leqslant \frac{4F}{r}((R - r)(2R + r) + \frac{\sqrt{3}F}{9}).$$

5) $$\Pi(b + c) \leqslant \frac{4F}{r}((R + r)(2R + r)^2 + \frac{4F^2}{27R^2}) \leqslant$$
$$\leqslant \frac{4F}{r}((R + r)(2R + r)^2 + \frac{\sqrt{3}F}{9}).$$

6) $$\Sigma \frac{1}{a} \leqslant \frac{1}{2RF}(2R^2 + 4Rr + r^2 + \frac{4F^2}{27R^2}) \leqslant \frac{1}{2RF}(2R^2 + 4Rr + r^2 + \frac{\sqrt{3}F}{9}).$$

7) $$\Sigma \frac{b + c}{a} \leqslant \frac{1}{Rr}\left(2R^2 + Rr + r^2 + \frac{4F^2}{27R^2}\right) \leqslant \frac{1}{Rr}(2R^2 + Rr + r^2 + \frac{\sqrt{3}F}{9}).$$

8) $$\Sigma x^2 \leqslant 4R^2 - 4Rr - r^2 + \frac{8F^2}{27R^2} \leqslant 4R^2 - 4Rr - r^2 + \frac{2\sqrt{3}F}{9}),$$

$(x = s - a$, etc.).

9) $$\Sigma x^3 \leqslant \frac{F}{r}(4R^2 - 8Rr + r^2 + \frac{8F^2}{27R^2}) \leqslant \frac{F}{r}(4R^2 - 8Rr + r^2 + \frac{2\sqrt{3}F}{9}).$$

10) $$\Sigma h_a \leqslant \frac{1}{R}(2R^2 + 4Rr + r^2 + \frac{4F^2}{27R^2}) \leqslant \frac{1}{R}(2R^2 + 4Rr + r^2 + \frac{\sqrt{3}F}{9}).$$

11) $$\Sigma h_b h_c \leqslant \frac{2r}{R}((2R + r)^2 + \frac{8F^2}{27R^2}) \leqslant \frac{2r}{R}((2R + r)^2 + \frac{2\sqrt{3}F}{9}).$$

12) $\Pi h_a \leq \frac{2r^2}{R}((2R+r)^2 + \frac{8F^2}{27R^2}) \leq \frac{2r^2}{R}((2R+r)^2 + \frac{2\sqrt{3}F}{9})$.

13) $$\Pi(h_b + h_c) \leq \frac{2rs^2}{R}\left((R+r)(2R+r) + \frac{4F^2}{27R^2}\right)$$

$$\leq \begin{Bmatrix} \frac{2r}{R}\left((R+r)(2R+r) + \frac{4F^2}{27R^2}\right)\left((2R+r)^2 + \frac{8F^2}{27R^2}\right) \\ \frac{2rs^2}{R}\left((R+r)(2R+r) + \frac{\sqrt{3}F}{9}\right) \end{Bmatrix} \leq$$

$$\leq \frac{2r}{R}\left((R+r)(2R+r) + \frac{\sqrt{3}F}{9}\right)\left((2R+r)^2 + \frac{2\sqrt{3}F}{9}\right).$$

14) $\Sigma \frac{1}{h_b h_c} \leq \frac{1}{2F^2}(2R^2 + 4Rr + r^2 + \frac{4F^2}{27R^2}) \leq \frac{1}{2F^2}(2R^2 + 4Rr + r^2 + \frac{\sqrt{3}F}{9})$.

15) $\Sigma \frac{1}{h_a^2} \leq \frac{1}{F^2}(2R^2 + \frac{4F^2}{27R^2}) \leq \frac{1}{F^2}(2R^2 + \frac{\sqrt{3}F}{9})$.

16) $\Sigma \frac{h_b + h_c}{h_a} \leq \frac{1}{Rr}(2R^2 + Rr + r^2 + \frac{4F^2}{27R^2}) \leq \frac{1}{Rr}(2R^2 + Rr + r^2 + \frac{\sqrt{3}F}{9})$.

17) $\Sigma r_b r_c \leq (2R+r)^2 + \frac{8F^2}{27R^2} \leq (2R+r)^2 + \frac{2\sqrt{3}F}{9}$.

18) $\Pi r_a \leq r(2R+r)^2 + \frac{8rF^2}{27R^2} \leq r(2R+r)^2 + \frac{2\sqrt{3}}{9} rF$.

19) $\Sigma r_a^2 \geq 8R^2 - r^2 - \frac{16F^2}{27R^2} \geq 8R^2 - r^2 - \frac{4\sqrt{3}F}{9}$.

20) $\Sigma r_a^3 \geq 16R^3 + r^3 - \frac{32F^2}{9R} \geq 16R^3 + r^3 - \frac{8\sqrt{3}}{3} RF$.

21) $\Pi(r_b + r_c) \leq 4R(2R+r)^2 + \frac{32F^2}{27R^2} \leq 4R(2R+r)^2 + \frac{8\sqrt{3}}{9} RF$.

22) $$\Sigma \frac{1}{r_b r_c} \geq (4R+r)/\left(r(2R+r)^2 + \frac{8rF^2}{27R^2}\right) \geq$$

$$\geq (4R+r)/(r(2R+r)^2 + \frac{2\sqrt{3}}{9} rF).$$

23) $\sum \frac{1}{r_a^2} \leqslant \frac{1}{F^2}(4R^2 - 4Rr - r^2 + \frac{8F^2}{27R^2}) \leqslant \frac{1}{F^2}(4R^2 - 4Rr - r^2 + \frac{2\sqrt{3}F}{9})$.

24) $$\sum \sin A \sin B \leqslant \frac{1}{2R^2}(2R^2 + 4Rr + r^2 + \frac{4F^2}{27R^2}) \leqslant$$

$$\leqslant \frac{1}{2R^2}(2R^2 + 4Rr + r^2 + \frac{\sqrt{3}F}{9}).$$

25) $\sum \sin^2 A \leqslant 2 + \frac{4F^2}{27R^4} \leqslant 2 + \frac{\sqrt{3}F}{9R^2}$.

This is a conversion of GI 2.3 and GI 11.27.2°.

26) $$\sum \sin^3 A \leqslant \frac{F}{4R^3 r}(4R^2 - 2Rr - 2r^2 + \frac{8F^2}{27R^2}) \leqslant$$

$$\leqslant \frac{F}{4R^3 r}(4R^2 - 2Rr - 2r^2 + \frac{2\sqrt{3}F}{9}).$$

27) $$\Pi(\sin B + \sin C) \leqslant \frac{F}{2R^3 r}\left((R + r)(2R + r) + \frac{4F^2}{27R^2}\right) \leqslant$$

$$\leqslant \frac{F}{2R^3 r}\left((R + r)(2R + r) + \frac{\sqrt{3}F}{9}\right).$$

28) $\sum \operatorname{cosec} A \leqslant \frac{1}{F}(2R^2 + 4Rr + r^2 + \frac{4F^2}{27R^2}) \leqslant \frac{1}{F}(2R^2 + 4Rr + r^2 + \frac{\sqrt{3}F}{9})$.

29) $$\sum \frac{\sin B + \sin C}{\sin A} \leqslant \frac{1}{Rr}(2R^2 + Rr + r^2 + \frac{4F^2}{27R^2}) \leqslant$$

$$\leqslant \frac{1}{Rr}(2R^2 + Rr + r^2 + \frac{\sqrt{3}F}{9}).$$

30) $\sum \cos B \cos C \leqslant \frac{1}{2R^2}\left(2Rr + r^2 + \frac{4F^2}{27R^2}\right) \leqslant \frac{1}{2R^2}(2Rr + r^2 + \frac{\sqrt{3}F}{9})$.

31) $\Pi \cos A \leqslant \frac{2F^2}{27R^4} \leqslant \frac{\sqrt{3}F}{18R^2}$.

32) $\sum \cos^2 A \geqslant 1 - \frac{4F^2}{27R^4} \geqslant 1 - \frac{\sqrt{3}F}{9R^2}$.

33) $\Sigma \sec B \sec C \geqslant 27R^3(R + r)/2F^2 \geqslant 6\sqrt{3}R(R + r)/F.$

34) $\Sigma \cot A \leqslant \frac{1}{F}(2R^2 + \frac{4F^2}{27R^2}) \leqslant \frac{1}{F}(2R^2 + \frac{\sqrt{3}F}{9}).$

35) $\Pi \cot A \leqslant \frac{4F}{27R^2} \leqslant \frac{\sqrt{3}}{9}.$

36) $\Sigma \cot^2 A \leqslant \frac{1}{F^2}\left(2R^2 + \frac{4F^2}{27R^2}\right)^2 - 2 \leqslant \frac{1}{F^2}(2R^2 + \frac{\sqrt{3}F}{9})^2 - 2.$

37) $\Pi(\tan B + \tan C) \geqslant \left(\frac{9R^2}{2F}\right)^3 = 54R^2/F.$

38) $\Pi \tan^2 \frac{A}{2} \geqslant (8R^2 - r^2 - \frac{16F^2}{27R^2})/((2R + r)^2 + \frac{8F^2}{27R^2}) \geqslant$

$\geqslant (8R^2 - r^2 - \frac{4\sqrt{3}F}{9})/((2R + r)^2 + \frac{2\sqrt{3}F}{9}).$

39) $\Sigma \cot^2 \frac{A}{2} \leqslant \frac{1}{r^2}\left(4R^2 - 4Rr - r^2 + \frac{8F^2}{27R^2}\right) \leqslant$

$\leqslant \frac{1}{r^2}(4R^2 - 4Rr - r^2 + \frac{2\sqrt{3}F}{9}).$

40) $\Sigma \cot^3 \frac{A}{2} \leqslant \frac{F}{r^4}(4R^2 - 8Rr + r^2 + \frac{8F^2}{27R^2}) \leqslant$

$\leqslant \frac{F}{r^4}(4R^2 - 8Rr + r^2 + \frac{2\sqrt{3}F}{9}).$

41) $\Sigma \sin^2 \frac{B}{2} \sin^2 \frac{C}{2} \leqslant \frac{1}{8R^2}\left(2R^2 - 2Rr + r^2 + \frac{4F^2}{27R^2}\right) \leqslant$

$\leqslant \frac{1}{8R^2}(2R^2 - 2Rr + r^2 + \frac{\sqrt{3}F}{9}).$

42) $\Sigma \cos^2 \frac{B}{2} \cos^2 \frac{C}{2} \leqslant \frac{1}{8R^2}(10R^2 + 6Rr + r^2 + \frac{4F^2}{27R^2}) \leqslant$

$\leqslant \frac{1}{8R^2}(10R^2 + 6Rr + r^2 + \frac{\sqrt{3}F}{9}).$

43) $\Sigma \cos 2A \geqslant -1 - \frac{8F^2}{27R^4} \geqslant -1 - \frac{2\sqrt{3}F}{9R^2}$.

44) $\Sigma \sin^4 \frac{A}{2} \geqslant \frac{1}{8R^2}(4R^2 - 4Rr - \frac{8F^2}{27R^2}) \geqslant \frac{1}{8R^2}(4R^2 - 4Rr - \frac{2\sqrt{3}F}{9})$.

45) $\Sigma \sec^2 \frac{A}{2} \geqslant 1 + (4R + r)^2/((2R + r)^2 + \frac{8F^2}{27R^2}) \geqslant$

$\geqslant 1 + (4R + r)^2/((2R + r)^2 + \frac{2\sqrt{3}F}{9})$.

46) $\Sigma \sec^2 \frac{B}{2} \sec^2 \frac{C}{2} \geqslant 8R(4R + r)/((2R + r)^2 + \frac{8F^2}{27R^2}) \geqslant$

$\geqslant 8R(4R + r)/((2R + r)^2 + \frac{2\sqrt{3}F}{9})$.

47) $\Sigma \operatorname{cosec}^2 \frac{A}{2} \leqslant \frac{2}{r^2}(2R^2 - 2Rr + r^2 + \frac{4F^2}{27R^2}) \leqslant$

$\leqslant \frac{2}{r^2}(2R^2 - 2Rr + r^2 + \frac{\sqrt{3}F}{9})$.

48) $\Sigma a \sin A \leqslant 4R + \frac{4F^2}{27R^3} \leqslant 4R + \frac{\sqrt{3}F}{9R}$.

49) $\cot an\, \omega \leqslant \frac{1}{F}(2R^2 + \frac{4F^2}{27R^2}) \leqslant \frac{\sqrt{3}}{9} + \frac{2R^2}{F}$, (ω is the Crelle-Brocard angle).

50) $\Sigma m_a^2 \leqslant 6R^2 + \frac{2F^2}{9R^2} \leqslant 6R^2 + \frac{\sqrt{3}F}{6}$.

51) $\Pi w_a \leqslant 8Rr^2((2R + r)^2 + \frac{8F^2}{27R^2})/((R + r)(2R + r) + \frac{4F^2}{27R^2}) \leqslant$

$\leqslant 8Rr^2((2R + r)^2 + \frac{2\sqrt{3}F}{9})/((R + r)(2R + r) + \frac{\sqrt{3}F}{9})$.

Remark. The above results are conversions of some results from 1.

2.3. An Inequality of Walker

A. W. Walker [3] gave the following result:
For ABC non-obtuse, the following inequality is valid

(10) $$s^2 \geq 2R^2 + 8Rr + 3r^2$$

or, equivalently,

(11) $$\Sigma a^2 \geq (\Sigma AH)^2$$

with equality if and only if the triangle is equilateral or right isosceles.

Proof. (M. S. Klamkin). Since $HA = 2R \cos A$, etc., for non-obtuse triangles, (10) and (11) are equivalent to

(12) $$\Sigma \sin^2 A \geq (\Sigma \cos A)^2.$$

Making the pedal triangle transformation $A = (\pi - A')/2$, etc., we get the equivalent inequality

$$\Sigma \cos^2 \frac{A'}{2} \geq (\Sigma \sin \frac{A'}{2})^2$$

for general triangles. The latter inequality, however, is known (GI 2.56).

It follows from the proof that for the equality case either the angles are $\pi/3$, $\pi/3$, $\pi/3$ or 0, $\pi/2$, $\pi/2$. Correspondingly, the equality case for (10) (i.e. (11)) is only for equilateral or right isosceles triangles.

Using (10), R. R. Janić [4] gave two new equivalent results. Similarly, we can prove the following results for non-obtuse triangles

1) $\Sigma bc \geq 2R^2 + 12Rr + 4r^2$; 2) $\Pi a^2 \geq 4Rr(2R^2 + 8Rr + 3r^2)$;

3) $\Sigma a^2 \geq 4(R + r)^2$; 4) $\Sigma a^3 \geq 4sR(R + r)$;

5) $\Pi(b + c) \geq 4s(R^2 + 5Rr + 2r^2)$; 6) $\Sigma \frac{1}{a} \geq \frac{R^2 + 6Rr + 2r^2}{2Rrs}$;

7) $\Sigma \frac{b + c}{a} \geq \frac{R^2 + 3Rr + 2r^2}{Rr}$; 8) $\Sigma x^2 \geq 2R^2 + r^2$;

9) $\Sigma x^3 \geq s(2R^2 - 4Rr + 3r^2)$; 10) $\Sigma \frac{1}{x^2} \leq \frac{12R^2 - 4Rr - 5r^2}{r^2(2R^2 + 8Rr + 3r^2)}$;

11) $\Sigma h_a \geq \frac{1}{R}(R^2 + 6Rr + 2r^2)$; 12) $\Sigma h_b h_c \geq \frac{2r}{R}(2R^2 + 8Rr + 3r^2)$;

13) $\Pi h_a \geq \frac{2r^2}{R}(2R^2 + 8Rr + 3r^2)$;

14) $\Pi(h_b + h_c) \geq \frac{2r}{R^2}(2R^2 + 8Rr + 3r^2)(R^2 + 5Rr + 2r^2)$;

15) $\Sigma \frac{1}{h_b h_c} \leqslant \frac{R^2 + 6Rr + 2r^2}{2r^2(2R^2 + 8Rr + 3r^2)}$;

16) $\Sigma \frac{1}{h_a^2} \geqslant \frac{(R + r)^2}{r^2(2R^2 + 8Rr + 3r^2)}$; and $\Sigma \frac{1}{h_a^2} \geqslant \frac{(R + r)^2}{2F^2}$;

17) $\Sigma \frac{h_b + h_c}{h_a} \geqslant \frac{R^2 + 3Rr + 2r^2}{2Rr}$; 18) $\Sigma r_b r_c \geqslant 2R^2 + 8Rr + 3r^2$;

19) $\Pi r_a \geqslant r(2R^2 + 8Rr + 3r^2)$; 20) $\Sigma r_a^2 \leqslant 12R^2 - 4Rr - 5r^2$;

21) $\Sigma r_a^3 \leqslant 40R^3 - 48R^2 r + r^3$; 22) $\Pi(r_b + r_c) \geqslant 4R(2R^2 + 8Rr + 3r^2)$;

23) $\Sigma \frac{1}{r_b r_c} \leqslant \frac{4R + r}{r(2R^2 + 8Rr + 3r^2)}$;

24) $\Sigma \frac{1}{r_a^2} \geqslant \frac{2R^2 + r^2}{r^2(2R^2 + 8Rr + 3r^2)}$ or $\Sigma \frac{1}{r_a^2} \geqslant \frac{2R^2 + r^2}{F^2}$;

25) $\Sigma \sin A \sin B \geqslant \frac{1}{2R^2}(R^2 + 6Rr + 2r^2)$;

26) $\Sigma \sin^2 A \geqslant \frac{(R + r)^2}{R^2}$; 27) $\Sigma \sin^3 A \geqslant \frac{s(R + r)}{R^2}$;

28) $\Pi(\sin A + \sin B) \geqslant \frac{s}{2R^3}(R^2 + 5Rr + 2r^2)$;

29) $\Sigma \operatorname{cosec} A \geqslant \frac{R^2 + 6Rr + 2r^2}{sr}$;

30) $\Sigma \frac{\sin B + \sin C}{\sin A} \geqslant \frac{R^2 + 3Rr + 2r^2}{Rr}$;

31) $\Sigma \cos B \cos C \geqslant \frac{1}{2R^2}(2Rr + r^2 - R^2)$;

32) $\Pi \cos A \geqslant \frac{1}{2R^2}(2Rr + r^2 - R^2)$;

33) $\Sigma \cos^2 A \leqslant \frac{1}{R^2}(2R^2 - 2Rr - r^2)$;

34) $\sum \cos^3 A \leqslant \frac{1}{2R^3}(4R^3 + 3R^2 r - 9Rr^2 - 4r^3)$;

35) $\prod(\cos B + \cos C) \geqslant \frac{r}{2R^3}(R^2 + 5Rr + 2R^2)$;

36) $\sum \cot A \geqslant \frac{(R + r)^2}{sr}$; 37) $\sum \tan^2 \frac{A}{2} \leqslant \frac{12R^2 - 4Rr - 5r^2}{2R^2 + 8Rr + 3r^2}$;

38) $\sum \tan^3 \frac{A}{2} \leqslant \frac{1}{s}\,\frac{40R^3 - 48R^2 r + r^3}{2R^2 + 8Rr + 3r^2}$; 39) $\sum \cot^2 \frac{A}{2} \geqslant \frac{2R^2 + r^2}{r^2}$;

40) $\sum \cot^3 \frac{A}{2} \geqslant \frac{s}{r^3}(2R^2 - 4Rr + 3r^2)$;

41) $\sum \sin^2 \frac{B}{2} \sin^2 \frac{C}{2} \geqslant \frac{R^2 + 2r^2}{16R^2}$;

42) $\sum \cos^2 \frac{B}{2} \cos^2 \frac{C}{2} \geqslant \frac{9R^2 + 8Rr + 2r^2}{8R^2}$;

43) $\sum \cos 2A \leqslant \frac{1}{R^2}(R^2 - 4Rr - 2r^2)$; 44) $\sum \operatorname{cosec}^2 \frac{A}{2} \geqslant 4 + \frac{2R^2}{r^2}$;

45) $\sum \sec^2 \frac{A}{2} \leqslant 1 + \frac{(4R + r)^2}{2R^2 + 8Rr + 3r^2}$;

46) $\sum \sec^2 \frac{A}{2} \sec^2 \frac{B}{2} \leqslant \frac{8R(4R + r)}{2R^2 + 8Rr + 3r^2}$;

47) $\sum \sin^4 \frac{A}{2} \leqslant \frac{3R^2 - 4Rr + 2r^2}{4R^2}$; 48) $\sum a \sin A \geqslant \frac{2}{R}(R + r)^2$;

49) $\cot \omega \geqslant \frac{(R + r)^2}{sr}$; 50) $\sum m_a^2 \geqslant 3(R + r)^2$;

51) $\prod w_a \geqslant \frac{8Rr^2(2R^2 + 8Rr + 3r^2)}{R^2 + 5Rr + 2r^2}$.

Remark. Inequality (10) is again given in [5].

References

1. T. Ono, G. Quijano, and F. Balitrand, 'Problem 4417', Intermed. Math. 21 (1914), 146; 22 (1915), 66 and 23 (1916), 86-87.
2. W. I. Gridasow and W. Mnich, 'Problem 1019', Matematyka (Warszawa) 32 (1972), 308-309.
3. A. W. Walker and M. G. Greening, 'Problem E 2388', Amer. Math. Monthly 79 (1972), 1135 and 80 (1973), 1142-1143.
4. R. R. Janić, Comments. In paper: B. M. Milisavljević: 'Some Inequalities Related to a Triangle', Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 498-541 (1975), 181-184.
5. J. Garfunkel, 'Problem 999*', Crux Math. 10 (1984), 319.
6. D. S. Mitrinović and J. E. Pečarić, 'Problem 1199*', Crux Math. 12 (1986), 283.

3. On Some Inequalities for Obtuse (Non-acute) Triangles

3.1. Emmerich's Inequality

Note that if d denotes the distance between the circumcentre and incentre of a triangle, then for non-acute triangles we have $d \geqslant r$, whence we get the A. Emmerich inequality ([1]):

(1) $$R \geqslant (1 + \sqrt{2})r, \quad \text{i.e.} \quad r \leqslant (\sqrt{2} - 1)R.$$

This result is better than an inequality of A. Lupaş [2], and for right triangles it can be found in SM, p. 11.

Now we give several results for non-acute triangles, which are simple consequences of (1) as well as of the results from Section 1, and also some further similar results.

1° $s \leqslant (1 + \sqrt{2})R$. 2° $\sum bc \leqslant 2R^2(1 + 2\sqrt{2})$.

3° $abc \leqslant 4R^2r(1 + \sqrt{2})$. 4° $\prod(b + c) \leqslant 4R^3(4 + 3\sqrt{2})$.

5° $\sum \frac{1}{a} \leqslant \frac{R}{2rs}(1 + 2\sqrt{2})$. 6° $\frac{1}{2R^2}(1 + \sqrt{2}) \leqslant \sum \frac{1}{bc} \leqslant \frac{1}{2R^2}(\sqrt{2} - 1)$.

7° $\sum \frac{b + c}{a} \leqslant \frac{R}{r}(4 - \sqrt{2})$. 8° $xyz \leqslant r^2R(1 + \sqrt{2})$.

9° $\sum \frac{1}{x} \geqslant \frac{1}{r}(2\sqrt{2} - 1)$. 10° $\sum \frac{a}{x} \geqslant 4\sqrt{2} + 2$.

11° $\sum \frac{a}{yz} \geqslant \frac{2}{r}(2\sqrt{2} - 1)$. 12° $4\sqrt{2}\,\frac{R}{r} \leqslant \sum \frac{a^2}{yz} \leqslant 4(\sqrt{2} + 2)$.

13° $F \leqslant Rr(1 + \sqrt{2})$. 14° $\sum h_a \leqslant R(1 + 2\sqrt{2})$.

15° $\sum h_bh_c \leqslant 2Rr(1 + \sqrt{2})^2$. 16° $\prod h_a \leqslant 2r^2R(1 + \sqrt{2})^2$.

17° $\sum \frac{1}{h_a^2} \leqslant \frac{2R^2}{r^4} \frac{1}{(2\sqrt{2}+3)^2}$. 18° $\sum \frac{h_b + h_c}{h_a} \leqslant \frac{R}{r}(4 - \sqrt{2})$.

19° $\prod \frac{h_b + h_c}{b + c} \leqslant 1$. 20° $r(4\sqrt{2} + 5) \leqslant \sum r_a \leqslant R(3 + \sqrt{2})$.

21° $\sum r_b r_c \leqslant R^2(1 + \sqrt{2})^2$. 22° $\prod (r_b + r_c) \leqslant 4R^3(1 + \sqrt{2})^2$.

23° $\sum \frac{r_b + r_c}{r_a} \geqslant 4\sqrt{2} + 2$. 24° $\sum \sin A \leqslant 1 + \sqrt{2}$.

25° $\sum \sin A \sin B \leqslant \frac{1}{2}(1 + 2\sqrt{2})$. 26° $\prod \sin A \leqslant \frac{1}{2}$.

27° $\prod (\sin A + \sin B) \leqslant \frac{\sqrt{2}}{2}(1 + \sqrt{2})^2$.

28° $\sum \operatorname{cosec} A \operatorname{cosec} B \geqslant 2(\sqrt{2} + 1)$. 29° $\sum \cos A \leqslant \sqrt{2}$.

30° $\sum \cos A \cos B \leqslant \frac{1}{2}$. 31° $\prod (\operatorname{cotan} A + \operatorname{cotan} B) \geqslant 2$.

32° $\sum \tan \frac{A}{2} \geqslant 2\sqrt{2} - 1$. 33° $\sum \tan^2 \frac{A}{2} \geqslant 7 - 4\sqrt{2}$.

34° $\prod (\tan \frac{B}{2} + \tan \frac{C}{2}) \geqslant 4(\sqrt{2} - 1)$.

35° $\prod \operatorname{cotan} \frac{A}{2} = \sum \operatorname{cotan} \frac{A}{2} = \frac{R}{r}(1 + \sqrt{2})$.

36° $4\sqrt{2} + 5 \leqslant \sum \operatorname{cotan} \frac{B}{2} \operatorname{cotan} \frac{C}{2} \leqslant \frac{R}{r}(3 + \sqrt{2})$.

37° $\prod (\operatorname{cotan} \frac{B}{2} + \operatorname{cotan} \frac{C}{2}) \leqslant \frac{4R^2}{r^2}(1 + \sqrt{2})$.

38° $\sum \frac{\operatorname{cotan} \frac{B}{2} + \operatorname{cotan} \frac{C}{2}}{\operatorname{cotan} \frac{A}{2}} \geqslant 4\sqrt{2} + 2$.

39° $\sum \sin^2 \frac{A}{2} \geqslant \frac{1}{2}(3 - \sqrt{2})$.

40° $\frac{r}{2R}(5 + 4\sqrt{2}) \leqslant \sum \cos^2 \frac{A}{2} \leqslant \frac{1}{2}(3 + \sqrt{2})$.

41° $\sum \cos^2 \frac{B}{2} \cos^2 \frac{C}{2} \leqslant \frac{1}{8}(7 + 4\sqrt{2})$. 42° $\sum \sin^4 \frac{A}{2} \geqslant \frac{1}{2}(2 - \sqrt{2})$.

43° $R + r \geqslant \sqrt{2F}$.

Proof. We start from $s \leqslant 2R + r$, i.e. $2rs \leqslant 4Rr + 2r^2$ or

(2) $$2F \leqslant 4Rr + 2r^2.$$

On the other hand, from (1), i.e. from $R - r \geqslant \sqrt{2}r$, we get

$$R^2 - 2Rr + r^2 \geqslant 2r^2, \quad \text{i.e.} \quad 2r^2 + 4Rr \leqslant (R + r)^2,$$

which together with (2) gives

$$2F \leqslant (R + r)^2.$$

Remark. The above result for right triangles is GI 11.23.

3.2. Inequalities of Bottema and Groenman

O. Bottema [3] gave the following inequality for obtuse triangles

(3) $$s > (3 + 2\sqrt{2})r.$$

It is obvious that (3) is a conversion of 3.1.1°.
J. T. Groenman [4] proved the following two inequalities for obtuse triangles

(4) $$(\sum a^2)/(\sum bc) \geqslant (8\sqrt{2} - 4)/7 \quad (= k_1),$$

(5) $$\sum bc \leqslant 2k_2 s^2 \quad (k_2 = 4\sqrt{2} - 5).$$

R. J. Stroeker [5] proved (5) by using non-linear programming.
It is known that the following identity is valid

$$\sum bc = s^2 + r^2 + 4Rr,$$

so (5) is equivalent to

$$k_3 s^2 \geqslant 4Rr + r^2, \quad k_3 = 8\sqrt{2} - 11,$$

i.e.

(6) $$s^2 \geqslant k_4(4Rr + r^2), \quad k_4 = (8\sqrt{2} + 11)/7.$$

Now, we shall show that the Bottema inequality (3) is a simple consequence of (1) and (6). Indeed,

$$s^2 \geqslant k_4(4(1 + \sqrt{2}) + 1)r^2 = (17 + 12\sqrt{2})r^2 = (3 + 2\sqrt{2})^2 r^2,$$

i.e. the following interpretation of (3) is valid

(7) $$s^2 \geqslant k_4(4Rr + r^2) \geqslant (3 + 2\sqrt{2})^2 r^2.$$

Of course, using (6) and the well-known identities for a triangle with r, R, s, we can get a lot of similar results. For example, the following inequalities are valid:

1) $\Sigma a^2 \geqslant 4k_5(4Rr + r^2)$, $k_5 = (4\sqrt{2} + 2)/7$, and

$\Sigma a^2 \geqslant 4k_6 s^2$, $k_6 = 6 - 4\sqrt{2}$.

2) $\Sigma \frac{1}{a} \geqslant \frac{k_7}{2} \frac{4R + r}{Rs}$, $k_7 = (4\sqrt{2} + 9)/7$, $\Sigma \frac{1}{a} \leqslant \frac{k_2 s}{2Rr}$.

3) $\Sigma \frac{1}{a^2} \geqslant \left(\frac{k_7}{2} \frac{4R + r}{Rs}\right)^2$, $\Sigma \frac{1}{a^2} \leqslant \left(\frac{k_2 s}{2Rr}\right)^2$.

4) $\Sigma \frac{1}{(s - a)^2} \leqslant k_3 \frac{4R + r}{r^3} - \frac{2}{r^2}$.

5) $\Sigma h_a \geqslant k_7(r^2 + 4Rr)/R$, $\Sigma h_a \leqslant k_2 s^2/R$.

6) $\Sigma h_b h_c \geqslant 2k_4 r^2(4R + r)/R$. 7) $\Pi h_a \geqslant 2k_4 r^3(4R + r)/R$.

8) $\Sigma \frac{1}{h_b h_c} \leqslant \frac{k_2}{2r^2}$. 9) $\Sigma \frac{1}{h_a^2} \geqslant \frac{k_6}{r^2}$.

10) $\Sigma r_b r_c \geqslant k_4(4Rr + r^2)$. 11) $\Pi r_a \geqslant k_4 r^2(4R + r)$.

12) $\Pi(r_b + r_c) \geqslant 4k_4 Rr(4R + r)$. 13) $\Sigma \frac{1}{r_b r_c} \leqslant \frac{k_3}{r^2}$.

14) $\Sigma \frac{1}{r_a^2} \geqslant \frac{k_8}{r^2}$, $k_8 = 23 - 16\sqrt{2}$.

15) $\Sigma \sin B \sin C \geqslant \frac{k_7}{2R^2}(4Rr + r^2)$, $\Sigma \sin B \sin C \leqslant \frac{k_2 s^2}{2R^2}$.

16) $\Sigma \sin^2 A \geq k_5(4Rr + r^2)/R^2$, $\Sigma \sin^2 A \geq k_6 s^2/R^2$.

17) $\Sigma \operatorname{cosec} A \geq k_7(4R + r)/s$, $\Sigma \operatorname{cosec} A \leq k_2 s/r$.

18) $\Sigma \cotan A \geq k_5(4R + r)/s$, $\Sigma \cotan A \geq k_6 s/r$.

19) $\Sigma \cotan^2 A \geq (k_6 s/r)^2 - 2$.

20) $\Sigma \cotan^2 \frac{A}{2} = k_9(4R + r)/r$, $k_9 = (8\sqrt{2} - 3)/7$,

$\Sigma \cotan^2 \frac{A}{2} \geq k_8 s^2/r^2$.

Note that the following result is valid (see I.2.46°):
If s, r and C are given, then there exists a triangle ABC if and only if

$$s \geq \frac{2(\sin C/2 + 1)^2 r}{\sin C}. \tag{8}$$

Since the function $C \to (\sin C/2 + 1)^2/\sin C$ is increasing for $C \geq \pi/2$, we directly get (3) from (8).

References

1. A. Emmerich, 'Problem 1291', Zeit. math. naturwiss. Unterricht 26 (1895), 104-105.
2. A. Lupaş, 'Problem 427', Mat. Vesnik 2 (15)(30)(1978), 193-194.
3. O. Bottema and G. R. Veldkamp, 'Problem 572', Nieuw Arch. Wisk. 28 (1980), 200 and 29 (1981), 109.
4. J. T. Groenman, 'Een Planimetrische Ongelijkheid Verscherpt', Nieuw Tijdschr. Wisk. 64/5 (1977), 256-259.
5. R. J. Stroeker, 'Application of Non-linear Programming to Plane Geometry', Elem. Math. 34 (1979), 1-7.

4. Some Other Classes of Special Triangles

4.0. Introduction

In 1. we considered the class of triangles satisfying

$$\Sigma a^2 \gtreqless kR^2, \quad \text{i.e.} \quad s^2 \gtreqless r^2 + 4Rr + \frac{1}{2} kR^2.$$

For $k \gtreqless 8$, we get classes of acute, right and obtuse triangles.
Of course, we can similarly consider the general class of triangles satisfying

$$s^2 \gtreqless \lambda R^2 + \mu Rr + \nu r^2, \quad (\lambda,\ \mu,\ \nu \text{ real numbers}),$$

and give several equivalent conditions, but we think that it is more interesting to consider some important special classes of triangles with very simple geometrical interpretations. In this Section we shall give several inequalities for such classes of triangles proved by Bager, Guba and Paasche.

4.1. Inequalities for Triangles of Bager's Type I or II

For every triangle we will assume without loss of generality that $A \leqslant B \leqslant C$. Every triangle falls into one of two types:

$$\text{type I:} \quad B \geqslant \pi/3, \qquad \text{type II:} \; B \leqslant \pi/3.$$

A. Bager [1] (see also [2]) has shown that the following result is valid:

If a triangle ABC is of $\begin{Bmatrix}\text{type I}\\ \text{type II}\end{Bmatrix}$, then

(1) $$\sqrt{3}\, \Sigma \cos A \lesseqgtr \Sigma \sin A.$$

Equality holds if and only if $B = \pi/3$.

Proof. Let x, y, z be real numbers satisfying $x + y + z = 0$. Then we have

$$\Sigma \sin x = 2 \sin \frac{x+y}{2} \cos \frac{x-y}{2} + 2 \sin \frac{z}{2} \cos \frac{z}{2} =$$

$$= -2 \sin \frac{z}{2} \cos \frac{x-y}{2} + 2 \sin \frac{z}{2} \cos \frac{x+y}{2} =$$

$$= 2 \sin \frac{z}{2}\left(\cos \frac{x+y}{2} - \cos \frac{x-y}{2}\right) =$$

$$= -4 \sin \frac{x}{2} \sin \frac{y}{2} \sin \frac{z}{2}.$$

As $\Sigma(\frac{\pi}{3} - A) = 0$, the formula just proved yields:

$$\frac{\sqrt{3}}{2} \Sigma \cos A - \frac{1}{2} \Sigma \sin A = \Sigma \sin(\frac{\pi}{3} - A) = -4\, \Pi \sin(\frac{\pi}{6} - \frac{A}{2}).$$

From this formula we easily derive (1).

In the same paper, Bager also gave the following equivalent results:

(2) $$\Sigma \cos \frac{A}{2} \lesseqgtr \sqrt{3} \Sigma \sin \frac{A}{2},$$

(3) $$s \gtreqless \sqrt{3}(R + r).$$

The following result is given in [3] (see also [4]):

For any triangle (other than equilateral) with $A \leqslant B \leqslant C$, we have

(4)
$$0 < HI/IO < 1 \quad \text{if} \quad B > \pi/3, \qquad HI = IO \quad \text{if} \quad B = \pi/3,$$
$$1 < HI/IO < 2 \quad \text{if} \quad B < \pi/3.$$

The constant 2 in the last inequality is best possible.

V. N. Murty [2] proved:

If a triangle ABC is of type I, or is an acute-angled triangle of type II, then

(5)
$$\sqrt{3}(r^2 + s^2 + 2Rr - 2R^2) - 8rs \geqslant 0.$$

J. Garfunkel and G. Tsintsifas [5] proved the following inequality for a triangle of type I:

(6)
$$\Sigma h_a \geqslant 7r + R.$$

Here we shall note that using the well-known identity

$$\Sigma h_a = \frac{1}{2R}(s^2 + r^2 + 4Rr)$$

and (3) we obtain

(7)
$$\Sigma h_a \gtrless \frac{1}{2R}(3R^2 + 10Rr + 4r^2),$$

which is a better result than (6) because

$$\frac{1}{2R}(3R^2 + 10Rr + 4r^2) \geqslant 7r + R.$$

Similarly, we can get series of analogous results for triangles of type I or II, and in what follows we shall only state such results without giving the proofs.

1. $\Sigma ab \gtrless 3R^2 + 10Rr + 4r^2$. 2. $abc \gtrless 4\sqrt{3}Rr(R + r)$.

3. $Q \gtrless 6R(R - 2r)$. 4. $\Sigma a^2 \gtrless 6R^2 + 4Rr + 4r^2$.

5. $\Sigma a^3 \gtrless 6sR^2 \gtrless 6\sqrt{3}R^2(R + r)$.

6. $\Pi(a + b) \gtrless 2s(3R^2 + 8Rr + 4r^2) \gtrless 2\sqrt{3}(R + r)(3R^2 + 8Rr + 4r^2)$.

7. $\Sigma \frac{1}{a} \gtrless \frac{1}{4Rrs}(3R^2 + 10Rr + 4r^2)$.

8. $\Sigma \frac{a + b}{c} \gtrless \frac{1}{2Rr}(3R^2 + 4Rr + 4r^2)$.

9. $xyz \gtreqless r^2\sqrt{3}(R + r) \quad (x = s - a, \text{ etc.})$

10. $\Sigma x^2 \gtreqless 3R^2 - 2Rr + r^2.$

11. $\Sigma x^3 \gtreqless 3s(R - r)^2 \gtreqless 3\sqrt{3}(R + r)(R - r)^2.$

12. $\Sigma \frac{1}{x} \lesseqgtr \frac{4R + r}{\sqrt{3}r(R + r)}.$ 13. $\Sigma \frac{a^2}{x} \gtreqless \frac{4\sqrt{3}}{r}(R^2 - r^2).$

14. $\Sigma \frac{c}{xy} \lesseqgtr \frac{2\sqrt{3}(4R + r)}{3r(R + r)}.$ 15. $\Sigma h_a h_b \gtreqless \frac{6r}{R}(R + r)^2.$

16. $\Pi h_a \gtreqless \frac{6r^2}{R}(R + r)^2.$

17) $\Pi(h_a + h_b) \gtreqless \frac{s^2 r}{R^2}(3R^2 + 8Rr + 4r^2) \gtreqless$

$\gtreqless \frac{3r}{R^2}(R + r)^2(3R^2 + 8Rr + 4r^2).$

18. $\Sigma \frac{1}{h_a h_b} \lesseqgtr \frac{3R^2 + 10Rr + 4r^2}{12r^2(R + r)^2}.$ 19. $\Sigma \frac{1}{h_a^2} \gtreqless \frac{3R^2 + 2Rr + 2r^2}{6r^2(R + r)^2}.$

20. $\Pi \frac{h_a + h_a}{a + b} \gtreqless \frac{r\sqrt{3}}{2R^2}(R + r).$

21. $\Sigma \frac{h_a + h_b}{h_c} \gtreqless \frac{1}{2Rr}(3R^2 + 4Rr + 4r^2).$ 22. $\Sigma r_a r_b \gtreqless 3(R + r)^2.$

23. $\Pi r_a \gtreqless 3r(R + r)^2.$ 24. $\Sigma r_a^2 \lesseqgtr 10R^2 - 4Rr - 5r^2.$

25. $\Sigma r_a^3 \lesseqgtr (4R + r)^3 - 36(R + r)^2 R.$ 26. $\Pi(r_a + r_b) \gtreqless 12R(R + r)^2.$

27) $\Sigma \frac{1}{r_a r_b} \lesseqgtr \frac{4R + r}{3r(R + r)^2}.$ 28. $\Sigma \frac{1}{r_a^2} \gtreqless \frac{3R^2 - 2Rr + r^2}{3r^2(R + r)^2}.$

29) $\Sigma \sin A \gtreqless \frac{\sqrt{3}}{R}(R + r) \quad (= \sqrt{3}\Sigma \cos A).$

30) $\Sigma \sin A \sin B \gtreqless \frac{1}{4R^2}(3R^2 + 10Rr + 4r^2).$

31. $\Pi \sin A \gtrless \frac{\sqrt{3}r}{2R^2}(R + r)$. 32. $\Sigma \sin^2 A \gtrless \frac{1}{2R^2}(3R^2 + 2Rr + 2r^2)$.

33. $\Sigma \sin^3 A \gtrless \frac{3s}{4R} \gtrless \frac{3\sqrt{3}(R + r)}{4R}$.

34. $\Pi (\sin A + \sin B) \gtrless \frac{s}{4R^3}(3R^2 + 8Rr + 4r^2) \gtrless$

$$\gtrless \frac{\sqrt{3}}{4R^3}(R + r)(3R^2 + 8Rr + 4r^2).$$

35. $\Sigma \operatorname{cosec} A \gtrless \frac{1}{2sr}(3R^2 + 10Rr + 4r^2)$.

36. $\Sigma \frac{\sin A + \sin B}{\sin C} \gtrless \frac{1}{2Rr}(3R^2 + 4Rr + 4r^2)$.

37. $\Sigma \cos A \cos B \gtrless \frac{1}{4R^2}(4r^2 + 6Rr - R^2)$.

38. $\Pi \cos A \gtrless \frac{1}{4R^2}(2r^2 + 2Rr - R^2)$.

39. $\Sigma \cos^2 A \lessgtr \frac{1}{2R^2}(3R^2 - 2Rr - 2r^2)$.

40. $\Sigma \cos^3 A \lessgtr \frac{1}{4R^3}(8R^3 + 3R^2r - 12r^2R - 8r^3) - 1$.

41. $\Pi(\cos A + \cos B) \gtrless \frac{r}{4R^3}(3R^2 + 8Rr + 4r^2)$.

42. $\Sigma \cos A \lessgtr \frac{s\sqrt{3}}{3R}$. 43. $\Pi \frac{\sin A + \sin B}{\cos A + \cos B} \gtrless \frac{\sqrt{3}}{r}(R + r)$.

44. $\Sigma \operatorname{cotan} A \gtrless \frac{1}{2sr}(3R^2 + 2Rr + 2r^2)$.

45. $\Pi \operatorname{cotan} A \gtrless \frac{1}{2sr}(2Rr + 2r^2 - R^2)$.

46. $\Sigma \operatorname{cotan}^2 A \gtrless \frac{1}{4s^2r^2}(3R^2 + 2Rr + 2r^2)^2 - 2$.

47. $\Pi(\operatorname{cotan} A + \operatorname{cotan} B) \lessgtr \frac{2R^2\sqrt{3}}{3r(R + r)}$.

48. $\Sigma \tan \frac{A}{2} \lessgtr \frac{\sqrt{3}(4R + r)}{3(R + r)}$.

49. $\Pi \tan \frac{A}{2} \lesseqgtr \frac{r\sqrt{3}}{3(R + r)}$. 50. $\Sigma \tan^2 \frac{A}{2} \lesseqgtr \frac{1}{s^2}(10R^2 - 4Rr - 5r^2)$.

51. $\Pi(\tan \frac{A}{2} + \tan \frac{B}{2}) \lesseqgtr \frac{4\sqrt{3}R}{3(R + r)}$.

52. $\Pi \cot \frac{A}{2} = \Sigma \cot \frac{A}{2} \gtreqless \frac{\sqrt{3}}{r}(R + r)$.

53. $\Sigma \cot^2 \frac{A}{2} \gtreqless \frac{1}{r^2}(3R^2 - 2Rr + r^2)$.

54. $\Sigma \cot^3 \frac{A}{2} \gtreqless \frac{3s}{r^3}(R - r)^2 \gtreqless \frac{3\sqrt{3}}{r^3}(R + r)(R - r)^2$.

55. $\Pi(\cot \frac{A}{2} + \cot \frac{B}{2}) \gtreqless \frac{4\sqrt{3}R}{r^2}(R + r)$.

56. $\Sigma \sin^2 \frac{A}{2} \sin^2 \frac{B}{2} \gtreqless \frac{1}{16R^2}(3R^2 - 2Rr + 4r^2)$.

57. $\Sigma \cos^2 \frac{A}{2} \cos^2 \frac{B}{2} \gtreqless \frac{1}{16R^2}(19R^2 + 14Rr + 4r^2)$.

58. $\Sigma \cos 2A \lesseqgtr - \frac{2R}{R^2}(2R + r)$. 59. $\Sigma \sin 2A \gtreqless \frac{2\sqrt{3}}{R^2} r(R + r)$.

60. $\Sigma \sin^4 \frac{A}{2} \lesseqgtr \frac{1}{8R^2}(5R^2 - 6Rr - 2r^2)$.

61. $\Sigma a \sin A \gtreqless \frac{1}{R}(3R^2 + 2Rr + 2r^2)$. 62. $\Sigma m_a^2 \gtreqless \frac{3}{2}(3R^2 + 2Rr + 2r^2)$.

63. $\cot \omega \gtreqless \frac{1}{2sr}(3R^2 + 2Rr + 2r^2)$ (ω is the Crelle-Brocard angle).

64. $\Pi w_a \gtreqless \frac{48Rr^2(R + r)^2}{3R^2 + 8Rr + 4r^2}$.

Comment by C. Tănăsescu. We shall give another characterisation of Bager's types. From

$$\Sigma \sin 3A = - 4\Pi \cos \frac{3A}{2} , \quad (\Sigma A = \pi;\ A, B, C \in R),$$

it follows that ΔABC is of Bager's type I, II, respectively, if and only if

$$\Sigma \sin 3A \lesseqgtr 0,$$

with equality if and only if $B = \pi/3$.

4.2. Inequalities of S. G. Guba for Special Triangles

S. G. Guba [6] (see also GI 14.6 or [7]) gave the following result: For a triangle with sides $a \leqslant b \leqslant c$, the inequalities

$$(8)\qquad \left(\frac{s}{3}\right)^2 \lesseqgtr 2Rr - r^2$$

are equivalent to $2b \lesseqgtr c + a$.

Proof. P. Nüesch [7] noted that the above result is a simple consequence of the following identity

$$(9)\qquad 2Rr - r^2 - \frac{s^2}{9} = -\frac{1}{18s}(-2a + b + c)(a - 2b + c)(a + b - 2c).$$

Note that using (8) and several known triangle identities we can show that the condition $2b \lesseqgtr c + a$ is equivalent to the following results:

1) $\Sigma ab \lesseqgtr 2r(11R - 8r)$; 2) $\Sigma a^2 \lesseqgtr 4r(7R - 5r)$;

3) $\Sigma a^3 \lesseqgtr 24sr(R - r)$; 4) $\Pi(a + b) \lesseqgtr 8rs(5R - 2r)$;

5) $\Sigma \frac{1}{a} \lesseqgtr \frac{11R - 4r}{2Rs}$; 6) $\Sigma \frac{a + b}{c} \lesseqgtr \frac{4}{R}(2R - r)$;

7) $\Sigma x^2 \lesseqgtr r(10R - 11r)$ $(x = s - a$, etc.);

8) $\Sigma x^3 \lesseqgtr 3sr(2R - 3r)$; 9) $\Sigma \frac{1}{x^2} \gtreqless \frac{16R^2 - 28Rr + 19r^2}{9r^3(2R - r)}$;

10) $\Sigma h_a \lesseqgtr \frac{r}{R}(11R - 4r)$; 11) $\Sigma h_a h_b \lesseqgtr \frac{18r^3}{R}(2R - r)$;

12) $\Pi(h_a + h_b) \lesseqgtr \frac{4s^2r^2}{R^2}(5R - 2r)$ $\left(\lesseqgtr \frac{36r^3}{R^2}(2R - r)(5R - 2r)\right)$;

13) $\Pi h_a \lesseqgtr \frac{18r^4}{R}(2R - r)$; 14) $\Sigma \frac{1}{h_a h_b} \gtreqless \frac{11R - 4r}{18r^2(2R - r)}$;

15) $\Sigma \frac{1}{h_a^2} \lesseqgtr \frac{7R - 5r}{9r^2(2R - r)}$; 16) $\Sigma \frac{h_a + h_b}{h_c} \lesseqgtr \frac{4}{R}(2R - r)$;

17) $\Sigma r_a r_b \lesseqgtr 9r(2R - r)$; 18) $\Pi r_a \lesseqgtr 9r^2(2R - r)$;

19) $\sum r_a^2 \gtreqless 16R^2 - 28Rr + 19r^2$; 20) $\prod (r_a + r_b) \lesseqgtr 36Rr(2R - r)$;

21) $\sum \frac{1}{r_a r_b} \gtreqless \frac{4R + r}{9r^2(2R - r)}$; 22) $\sum \frac{1}{r_a^2} \lesseqgtr \frac{10R - 11r}{9r^2(2R - r)}$;

23) $\sum \sin A \sin B \lesseqgtr \frac{r}{2R^2}(11R - 4r)$;

24) $\sum \sin^2 A \lesseqgtr \frac{r}{R^2}(7R - 5r)$; 25) $\sum \sin^3 A \lesseqgtr \frac{3rs}{R^3}(R - r)$;

26) $\prod(\sin A + \sin B) \lesseqgtr \frac{rs}{R^3}(5R - 2r)$; 27) $\sum \operatorname{cosec} A \lesseqgtr \frac{1}{s}(11R - 4r)$;

28) $\sum \frac{\sin A + \sin B}{\sin C} \lesseqgtr \frac{4}{R}(2R - r)$;

29) $\sum \cos A \cos B \lesseqgtr \frac{1}{2R^2}(9Rr - 4r^2 - 2R^2)$;

30) $\prod \cos A \lesseqgtr \frac{1}{2R^2}(7Rr - 2R^2 - 5r^2)$;

31) $\sum \cos^2 A \gtreqless \frac{1}{R^2}(3R^2 - 7Rr + 5r^2)$;

32) $\sum \cos^3 A \gtreqless \frac{1}{R^3}(R^3 + 3R^2 r - 12Rr^2 + 7r^3)$;

33) $\prod(\cos A + \cos B) \lesseqgtr \frac{r^2}{R^3}(5R - 2r)$; 34) $\sum \operatorname{cotan} A \lesseqgtr \frac{1}{s}(7R - 5r)$;

35) $\prod \operatorname{cotan} A \lesseqgtr \frac{1}{sr}(7Rr - 5r^2 - 2R^2)$;

36) $\sum \operatorname{cotan}^2 A \lesseqgtr \frac{1}{s^2}(7R - 5r)^2 - 2$;

37) $\sum \tan^2 \frac{A}{2} \gtreqless \frac{16R^2 - 28Rr + 19r^2}{9r(2R - r)}$;

38) $\sum \operatorname{cotan}^2 \frac{A}{2} \lesseqgtr \frac{1}{r}(10R - 11r)$; 39) $\sum \operatorname{cotan}^3 \frac{A}{2} \lesseqgtr \frac{3s}{r^2}(2R - 3r)$;

40) $\sum \sin^2 \frac{A}{2} \sin^2 \frac{B}{2} \lesseqgtr \frac{r(5R - 4r)}{8R^2}$;

41) $\Sigma \cos^2 \frac{A}{2} \cos^2 \frac{B}{2} \lesseqgtr \frac{8R^2 + 13Rr - 4r^2}{8R^2}$;

42) $\Sigma \cos 2A \gtreqless \frac{1}{R^2}(3R^2 - 14Rr + 12r^2)$;

43) $\Sigma \sin^4 \frac{A}{2} \gtreqless \frac{1}{4R^2}(4R^2 - 9Rr - 4r^2)$; 44) $Q \lesseqgtr 12r(R - 2r)$;

45) $\Sigma m_a^2 \lesseqgtr 3r(7R - 5r)$;

46) $\cot \omega \lesseqgtr \frac{1}{s}(7R - 5r)$ (ω is the Crelle-Brocard angle);

47) $\Pi w_a \lesseqgtr \frac{36Rr^2(2R - r)}{5R - 2r}$;

48) $r/h_b \lesseqgtr 1/3$.

4.3. Inequalities of I. Paasche for Special Triangles

I. Paasche [7] gave the following result as a problem:
For a triangle with sides $a \leqslant b \leqslant c$ the following

(10) $$(\frac{s}{2})^2 \lesseqgtr 4Rr - r^2$$

is equivalent to $b + c \lesseqgtr 3a$.

P. Nüesch [7] noted that a similar identity to (9) could be used in the proof of this result.

Using (10) and known triangle identities we can show that the following results are equivalent to $b + c \lesseqgtr 3a$:

1) $\Sigma ab \lesseqgtr r(20R - 3r)$; 2) $\Sigma a^2 \lesseqgtr 2r(12R - 5r)$;

3) $\Sigma a^3 \lesseqgtr 2sr(10R - 7r)$; 4) $\Pi(a + b) \lesseqgtr 6sr(6R - r)$;

5) $\Sigma \frac{1}{a} \lesseqgtr \frac{20R - 3r}{4Rs}$; 6) $\Sigma \frac{a + b}{c} \lesseqgtr \frac{14R - 3r}{2R}$;

7) $\Sigma x^2 \lesseqgtr 2r(4R - 3r)$; 8) $\Sigma x^3 \lesseqgtr 4sr(R - r)$;

9) $\Sigma \frac{1}{x^2} \gtreqless \frac{(4R - 3r)^2}{4r^3(4R - r)}$; 10) $\Sigma h_a \lesseqgtr \frac{r}{2R}(20R - 3r)$;

11) $\Sigma h_a h_b \lesseqgtr \frac{8r^2}{R}(4R - r)$; 12) $\Pi h_a \lesseqgtr \frac{8r^3}{R}(4R - r)$;

13) $\Pi(h_a + h_b) \lesseqgtr \frac{3s^2r^2}{R^2}(6R - r)$ $\left(\lesseqgtr \frac{12r^3}{R^2}(4R - r)(6R - r)\right)$;

14) $\Sigma \frac{1}{h_a h_b} \gtreqless \frac{20R - 3r}{16r^2(4R - r)}$; 15) $\Sigma \frac{1}{h_a^2} \lesseqgtr \frac{12R - 5r}{8r^2(4R - r)}$;

16) $\Sigma \frac{h_a + h_b}{h_c} \lesseqgtr \frac{14R - 3r}{2R}$; 17) $\Sigma r_a r_b \lesseqgtr 4r(4R - r)$;

18) $\Pi r_a \lesseqgtr 4r^2(4R - r)$; 19) $\Sigma r_a^2 \gtreqless (4R - 3r)^2$;

20) $\Pi(r_a + r_b) \lesseqgtr 16Rr(4R - r)$; 21) $\Sigma \frac{1}{r_a r_b} \gtreqless \frac{4R + r}{4r^2(4R - r)}$;

22) $\Sigma \frac{1}{r_a^2} \lesseqgtr \frac{4R - 3r}{2r^2(4R - r)}$; 23) $\Sigma \sin A \sin B \lesseqgtr \frac{1}{4R^2}(20Rr - 3r^2)$;

24) $\Sigma \sin^2 A \lesseqgtr \frac{1}{2R^2}(12Rr - 5r^2)$; 25) $\Sigma \sin^3 A \lesseqgtr \frac{sr}{4R^3}(10R - 7r)$;

26) $\Pi(\sin A + \sin B) \lesseqgtr \frac{sr}{4R^3}(18R - 3r)$;

27) $\Sigma \operatorname{cosec} A \lesseqgtr \frac{20R - 3r}{2s}$; 28) $\Sigma \frac{\sin A + \sin B}{\sin C} \lesseqgtr \frac{14R - 3r}{2R}$;

29) $\Sigma \cos A \cos B \lesseqgtr \frac{16Rr - 3r^2 - 4R^2}{4R^2}$;

30) $\Pi \cos A \lesseqgtr \frac{12Rr - 4R^2 - 5r^2}{4R^2}$;

31) $\Sigma \cos^2 A \gtreqless \frac{6R^2 - 12Rr + 5r^2}{2R^2}$;

32) $\Sigma \cos^3 A \gtreqless \frac{1}{4R^3}(4R^3 + 12R^2r - 42Rr^2 + 13r^3)$;

33) $\Pi(\cos A + \cos B) \lesseqgtr \frac{3r^2}{4R^3}(6R - r)$;

34) $\Sigma \cot A \lesseqgtr \frac{1}{2s}(12R - 5r)$;

35) $\Pi \cot A \lesseqgtr \frac{1}{2sr}(12Rr - 5r^2 - 4R^2)$;

36) $\Sigma \cot^2 A \lesseqgtr \frac{1}{4s^2}(12R - 5r)^2 - 2$;

37) $\Sigma \tan^2 \frac{A}{2} \gtreqless \frac{(4R - 3r)^2}{4r(4R - r)}$; 38) $\Sigma \cot^2 \frac{A}{2} \lesseqgtr \frac{2}{r}(4R - 3r)$;

39) $\Sigma \cot^3 \frac{A}{2} \lesseqgtr \frac{4s(R - r)}{r^2}$;

40) $\Sigma \sin^2 \frac{A}{2} \sin^2 \frac{B}{2} \lesseqgtr \frac{r}{16R^2}(8R - 3r)$;

41) $\Sigma \cos^2 \frac{A}{2} \cos^2 \frac{B}{2} \lesseqgtr \frac{16R^2 + 24Rr - 3r^2}{16R^2}$;

42) $\Sigma \cos 2A \gtreqless \frac{3R^2 - 12Rr + 7r^2}{R^2}$;

43) $\Sigma \sin^4 \frac{A}{2} \gtreqless \frac{8R^2 - 16Rr + 5r^2}{8R^2}$; 44) $Q \lesseqgtr 2r(4R - 7r)$;

45) $\Sigma m_a^2 \lesseqgtr \frac{3r}{2}(12R - 5r)$; 46) $\cot \omega \lesseqgtr \frac{1}{2s}(12R - 5r)$;

47) $\Pi w_a \lesseqgtr \frac{64Rr^2(4R - r)}{3(6R - r)}$; 48) $\frac{r}{h_a} \geqslant \frac{1}{4}$.

4.4. Some Further Remarks

Of course, we can consider several other classes of triangles. For example, the following results are valid:

(i) For a triangle with sides $a \leqslant b \leqslant c$ each of the three following conditions

$$(11) \qquad (s^2 + 4Rr + r^2)^3 \lesseqgtr 32s^4Rr$$

is equivalent to the corresponding $b^2 \lesseqgtr ac$.

From (11) and some known identities for triangles we obtain the following conditions equivalent to $b^2 \lesseqgtr ac$:

1) $\Sigma bc \lesseqgtr (32s^4Rr)^{1/3}$, 2) $\Sigma h_a \lesseqgtr (4s^4r/R^2)^{1/3}$,

3) $\Sigma 1/h_bh_c \lesseqgtr (R/2s^2r^5)^{1/3}$, 4) Σ cosec $A \lesseqgtr (4sR/r^2)^{1/3}$,

5) $\Sigma \sin B \sin C \lesseqgtr (s^4r/2R^5)^{1/3}$.

(ii) (W. Janous [8]). For a triangle with sides $a \leqslant b \leqslant c$ the condition

$$(12) \qquad \left(\frac{k-1}{k+1}\right)^2 s^2 \lesseqgtr 2(k-1)Rr - r^2$$

is equivalent to

(a) $2b \lesseqgtr a + c$ if $k = 2$, or

(b) $b + c \lesseqgtr ka$ if $k \geqslant 3$.

Proof. These equivalences are consequences of the identity

$$(13) \qquad 2(k-1)Rr - r^2 - \left(\frac{k-1}{k+1}\right)^2 s^2 = -\frac{1}{2(k+1)^2 s}\Pi(-ka + b + c),$$

which was given by P. Nüesch (see (9)) for $k = 2$.

Remarks. 1° It is obvious that Janous' result is a generalization of Paasche's and of Guba's results (see 4.2 and 4.3).

2° Janous gave generalizations of other results from 4.2 and 4.3. For instance, (a) and (b) are equivalent to

1) $$\Sigma bc \lesseqgtr 2r\left(\frac{k^2 + 4k - 1}{k - 1} R - \frac{2k}{(k-1)^2} r\right),$$

2) $$\Pi h_a \lesseqgtr \frac{2r^3}{R}\left(\frac{k+1}{k-1}\right)^2 (2(k-1)R - r),$$

3) $$\Sigma m_a^2 \lesseqgtr 3r\left(\frac{k^2+3}{k-1} R - \frac{k^2+1}{(k-1)^2} r\right).$$

Comment by S. Tănăsescu. Let $J = [2 + d/R, (R + d)/r]$, where $d = (R^2 - 2Rr)^{1/2}$. Notice incidentaly that we always have $2 \leqslant 2 + d/R < 3$, but $3 \leqslant (R + d)/r$ if and only if $R \geqslant 9r/4$. The finest statement for Janous' result (b) would be:

For any $k \in J$, (12) is equivalent to $\frac{b + c}{a} \lesseqgtr k$.

Of course, for $k > (R + d)/r$, (12) is still true but it is not so sharp.

References

1. A. Bager, 'A Family of Goniometric Inequalities', Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 338-352 (1971), 5-25.
2. V. N. Murty, 'A New Inequality for R, r and s', Crux Math. 8 (1982), 62-68.
3. W. J. Blundon and A. Bager, 'Problem E 2282', Amer. Math. Monthly 78 (1971), 196, 542 and 79 (1972), 397-398.
4. G. C. Giri and B. Prielipp, 'Problem 739', Crux Math. 9 (1983), 210-211.
5. J. Garfunkel and G. Tsintsifas, Inequalities through R-r-s Triangles. (unpublished).
6. S. G. Guba, Matematika v škole, 1965, No. 5, 69 and 1966, No. 4, 77-78.
7. I. Paasche and P. Nüesch, 'Problem 640', Elem. Math. 26 (1971), 19 and 27 (1972), 16-17.
8. W. Janous, Private communication.

5. Some Other Results for Special Triangles

In this part of the book we also use the following notation:
(Δ_a) for acute triangles, (Δ_{na}) for non-acute triangles,
(Δ_o) for obtuse triangles, (Δ_{no}) for non-obtuse triangles,
(Δ_r) for right triangles, (Δ_i) for isosceles triangles.

5.1. $$\Sigma a^8 < 2\Sigma b^4c^4 \quad (\Delta_a).$$

Ş. Gheorghiu, 'Problem 17693', Gaz. Mat. (Bucharest) 84 (1979), 153.

5.2. Let $c > a, b$. If $\lambda > 2$, then for non-acute triangles

$$a^\lambda + b^\lambda < c^\lambda.$$

If $0 < \lambda < 2$, then the reverse inequality is valid for non-obtuse triangles.

Remark. This is a generalization of GI 11.19.

5.3. Let $c > a, b$. If $0 < \lambda \leqslant 2$, then for non-acute triangles

$$a^\lambda + b^\lambda \leqslant 2^{1-\lambda/2}c^\lambda.$$

The reverse inequality holds for non-obtuse triangles if $\lambda \geqslant 2$.

Remark. This is a generalization of GI 11.20.

5.4. $$\Sigma \frac{1}{-\sin^2 A + \sin^2 B + \sin^2 C} \geqslant 4 \quad (\Delta_a).$$

D. Buşneag, 'Problem 9270', Gaz. Mat. (Bucharest) B 19 (1968), 674.

5.5. If all angles of the triangle are $\leqslant \pi - 4\cos^{-1}(\sqrt{3}t)$, where t is the unique real zero of polynomial $12x^3 + 12x^2 - 3x - 4$, then

$$\sum \cos \frac{A}{2} \geqslant \frac{4}{\sqrt{3}}(1 + \prod \sin \frac{A}{2}).$$

J. Garfunkel, B. Rennie, and G. P. Henderson, 'Problem 987', Crux Math. **10** (1984), 292 and **12** (1986), 33-35.

5.6. $$\prod(\tan^2 A)^{\tan^2 A} \geqslant 19683 \quad \{E\} \quad (\Delta_a).$$

C. M. Goodyear, 'Problem 9402', Math. Questions **59** (1893), 93-94.

5.7. $$\frac{\sum \tan^5 A}{\sum \tan A} \geqslant \left(\frac{\sum \tan^2 A}{\sum \tan A}\right)^4 \geqslant \left(\frac{\sum \tan A}{3}\right)^4 \geqslant 9 \quad (\Delta_a).$$

This result, due to C. Tănăsescu, is an interpolation of 'Problem 2739', Mat. v škole 1985, No. 2, 65-66.

5.8. If in an acute triangle we have $a > b > c$ or $b > c > a$, or $c > a > b$, then

$$\prod\left(\frac{\cot B}{\cot C}\right)^{\sec 2A} < 1.$$

H. W. Segar, 'Problem 10615', Educ. Times **59** (1893), 93-94.

5.9. If $f(x_i), g(x_i) > 0$ ($x_i \in R$; $i = 1, 2, 3$), $\lambda \geqslant 0$, p, q, r natural numbers and

$$F = \sum f(x_1)^p (1 + \lambda g(x_1)^q)^{1/r},$$

then

(1) $$F \geqslant 3(\prod f(x_1))^{p/3}(1 + \lambda(\prod g(x_1))^{1/3})^{1/r},$$

with equality if and only if $f(x_1) = f(x_2) = f(x_3)$ and $g(x_1) = g(x_2) = g(x_3)$. Using this result and GI 2.8, 2.12, 2.24, 2.28, 2.32, 2.42 one can obtain thirty-six inequalities by letting

$$f(x),\ g(x) = \operatorname{cosec} x,\ \sec x,\ \tan x,\ \operatorname{cosec}\frac{x}{2},\ \sec\frac{x}{2},\ \cot\frac{x}{2}.$$

For example, for acute triangles,

$$\sum \sec^p A(1 + \lambda \tan^q A)^{1/r} \geqslant 3 \cdot 2^p(1 + \lambda(\sqrt{3})^q)^{1/r}$$

or in the case of $\lambda = 0$:

$$\sum \operatorname{cosec}^p A \geq 3\left(\frac{2}{\sqrt{3}}\right)^p, \quad \sum \sec^p \frac{A}{2} \geq 3\left(\frac{2}{\sqrt{3}}\right)^p, \quad \sum \sec^p A \geq 3 \cdot 2^p,$$

$$\sum \operatorname{cosec}^p \frac{A}{2} \geq 3 \cdot 2^p, \quad \sum \tan^p A \geq 3(\sqrt{3})^p, \quad \sum \cot^p \frac{A}{2} \geq 3(\sqrt{3})^p.$$

Ž. M. Mitrović and M. S. Stanković, 'Some Inequalities for the Angles of an Acute Triangle', Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 357-380 (1971), 97-99.

5.10. For acute triangles, the inequality

$$\sum \sin \frac{A}{2} \geq \frac{4}{3}\left(1 + \prod \sin \frac{A}{2}\right)$$

is super strong (see X.6.61).

J. Garfunkel, Private communication.

5.11. $\sum \sec A \geq 2\sqrt{3}\sum \cot A \geq 6 \quad (\Delta_a).$

This is an interpolating inequality, due to W. Janous, of the corrected version of GI 2.45:

(1) $\sum \sec A \geq 6 \quad (\Delta_{no}).$

(1) is also proved by L. V. Skvorcova.

Janous also proved the following inequality

(2) $\sum \sec A > 1 \quad (\Delta_o).$

L. V. Skvorcova, Mat. v škole 1968, No. 5, 84 and 1969, No. 3, 78.

5.12. $\sum \sin^2 A \tan A + 3\prod \sin A < 0 \quad (\Delta_o),$

$$\sum \sin^2 A \tan A \geq 6\prod \sin A \quad (\Delta_a).$$

The first inequality is due to W. Janous, and the second to V. N. Murty and B. Prielipp. See:

M. S. Klamkin and W. Janous, 'Problem 1060', Crux Math. 11 (1985), 189 and 12 (1986), 291-293.

5.13. $\sum \tan A \leq 2\sum \sin 2A \quad (\Delta_o),$

$$\sum \tan A \geq 2\sum \sin A \geq 2\sum \sin 2A \quad \{E\} \quad (\Delta_a).$$

This is an answer of J. Garfunkel to a problem of M. S. Klamkin.

M. S. Klamkin and J. Garfunkel, 'Problem 958', Crux Math. 10 (1984), 196 and 11 (1985), 263.

5.14. $F^2 \geq \dfrac{9(\prod a^2)(\prod \cos A)}{2(\sum a)^2} \quad (\Delta_a),$

$$F^2 \geqslant \frac{(\Pi a)^2 (\Sigma a)^2 (\Pi \cos A)}{\Sigma a^4 + 2\Sigma b^2 c^2 \cos A} \quad (\Delta_a).$$

A. Nenov and D. Nikolov, 'Problem 3', Matematika (Sofija) 1981 No. 8, 37 and 1982, No. 2, 32-33.

5.15. $\Sigma \frac{A}{a} > \frac{3\sqrt{3}}{2 \max (a, b, c)} \quad (\Delta_a).$

Proof. (W. Janous) Sinc $\sin x < x$ for $x > 0$, we have

$$\Sigma A/a > \Sigma \sin A/a = 3/(2R).$$

The proof is completed if we have

$$\frac{3}{2R} \geqslant \frac{3\sqrt{3}}{2 \max (a, b, c)},$$

i.e.

$$\max (a, b, c) \geqslant \sqrt{3}R.$$

Suppose that this inequality is not true, i.e. that the following is valid

$$\max (a, b, c) < \sqrt{3}R.$$

Then

$$R = \frac{a}{2 \sin A} < \frac{\sqrt{3}R}{2 \sin A},$$

i.e.

$$\sin A < \sqrt{3}/2.$$

Since the triangle is acute, we have $A < \pi/3$, and similarly $B, C < \pi/3$. Therefore $\Sigma A < \pi$, which is a contradiction.

D. Anca, 'Problem 17707', Gaz. Mat. (Bucharest) 84 (1979), 155.

5.16. $\frac{\pi^2}{3} < \frac{\Sigma (aA)^2}{\Sigma a^2} + \Sigma BC < \frac{\pi^2}{2} \quad (\Delta_a).$

N. Saganai, 'Problem O:36', Gaz. Mat. (Bucharest) 84 (1979), 210-211.

5.17. $3\Sigma a \leqslant \pi\Sigma(a/A)$, and $3\Sigma a^2 \geqslant \pi\Sigma(a^2/A) \quad (\Delta_{no}).$

A. Oppenheim and L. E. Mattics, 'Problem E 2649', Amer. Math. Monthly 84 (1977), 294 and 85 (1978), 596-597.

5.18. $\frac{r}{R} \geqslant \frac{11 - 2k}{2(k - 1)} \quad (\Delta_a)$,

where $k = F/F'$ and F' - area of the orthic triangle. Equality holds if and only if the triangle is equilateral.

A. Bănică, 'Problem 11927', Gaz. Mat. (Bucharest) B 23 (1972), 724-725.

5.19. If ϕ denotes the orthic triangle inradius of an acute triangle then

$$\phi R \leqslant 4F^2/(27R^2) \leqslant r^2.$$

This is an interpolating inequality for a result from:

L. Bankoff and Z. Katz, 'Problem 337', Pi Mu Epsilon J. 6 (1975), 191.

5.20. $(\Sigma h_a)/(\Sigma a) > \frac{1}{2}(1 + \frac{r}{s})^2 \quad (\Delta_a)$.

Remark. This inequality is better than the inequality from a problem of S. Reich, and it is due to M. S. Klamkin.

S. Reich, S. M. Diano, and L. Bankoff, 'Problem 749', Math. Mag. 43 (1970), 48 and 228-229.

5.21. $\Sigma m_a/h_a \leqslant 1 + (R/r) \quad (\Delta_a)$.

D. M. Milošević, 'Problem M 811', Kvant 1983, No. 10, 46-47.

5.22. $\min (h_a, h_b, h_c) \leqslant R + r \leqslant \max (h_a, h_b, h_c) \quad (\Delta_a)$.

Remark. See also GI 6.11 and 11.16.

V. Gridasov, Matematika i Fizika (Sofija) 11 (1968), 33.

5.23. $F \geqslant \frac{\sqrt{3}}{4} \min (ab, bc, ca)$ and $R \leqslant \frac{\sqrt{3}}{3} \max (a, b, c) \quad (\Delta_a)$.

E. C. Popa, 'On some Geometric Inequalities (Romanian)', Gaz. Math. (Bucharest) 87 (1982), 256-257.

5.24. For non-obtuse triangles

(1) $$2(4/5)^\lambda \leqslant \Sigma\left(\frac{a^2}{m_b^2 + m_c^2}\right)^\lambda \leqslant 3(2/3)^\lambda \quad (0 < \lambda \leqslant 1).$$

(2) $$\Sigma\left(\frac{a^2}{m_b^2 + m_a^2}\right)^\lambda \geqslant 3(2/3)^\lambda \quad (\lambda \leqslant 0).$$

Proof. It is known that if f is a concave function ($f(0) = 0$) and if $0 < x_1, x_2, x_3 < \frac{1}{2} \Sigma x_1$, then

(3) $$2f(\frac{1}{2}\sum x_1) \leq \sum f(x_1) \leq 3f(\frac{1}{3}\sum x_1).$$

Start with the function $x \to f(x) = (\frac{x}{mx + n})^{\lambda}$ $(m, n > 0, x \geq 0)$. Since $f''(x) = \lambda mx^{\lambda-2} \frac{-2mx + (\lambda - 1)n}{(mx + n)^{\lambda+2}}$, we infer that f is concave for $0 < \lambda \leq 1$ and f is convex for $\lambda \leq 0$. For non-obtuse triangle $a^2 \leq b^2 + c^2$, etc. i.e. $a^2 \leq \frac{1}{2}\sum a^2$, etc., so using the substitutions $m = 3$, $n = \sum a^2$, $x_1 = a^2$, etc., from (3) we obtain (1) because $m_b^2 + m_c^2 = \frac{1}{4}(4a^2 + b^2 + c^2)$. Analogously, one proves (2).

5.25. Let $S = \sum \tan^2 \frac{A}{2}$. Then

a) $S \leq 2$ for non-obtuse triangles;

b) $S \geq 2$ for every obtuse triangle such that $\max(A, B, C) \geq 2\arctan \frac{4}{3}$;

c) If $\frac{\pi}{2} < \max(A, B, C) < 2\arctan \frac{4}{3}$, there are triangles for which $S > 2$, and also triangles for which $S < 2$.

M. L. Gerver, 'Problem M 209', Kvant 1973, No. 6, 12, 1974, No. 1, 21 and 1974, No. 3, 39.

5.26. If A is an obtuse angle of a triangle, then

$$\tan B \tan C \leq \cot^2 \frac{A}{2}$$

with equality if and only if B = C.

M. Tena and M. Ilie, 'Problem 17444*', Gaz. Mat. (Bucharest) 84 (1979), 150.

5.27. $$4(ac + b^2) \leq 5c^2 \quad (\Delta_r).$$

Remark. Here and in the other results for right triangles c denotes the length of the hypotenuse.

J. Garfunkel, 'Problem 431', Pi Mu Epsilon J. 6 (1978), 540.

5.28. $$(a^2(b + c) + b^2(a + c))/abc > \pi \quad (\Delta_r).$$

Proof. More generally, let a, b, c denote the lengths of the sides of a triangle ABC with $c \geq a$, $c \geq b$, then

(1) $$(a^2(b + c) + b^2(a + c))/abc \geq 2 + \operatorname{cosec} \frac{C}{2},$$

with equality if and only if a = b. This result will be proved with the aid of the inequality

(2) $$\frac{c^2}{ab} + \frac{2\sqrt{ab}}{c} \geq 2 - 2\cos C + \operatorname{cosec} \frac{C}{2}$$

which may be seen as follows: The Law of Cosines gives $c^2 = a^2 + b^2 - 2ab \cos C \geqslant 2ab - 2ab \cos C$, so $c/\sqrt{ab} \geqslant \sqrt{2(1 - \cos C)}$. Also $\sqrt{2(1 - \cos C)} \geqslant 1$ since $C \geqslant \pi/3$. Since the function $x \to f(x) = x^2 + (2/x)$ is increasing for $x > 1$, we have $f(c/\sqrt{ab}) \geqslant f(\sqrt{2(1 - \cos C)})$, which is equivalent to (2).

The proof of (1) now proceeds using the Arithmetic-Geometric Means Inequality and the Law of Cosines:

$$(a^2(b + c) + b^2(a + c))/abc = \frac{a^2 + b^2}{ab} + \frac{a + b}{c} =$$

$$= \frac{c^2}{ab} + 2 \cos C + \frac{a + b}{c} \geqslant$$

$$\geqslant \frac{c^2}{ab} + 2 \cos C + \frac{2\sqrt{ab}}{c} \geqslant$$

$$\geqslant 2 - 2 \cos C + \operatorname{cosec} \frac{C}{2} + 2 \cos C =$$

$$= 2 + \operatorname{cosec} \frac{C}{2},$$

with equality in all instances if and only if $a = b$. The required result follows when $C = \pi/2$ since $2 + \sqrt{2} > \pi$.

V. N. Murty, W. Blumberg, and M. Fraser, 'Problem 238', The College Math. J. 15 (1984), 352-353.

5.29. $s/h_c \geqslant 1 + \sqrt{2}$ (Δ_r).

S. T. Berkolajko, 'Problem 1088', Mat. v Škole, 1972, No. 4, 81 and 1973, No. 2, 78.

M. Gruda, 'Problem 3', Matematyka (Warszawa) 27 (1974), 311.

5.30. $c^2 + 1/h_c^2 \geqslant 4$ (Δ_r).

N. Schaumberger and A. J. Berlau, 'Problem 104', New York St. Math. Teac. J. 1979, 54 and 1981, 42-43.

5.31. $c + h_c > a + b$ (Δ_r).

Proof (Š. Arslanagić). From inequality $(c - a)(c - b) > 0$, we get $c + ab/c > a + b$, which is equivalent to the required inequality, because $h_c = ab/c$.

5.32. $1/2 < m_a/m_b < 2$ (Δ_r).

G. B. Hasin, 'Problem 381', Mat. v Škole 1967, No. 4, 75 and 1968, No. 3, 66.

5.33. $$\frac{r^2}{m_a^2 + m_b^2} \leqslant \frac{3 - \sqrt{8}}{5} \quad (\Delta_r).$$

T. A. Ivanova, 'Problem 1614', Mat. v škole 1975, No. 6, 79 and 1976, No. 4.

5.34. $$w_c^4 \leqslant m_c h_c^3 \quad (\Delta_r).$$

Ja. N. Temraliev, 'Problem 1638', Mat. v škole 1976, No. 6, 70.

5.35. $$w_a \geqslant 3h_c\sqrt{3}/4 \quad (\Delta_r),$$

with equality if and only if $\cos A = 1/3$.
Zeitschr. math. naturwiss. Unterr. 28 (1897), 34-35.

5.36. $$aw_a \leqslant \frac{2\sqrt{6}}{9} c^2 \quad (\Delta_r),$$

with equality if and only if $\sin A = \sqrt{5}/3$.
Zeitschr. math. naturwiss. Unterr. 28 (1897), 35.

5.37. $$w_a^2 + w_b^2 \geqslant 5r^2(3 + 2\sqrt{2}) \quad (\Delta_r),$$

with equality if and only if $a = b$.
N. Mihajlovska and I. Kujundžić, 'Problem 76', Matematika 1979, No. 4, 94-95.

5.38. If ϕ is the angle between m_a and m_b for a right triangle, then

$$\cos \phi > 4/5.$$

Š. Arslanagić, I. Kujundžić, and D. Jocić, 'Problem 65', Matematika 1979, No. 2, 61-62.

5.39. If $p = (\Pi m_a)/abc$, then

$$0 < p < +\infty \quad (\Delta_o), \quad \frac{5}{8} \leqslant p < +\infty \quad (\Delta_r), \quad \frac{5}{8} < p < +\infty \quad (\Delta_a).$$

These inequalities cannot be improved. Equality holds for a right isosceles triangle.
O. Bottema and M. Jovanović, 'On the Ratio $m_a m_b m_c/(abc)$ for the Triangle ABC', Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 412-460 (1973), 197-199.

5.40. $$\frac{\sqrt{2}}{2} < \frac{a}{w_b} \leqslant \frac{3}{2} \quad (\Delta_i),$$

where a is the base of a triangle.

V. I. Gridasov, 'Problem 566', Mat. v škole 1969, No. 1, 74 and No. 5, 77.

5.41. Let $h_b = h_c$ in a non-obtuse triangle. Then

$$3R \geqslant h_a + h_b.$$

Ju. I. Gerasimov, 'Problem 557', Mat. v škole 1968, No. 6, 73 and 1969, No. 4, 80.

5.42. If $a^2 + b^2 > 2c^2$, then $C < \pi/3$.

Remark. This result is due to D. M. Milošević.

5.43. If $A = 2B$, then

$$1/2 < m_a/b \leqslant \sqrt{5}/2.$$

E. A. Bokov, 'Problem 720', Mat. v škole 1970, No. 1, 82 and No. 5, 76.

5.44. If $b + c = 2R\sqrt{3}$, then

$$\frac{\sin 2A}{\sin 3A} \geqslant \frac{2}{3}.$$

'Problem 1224', Mat. v škole 1973, No. 3, 80 and 1974, No. 1, 72.

5.45. If the medians m_a and m_c are perpendicular to each other, then

$$\cos B \leqslant 4/5.$$

Proof. Since $\vec{m}_a = \frac{1}{2}\vec{a} - \vec{c}$, $\vec{m}_c = \frac{1}{2}\vec{c} - \vec{a}$, and $m_a \perp m_c$, we have

$$(\frac{1}{2}\vec{a} - \vec{c})(\frac{1}{2}\vec{c} - \vec{a}) = \vec{0}$$

i.e.

$$\vec{a} \cdot \vec{c} = \frac{2}{5}(a^2 + c^2).$$

On the other hand

$$\cos B = \frac{\vec{a} \cdot \vec{c}}{ac} = \frac{2}{5}\frac{a^2 + c^2}{ac} = \frac{4}{5}\frac{a^2 + c^2}{2ac} \leqslant \frac{4}{5},$$

because $x^2 + y^2 \geqslant 2xy$.

T. M. Korikova, 'Some Geometric Inequalities and Their Vectorial Proofs' (Russian), Mat. v škole 1977, No. 3, 64-67.

Š. Arslanagić, 'Dokazivanje nejednakosti pomoću vektora', Matematika 1983, No. 1, 65-72.

5.46. If $a + b = 2c$, then $c^2 \geqslant 6Rr$.

Gh. Popescu and G. Muresan, 'Problem 9368', Gaz. Mat. (Bucharest) B 20 (1969), 40 and B 21 (1970), 602.

5.47. If $AI = (2Rr)^{1/2}$, then

$$\sin B \sin C \leqslant 3/4.$$

I. A. Kušnir, 'Problem 383', Mat. v škole 1967, No. 4, 75 and 1968, No. 3, 66-67.

5.48. $$\sum \frac{a}{b^2 + c^2} \sec A \geqslant s/(3Rr) \quad (\Delta_a).$$

This inequality is due to D. M. Milošević.

5.49. The length of a side of the Morley triangle of a given triangle T is less than one third the length of the smallest side of T.

Remark. The Morley triangle of T is the equilateral triangle formed by the intersection in pairs of the angle trisectors of T.

M. S. Klamkin and R. Spira, 'Problem 908', Elem. Math. 39 (1984), 80.

5.50. $$\sum w_a^{-k} < K\sum a^{-k} \quad (\Delta_a),$$

where $K = 2^{1-k/2}$ for $k \in (0, 1)$ and $K = 2^{k/2}$ for $k \geqslant 1$.

This result is due to D. M. Milošević.

5.51. $$(\sum \cos A)/(\sum \cos B \cos C) \geqslant 2 \quad (\Delta_a).$$

S. Iwata, 'Encyclopaedia of Geometry (Japanese)', Tokyo 1971, Vol. 5, p. 334

Comment by C. Tănăsescu. The function $F(A, B, C) = \sum \cos A - 2\sum(\cos B \cdot \cos C)$ is strictly Schur-convex for $A, B, C \in [0, \pi/2]$, so we have

$$0 \leqslant \sum \cos A - 2\sum \cos B \cos C \leqslant 1. \quad (\Delta_{no}).$$

5.52. $$\sum \cos \frac{B}{2} \cos \frac{C}{2} > \frac{1}{2} + \sqrt{2} \quad (\Delta_a).$$

This inequality was conjectured by J. T. Groenman. W. Janous gave the proof of this inequality.

J. T. Groenman, 'Problem 1111', Crux Math. 12 (1986), 26.

5.53. $$\sum\sqrt{\operatorname{cotan} A} \geqslant 2 \quad (\Delta_{no}).$$

V. M. Popa, 'Problem 20818', Gaz. Mat. (Bucharest) 91 (1986), 262.

5.54. Let the functions f and g be defined by

$$f(x) = \frac{\pi^2 x}{2\pi^2 + 8x^2} \quad \text{and} \quad g(x) = \frac{8x}{4\pi + \pi x^2} \quad \text{for all real } x.$$

(a) For acute triangles the following inequality is valid

(1) $$\Sigma f(A) < \frac{a + b + c}{4R} < \Sigma g(A).$$

(b) Problem: Determine functions f and g, where f(x) and g(x) have the form $x/(u + vx^2)$, with u and v real constants, for which the inequalities in (1) are best possible.

M. Bencze, 'Problem 1236', Math. Mag. 59 (1986), 44.

5.55. $$\Sigma (bc)^{\cos A} \leqslant 2 + \frac{1}{2}\Sigma a^2 - \frac{r}{R} \quad (\Delta_a)$$

with equality if and only if a = b = c = 1.

A. Lupaş, 'Problem 689', Math. Mag. 41 (1968), 95-96 and 289-290.

Chapter XI

TRIANGLE AND POINT

1. Some Applications of a Generalization of the Leibniz Identity

1.1. Let P be a point in the plane of a triangle ABC, and let M be an arbitrary point in space. Then $\vec{MP} = (\Sigma x_1 \vec{MA})/(\Sigma x_1)$, where x_1, x_2, x_3 are real numbers, and the following generalization of the well-known Leibniz identity is valid

$$(1) \qquad (\Sigma x_1)^2 MP^2 = \Sigma x_1 \Sigma x_1 MA^2 - \Sigma a^2 x_2 x_3 .$$

For a generalization of this identity see XVIII.2.20. A simple consequence of (1) is the following inequality of Klamkin ([1]):

$$(2) \qquad \Sigma x_1 \Sigma x_1 MA^2 \geqslant \Sigma a^2 x_2 x_3 .$$

For M = A, (1) becomes

$$(3) \qquad AP^2 = \frac{1}{\Sigma x_1}(x_2 c^2 + x_3 b^2) - \frac{1}{(\Sigma x_1)^2} \Sigma a^2 x_2 x_3 .$$

Additionally, by defining the points P and M as special points of the triangle, we easily obtain numerous identities with characteristic points of the triangle.

For example, we have for

1) $x_1 = x_2 = x_3 = 1$ - the centroid G of a triangle;

2) $x_1 = a$, etc. - the incentre I;

3) $x_1 = \sin 2A$, etc. - the circumcentre O;

4) $x_1 = \tan A$, etc. - the orthocentre H;

5) $x_1 = a^2$, etc. - the Lhuilier-Lemoine point K;

6) $x_1 = 1/(s - a)$, etc. - the Gergonne point Γ;

7) $x_1 = (s - a)$, etc. - the Nagel point N;

8) $(x_1, x_2, x_3) = \left(\frac{1}{c^2}, \frac{1}{a^2}, \frac{1}{b^2}\right)$ - the Crelle-Brocard point Ω_1;

9) $(x_1, x_2, x_3) = \left(\frac{1}{b^2}, \frac{1}{c^2}, \frac{1}{a^2}\right)$ - the Crelle-Brocard point Ω_2.

Remark. The points Ω_1 and Ω_2 were defined in 1816 by A. L. Crelle [8], and in 1875 H. Brocard revived the subject.

For these points (3) becomes

$$AG^2 = \frac{2}{9}(\Sigma a^2) - \frac{a^2}{3}, \quad AI^2 = \frac{bc(s-a)}{s},$$

$$AK^2 = \frac{b^2c^2(2\Sigma a^2 - 3a^2)}{(\Sigma a^2)^2}, \quad AO = R,$$

$$A\Gamma^2 = \frac{sr}{4R+r}\left(\frac{c^2}{s-b} + \frac{b^2}{s-c}\right) - \frac{4s^2r(R+r)}{(4R+r)^2},$$

$$AN^2 = (b-c)^2 + 4r^2, \quad AH = 2R|\cos A|, \quad A\Omega_1^2 = \frac{c^4b^2}{\Sigma b^2c^2},$$

$$A\Omega_2^2 = \frac{b^4c^2}{\Sigma b^2c^2}.$$

Using these identities, (1) and some other identities from Section IV.2, we can get several important identities.

A. For M = O, we get

$$OP^2 = R^2 - (\Sigma a^2x_2x_3)/(\Sigma x_1)^2 \tag{4}$$

and as a simple consequence of GI 14.1: $R^2(\Sigma x_1)^2 \geqslant \Sigma a^2x_2x_3$, where x_1, x_2, x_3 are real numbers. Equality is valid if and only if P = O.

Further, by specializing the point P we can get the following results from (4):

1) $OI^2 = R(R - 2r)$,

and as a consequence the Chapple-Euler inequality $R \geqslant 2r$ (see GI 5.1 and [10]);

2) $OG^2 = R^2 - \frac{1}{9}\Sigma a^2 = \frac{1}{9}(9R^2 + 8Rr + 2r^2 - 2s^2)$;

3) $OK^2 = R^2 - 3(\Pi a^2)/(\Sigma a^2)^2$;

4) $O\Gamma^2 = R^2 - 4s^2r(R + r)/(4R + r)^2$,

and as a consequence

$$s \leqslant R(4R + r)/2\sqrt{r(R + r)};$$

5) $ON = R - 2r$;

6) $OH^2 = 9R^2 - \Sigma a^2 = 9R^2 + 2r^2 + 8Rr - 2s^2$, i.e. $OH = 3 \cdot OG$;

7) $O\Omega_1^2 = O\Omega_2^2 = R^2 - 1/(\Sigma a^{-2}) = R^2(1 - 4\sin^2\omega)$.

B. For $P = I$, we get:

(5) $$\Sigma aMA^2 = 2sMI^2 + abc,$$

and as a consequence we have the following inequality ([2], [1]):

$$\Sigma aMA^2 \geqslant abc \quad \text{(equality holds only for } M = I\text{)}.$$

Some further applications are:

1) $\Sigma aGA^2 = \frac{2s}{9}(s^2 + 5r^2 + 2Rr)$, $GI^2 = \frac{1}{9}(s^2 + 5r^2 - 16Rr)$.

Proof. $\Sigma aGA^2 = \Sigma a(\frac{2}{9}(\Sigma a)^2 - \frac{a^2}{3}) = \frac{1}{9}(2(\Sigma a^2) \cdot 2s - 3\Sigma a^3) =$

$$= \frac{2s}{9}(s^2 + 5r^2 + 2Rr),$$

i.e. we get the first identity. The second now easily follows from (5).

2) $NI^2 = s^2 - 16Rr + 5r^2$, i.e. $NI = 3 \cdot GI$.

3) $\Sigma aKA^2 = 16sRr(s^4 - 6s^2Rr - r^2(4R + r)^2)/(\Sigma a^2)^2$,

$$KI = \frac{2rs\sqrt{R(R + r) - rR(4R + r)^2/s^2}}{s^2 - r^2 - 4Rr}.$$

4) $\Sigma aHA^2 = 2s(4R^2 + 6Rr + 3r^2 - s^2)$, $HI^2 = 4R^2 + 4Rr + 3r^2 - s^2$.

Proof. $\Sigma aHA^2 = 4\Sigma aR^2\cos^2 A = 4R^2\Sigma a(1 - a^2/4R^2) = 8R^2s - \Sigma a^3 =$

$$= 8R^2s - 2s(s^2 - 3r^2 - 6Rr) = 2s(4R^2 + 6Rr + 3r^2 - s^2),$$

and from (5) we get the second identity.

C. For P = N, we get

(6) $$\Sigma(s - a)MA^2 = sMN^2 + 4sr(R - r),$$

and as a consequence

$$\Sigma(s - a)MA^2 \geqslant 4F(R - r).$$

Some special cases of (6) are:

1) $\Sigma(s - a)GA^2 = \frac{4s}{9}(s^2 - 7Rr - 4r^2)$, $GN^2 = \frac{4}{9}(s^2 + 5r^2 - 16Rr)$,

i.e. $GN = \frac{2}{3} NI$.

2) $$KN^2 = \frac{2}{(\Sigma a^2)}(s^4 - 2s^2 r(6R - r) + r^3(4Rr + r)) - \\ - \frac{48s^2r^2R^2}{(\Sigma a^2)^2} - 4r(R - r).$$

3) $\Sigma(s - a)HA^2 = 4s(R^2 - Rr - r^2)$, $NH^2 = 4R(R - 2r)$, i.e.

$NH = 2 \cdot OI$.

D. For P = K, we get

(7) $$\Sigma a^2MA^2 = (\Sigma a^2)MK^2 + 3(\Pi a^2)/(\Sigma a^2),$$

and as a consequence

$$\Sigma a^2MA^2 \geqslant 3(\Pi a^2)/(\Sigma a^2),$$

with equality if and only if M = K.

We shall also give the following consequence of (7):

1) $\Sigma a^2AH^2 = 4R^2(\Sigma a^2) - (\Sigma a^4)$,

$HK^2 = 4R^2 - \Sigma a^2 + (2(\Sigma a^2)(\Sigma b^2c^2) - 3\Pi a^2)/(\Sigma a^2)^2;$

2) $\Omega_1K^2 = 1/(\Sigma a^{-2}) - 3\Pi a^2/(\Sigma a^2)^2 = 4R^2\left(\sin^2 \omega - 3\left(\frac{\Pi \sin A}{\Sigma \sin^2 A}\right)^2\right)$.

This gives the inequality $\Sigma a^4 \geqslant \Sigma b^2c^2$.

We have also $O\Omega_1^2 + O\Omega_2^2 = OK^2$ (by A.3) and thus the well-known relation $\Omega_1\Omega_2 \perp OK$. Inequality $\Omega_1K^2 \geqslant 0$ gives the pretty inequality

$$\sin\omega \geqslant \sqrt{3}\,\frac{\Pi \sin A}{\Sigma \sin^2 A} = \sqrt{3}\,\frac{rs}{s^2 - r(4R + r)} \geqslant \frac{\sqrt{3}}{2}\,\frac{rs}{2R^2 + r^2} \geqslant 2\left(\frac{r}{R}\right)^2,$$

for the Crelle-Brocard angle ω.

E. For $P = \Gamma$, we get

$$\Sigma \frac{MA^2}{s - a} = \frac{4R + r}{sr} M\Gamma^2 + \frac{4s(R + r)}{4R + r} \tag{8}$$

and as a consequence

$$\Sigma \frac{MA^2}{s - a} \geqslant \frac{4s(R + r)}{4R + r},$$

with equality if and only if $M \equiv \Gamma$.

As some applications of (8) we shall give the following results:

1) $$\Sigma \frac{GA^2}{s - a} = \frac{4}{9sr}(s^2(R + 4r) - r(4R + r)^2),$$

$$\Gamma G^2 = \frac{4}{9}\,\frac{s^2(4R^2 + 8Rr - 5r^2) - r(4R + r)^3}{(4R + r)^2}.$$

2) $$\Sigma \frac{AH^2}{s - a} = \frac{4(R^2(4R + r) - s^2(R - r))}{sr},$$

$$H\Gamma^2 = 4R^2\left(1 - \frac{2s^2(2R - r)}{R(4R + r)^2}\right).$$

3) $$\Sigma \frac{AI^2}{s - a} = \frac{s^2 + r^2 + 4Rr}{s}, \qquad I\Gamma^2 = r^2\left(1 - \frac{3s^2}{(4R + r)^2}\right).$$

4) $$K\Gamma^2 = \frac{2(-s^2(\Sigma b^2c^2) + 4Rrs^2(\Sigma bc) + \Sigma b^3c^3)}{r(4R + r)(\Sigma a^2)} - \frac{3\Pi a^2}{(\Sigma a^2)^2} - \frac{4s^2r(R + r)}{(4R + r)^2}.$$

5) $$\Sigma \frac{AN^2}{s - a} = \frac{4}{sr}(s^2(R + r) - 4rR(4R + r)),$$

$$N\Gamma^2 = \frac{16R}{(4R + r)^2}(s^2(R + r) - r(4R + r)^2).$$

F. For $P = G$, we get the Leibniz identity:

$$\Sigma MA^2 = 3MG^2 + \frac{1}{3}(\Sigma a^2), \tag{9}$$

wherefrom we get

$$\Sigma MA^2 \geqslant \frac{1}{3}\,\Sigma a^2$$

with equality if and only if $M \equiv G$.

1) $\Sigma AH^2 = 2(6R^2 + 4Rr + r^2 - s^2)$,

$GH^2 = \frac{4}{9}(9R^2 + 8Rr + 2r^2 - 2s^2)$, i.e. $GH = \frac{2}{3}\,OH$.

2) $KG^2 = \frac{1}{9(\Sigma a^2)^2}(6(\Sigma a^2)(\Sigma b^2c^2) - (\Sigma a^2)^3 - 27\Pi a^2)$.

G. For $P = \Omega_i$ $(i = 1, 2)$ we get from (1):

$$(\Sigma a^{-2})M\Omega_1^2 = \Sigma MA^2/c^2 - 1, \quad (\Sigma a^{-2})M\Omega_2^2 = \Sigma MA^2/b^2 - 1.$$

For instance, we obtain

1) $G\Omega_1^2 = \frac{2}{9}\,\Sigma a^2 - 1/(\Sigma a^{-2}) - \frac{1}{3}(\Sigma a^4b^2)/(\Sigma a^2b^2)$,

2) $G\Omega_2^2 = \frac{2}{9}\,\Sigma a^2 - 1/(\Sigma a^{-2}) - \frac{1}{3}(\Sigma a^2b^4)/(\Sigma a^2b^2)$,

3) $G\Omega_1^2 + G\Omega_2^2 = (1/9)\Sigma a^2 - 1/(\Sigma a^{-2})$,

4) $OG^2 + G\Omega_1^2 + G\Omega_2^2 = O\Omega_1^2 = OK^2 - K\Omega_1^2$,

5) $OG^2 + G\Omega_1^2 + G\Omega_2^2 \leqslant OK^2$ {E}.

Further,

6) $H\Omega_1^2 = 4R^2 - 1/(\Sigma a^{-2}) - (\Sigma a^4b^2)/(\Sigma a^2b^2)$,

7) $H\Omega_2^2 = 4R^2 - 1/(\Sigma a^{-2}) - (\Sigma a^2b^4)/(\Sigma a^2b^2)$,

8) $H\Omega_1^2 + H\Omega_2^2 = 8R^2 + 1/(\Sigma a^{-2}) - \Sigma a^2 \geqslant 0$:

now, since $(\Sigma a^{-2})^{-1} = 4R^2 \sin^2 \omega$, we get the inequality

9) $(8 + 4 \sin^2 \omega)R^2 \geqslant \Sigma a^2$,

that is finer than $\Sigma a^2 \leqslant 9R^2$.

A last interesting identity:

10) $(G\Omega_1^2 + \Omega_1 H^2) + (G\Omega_2^2 + \Omega_2 H^2) = 2GH^2$

(one of the angles $G\Omega_1 H$, $G\Omega_2 H$ is obtuse, the other is acute).

H. Here, we shall note that S. Bilčev and E. Velikova [9] gave the following simple inequalities

$$A\Omega_2^2 \leqslant \frac{b^2}{a^2} R^2, \quad B\Omega_2^2 \leqslant \frac{c^2}{b^2} R^2, \quad C\Omega_2^2 \leqslant \frac{a^2}{c^2} R^2.$$

So, using (2) in the case $x_1, x_2, x_3 > 0$, we have

$$\Sigma a^2 x_2 x_3 \leqslant R^2 \Sigma x_1 \Sigma x_1 b^2/a^2.$$

Similarly, we can get

$$\Sigma a^2 x_2 x_3 \geqslant R^2 \Sigma x_1 \Sigma x_1 c^2/a^2.$$

Remark. Bilčev and Velikova proved these inequalities in the case $x_1 = x_2 = x_3 = 1$.

Of course, if in (2) we directly substitute the expressions for $A\Omega_1^2$, etc. and $A\Omega_2^2$ etc., we get the following inequalities for real x_1, x_2, x_3:

$$\Sigma x_1 \Sigma x_1 b^2/a^2 \geqslant \Sigma a^{-2} \Sigma a^2 x_2 x_3,$$
$$\Sigma x_1 \Sigma x_1 c^2/a^2 \geqslant \Sigma a^{-2} \Sigma a^2 x_2 x_3.$$

Remark. The cases with $x_1 = x_2 = x_3$ were proved by Bilčev and Velikova. The first inequality is a generalization of IX.1.1.

In their paper, Bilčev and Velikova also used relation

$$\Omega_1 \Omega_2^2 = \frac{\Sigma a^4 - \Sigma b^2 c^2}{(\Sigma bc/a)^2}$$

in the proof of the well-known inequality $\Omega_1 \Omega_2 \leqslant R/2$ (GI 14.25).

Comment by V. Mascioni. Since $K O \Omega_1 = \omega$, we get $\Omega_1 \Omega_2 = \sin\omega \cdot O\Omega_1 = 2R \sin\omega\sqrt{1 - 4\sin^2\omega}$. It is very easy to derive $\Omega_1\Omega_2 \leqslant R/2$ from this.

Here we shall also note that using the relation for $A\Omega_1$, etc. and for $A\Omega_2$, etc., and other inequalities involving R_1, R_2, R_3 we can obtain other interesting results. For example, the following result is due to Bilčev and Velikova:

Using the inequality: $R_1 + R_2 + R_3 \geqslant 2\sqrt{F\sqrt{3}}$ in the case $P = \Omega_2$, we get

$$(\Sigma b^2 c)^2/(\Sigma b^2 c^2) \geqslant 4F\sqrt{3}$$

which is better than GI 4.4.: $\Sigma a^2 \geqslant 4F\sqrt{3}$.

Similarly, they got

$$(\Sigma bc^2)^2/(\Sigma b^2 c^2) \geqslant 4F\sqrt{3}.$$

J. The centre S of Spieker's circle has barycentric coordinates $x_1 = 2s - a$, etc. It follows that S is the midpoint between N and I. We have then

$$MS^2 = \frac{1}{4s}\Sigma MA^2(2s - a) - \frac{1}{4}(s^2 - 3r^2 - 4Rr) = \frac{1}{2}(MN^2 + MI^2) - NS^2,$$

and, for example,

1) $4OS^2 = 4R^2 + 3r^2 + 4Rr - s^2$,

2) $4HS^2 = 16R^2 + 8Rr + r^2 - 3s^2$,

3) $4NS^2 = s^2 + 5r^2 - 16Rr = NI^2$.

This proves the important inequalities:

$$s^2 \leqslant 4R^2 + 4Rr + 3r^2, \quad 3s^2 \leqslant (4R + r)^2, \quad 16Rr \leqslant s^2 + 5r^2.$$

This shows that the point S is crucial for inequality theory: in fact almost all of the inequalities from 1.2, 1)-13) reduce, i.e. they are equivalent to $OS^2 \geqslant 0$, $HS^2 \geqslant 0$, $NS^2 \geqslant 0$.

Remark. The above results for Ω_1, Ω_2 (except the part H.) and S are given by V. Mascioni. The other results can be found in [3] (we only gave some simple extensions). For example, only the case $x_1, x_2, x_3 > 0$ was considered in [3], i.e. the case when the point P is in the interior of the triangle ABC. Let the corresponding cevians be AA_1, BB_1, CC_1 ($A_1 \in BC$, $B_1 \in CA$, $C_1 \in AB$), i.e. let

$$\frac{C_1A}{C_1B} = \frac{x_2}{x_1}, \quad \frac{B_1C}{B_1A} = \frac{x_1}{x_3}, \quad \frac{A_1B}{A_1C} = \frac{x_3}{x_2};$$

then

$$PA = \frac{(x_2 + x_3)AA_1}{x_1 + x_2 + x_3}, \quad PA_1 = \frac{x_1 AA_1}{x_1 + x_2 + x_3}, \quad \text{and}$$

$$AA_1^2 = \frac{c^2x_2 + b^2x_3}{x_2 + x_3} - \frac{a^2x_2x_3}{(x_2 + x_3)^2}; \quad \text{etc.}$$

Comment by S. J. Bilčev, H. Lesov, and E. A. Velikova. Since

$$4AS^2 = 3s^2 - r^2 - 8Rr - 2(a^2 + bc)$$

we can obtain the following inequalities

$$\Sigma AS \geq \frac{1}{4R}(3s^2 - 5r^2 - 14Rr) \geq \frac{r}{2R}(17R - 10r) \geq \frac{12r^2}{R},$$

$$\Sigma \frac{AS^2}{a} \geq \frac{r}{Rs}(16R^2 - 15Rr + 2r^2) \geq \frac{36r^3}{Rs},$$

$$\Sigma \frac{AS^2}{a} \leq \frac{R}{rs}(R^2 + 3Rr - r^2), \quad 4sRr \leq \Sigma aAS^2 \leq 2s(R^2 - Rr + 2r^2).$$

The following identities are also valid:

$$S\Gamma^2 = \frac{(4R - r)(4R + 7r)}{4(4R + r)^2} s^2 - \frac{r(16R + 3r)}{4},$$

$$I\Omega_1^2 = (\Sigma \frac{a}{b} - 1)/(\Sigma \frac{1}{a^2}) - 4Rr, \quad I\Omega_2^2 = (\Sigma \frac{b}{a} - 1)/(\Sigma \frac{1}{a^2}) - 4Rr,$$

$$G\Omega_1^2 = \frac{1}{9\Sigma \frac{1}{a^2}} (2\Sigma \frac{a^2}{b^2} - \Sigma \frac{b^2}{a^2} - 3),$$

$$G\Omega_2^2 = \frac{1}{9\Sigma \frac{1}{a^2}} (2\Sigma \frac{b^2}{a^2} - \Sigma \frac{a^2}{b^2} - 3),$$

$$H\Omega_2^2 = 4R^2 - (1 + \Sigma \frac{a^2}{b^2})/(\Sigma \frac{1}{a^2}), \quad H\Omega_1^2 = 4R^2 - (1 + \Sigma \frac{b^2}{a^2})/(\Sigma \frac{1}{a^2}),$$

$$N\Omega_1^2 = 2s^2 + 2r^2 - 8Rr - (1 + (\Sigma \frac{b}{a})^2)/(\Sigma \frac{1}{a^2}),$$

$$N\Omega_2^2 = 2s^2 + 2r^2 - 8Rr - (1 + (\Sigma \frac{a}{b})^2)/(\Sigma \frac{1}{a^2}),$$

$$\Gamma\Omega_1^2 = \frac{rs}{4R + r}\left[\frac{\Sigma \frac{b^2c^4}{s - a}}{\Sigma b^2c^2} - 4s\frac{R + r}{4R + r}\right],$$

$$\Gamma\Omega_2^2 = \frac{rs}{4R + r}\left[\frac{\Sigma \frac{b^4c^2}{s - a}}{\Sigma b^2c^2} - 4s\frac{R + r}{4R + r}\right],$$

whence we can obtain several inequalities, as, for example,

$$\sin^2 \omega \geqslant \frac{4r^2}{s^2 + r^2 - 6Rr} \geqslant \frac{2r^2}{2R^2 - Rr + 2r^2},$$

$$\operatorname{cosec}^2 \omega = 4R^2\Sigma \frac{1}{a^2} \geqslant 1 + \Sigma \frac{a^2}{b^2}, \quad 2\Sigma \frac{a^2}{b^2} \geqslant \Sigma \frac{b^2}{a^2} + 3,$$

$$2\Sigma \frac{b^2}{a^2} \geqslant \Sigma \frac{a^2}{b^2} + 3, \quad \Sigma a^2 \leqslant 4R^2(2 + \sin^2 \omega) \leqslant 9R^2,$$

etc.

Of course, using the method from Chapter V we can give several applications of the above identities. Here, we shall give some of these applications.

1.2. First, we shall give some inequalities which are valid for every triangle.

1) $$\frac{1}{9}(R - 2r)(R + 2r) \leqslant OG^2 = \frac{1}{9} OH^2 = \frac{1}{4} GH^2 \leqslant$$
$$\leqslant \frac{1}{9}(9R^2 - 24Rr + 12r^2) \quad \{E\};$$

2) $$R^2 - \frac{4}{3} r(R + r) \leqslant O\Gamma^2 \leqslant R^2 - \frac{12r^2(R + r)}{4R + r} \quad \{E\};$$

3) $$4RF \leqslant \Sigma aGA^2 \leqslant \frac{4s}{9}(2R^2 + 3Rr + 4r^2) \quad \{E\},$$

4) $$GI^2 = \frac{1}{9} NI^2 = \frac{1}{4} GN^2 \leqslant \frac{4}{9}(R^2 - 3Rr + 2r^2) \quad \{E\},$$

5) $$4RF \leqslant \Sigma aHA^2 \leqslant 4s(2R^2 - 5Rr + 4r^2) \quad \{E\},$$

6) $$HI^2 \leqslant 4(R^2 - 3Rr + 2r^2) \quad \{E\},$$

7) $$4F(R - r) \leqslant \Sigma(s - a)GA^2 \leqslant \frac{4s}{9}(4R^2 - 3Rr - r^2) \quad \{E\},$$

8) $$\frac{4r}{s}(5R - r) \leqslant \Sigma \frac{AI^2}{s - a} \leqslant \frac{4}{s}(R + r)^2 \quad \{E\},$$

9) $$4r^2 \frac{R^2 - Rr - 2r^2}{(4R + r)^2} \leqslant I\Gamma^2 \leqslant 8r^2 \frac{2R^2 - 5Rr + 2r^2}{(4R + r)^2} \quad \{E\},$$

10) $$4(R^2 - r^2) \leqslant \Sigma AH^2 \leqslant 12(R - r)^2 \quad \{E\},$$

11) $$GH^2 \geqslant GI^2 + IH^2 \quad \{E\},$$

so the angle GIH is obtuse.

Proof. L. Bankoff (in an unpublished note, dated May 5, 1967) and W. J. Blundon [4] (see [5]) independently proved that

$$GH^2 - GI^2 - IH^2 = \frac{4r}{3R} OI^2 = \frac{4}{3} r(R - 2r),$$

wherefrom we easily get 11).

12) $$OH^2 \geqslant IO^2 + 2IH^2 \quad \{E\}.$$

Remark. This result is due to W. J. Blundon.

13) $$1 < HO/IO < 3 \quad \text{and} \quad 0 < HI/HO < 2/3.$$

Remark. This is a result from [6].

14) $$12r^2 \leqslant \Sigma IA^2 \leqslant 4(R^2 - Rr + r^2) \quad \{E\}.$$

The first inequality is given in [11].

1.3. Depending upon the fact whether a triangle is acute, right, or obtuse, the following statements hold:

1) $OH \lesseqgtr R$; 2) $IH \lesseqgtr r\sqrt{2}$; 3) $\Sigma HA^2 \lesseqgtr 4R^2$;

4) $\Sigma IA^2 \gtreqless 4R^2 - 4Rr + 2r^2$; 5) $\Sigma aGA^2 \gtreqless \frac{4s}{9}(2R^2 + 3Rr + 3r^2)$;

6) $GI^2 \gtreqless \frac{2}{9}(2R^2 - 6Rr + 3r^2)$; 7) $\Sigma aHA^2 \lesseqgtr 4F(R + r)$;

8) $\sum(s-a)GA^2 \gtreqless \frac{4s}{9}(4R^2 - 3Rr - 3r^2)$;

9) $\sum(s-a)HA^2 \gtreqless 4(2R+r)(R^2 - Rr - r^2)$;

10) $\sum \frac{AI^2}{s-a} \gtreqless \frac{1}{s}(4R^2 + 8Rr + 2r^2)$; 11) $I\Gamma^2 \lesseqgtr \frac{2r^2(2R^2 - 2Rr - r^2)}{(4R+r)^2}$;

12) $OH \lesseqgtr \frac{1}{2}\max\{a, b, c\}$.

Remark. The first four results are consequences of analogous results from Section X.1, where triangles satisfying $a^2 + b^2 + c^2 \gtreqless kR^2$ were considered. Of course, we can similarly give generalizations of the above results 5-11) for these triangles. 12) is due to C. Tănăsescu.

1.4. Now, we shall give some results for triangles of Bager's type I (see Section X.4.1). The inequalities have to be reversed for triangles of Bager's type II.

1) $OH^2 \leqslant 3R^2 - 4Rr - 4r^2$; 2) $IH^2 \leqslant R(R - 2r)$;

3) $\sum aGA^2 \geqslant \frac{2s}{9}(3R^2 + 8Rr + 8r^2)$; 4) $GI^2 \geqslant \frac{1}{9}(3R^2 - 10Rr + 8r^2)$;

5) $\sum aHA^2 \leqslant 2sR^2$; 6) $\sum(s-a)GA^2 \geqslant \frac{4s}{9}(3R^2 - Rr - r^2)$;

7) $\sum(s-a)HA^2 \geqslant 4(R+r)\sqrt{3}(R^2 - Rr - r^2)$;

8) $\sum \frac{AI^2}{s-a} \geqslant \frac{1}{s}(3R^2 + 10Rr + 4r^2)$;

9) $I\Gamma^2 \leqslant \frac{r^2(7R^2 - 10Rr - 8r^2)}{(4R+r)^2}$; 10) $\sum AH^2 \leqslant 2(3R^2 - 2Rr - 2r^2)$.

Remark. Of course, 2) follows from the following result (see [6]):

$0 < HI/IO < 1$ if $B > \pi/3$, $HI = IO$ if $B = \pi/3$, and

$1 < HI/IO < 2$ if $B < \pi/3$,

where $A \leqslant B \leqslant C$.

Comment by V. Mascioni. Further, we have

$$HI^2 = \left(4 - \frac{s^2 - 12Rr - 3r^2}{R(R-2r)}\right)IO^2 \leqslant 4 \cdot IO^2 \quad \{E\},$$

since

$$s^2 - 12Rr - 3r^2 = (s^2 - 16Rr + 5r^2) + 4r(R - 2r) \geqslant$$

$$\geqslant 4r(R - 2r) \geqslant 0. \quad \{E\}$$

We may thus state the following refinement of $HI/IO < 2$:

$$HI \leqslant 2\sqrt{1 - \frac{r}{R}} \cdot IO \quad \{E\}$$

i.e.

$$\frac{HI}{IO} < 2\sqrt{1 - \frac{r}{R}}$$

if the triangle is not equilateral.

Since for acute triangles of Bager's type II we have $\frac{r}{R} \geqslant \frac{\sqrt{3} - 1}{2}$ (see Chapter VIII, 2.2, Example 10°), it follows that

$$\frac{HI}{IO} < \sqrt{6 - 2\sqrt{3}} \quad \text{(acute, Bager II)}.\square$$

1.5. In this Section we shall consider special triangles of S. G. Guba (see Section XI.4.2), i.e. if $a \leqslant b \leqslant c$, the condition $2b \lesseqgtr a + c$ is equivalent to:

1) $OH^2 \gtreqless 9R^2 - 28Rr + 20r^2$; 2) $HI^2 \gtreqless 2(2R^2 - 7Rr + 6r^2)$;

3) $\Sigma aGA^2 \lesseqgtr \frac{8sr}{9}(5R - r)$; 4) $\Sigma aHA^2 \gtreqless 8s(R^2 - 3Rr + 3r^2)$;

5) $GI^2 \lesseqgtr \frac{2r}{9}(R - 2r)$; 6) $\Sigma(s - a)GA^2 \lesseqgtr \frac{4F}{9}(11R - 13r)$;

7) $\Sigma \frac{AI^2}{s - a} \lesseqgtr \frac{2r}{s}(11R - 4r)$;

8) $I\Gamma^2 \gtreqless \frac{2r^2(8R^2 - 23Rr + 14r^2)}{(4R + r)^2}$; 9) $\Sigma AH^2 \gtreqless 4(3R^2 - 7Rr + 5r^2)$.

1.6. Now, we shall consider the special triangles of I. Paasche, i.e. if $a \leqslant b \leqslant c$, the condition $3a \gtreqless b + c$ is equivalent to:

1) $OH^2 \gtreqless 9R^2 - 24Rr + 10r^2$; 2) $HI^2 \gtreqless 4R^2 - 12Rr + 7r^2$;

3) $GI \lesseqgtr r/3$; 4) $\Sigma aGA^2 \lesseqgtr \frac{2F}{9}(18R + r)$;

5) $\Sigma aHA^2 \gtreqless 2s(4R^2 - 10Rr + 7r^2)$; 6) $\Sigma(s - a)GA^2 \lesseqgtr \frac{4F}{9}(9R - 8r)$;

7) $\sum \frac{AI^2}{s-a} \lesseqgtr \frac{r}{s}(20R - 3r)$; 8) $I\Gamma^2 \gtreqless \frac{r^2(16R^2 - 40Rr + 13r^2)}{(4R + r)^2}$;

9) $\sum AH^2 \gtreqless 2(6R^2 - 12Rr + 5r^2)$.

1.7. In this Section we shall give some conversions of results from 1.3, but only for an acute triangle:

1) $HI^2 \geqslant 2r^2 - \frac{8F^2}{27R^2} \geqslant 2r^2 - \frac{2\sqrt{3}\,F}{9}$;

2) $OH^2 \geqslant R^2 - \frac{16F^2}{27R^2} \geqslant R^2 - \frac{4\sqrt{3}\,F}{9}$;

3) $\sum AH^2 \geqslant 4\left(R^2 - \frac{4F^2}{27R^2}\right) \geqslant 4(R^2 - \frac{\sqrt{3}\,F}{9})$;

4) $\sum aGA^2 \leqslant \frac{4s}{9}\left(2R^2 + 3Rr + 3r^2 + \frac{4F^2}{27R^2}\right) \leqslant$

$\leqslant \frac{4s}{9}(2R^2 + 3Rr + 3r^2 + \frac{\sqrt{3}\,F}{9})$;

5) $GI^2 \leqslant \frac{2}{9}\left(2R^2 - 6Rr + 3r^2 + \frac{4F^2}{27R^2}\right) \leqslant \frac{2}{9}(2R^2 - 6Rr + 3r^2 + \frac{\sqrt{3}\,F}{9})$;

6) $\sum aHA^2 \geqslant 4s\left(Rr + r^2 - \frac{4F^2}{27R^2}\right) \geqslant 4s(Rr + r^2 - \frac{\sqrt{3}\,F}{9})$;

7) $\sum (s-a)GA^2 \leqslant \frac{4s}{9}\left(4R^2 - 3Rr - 3r^2 + \frac{8F^2}{27R^2}\right) \leqslant$

$\leqslant \frac{4s}{9}(4R^2 - 3Rr - 3r^2 + \frac{2\sqrt{3}\,F}{9})$;

8) $\sum \frac{AI^2}{s-a} \leqslant \frac{2}{s}\left(2R^2 + 4Rr + r^2 + \frac{4F^2}{27R^2}\right) \leqslant \frac{2}{s}(2R^2 + 4Rr + r^2 + \frac{\sqrt{3}\,F}{9})$.

1.8. The following results are also valid for non-obtuse triangles:

1) $IH^2 \leqslant 2R(R - 2r)$; 2) $OH^2 \leqslant 5R^2 - 8Rr - 4r^2$;

3) $GI^2 \geqslant \frac{2}{9}(R^2 - 4Rr + 4r^2)$ 4) $\sum aGA^2 \geqslant \frac{4s}{9}(R^2 + 5Rr + 4r^2)$;

5) $\sum aHA^2 \leqslant 4sR(R - r)$; 6) $\sum (s - a)GA^2 \geqslant \frac{4s}{9}(2R^2 - Rr - r^2)$;

7) $\sum \frac{AI^2}{s - a} \geqslant \frac{2}{s}(R^2 + 6Rr + 2r^2)$; 8) $I\Gamma^2 \leqslant \frac{2r^2(5R^2 - 8Rr - 4r^2)}{(4R + r)^2}$;

9) $\sum AH^2 \leqslant 4(2R^2 - 2Rr - r^2)$.

Remark. Note that the following result is also given in [7]:

$$\sum a^2 \geqslant (\sum AH)^2,$$

with equality if and only if ABC is equilateral or right isosceles. The same equality conditions are valid for the other results from this Section.

Comment by V. Mascioni. Inequalities

1.2. 14) $\frac{1}{4} GH^2 \leqslant \frac{1}{9}(9R^2 - 8Rr + 7r^2 - s^2)$,

1.5. 10) $I\Gamma^2 \gtreqless \frac{2r^2(8R^2 - 5Rr + 5r^2 - s^2)}{(4R + r)^2}$,

1.5. 11) $\sum AH^2 \gtreqless 12R^2 - 10Rr + 11r^2 - s^2$,

1.6. 10) $I\Gamma^2 \gtreqless \frac{r^2(16R^2 - 8Rr + 5r^2 - 2s^2)}{(4R + r)^2}$

1.8. 10) $I\Gamma^2 \leqslant \frac{r^2(12R^2 - 8Rr - 5r^2 - s^2)}{(4R + r)^2}$

are better than those in the text, i.e. 1.2. 1), 1.5. 8) and 9), 1.6. 8) and 1.8. 8), respectively.

References

1. M. S. Klamkin, 'Geometric Inequalities via the Polar Moment of Inertia', Math. Mag. 48 (1975), 44-46.
2. Z. A. Skopec, 'Problem 2340', Mat. v škole 1981, No. 1, 66 and 1981, No. 5, 61.
3. V. Băndilă, 'O generalizare a unei relatii a lui Leibniz si aplicarea ei la calculul distantelor dintre unele puncte remarcabile ale unui triunghi', Gaz. Mat. (Bucharest) 90 (1985), 35-41.
4. W. J. Blundon, 'Problem 260', Crux Math. 4 (1978), 58-60.
5. H. S. M. Coxeter, 'The Lehmus Inequality', Aeq. Math. 28 (1985), 4-12.
6. W. J. Blundon and A. Bager, 'Problem E 2282', Amer. Math. Monthly 78 (1971), 196 and 542, and 79 (1972), 397-398.

7. A. W. Walker and M. G. Greening, 'Problem E 2388', Amer. Math. Monthly 79 (1972) 1135 and 80 (1973), 1142-1143.
8. A. L. Crelle, Ueber einige Eigenschaften des ebenen geradlinigen Dreiecks rücksichtlich dreier durch die Winkelspitzen gezogenen geraden Linien, Berlin 1816.
9. S. Bilčev and E. Velikova, Točki na Krel-Brokar i sv'rzanite s tjah geometrični neravenstva. To appear.
10. J. S. Mackay, Historical Notes on a Geometrical Theorem and Its Developments. Proc. Edinburgh Math. Soc. 5 (1887), 62-63.
11. A. Nemet-Buhin, 'Problem F 2435', Köz. Mat. Lapok 1984, No. 4, 155-157.

2. Some Transformations

2.1. Inversion

Let P be an interior point of a triangle $A_1A_2A_3$. In Figure 1 we invert the vertices A_1, A_2, A_3 to A_1', A_2', A_3' by means of a circle of radius ρ (we shall consider the case $\rho = \sqrt{K}$, $K = R_1R_2R_3$).

($R_i = PA_i$; a_i, side opposite to A_i; h_i, altitude from A_i; r_i, distance from P to a_i; w_i, angle bisectors from P to a_i; $R_i' = PA_i'$, etc.)

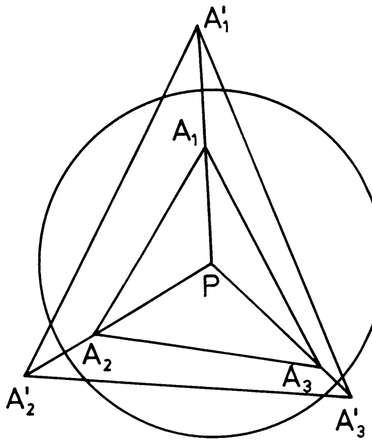


Fig. 1.

In [1], A. Oppenheim only considered the transformation of R_i, r_i into R_i', r_i'. In [2], M. S. Klamkin also gave the transformations for some other elements of a triangle. First, we have

$$R_i' = K/R_i.$$

Since

$$a_1^2 = R_2^2 + R_3^2 - 2R_2R_3 \cos A_2PA_3,$$

$$a_1'^2 = R_2'^2 + R_3'^2 - 2R_2'R_3' \cos A_2'PA_3', \quad \text{etc.}$$

we then obtain

$$a_i' = a_iR_i.$$

Remark. It is to be noted that a_1R_1, etc. are sides of a triangle whenever a_1, etc. are sides of a triangle. This corresponds to the Möbius-Neuberg inequality (see XIII). Note that the above proof is also valid if P is an exterior point.

Further, since

$$R_2R_3 \sin A_2PA_3 = r_1a_1, \quad R_2'R_3' \sin A_2'PA_3' = r_1'a_1', \quad \text{etc.}$$

we get

$$r_i' = r_iR_i,$$

and from

$$w_1^2 = R_2R_3(1 - a_1^2/(R_2 + R_3)^2), \quad w_i' = w_iR_i.$$

The following results are also valid:

$$R' = KRF/F', \quad h_i' = h_iF'/FR_i, \quad r' = 2F'/\Sigma a_1R_1.$$

Consequently, for any triangle inequality (or equality) of the form

(1) $$F(a_i, h_i, R_i, r_i, w_i, R, r) \geqslant 0$$

we also have the dual inequality

(2) $$F(a_iR_i, h_iF'/FR_i, K/R_i, r_iR_i, w_iR_i, KRF/F', 2F'/\Sigma a_1R_1) \geqslant 0.$$

EXAMPLES. 1° [1], GI 12.13 ⇒ GI 12.30.

2° [2] GI 12.7 ⇒ $\Sigma a_1R_1 > \Sigma R_2R_3 > \frac{1}{2}(\Sigma a_1R_1)$,

GI 12.8 ⇒ $\Sigma R_2R_3 > \Sigma a_1R_1 - \max(a_iR_i)$,

$$\text{GI } 12.19 \Rightarrow R_1R_2R_3(\Sigma a_1) \geqslant 2(\Sigma a_1r_1R_1^2),$$

$$\text{GI } 12.36 \Rightarrow \Sigma(1 + r_1R_1^2/K)^{-1} \geqslant 2,$$

$$\text{GI } 12.48 \Rightarrow \Sigma R_2R_3 \geqslant 2(\Sigma w_1R_1),$$

$$\text{GI } 12.50 \Rightarrow \Sigma R_1 \geqslant 4w_1w_2w_3(\Sigma 1/(w_1R_1)),$$

$$\text{GI } 12.52 \Rightarrow (R_1R_2R_3)^2 \geqslant \Pi(w_2R_2 + w_3R_3),$$

$$\text{GI } 12.53 \Rightarrow 3(\Sigma R_2^2R_3^2) \geqslant \Sigma a_1^2R_1^2.$$

2.2. Reciprocation

By reciprocation we obtain a triangle $A_1'A_2'A_3'$ so that P is still an internal point. The distances of P from the vertices of $A_1'A_2'A_3'$ are inversely proportional to its distances from the sides of $A_1A_2A_3$, i.e.

$$R_i' = k/r_i, \quad r_i' = k/R_i.$$

This result is given in [1]. Let $k = r_1r_2r_3$ and $K = R_1R_2R_3$. Then the following results are also valid ([2]):

$$a_i' = a_ir_iR_i/2R, \quad F' = kF/2R, \quad R' = K/4R, \quad h_i' = kh_i/r_iR_i,$$

$$r' = 2kF/(\Sigma a_1r_1R_1).$$

So, using the reciprocation we can transform any inequality of the form: $F(a_i, h_i, R_i, r_i, R, r, F) \geqslant 0$, to its reciprocation-dual.

EXAMPLE. 1° [1] GI 12.13 ⇒ GI 12.23.

2.3. Isogonal Conjugates

If P and P' are such that A_1P, A_1P' are equally inclined to the bisector of A_1; A_2P, A_2P' equi-inclined to the bisector of A_2; A_3P, A_3P' to those of the angle A_3, then P and P' are isogonal conjugates. Then [1]:

$$R_i' = 2Fr_iR_i/k(\Sigma a_1/r_1), \quad r_i' = 2F/r_i(\Sigma a_1/r_1), \quad (k = r_1r_2r_3).$$

Note that the following identities are also valid [2]:

$$\frac{1}{2kR}\,\Sigma a_1 r_1 R_1^2 = \Sigma a_1/r_1 = \frac{F}{kR}(R^2 - OP^2).$$

It will be observed that if P is an internal point of the triangle so also is P' and vice-versa. Note that now we have no inversion of the triangle. Hence, any inequality of the form

$$F(R_i,\ r_i,\ a_i,\ \ldots) \geqslant 0,$$

where only r_i and R_i depend on P, becomes

$$F(2Fr_iR_i/k(\Sigma a_1/r_1),\ 2F/r_i(\Sigma a_1/r_1),\ a_i,\ \ldots) \geqslant 0.$$

EXAMPLES. 1° [1] GI 12.23 ⇒ GI 12.32,

GI 12.24 ⇒ GI 12.33.

2.4. Pedal Triangle

For any inequality of the form $F(R_i, a_i) \geqslant 0$, there is a dual inequality

$$F(r_i,\ R_i \sin A_i) \geqslant 0,$$

obtained by considering the pedal triangle of P (see for example Klamkin's solutions of Carlitz's problem [3]). Here the distances from P to the vertices of the pedal triangle are r_1, r_2, r_3 and the sides of the pedal triangle are $R_1 \sin A_1$, etc.

References

1. A. Oppenheim, 'The Erdös Inequality and Other Inequalities for a Triangle', Amer. Math. Monthly 68 (1961), 226-230.
2. M. S. Klamkin, 'Triangle Inequalities via Transforms', Notices of Amer. Math. Soc., Jan. 1972, A-103, 104.
3. L. Carlitz, R. Garfield, P. Tracy, L. Bankoff, and M. S. Klamkin, 'Problem E 2347', Amer. Math. Monthly **79** (1972), 303; **80** (1973), 321, and **81** (1974), 904-905.

3. Some Further Remarks on the Polar Moment of Inertia Inequality

Let $A_1A_2A_3$ be an arbitrary triangle of sides a_1, a_2, a_3 and let P be an arbitrary point. If $R_i = PA_i$ (i = 1, 2, 3) and r_i is the distance from P to the side a_i (i = 1, 2, 3), then Klamkin's inequality (2) from 1.1, i.e. the polar moment of inertia inequality, can be written in the following form [1]

$$(1) \qquad \Sigma x_1 \Sigma x_1 R_1^2 \geqslant \Sigma x_2 x_3 a_1^2,$$

where x_1, x_2, x_3 are real numbers. There is equality if and only if $x_1/F_1 = x_2/F_2 = x_3/F_3$ (F_1 denotes the area of A_2PA_3, etc.). (A physical interpretation of (1) is that the polar moment of inertia of three masses x_1, x_2, x_3 located at A_1, A_2, A_3, respectively, is minimized by taking the axis through the centroid of the masses).

Suppose that P is an internal point. Note that the equality case for (1) can be written in the following form

$$a_1r_1/x_1 = a_2r_2/x_2 = a_3r_3/x_3. \tag{2}$$

The dual of (1) via inversion is [2]:

$$(\Sigma x_1)(\Sigma x_1R_2^2R_3^2) \geqslant \Sigma a_1^2R_1^2x_2x_3 \tag{3}$$

with equality if and only if

$$a_1r_1R_1^2/x_1 = a_2r_2R_2^2/x_2 = a_3r_3R_3^2/x_3.$$

If, in (3) we put $x_1 = a_1R_1$, etc., we get

$$\Sigma a_1R_2R_3 \geqslant a_1a_2a_3 = 4RF, \tag{4}$$

or

$$\Sigma \frac{R_2R_3}{a_2a_3} \geqslant 1, \tag{4'}$$

with equality if and only if P is either one of the vertices or the orthocenter H.

Remark. Inequality (4) was proved by T. Hayashi [3], and was conjectured much later as (4') by K. Stolarsky [4] and proved by M. S. Klamkin [5].

The reciprocation-dual of (1) is [2]:

$$4R^2(\Sigma x_1)(\Sigma x_1r_2^2r_3^2) \geqslant \Sigma(a_1r_1R_1)^2x_2x_3 \tag{5}$$

with equality if and only if

$$a_1r_1/x_1 = a_2r_2/x_2 = a_3r_3/x_3.$$

If we put $x_1 = a_1r_1R_1$, etc., we get the following inequality from [6]:

$$\Sigma a_1R_1r_2r_3 \geqslant FR_1R_2R_3/R, \tag{6}$$

with equality when P = O.

The isogonal conjugate dual of (1) is [6]:

$$(\Sigma x_1)(\Sigma x_1(r_1R_1)^2) \geqslant \frac{(R^2 - PO^2)^2}{4R^2}(\Sigma a_1^2 x_2 x_3) \tag{7}$$

with equality if and only if

$$r_1x_1/a_1 = r_2x_2/a_2 = r_3x_3/a_3.$$

If we put $x_1 = a_1$, etc., we get

$$\Sigma a_1(r_1R_1)^2 \geqslant F(R^2 - PO^2)^2/R, \tag{8}$$

with equality when P = I.

The pedal dual of (1) is [7]:

$$(\Sigma x_1)(\Sigma x_1 r_1^2) \geqslant \Sigma R_1^2 x_2 x_3 \sin^2 A_1. \tag{9}$$

Remark. For $x_1 = x_2 = x_3$ we get Carlitz' inequality

$$\Sigma R_1^2 \sin^2 A_1 \leqslant 3\Sigma r_1^2. \tag{10}$$

Note that equality holds in (10) if and only if P is the Lhuilier-Lemoine point of the triangle.

By applying (1) to the right hand side of (9), we have in the case $\Sigma x_1 > 0$,

$$\frac{\Sigma x_2 x_3 R_1^2 \sin^2 A_1}{\Sigma x_1} \geqslant \frac{4F^2}{\Sigma a_1^2/x_1}. \tag{11}$$

Inequalities (10) and (11) also provide a strengthening and a generalization of GI 12.54:

$$\Sigma x_1 r_1^2 \geqslant 4F^2/(\Sigma a_1^2/x_1) \quad (\Sigma x_1 > 0). \tag{12}$$

As a consequence of (1) we got in 1.1.B the following inequality

$$\Sigma a_1 R_1^2 \geqslant a_1 a_2 a_3. \tag{13}$$

There is equality in (13) if and only if P = I.

A generalization of (13) is given in [6] and [8], i.e. the following theorem is valid:

THEOREM A. Let P and P' be two arbitrary points and let $R_i = PA_i$, $R_i' =$

$P'A_i$ (i = 1, 2, 3). Then the following inequality is valid

(14) $$\Sigma a_1 R_1 R_1' \geqslant a_1 a_2 a_3$$

with equality if and only if P and P' are isogonal conjugates with respect to the given triangle.

Remarks. 1° G. Bennett [6] gave a geometric proof of (14) but only for interior points P and P'. In his proof, Klamkin [8] used an identity for complex numbers and the basic triangle inequality (see 4.2 of this Chapter).

2° It is a known result [9] that if P and P' are foci of an ellipse inscribed in the triangle $A_1 A_2 A_3$, then we have the equality condition of inequality (14). The proof given was geometric. M. Fujiwara [10] using the same complex identity as M. S. Klamkin [8] easily established the equality conditions of inequality(14) that P and P' must be isogonal conjugates. The ellipse result of [9] also follows easily from the noted identity by using the known general angle property of an ellipse that two tangents to an ellipse from a given point make equal angles with the focal radii to the given point ([8]).

Now, we shall show that (14) can be proved by using (1).

Indeed, for $x_1 = a_1 R_1'/R_1$, etc., (1) becomes

$$(\Sigma a_1 R_1'/R_1)(\Sigma a_1 R_1 R_1') \geqslant (\Sigma a_1^2 (a_2 R_2'/R_2)(a_3 R_3'/R_3))$$

i.e.

(15) $$(\Sigma a_1 R_1' R_2 R_3)(\Sigma a_1 R_1 R_1') \geqslant a_1 a_2 a_3 (\Sigma a_1 R_1 R_2' R_3')$$

with equality if and only if

(16) $$r_1 R_1/R_1' = r_2 R_2/R_2' = r_3 R_3/R_3'.$$

Similarly, we get

(17) $$(\Sigma a_1 R_1 R_2' R_3')(\Sigma a_1 R_1 R_1') \geqslant a_1 a_2 a_3 (\Sigma a_1 R_1' R_2 R_3)$$

with equality if and only if

(18) $$r_1' R_1'/R_1 = r_2' R_2'/R_2 = r_3' R_3'/R_3.$$

By summing the inequalities (15) and (17) we get (14), with equality if and only if (16) and (18) are valid, i.e. if and only if P and P' are isogonal conjugates (see 2.3 of this Chapter).

Now, we shall give some converse results. First, using the substitutions $x_1 \to yz$, etc. we get from (1)

$$\Sigma yz \Sigma yz R_1^2 \geqslant xyz \Sigma x a_1^2$$

and if we set $x \to x_1R_1^2/a_1^2$ etc., we get

$$(\Pi x_1a_1^2)(\Sigma x_1R_1^2) \leqslant (\Sigma x_2x_3a_1^2R_2^2R_3^2)(\Sigma x_2x_3a_1^2). \tag{19}$$

Remark. It is a generalization of an inequality of A. Oppenheim (see VI.1.(22')).

Now, we shall give some dual inequalities of (19).

Dual via inversion is:

$$(\Pi x_1a_1^2)(\Sigma x_1R_2^2R_3^2) \leqslant (\Sigma x_2x_3a_1^2R_1^4)(\Sigma x_2x_3a_1^2R_1^2); \tag{20}$$

dual via reciprocation is:

$$(\Pi x_1a_1^2R_1^2)(\Sigma x_1r_2^2r_3^2) \leqslant 4R^2(\Sigma x_2x_3a_1^2R_1^2r_1^4)(\Sigma x_2x_3a_1^2R_1^2r_1^2); \tag{21}$$

and dual via isogonal conjugates is:

$$\frac{(R^2 - OP^2)^2}{4R^2}(\Pi x_1a_1^2)(\Sigma x_1r_1^2R_1^2) \leqslant (\Sigma x_2x_3(a_1r_2r_3R_2R_3)^2)(\Sigma x_2x_3a_1^2). \tag{22}$$

Remark. Of course, we can also give the pedal dual of (19).

References

1. M. S. Klamkin, 'Nonnegative Quadratic Forms and Triangle Inequalities'. Notices of A.M.S., Oct. 1971, p. 966.
2. D. S. Mitrinović, J. E. Pečarić, and V. Volenec, 'On the Polar Moment of Inertia Inequality'. Rad JAZU (Zagreb). To appear.
3. T. Hayashi, 'Two Theorems on Complex Numbers', Tôhoku Math. J. 4 (1913-1914), 68-70.
4. K. Stolarsky, 'Cubic Triangle Inequalities', Amer. Math. Monthly 78 (1971), 879-881.
5. M. S. Klamkin, 'Triangle Inequalities via Transforms', Notices of A.M.S. Jan. 1972, A-103, 104.
6. G. Bennett, 'Multiple Triangle Inequalities', Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 577-598 (1977), 39-44.
7. M. S. Klamkin, 'Solution of Problem E 2347', Amer. Math. Monthly 81 (1974), 904-905.
8. M. S. Klamkin, 'Problem 77-10', SIAM Rev. 20 (1978), 400-401.
9. W. J. Miller, Mathematical Questions and Solutions from the Educ. Times, Vol. 7, London, 1876, Problem 210, p. 43.
10. M. Fujiwara, 'On the Deduction of Geometrical Theorems from Algebraic Identities', Tôhoku Math. J. 4 (1913-1914), 75-77.

4. Two Simple Methods for Generating Inequalities Involving Triangle and Point

4.1. Applications of Jensen's Inequality for Convex Functions and of the Inequality for Means

First, we shall give the following theorem:

THEOREM 1. Let x_1, x_2, x_3 be positive numbers such that $ar_1/x_1 \leqslant d$, etc. and let $f:[0, d] \to R$ be a convex function. Then

(1) $$(\Sigma x_1) f(2F/\Sigma x_1) \leqslant \Sigma x_1 f(ar_1/x_1).$$

If f is strictly convex, then equality holds in (1) if and only if

(2) $$ar_1/x_1 = br_2/x_2 = cr_3/x_3.$$

The inequality is reversed if f is a concave function.

Proof. Using Jensen's inequality for convex functions and the identity

$$\Sigma ar_1 = 2F,$$

we get

$$(\Sigma x_1) f(2F/\Sigma x_1) = (\Sigma x_1) f\left(\frac{\Sigma x_1 (ar_1/x_1)}{\Sigma x_1}\right) \leqslant \Sigma x_1 f(ar_1/x_1).$$

Corollary 1. For $x_1 = x_2 = x_3 = 1$, (1) becomes

(3) $$3f(2F/3) \leqslant \Sigma f(ar_1).$$

with equality if and only if $P = G$.

EXAMPLES. 1° For $f(t) = t^p$, we get [8]:

(4) $$\Sigma (ar_1)^p \geqslant 3^{1-p} 2^p F^p$$

for $p < 0$ and $p > 1$, and reverse inequality for $0 < p < 1$.

Remark. The cases $p = -1, -2, \ldots$ are given in [1]; $p = 3$ in [2]; $p = 1, 2, 3, \ldots$ in [3]; $P = -1$ in [4].

2° For $f(t) = \log t$, we get GI 12.29.

Corollary 2. For $x_1 = a$, $x_2 = b$, $x_3 = c$, (1) becomes

(5) $$2sf(r) \leqslant \Sigma af(r_1).$$

The equality is valid if and only if $P = I$.

3° For $f(t) = t^p$ we get [8]:

(6) $$2sr^p \leqslant \Sigma ar_1^p$$

for $p < 0$ and $p > 1$. For $0 < p < 1$, the reverse inequality is valid.

Remark. The cases $p = -1$ and $p = 2$ were given in [5]; the case $p = -1$ was also set as a problem of the XXII International Olympiad, Washington, 1981, and in [6] and [18].

4° For $f(t) = \log t$ we get

(7) $$\Pi r_1^a \leqslant r^{2s}.$$

Corollary 3. For $f(t) = t^m$, $x_1 = (a^m/x)^{1/(m-1)}$, etc., (1) becomes for $m > 1$ and $m < 0$

(8) $$\Sigma x r_1^m \geqslant (2F)^m (\Sigma (a^m/x)^{1/(m-1)})^{1-m} \quad (= U),$$

with equality if and only if

(9) $$x r_1^{m-1}/a = y r_2^{m-1}/b = z r_3^{m-1}/c,$$

For $0 < m < 1$, the reverse inequality is valid.

Remark. The case $m > 1$ was given by M. S. Klamkin in [5].

Corollary 3 gives an answer to the following problem (see [7]): Locate a point P in the interior of a triangle which minimizes

$$S = \Sigma x r_1^m,$$

where x, y, z are positive numbers and $m > 1$.

It is obvious that $S_{min} = U$ for the points P satisfying (9).

The maximum of S is easily obtained in the following manner [7]:

$$S/(2F)^m = \Sigma \frac{x}{a^m}\left(\frac{ar_1}{2F}\right)^m \leqslant \max\left(\frac{x}{a^m}, \frac{y}{b^m}, \frac{c}{c^m}\right)\left(\frac{\Sigma ar_1}{2F}\right)$$

since $1 \geqslant ar_1/2F \geqslant 0$, etc. Thus

(10) $$S_{max} = (2F)^m \max (x/a^m, y/b^m, z/c^m)$$

and which is taken on for P coinciding with one of the vertices of the triangle.

Remarks. 1° Corollary 3 gives $S_{min} = U$ for $m < 0$, and $S_{max} = U$ for $0 < m < 1$.

2° P. Fermat had given the suggestion to Torricelli to find the point for which the minimum of ΣR_1 is attained. Torricelli found three solutions and he gave the same problem to Viviani. Viviani published a solution of his own in 1658. This important point, known as the Fermat-Torricelli point, was studied by T. Simpson in 1750, Fuss in 1798, Tedenat and Lhuilier in 1810, Gruson in 1816, Bertrand in 1843, Lehmus in 1854, Grunert in 1867, and others [19].

S. Lhuilier, in 1809, investigated the point in a triangle (or tetrahedron) having the minimal sum of squares of distances to the sides (or faces). E. W. Grebe rediscovered this point in 1847, and E. Lemoine again in 1873. It became generally known as the "Lemoine point" (or occasionally the "Grebe point"), but its discovery by Lhuilier was forgotten.

3° (H. S. M. Coxeter). For a given scalene triangle $A_1A_2A_3$ and a variable point P, let F_t denote the position of P for which ΣR_1^t is extremal, and let L_t denote the position for which Σr_1^t is extremal.

P. Penning [20] shows that the locus of F_t, when t varies over all real values, is a curve (having two branches) which lies entirely inside the triangle and passes through the midpoints of the sides, two distinct points F_{+0} and F_{-0}, the Fermat-Torricelli point F_1, the centroid F_2, and the circumcentre F_∞. (The letter F is appropriate because it was Fagnano who first observed that F_2 is centroid.)

H. S. M. Coxeter noticed that the locus of L_t coincides with the locus of the point whose trilinear coordinates r_ν are proportional to a_ν^u ($\nu = 1, 2, 3$) when u varies over all real values ($u = 1/(t - 1)$). Thus it is a curve passing through the centroid L_0, the vertex $L_1 = A_1$ ($a_1 > a_2 > a_3$), the Lhuilier-Lemoine point L_2 (hence the use of the letter L), and the incentre L_∞. J. L. Synge has given the following proof:

Since $\Sigma a_1r_1 = 2F$, $\Sigma a_1dr_1 = 0$. If Σr_1^t is extremal, $\Sigma r_1^{t-1}dr_1 = 0$. Thus $r_\nu^{t-1} = \lambda a_\nu$ ($\nu = 1, 2, 3$) and $r_\nu = \mu a_\nu^u$ ($u = 1/(t - 1)$).

The point whose trilinear coordinates are proportional to a_ν^u is, of course, the point whose areal coordinates are proportional to a_ν^{u+1}. Its locus was thoroughly investigated by O. Bottema and P. Penning [21].

In problem 801 [9], S. Reich asked for a proof or disproof of the triangle inequality

$$\Sigma 1/r_1 \geqslant 3/r. \tag{11}$$

L. Carlitz noted that the inequality is invalid and also showed that

$$\Sigma 1/r_1 \geqslant (\Sigma\sqrt{a})^2/2F. \tag{12}$$

Generalizations of this inequality were also given by M. S. Klamkin [7] and [10]), i.e. the following inequality is given in [10]:

(13) $$\Sigma x/r_1^p \geqslant \frac{1}{(2F)^p}(\Sigma(xa^p)^{1/(p+1)})^{p+1} \qquad (p > 0)$$

with equality if and only if

$$ar_1^{p+1}/x = br_2^{p+1}/y = cr_3^{p+1}/z.$$

Of course, this inequality is a special case of (8) ($m = -p$).

Klamkin [10] also showed that instead of (11) the following analogous inequality is valid

(14) $$\Sigma 1/r_1 \geqslant 2/r$$

with equality for a degenerate triangle.

He also gave the following generalization of (14):

(15) $$\Sigma \frac{x_i^{p+1}}{r_1^p} \geqslant \frac{(\Sigma x_i a^{p/(p+1)})^{p+1}}{r^p(\Sigma a)^p} \geqslant \frac{(x_1 + x_2)^{p+1}}{(2r)^p}$$

where it is assumed that $x_1 \leqslant x_2 \leqslant x_3$.

Proof. To prove (15), it suffices to show that

$$(x_1 a^m + x_2(b^m + c^m))^{p+1} \geqslant (x_1 + x_2)^{p+1}\left(\frac{a+b+c}{2}\right)^p$$

where $m = p/(p + 1)$, or

(16) $$\left(b^m + c^m - \frac{x_1}{x_1 + x_2}(b^m + c^m - a^m)\right)^{p+1} \geqslant \left(\frac{a+b+c}{2}\right)^p.$$

Since the minimum of the left hand side of (16) is taken on for $x_1 = x_2$, (16) becomes

(17) $$\frac{\Sigma a^m}{2} \geqslant \left(\frac{\Sigma a}{2}\right)^m.$$

Klamkin proved (17), but we shall note that it is a simple consequence of Petrović's inequality for concave function (see VIII, 1.1, Theorem 1):

$$\Sigma f(a) \geqslant 2f(s).$$

Remark. Similar extensions for simplexes are also valid. For extensions of (13) and (15), see [10] and [11].

Corollary 4. (i) $(\Sigma\sqrt{r_1})^2 \leqslant (s^2 + r^2 + 4Rr)/2R$

with equality if and only if $P = I$.

(ii) $\Sigma\sqrt{r_1} \leqslant (R + r)\sqrt{2/R} \leqslant 3\sqrt{R/2}$

with equality if and only if ABC is equilateral and P is the center of ABC.

Proof. For x = y = z and m = 1/2, Corollary 3 becomes

$$(18) \qquad \Sigma\sqrt{r_1} \leqslant (2F\Sigma 1/a)^{1/2} = \left(\frac{s^2 + r^2 + 4Rr}{2R}\right)^{1/2}.$$

Further, using IX.5.7 we get (ii).

Remark. (ii) is an interpolation of an inequality of L. Carlitz [17].

Corollary 5. (i) $\Sigma bcr_2r_3 \leqslant 4F^2/3$,

(ii) $\Sigma bcr_2r_3 + \Sigma a^2r_1^2 \geqslant 8F^2/3$.

In each case there is equality if and only if P is the centroid of ABC.

Proof. Using the identity $\Sigma ar_1 = 2F$ we get

$$\Sigma bcr_2r_3 = \frac{1}{2}(4F^2 - \Sigma a^2r_1^2).$$

Now, using the well-known inequality (see Example 1°): $\Sigma a^2r_1^2 \geqslant 4F^2/3$, we get (i). Similarly, we can prove (ii).

Remark. (i) is given in [12]. (ii) is due to L. Carlitz.

Corollary 6. Let the conditions of Theorem 1 be fulfilled. Then

$$(19) \qquad (\Sigma x_1)f(1/(\Sigma x_1)) \leqslant \Sigma x_1 f(r_1/h_a x_1)$$

with equality if and only if (2) is valid.

EXAMPLES 5° For $f(t) = t^p$ $(p > 1,\ p < 0)$, we get

$$(20) \qquad (\Sigma x_1)^{1-p} \leqslant \Sigma x_1^{1-p}(r_1/h_a)^p.$$

The reverse inequality is valid for $0 < p < 1$.

Remark. For $x_1 = x_2 = x_3 = 1$, $p = -1$, we have GI 12.12.

6° For $f(t) = \log t$, we get

$$(20') \qquad (\Pi h_a^{x_1})/(\Pi r_1^{x_1}) \geqslant (\Sigma x_1)^{(\Sigma x_1)}/(\Pi x_1^{x_1}).$$

For $x_1 = x_2 = x_3 = 1$, we get GI 12.11.

Corollary 7. Let $f:[0, d] \to (-\infty, +\infty)$ be a convex function and let r_1, r_2, $r_3 \leqslant d$. Then

(21) $$f(r)/r \leqslant \Sigma f(r_1)/h_a$$

with equality if and only if $P = I$.

Proof. For $x_1 = 1/h_a$, etc., (19) becomes

$$(\Sigma 1/h_a) f(1/(\Sigma 1/h_a)) \leqslant \Sigma f(r_1)/h_a.$$

Now, using the identity $\Sigma 1/h_a = 1/r$, we get (21).

The following theorem is also valid:

THEOREM 2. Let $f:[0, 1] \to (-\infty, +\infty)$ be a convex function. Then

$$\Sigma f\left(\frac{r_1'}{R_1 + r_1'}\right) \geqslant 3f(1/3) \quad \text{and} \quad \Sigma f\left(\frac{R_1}{R_1 + r_1'}\right) \geqslant 3f(2/3).$$

Proof. This is a simple consequence of Jensen's inequality, since the following identities are known:

$$\Sigma \frac{r_1'}{R_1 + r_1'} = 1 \quad \text{and} \quad \Sigma \frac{R_1}{R_1 + r_1'} = 2.$$

Corollary 8. For $f(t) = t^p$ ($p > 1$ and $p < 0$), we get

(22) $$\Sigma\left\{\frac{r_1'}{R_1 + r_1'}\right\}^p \geqslant 3^{1-p} \quad \text{and} \quad \Sigma\left\{\frac{R_1}{R_1 + r_1'}\right\}^p \geqslant 2^p 3^{1-p}.$$

For $0 < p < 1$ we have the reverse inequalities.

Corollary 9. For $f(t) = \log t$, we get

$$27\Pi r_1' \geqslant \Pi(R_1 + r_1') \quad \text{and} \quad 27\Pi R_1 \geqslant 8\Pi(R_1 + r_1').$$

Remark. For $p = -1$, (22) becomes

$$\Sigma \frac{R_1 + r_1'}{r_1'} \geqslant 9 \text{ ([13])} \quad \text{and} \quad \Sigma \frac{R_1 + r_1'}{R_1} \geqslant \frac{9}{2} \quad \text{(GI 12.40)}$$

whence using the identities ([13]): $\Sigma \frac{R_1}{r_1'} = \Sigma \frac{R_1 + r_1'}{r_1'} - 3$ and $\Sigma \frac{r_1'}{R_1} = \Sigma \frac{R_1 + r_1'}{R_1} - 3$, we get

(23) $\quad \sum \frac{R_1}{r_1'} \geqslant 6 \quad$ (GI 12.38)

and

(24) $\quad \sum \frac{r_1'}{R_1} \geqslant \frac{3}{2} \quad$ ([13], [14]).

Note that (23) is also a simple consequence of GI 12.39, i.e.

(25) $\quad \Pi(R_1/r_1') \geqslant 8$

i.e.

(25') $\quad M_o(R_1/r_1') \geqslant 2$

whence we get, for $k \geqslant 0$,

(26) $\quad M_k(R_1/r_1') \geqslant 2, \quad$ ([15]).

Note that (25) can be written in the following forms:

$$M_0(R_2R_3/r_2'r_3') \geqslant 4 \quad \text{and} \quad M_0(R_1^2/r_2'r_3') \geqslant 4,$$

so, the following inequalities are also valid:

(27) $\quad M_k(R_2R_3/r_2'r_3') \geqslant 4 \quad$ ([15]), $k = 1$: GI 12.47),

and

(28) $\quad M_k(R_1^2/r_2'r_3') \geqslant 4.$

Similarly, we can use GI 12.25, 12.26, 12.27, 12.28, 12.51, 12.52, and get the following results for $k \geqslant 0$:

(29) $\quad M_k(R_1/r_1) \geqslant S^{-1/3} \quad (S = \Pi \sin \frac{A}{2},\ 0 < 8S \leqslant 1),$

(30) $\quad M_k(R_2R_3/r_2r_3) \geqslant S^{-2/3},$

(31) $\quad M_k(R_1^2/r_2r_3) \geqslant S^{-2/3},$

(32) $\quad M_k\left(\frac{R_1}{r_2 + r_3}\right) \geqslant (8S)^{-1/3},$

(33) $\quad M_k(R_1/w_1) \geqslant 2,$

(34) $$M_k(R_2R_3/w_2w_3) \geq 4,$$

(35) $$M_k(R_1^2/w_2w_3) \geq 4,$$

(36) $$M_k\left(\frac{R_1}{w_2 + w_3}\right) \geq 1.$$

Remark. Inequalities (31) and (32) were given by W. Janous as generalizations of results from [16], in a private communication.

Finally, note that using majorization of vectors, we can give generalizations of Corollary 1 and Theorem 2, i.e. we have

(37) $$(ar_1, br_2, cr_3) \succ (2F/3, 2F/3, 2F/3),$$

(38) $$\left(\frac{r_1'}{R_1 + r_1'}, \frac{r_2'}{R_2 + r_2'}, \frac{r_3'}{R_3 + r_3'}\right) \succ \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

and

(39) $$\left(\frac{R_1}{R_1 + r_1'}, \frac{R_2}{R_2 + r_2'}, \frac{R_3}{R_3 + r_3'}\right) \succ \left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right).$$

So, since the elementary symmetric function $x \to T_2(x)$ is Schur-concave, using (37) we get Corollary 5 (i).

References

1. I. Tomescu, 'Problem 13248', Gaz. Mat. (Bucharest) B 24 (1973), 372.
2. I. Tomescu, 'Problem 13027', Gaz. Mat. (Bucharest) B 24 (1973), 307.
3. D. Mihet, 'Problem 13668', Gaz. Mat. (Bucharest) B 24 (1973), 740.
4. I. Liviu, 'Problem 17055', Gaz. Mat. (Bucharest) 83 (1978), 82.
5. L. Carlitz and M. S. Klamkin, 'Problem 910', Math. Mag. 48 (1975), 242-243.
6. H. H. Hamzin, 'Problem 749', Mat. v škole 1970, No. 2, 88 and 1970, No. 6, 75.
7. N. Schaumberger and M. S. Klamkin, 'Problem 140', Two-year Coll. Math. J. 11 (1980), 279-280.
8. I. Tomescu, 'Generalizari ale problemei 13027', Gaz. Mat. (Bucharest) B 24 (1973), 583-586.
9. S. Reich and L. Carlitz, 'Problem 801', Math. Mag. 44 (1971), 166 and 45 (1972), 107-109.
10. M. S. Klamkin, 'Extensions of a Triangle Inequality of Carlitz', Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 602-633 (1978), 147-149.
11. M. S. Klamkin, 'Extensions of a Triangle Inequality of Carlitz II', Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 678-715

(1980), 159-160.
12. I. Tomescu, 'Problem 9290', Gaz. Mat. (Bucharest) B 20 (1969), 355-356.
13. V. Stoican, 'Asupra unor relatii intr-un triunghi', Gaz. Mat. (Bucharest) B 20 (1969), 144-147.
14. V. Stoican, 'Problem 9720', Gaz. Mat. (Bucharest) B 21 (1970),, 148-150.
15. M. Stan, 'Problem 8793', Gaz. Mat. (Bucharest) B 19 (1968), 170-171..
16. G. Tsintsifas, W. Janous, and M. S. Klamkin, 'Problem 1045', Crux Math. 11 (1985), 147 and 189,and 12 (1986), 222-224.
17. L. Carlitz and M. S. Klamkin, 'Problem 959', Math. Mag. 50 (1977), 212-213.
18. R. Bán, 'Problem F 2425', Köz. Mat. Lapok 1983, No. 10, 202-203..
19. E. Vigarié, 'Esquisse historique sur la marche du dévelopment de la géométrie du triangle'. Association française pour l'avancement des sciences. Congrès de Paris 1889, p. 1-25.
20. P. Penning, 'Expoints', Nieuw Arch. Wisk. (3) 4 (1986), 19-31.
21. O. Bottema and P. Penning, 'De verzameling van de punten welker oppervlaktecoordinaten ten opzichte van een driehoek zich verhouden als machten van de zijdelengten', Nieuw Tijdsch. Wisk. 72 (1985), 157-162.

4.2. Triangle Inequalities from the Triangle Inequality

Although the title here may appear somewhat paradoxical, all inequalities of the triangle depend ultimately on the basic triangle inequality: $|u| + |v| \geqslant |u + v|$, where u and v may be complex numbers or vectors. More generally, if n + 1 vectors $a_0, a_1, \ldots, a_n$ ($a_i \neq 0$ for all i) satisfy the equality [3]:

$$a_0 = a_1 + a_2 + \ldots + a_n,$$

then

$$|a_0| \leqslant |a_1| + |a_2| + \ldots + |a_n|. \tag{1}$$

If, in addition, (1) holds with equality then all vectors are positive multipliers of a fixed vector. In the complex plane, now, division of a_i and a_j always results in real positive numbers; as a special case

$$\frac{a_1}{a_0} + \frac{a_2}{a_0} + \ldots + \frac{a_n}{a_0} = 1 \qquad \left(0 \leqslant \frac{a_i}{a_0} \leqslant 1,\ i = 1, \ldots, n\right).$$

Here we shall give some results from [1] and [2]. The summations and products to be indicated are cyclic ones. First we shall give identities which were used in the proofs.

$$\Sigma(v - w)(v + w - u) = 0. \tag{2}$$

(3) $\quad \Sigma vw(v - w) = \Sigma u^2(v - w) = \Sigma u(w^2 - v^2) = \Sigma(w - v)(w^2 + v^2) =$

$$= \Sigma(v - w)(v + w)^2 = \Sigma(v - w)(v + w - 2u)^2/9 = \Pi(w - v).$$

(4) $$\Sigma vw(v + w) + 2\Pi u = \Sigma u(v + w)^2 - 4\Pi u = \Sigma u^2(v + w) + 2\Pi u =$$

$$= \Sigma u(v^2 + w^2) + 2\Pi u = \Sigma u(v - w)^2 + 8\Pi u =$$

$$= \{\Sigma u\}^3/3 - \Sigma u^3/3 = \Pi(v + w).$$

(5) $$\Sigma(v - w)^2(v + w - 2u) = \Pi(2u - v - w).$$

(6) $$\Sigma(v^3 - w^3) = 0.$$

(7) $$\Sigma(v - w)^3(v + w - 2u) = 0.$$

(8) $$\Sigma(v - w)(v + w - 2u)^3 = 0.$$

(9) $$\Sigma u(v - w)^3 = \Sigma u^3(w - v) = \Sigma(w - v)(v + w - u)^3/4 =$$

$$= \Sigma u(v - w)(v + w)^2 = \Sigma vw(w^2 - v^2) = \Sigma u \cdot \Pi(v - w).$$

(10) $$\Sigma(v^4 - w^4) = 0.$$

(11) $$\Sigma u^2(v^2 - w^2) = 0.$$

(12) $$\Sigma u^3(v^2 - w^2) = \Sigma v^2w^2(v - w) = \Sigma vw \cdot \Pi(v - w).$$

(13) $$\Sigma u^3(v - w)^3 = \Sigma u^3(w - v)(w + v)^2 = \Sigma u^3(v - w)(v^2 + w^2) =$$

$$= \Sigma u(v - w)(v + w - u)^4/16 = \Pi u(v - w).$$

(14) $$\Sigma v^2w^2(v^2 - w^2) = \Sigma u^4(v^2 - w^2) = \Sigma u^2(w^4 - v^4) =$$

$$= \Sigma(v^2 - w^2)(v^2 + w^2)^2 = \Pi(w^2 - v^2).$$

(15) $$\Sigma u(v - w)(u - z) = -\Pi(v - w)\ (z \in C).$$

(16) $$\Sigma \frac{(u - b_1)(u - b_2)}{(u - v)(u - w)} \cdot \frac{1}{u} = \frac{b_1b_2}{uvw}.$$

(17) $$\Sigma(u' - u)(v - w) = \Sigma u'(v - w).$$

(18) $$\Sigma\{u(v - w)^5 + u^5(v - w)\} = 10uvw\Pi(v - w).$$

Our numbering of the inequalities here (as in [1]) e.g., (2.1), (2.2), ... will indicate that they were derived from identity number (2) by applying $|u| + |v| \geq |u + v|$ in various ways. Further, u, v, w are complex numbers from a common origin P to the vertices A_1, A_2, A_3, respectively, and $R_i = PA_i$, b_j, j = 1, 2, ... and z are complex numbers from P to arbitrary points B_j and Z, respectively, $R_i' = A_iZ$. O, G, H and I denote the circumcenter, centroid, orthocenter and incenter, respectively, of $a_1a_2a_3$. Finally, M_1, etc., R, r, Δ denote the median from A_1, the circumradius, the inradius, the area, respectively, of $A_1A_2A_3$ and m_1, Δ_1, etc., denote the median from P and area of PA_2A_3.

We now essentially list the inequalities that follow immediately from some of the above identities:

(3.1) $\Sigma a_1R_2R_3 \geq a_1a_2a_3.$

This is Hayashi's inequality (see 3.(4)).

(3.2) $\Sigma a_1R_1^2 \geq a_1a_2a_3$ (see 3.(13)).

(3.3) $2\Sigma a_1m_1R_1 \geq a_1a_2a_3.$

(3.4) $4\Sigma a_1m_1^2 > a_1a_2a_3.$

(3.5) $4\Sigma a_1M_1^2 \geq 9a_1a_2a_3.$

Or equivalently,

(3.6) $2\Pi(a_2 + a_3) \geq 13a_1a_2a_3 + \Sigma a_1^3.$

(4.1) $\Sigma m_1R_2R_3 \geq |4m_1m_2m_3 - R_1R_2R_3|.$

(4.2) $\Sigma R_1m_1^2 \geq |2m_1m_2m_3 - R_1R_2R_3|.$

(4.3) $\Sigma m_1R_1^2 \geq |4m_1m_2m_3 - R_1R_2R_3|.$

(4.4) $\Sigma R_1a_1^2 \geq 8|m_1m_2m_3 - R_1R_2R_3|.$

(4.5) $2\Sigma m_1R_2R_3 + \Sigma R_1a_1^2 \geq 6R_1R_2R_3.$

(4.6) $2\Sigma R_1m_1^2 + \Sigma m_1R_1^2 \geq 3R_1R_2R_3.$

(4.7) $4\Sigma R_1m_1^2 + \Sigma R_1a_1^2 \geq 12R_1R_2R_3.$

(5.1) $\Sigma M_1 a_1^2 \geqslant 4\Pi M_1$.

(9.1) $\Sigma R_1 a_1^3 \geqslant 3a_1a_2a_3\overline{PG}$.

(9.2) $\Sigma R_1^3 a_1 \geqslant 3a_1a_2a_3\overline{PG}$.

(9.3) $4\Sigma a_1 R_1 m_1^2 \geqslant 3a_1a_2a_3\overline{PG}$.

(12.1) $\Sigma a_1 R_2^2 R_3^2 \geqslant R_1R_2R_3\{\Sigma(3R_2^2R_3^2 - a_1^2R_1^2)\}^{1/2}$.

(13.1) $4\Sigma a_1 m_1^2 R_1^3 \geqslant \Pi a_1 R_1$.

(14.1) $\Sigma a_1 m_1 R_2^2 R_3^2 \geqslant 4\Pi a_1 m_1$.

(14.2) $\Sigma a_1 m_1 R_1^4 \geqslant 4\Pi a_1 m_1$.

(15.1) $\Sigma a_1 R_1 R_1' \geqslant a_1a_2a_3$.

This is the two point triangle inequality (see 3.(14)).

(16.1) $a_1R_{11}R_{12}R_2R_3 + a_2R_{21}R_{22}R_3R_1 + a_3R_{31}R_{33}R_1R_2 \geqslant |b_1||b_2|a_1a_2a_3$.

This is a variation of Hayashi's inequality (2.1) and reduces to it if b_1 and b_2 coincide with the circumcenter and P is on the circumcircle.

(18.1) $\Sigma a_1 R_1 (a_1^4 + R_1^4) \geqslant 10\Pi a_1 R_1$.

This is a result from [2]. It is easily seen that equality holds if the triangle is equilateral. Klamkin conjectured that this is the only case for equality (assuming the triangle is non-degenerate).

Remark. Note that the well-known Möbius-Neuberg and Möbius-Pompeiu theorems are obtained by the above method (see Chapter XIII). Identities (2), (6-8), (10-11) and (17) have as consequences similar results, and they are also given in Chapter XIII. For two other results see for example Comment by K. Post in Chapter XV.

References

1. M. S. Klamkin, 'Triangle Inequalities from the Triangle Inequality', Elem. Math. 34 (1979), 49-55.
2. M. S. Klamkin and L. Kuipers, 'Aufgabe 896', Elem. Math. 38 (1983), 73 and 39 (1984), 79.
3. K. Post. Private communication.

5. Erdős-Mordell's and Related Inequalities

5.1. An important geometric inequality is the well-known Erdős-Mordell inequality (GI 12.13):

THEOREM 1. If P is a point in a triangle, then

$$\Sigma R_1 \geqslant 2\Sigma r_1. \tag{1}$$

Equality holds if and only if the triangle is equilateral and P is its centre.

The following generalization of Theorem 1 is valid (GI 12.24):

THEOREM 2. If k is a real number such that $0 < k \leqslant 1$, then

$$\Sigma R_1^k \geqslant 2^k \Sigma r_1^k. \tag{2}$$

The reverse inequality is valid for $-1 \leqslant k < 0$. Equality holds as for (1). If $k > 1$, then

$$\Sigma R_1^k > 2\Sigma r_1^k, \tag{3}$$

and the reverse inequality is valid for $k < -1$.

Now, we shall prove:

THEOREM 3. For $0 < k \leqslant 1$, the following inequality is valid:

$$\Sigma R_1^k \geqslant 2^{k-1}\Sigma((b/c)^k + (c/b)^k)r_1^k \tag{4}$$

and the reverse inequality is valid for $k < 0$. For $k = 1$, equality holds if and only if P is the circumcentre, and for $k < 1$ as for (1).

For $k > 1$,

$$\Sigma R_1^k > \Sigma((b/c)^k + (c/b)^k)r_1^k. \tag{5}$$

Proof. It is known that

$$R_1 \geqslant (b/a)r_3 + (c/a)r_2, \quad R_2 \geqslant (c/b)r_1 + (a/b)r_3, \tag{6}$$

$$R_3 \geqslant (b/c)r_1 + (a/c)r_2$$

with equality in all inequalities if and only if P is the circumcentre.

To prove (4) we shall apply to (6) the elementary inequality

$$\left(\frac{u + v}{2}\right)^k \geqslant \frac{u^k + v^k}{2} \quad (u, v > 0, 0 < k \leqslant 1) \tag{7}$$

equality occurring if and only if $u = v$ (for $k \neq 1$ of course); the reverse inequality is valid for $k < 0$.

Thus, for $0 < k \leqslant 1$, (4) yields

$$(8) \qquad R_1^k \geqslant 2^{k-1}((b/a)^k r_3^k + (c/a)^k r_2^k), \quad \text{etc.}$$

and the reverse inequalities are valid for $k < 0$. By adding these inequalities we obtain (4).

Similarly, using the elementary inequality

$$(9) \qquad (u + v)^k > u^k + v^k \quad (k > 1),$$

we obtain (5).

Corollary 1.

$$(10) \qquad \Sigma R_1^k \geqslant 2^{k-1}\Sigma((b/c)^k + (c/b)^k) r_1^k \geqslant 2^k \Sigma r_1^k \quad (0 < k \leqslant 1),$$

$$(11) \qquad \Sigma R_1^k > \Sigma((b/c)^k + (c/b)^k) r_1^k \geqslant 2\Sigma r_1^k \quad (k > 1).$$

Proof. Since $(u/v) + (v/u) \geqslant 2$ $(u, v > 0)$, with equality for $u = v$, for Theorem 3 we get Corollary 1.

Remarks. 1° This is a generalization of GI 12.16.

2° From (10) and (11) we get (2) and (3). The cases for $k < 0$ follow by reciprocation.

THEOREM 4. For $0 < k \leqslant 1$, the following inequality is valid

$$(12) \qquad \Sigma(r_1 R_1)^k \geqslant 2^{k-1}\Sigma((b/c)^k + (c/b)^k)(r_2 r_3)^k$$

and the reverse inequality is valid for $k < 0$. For $k > 1$, we have

$$(13) \qquad \Sigma(r_1 R_1)^k > \Sigma((b/c)^k + (c/b)^k)(r_2 r_3)^k.$$

Proof. From (6) we obtain

$$(14) \qquad R_1 r_1 \geqslant (b/a) r_3 r_1 + (c/a) r_2 r_1, \text{ etc.}$$

so, similarly to the proof of Theorem 3 we obtain Theorem 4.

Corollary 2.

$$(15) \qquad \Sigma(r_1 R_1)^k \geqslant 2^{k-1}\Sigma((b/c)^k + (c/b)^k)(r_2 r_3)^k \geqslant$$

$$\geqslant 2^k \Sigma(r_2 r_3)^k \quad (0 < k \leqslant 1),$$

(16) $$\Sigma(r_1R_1)^k > \Sigma((b/c)^k + (c/b)^k)(r_2r_3)^k \geqslant 2\Sigma(r_2r_3)^k \quad (k > 1).$$

3° This is a generalization of GI 12.31.

THEOREM 5. For $0 < k \leqslant 1$,

(17) $$\Sigma(r_1R_1)^k \geqslant 2^k\Sigma(r_2r_3)^k.$$

The reverse inequality is valid for $-1 \leqslant k < 0$. For $k > 1$,

(18) $$\Sigma(r_1R_1)^k > 2\Sigma(r_2r_3)^k,$$

and the reverse inequality is valid for $k < -1$.

Proof. (17) and (18) are given in Corollary 2. The reverse results follow from these inequalities by isogonal conjugate transformations.

4° The case $0 < k \leqslant 1$, i.e. the inequality (17) is given in GI 12.33. For $k = -1$ we have GI 12.32.

THEOREM 6. For $0 < k \leqslant 1$,

(19) $$\Sigma(R_2R_3)^k \geqslant 2^k\Sigma(r_1R_1)^k.$$

The reverse inequality is valid for $-1 \leqslant k < 0$. For $k > 1$,

(20) $$\Sigma(R_2R_3)^k > 2\Sigma(r_1R_1)^k,$$

and the reverse inequality is valid for $k < -1$.

Proof. This follows from Theorem 2 by inversion.

5° For $k = 1$ we get GI 12.30 and for $k = -1$ GI 12.34.

THEOREM 7. For $0 < k \leqslant 1$,

(21) $$\Sigma(R_2R_3)^k \geqslant 4^k\Sigma(r_2r_3)^k.$$

The reverse inequality is valid for $-1 \leqslant k < 0$. For $k > 1$,

(22) $$\Sigma(R_2R_3)^k > 4\Sigma(r_2r_3)^k$$

and the reverse inequality is valid for $k < -1$.

Proof. This is a simple consequence of Theorems 5 and 6.

6° For $k = 1$ we get GI 12.21 and, for $k = -1$, GI 12.35.

5.2. Recently, as answer to a problem of G. Tsintsifas, M. S. Klamkin [2] proved the following generalization of Theorem 1:

THEOREM 8. Let P_j $(j = 1, \ldots, n)$ denote any set of n points lying in the interior or on the boundary of a given triangle ABC, and let R_{1j} and r_{1j} denote the distances from P_j to the vertex A and to the side a (similarly we define R_{2j}, r_{2j}, R_{3j}, r_{3j}). If λ_j (j = 1, ..., n) are positive numbers with $\sum_{j=1}^{n} \lambda_j = 1$, and $R_i = \prod_{j=1}^{n} R_{ij}^{\lambda_j}$ and $r_i = \prod_{j=1}^{n} r_{ij}^{\lambda_j}$ (i = 1, 2, 3), then (1) is valid with equality if and only if the triangle is equilateral and all the points P_j coincide with its centre.

Proof. (M. S. Klamkin) Now, (6) becomes

$$R_{1j} \geqslant r_{2j}(c/a) + r_{3j}(b/a), \quad R_{2j} \geqslant r_{3j}(a/b) + r_{1j}(c/b),$$

$$R_{3j} \geqslant r_{1j}(b/c) + r_{2j}(a/c).$$

We now employ the following case of Hölder's inequality

$$\prod_{j=1}^{n} (u_j + v_j)^{\lambda_j} \geqslant \prod_{j=1}^{n} u_j^{\lambda_j} + \prod_{j=1}^{n} v_j^{\lambda_j}$$

where all u_j, $v_j \geqslant 0$. With all products running from j = 1 to j = n, we then have

$$\Pi R_{1j}^{\lambda_j} \geqslant \Pi(r_{2j}(c/a) + r_{3j}(b/a))^{\lambda_j} \geqslant (c/a)\Pi r_{2j}^{\lambda_j} + (b/a)\Pi r_{3j}^{\lambda_j},$$

i.e.

$$R_1 \geqslant (c/a)r_2 + (b/a)r_3, \quad \text{etc.}$$

Since $(b/c) + (c/b) \geqslant 2$, it follows by addition that (1) is valid.

Similarly to the proofs from 5.1 we can get the following result:

THEOREM 9. With the same notation as in Theorem 8, the inequalities from Corollaries 1 and 2 are also valid.

7° M. S. Klamkin [2] noted that the following generalization of Theorem 8 is also valid:

Consider n triangles $A_{1j}A_{2j}A_{3j}$ of sides a_{1j}, a_{2j}, a_{3j}, and n points $P_1, \ldots, P_n$, where P_j is an interior or boundary point of triangle $A_{1j}A_{2j}A_{3j}$ for each j. Then if R_{ij} and r_{ij} denote the distances from P_j to the vertex A_{ij} and the side a_{ij}, respectively, and if λ_j, R_i, r_i are defined as in Theorem 8, inequality (1) is still valid.

Of course, we can show that Theorem 9 is also valid, but in this case we should put

$$a = \prod_{j=1}^{n} a_{1j}^{\lambda_j}, \quad \text{etc.}$$

5.3. In 1973, H. Demir [3] gave the following result:

THEOREM 10. Erdős-Mordell's inequality (1) holds for every point P in the plane of the triangle ABC when we make the interpretation $R_1 \geqslant 0$, etc. always and r_1 is positive or negative depending on whether P and A are on the same side of a or on opposite sides (the same for r_2 and r_3).

The proof of this result was given in 1984 by C. W. Dodge [4].

With the same notation, the following result is given in [8]

$$\Sigma R_1 \geqslant \Sigma \left| \frac{pr_1 r_2 \cdot \operatorname{sgn} r_2 + pr_1 r_3 \cdot \operatorname{sgn} r_3}{\sin A} \right|$$

with equality for the centre of the equilateral triangle.

5.4. Let w_1 be the angle-bisector of the angle BPC $(= 2\delta_1)$, etc., where P is an internal point of the triangle ABC. Of course, $w_1 \geqslant r_1$, etc., so the following inequality of D. F. Barrow (GI 12.48) is better than (1):

THEOREM 11. $\Sigma R_1 \geqslant 2\Sigma w_1$

with equality as for (1).

A. Oppenheim [5] gave a generalization of Barrow's inequality. Here, we shall give an extension of his result:

THEOREM 12. Let x, y, z be real numbers. Then

(23) $$\Sigma w_1 \left(\frac{1}{R_2} + \frac{1}{R_3}\right) yz \leqslant \Sigma x^2$$

with equality if and only if

(24) $$x/\sin \delta_1 = y/\sin \delta_2 = z/\sin \delta_3.$$

<u>Proof</u>. As in [5], we have

$$w_1(R_2 + R_3) = 2R_2R_3 \cos \delta_1$$

i.e.

(25) $$w_1\left(\frac{1}{R_2} + \frac{1}{R_3}\right) = 2 \cos \delta_1, \text{ etc.}$$

Hence,

$$\Sigma w_1\left(\frac{1}{R_2} + \frac{1}{R_3}\right) yz = 2\Sigma yz \cos \delta_1 \leqslant \Sigma x^2$$

where we used the well-known asymmetric trigonometric inequality of

J. Wolstenholme (see Chapter VI), because $\delta_1 + \delta_2 + \delta_3 = \pi$.

8° Oppenheim proved (23) only for positive numbers x, y, z. He did not give the necessary and sufficient condition for equality, i.e. (24). He only notes that equality in (23) is valid if $x = y = z$ and $\delta_1 = \delta_2 = \delta_3$, which is a trivial case.

THEOREM 13. Let $R_1^* = \sqrt{R_2R_3}\cos\delta_1$, etc. Then

$$\Sigma x^2R_1 \geqslant 2\Sigma yzR_1^* \tag{26}$$

with equality if and only if

$$x\sqrt{R_1}/\sin\delta_1 = y\sqrt{R_2}/\sin\delta_2 = z\sqrt{R_3}/\sin\delta_3. \tag{27}$$

Proof. Using the substitutions $x \to x\sqrt{R_1}$, etc. and identity (25), we get Theorem 13 from Theorem 12.

THEOREM 14. The following inequality is valid

$$\Sigma x^2R_1 \geqslant 2\Sigma yzw_1 \quad (x, y, z \geqslant 0) \tag{28}$$

with equality if and only if P is the circumcentre and (24) is valid.

Proof. It is known that (see GI 16.9):

$$R_1^* \geqslant w_1 \tag{29}$$

with equality if and only if $R_2 = R_3$. Hence we get (28) from (26).

THEOREM 15. If $x, y, z \geqslant 0$, then

$$\Sigma x^2R_1 \geqslant 2\Sigma yzr_1 \tag{30}$$

with equality if and only if the triangle is equilateral, P is its centre and $x = y = z$.

The following two theorems are equivalent to Theorem 12:

THEOREM 16. Let x, y, z be real numbers. Then

$$\Sigma x^2w_1^2 \geqslant w_1w_2w_3\Sigma yz\left(\frac{1}{R_2} + \frac{1}{R_3}\right) \tag{31}$$

with equality if and only if

$$xw_1/\sin\delta_1 = yw_2/\sin\delta_2 = zw_3/\sin\delta_3. \tag{32}$$

THEOREM 17. Let x, y, z be real numbers. Then

(33) $\sum x^2 w_1^2 R_1^2 \geqslant w_1 w_2 w_3 \sum yz(R_2 + R_3)$

with equality if and only if

(34) $x w_1 R_1/\sin \delta_1 = y w_2 R_2/\sin \delta_2 = z w_3 R_3/\sin \delta_3.$

9° Theorems 16 and 17 are generalizations of some results from [5]. The case x = y = z = 1 of Theorem 13 is generalized for polygons in [6] (see XV.27).

Finally, we shall note that, using inversion, reciprocation and isogonal conjugate transformations, we can obtain from (30) and (28):

(35) $\sum x^2 r_1 R_1 \geqslant 2\sum yz r_2 r_3,$

(36) $\sum x^2 R_2 R_3 \geqslant 2\sum yz r_1 R_1,$

(37) $\sum x^2 r_1^{-1} \geqslant 2\sum yz R_1^{-1},$

(38) $\sum x^2 R_2 R_3 \geqslant 2\sum yz r_1 w_1,$

where $x, y, z \geqslant 0$.

References

1. A. Oppenheim, 'The Erdős Inequality and Other Inequalities for a Triangle', Amer. Math. Monthly 68 (1961), 226-230.
2. G. Tsintsifas and M. S. Klamkin, 'Problem 982', Crux Math. 10 (1984), 291 and 12 (1986), 28-31.
3. H. Demir, 'Problem E 2462', Amer. Math. Monthly 81 (1974), 281.
4. C. W. Dodge, 'The Extended Erdős-Mordell Inequality', Crux Math. 10 (1984), 274-281.
5. A. Oppenheim, 'Some Inequalities for a Triangle', Math. Gaz. 53 (1969), 38-40.
6. H. Vogler, 'Eine Bemerkung zum Erdős-Mordellschen Satz für Polygone', Anz. Österr. Akad. Wiss. Math.-natur-wiss. Kl. 103, No. 1-14 (1966), 241-251.
7. D. S. Mitrinović and J. E. Pečarić, 'Erdős-Mordell's and Related Inequalities', C. R. Math. Rep. Acad. Sci. Canada, 8 (1986), 381-386.
8. B. J. Malešević, 'Erdős-eva teorema u ravni trougla', Zbornik radova Primatijada-85 (Yugoslavia).

6. Miscellaneous Inequalities Involving Characteristic Points

6.1. $\sum bc \geqslant (\sum AI)^2.$ {E}

Proof. (V. Mascioni). Since $AI^2 = \frac{bc(s-a)}{s}$ and $AI \cdot BI = \frac{abc}{s}\sin\frac{C}{2}$, we immediately get

$$(\Sigma AI)^2 = \Sigma bc - \frac{2abc}{s}\left(\frac{3}{2} - \Sigma \sin\frac{A}{2}\right) \leqslant \Sigma bc,$$

where we used GI 2.10.

A. W. Walker and M. G. Greening, 'Problem E 2388', Amer. Math. Monthly 79 (1972), 1135 and 80 (1973), 1142-1143.

6.2. $$\frac{1}{2s^3}\Sigma a^3 + 16\left(\Pi \frac{IA}{w_a}\right)^{1/3} \leqslant \frac{100}{9}. \quad \{E\}$$

P. Leuenberger and A. Bager, 'Aufgabe 678', Elem. Math. 27 (1972), 115, and 28 (1973), 131.

6.3. $$Rr\Sigma aAI \geqslant s\Pi AI.$$

We were informed of this result by W. Janous.

Comment by V. Mascioni. One could derive a number of inequalities from the following identities:

$$\Sigma aAI = 4rR\Sigma\cos\frac{A}{2}, \quad \Sigma AI \cdot BI = 4rR\Sigma\sin\frac{A}{2}, \quad \Pi AI = 4r^2R,$$

$$\Sigma aAI^2 = 4srR.$$

Janous' inequality is equivalent to

$$\Sigma\cos\frac{A}{2} \geqslant s/R = \Sigma\sin A,$$

and thus to

(1) $$2\sqrt{r/R} \leqslant \Sigma\sqrt{(1-a/s)a/s}.$$

It is therefore an improvement of $2r \leqslant R$, by Cauchy's inequality. Since $s\Sigma AI = r\Sigma aAI^2$, Janous' inequality is also equivalent to

$$R\Sigma aAI \geqslant \Sigma aAI^2.$$

A curious proof of Janous' inequality in the form (1) follows: by Čebyšev's inequality we first get

$$\Sigma\sqrt{(1-a/s)a/s} \geqslant \frac{1}{3s}(\Sigma\sqrt{a})(\Sigma\sqrt{s-a}).$$

Then, by the mean inequality $M_{1/2} \geqslant M_0$, we get

$$\Sigma\sqrt{(1-a/s)a/s} \geqslant 3 \cdot 2^{1/3}(R/s)^{1/6}(r/s)^{1/2}.$$

Now, $3 \cdot 2^{1/3}(R/s)^{1/6}(r/s)^{1/2} \geqslant 2(r/R)^{1/2}$ is equivalent to $s \leqslant \frac{3\sqrt{3}}{2}R$, which is well-known and thus proves (1).

6.4. If the lines AI, BI, CI meet the circumcircle of the triangle again in D, E, F, respectively, then

$$\Sigma \frac{AI}{ID} \geqslant 3. \quad \{E\}$$

Proof. (R. H. Eddy). We will sharpen the inequality and also find an upper bound for the sum involved. With

$$AI = r \operatorname{cosec} \frac{A}{2}, \quad ID = 2R \sin \frac{A}{2}, \quad \sin^2 \frac{A}{2} = \frac{(s-b)(s-c)}{bc}, \text{ etc.}$$

we obtain

$$(1) \qquad \Sigma \frac{AI}{ID} = \frac{r}{2R}(s\Sigma bc - 3abc)/\Pi(s-a).$$

With the known relations

$$\Sigma bc = r^2 + 4Rr + s^2, \quad abc = 4Rrs, \quad r^2 s = \Pi(s-a),$$

the right member of (1) becomes

$$(2) \qquad \frac{r^2 - 8Rr + s^2}{2Rr}.$$

With the help of (1) and (2), the Gerretsen inequalities

$$16Rr - 5r^2 \leqslant s^2 \leqslant 4R^2 + 4Rr + 3r^2,$$

are easily shown to be equivalent to

$$(3) \qquad 2(2 - r/R) \leqslant \Sigma \frac{AI}{ID} \leqslant 2(R/r + r/R - 1).$$

We now use the familiar $R \geqslant 2r$ to obtain our final result:

$$(4) \qquad 3 \leqslant 2(2 - \frac{r}{R}) \leqslant \Sigma \frac{AI}{ID} \leqslant 2(\frac{R}{r} + \frac{r}{R} - 1).$$

The lower bound is attained just when $R = 2r$, that is, just when the triangle is equilateral, and then equality holds throughout in (4).
Comment by M. S. Klamkin. We have simply

$$(5) \qquad \Pi \frac{AI}{ID} = \frac{2r}{R} \leqslant 1.$$

This follows immediately from

$$\frac{AI}{ID} = \frac{2 \sin \frac{B}{2} \sin \frac{C}{2}}{\sin \frac{A}{2}}, \quad \text{etc.}, \quad r = 4R\Pi \sin \frac{A}{2},$$

and $R \geqslant 2r$.

J. Garfunkel, S. C. Chan, R. H. Eddy, M. S. Klamkin, and J. Dou, 'Problem 644', Crux Math. 7 (1981), 147 and 8 (1982), 154-157.

6.5. With the same notation as in 6.4,

$$(\Sigma AI)/(\Sigma ID) \leqslant 1.$$

L. Bankoff, J. Garfunkel, and O. P. Lossers, 'Problem S 23', Amer. Math. Monthly 88 (1981), 536-537.

Comment by V. Mascioni. Since $ID = 2R\Sigma \sin \frac{A}{2} = \frac{1}{2r} \Sigma AI \cdot BI$, we get

$$\frac{\Sigma AI}{\Sigma ID} = 2r \frac{\Sigma AI}{\Sigma AI \cdot BI} = \frac{2r}{\Pi AI} \frac{\Sigma AI}{\Sigma 1/AI}.$$

Now, by $\Pi AI = 4r^2R$ and by the mean inequality $M_1 \cdot M_{-1} \geqslant M_0^2$ we get

$$(\Sigma AI)/(\Sigma ID) \geqslant 2r(\Pi AI)^{-1/3} = (2r/R)^{1/3},$$

which is a good companion to 6.5.

6.6. $\quad \Sigma bc/AI^2 \geqslant 9. \quad \{E\}$

Proof (V. Mascioni).

$$\Sigma bc/AI^2 = \Sigma s/(s - a) \geqslant 9/(\Sigma(s - a)/s = 9.$$

P. Balev and V. Gridasov, 'Problem 3', Mat. i Fiz. (Sofija) 11 (1968), No. 3, 57 and No. 6, 53-54.

6.7. Let AA_1, BB_1, CC_1 be the angle-bisectors of a triangle. Then

$$\min (IA/IA_1, IB/IB_1, IC/IC_1) \leqslant 2 \leqslant \max (IA/IA_1, IB/IB_1, IC/IC_1).$$

G. Iureş, 'Problem 20233', Gaz. Mat. (Bucharest) 89 (1984), 380.

6.8. Let x, y, z be the distances from O to the sides of a triangle. Then

$$\Sigma R/x > 3 \quad \text{and} \quad \Sigma a/x \geqslant 18 \tan \omega \quad (\omega \text{ is the Crelle-Brocard angle}).$$

The first inequality is Janous' correction of an inequality of R. Grozdanov, and the second is Mascioni's correction of another inequality of Grozdanov.

R. Grozdanov, 'Zadatak 5', Mat. i Fiz. (Sofija) 11 (1968), No. 4, 56.

6.9. For acute triangles

$$\Sigma a/HA \geqslant 27R^2/4F \geqslant 3\sqrt{3}. \quad \{E\}$$

This is an interpolating inequality for a result from 'Problem F 2540', Kőz. Mat. Lapok 35 (1985), 271-272 and 36 (1986), 103-106.

6.10. (a) $\sqrt{3} \leqslant s/3r \leqslant \Sigma HA/a.$ {E}

(b) For acute triangles,

$$2R^2/rs \leqslant \Sigma HA/a \leqslant (2R^2 + r^2)/rs.$$

Proof. (V. Mascioni and J. E. Pečarić) Since $a/HA = |\tan A|$, we have

$$\Sigma HA/a = \Sigma|\text{cotan } A| \geqslant \Sigma \text{ cotan } A = \text{cotan } \omega \geqslant s/3r \geqslant \sqrt{3}.$$

This proves (a).

For acute triangles we have

$$\Sigma HA/a = \Sigma \text{ cotan } A \leqslant (2R^2 + r^2)/(sr) \quad \text{or} \quad \geqslant 2R^2/rs,$$

where we used (IX.6.15) and inequality 37) from Section X.1.

Remark. (a) is due to G. Kalajdžić. (b) is due to J. E. Pečarić and V. Mascioni.

G. Kalajdžić, 'Some Inequalities for the Triangle', Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 247-273 (1969), 171-174.

6.11. Let AL, BM, CN be the altitudes of a triangle. Then

$$\Sigma a \geqslant 2\sqrt{3}(\Sigma HL).$$

L. D. Kurljandčik, 'Problem 2699', Mat. v Škole 1984, No. 1, 67 and 1984, No. 6, 66.

6.12. Let A_1, B_1, C_1 be the midpoints of the sides BC, CA, AB, respectively. Then

$$\Sigma HA_1 \geqslant 2R - r.$$

D. Dimitrov, 'Problem 2', Matematika (Sofija) 1975, No. 5, 31 and 1976, No. 2, 36.

6.13. With the same notations as in 6.12 we have

$$\Sigma(\overrightarrow{A_1A}, \overrightarrow{A_1H})^n \geqslant 3\left(\frac{F}{\sqrt{3}}\right)^n \quad (n \in N).$$

S. Bilčev, 'Zadatak 2', Ob. po matematika 1984, No. 5, 60.

6.14. If P is an arbitrary point, then

$$3PG \leqslant \Sigma PA.$$

Proof. Since

$$\Sigma\overrightarrow{PA} = 3\overrightarrow{PG}$$

we have

$$3|\overrightarrow{PG}| = |\Sigma\overrightarrow{PA}| \leqslant \Sigma|\overrightarrow{PA}|.$$

'Problem 461', Mat. v škole 1968, No. 2, 89 and 1968, No. 6, 82. Comment by V. Mascioni. By 1.1.(9) we get

$$\Sigma PA \leqslant \sqrt{9PG^2 + \Sigma a^2} \quad \text{(a companion to 6.14).}$$

6.15.(a) $12r^2 \leqslant 6Rr \leqslant 4r(2R - r) \leqslant \Sigma GA^2 \leqslant \frac{8}{3} R^2 + \frac{4}{3} r^2 \leqslant 3R^2.$ {E}

(b) Depending upon the fact whether a triangle is acute, right, or obtuse, the following statements hold

$$\Sigma GA^2 \gtreqless 8R^2/3$$

These results are due to J. E. Pečarić and V. Mascioni, and they are an extension of a result from

'Problem F 2299', Köz. Mat. Lapok 62 (1981), 78 and 63 (1981), 194-196.

6.16. Let $\Theta_a = \measuredangle GBC$, etc., then

$$\Sigma \sin \Theta_a \leqslant 3/2.$$

J. Garfunkel, B. J. Venkatachala, C. R. Pranesachar, and C. S. Gardner, 'Problem E 2715', Amer. Math. Monthly 85 (1978), 384; 86 (1979), 705-706, and 87 (1980), 304.

6.17. Suppose that AG, BG, CG meet the circumcircle of the triangle in A', B', C', respectively. Then the following results are valid

(a) $\Sigma AG \leqslant \Sigma GA'$; (b) $\Sigma AG/GA' = 3$; (c) $\Pi AG \leqslant \Pi GA'$.

Proof. From $4m_a^2 = 2\Sigma a^2 - 3a^2$, etc., it follows that $m_b \gtreqless m_c$ according as $b \lesseqgtr c$, etc., i.e.

(1) $$(b^2 - c^2)(m_b^{-1} - m_c^{-1}) \geqslant 0.$$

If D is the midpoint of BC, then $m_a \cdot DA' = (a/2)^2$, so $DA' = a^2/4m_a$ and

$$GA' = GD + DA' = (\Sigma a^2)/6m_a.$$

Thus

$$GA' - AG = \frac{a^2 + b^2 + c^2}{6m_a} - \frac{2}{3} m_a = \frac{(a^2 - b^2) + (a^2 - c^2)}{6m_a}.$$

With this and two similar results we obtain

(2) $$\Sigma(GA' - AG) = \frac{1}{6} \Sigma(b^2 - c^2)(m_b^{-1} - m_c^{-1}).$$

Now the right-hand side of (2) is non-negative by (1), and (a) follows, with equality just when the triangle is equilateral.

(b) Applying some results used in the solution of part (a) again, we have

$$\frac{AG}{GA'} = \frac{4m_a^2}{a^2 + b^2 + c^2} = 2 - \frac{3a^2}{a^2 + b^2 + c^2}, \quad \text{etc.}$$

Hence

$$\Sigma \frac{AG}{GA'} = 6 - 3 = 3.$$

(c) From (b) and the A.M.-G.M. inequality, we have

$$1 = \frac{1}{3} \Sigma AG/GA' \geqslant (\Pi AG/GA')^{1/3},$$

and (c) follows, with equality just when the triangle is equilateral.

Remarks. 1° The above results and proofs are given as Problem 723 in Crux Math. Part (a) is also given as Problem E 2959 in Amer. Math. Monthly, and part (b) is given as Problem 1119 in Math. Mag.

2° Note that the following generalization of (c) is a simple consequence of (b):

$$M_r(AG/GA', BG/GB', CG/GC') \leqslant 1 \quad (r < 1).$$

3° Further, note that the triplets (GA', GB', GC') and (AG/GA', BG/GB', CG/GC') are oppositely ordered, so using (b) and the Čebyšev inequality gives (a).

G. Tsintsifas and K. Satyanarayana, 'Problem 723', Crux Math. 8 (1982), 77 and 9 (1983), 91-92.
J. Garfunkel, 'Problem E 2959', Amer. Math. Monthly 89 (1982), 498.
K. R. S. Sastry and T. Pham, 'Problem 1119', Math. Mag. 55 (1982), 180-182.

6.18. If I_a, I_b, I_c are the excentres of a triangle, then

(a) $12r \leqslant \frac{4\sqrt{3}}{3} s \leqslant \Sigma II_a \leqslant 7R - 2r$;

(b) $4s \leqslant \Sigma I_b I_c \leqslant \frac{\sqrt{3}}{3}(17R + 2r)$.

Proof. (a) We have $II_a = a \sec \frac{A}{2}$, etc. If $a \geqslant b \geqslant c$, then $\sec \frac{A}{2} \geqslant \sec \frac{B}{2} \geqslant \sec \frac{C}{2}$. By the Čebyšev inequality

$$(\Sigma a_1)(\Sigma a_2) \leqslant 3\Sigma a_1 a_2 \qquad (a_1 < b_1 < c_1,\ a_2 < b_2 < c_2),$$

we have

$$\Sigma II_a \geqslant \frac{1}{3}(\Sigma a)(\Sigma \sec \frac{A}{2}) \geqslant \frac{1}{3}\, 2s2\sqrt{3} = \frac{4\sqrt{3}}{3}\, s.$$

On the other hand $II_a^2 = 4R(r_a - r)$, etc. Hence

$$\Sigma II_a \leqslant \sqrt{3}(\Sigma II_a^2)^{1/2} = 4(\frac{3}{2}\, R(2R - r))^{1/2} \leqslant 7R - 2r.$$

(b) Since $I_b I_c^2 = 4R(r_b + r_c)$, etc., we have

$$\Sigma I_b I_c \leqslant \sqrt{3}(\Sigma I_b I_c^2)^{1/2} = \frac{4\sqrt{3}}{3}(\frac{9}{2}\, R(4R + r))^{1/2} \leqslant \frac{\sqrt{3}}{3}(17R + 2r).$$

Since

$$\Sigma I_b I_c \geqslant 3(\Pi I_b I_c)^{1/3}$$

and

$$\Pi I_b I_c = 16R^2 s, \quad R^2 \geqslant 4s^2/27$$

we obtain

$$\Sigma I_b I_c \geqslant 4s.$$

Remark. (V. Mascioni). The later inequality is equivalent to 6.3. (use $I_b I_c = 4R \cos \frac{A}{2}$).

G. Kalajdžić, 'Some Inequalities for the Triangle', Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 247-273 (1969), 171-174.

6.19. Denote the Gergonne and Nagel cevians from vertex A by g_a and n_a, respectively. Then

$$\Sigma h_a \leqslant \Sigma g_a \leqslant \Sigma w_a \leqslant \Sigma m_a \leqslant \Sigma n_a. \quad \{E\}$$

R. H. Eddy, 'A Sequence of Inequalities for Certain Sets of Concurrent Cevians', Elem. Math. 35 (1980), 145-146.

6.20. With the same notations as in 6.19 we have

$$8Rr + 11r^2 \leqslant \Sigma g_a^2 \leqslant 4R^2 + 11r^2. \quad \{E\}$$

Proof. It follows from the law of cosines that

$$g_a^2 = (s - b)^2 + c^2 - 2c(s - b)(a^2 + c^2 - b^2)/2ac.$$

Putting $2(s - b) = a + c - b$, we obtain

$$4g_a^2 = -a^2 + 3b^2 + 3c^2 - 2bc + 2(b^3c^2 + b^2c^3 - b^4c - bc^4)/abc.$$

Thus,

$$4\Sigma g_a^2 = 5\Sigma a^2 - 2\Sigma bc + 2(\Sigma b^3c^2 - \Sigma b^4c)/abc.$$

Put $t_1 = a + b + c = 2s$, $t_2 = ab + bc + ca = s^2 + 4Rr + r^2$, $t_3 = abc = 4Rrs$. Then $\Sigma a^2 = t_1^2 - 2t_2 = 2s^2 - 8Rr - 2r^2$, and $\Sigma b^3c^2 - \Sigma b^4c = 4t_1t_2^2 - 6t_2t_3 - t_1^2t_3 - t_1^3t_2 = -4Rrs^3 + 8R^2s^3 + 32R^2r^2s + 40Rr^3s + 8r^4s$. Hence, after some simplifications, we obtain

$$\Sigma g_a^2 = s^2 - 8Rr + 2r^2 + \frac{r}{R}(s^2 + r^2). \tag{1}$$

Using (1) and the well-known Gerretsen inequality $s^2 \leqslant 4R^2 + 4Rr + 3r^2$, we have

$$\Sigma g_a^2 \leqslant 4R^2 + 9r^2 + 4r^3/R$$

so that

$$(4R^2 + 11r^2) - \Sigma g_a^2 \geqslant \frac{2r^2}{R}(R - 2r) \geqslant 0,$$

with equality only for the equilateral triangle. Again, using the Gerretsen inequality $s^2 \geqslant 16Rr - 5r^2$, we have

$$\Sigma g_a^2 \geqslant 8Rr + 13r^2 - 4r^3/R,$$

so that

$$\Sigma g_a^2 - (8Rr + 11r^2) \geqslant \frac{2r}{R}(R - 2r) \geqslant 0,$$

with equality only for the equilateral triangle. This completes the proof.

W. J. Blundon and R. H. Eddy, 'Problem 478', Nieuw Arch. Wisk. 25 (1977), 423 and 26 (1978), 354-355.

6.21. According to Feuerbach, in a triangle ABC the incircle and Euler's nine-point-circle are tangent at a point ϕ. If A', B', C' are the mid-points of the sides and the triangle is not equilateral, then

$$2R^2 + 4Rr - r^2 - 2(R - r)\sqrt{R(R - 2r)} < \Sigma\phi A^2 < 2R^2 + 4Rr - r^2 + 2(R - r)\sqrt{R(R - 2r)},$$

$$R(R + r - \sqrt{R(R - 2r)}) < \Sigma\phi A'^2 < R(R + r + \sqrt{R(R - 2r)}).$$

Proof. It is known that (see for instance V. Thébault, Mathesis 1929, p. 354; V. Thébault 'Sur les points de Feuerbach', Gaz. Mat. 39 (1933), p. 42 or Problem 3396 from Gaz. Mat. 30 (1924)):

$$\phi A^2 = \frac{2R(bc - 4Rr - r^2) - r(b^2 + c^2 - a^2)}{2(R - 2r)},$$

$$\phi A'^2 = \frac{R(b - c)^2}{4(R - 2r)}.$$

By means of these equalities,

$$(1) \qquad \Sigma\phi A^2 = \frac{R - r}{2(R - 2r)} Q + r(r + 4R), \qquad \Sigma\phi A'^2 = \frac{RQ}{4(R - 2r)}$$

where

$$Q = \Sigma(b - c)^2 = 2s^2 - 6r(4R + r).$$

Note that the following inequalities are valid (see IX.5.3)

$$(2) \qquad 4(R - 2r)(R + r - \sqrt{R(R - 2r)}) \leqslant Q \leqslant 4(R - 2r)(R + r + \sqrt{R(R - 2r)}).$$

Combining (2) and (1) one finds the desired inequalities.

Remark. We note that we can obtain extensions of the above inequalities similar to those of Lupaş' inequalities (2) (see IX.5.3).

A. Lupaş, 'Problem 343', Mat. Vesnik 13 (28) (1976), 365-366.

6.22. A finite point X in the plane of the triangle ABC is called an isoperimetric point of this triangle if the triangle XBC, XCA and XAB all have the same perimeter 2σ. For this point the following inequality is valid

$$\frac{\sigma}{s} \geq \frac{1}{9}(3 + 2\sqrt{3}). \quad \{E\}$$

G. R. Veldkamp, 'The Isoperimetric Point and the Point(s) of Equal Detour in a Triangle', Amer. Math. Monthly 92 (1985), 546-558.

6.23. Let I_o be any tritangent centre (i.e., either the incentre or an excentre of a triangle), let N be the nine-point centre and let K and L be the points on the Euler line which are respectively the reflections of H and G in O; that is to say KL:LO:OG:GN:NH = 4:2:2:1:3. Then

$$IK \leq 4R + r, \quad I_aK \leq 4R - r_a, \quad 9OI_o^2 - 4I_oN^2 \geq 4I_oN \cdot I_oK.$$

A. P. Guinard, 'Incentres and Excentres Viewed from the Euler Line', Math. Mag. 58 (1985), 89-92.

6.24. GI 14.28 gives the following well-known inequality

(1) $\omega \leq \pi/6 \quad \{E\},$

in which ω denotes the Crelle-Brocard angle.

Refinements of this inequality were subject of many papers and problems. These results will be discussed in the following lines.

First, we shall give the inequality

(2) $\omega^3 \leq \Pi(A - \omega). \quad \{E\}$

Proof. Inequality (2) is given by F. Abi-Khuzam. But, G. R. Veldkamp noted that there was some doubt as to the correctness of the method used, and he gave a simpler proof. Here we shall give the very simple proof of R. J. Stroeker and H. J. T. Hoogland (a modification of Veldkamp's proof).

The function $x \to f(x) = \log(x/\sin x)$ is strictly convex on the interval $(0, \pi)$, because $f''(x) = \sin^{-2} x - x^{-2} > 0$ for $0 < x < \pi$. Hence by Jensen's inequality for a convex function f

$$f(\pi/6) = f(\frac{1}{6}\Sigma(\omega + (A - \omega)) \leq \frac{1}{6}\Sigma(f(\omega) + f(A - \omega))$$

so that

$$\log(\frac{\pi/6}{\sin \pi/6}) \leq \frac{1}{6} \log\Pi (\frac{\omega}{\sin \omega} \cdot \frac{A - \omega}{\sin(A - \omega)}).$$

Using the trigonometric version of Ceva's theorem, i.e.

$$\sin^3 \omega = \Pi \sin(A - \omega),$$

inequality (1) and the fact that the function x/sin x is increasing on the interval $(0, \pi/6]$, we deduce

$$\left(\frac{\omega}{\sin \omega}\right)^6 \leqslant \left(\frac{\pi/6}{\sin \pi/6}\right)^6 \leqslant (\sin \omega)^{-6}\Pi\omega(A - \omega),$$

which implies (2).

Remark. L. Kuipers gave a generalization of the above results: Let be given n positive angles A_j subject to $\Sigma_{j\neq k}$ cotan A_j cotan A_k = 1. Define $\omega < \pi/2$ by cotan $\omega = \Sigma$ cotan A_j. Then

$$\omega \leqslant \text{arccotan}\sqrt{2n/(n - 1)}, \quad \sin^n \omega = \Pi \sin(A_j - \omega) \quad \text{and}$$

$$\omega^n < \Pi(A_j - \omega).$$

In 1963, Yff conjectured that ω satisfies the following inequality

(3) $\qquad 8\omega^3 \leqslant ABC. \quad \{E\}$

Proof. The first proof was given by F. Abi-Khuzam. Simpler proofs were given by O. Bottema and M. S. Klamkin. G. R. Veldkamp and R. J. Stroeker and H. J. T. Hoogland showed that (3) is a consequence of (2). Indeed

$$64\omega^6 \leqslant \Pi 4\omega(A - \omega) \leqslant \Pi A^2,$$

because $4\omega(A - \omega) \leqslant A^2$ is equivalent to $(A - 2\omega)^2 \geqslant 0$.

Inequality (3) follows from (2). This is an immediate consequence of the inequalities

$$2\omega \leqslant M_0(A, B, C) \leqslant M_1(A, B, C) = \pi/3.$$

R. J. Stroeker conjectured the following inequalities

(4) $\qquad \Sigma A^{-1} \leqslant \frac{3}{2}\,\omega^{-1} \quad \{E\},$

(5) $\qquad \Sigma A^{-2} \geqslant \frac{3}{4}\,\omega^{-2} \quad \{E\}.$

Also interesting are

(6) $\qquad \omega^{-1} < \Sigma A^{-1}$

and

(7) $\qquad \Sigma A^{-2} < \omega^{-2},$

proved in his joint paper with Hoogland.

Inequality (3) would be a consequence of (4). This is evident from the following equivalent form of (4):

$$2\omega \leqslant M_{-1}(A, B, C).$$

R. J. Stroeker and H. J. T. Hoogland noted that (4) and (5) would imply the inequality

$$(8) \qquad \frac{4}{3}\pi\omega^2 \leqslant \Pi A \quad \{E\},$$

which is a sharpening of (3). Since (5) is equivalent to

$$M_{-2}(A, B, C) \leqslant 2\omega,$$

this inequality implies that

$$(9) \qquad \min (A, B, C) \leqslant 2\omega. \quad \{E\}$$

R. J. Stroeker and H. J. T. Hoogland provided substantial numerical evidence in support of conjectures (4) and (5) and gave an analytical proof for (4) in the case of an isosceles triangle.

Finally, V. Mascioni proved (4) in his first paper. In his second paper he proved the following results:

(i) The conjecture (5) is true.

(ii) Let r (i.e. s) be the least (i.e. the largest) positive real number such that

$$M_{-r}(A, B, C) \leqslant 2\omega \leqslant M_{-s}(A, B, C)$$

is valid. Then

$$1 \leqslant s \leqslant \frac{\log(3/2)}{\log(4/3)} < \frac{\log 3}{\log 2} \leqslant r \leqslant 2.$$

F. Abi-Khuzam, 'Inequalities of Yff type in the triangle', Elem. Math. 35 (1980), 80-81.

G. R. Veldkamp, 'Elementair bewijs van een Yff-achtige ongelijkheid', Nieuw Tijdschr. voor Wisk. 69 (1981), 47-51.

R. J. Stroeker and H. J. T. Hoogland, 'Brocardian Geometry Revisited or Some Remarkable Inequalities', Nieuw Arch. Wisk. (4), vol. 2 (1984), 281-310.

L. Kuipers, 'Uitbreiding van de ongelijkheid van Abi-Khuzam tot meer dan vier hoekrootten', Nieuw Tijdschr. voor Wisk. 69 (1981-1982), No. 4, 166-169.

P. Yff, 'An Analogue of the Brocard Points', Amer. Math. Monthly 70 (1963), 500.

F. Abi-Khuzam, 'Proof of Yff's Conjecture of the Brocard Angle of a Triangle', Elem. Math. 29 (1974), 141-142.

O. Bottema, 'On Yff's Inequality for the Brocard Angle of a Triangle', Elem. Math. 31 (1976), 13-14.

M. S. Klamkin, 'On Yff's Inequality for the Brocard Angle of a Triangle', Elem. Math. 32 (1977), 118.

V. Mascioni, 'Zur Abschätzung des Brocardschen Winkels', Elem. Math. 41 (1986), 98-101.
R. J. Stroeker, 'Problem E 2905*', Amer. Math. Monthly 88 (1981), 619-620.
V. Mascioni, 'Zur Abschätzung des Brocardschen Winkels, II', Elem. Math. (to appear).

6.25. $$\cot \omega \geq \Sigma \tan \frac{A}{2} \geq \sqrt{3}.$$

Proof. (G. R. Veldkamp) The function $x \to \tan x$ is convex on $(0, \pi/2)$. Hence $\Sigma \tan \frac{A}{2} \geq 3 \tan \frac{\pi}{6} = \sqrt{3}$, i.e. the second inequality is true. Further

$$\cot A + \cot B = \frac{\sin(A + B)}{\sin A \sin B} = \frac{2 \sin C}{\cos C + \cos(A - B)} \geq$$

$$\geq \frac{2 \sin C}{1 + \cos C} = 2 \tan \frac{A}{2},$$

etc.
Hence:

$$\cot \omega = \Sigma \cot A \geq \Sigma \tan \frac{A}{2} \geq \sqrt{3}.$$

Emmerich-Anderson and Zerr, 'Problem 10656', Math. Quest. 54 (1891), 100.

6.26.(1) $\Sigma a \sin A \geq 18R \sin^2 \omega$ {E},

(2) $\Sigma \sin^2 A \geq 9 \sin^2 \omega$. {E}

Remark. Inequality (2) is a simple consequence of (1).
S. Terzijan, 'Problem 2', Matematika (Sofija) 1984, No. 1, 37 and 1984, No. 6, 38.

6.27. $$3 \tan^2 \omega \geq 8\Pi \cos A. \quad \{E\}$$

Proof. (M. S. Klamkin) Since the r.h.s. is less than zero for non-acute triangles, we can take the triangle to be acute. This being the case, we obtain a stronger intermediate inequality, i.e.

$$3 \tan^2 \omega \geq 6\sqrt{6} abc (\Pi \cos A)^{1/2} / (\Sigma a^2)^{3/2} \geq 8\Pi \cos A$$

with equality if and only if $a = b = c$. Since

$$\tan^2 \omega = 16F^2 (\Sigma a^2)^2 \quad \text{and} \quad \cos A = (b^2 + c^2 - a^2)/2bc, \quad \text{etc.,}$$

6.27 is equivalent to

(1) $$48F^2 \geq (\Sigma a^2/abc)^2 \Pi (b^2 + c^2 - a^2).$$

Letting $(a^2, b^2, c^2) = (u, v, w)$, (1) reduces to

$$3F_1^2 \geqslant \frac{u + v + w}{uvw} F^2,$$

where F and F_1 denote the areas of triangles of sides (u, v, w) and $(\sqrt{u}, \sqrt{v}, \sqrt{w})$, respectively. Since $4F_1^2 \geqslant \sqrt{3}\, F$ by the Finsler-Hadwiger inequality, it suffices to show that

$$\frac{3\sqrt{3}}{4} \geqslant \frac{u + v + w}{uvw} F$$

or that $3\sqrt{3}\, R \geqslant u + v + w$. But the latter inequality is just GI 5.3 with equality if and only if $a = b = c$.

A. W. Walker and M. S. Klamkin, 'Problem 322', Nieuw Arch. Wisk. 20 (1972), 161 and 21 (1973), 113-114.

6.28. If ω is the Crelle-Brocard angle, and C any acute angle of a triangle, then

$$\cot \omega \sin 2C > 2 \quad \text{in an obtuse-angled triangle,}$$

$$\cot \omega \sin 2C < 2 \quad \text{in an acute-angled triangle.}$$

Proof. Since $\cot \omega = \sum \cot A$, we have

$$\cot \omega \sin 2C = 2 \cos C(\sin(A + B)(\cot A + \cot B) + \cos C)$$

$$= 2 \cos^2(A + B) + 2 \sin^2(A + B) \cos C/\sin A \sin B.$$

This is > 2 or < 2 according as $\cos C/\sin A \sin B$ is > 1 or < 1. But $\cos C/\sin A \sin B = 1 - \cot A \cot B$, which is < 1 if A and B are both acute, and > 1 if one of them is obtuse.

A. Emmerich-Anderson and R. Knowles, Math. Quest. 53 (1890), 112.

6.29. If ω is the Crelle-Brocard angle, then

$$\frac{3}{2} \frac{(\Pi \operatorname{cosec} A)^{1/3}}{(\Sigma \operatorname{cosec}^2 A)^{1/2}} \leqslant \cos \omega \leqslant \frac{9}{8} \frac{\Pi \operatorname{cosec} A}{(\Sigma \operatorname{cosec}^2 A)^{1/2}} .$$

This result is due to W. Janous.

7. Miscellaneous Inequalities Involving Arbitrary Points

7.1. $\quad \Sigma a r_2 r_3 \leqslant abc/4$

with equality if and only if $P = O$.

Ju. I. Gerasimov, 'Problem 848', Mat. v Škole 1971, No. 4, 86.

I. Tomescu, 'Problem 13232', Gaz. Mat. (Bucharest) 24 (1973), 370.

7.2. (1) $\Sigma(s - a)r_2r_3 \leqslant r^2s$,

(2) $$\Sigma ar_1^2 + \Sigma(s - a)r_2r_3 \leqslant 3r^2s.$$

In each case there is equality if and only if $P = I$.

Proof. The following inequality is given by J. Wolstenholme (this is a simple consequence of the well-known trigonometric asymmetric inequality from Chapter VI):

(3) $$\Sigma a^2x^2 \geqslant \Sigma(b^2 + c^2 - a^2)yz.$$

By multiplying (1) and (2) by 4s and using $2rs = \Sigma ar_1$, they can be rewritten, respectively, as

(4) $$\Sigma(a^2r_1^2 - (b^2 + c^2 - a^2)r_2r_3) \geqslant 0,$$

(5) $$\Sigma(a(2b + 2c - a)r_1^2 + ((b + c)^2 - a^2 - 6bc)r_2r_3) \geqslant 0.$$

Now noting that if $a_1^2 = a(2b + 2c - a)$, $b_1^2 = b(2c + 2a - b)$, and $c_1^2 = c(2a + 2b - c)$, then a_1, b_1, c_1 are sides of a triangle, it follows that (4) and (5) are valid for all real r_1, r_2, r_3 with equality if and only if $r_1 = r_2 = r_3$.

L. Carlitz and M. S. Klamkin, 'Problem 910', Math. Mag. Sept.-Oct. 1975, 242-243.

7.3. $$12R^2\Sigma r_1^2 \geqslant \Sigma a^2R_1^2.$$

M. S. Klamkin, 'Problem P 228', Can. Math. Bull. 17 (1974), 309.

7.4. Let $S_i := R_1 + R_2 + R_3 - R_i$ $(i = 1, 2, 3)$. Then

$$2F\sqrt{3} + \frac{5}{6}\Sigma a^2 \leqslant \Sigma s_1^2 \leqslant \Sigma a^2 + 4R_1R_2R_3 \max(1/R_i).$$

M. S. Klamkin and M. H. Hendricks, 'Problem 377', Nieuw Arch. voor Wisk. 23 (1975), 90-91.

7.5. (a) $\Sigma R_1(\frac{1}{2}\sin A) \leqslant \Sigma r_1 \leqslant \Sigma R_1 \sin\frac{A}{2}$,

(b) $r\Sigma R_2R_3 \geqslant 8\Pi r_1$.

H. Demir and M. G. Greening, 'Problem E 2160', Amer. Math. Monthly 76 (1969), 300 and 1146-1147.

7.6. $\Sigma r_1R_2R_3 \geqslant 12r_1r_2r_3,$

$$\Sigma r_1R_1^2 \geqslant 12r_1r_2r_3,$$

$$\Sigma r_2^2r_3^2R_2R_3 \geqslant 12r_1^2r_2^2r_3^2.$$

In each case there is equality if and only if ABC is equilateral and P is the centre of ABC.

L. Carlitz, Math. Mag. 46 (1973), 166.

7.7. $\Sigma R_1(r_1 + r_3) \geqslant \Sigma(r_1 + r_2)(r_1 + r_3),$

$$\Sigma(R_1 + R_2)(R_1 + R_3) \geqslant 4\Sigma(r_1 + r_2)(r_1 + r_3),$$

with equality if and only if ABC is equilateral and P is its centre.

L. Carlitz and M. S. Klamkin, 'Problem E 2348', Amer. Math. Monthly 79 (1972), 304 and 81 (1974), 323-324.

7.8. $\Sigma \frac{r_2 + r_3}{r_2 + 2R_1 + r_3} \leqslant 1 \leqslant \frac{1}{3} \Sigma \frac{R_1}{r_2 + r_3}$

with equality if and only if the triangle is equilateral and P is its centre.

F. Leuenberger and L. Goldstone, 'Problem 2445', Amer. Math. Monthly 80 (1973), 1138 and 81 (1974), 1117-1119.

7.9. $\Sigma R_1w_1^2 + 12w_1w_2w_3 \leqslant \frac{9}{4} R_1R_2R_3,$

with equality if and only if the triangle is equilateral and P is its centre.

Remark. This result is is given in:

L. Carlitz and F. Gotze, 'Aufgabe 663', Elem. Math. 27 (1972), 19 and 28 (1973), 17-18.

Editor noted that the following generalization of the above inequality was given by J. Schopp:

$$\Sigma R_1w_1^2 + (2 + \lambda)\Pi w_1 \leqslant \frac{8 + \lambda}{8} \Pi R_1 \quad (\lambda \geqslant 0).$$

A proof of 7.9 is also given in:

S. Horák, 'O jedne nerovnosti v trojuhelniku', Rozh. Mat. Fyz. 56 (1977-1978), 433-437.

7.10. $\Sigma R_1\sqrt{a(s - a)} \geqslant 2s\sqrt{Rr}.$

This inequality is due to S. J. Bilčev.

7.11. If $A \geqslant 90°$, then

$$\min_i(R_i) \leqslant \frac{a}{2} \leqslant \max_i(R_i)\,.$$

D. Mihet, Gaz. Mat. (Bucharest) 88 (1983), 149.

7.12. Let AA', BB' and CC' be concurrent cevians of a triangle ABC, and let $\alpha = AC'/BC'$, $\beta = BA'/CA'$ and $\gamma = CB'/AB'$. Let $p \neq 0$ be a real number. Then

$$\Sigma\alpha^p \geqslant 3.$$

This inequality is due to W. Janous. For $p = 2$ we get an inequality of Filip.

A. Filip, 'Problem 19279*', Gaz. Mat. (Bucharest) 88 (1983), 250.

7.13. With the same notation as in 7.12

$$\Sigma \frac{\alpha}{\alpha\beta + 1} \geqslant \frac{3}{2}, \quad \Sigma \frac{\alpha}{(1 + \alpha)(1 + \beta)} \geqslant \frac{3}{4}.$$

The second inequality is due to W. Janous. The first one is given in:

V. Stoican, 'Problem 9720', Gaz. Mat. (Bucharest) B 20 (1969), 441 and B 21 (1970), 148-150.

7.14. Let AA', etc. be concurrent cevians of a triangle ABC. Then

$$(4A'C^2 + BC^2)(4B'A^2 + CA^2)(4C'B^2 + AB^2) \geqslant 512A'C^2\cdot B'A^2\cdot C'B^2.$$

E. Popa, 'Problem 10551', Gaz. Mat. (Bucharest) B 21 (1970), 492.

7.15. Let $D \in BC$ and $k = CD/CB$. Then

$$AD < kAB + (1 - k)AC.$$

T. Andrescu, D. M. Batinetu, I. V. Maftei, and M. Tena, Gaz. Mat. (Bucharest) 88 (1983) 286 and 89 (1984), 148.

7.16. Let P be a point in the interior of a triangle ABC, and D, E and F the feet of the normal-projections of P to sides a, b and c, respectively. Furthermore, $r_1 = PD$, $r_2 = PE$, $r_3 = PF$, $x = BD$, $y = CE$, $z = AF$. Then

$$(\Sigma x^2)(\Sigma r_1^2) \geqslant F^2.$$

We were informed of this result by W. Janous.

7.17. With the same notation as in 7.16.

$$\Sigma x^2 \geqslant \frac{1}{4}\,\Sigma a^2.$$

This result is due to W. Janous. Note that 7.16. is a simple consequence of 7.17 and of GI 12.54.

7.18. Conjecture: Let the point P be the feet of the altitude from A. Then

$$r_2 + r_3 < (s + h_a)/2.$$

C. Caragea, 'Problem 9307', Gaz. Mat. (Bucharest) B 19 (1968), 751.

7.19. In the triangle ABC let M be any point on side BC. Prove that

$$(AM - AC) \cdot BC \leqslant (AB - AC) \cdot MC.$$

J. Vukmirović, L. Bankoff, M. G. Greening, M. A. Bershad, 'Problem E 2128', Amer. Math. Monthly 75 (1968), 1007 and 76 (1969), 941-942.

7.20. Let A_i ($i = 0, 1, 2 \pmod 3$) be the vertices of a triangle, Γ its inscribed circle with centre I. Let B_i be the intersection of the segment A_iI with Γ and let C_i be the intersection of line A_iI with the side $A_{i-1}A_{i+1}$. Then

$$(1) \qquad \Sigma A_iC_i \leqslant 3\Sigma A_iB_i.$$

Editors noted that L. Goldstone proved

$$\Sigma(3A_iB_i - A_iC_i) \geqslant 2r(1 - 2(r/R)).$$

Since $R \geqslant 2r$, this inequality is also an improvement of (1).
J. Garfunkel and O. P. Lossers, 'Problem E 2634', Amer. Math. Monthly 84 (1977) 58 and 85 (1978), 281-282.

7.21. In an acute triangle ABC, angle-bisector BT_1 intersects altitude AH_1 in D. Angle-bisector CT_2 intersects altitude BH_2 in E, and angle-bisector AT_3 intersects altitude CH_3 in F. Then

$$\Sigma \frac{DH_1}{AH_1} \leqslant 1.$$

J. Garfunkel, 'Problem 222', Pi Mu Epsilon J. 1969, No. 5, 24 and 1970, No. 5, 136-137.

7.22. A_1, B_1 and C_1 are points on the respective sides, opposite the vertices A, B and C of the triangle ABC. Then

$$\frac{3}{4}\Sigma a^2 \leqslant \Sigma AA_1^2 \leqslant \Sigma a^2$$

if $AC_1:C_1B = BA_1:A_1C = CB_1:B_1A$ holds true.

This result, similar to 7.17, is given in:
'Problem F 2142', Köz Mat. Lap. 56 (1978), 126 and 57 (1978), 61-62.

7.23. Let ABC be a triangle with P an interior point. Let A', B', C' be the points where the perpendiculars drawn from P meet the sides of ABC. Let A'', B'', C'' be the points where the lines joining P to A, B, C meet the corresponding sides of ABC. Then

$$\Sigma B'C' \leqslant \Sigma B''C''.$$

J. Garfunkel and C. S. Gardner, 'Problem E 2716', Amer. Math. Monthly 89 (1982), 594-598.

7.24. Let AA_1, BB_1, CC_1 be three arbitrary cevians of the triangle ABC. If $AA_1 \cap B_1C_1 = P$, $BB_1 \cap A_1C_1 = Q$, $CC_1 \cap A_1B_1 = R$, then

$$\Sigma \frac{AP}{PA_1} \geqslant 3 \quad \text{(conjecture)}.$$

I. Safta, 'Problem C:114', Gaz. Mat. (Bucharest) 86 (1981), 224.

7.25. Let A_iB_i $(i = 1, 2, 3)$ be the arbitrary cevians of the triangle $A_1A_2A_3$ and let P be an interior point. If M_i is the point where the line parallel to A_iB_i drawn from P meets the side opposite to A_i, then

$$\min_i (A_iB_i) \leqslant \Sigma MM_1 \leqslant \max_i (A_iB_i).$$

F. S. Pîrvănescu, 'Asupra problemei 9539 din G. M. B. 4/1969', Gaz. Mat. (Bucharest) 86 (1981), 56-57.

7.26. Let $m_b = BB_1$, $w_c = CC_1$ and $O = BB_1 \cap CC_1$. Then

a) $\Sigma \dfrac{AO}{OA_1} \geqslant 6;$

b) if $BA_1 \geqslant CA_1$, then:

$$\frac{OA}{OA_1} + \frac{BO}{OB_1} \geqslant 2 + \frac{CO}{OC_1}.$$

D. Seclăman, 'Problem 18974', Gaz. Mat. (Bucharest) 86 (1981), 419.

7.27. $$12\sqrt{3}\, F \leqslant 3D^2 \leqslant 4s^2,$$

where

$$D = \min (\Sigma R_1).$$

H. Toepken, 'Aufgabe 301', Jahresb. d. Deutsch. Math.-Verein. LI. 2. Abt. Heft 2 (1941), 3.

7.28. $\Sigma R_1R_2/(ab) \geq 1 \geq 4\Sigma r_1r_2/(ab)$.

This is Janous' interpolation of an inequality from:
G. Tsintsifas, 'Problem 1144', Crux Math. 12 (1986), 107.

7.29. $\Sigma(s - a)R_1 \geq 2F$.

This inequality of W. Janous is stronger than Tsintsifas' inequality:

$$\Sigma(b + c)R_1 \geq 8F.$$

G. Tsintsifas, 'Problem 1159', Crux Math. 12 (1986), 140.

7.30. $\Sigma \dfrac{PA^3}{bc} \geq 3PG$.

I. Iwata, Encyclopaedia of Geometry, (Japanese), Tokyo, 1971, Vol. 5, p. 347.

7.31. $\Sigma 1/r_1' > 9/\max (a, b, c)$.

D. M. Bătinetu, 'Problem O:150', Gaz. Mat. (Bucharest) 85 (1980), 325.

7.32. $\Sigma R_2R_3\Sigma R_1 > \Sigma a^2R_1$.

Remark. This inequality is a simple consequence of (2) in 1.1.
C. Cîrtoaje, 'Problem 10732', Gaz. Mat. (Bucharest) B 21 (1970), 682.

7.33. $\Pi R_1 \geq 4 \dfrac{R}{r} \Pi r_1, \quad \Pi R_1 \geq \dfrac{R}{2r} \Pi(r_2 + r_3)$.

In each case there is equality if and only if $P = I$.

Proof. Since $\Pi \sin \dfrac{A}{2} = \dfrac{r}{4R}$ these inequalities are equivalent to GI 12.26 and 12.28.

The first inequality is given in:
'Problem 2920', Mat. v škole 1985, No. 6, 64.

7.34. The straight line through the incentre of a triangle ABC passes through the points P and L on sides AC and CB respectively. Then

$$CP + CL \geq 2ab/s.$$

L. Ljubenov, 'Problem 2', Obučenieto po Matematika 1984, No. 1, 59.

Chapter XII

INEQUALITIES WITH SEVERAL TRIANGLES

1. Inequalities Related to Two Triangles Inscribed One in the Other

1.1. Let ABC be a triangle. Let D be a point between B and C, let E be a point between C and A, and let F be a point between A and B. Denote the areas of triangles DEF, AEF, BFD, CDE by G, F_1, F_2, F_3, respectively, and assume without loss of generality that $F_1 \leq F_2 \leq F_3$.

J. F. Rigby proved the following results:

(a) $G^3 + (F_1 + F_2 + F_3)G^2 - 4F_1F_2F_3 \geq 0$

with equality if and only if AD, BE and CF are concurrent.

Proof. Let BC, CA and AB be divided at D, E and F respectively in ratios $x:x'$, $y:y'$ and $z:z'$, where $x + x' = y + y' = z + z' = 1$, and let ABC have area F_0. Then $F_1 = y'zF_0$, $F_2 = z'xF_0$, $F_3 = x'yF_0$, and $G = (1 - y'z - z'x - x'y)F_0 = (xyz + (1 - z + yz - x + zx - y + xy - xyz))F_0 = (xyz + x'y'z')F_0$. Hence

$$G^3 + (\Sigma F_1)G^2 - 4\Pi F_1 = G^2F_0 - 4\Pi F_1 = (xyz - x'y'z')^2F_0^3 \geq 0.$$

Equality occurs if and only if $\frac{x}{x'}\frac{y}{y'}\frac{z}{z'} = 1$, i.e. if AD, BE and CF are concurrent, by Ceva's theorem.

(b) If G, F_1, F_2, F_3 are any positive real numbers satisfying the inequality (a), and if ABC is any triangle with area $G + F_1 + F_2 + F_3$, then there exist just k different positions for an inscribed triangle DEF such that the triangles DEF, AEF, BFD and CDE have areas G, F_1, F_2, F_3, where $k = 1$ or 2 according as we have equality or strict inequality in (a).

(c) $G \geq \frac{1}{2}(\sqrt{(F_1^2 + 8F_1F_2)} - F_1)$.

Proof. (a) is equivalent to

$$F_3(4F_1F_2 - G^2) \leq G^3 + (F_1 + F_2)G^2.$$

Hence, if $4F_1F_2 - G^2 \geq 0$,

$$F_2(4F_1F_2 - G^2) \leq G^3 + (F_1 + F_2)G^2.$$

This last inequality is trivially true if $4F_1F_2 - G^2 < 0$, and hence in all cases

$$G^3 + (F_1 + 2F_2)G^2 - 4F_1F_2^2 \geq 0,$$

i.e.

$$(G + 2F_2)(G^2 + F_1G - 2F_1F_2) \geq 0,$$

so that

$$G^2 + F_1G - 2F_1F_2 \geq 0.$$

Hence

$$(2G + F_1)^2 \geq (F_1^2 + 8F_1F_2),$$

from which we easily deduce that

$$G \geq \frac{1}{2}(\sqrt{(F_1^2 + 8F_1F_2)} - F_1).$$

(d) Let G, F_1 and F_2 be any positive real numbers such that $F_2 \geq F_1$ and $G \geq \frac{1}{2}(\sqrt{(F_1^2 + 8F_1F_2)} - F_1)$. If (i) $G^2 \geq 4F_1F_2$ and F_3 is any number such that $F_1 \leq F_2 \leq F_3$, or if (ii) $G^2 < 4F_1F_2$ and F_3 is any number such that $F_1 \leq F_2 \leq F_3 \leq (G^3 + G^2(F_1 + F_2))/(4F_1F_2 - G^2)$, then there exists a triangle ABC and an inscribed triangle DEF such that the triangles DEF, AEF, BFD and CDE have areas G, F_1, F_2 and F_3 (where $F_1 \leq F_2 \leq F_3$).

Remark. Inequality (c) is better than inequalities $G \geq F_1$ and $G \geq \sqrt{F_1F_2}$ (see GI 9.1).

O. Bottema proved the following result:

(e) In a triangle ABC with area F_0 we can inscribe a triangle DEF such that the remaining triangles have given areas F_1, F_2, F_3 if and only if

$$F_0 \geq F_M \equiv 2M(1 + \cos(\frac{1}{3} \arccos(\frac{2P}{M^3} - 1))),$$

where $3M = F_1 + F_2 + F_3$, $P = F_1F_2F_3$.

Remark. For a fixed value of M the minimum F_M is an increasing function of P; as $0 < P \leq M^3$, we have

$$3M < F_M \leqslant 4M.$$

J. F. Rigby, 'Inequalities Concerning the Areas Obtained when One Triangle is Inscribed in Another', Math. Mag. 45 (1972), 113-116.
O. Bottema, 'A Theorem on an Inscribed Triangle', Math. Mag. 47 (1974), 34-36.

1.2. If F_1 is the area of the least triangle with given angles inscribed in ABC, then

$$F_1 = \frac{1}{2}\frac{F}{K+1},$$

where $K = (\Sigma d_0^2(b^2 + c^2 - a^2))/(16FF_0) \geqslant 1$ (by GI 10.8), with equality if and only if ABC and $D_0E_0F_0$ are similar. Here $D_0E_0F_0$ and inscribed triangle DEF are similar.

Remark. This result is due to A. Oppenheim, and it is a generalization of GI 9.17.

1.3. Let AA_1, BB_1, CC_1 be concurrent cevians of a triangle ABC. If F_1 is the area of $A_1B_1C_1$ (A_1, B_1, C_1, on the sides BC, CA, AB, respectively), then

$$F_1 \leqslant \frac{1}{4} F,$$

with equality if and only if A_1, B_1, C_1 are the midpoints of the sides.

Proof. (C. Tănăsescu) If $x = BA_1:A_1C$, $y = CB_1:B_1A$, $z = AC_1:C_1B$, then $xyz = 1$ and

$$F_1 = \frac{2}{\Pi(x+1)} F.$$

So the result follows from $x + 1 \geqslant 2\sqrt{x}$, etc.

This is an elementary, but crucial property. Although it appears as Problem M 805* in 'Kvant', 1983, No. 8, 52-53, it seems more plausible to consider this as an old, traditional result. Many special problems are manifestations of the above principle. Here we shall give the following result:

If AA_1, BB_1, CC_1 are the internal angle bisectors, then

$$F_1 = \frac{2abc}{\Pi(b+c)} F, \tag{1}$$

so we get as a result

$$\frac{3abc}{4\Sigma a^3} \leqslant \frac{F_1}{F} \leqslant \frac{1}{4}.$$

J. T. Groenman and M. S. Klamkin, 'Problem 762', Crux Math. 8 (1982), 209, 278 and 9 (1983), 249-250.

Remark. Using (1) we get

$$F_1 = \frac{4RrF}{s^2 + r^2 + 2Rr},$$

so the following inequalities are also valid

$$\frac{4Rr}{2R^2 + 3Rr + 2r^2} \leqslant \frac{F_1}{F} \leqslant \frac{2R}{9R - 2r} \leqslant \frac{1}{4}.$$

Similarly, for the orthic triangle of an acute triangle we have

$$F_1 = \frac{2\Pi \tan A}{\Pi(\tan B + \tan C)} F = \frac{s^2 - (2R + r)^2}{2R^2} F,$$

i.e.

$$F_1 \leqslant 4F^3/27R^4 \leqslant \sqrt{3}F^2/9R^2.$$

For two similar results see 1.14 and 1.16.

1.4. Let ABC be an arbitrary triangle and let P denote a point in the interior of ABC. Let D, E, F denote the feet of the perpendiculars from P to BC, CA, AB, respectively. Let F_0 and R_0 denote the area and the circumradius of DEF, respectively. Then the following results were proved by L. Carlitz:

1) $2FF_0R \geqslant r_1r_2r_3s^2$

with equality if and only if P is the incentre of ABC.

2) $R_1R_2R_3F = 8F_0R_0R^2$.

3) $R_1R_2R_3r^2 \geqslant 4R_0Rr_1r_2r_3$

with equality if and only if P is the incentre of ABC.

4) $16F_0^2R_0R^3 \geqslant R_1R_2R_3r_1r_2r_3s^2$

with equality if and only if P is the incentre of ABC.

5) $2F_0^2R_0R^3 \geqslant (r_1r_2r_3s)^2$

with equality if and only if ABC is equilateral and P is the incentre.

6) $$R_1R_2R_3 \leqslant 2R_0R^2$$

with equality if and only if P is the circumcentre of ABC.

7) $$R_1R_2R_3r_1r_2r_3 \leqslant r^2R_0R^3$$

with equality if and only if ABC is equilateral and P is the incentre.

L. Carlitz, 'Some Inequalities for the Triangle', Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 357-380 (1971), 7-10.

1.5. Let P be an interior point of an acute triangle ABC, with pedal triangle DEF. Then

$$\Sigma R_1 \geqslant 6r \geqslant \frac{2}{\sqrt{3}} \Sigma DE$$

with equality if and only if ABC is equilateral and P is its centroid.

J. Garfunkel and L. Bankoff, 'Problem E 2038', Amer. Math. Monthly 74 (1967), 1262 and 76 (1969), 87.

1.6. Let DEF be a triangle inscribed in ABC (with area F_0). Then

$$\Sigma DE^2 \geqslant 12R^2 \frac{\Pi \sin^2 A}{\Sigma \sin^2 A} = 12F_0^2/(\Sigma a^2)$$

with equality if and only if DEF is the pedal triangle of the Lhuilier-Lemoine point.

E. Egerváry, 'Über ein Minimumproblem der Elementargeometrie', J. reine angew. Math. 178 (1938), 174-186.

1.7. Let m = max (DE, EF, FD), where DEF is a triangle inscribed in ABC. If A, B, C < 2π/3, then

$$m \geqslant 2R\left(\frac{\Pi \sin A}{\Sigma \cotan A + \sqrt{3}}\right)^{1/2}$$

with equality if and only if DEF is the pedal triangle of the isodynamic point P, i.e. if and only if for this point $aR_1 = bR_2 = cR_3$.

If A ⩾ 2π/3, then

$$m \geqslant 2R \frac{\Pi \sin A}{\sin B + \sin C} = \frac{2F_0}{b + c} \quad (F_0 \text{ is area of ABC}),$$

with equality if and only if DEF is the pedal triangle of the mid-point A_1 of the side BC.

E. Egerváry, The same reference as in 1.6.

1.8. Let ABC be a given triangle and let DEF be the pedal triangle of a point M inside of triangle ABC, with the sides a_1, b_1, c_1. For an arbitrary point P inside of triangle ABC, let $R_1 = AP$, $R_2 = BP$, $R_3 = CP$. Then $\max\left(\frac{aR_1}{a_1}, \frac{bR_2}{b_1}, \frac{cR_3}{c_1}\right)$ has a minimum if and only if P = M. Also, Σa_1R_1 has a minimum if and only if P is an isogonal point to the point M with respect to the triangle ABC.

E. Egerváry, The same reference as in 1.6.

1.9. If p_a, p_b, p_c denote the lengths of the perpendiculars from the vertices B, C, A to the medians m_a, m_b, m_c, respectively, of an acute triangle ABC, then Σp_a is not less than the sum of the sides of the triangle of minimum perimeter which can be inscribed in ABC.

J. Garfunkel and L. Bankoff, 'Problem E 2037', Amer. Math. Monthly **74** (1967), 1262 and **76** (1969), 86.

1.10. Let L, M, N be the feet of the altitudes of triangle ABC and let F_1 be the area of triangle LMN. Then

$$\Sigma BL \cdot CL \geqslant 4\sqrt{3}F_1.$$

Proof. Since quadrilateral BCMN has a circumscribed circle, we have

$$BL \cdot CL = LM \cdot LN, \tag{1}$$

and similarly

$$CM \cdot AM = LM \cdot MN, \quad AN \cdot BN = LN \cdot MN. \tag{2}$$

If A_1, B_1, C_1 are the angles of triangle LMN, then by (1) and (2) we obtain

$$\Sigma BL \cdot CL = 2F_1 \Sigma \operatorname{cosec} A_1 \geqslant 4F_1\sqrt{3},$$

since

$$\Sigma \operatorname{cosec} A_1 \geqslant 2\sqrt{3}.$$

G. Kalajdžić, 'Some Inequalities for the Triangle', Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. **247-273** (1969), 171-174.

Remark. The following inequality is also valid:

$$\Sigma a/HL \leqslant 3\sqrt{3}F/F_1.$$

R. Somesan, 'Problem 20401', Gaz. Mat. (Bucharest) **90** (1985), 124 and **91** (1986), 128.

1.11. With the notation as in 1.10, the perimeter of the triangle LMN is greater than the chord of triangle ABC through the orthocentre H parallel to LM.

This result is due to G. D. Chakerian and M. S. Klamkin.

1.12. Let D, E, F be points on BC, CA and AB, respectively, and let L and L_1 be the perimeters of triangles ABC and DEF, respectively. If

$$AF:FB = BD:DC = CE:EA = 1:k \quad (k > 1)$$

then

$$\frac{k^2 - k + 1}{(k + 1)k} < \frac{L_1}{L} < \frac{k}{k + 1} .$$

Both bounds are best possible.

This is a result of W. Janous, and it is a generalization and refinement of a result from:

V. Turčaninov, 'Problem M 692', Kvant 1981, No. 7, 19 and 1982, No. 3, 31-32.

1.13. Given are a triangle ABC, its centroid G, and the pedal triangle PQR of its incentre I. The segments AI, BI, CI meet the incircle in U, V, W. Let F_1, F_2 be the areas of triangles PQR and UVW. Then

$$F_1 \leqslant F_2.$$

J. Garfunkel and M. S. Klamkin, 'Problem 648', Crux Math. 7 (1981), 178 and 8 (1982), 180-181.

1.14. In the triangle ABC let $P \in AB$, $M \in BC$, $N \in CA$ be such that the lines through A and M, B and N and C and P divide the circumference of ABC in two equal parts. Then, area (MNP) $\leqslant \frac{1}{4}$ area (ABC).

We were informed of this result by W. Janous.

1.15. In the triangle ABC let M, N, P be the midpoints of AB, BC and CA, respectively. Furthermore, let $U \in MB$, $V \in NC$, $W \in PA$. Then

$$\text{area (UVW)} \geqslant \text{area (MNP)}$$

L. Panaitopol, Gaz. Mat. (Bucharest) 85 (1980), 41 and 434.

1.16. Let P be in the plane of ABC (not a vertex), PU internal bisector of BPC, U on BC, so for V, W. Then

$$\text{area (UVW)} \leqslant \frac{1}{4} F$$

with equality if and only if P is the circumcentre.

Proof. We have in fact

$$\text{area (UVW)} = \frac{2\Pi R_1}{\Pi(R_2 + R_3)} \cdot F \leqslant \frac{1}{4} F.$$

This result is due to A. Oppenheim.

1.17. Equilateral triangle ABC is inscribed in triangle XYZ, with A between Y and Z, B between Z and X, and C between X and Y. Show that $XA + YB + ZC < XY + YZ + ZX$. Is equality possible?

C. P. Popescu, 'Problem E 3091', Amer. Math. Monthly 92 (1985), 360.

1.18. Let P be an interior point of a triangle ABC, with pedal triangle DEF. Then

$$\Sigma(r_2 + r_3)\cos\frac{A}{2} \leqslant \Sigma EF.$$

'Problem E 2517', Amer. Math. Monthly 83 (1976).

1.19. Let A_1, B_1, C_1 be points on ths sides BC, CA, AB of the triangle ABC, respectively, and a_1, b_1, c_1 the sides of the triangle $A_1B_1C_1$. If F is the area of ABC, then

$$\Sigma a^2 b_1 c_1 \geqslant 4F^2.$$

G. A. Tsintsifas, 'Problem E 3154', Amer. Math. Monthly 93 (1986), 482.

1.20. Conjecture: Let three points P, Q, R be on the sides AC, BC, AB of a triangle ABC, respectively. If $PA + AR = RB + BQ = QC + CP$, then

$$PR + RQ + QP \geqslant s. \quad \{E\}$$

This conjecture is due to Tao Maoqi and Zhang Jingzhong. They provided substantial numerical evidence in support of their result.

Private communication.

2. Some Other Inequalities with Triangles Connected to the Given Triangle

2.1. Let AI, BI, CI cut the circumcircle of the triangle $\Delta = ABC$ in A', B', C', respectively. It is easily verified that the angles of $\Delta' = A'B'C'$ are $\frac{\pi - A}{2}$, $\frac{\pi - B}{2}$, $\frac{\pi - C}{2}$. We may call Δ' the first derived triangle of ABC, and similarly the n-th derived triangle $\Delta^{(n)}$ of ABC recursively as the first derived triangle of $\Delta^{(n-1)}$. If $A^{(n)}$, $B^{(n)}$, $C^{(n)}$ are the angles of $\Delta^{(n)}$, we have

$$A^{(n)} = \frac{2^n - (-1)^n}{3 \cdot 2^n} \pi + \frac{(-1)^n}{2^n} A, \quad \text{etc.}$$

All the triangles $\Delta^{(n)}$ have a common circumcircle, namely the circumcircle of Δ, and the following inequalities are valid:

$$(1)\quad r \leqslant r', \qquad (2)\quad F \leqslant F', \qquad (3)\quad s \leqslant s',$$

$$(4)\quad r_a \leqslant r'_a \quad (A \leqslant \tfrac{\pi}{3}), \qquad (5)\quad \Sigma A'I \leqslant \Sigma A''I'.$$

For each of the first three inequalities there is equality if and only if the triangle Δ is equilateral.

L. Carlitz, 'Some Inequalities Related to Euler's Theorem $R \geqslant 2r$', Publ. Inst. Math. N. S. **12** (26) (1971), 11-17.

2.2. Let ABC be a triangle with centroid G inscribed in a circle with centre O. A point M lies on the disk ω with diameter OG. The lines AM, BM, CM meet the circle again in A', B', C', respectively. Then

$$\text{area (ABC)} \leqslant \text{area (A'B'C')}.$$

G. Tsintsifas, 'Problem 743', Crux Math. **8** (1982), 135 and **9** (1983), 182-184.

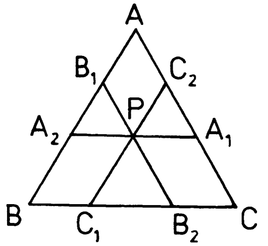
2.3. Let a triangle ABC be given and M be a point of its interior. The lines through A and M, B and M, C and M shall intersect the respective sides in A_1, B_1 and C_1. Let $s_1, s_2, \ldots, s_6$ be the areas of triangles MA_1B, MA_1C, MB_1C, MB_1A, MC_1A and MC_1B, respectively. Then

$$\frac{s_1}{s_2} + \frac{s_3}{s_4} + \frac{s_5}{s_6} \geqslant 3$$

with equality if and only if M is the centroid of ABC.

We were informed of this result by W. Janous.

2.4. P is an interior point of a triangle ABC. Lines through P parallel to the sides of the triangle meet those sides in the points A_1, A_2, B_1, B_2, C_1, C_2 as shown in the figure. The areas of the triangles A_1C_2P, etc. are s_1, s_2 and s_3, and the areas of the parallelograms AB_1PC_2, etc. are S_1, S_2, S_3.



Then the following inequalities are valid:

(a) $9M_s(s_1, s_2, s_3) \leqslant F \leqslant 9M_r(s_1, s_2, s_3)$

where $s \leqslant 1/2 \leqslant r$.

(b) (Gelă)

$$S_1S_2S_3 = 8s_1s_2s_3; \quad \frac{1}{2}\sqrt[3]{S_1S_2S_3} + 2\sqrt[3]{s_1s_2s_3} \leqslant \frac{1}{3}F.$$

(c) (Tsintsifas)

$$3 \cdot \text{area}\,(A_1B_1C_1) \leqslant \text{area}\,(ABC);$$

$$3 \cdot \text{area}\,(A_1C_2B_1A_2C_1B_2) \geqslant 2 \cdot \text{area}\,(ABC).$$

Remarks. (a) is a simple generalization of a result of Gerasimov and Kalajdžić.

(b) If $P = I$, then $A_1C_2 + B_1A_2 + C_1B_2 \geqslant (a + b + c)/3$. This is a result from:

S. Iwata, Encyclopaedia of Geometry (Japanese), Tokyo, 1971, Vol. 5, p. 344.

I. Gerasimov, 'Problem 701', Mat. v Škole 1969, No. 6, 65 and 1970, No. 4, 86.

G. Kalajdžić, 'Some Inequalities for the Triangle', Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 247-273 (1969), 171-174.

R. Gelă, 'Problem 19955*', Gaz. Mat. (Bucharest) 88 (1983), 464 and 89 (1984), 373.

G. Tsintsifas and S. Wagon, 'Problem 862', Crux Math. 9 (1983), 208 and 10 (1984), 322-323.

2.5. In a triangle ABC the tangents to the incircle and parallel to the sides of the triangle are drawn. They generate three small triangles with areas F_1, F_2 and F_3. Then

$$F_1 + F_2 + F_3 \geqslant \frac{1}{3}F.$$

Proof. (W. Janous) All triangles are similar, so we have

$$\frac{F_1}{F} = \left(\frac{h_c - 2r}{h_c}\right)^2 = (1 - c/s)^2, \quad \text{etc.}$$

Since

$$\left(\Sigma\,\frac{1}{3}\,\frac{F_1}{F}\right)^{1/2} = \left(\frac{1}{3}\,\Sigma(1 - a/s)^2\right)^{1/2} \geqslant \frac{1}{3}\,\Sigma(1 - a/s) = 1/3,$$

we get the desired inequality.

2.6. Let P be an interior point of a triangle ABC and let AA_1 and BB_1 be two of its cevians. If $F \in AB_1$ and $G \in BC$ such that F, P and G lie on the same line, then

$$\text{area } (B_1PA_1) \leqslant \frac{1}{4} \text{ area } (FGC).$$

N. Nenov and J. Tabov, 'Problem 2', Matematika (Sofija) 1979, No. 1, 35 and 1979, No. 4, 35-36.

2.7. The points, D, E, F lie on the hypotenuse AB of a right triangle ABC such that CD is normal to AB, CE is the angle-bisector of C and CF is the angle-bisector of the angle BCD. Then

$$F \geqslant (\sqrt{2} + 1) \cdot \text{area } (CEF).$$

E. A. Bokov, 'Problem 1163', Mat. v Škole 1973, No. 1, 75 and 1973, No. 5, 87.

2.8. Let in the exterior of a triangle ABC be constructed a triangle $A_1B_1C_1$ (with area F_1), the sides of which are a distance d apart from the sides of ABC. Then

$$F_1 = (d + r)^2 \sum \cot\text{an} \frac{A}{2} \quad \text{and} \quad F_1 - F \geqslant 3\sqrt{3}\, d^2.$$

N. Teodorescu, 'Problem 3', Gaz. Mat. (Bucharest) 89 (1984), 409-411.

2.9. Given a triangle ABC with its inscribed circle (I), lines AI, BI, CI cut the circle in points D, E and F, respectively. Then

$$\Sigma AB \geqslant (\Sigma DE)/\sqrt{3}.$$

J. Garfunkel and W. Dodge, 'Problem 368', Pi Mu Epsilon J. 6 (1976), 227 and 6 (1977), 376.

2.10. Let a triangle ABC be given. Let midpoints of its sides BC, CA, AB be A_1, B_1, C_1, respectively. Let the angle-bisectors of the triangle $A_1B_1C_1$ meet the sides of the triangle ABC in points M, N, P, respectively. If $BC < CA < AB$, then

$$\Sigma BM = 2s \quad \text{and} \quad \Sigma A_1M \geqslant 3\sqrt[3]{s^2 r}.$$

D. Cvetkov, 'Problem 4', Ob. po Matematika (Sofija) 1980, No. 1, 41 and 1980, No. 5, 46-47.

2.11. A_1 lies on the side BC of a triangle ABC. Let the radii of incircles of triangles ABA_1 and AA_1C be equal. Then

$$w_a \leqslant AA_1 \leqslant m_a.$$

M. Marčev, 'Problem 1', Matematika (Sofija) 1976, No. 5, 34 and 1977, No. 2, 37.

2.12. Let I be the incentre of a triangle ABC, and let R, R_1, R_2, R_3 be the radii of circumcircles of triangles ABC, BIC, CIA and AIB, respectively. Then

$$3R^2 \leqslant \Sigma R_1^2 < 4R^2.$$

Proof. Using the sine law we get $a = 2R_1 \sin(\pi - \frac{B + C}{2}) = 2R \sin A$, i.e. $2R_1 \cos \frac{A}{2} = 4R \sin \frac{A}{2} \cos \frac{A}{2}$. Hence $R_1 = 2R \sin \frac{A}{2}$, etc. and

$$\Sigma R_1^2 = 4R^2 \Sigma \sin^2 \frac{A}{2} = 2R(2R - r).$$

The above inequalities are now a simple consequence of this identity.

Remark. The first inequality was proved by K. V. Vetrov. Further, since for non-acute triangles $r \leqslant (\sqrt{2} - 1)R$, we have

$$\Sigma R_1^2 \geqslant 2(3 - \sqrt{2})R^2.$$

K. V. Vetrov, 'Problem 419', Mat. v Škole 1967, No. 6, 77 and 1968, No. 4, 81-82.

2.13. Let P be a point inside a non-obtuse triangle ABC, and let D, E, F be the points situated symmetrically to P with respect to the sides BC, CA, AB. Then the following generalization of GI 10.4 is valid

(1) $\quad$ area DEF $\leqslant$ area ABC.

If ABC is equilateral and s' is the semi-perimeter of DEF, then

(2) $\quad 2s/\sqrt{3} \geqslant s' \geqslant s.$

The left-hand-side equality occurs if and only if the point P is located at one of the vertices of ABC. The right-hand-side equality occurs if and only if P is located at the centroid of ABC.

M. S. Klamkin, 'Notes on Inequalities Involving Triangles or Tetrahedrons', Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 330-337 (1970), 1-15.

2.14. Let the side BC of a triangle ABC be subdivided using the points $B = P_0, P_1, \ldots, P_{n-1}, P_n = C$ in order. If r_i is the inradius of triangle $AP_{i-1}P_i$ for $i = 1, \ldots, n$, then

$$r_1 + \ldots + r_n < \frac{1}{2} h_a \log \frac{s}{s - a}.$$

Proof (V. Mascioni). We prove the following refinement:

(1) $$r_1 + \ldots + r_n \leqslant \frac{1}{2} h_a \cdot n\left(1 - \left(\frac{s - a}{s}\right)^{1/n}\right),$$

with equality if and only if $r_1 = \ldots = r_n$.

Define $F(x, y) := \frac{x - y}{x + y}$ and note that

(2) $$F(r + s, t)F(r + u, v) = F(s + u, t + v)$$

is equivalent to

(3) $$(r^2 + v^2 - u^2)t + (r^2 + t^2 - s^2)v = 0.$$

If we consider the following figure,

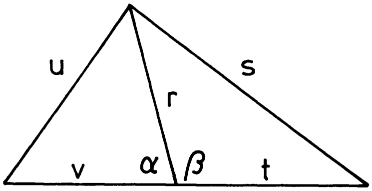


Fig. 1.

we find that (3) holds true (by the Theorem of Carnot, we have that (3) is equivalent to $\cos \alpha + \cos \beta = 0$, which is evidently satisfied since $\alpha + \beta = \pi$), hence (2) is true in this case, too. As regards Demir's inequality, we obtain thus (with complete induction)

(4) $$\frac{s - a}{a} = \prod_{i=1}^{n} \frac{s_i - a_i}{s_i},$$

where the index $\underline{i}$ refers to triangle $AP_{i-1}P_i$. Now, the geometric-arithmetic-mean inequality gives

$$\left(\frac{s - a}{s}\right)^{1/n} \leqslant \frac{1}{n} \sum_{i=1}^{n} \left(\frac{s_i - a_i}{s_i}\right) = 1 - \frac{2}{nh_a} \sum_{i=1}^{n} r_i,$$

which is nothing but (1), and which is a refinement of Demir's inequality since $1 - \left(\frac{s - a}{s}\right)^{1/n} < \frac{1}{n} \log \frac{s}{s - a}$ (following from $1 - \frac{1}{x} < \log x$ for $x > 1$).

H. Demir, 'Problem 1206', Math. Mag. 58 (1985), 46.

2.15. Let P be an interior or boundary point of a triangle ABC. If the triangles PBC, PCA, PAB have the inradii ρ_a, ρ_b, ρ_c, respectively, then

$$r < \Sigma\rho_a < \min (a, b, c).$$

H. Demir, 'Incircles within', Math. Mag. 59 (1986), 77-83.

2.16. $3\sqrt{3}Rr \leqslant \text{area } (I_aI_bI_c) \leqslant 4R^2 + 2Rr(3\sqrt{3} - 4)$. {E}

2.17. Let T_0 = ABC be a given triangle. Consider a sequence of triangles T_k (k = 1, 2, ...) where the sides a_k, b_k, c_k of T_k are equal to the exterior angle-bisectors of T_{n-1}. Let F_k be the area of T_k and $Q_k = \Sigma(b_k - a_k)^2$. Then

$$\Sigma a^2 \geqslant 4F\sqrt{3} + Q + F \sum_{k=1}^{n} Q_k/F_k.$$

This inequality is due to G. R. Veldkamp, and it is a refinement of an inequality of P. Finsler and H. Hadwiger (see GI 4.7).

2.18. Let D, E, F be the centroids of the equilateral triangles constructed outwardly on the sides of an arbitrary triangle ABC. Then

$$\text{area (DEF)} \geqslant \text{area (ABC)}.$$

A. Ermilov, Kvant 1977, No. 5, 16.

2.19. Let, on the elongations of the side BC of a triangle ABC, be projected the lengths CA' = BA'' = b + c, on elongations of the side AC: AB' = CB'' = a + c and on the elongations of the side AB:BC' = AC'' = a + b. If F is the area of the triangle ABC and F_1, F_2, F_3, F_1', F_2', F_3' are areas of the triangles CA'B', AB'C', BC'A', BA''C'', AB''C'', CA''B'', respectively, then

$$\text{(a)} \quad \frac{1}{F} = \Sigma\left(\frac{1}{F_1} + \frac{1}{F_1'}\right), \qquad \text{(b)} \quad \Pi\left(\frac{1}{F_1} + \frac{1}{F_1'}\right) \leqslant \frac{1}{27F^3}.$$

Remark. This is a result of I. Georgescu. Of course, (b) is a simple consequence of (a) and the geometric-arithmetic mean inequality. Therefore, using (a) and Jensen's inequality for convex functions we can give a generalization of (b) (see Section VIII.1).

2.20. Let be given a triangle ABC of area F. Three lines parallel to the sides of the triangle and intersecting them give the new triangle MNK of area F'. The sides of these triangles form six new triangles of areas $F_1, F_2, \ldots, F_6$. Then

$$9M_0(F_1, \ldots, F_6) \leqslant \sqrt{FF'} \leqslant 9M_1(F_1, \ldots, F_6).$$

2.21. Let BCP_1 and P_2CB be the two equilateral triangles constructed on the side BC of an arbitrary triangle ABC. If $\alpha = AP_2$ and $\beta = AP_1$, then

(i) $\alpha\beta \leqslant \Sigma bc$,

(ii) $2s/\sqrt{3} \leqslant \alpha + \beta \leqslant 2s$,

(iii) $\alpha^3 + \beta^3 \geqslant 8\Pi m_a$.

If P is a point in the interior of the triangle ABC, then

(iv) $cPA + aPB + bPC \geqslant \alpha\beta$,

(v) $bPA + cPB + aPC \geqslant \alpha\beta$.

S. Iwata, Encyclopaedia of Geometry (Japanese), Tokyo, 1971, Vol. 5, pp. 343-344 and 347-348.

2.22. Let P be a point in the plane of a triangle ABC. If F_1, F_2, F_3 are areas of the triangles PBC, PCA and PAB, respectively, then

$$\Sigma F_1^2 \leqslant \frac{1}{16}(\Sigma PA^2)^2.$$

V. V. Nákladem, 'Problem E 558', Amer. Math. Monthly 50 (1943), 12 and 567-568.

2.23. If A_1, B_1, C_1 are the second points of intersection of the altitudes and the circumcircle of an acute triangle ABC, then

$$\text{area } A_1B_1C_1 \leqslant \text{area } ABC.$$

S. Iwata, Encyclopaedia of Geometry (Japanese), Tokyo 1971, Vol. 5, p. 355.

3. The Neuberg-Pedoe and the Oppenheim Inequalities

3.0. Introduction

This Section concerns results about the well-known Neuberg-Pedoe and the Oppenheim inequalities for the areas of two triangles. Of course, these inequalities were considered in the book 'Geometric Inequalities' [1], but after the appearance of this book many new results connected with these inequalities appeared.

Firstly, it was noted that Pedoe's result from GI 10.8 was partly proved in 1891 by J. Neuberg, and many authors call this inequality the Neuberg-Pedoe inequality.

Further, A. Oppenheim noted that his inequality from GI 10.12 is equivalent to the Neuberg-Pedoe inequality.

Several papers contain new proofs of the Neuberg-Pedoe inequality. In 3.1. we shall show that the idea of a very simple Carlitz proof can be used in the proof of the Oppenheim inequality, too.

In 3.2. we give the comments of O. Bottema on the mixed area of two triangles. It is interesting that one of his results is connected with a result of L. Carlitz, which we give in 3.3. An extension of the Carlitz results is also given.

Refinements of the Neuberg-Pedoe inequality were given in several papers. We give some new proofs and some extensions of these results. We also give some similar results for the Oppenheim inequality.

Further, we give some generalizations of the Neuberg-Pedoe inequality.

In 3.6. we shall consider some further generalizations of the Oppenheim inequality for triangles, quadrilaterals and tetrahedra. We shall give some extensions of these results.

3.1. The Neuberg-Pedoe and the Oppenheim Inequality

THEOREM A. Let a_i, b_i, c_i denote the sides of the triangle $A_iB_iC_i$ $(i = 1, 2)$ with areas F_i. Then

(1) $$H \geqslant 16F_1F_2$$

where $H = \Sigma a_1^2(-a_2^2 + b_2^2 + c_2^2)$, with equality if and only if the triangles are similar.

This is the well-known Neuberg-Pedoe inequality (see [2-6] or GI 10.8).

There exists several proofs of the Neuberg-Pedoe inequality. For some of these proofs see also [7-11].

L. Carlitz [8] gave a very simple proof of this inequality by using the well-known Aczél inequality, i.e. the following special case of the Aczél inequality:

LEMMA 1. Let $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ be two sequences of real numbers, such that

(2) $$a_1^2 - a_2^2 - \ldots - a_n^2 > 0 \quad \text{and} \quad b_1^2 - b_2^2 - \ldots - b_n^2 > 0.$$

Then

(3) $$(a_1^2 - a_2^2 - \ldots - a_n^2)(b_1^2 - b_2^2 - \ldots - b_n^2) \leqslant$$
$$\leqslant (a_1b_1 - a_2b_2 - \ldots a_nb_n)^2$$

with equality if and only if the sequences a and b are proportional.

A. Oppenheim [12] (see also GI 10.12) gave the following result:

THEOREM B. Suppose that $A_iB_iC_i$ $(i = 1, 2)$ are triangles with sides a_i, b_i, c_i and areas F_i. Define numbers a_3, b_3, c_3 by $a_3 = (a_1^2 + a_2^2)^{1/2}$, etc. then a_3, b_3, c_3 are the sides of a triangle with area F_3, and the following inequality is valid

(4) $$F_3 \geqslant F_1 + F_2,$$

with equality if and only if the triangles are similar.

Remark. Note that a necessary and sufficient condition for triangularity is $F^2 > 0$.

Now we shall show that the Oppenheim inequality is a simple consequence of the well-known Bellman inequality [13] (see also [14, p. 38] or [15, p. 58]), i.e. of the following consequence of Bellman's inequality:

LEMMA 2. If a and b are sequences of non-negative real numbers which satisfy (2), then

$$(5)\qquad (a_1^2 - a_2^2 - \dots - a_n^2)^{1/2} + (b_1^2 - b_2^2 - \dots - b_n^2)^{1/2} \leqslant$$

$$\leqslant ((a_1 + b_1)^2 - (a_2 + b_2)^2 - \dots - (a_n + b_n)^2)^{1/2}$$

with equality if and only if the sequences a and b are proportional.

Proof of Theorem B. To prove (4) we take

$$4F_1 = ((a_1^2 + b_1^2 + c_1^2)^2 - 2(a_1^4 + b_1^4 + c_1^4))^{1/2}$$

and similarly for F_2 and F_3. Now put

$$a_1 \to a_1^2 + b_1^2 + c_1^2, \quad a_2 \to 2^{1/2}a_1^2, \quad a_3 \to 2^{1/2}b_1^2,$$

$$a_4 \to 2^{1/2}c_1^2, \quad b_1 \to a_2^2 + b_2^2 + c_2^2, \quad b_2 \to 2^{1/2}a_2^2,$$

$$b_3 \to 2^{1/2}b_2^2, \quad b_4 \to 2^{1/2}c_2^2.$$

Then (2) holds and (5) becomes

$$4F_1 + 4F_2 \leqslant ((\Sigma a_1^2 + \Sigma a_2^2)^2 - 2\Sigma(a_1^2 + a_2^2)^2)^{1/2} =$$

$$= ((\Sigma a_3^2)^2 - 2\Sigma a_3^4)^{1/2} = 4F_3$$

with equality if and only if the triangles are similar.

Remarks. 1° By the same substitution which converts Lemma 2 into Theorem B, one also derives Theorem A from Lemma 1.

2° Oppenheim noted in [16] that Theorems A and B are equivalent. He also noted that Theorem B (and, therefore, Theorem A) is equivalent to GI 14.1. Analogously, we can easily show that Lemmas 1 and 2 are also equivalent.

3° Further, it is known that Lemmas 1 and 2 can be easily proved by using the Cauchy and the Minkowski inequalities, respectively (see [17]). Therefore, we can give a simple proof of Theorems A and B by using these inequalities, too.

4° A generalization to several dimensions of the Neuberg-Pedoe inequality is given in [18]. The same is also valid for the Oppenheim inequality (see 3.6. of this Section or [16]). A generalization for two n-gons of the Neuberg-Pedoe inequality is given in [19].

3.2. Comment by O. Bottema: On the Mixed Area of Two Triangles

3.2.1. For the area F_1 of the triangle with sides a_1, b_1, c_1 and the angles A_1, B_1, C_1 we have

$$16F_1^2 = -a_1^4 - b_1^4 - c_1^4 + 2b_1^2c_1^2 + 2c_1^2a_1^2 + 2a_1^2b_1^2,$$

a homogeneous quadratic form of a_1^2, b_1^2, c_1^2. For a second triangle with sides a_2, b_2, c_2 the analogous formula holds. We consider the expression

$$16F_{12}^2 = -a_1^2a_2^2 - b_1^2b_2^2 - c_1^2c_2^2 + (b_1^2c_2^2 + b_2^2c_1^2) + + (c_1^2a_2^2 + c_2^2a_1^2) + (a_1^2b_2^2 + a_2^2b_1^2), \tag{6}$$

that is the 'polar form' of the two quadratics. We shall call F_{12} the 'mixed area' of the two triangles. We have

$$8F_{12}^2 = b_1^2c_2^2 + b_2^2c_1^2 - 2b_1c_2b_2c_1 \cos A_1 \cos A_2. \tag{7}$$

Hence $F_{12}^2 > 0$: the mixed area is a real number.

Obviously, $F_{12} = F_{21}$, $F_{11} = F_1$, $F_{22} = F_2$.

Furthermore,

$$8(F_{12}^2 - F_{11}F_{22}) = b_1^2c_2^2 + b_2^2c_1^2 - 2b_1c_2b_2c_1 \cos(A_1 - A_2) \tag{8}$$

and therefore

$$F_{12}^2 \geqslant F_{11}F_{22},$$

that is the Neuberg-Pedoe inequality.

3.2.2. Bottema met the concepts in 3.2.1. dealing with the following problems. In a plane two triangles $A_1B_1C_1$ and $A_2B_2C_2$ are given. Question: is it possible to place them in such a way that A_1A_2, B_1B_2, C_1C_2 are parallel?
Answer: it is possible if and only if

$$2F_{12}^2 \geqslant F_{11}^2 + F_{22}^2. \tag{9}$$

The same condition holds for two triangles in space if one is a parallel projection of the other.

A second problem: once more two triangles $A_1B_1C_1$ and $A_2B_2C_2$ are given. Is it possible to place $A_2B_2C_2$ in such a way that it is an inscribed triangle of $A_1B_1C_1$? (A_2, B_2, C_2 on the corresponding sides - if necessary extended - of $A_1B_1C_1$)?

Answer: the necessary and sufficient condition reads

$$2F_{12}^2 + 2F_1F_2 - F_1^2 \geq 0.$$

The above lines are a summary of two papers: [20] and [21].

3.3. On a Result of Carlitz

L. Carlitz [22] proved the following result:

THEOREM C. If the differences

$$(10) \qquad a_1^2 - a_2^2, \quad b_1^2 - b_2^2, \quad c_1^2 - c_2^2$$

are all positive or all negative and in addition the numbers

$$(11) \qquad |a_1^2 - a_2^2|^{1/2}, \quad |b_1^2 - b_2^2|^{1/2}, \quad |c_1^2 - c_2^2|^{1/2}$$

form the sides of a triangle Δ (possibly degenerate), then

$$(12) \qquad 8(F_1^2 + F_2^2) - H = 8F^2(\Delta),$$

where $F(\Delta)$ denotes the area of Δ. Otherwise,

$$(13) \qquad H \geq 8(F_1^2 + F_2^2).$$

Remark. Inequality (13) is the same as (9), so the above result gives more information on triangles satisfying the first Bottema question from 3.2.2.

Now we shall give an extension of Theorem C, i.e. the following theorem is valid:

THEOREM D. If the differences (10) are all positive or all negative and in addition the numbers (11) form the sides of a triangle Δ (possibly degenerate), then equality (12), and equalities

$$(14) \qquad 2F_1^2 + 2F_2^2 - F_3^2 = F^2(\Delta),$$

and

(15) $$4F_3^2 - H = 4F^2(\Delta)$$

are valid. Otherwise,

(16) $$H > 4F_3^2 > 8(F_1^2 + F_2^2).$$

Proof. The following identity is given in [22]:

(17) $$16(F_1^2 + F_2^2) - 2H = T$$

where

$$T = 2\Sigma(a_1^2 - a_2^2)(b_1^2 - b_2^2) - \Sigma(a_1^2 - a_2^2)^2.$$

Note that the following identity is also valid

(18) $$16(F_1^2 + F_2^2) + 2H = 16F_3^2.$$

So, we have

(19) $$32(F_1^2 + F_2^2) = T + 16F_3^2$$

(20) $$16F_3^2 - 4H = T.$$

Note that if the differences (10) are all positive or all negative and in addition the numbers (11) form the sides of a triangle Δ, then $T = 16F^2(\Delta)$. In this case we have

(21) $$8(F_1^2 + F_2^2) \geqslant 4F_3^2 \geqslant H.$$

Otherwise $T < 0$, so from (19) and (20) we get (16).

Remark. Obviously, (16) is the refinement of (13).

3.4. Sharpening the Neuberg-Pedoe and the Oppenheim Inequality

K. S. Poh [23] proved the following refinement of (1):

THEOREM E. Let conditions of Theorem A be satisfied, and let

$$E = (\Sigma a_1^2)(\Sigma a_2^2) - 2((\Sigma a_1^4)(\Sigma a_2^4))^{1/2}.$$

Then

(22) $$H \geqslant E \geqslant 16F_1F_2,$$

and $E = 16F_1F_2$ if and only if the two triangles are equibrocardian, i.e. if and only if the two triangles have equal Crelle-Brocard angles. Moreover, the following are equivalent

(i) $H = 16F_1F_2$, (ii) $H = E$, (iii) $\Delta A_1B_1C_1 \sim \Delta A_2B_2C_2$.

J. F. Rigby [24] gave a short proof of Theorem C. In the proof of inequality

(23) $$E \geq 16F_1F_2$$

he starts from the fact that $4F_i = \sqrt{X_i^2 - 2Y_i}$, where $X_i = \Sigma a_i^2$, $Y_i = \Sigma a_i^4$ $(i = 1, 2)$.

We shall note that (23) is also a simple consequence of Lemma 1. Indeed, we have

$$16F_1F_2 = \sqrt{X_1^2 - 2Y_1}\sqrt{X_2^2 - 2Y_2} \leq X_1X_2 - 2\sqrt{Y_1Y_2} = E$$

where we used (3) for $n = 2$. Equality occurs if and only if $X_1^2/Y_1 = X_2^2/Y_2$. Since [23]:

$$\Sigma a^4 = 8F^2((\Sigma \text{ cotan } A)^2 - 1) \quad \text{and} \quad \Sigma a^2 = 4F\Sigma \text{ cotan } A,$$

condition for equality is equivalent with

$$(\Sigma \text{ cotan } A_1)^2 = (\Sigma \text{ cotan } A_2)^2,$$

i.e.

(24) $$\Sigma \text{ cotan } A_1 = \Sigma \text{ cotan } A,$$

since Σ cotan $A_i \geq \sqrt{3}$ $(i = 1, 2)$ (GI 2.38).

In the formulation of his theorem, Poh gave equation (24) for $E = 16F_1F_2$. We remark that cotan $\omega = \Sigma$ cotan A, ω being the Crelle-Brocard angle of the triangle, so (24) becomes cotan ω_1 = cotan ω_2, i.e. $\omega_1 = \omega_2$. Therefore, $E = 16F_1F_2$ if and only if the two triangles are equibrocardian.

Further, we shall note that inequalities (22) can be proved by the method of Carlitz, if instead of Lemma 1 we use the following refinement of Aczel's inequality:

LEMMA 3. If the conditions of Lemma 1 are satisfied then

(25) $$(a_1^2 - a_2^2 - \dots - a_n^2)(b_1^2 - b_2^2 - \dots - b_n^2) \leq$$

$$\leq (a_1b_1 - (a_2^2 + \dots + a_1^2)^{1/2}(b_2^2 + \dots + b_n^2)^{1/2})^2 \leq$$

$$\leqslant (a_1b_1 - a_2b_2 - \dots - a_nb_n)^2.$$

Proof. By substitution

$$n = 2, \quad a_1 \to a_1, \quad b_1 \to b_1, \quad a_2 \to (a_2^2 + \dots + a_n^2)^{1/2},$$

$$b_2 \to (b_2^2 + \dots + b_n^2)^{1/2}$$

from Lemma 1 we get the first inequality, and using the Cauchy inequality we get the second inequality.

Of course, we can give the similar extension of Theorem B:

THEOREM F. Let the condition of Theorem B be satisfied. Then

$$(26) \qquad F_3 \geqslant G \geqslant F_1 + F_2$$

where $G = \frac{1}{4}((\Sigma a_3^2)^2 - 2((\Sigma a_1^4)^{1/2} + (\Sigma a_2^4)^{1/2})^2)^{1/2}$, and $F_1 + F_2 = G$ if and only if the two triangles are equibrocardian. Moreover, the following are equivalent

(i) $F_3 = F_1 + F_2$, (ii) $F_3 = G$, (iii) $\Delta A_1B_1C_1 \sim \Delta A_2B_2C_2$.

The proof is similar to the proof of Theorem E. We only use Lemma 2 instead of Lemma 1, for the second inequality, and Minkowski's inequality instead of Cauchy's in the first inequality.

We shall also note that inequalities (26) can be proved if instead of Lemma 2 we use the following result:

LEMMA 4. If the conditions of Lemma 2 are satisfied then

$$(a_1^2 - a_2^2 - \dots - a_n^2)^{1/2} + (b_1^2 - b_2^2 - \dots - b_n^2)^{1/2} \leqslant$$

$$\leqslant ((a_1 + b_1)^2 - ((a_2^2 + \dots + a_n^2)^{1/2} + (b_1^2 + \dots + b_n^2)^{1/2})^2)^{1/2} \leqslant$$

$$\leqslant ((a_1 + b_1)^2 - (a_2 + b_2)^2 - \dots - (a_n + b_n)^2)^{1/2}.$$

Proof is similar to the proof of Lemma 3.

Chia-Kuei Peng [25] proved the following sharpening of the Neuberg-Pedoe inequality:

THEOREM G. Let the conditions of Theorem A be satisfied. Then

$$(27) \qquad H \geqslant 8\left(\frac{S_2}{S_1} F_1^2 + \frac{S_1}{S_2} F_2^2\right) \geqslant 16F_1F_2$$

where $S_1 = \Sigma a_1^2$ and $S_2 = \Sigma a_2^2$.

Here, we shall show that the following result is valid:

THEOREM H. With the same conditions as in previous theorem,

$$H \geqslant E \geqslant 8\left(\frac{S_2}{S_1} F_1^2 + \frac{S_1}{S_2} F_2^2\right) \geqslant 16F_1F_2. \tag{28}$$

Proof. Of course, we need only prove

$$E \geqslant 8\left(\frac{S_2}{S_1} F_1^2 + \frac{S_1}{S_2} F_2^2\right).$$

We have

$$E = S_1S_2 - 2((\Sigma a_1^4)(\Sigma a_2^4))^{1/2} =$$

$$= S_1S_2(1 - 2((\frac{1}{S_1^2} \Sigma a_1^4)(\frac{1}{S_2^2} \Sigma a_2^4))^{1/2}) \geqslant$$

$$\geqslant S_1S_2(1 - \frac{1}{S_1^2} \Sigma a_1^4 - \frac{1}{S_2^2} \Sigma a_2^4) = 8S_1S_2\left(\frac{F_1^2}{S_1^2} + \frac{F_2^2}{S_2^2}\right),$$

where we used the arithmetic-geometric mean inequality and the following formula

$$16F^2 = S^2 - 2\Sigma a^4, \quad \text{i.e.} \quad \Sigma a^4 = \frac{1}{2} S^2 - 8F^2.$$

Gao Ling [26] gave two refinements of the Neuberg-Pedoe inequality. Here we shall give some extensions of his results.

THEOREM I. Let $A_iB_iC_iD_i$ $(i = 1, 2)$ be two quadrilaterals inscribed in circles, let $B_iC_i = a_i$, $C_iD_i = b_i$, $D_iA_i = c_i$ and $A_iB_i = d_i$ $(i = 1, 2)$ and let F_i denote the areas of $A_iB_iC_iD_i$ $(i = 1, 2)$. If

$$K = 4(b_1c_1 + d_1a_1)(b_2c_2 + d_2a_2) -$$

$$- (a_1^2 - b_1^2 - c_1^2 + d_1^2)(a_2^2 - b_2^2 - c_2^2 + d_2^2),$$

then

$$0 \leqslant K - 16F_1F_2 < 8(b_1c_1 + d_1a_1)(b_2c_2 + d_2a_2) \tag{29}$$

with equality if and only if corresponding angles B_1 and B_2 are equal.

Proof. Since angles B_1 and D_1 are supplementary, we have

$$2F_1 = (b_1c_1 + d_1a_1) \sin B_1.$$

On the other hand, from the cosine law

$$A_1^2C_1^2 = b_1^2 + c_1^2 + 2b_1c_1 \cos B_1 = d_1^2 + a_1^2 - 2d_1a_1 \cos B_1$$

and hence

$$a_1^2 - b_1^2 - c_1^2 + d_1^2 = 2(b_1c_1 + d_1a_1) \cos B_1.$$

Using these equalities and similar equalities for the quadrilateral $A_2B_2C_2D_2$, we obtain

$$K - 16F_1F_2 = 4(b_1c_1 + d_1a_1)(b_2c_2 + d_2a_2)(1 - \cos(B_1 - B_2))$$

and (29) follows immediately since

$$-1 < \cos(B_1 - B_2) \leqslant 1$$

with equality if and only if $B_1 = B_2$.

Note that the following result is a simple consequence of Theorem I:

THEOREM J. (i) Let the conditions of Theorem A be satisfied. Then

$$(30) \qquad 2(b_1c_2 - b_2c_1)^2 \leqslant H - 16F_1F_2 < 2(b_1c_2 + b_2c_1)^2$$

with equality if and only if $A_1 = A_2$, and

$$\frac{2}{3}\,\Sigma(b_1c_2 - b_2c_1)^2 \leqslant H - 16F_1F_2 < \frac{2}{3}\,\Sigma(b_1c_2 + b_2c_1)^2$$

with equality if and only if the two triangles are similar.

(ii) Let the conditions of Theorem B be satisfied. Then

$$\frac{1}{4}(b_1c_2 - b_2c_1)^2 \leqslant F_3^2 - (F_1 + F_2)^2 < \frac{1}{4}(b_1c_2 + b_2c_1)^2$$

with equality if and only if $A_1 = A_2$, and

$$\frac{1}{12}\,\Sigma(b_1c_2 - b_2c_1)^2 \leqslant F_3^2 - (F_1 + F_2)^2 < \frac{1}{12}\,\Sigma(b_1c_2 + b_2c_1)^2$$

with equality if and only if the two triangles are similar.

Remarks. 1° Gao Ling [26] proved only the first inequalities in (29) and (30). The above proof is only a simple extension of his proof.

2° Now, we shall note that (30) follows directly from Bottema's identity (8), i.e.

$$H - 16F_1F_2 = 2(b_1^2c_2^2 + b_2^2c_1^2) - 4b_1c_2b_2c_1 \cos(A_1 - A_2)$$

since $-1 < \cos(A_1 - A_2) \leqslant 1$.

3.5. Further Generalizations of the Neuberg-Pedoe Inequality

We now define

$$S_i = \Sigma a_i^2 \quad (i = 1, 2);$$

$$H_p = S_1^{2/p}S_2^{2/q} - 2\Sigma a_1^{4/p}a_2^{4/q}, \quad p > 1, \quad q = p/(p - 1);$$

$$E_p = S_1^{2/p}S_2^{2/q} - 2(\Sigma a_1^4)^{1/p}(\Sigma a_2^4)^{1/q}.$$

Then the following generalization of Theorem H is valid [27]:

THEOREM K.

$$H_p \geqslant E_p \geqslant 16S_1^{2/p}S_2^{2/q}\left(\frac{1}{p}\frac{F_1^2}{S_1^2} + \frac{1}{q}\frac{F_2^2}{S_2^2}\right) \geqslant 16F_1^{2/p}F_2^{2/q}.$$

The cases of equality are:

(a) In each pair of neighboring inequalities there occurs inequality if and only if the two triangles are similar.

(b) $E_p = 16F_1^{2/p}F_2^{2/q}$ holds if and only if the two triangles have equal Crelle-Brocard's angles (i.e. $\omega_1 = \omega_2$).

Now we shall prove the following theorem:

THEOREM L. Let $p, q \geqslant 1$. Then

$$\Sigma a_1^{2/p}(-a_2^{2/q} + b_2^{2/q} + c_2^{2/q}) \geqslant 3(4/\sqrt{3})^{1/p+1/q}F_1^{1/p}F_2^{1/q}.$$

Proof. If we apply the Neuberg-Pedoe inequality to triangles with sides $a_1^{1/p}$, etc. and $a_2^{1/q}$, etc. $(p, q > 1)$, we get

$$\Sigma a_1^{2/p}(-a_2^{2/q} + b_2^{2/q} + c_2^{2/q}) \geqslant 16F_pF_q$$

where F_p and F_q are the areas of these triangles.

Now, using the Oppenheim inequality VII.2.(1) we get the desired inequality.

Remarks. 1° For p = q, we get

$$\Sigma a_1^{2/p}(-a_2^{2/p} + b_2^{2/p} + c_2^{2/p}) \geqslant 3(\frac{16}{3} F_1F_2)^{1/p}.$$

For p = 2 we get an inequality of G. Ling [28] (see VII.2). For $p = 2^{m+1}$ we have a result from [29].

2° Similarly, we can give generalizations of some other inequalities of S. Bilčev and H. Lesov [29]. The same is valid for Theorem H. A further extension of Theorem L can be given using Carroll's generalization of Oppenheim's inequality (see VII.2).

Finally, we shall prove the following result:

THEOREM M. Let $a_1 \geqslant a_2 \geqslant \ldots \geqslant a_n$ and $b_1 \leqslant b_2 \leqslant \ldots \leqslant b_n$ be the sides of two convex n-gons with semi-perimeters s and s' and areas F and F', respectively. Further, let f and g be two increasing convex functions on $[0, +\infty)$. Then

$$\Sigma g(b_1)(-f(a_1) + f(a_2) + \ldots + f(a_n)) \geqslant$$

$$\geqslant n(n - 2)f((\frac{4}{n} \tan \frac{\pi}{n} F)^{\frac{1}{2}})g((\frac{4}{n} \tan \frac{\pi}{n} F')^{\frac{1}{2}}).$$

Proof. Since

$$-f(a_1) + f(a_2) + \ldots + f(a_n) \leqslant f(a_1) - f(a_2) + \ldots + f(a_n) \leqslant$$

$$\leqslant \ldots \leqslant f(a_1) + \ldots + f(a_{n-1}) - f(a_n),$$

using Čebyšev's inequality for monotone sequences and Jensen's inequality for convex functions we get

$$\Sigma g(b_1)(-f(a_1) + f(a_2) + \ldots + f(a_n)) \geqslant \frac{n - 2}{n} \Sigma g(b_1)\Sigma f(a_1) \geqslant$$

$$\geqslant n(n - 2)g(\frac{1}{n} \Sigma b_1)f(\frac{1}{n} \Sigma a_1) =$$

$$= n(n - 2)g(2s'/n)f(2s/n).$$

It is known that of all n-gons with the same perimeter the regular n-gon has greatest area. Thus

$$F \leqslant (\Sigma a_1)^2/(4n \tan \frac{\pi}{n}),$$

i.e.

$$s^2 \geqslant n \tan \frac{\pi}{n} F.$$

Of course, the desired inequality is a simple consequence of these results.

3° For $f(x) = g(x) = x^r$ $(r > 1)$, we get

$$\Sigma b_1^r(-a_1^r + a_2^r + \ldots + a_n^r) \geqslant n(n-2)\left(\frac{16}{n^2}\tan^2\frac{\pi}{n}FF'\right)^{r/2}.$$

For n = 3, we have a result from [29].

4° The above results are given in [37]. Some similar results for two triangles are given by W. Janous [38]. For example, he proved:

Let $a_i \geqslant b_i \geqslant c_i$ $(i = 1, 2)$ be the sides of triangle Δ_i $(i = 1, 2)$. Then

$$\Sigma a_1^p(-a_2^q + b_2^q + c_2^q) \leqslant 3\left(\frac{4}{\sqrt{3}}F_1 + k\Sigma(b_1 - c_1)^2\right)^{\frac{p}{2}}\left(\frac{4}{\sqrt{3}}F_2 + k\Sigma(b_2 - c_2)^2\right)^{\frac{q}{2}},$$

where $k = 8/9$ if $0 < p, q \leqslant 1$, and $k = 1$ if $0 < p, q \leqslant 2$.

3.6. Further Generalizations of the Oppenheim Inequality

Now, we shall give some generalizations of the Oppenheim inequality for triangles, quadrilaterals and tetrahedra.

First, we shall give the following Oppenheim generalization of his Theorem B:

THEOREM N. Suppose that $A_iB_iC_i$ $(i = 1, 2)$ are two triangles. Define for any $p \geqslant 1$, $a = (a_1^p + a_2^p)^{1/p}$, etc. Then a, b, c are the sides of a triangle. The three areas are connected by the inequality (if p = 1 or 2 or 4)

$$F^{p/2} \geqslant F_1^{p/2} + F_2^{p/2} \tag{31}$$

with equality if and only if the triangles are similar.

Oppenheim [16] also showed that the inequality does not hold for $p > 4$, and he also gave the conjecture that Theorem K holds for $1 \leqslant p \leqslant 4$.

The case p = 2 is Theorem B. Note that a generalization of this case for n triangles was given in [30], and one result similar to Theorem N was given in [31, p. 39]. The case p = 1 is again given in Math. Magazine 56 (1983), 19 and Amer. Math. Monthly 90 (1983), 522-523. But one similar result for n triangles was given first by M. S. Klamkin in [30]. The proof from the Monthly is similar to Klamkin's proof. Note that Oppenheim's proof is simpler. He used another inequality of Minkowski (see [32, p. 88] or [14, p. 26]).

A. Oppenheim [33] also proved the following result:

THEOREM O. Suppose that $A_1B_1C_1D_1$, $A_2B_2C_2D_2$ are two inscribable quadrilaterals of sides $a_1, \ldots, d_1$ and $a_2, \ldots, d_2$. Define a, etc. by

$$a = (a_1^p + a_2^p)^{1/p}, \quad \text{etc.} \quad (p \geqslant 1).$$

Then there is an inscribable quadrilateral of sides a, etc.; the areas satisfy (for p = 1, 2, 4) the inequality (31) with equality if and only if the given polygons are similar.

The Oppenheim conjectures that Theorems N and O are also valid for $1 \leqslant p \leqslant 4$ were proved by C. E. Carroll [34], i.e. the following results are valid:

THEOREM P. Let the conditions for Theorem N be fulfilled. Inequality (31) is valid in the case when $1 \leqslant p \leqslant 4$, too. Apart from trivial cases with $p = 1$ and $F_1 = F_2 = 0$, equality holds if and only if

$$a_1/a_2 = b_1/b_2 = c_1/c_2. \tag{32}$$

THEOREM Q. If $p \geqslant 1$, if the triangles having areas F_1 and F_2 are acute or right triangles, and if a, b, c, F are as in Theorem N, then (31) holds with equality if and only if (32) holds.

THEOREM R. If $1 \leqslant p \leqslant 4$ and if two quadrilaterals have sides $a_1, \ldots, d_1$ and $a_2, \ldots, d_2$, then $a = (a_1^p + a_2^p)^{1/p}$, etc. are the sides of a quadrilateral, and the maximum areas satisfy (31). Equality holds only if the sets $a_1, \ldots, d_1$ and $a_2, \ldots, d_2$ are proportional; but there are trivial exceptions with $p = 1$ and $F_1 = F_2 = 0$.

Now, we shall show that we can give generalizations of these results for the case of n triangles or quadrilaterals. In our results trivial equality cases are not included.

THEOREM S. Suppose that $A_iB_iC_i$ $(i = 1, \ldots, n)$ are n triangles. Define for any $p \geqslant 1$

$$a = \left(\sum_{i=1}^{n} w_i a_i^p \right)^{1/p}, \quad \text{etc.}$$

where $w_i \geqslant 0$ $(i = 1, \ldots, n)$. Then a, b, c are the sides of a triangle. If $1 \leqslant p \leqslant 4$, the n + 1 areas are connected by the inequality

$$F^{p/2} \geqslant \sum_{i=1}^{n} w_i F_i^{p/2} \tag{33}$$

with equality if and only if the given n triangles are similar.

Proof. In the case $w_i = 1$ $(i = 1, \ldots, n)$ we shall use the mathematical induction method. Indeed, for n = 2, this is Theorem P. Suppose that Theorem S for $w_i = 1$ $(i = 1, \ldots, n)$ is true for n - 1 and put

$$a_i' \to a_i \quad (i = 1, \ldots, n-2), \quad a_{n-1}' \to (a_{n-1}^p + a_n^p)^{1/p}, \quad \text{etc.}$$

Then from $F'^{p/2} \geq \sum_{i=1}^{n-1} F_i'^{p/2}$ follows $F^{p/2} \geq \sum_{i=1}^{n} F_i^{P/2}$, because Theorem P gives $F'^{p/2}_{n-1} \geq F_{n-1}^{p/2} + F_n^{p/2}$. Further, by substitution $a_i \to w_i^{1/p} a_i$, etc. $(i = 1, \ldots, n)$, we get our result.

Similarly, we can prove the following two theorems:

THEOREM T. In the previous theorem let $p \geq 1$, and let the given n triangles be acute or right triangles. Then (33) holds with equality if and only if the given n triangles are similar.

THEOREM U. If $1 \leq p \leq 4$ and if n quadrilaterals have sides a_i, b_i, c_i, d_i $(i = 1, \ldots, n)$, then $a = \left(\sum_{i=1}^{n} w_i a_i^p\right)^{1/p}$, etc. are the sides of a quadrilateral, and the maximum areas satisfy (33). Equality holds if and only if the sets a_i, b_i, c_i, d_i $(i = 1, \ldots, n)$ are proportional.

Remark. It is known that, if a quadrilateral has sides of fixed length, the area is maximum when the vertices lie on a circle.

Now, we shall give two applications of Theorems S and U.

COROLLARY 1. Of all triangles with the same perimeter, the equilateral triangle has the greatest area.

Proof. Put $\sum_{i=1}^{n} w_i = 1$, then from (33) for $p = 1$ follows that the symmetric function $F^{1/2}$ is concave. Therefore, $F^{1/2}$ is a Schur-concave function (see [35, p. 64]), and the same is valid for F (see [35, p. 61]), a known result ([35, p. 209]). A simple consequence of this result is

$$F\left(\frac{1}{3}\Sigma a, \frac{1}{3}\Sigma a, \frac{1}{3}\Sigma a\right) \geq F(a, b, c),$$

i.e. Corollary 1.

Similarly, we can prove (see for example [35, p. 209]):

COROLLARY 2. Of all quadrilaterals with a given perimeter, the square has the greatest area.

As we said in 3.1., A. Oppenheim noted that the generalization of Theorem B in several dimensions is also valid. As an example, he gave the following result:

THEOREM V. If tetrahedra T, T_1, T_2 have edges connected by the equations $a^2 = a_1^2 + a_2^2$, etc., then their volumes satisfy the inequality

$$V^{2/3} \geq V_1^{2/3} + V_2^{2/3}$$

with equality if and only if T_1 and T_2 are similar.

Similarly to the proof of Theorem S we can prove:

THEOREM W. If tetrahedra T, T_i (i = 1, ..., n) have edges connected by the equations $a^2 = \sum_{i=1}^{n} w_i a_i^2$, etc., where w_i (i = 1, ..., n) are positive numbers, then their volumes satisfy the inequalities

$$V^{2/3} \geqslant \sum_{i=1}^{n} w_i V_i^{2/3}$$

with equality if and only if the T_i (i = 1, ..., n) are similar.

Remark. The method from the proof of Theorem S we can use for generalization of some other similar results:

1) Let the conditions of Theorem S be satisfied for p = 2. Then

$$h^2 \geqslant \sum_{i=1}^{n} w_i h_i^2 \tag{34}$$

where h_i is the altitude of the triangle $A_iB_iC_i$ (i = 1, ..., n), and h is the altitude of the triangle with sides a, b, c, with equality only for similar triangles.

2) Let the conditions from 1) be satisfied, then the analogous results for the circumradii are valid

$$R^2 \leqslant \sum_{i=1}^{n} w_i R_i^2$$

with equality if the given triangles are similar or if the triangles are right angled, with vertices corresponding.

3) Let the conditions of Theorem W be satisfied, then corresponding altitudes satisfy (34).

We shall now give a conjecture, due to D. Blagojević:

Let $A_iB_iC_i$, $i \in \{1, 2, \ldots, n\}$, be a finite set of triangles with sides a_i, b_i, c_i. It is known that for all such sets, the triangles with the sides $M_r(a)$, $M_r(b)$, $M_r(c)$ exists if and only if $r \geqslant 1$. (Here $M_r(a)$ is the mean order r of $a_1, a_2, \ldots, a_n$.)

Conjecture. If angles of the triangles are bounded from below by an angle $\alpha < \frac{\pi}{3}$, then the bound $r = r(\alpha)$ could be moved to be less than 1. (If necessary, add the conditions $a_i \leqslant b_i \leqslant c_i$ for all i.) If so, find some properties of the function $r = r(\alpha)$.

References

1. O. Bottema, R. Ž. Djordjević, R. R. Janić, D. S. Mitrinović, and P. M. Vasić, Geometric Inequalities, Groningen, 1969.
2. J. Neuberg, 'Sur les projections et contre-projections d'un triangle fixe', Acad. Roy. de Belgique 44 (1891), 31-33.
3. D. Pedoe, 'An Inequality for Two Triangles', Proc. Cambridge Philos. Soc. 38 (1943), 397-398.
4. D. Pedoe, 'On Some Geometrical Inequalities', Math. Gaz. 26 (1942), 202-208.
5. D. Pedoe, 'Thinking Geometrically', Amer. Math. Monthly 77 (1970), 711-721.
6. D. Pedoe, 'Inside-Outside: The Neuberg-Pedoe Inequality', Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 544-576 (1976), 95-97.
7. L. Carlitz, 'An Inequality Involving the Area of Two Triangles', Amer. Math. Monthly 78 (1971), 772.
8. L. Carlitz, 'Some Inequalities for Two Triangles', Math. Mag. 45 (1972), 43-44.
9. G. Chang, 'Proving Pedoe's Inequality by Complex Number Computation', Amer. Math. Monthly 89 (1982), 692.
10. O. Bottema and M. S. Klamkin, 'Joint Triangle Inequalities', Simon Stevin 48 (1974), I-II, 3-8.
11. L. Boček, 'O jedne nerovnosti pro součin obsahu dvou trojuhelniku', Rozhledy Mat. Fyz. Praha 59 (1980-1981), 244-249.
12. A. Oppenheim, 'Problem 5092', Amer. Math. Monthly 70 (1963), 444 and 71 (1964), 444.
13. R. Bellman, 'On an Inequality Concerning an Indefinite Form', Amer. Math. Monthly 63 (1956), 108-109.
14. E. F. Beckenbach and R. Bellman, Inequalities, Berlin-Heidelberg-New York, 1965.
15. D. S. Mitrinović, Analytic Inequalities, Berlin-Heidelberg-New York, 1970.
16. A. Oppenheim, 'Some Inequalities for Triangles', Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 357-380 (1971), 21-28.
17. P. M. Vasić and J. E. Pečarić, 'On the Hölder and Some Related Inequalities', Matematica (Cluj), 24 (47) (1982), 95-103.
18. Yang Lu and Zhang Jing-Zhong, 'A Generalization to Several Dimensions of the Neuberg-Pedoe Inequality, with Applications', Bull. Australian Math. Soc. 27 (1983), 203-214.
19. J. Shilleto, 'Pedoe's Triangle Inequality and Hermitian Forms', Coxeter Festschrift III, Mitt. Math. Semin. Giessen 165 (1984), 199-204.
20. O. Bottema, 'The Mixed Area of Two Triangles (Dutch)', Euclides 44 (1968-1969), 304-310.
21. O. Bottema, 'Inscribed Triangles (Dutch)', Euclides 50 (1974-1975), 363-367.
22. L. Carlitz, 'Inequalities for the Area of Two Triangles', Amer. Math. Monthly 80 (1973), 910-911.
23. K. S. Poh, 'A Short Note on a Pedoe's Theorem about Two Triangles', Math. Medley (Singapore) 11 (1983), 57-61.
24. J. F. Rigby, On a Theorem of Pedoe, Private communication.
25. Chia-Kuei Peng, 'Sharpening the Neuberg-Pedoe Inequality, I', Crux. Math. 10 (1984), 68-69.

26. Gao Ling, 'Sharpening the Neuberg-Pedoe Inequality, II', Crux. Math. 10 (1984), 70-71.
27. W. Janous, Another Extension of the Neuberg-Pedoe Inequality, Private communication.
28. G. Ling and V. D. Mascioni, 'Problem 1156', Math. Mag. 57 (1984), 47-48.
29. S. Bilčev and H. Lesov, V'rhu edno obobščenie na neravenstvoto na Neuberg-Pedoe. Ses. "100 god. roz. L. Čakalov" (Sofija), 1986.
30. M. S. Klamkin, 'Notes on Inequalities Involving Triangles or Tetrahedrons', Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 330-337 (1970), 1-15.
31. D. O. Škljarskij, N. N. Čencov, and I. M. Jaglom, Geometričeskie ocenki i zadači iz kombinatornoj Geometrii, Moskva, 1974.
32. D. S. Mitrinović, P. S. Bullen, and P. M. Vasić, 'Sredine i sa njima povezane nejednakosti', Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 600 (1977), 1-232.
33. A. Oppenheim, 'Inequalities Involving Elements of Triangles, Quadrilaterals or Tetrahedra', Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 461-497 (1974), 257-263.
34. C. E. Carroll, 'Proof of Oppenheim's Area Inequalities for Triangles and Quadrilaterals', Aequationes Math. 24 (1982), 97-109.
35. A. W. Marshall and I. Olkin, Inequalities: Theory of Majorization and Its Applications, New York-London-Toronto-Sydney-San Francisco, 1979.
36. D. S. Mitrinović and J. E. Pečarić, 'About the Neuberg-Pedoe and the Oppenheim Inequalities', J. Math. Anal. Appl. 129 (1988), 196-210.
37. D. S. Mitrinović and J. E. Pečarić, 'Inequality Among Sides of Triangles with Given Areas' (Bulgarian), Ob. po Matematika (Sofija) 1987, 4, 39-40.
38. W. Janous, Some Generalizations of the Neuberg-Pedoe Inequality. To appear.

4. On O. Bottema's Inequality for Two Triangles.

In this Section we shall show that the known Bottema inequality for two triangles and an interior point of one of these triangles is also valid for any point. We shall also give some related results.

The following result is well-known ([1], GI 12.18):

THEOREM A. Let x, y, z be the distances from any point P from the plane of a triangle ABC of area F to the respective vertices. Then

$$\Sigma x \geqslant 2\sqrt{F\sqrt{3}}. \tag{1}$$

Equality occurs if and only if the triangle is equilateral and P is its centre.

Inequality (1) is stronger than ([2], GI 12.14): $\Sigma x \geqslant 6r$, where r is the radius of incircle, because (GI 5.11) $6r \leqslant 2\sqrt{F\sqrt{3}}$.

The following result is also known ([3], [4], GI 12.55):

THEOREM B. If all angles are less than 120°, then

$$\Sigma x \geq (\frac{1}{2}\Sigma a^2 + 2F\sqrt{3})^{1/2}.$$

If $A \geq 120°$, then

$$\Sigma x \geq b + c.$$

Here x, y, z are the distances from an interior point P of a triangle to the respective vertices.

Equality holds if P coincides with Fermat-Torricelli's point.

Remark. Let $A_1A_2B_3$, $A_2A_3B_1$, $A_3A_1B_2$ be the equilateral triangles erected outwardly on the sides of an arbitrary triangle $A_1A_2A_3$.

The segments A_1B_1, A_2B_2, A_3B_3 are concurrent at a point T, the Fermat-Torricelli point of the triangle (see for example [8]).

O. Bottema [5] proved the following inequality for two triangles (see also GI 12.56):

THEOREM C.

$$(2) \qquad \Sigma a_1 x \geq (M/2 + 8FF_1)^{1/2},$$

where $M = \Sigma a_1^2(b^2 + c^2 - a^2)$; a_1, b_1, c_1, and a, b, c are sides of two arbitrary triangles of areas F_1 and F, respectively, and x, y, z are the distances from an interior point P of triangle ABC to the respective vertices.

In his book, when dealing with the Fermat-Torricelli point (being the case $a_1 = b_1 = c_1$), Bottema generalized a known proof for the case that a_1, b_1, c_1 are the sides of an arbitrary triangle. The proof was complicated and has been much simplified by Bottema and Klamkin in [6]. A new proof of theorem C is given in [7], too.

Now we shall show that (2) is also valid in the case when x, y, z are the distances from an arbitrary point P of the space to the respective vertices (see [12]).

Of course, it is sufficient to consider the case when the point P belongs to the plane of the triangle ABC.

We shall use the idea of the proof from [6]. Let A', B', C' be defined as in the figure

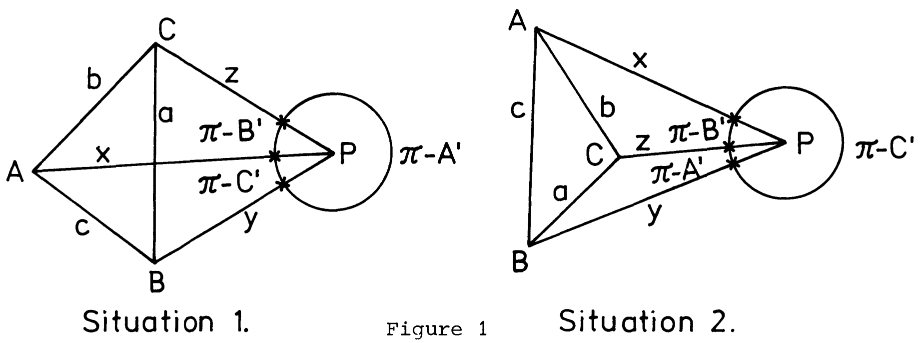


Figure 1

Note that

$$A' + B' + C' = \pi, \quad \text{and} \quad a^2 = y^2 + z^2 + 2yz \cos A', \quad \text{etc.}$$

Thus

$$b^2 + c^2 - a^2 = 2x^2 + 2x(z \cos B' + y \cos C') - 2yz \cos A', \text{ etc.}$$

and depending upon whether situation 1 or 2 holds, we have

$$8F_1F = \pm 4F_1\Sigma yz \sin A'.$$

On substituting back in (2), we obtain

$$(\Sigma a_1 x)^2 \geqslant \Sigma a_1^2(x^2 + x(z \cos B' + y \cos C') - - yz \cos A') \pm 4F_1\Sigma yz \sin A',$$

or

$$\Sigma yz(2b_1c_1 + (a_1^2 - b_1^2 - c_1^2)\cos A' \mp 4F_1 \sin A') \geqslant 0,$$

or finally,

$$4\Sigma yzb_1c_1 \sin^2(A_1 \mp A')/2 \geqslant 0.$$

This is obviously true because the terms on the left-hand side are all non-negative. This implies that (2) holds not only for interior points but for exterior points P as well.

There is equality only if each term of the sum vanishes, i.e.:
Case 1: $xyz \neq 0$, $A_1 = A'$, $B_1 = B'$, $C_1 = C'$,
Case 2: $xyz = 0$ (say $x = 0$), $A_1 = A'$.

It is obvious that equality does not exist in the case when P is an exterior point of a triangle ABC.

By coupling the above result and the Neuberg-Pedoe inequality (GI 10.8) we get a similar extension of a result from [6]:

$$(3) \qquad \Sigma a_1 x \geqslant (M/2 + 8FF_1)^{1/2} \geqslant 4\sqrt{FF_1},$$

wherefrom in the case $a_1 = b_1 = c_1$, we get

$$(4) \qquad \Sigma x \geqslant (\tfrac{1}{2} \Sigma a^2 + 2F\sqrt{3})^{1/2} \geqslant 2\sqrt{F\sqrt{3}}.$$

Using (3) we can obtain the following generalization of a result of Bottema and M. S. Klamkin [9]:

If a, b, c and x, y, z denote the sides of a triangle ABC of area F and the distances from any point P to A, B, C, respectively, than either of the pair of inequalities

$$\Sigma a^2 \geqslant 4F\sqrt{3}, \quad \Sigma x^2 \geqslant 4F/\sqrt{3},$$

with equality if ABC is equilateral and P is the centroid, implies the other one.

Proof. Using (3) we get

$$(5) \qquad (\Sigma a_1 x)^2 \geqslant 16FF_1.$$

Then by Cauchy's inequality,

$$(\Sigma a_1^2)/4F_1\sqrt{3} \geqslant 4F/\sqrt{3}(\Sigma x^2),$$

which gives the desired result.

Remark. In [10] the following result is given:

1) Let be given a triangle ABC with sides a, b, c. Let $f(X) = \Sigma aXA$, where X ranges over the plane of the triangle. Then

$$(6) \qquad f(X) \geqslant 4F.$$

Let A be the largest angle of the triangle. If angle $A \leqslant 90°$, then the orthocentre H does not lie outside the triangle, and equality holds in (6) when X = H, i.e. min f(X) = 4F, attained when X = H. The situation is different, however, if angle $A > 90°$. Then

$$\min f(X) = 2bc > 4F$$

attained when X = A.

Now, we shall give a similar result from [11]:

2) Let be given the triangle $A_1A_2A_3$, let $\lambda_1, \lambda_2, \lambda_3 \in (0, +\infty)$, and let (i, j, k) be any cyclic permutation of (1, 2, 3). Let

$$L_i = \lambda_j^2 + \lambda_k^2 + 2\lambda_j\lambda_k \cos A_i - \lambda_i^2.$$

At least two of the numbers L_1, L_2, L_3 are positive.

If $L_1, L_2, L_3 > 0$, then $\Sigma\lambda_1R_1$ ($R_1 = PA_1$, etc.) has a minimum for an internal point P if $\sin A_2PA_3 : \sin A_3PA_1 : \sin A_1PA_2 = \lambda_1 : \lambda_2 : \lambda_3$.

If $L_i \leqslant 0$, then $\Sigma\lambda_1R_1$ has a minimum for A_i.

For $\lambda_1 = \lambda_2 = \lambda_3$ we get Theorem B (Fermat's problem).

Finally, we shall give an inequality of G. Bennett [7], similar to 2):

If w_1, w_2, w_3 are real numbers, then

$$(7) \qquad (\Sigma w_1)(\Sigma w_1a_1^2b^2c^2) \geqslant (M/2 + 8FF_1)(\Sigma a^2w_2w_3),$$

with equality if and only if

$$ab_1c_1 \sin(A + A_1)/w_1 = bc_1a_1 \sin(B + B_1)/w_2 =$$
$$= ca_1b_1 \sin(C + C_1)/w_3.$$

(7) can be viewed as a three-triangle inequality by restricting the w_i to be the sides of a third arbitrary triangle.

References

1. U. T. Bödewadt, Jber. Deutsch. Math.-Verein. 46 (1936), 7 kursiv.
2. M. Schreiber, 'Aufgabe 196', Ibid. 45 (1935), 63 kursiv.
3. Encyklopädie der Mathematischen Wissenschaften, Bd. III', Heft 6, p. 1118, and Heft 7, p. 1185.
4. T. Lalesco, La Géométrie du Triangle, Paris 1937, p. 41.
5. O. Bottema, Hoofdstukken uit de elementaire meetkunde, Den Haag 1944, pp. 97-99.
6. O. Bottema and M. S. Klamkin, 'Joint Triangle Inequalities', Simon Stevin 48 (1974-1975), I-II (1974), 3-8.
7. G. Bennett, 'Multiple Triangle Inequalities', Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 577-598 (1977), 39-44.
8. J. Naas and H. L. Schmid, Mathematisches Wörterbuch, Bd. II, Berlin-Stuttgart, 1961, p. 732.
9. O. Bottema and M. S. Klamkin, A Pair of Triangle Inequalities. Notes Canadian Math. Congress 7 (1975), No. 6, p. 8.
10. J. Dou and J. B. Tabov, 'Problem 866', Crux Math. 9 (1983), 209 and 10 (1984), 327-329.
11. E. Egerváry, 'Ueber ein Minimumproblem der Elementargeometrie', J. reine angew. Math. 1978 (1938), 174-186.
12. D. S. Mitrinović and J. E. Pečarić, 'Note on O. Bottema's Inequality for Two Triangles', C. R. Math. Rep. Acad. Sci. Canada 8, No. 2 (1986), 141-144.

5. Miscellaneous Inequalities Involving Elements of Several Triangles

5.1. For any two triangles the following inequalities are valid

(1) $$\sum \frac{1}{s - a} \geq 4\sum \frac{1}{a} \cos A',$$

(2) $$\sum r_a \geq 2\sum h_a \cos A',$$

with equality if and only if both triangles are equilateral.

A. Oppenheim and L. Bankoff, 'Problem E 2059', Amer. Math. Monthly 75 (1968), 189 and 76 (1969), 196-197.

5.2. For any two triangles the following inequalities are valid

(1) $\sum aa' \geq 4\sqrt{3}\sqrt{FF'}$; (2) $\sum m_a m_a' \geq 3\sqrt{3}\sqrt{FF'}$;

(3) $\Sigma a^2 a'^2 \geq 16FF'$; (4) $9RR' \geq \Sigma aa'$;

(5) $\Sigma a^2/r'_b r'_c \geq 4F/F'$; (6) $\Sigma a^2 r'_a \geq 2Fs'$;

(7) $8\sqrt{3}\sqrt{F/F'} \leq \Sigma(b + c)/r'_a \leq 18RR'/F'$; (8) $\Sigma 1/aa' \geq 1/RR'$;

(9) $\Sigma r_a/h'_a \geq 3\sqrt{F/F'}$; (10) $\Sigma r_a r'_a \geq 3\sqrt[3]{F^2 F'^2/rr'}$;

(11) $\Sigma a'/(b^2 + c^2) \leq 9R'/8F$; (12) $\Sigma a' r_a^2 \geq 18FF'/s'$.

G. A. Tsintsifas, Private communication.

Comment by W. Janous. a) The following n triangles inequality is a generalization of the results from (1-3):

If $t_1, \ldots, t_n$ are positive real numbers and $0 \leq k \leq n$, then

$$\Sigma a_1^{t_1} \ldots a_k^{t_k} m_{a_{k+1}}^{t_{k+1}} \ldots m_{a_n}^{t_n} \geq$$

$$\geq 3(4/\sqrt{3})^{\sum_{i=1}^{k} t_i/2} (\sqrt{3})^{\sum_{i=k+1}^{n} t_i/2} \prod_{i=1}^{n} F_i^{t_i/2}.$$

b) As $k = \log(9/4)/\log(4/3) > 2$, we have on grounds of GI 5.28, i.e. $R\sqrt{3} \geq M_k(a, b, c)$ the following interpolation of (4):

$$9RR' \geq 3(\tfrac{1}{3}\,\Sigma a^{k/2} a'^{k/2})^{2/k} \geq \Sigma aa'$$

(where we used firstly the Cauchy-Schwarz-inequality and secondly the $M_{k/2} - M_1$ inequality).

5.3. The following two-triangles-inequalities are valid

(1) $$\left(\frac{s}{s'}\right)^2 \geq \Sigma \frac{ab}{a'b'} \left(1 - \frac{c'}{s'}\right),$$

(2) $$\left(\frac{s}{s'}\right)^2 \geq \Sigma \frac{ab}{b'c'} \left(1 - \frac{a'}{s'}\right),$$

(3) $$\left(\frac{\Sigma \cos^u(A/\lambda)}{\Sigma \cos^v(A'/\delta)}\right)^2 \geq \Sigma \frac{\cos^u(A/\lambda)\cos^u(B/\lambda)}{\cos^v(A'/\delta)\cos^v(B'/\delta)} \left(1 - \frac{2\cos^u(C'/\lambda)}{\Sigma \cos^v(A'/\delta)}\right),$$

(4) $$(\Sigma a' \cos^u(A/\lambda))^2 \geq 4s'\Sigma(s' - a')\cos^u(B/\lambda)\cos^u(C/\lambda),$$

where $0 < u \leqslant \lambda$, $\lambda \geqslant 2$, $0 < v \leqslant \delta$ and $\delta \geqslant 2$.

W. Janous, Private communication.

5.4. The following two-triangle inequality is valid

$$(1) \qquad \left(\frac{s_1}{r_1R_1}\frac{s_2}{r_2R_2}\right)^{1/2} \geqslant 3\Sigma 1/\sqrt{a_1a_2}$$

with equality if and only if the two triangles are equilateral.

The analogous three-triangle inequality

$$(2) \qquad \left(\frac{s_1}{r_1R_1}\frac{s_2}{r_2R_2}\frac{s_3}{r_3R_3}\right)^{1/2} \geqslant 9\Sigma 1/\sqrt{a_1a_2a_3}$$

does not hold.

Proof. We prove first that

$$(3) \qquad (\Sigma a_1)(\Sigma a_2) \geqslant 3(\Sigma\sqrt{a_1a_2b_1b_2}).$$

Apply the Cauchy inequality to $L = \sqrt{\Sigma a_1}\sqrt{\Sigma a_2}$ to obtain

$$L \geqslant \Sigma\sqrt{a_1a_2} \quad \text{and} \quad L^2 \geqslant \Sigma a_1a_2 + 2(\Sigma\sqrt{a_1a_2b_1b_2}).$$

As for positive x, y, z we know that $\Sigma x^2 \geqslant \Sigma yz$, (3) immediately follows.

Now, L^2 is $4s_1s_2$, and we have the well-known relation $4r_iR_i = (\Pi a_i)/s_i$, $(i = 1, 2)$. Hence

$$4s_1s_2 = \sqrt{s_1s_2/(r_1R_1r_2R_2)} \cdot \sqrt{a_1a_2b_1b_2c_1c_2}.$$

By setting this in (3) and dividing both sides by $\sqrt{a_1a_2b_1b_2c_1c_2}$, (1) is proved. Equality will hold in (3) if and only if we have $a_1 = a_2 = b_1 = b_2 = c_1 = c_2$; that is, if both triangles are equilateral and congruent since only then the equality holds in both of the inequalities that were used above.

By using $4r_iR_i = a_ib_ic_i/s_i$, equation (2) becomes analogous to (3):

$$p_1p_2p_3 \geqslant 9(\Sigma\sqrt{a_1a_2a_3b_1b_2b_3}).$$

Now, the last relation may fail for triangles, e.g. consider sides a_1, $a_1/100$, $99a_1/100$ as a_1, b_1, c_1; a_2, $a_2/100$, $99a_2/100$ as a_2, b_2, c_2; a_3, $a_3/100$, $99a_3/100$ as a_3, b_3, c_3. Then one left-hand side is $8a_1a_2a_3$. The right side is greater than $(8.8)a_1a_2a_3$.

Remark. The above solution of a problem of M. S. Klamkin was given by P. Bracken. M. S. Klamkin also proved a companion inequality to (1):

$$4(\Sigma 1/\sqrt{a_1})(\Sigma 1/\sqrt{a_2}) \geqslant \left(\frac{s_1}{r_1R_1}\frac{s_2}{r_2R_2}\right)^{1/2}.$$

M. S. Klamkin and P. Bracken, 'Problem 1043', Math. Mag. 51 (1978), 193 and 52 (1979), 320-321.

5.5. Let the triangles $A_1A_2A_3$ and $B_1B_2B_3$ be inscribed in a circle of radius R. Let N_1 and N_2 be the centres of their nine-point-circles, H_1 and H_2 - the orthocentres, G_1 and G_2 - the centroids. Then the following inequalities are valid:

(1) $$N_1N_2 < \frac{1}{2}(4R + \min A_iB_j), \quad i, j = 1, 2, 3.$$

(2) $$H_1N_2 < \frac{1}{2}(6R + \sqrt{R^2 + 2(\min A_iB_j)^2}), \quad i, j = 1, 2, 3.$$

(3) $$G_1N_2 < \frac{11}{6} R + \frac{1}{6}\sqrt{6} \min A_iB_j, \quad i, j = 1, 2, 3.$$

(4) $$H_1G_2 < \frac{1}{3}(8R + \sqrt{4R^2 + 3(\min A_iB_j)^2}), \quad i, j = 1, 2, 3.$$

E. Jucovič, Private communication.

5.6. Let A, B, C and A_1, B_1, C_1 be angles of the triangles ABC and $A_1B_1C_1$, respectively. Suppose that $A \geqslant B \geqslant C$, $A_1 \geqslant B_1 \geqslant C_1$, and also that $A \geqslant A_1$ and $A + B \geqslant A_1 + B_1$. Then

$$\frac{s}{r} = \Sigma \cotan \frac{A}{2} \geqslant \Sigma \cotan \frac{A_1}{2} = \frac{s_1}{r_1}.$$

J. Garfunkel and M. S. Klamkin, 'Problem 1049', Crux Math. 11 (1985), 148 and 12 (1986), 249.

5.7. Let P, P_1 be interior points of the triangles ABC and $A_1B_1C_1$, respectively. Let r_1, r_2, r_3 be the distances from P to the sides of ABC and ρ_1, ρ_2, ρ_3 the distances from P_1 to the sides of $A_1B_1C_1$. Also let F, F_1 denote the respective areas. The extreme value of $\Sigma r_1\rho_1$ is $4FF_1/(\Sigma aa_1)$, and the extreme value of $\Sigma aa_1r_1\rho_1$ is $\frac{4}{3} FF_1$.

This result is due to L. Carlitz.

5.8. Let T, T' be two triangles such that $a \leqslant a'$, $b \leqslant b'$, $c \leqslant c'$.

(i) If T' is non-obtuse, then $F \leqslant F'$. If T' is obtuse, the inequality need not be true.

(ii) If T is non-obtuse, then $R \leqslant R'$. If T is obtuse, the inequality need not be true.

(iii) No isotone relation holds between the inradii. It is possible to have two acute angled triangles T, T' such that

$$a < a', \quad b < b', \quad c < c', \quad \text{but} \quad r > r'.$$

A. Oppenheim, 'Some Inequalities for Triangles', Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 357-380 (1971), 21-28.

5.9. Suppose that a triangle T is formed from two triangles T_1 and T_2 by taking

$$a = \max(a_1, a_2), \quad b = \max(b_1, b_2), \quad c = \max(c_1, c_2).$$

Then

$$F \geqslant \min(F_1, F_2).$$

If one of T_1, T_2 is non-obtuse, then $R \geqslant \min(R_1, R_2)$.

If both T_1, T_2 are obtuse, then

$$F \geqslant \max(F_1, F_2), \qquad R \geqslant \max(R_1, R_2).$$

If both T_1 and T_2 are obtuse, no conclusion can be drawn about R.

A. Oppenheim, The same reference as in 5.8.

5.10. Let a_i, b_i, c_i denote the sides of n triangles $A_iB_iC_i$ $(i = 1, \ldots, n)$. If $w_i \geqslant 0$ $(i = 1, \ldots, n)$, then the three numbers

$$a = \Sigma w_i a_i, \quad b = \Sigma w_i b_i, \quad c = \Sigma w_i c_i$$

are possible lengths of sides for a triangle ABC. Then

$$(1) \qquad (r^2 s)^{1/3} \geqslant \Sigma w_i (r_i^2 s_i)^{1/3},$$

$$(2) \qquad (FR)^{1/3} \geqslant \Sigma w_i (F_i R_i)^{1/3},$$

with equality if and only if the triangles are directly similar.

Proof. (The above result with $\Sigma w_i = 1$, was proved by M. S. Klamkin. Here we shall give a new proof.) Note that $s = \Sigma w_i s_i$, $(4RF)^{1/3} = (abc)^{1/3}$, $(r^2 s)^{1/3} = ((s - a)(s - b)(s - c))^{1/3}$, and then inequalities (1) and (2) are direct consequences of the following inequality

(3) $$G_m\left(\sum_{i=1}^{n} w_i\bar{a}_i\right) \geqslant \sum_{i=1}^{n} w_i G_m(\bar{a}_i),$$

where G_m is the geometric mean and $\bar{a} = (a_1, \ldots, a_m)$. This inequality.is a simple generalization of the well-known Minkowski inequality.

M. S. Klamkin, 'Notes on Inequalities Involving Triangles or Tetrahedrons', Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 330-337 (1970), 1-15.

5.11. Let A_i, B_i, C_i denote the angles of n triangles $A_iB_iC_i$ (i = 1, ..., n). Let the angles of ABC be given by

$$A = \Sigma w_i A_i, \quad B = \Sigma w_i B_i, \quad C = \Sigma w_i C_i,$$

where $\Sigma w_i = 1$, $w_i \geqslant 0$ (i = 1, ..., n). Then the following inequalities are valid

(1) $s^2/F \leqslant \Sigma w_i s_i^2/F_i$; (2) $(r/R)^{1/3} \geqslant \Sigma w_i (r_i/R_i)^{1/3}$;

(3) $s/R \geqslant \Sigma w_i s_i/R_i$; (4) $(F/R^2)^{1/3} \geqslant \Sigma w_i (F_i/R_i^2)^{1/3}$;

with equality if and only if the n triangles are directly similar.

Remark. This result is given in the reference quoted in 5.10. Note that of special interest is the case n = 3 (for 5.10 and 5.11) if

$$a' = ua + vb + wc, \quad b' = va + wb + uc, \quad c' = wa + ub + vc,$$

where $u + v + w = 1$, $u, v, w \geqslant 0$, for 5.10, and similarly defined triangle with angles for 5.11.

5.12. L. Carlitz proved the following results:

(i) If O_i is the circumcentre, G_i the centroid and R_i the circumradius of $A_iB_iC_i$ (i = 1, 2), then

(1) $$R_1R_2 - \frac{1}{9}\Sigma a_1a_2 \geqslant O_1G_1 \cdot O_2G_2,$$

with equality if and only if the triangles are similar.

(ii) If I_i denote the incentre, I_{ai}, I_{bi}, I_{ci} the excentres of $A_iB_iC_i$ (i = 1, 2), then

(2) $$12R_1R_2 - \Sigma O_1I_{a1} \cdot O_2I_{a2} \geqslant O_1I_1 \cdot O_2I_2,$$

with equality if and only if the triangles are similar.

Carlitz proved (1) and (2) using Aczél's inequality (i.e. Lemma 1 from Subsection 3.1.) and the following known identities

(3) $$R^2 - \frac{1}{9}\Sigma a^2 = OG^2 \quad \text{and} \quad 12R^2 - \Sigma OI_a^2 = OI^2.$$

Note that (1) and (2) can also be proved by using Cauchy's inequality. Further, (1) and (2) are equivalent to the following inequalities

(1') $$(R_1 + R_2)^2 - \frac{1}{9}\Sigma(a_1 + a_2)^2 \geq (O_1G_1 + O_2G_2)^2,$$

(2') $$12(R_1 + R_2)^2 - \Sigma(O_1I_{a1} + O_2I_{a2})^2 \geq (O_1I_1 + O_2I_2)^2.$$

These inequalities can be proved by using Bellman's inequality (lemma 2, Subsection 3.1), or the well-known Minkowski inequality. Of course, we can give simple generalizations of these inequalities for the case of n triangles:

If $w_i \geq 0$ $(i = 1, \ldots, n)$, then

$$(\sum_i w_iO_iG_i)^2 \leq (\sum_i w_iR_i)^2 - \frac{1}{9}\Sigma(\sum_i w_ia_i)^2,$$

$$(\sum_i w_iO_iI_i)^2 \leq 12(\sum_i w_iR_i)^2 - \Sigma(\sum_i w_iO_iI_{ai})^2,$$

with equality if and only if the given triangles are similar.

L. Carlitz, 'Some Inequalities for Two Triangles', Math. Mag. **45** (1972), 43-44.

Comment by V. Mascioni. Following Carlitz' ideas we get, for instance,

$$O\Omega_{11} \cdot O\Omega_{12} \leq R_1R_2(1 - 4\sin\omega_1 \sin\omega_2)$$

(using XI, 1.1, (6, 3)), We have also

$$G_1\Omega_{11} \cdot G_2\Omega_{12} + G_1\Omega_{21} \cdot G_2\Omega_{22} \leq \sqrt{(G_1\Omega_{11}^2 + G_1\Omega_{21}^2)(G_2\Omega_{12}^2 + G_2\Omega_{22}^2)}$$

$$\leq \frac{1}{9}(\Sigma a_1^2 \cdot \Sigma a_2^2)^{1/2} - (\Sigma a_1^{-2} \cdot \Sigma a_2^{-2})^{-1/2} \leq$$

$$\leq R_1R_2(1 - 4\sin\omega_1 \sin\omega_2)$$

(using XI.1.1, (6, 3)).

Starting from XI, 1.1, (5) (i.e. $\Sigma aMA^2 - abc = 2sMI^2$), we get by Aczél's inequality

$$2\sqrt{\frac{s_1s_2}{R_1R_2}} \leq \frac{1}{PI_1 \cdot QI_2}\left(3\sqrt[4]{\Sigma PA_1^4} \cdot \sqrt[4]{\Sigma QA_2^4} - 4\sqrt{F_1F_2}\right),$$

for all points P, Q in the plane. If we take $P = O_1$, $Q = O_2$, we get

$$2\sqrt{(R_1 - 2r_1)(R_2 - 2r_2)s_1s_2} \leqslant 3\sqrt{3}R_1R_2 - 4\sqrt{F_1F_2}$$

(since $O_iI_i^2 = R_i(R_i - 2r_i)$).

Comment by W. Janous. For the last inequality of Mascioni's comment we could proceed a little bit sharper as follows:

Starting again from XI, 1.1, (5), we get, via Aczél's inequality,

$$2\sqrt{s_1s_2}M_1I_1 \cdot M_2I_2 \leqslant \sqrt{(\Sigma a_1M_1A_1^2)(\Sigma a_2M_2A_2^2)} - \sqrt{a_1b_1c_1a_2b_2c_2}.$$

Putting $M_i = O_i$, $i = 1, 2$, we get

$$2\sqrt{s_1s_2R_1R_2(R_1 - 2r_1)(R_2 - 2r_2)} \leqslant 2R_1R_2\sqrt{s_1s_2} - 4\sqrt{F_1F_2R_1R_2}.$$

Using now GI 5.3, i.e. $2s_i \leqslant 3R_i\sqrt{3}$, $i = 1, 2$, we finally arrive at

$$2\sqrt{s_1s_2(R_1 - 2r_1)(R_2 - 2r_2)} \leqslant 2\sqrt{R_1R_2s_1s_2} - 4\sqrt{F_1F_2} \leqslant$$
$$\leqslant 3\sqrt{3}R_1R_2 - 4\sqrt{F_1F_2}.$$

Further, starting from XI.1.1, C(6), i.e. $\Sigma(s - a)MA^2 - 4F(R - r) = sMN^2$, we get

$$\sqrt{s_1s_2}M_1N_1 \cdot M_2N_2 \leqslant \sqrt{(\Sigma(s_1 - a_1)M_1A_1^2)(\Sigma(s_2 - a_2)M_2A_2^2)} -$$
$$- 4\sqrt{F_1F_2(R_1 - r_1)(R_2 - r_2)}.$$

Putting $M_i = O_i$ we get, because of $O_iN_i = R_i - 2r_i$,

$$\sqrt{s_1s_2}(R_1 - 2r_1)(R_2 - 2r_2) \leqslant R_1R_2\sqrt{s_1s_2} -$$
$$- 4\sqrt{F_1F_2(R_1 - r_1)(R_2 - r_2)}.$$

5.13. For any two triangles the following inequalities are valid:

(1) $4F(\Sigma a'/\Pi a')^{1/2} \leqslant \Sigma a^2/a' \leqslant R^2(\Sigma a')^2/\Pi a'$,

(2) $2\sqrt{2}s(r^2/R'r')^{1/2} \leqslant \Sigma a^2/a' \leqslant s'R^2/R'r'$,

(3) $F(\Sigma a'^2)^{1/2}/R'F' \leqslant \Sigma a^2/a'^2 \leqslant (R(\Sigma a'^2)/4R'F')^2$,

(4) $\frac{2F}{R'}(\frac{\sqrt{3}}{F'})^{1/2} \leqslant \Sigma a^2/a'^2$ (M. S. Klamkin),

(5) $16FF'R'/R \leqslant (\Sigma aa')^2/(\Sigma a/a')$,

(6) $(\Sigma aa')/(\Sigma a/a')^2 \leqslant F'RR'/F$.

S. Bilčev and H. Lesov, 'V'rhu edno ravenstvo i sv'rzanite s nego geometrični neravenstva', Ob. po matematika (1986), 4 (1986), 33-42.

5.14. $(R/R')^2 \leqslant \Sigma(4R^2 - a^2)/a'^2$

with equality if tan A tan A' = tan B tan B' = tan C tan C'.

S. Bilčev and H. Lesov, The same reference as in 5.13.

5.15. $(s'/R)\sqrt{2R'r'} \leqslant \Sigma b'c'(s - a)/bc \leqslant s'^2/s$.

These inequalities are due to S. Bilčev and E. Velikova.

5.16. Let r be a real number. Then the following two-triangle inequalities are valid

(1) $\Sigma(-a'^r + b'^r + c'^r)\dfrac{a^4}{a'^r} \geqslant 16F^2$,

(2) $\Sigma a^2/x^r \geqslant (\Sigma a)^2/(\Sigma a'^r)$,

where (x, y, z) is any cyclic permutation of (a', b', c').

S. J. Bilčev and E. A. Velikova, 'Ob odnoj probleme G. Tsintsifasa', Matematika i matematičko obrazovanie, BAN, 1987, 586-593.

5.17. Let P denote a point interior to the triangle ABC, and let r_1, r_2, r_3 denote the distances from P to the sides of the triangle. If p denotes the perimeter of the pedal triangle, then

$$\Sigma(r_1 + r_2)\cos\frac{C}{2} \leqslant p.$$

Equality holds if and only if $r_1 = r_2 = r_3$.

A. G. Ferrer, L. Walker, and C. Abungu, 'Problem E 2517', Amer. Math. Monthly 83 (1976), 204.

5.18. Let a_r, b_r, c_r, r = 1, ..., n, be the sides of n obtuse triangles ($n \geqslant 2$) with $c_r > a_r, b_r$ for each r. Then

$$c_1 \ldots c_n > a_1 \ldots a_n + b_1 \ldots b_n.$$

This is Klamkin's generalization of a problem given by Groenman.

J. T. Groenman, M. S. Klamkin, and J. Dou, 'Problem 1030', Crux Math. 11 (1985), 83 and 12 (1986), 193-194.

5.19. Let a_i, b_i, c_i, R_i, F_i be the sides, radii and areas of n triangles and x_i, y_i, z_i be non-negative numbers ($i = 1, \ldots, n$). Then

$$(R_1 \ldots R_n)^{2/n} \prod_{i=1}^{n} (x_i^2 + y_i^2 + z_i^2)^{2/n} \geqslant \Sigma\left(\prod_{i=1}^{n} y_i z_i a_i\right)^{2/n}$$

and

$$16(F_1 \ldots F_n)^{2/n} \leqslant \prod_{i=1}^{n} (\Sigma x_i^2 b_i^2 c_i^2)^{2/n} / \Sigma\left(\prod_{i=1}^{n} y_i z_i a_i^2\right)^{2/n}.$$

This result is due to W. Janous.

CHAPTER XIII

THE MÖBIUS-NEUBERG AND THE MÖBIUS-POMPEIU THEOREMS

1. The literature devoted to the following theorem and its variations and generalizations is great, and here we cite 89 references.

THEOREM 1.1. If A, B, C and D are arbitrary points in a plane, then AD • BC, BD • CA and CD • AB are proportional to the sides of a triangle FGH. This triangle is degenerate if and only if the points A, B, C, D lie on a circle (in this case we have Ptolemy's well-known theorem).

Many articles have been written on this subject and some of them are not readily available. This chapter offers a history of this theorem and some related results. Certain facts concerning some priorities are brought to light. Some results have been rediscovered several times. There is also a considerable number of papers which offer apparent generalizations as new results.

First, we shall note that Theorem 1.1 is known as the Neuberg-Pompeiu theorem: the "Neuberg" because many authors thought that he had discovered this theorem ([1], 1891), and "Pompeiu" because he had given the following theorem ([2], 1936), known as Pompeiu's theorem:

THEOREM 1.2. Let ABC be an equilateral triangle and D any point in its plane. Then distances AD, BD, CD are lengths of the sides of a triangle.

Of course, Theorem 1.2 is a special case of Theorem 1.1, but after the appearance of Pompeiu's paper, his result preoccupied many mathematicians and many new contributions were given.

However, Theorem 1.1 was proved in 1852 by A. F. Möbius [3]. In his paper, Möbius also gave Theorem 1.2 and the following result:

The angles of the triangle FGH in Theorem 1.1 are:

$$HFG = BDC - BAC = DBA - DCA,$$

$$FGH = CDA - CBA = DCB - DAB,$$

$$GHF = ADB - ACB = DAC - DBC.$$

Möbius, in the proofs, used the following identity

$$[AD] \cdot [BC] + [BD] \cdot [CA] + [CD] \cdot [AB] = 0,$$

where $[AD] = z_1 - z_4$, etc., z_1, z_2, z_3, z_4 are complex numbers corresponding to A, B, C, D, i.e. the identity

$$(1) \qquad (z_1 - z_4)(z_2 - z_3) + (z_2 - z_4)(z_3 - z_1) + (z_3 - z_4)(z_1 - z_2) = 0.$$

Throughout this chapter we will call Theorem 1.1 the Möbius-Neuberg theorem and Theorem 1.2 the Möbius-Pompeiu theorem.

2. In [5] (1903-1904) C. Tweedie gave several very interesting generalizations of the above results:

THEOREM 2.1. If ABC, A'B'C' are any two equilateral triangles in a plane, their vertices being taken in the same sense of rotation, of the three lines AA', BB', CC', the sum of any two is not less than the third.

Remark. If the triangle A'B'C' becomes infinitesimal, the theorem still applies, i.e. in this case we get Theorem 1.2 (see Cor. 2 from [5]).

THEOREM 2.2. If, instead of being equilateral, the triangles ABC and A'B'C' are directly similar, then BC • AA', CA • BB', AB • CC' are proportional to the sides of a triangle. This triangle is degenerate if and only if the centre of similitude lies on the circumcircle of one (and therefore of both) of the given triangles ABC and A'B'C'.

Remark. In particular, if the triangle A'B'C' becomes infinitesimal, we get Theorem 1.1 (see [5], p. 23 and [6]).

In part 5 of his paper, C. Tweedie also noted that the inequality theorem admits an extension to two similar triangles in parallel planes, and there is no case of equality, so long as the planes are distinct. So the following theorems hold:

THEOREM 2.3. Theorem 2.1 is also valid if ABC and A'B'C' are two equilateral triangles in parallel planes, and there is no case of equality, so long as the planes are distinct.

THEOREM 2.4. Theorem 2.2 is also valid if ABC and A'B'C' are two directly similar triangles in parallel planes, and there is no case of equality, so long as the planes are distinct.

Remark. Tweedie proved only Theorem 2.3.

Of course, in the infinitesimal case one has the following two theorems:

THEOREM 2.5. Theorem 1.2 is also valid in the case when D is any point in space.

THEOREM 2.6. Theorem 1.1 is also valid in the case when A, B, C, D are arbitrary points in space.

3. The result of Theorem 1.1 could be found in such classical books as [7] and [8], and then again in [9], [10], [13], [15], [20], [24], [29], [38], [40], [46], [47], [48], [55], [57], [60], [70] and [71].

The proof of Theorem 1.2 could be found in [4] and again in [2], [16], [17], [18], [19], [22], [28], [37], [39], [41], [42], [44], [45], [50], [53], [54], [57] and [64].

Theorem 2.1 was proved in [20], Theorem 2.2 in [12], [20], [55], [69], [72], Theorem 2.5 in [23] and Theorem 2.6 in [31], [46], [49], [68], [77], [78], [84] and [89].

As we said, D. Pompeiu [2], [54] discovered Theorem 1.2. In the proof he used identity (1). After that, this result preoccupied many mathematicians, and in the literature Theorem 1.2 is quoted as Pompeiu's theorem.

N. G. Botea [20] proved Theorem 2.1 by the identity

$$(z_1' - z_1)(z_2 - z_3) + (z_2' - z_2)(z_3 - z_1) + \\ + (z_3' - z_3)(z_1 - z_2) = 0,$$

where z_1, z_2, z_3 and z_1', z_2', z_3' are the affixes of the vertices of two directly similar triangles.

D. Pompeiu [12] proved also the following result:

If ABC and A'B'C' are directly similar triangles and $\overrightarrow{OA_1} = \overrightarrow{AA'}$, $\overrightarrow{OB_1} = \overrightarrow{BB'}$, $\overrightarrow{CC_1} = \overrightarrow{CC'}$, then the triangle $A_1B_1C_1$ is directly similar to the two given ones.

This result and Theorem 1.1 imply Theorem 2.2.

V. Thébault [31] proved Theorem 2.6 as a simple consequence of von Staudt's well-known formula

$$6RV = \sqrt{s(s - aa')(s - bb')(s - cc')},$$

where $a = BC$, $b = CA$, $c = AB$, $a' = AD$, $b' = BD$, $c' = CD$, $s = \frac{1}{2}(aa' + bb' + cc')$ and V and R are the volume and circumradius of a tetrahedron ABCD.

J.-P. Sydler [43] and A. K. Humal [52] proved that the Möbius-Pompeiu theorem is also valid in n-dimensional space.

4. In [9], [10], [49], [50], [60] and [61] more precise versions of Theorems 1.1 and 1.2 are proved.

De Lapierre [9], J. Lhermitte [10] and W. Boomstra [49] proved the following result:

Let PQR be the pedal triangle of a point D with respect to the given triangle ABC. The products AD • BC, BD • CA, CD • AB are proportional to the sides QR, RP, PQ of the triangle PQR.

Z. A. Skopec [50] proved the following statement:

Let ABC be an equilateral triangle and D any point in its plane. There is a triangle A'B'C' with the vertices A', B', C' on the lines BC, CA, AB, respectively, and the sides B'C' = AD, C'A' = BD, A'B' = CD.

This result is a consequence of the constructions DA' ∥ AB, DB' ∥ BC, DC' ∥ CA and the fact that DB'AC', DC'BA', DA'CB' are isosceles trapezoids (possibly with intersecting diagonals).

In [61] Z. A. Skopec proved that the triangle A'B'C' has the area $\frac{\sqrt{3}}{4}|p|$, where p is the potency of point D with respect to the circumcircle of the triangle ABC.

O. Bottema [60] gave the following result:

If the triangles ABC and PQR are given, a point D exists with the property AD : BD : CD = p : q : r if and only if

$$(-ap + bq + cr)(ap - bq + cr)(ap + bq - cr) \geqslant 0.$$

The two triangles may be interchanged.

This result is given in the well-known book [67] (referred to as GI in the text) as Corollary 2 in 15.5. In this book we can also find

Theorem 1.1 as GI 15.4 and 15.5, Theorem 1.2 as Corollary 1 in GI 15.5 and Theorem 2.5 as GI 12.4.

5. Generalizations of the Möbius-Pompeiu theorem for polygons and more general sets of points were proved in [32], [33], [34], [35], [36], [58], [73], [74], [76], [79], [86], [87], [88] and [89].

D. Pompeiu [32] (see also [86]) gave a new direction in the generalizations of his theorem, by proving the following result:

THEOREM 5.1. Let ABCD be a square and P any point in its plane. Then AP, BP, CP, DP are lengths of the sides of a quadrangle.

He also noted that one could prove a generalization of this result, i.e. the following statement:

THEOREM 5.2. Let $A_1A_2 \dots A_n$ be a regular n-gon and P any point in its plane. Then the distances A_1P, A_2P, ..., A_nP are lengths of the sides of an n-gon.

Proofs of Theorem 5.2 were given in [33] and [34].

In [35] D. Pompeiu proved a new generalization:

THEOREM 5.3. Let $A_1A_2 \dots A_n$ be an n-gon with equal sides and P any point in its plane. If B_1, B_2, ..., B_n are the midpoints of the sides A_1A_2, A_2A_3, ..., A_nA_1, then B_1P, B_2P, ..., B_nP are the sides of an n-gon.

The proof is very simple. Let O and z_1, z_2, ..., z_n be the affixes of the points P and A_1, A_2, ..., A_n, respectively, and let $z_{j+1} - z_j = a \cdot \exp(i\phi_j)$, $(j = 1, 2, \dots, n)$, where $z_{n+1} = z_1$. Then from the identity

$$\sum_{j=1}^{n} (z_{j+1}^2 - z_j^2) = 0$$

it follows

$$\sum_{j=1}^{n} (z_j + z_{j+1})e^{i\phi_j} = 0,$$

i.e. the vectors $\frac{1}{2}(z_j + z_{j+1})$ $(j = 1, 2, \dots, n)$ rotated through the angles ϕ_j $(j = 1, 2, \dots, n)$ are parallel to the sides of an n-gon.

M. Zacharias [36] proved that Theorem 5.3 is valid for any n-gon with n = 2m and any point P (in the plane or in the space). This is a consequence of the obvious equality

$$\vec{PB}_1 - \vec{PB}_2 + \vec{PB}_3 - \dots - \vec{PB}_{2n} = \vec{0},$$

where

$$\vec{PB}_j = \frac{1}{2}(\vec{PA}_j + \vec{PA}_{j+1}) \quad (j = 1, 2, \ldots, 2n)$$

and $A_{2n+1} = A_1$.

S. V. Pavlović [58] proved a generalization of these results for two n-gons.

M. Dincă [73], [74] gave the following results:

THEOREM 5.4. Let $A_1A_2 \ldots A_n$ be a regular polygon and P any point in the space. The distances A_jP ($j = 1, 2, \ldots, n$) are the lengths of the sides of a convex polygon.

THEOREM 5.5. Let $A_1A_2 \ldots A_n$ be a regular polygon and P any point in its plane. The powers with exponents $p \in \{1, 2, \ldots, n - 2\}$ of the distances A_jP ($j = 1, 2, \ldots, n$) are the lengths of the sides of a convex polygon.

We shall give Dincă's nice proof of Theorem 5.5.

Let z and the solutions $s_1, s_2, \ldots, s_n$ of the equation $x^n - 1 = 0$ be the affixes of the points P and $A_1, A_2, \ldots, A_n$. Then one has the identity

$$\sum_{j=1}^{n} s_j(z - s_j)^p = 0, \tag{2}$$

which implies Theorem 5.5. Indeed,

$$\sum_{j=1}^{n} s_j(z - s_j)^p = \left(\sum_{j=1}^{n} s_j\right)z^p - \left(\sum_{j=1}^{n} s_j^2\right)C_p^1 z^{p-1} + \ldots + (-1)^p \sum_{j=1}^{n} s_j^{p+1}.$$

The homogeneous symmetric polynomials $\sum_{j=1}^{n} s_j^k$ ($1 \leqslant k \leqslant n - 1$) can be represented by the elementary symmetric polynomials

$$\Sigma s_1, \Sigma s_1 s_2, \ldots, \Sigma s_1 s_2 \ldots s_{n-1},$$

which are, in our case, identically equal zero. Now, Theorem 5.5 is a simple consequence of (2) and the inequality

$$|z_1 + \ldots + z_{n-1}| \leqslant |z_1| + \ldots + |z_{n-1}|.$$

THEOREM 5.6. Theorem 5.5 is also valid if P is any point in the space.

V. Bazon [76] proved the following two theorems:

THEOREM 5.7. Let $\{A_1, A_2, \ldots, A_n\}$ ($n \geqslant 4$) be a set of points in space such that for every $i \in \{1, 2, \ldots, n\}$ there are $j, k, l \in \{1, 2, \ldots, n\}$

with the property that i, j, k, l are mutually different and that $A_iA_j = A_kA_l$. If P is any point, then the distances A_hP (h = 1, 2, ..., n) are the lengths of the sides of an n-gon.

THEOREM 5.8. Let $\{A_1, A_2, \ldots, A_n\}$ ($n \geq 6$) be a set of points on a sphere with radius r such that for every i, j $\in \{1, 2, \ldots, n\}$ ($i \neq j$) it holds $A_iA_j \geq r$. If P is any point, then the distances A_hP (h = 1, 2, ..., n) are the lengths of the sides of an n-gon.

Theorem 5.7 can be applied to the set of vertices of a parallelogram, a parallelepiped or a tetrahedron (and specially a quadrangle) with a pair of equal opposite edges. Theorem 5.8 can be applied to the set of vertices of a regular octahedron or dodecahedron.

Proof of Theorem 5.7: For every i $\in \{1, 2, \ldots, n\}$ we have successively

$$A_iP \leq A_iA_j + A_jP = A_jP + A_kA_l \leq A_jP + A_kP + A_lP \leq \sum_{\substack{h=1\\h\neq i}}^{n} A_hP.$$

Remark. The condition $A_iA_j = A_kA_l$ can be weakened to $A_iA_j \leq A_kA_l$.

Proof of Theorem 5.8: For every i $\in \{1, 2, \ldots, n\}$ let j, k, l, p, q $\in \{1, 2, \ldots, n\}$ be any indices such that i, j, k, l, p, q are mutually different. Then we have $A_jO \leq A_kA_l$ and $OA_j \leq A_pA_q$, where O is the centre of our sphere. Therefore,

$$A_iP \leq A_iO + OA_j + A_jP \leq A_jP + A_kA_l + A_pA_q \leq$$

$$\leq A_jP + A_kP + A_lP + A_pP + A_qP \leq \sum_{\substack{h=1\\h\neq i}}^{n} A_hP.$$

Remark. The conditions of Theorem 5.8 can be weakened so that for every i there are k, l, p, q such that i, k, l, p, q are mutually different and $A_kA_l \geq r$, $A_pA_q \geq r$.

The following generalization of Theorem 2.1 is given in [79]:

THEOREM 5.9. Let $A_1A_2 \ldots A_n$ and $A_1'A_2' \ldots A_n'$ be any two regular polygons, which vertices are taken in the same sense of rotation, situated in parallel planes. Let $A_iA_i' = a_i$ (i = 1, 2, ..., n) and let p be an integer with $1 \leq p \leq n - 2$. Then a_i^p (i = 1, 2, ..., n) are lengths of the sides of an n-gon (possible degenerate).

I. Pop also proved the following theorem.

THEOREM 5.10. Let $A_1A_2A_3$ be an equilateral triangle with the sides a, P any point in space on distance h from its plane, and $a_i = A_iP$ (i = 1, 2, 3). If $h > a$, then a_i^2 $(i = 1, 2, 3)$ are lengths of the sides of a triangle.

For the proof of Theorem 5.10, I. Pop proved the following lemma, for which we shall later give a new proof:

LEMMA 5.11. Let $A_1A_2A_3$ be an equilateral triangle with the side a and N a point in its plane. Then

$$NA_j^2 - \sum_{i \neq j} NA_i^2 \leqslant a^2 \quad (j = 1, 2, 3).$$

Proof of Theorem 5.10: Let N be the projection of P onto the plane $A_1A_2A_3$. Then for every $i \in \{1, 2, 3\}$ we have according to the Lemma 5.11,

$$\sum_{i \neq j} PA_i^2 - PA_j^2 = \sum_{i \neq j} NA_i^2 - NA_j^2 + h^2 \geqslant h^2 - a^2 \geqslant 0.$$

6. In [9], [10], [11], [25], [27], [45], [55], [57] and [75] there were considered some inverse problems for Theorems 1.1 and 1.2.

De Lapierre [9], J. Lhermitte [10] and Huisman [11] gave the construction of a point D such that its distances DA, DB, DC from the vertices of a triangle ABC are proportional to given three numbers.

S. V. Pavlović [45], [55] and [57] proved that the conversion of the Möbius-Pompeiu theorem is possible, and he gave a negative answer to this question for the generalized Möbius-Pompeiu theorem in the case of regular n-gons.

N. Schaumberger [75] proposed the following problem:

Find the length of a side of an equilateral triangle in which the distances from its vertices to an interior point are 5, 7 and 8.

In his solution, H. Eves [75] proved the following two theorems:

THEOREM 6.1. Let s be the length of one side of an equilateral triangle ABC with circumcircle γ. Let D be a point in the same plane and let $a = DA$, $b = DB$, $c = DC$. Then

$$s^2 = \frac{1}{2}(a^2 + b^2 + c^2) \pm \tag{3}$$
$$\pm \frac{1}{2} \sqrt{3(a + b + c)(-a + b + c)(a - b + c)(a + b - c)},$$

where the radicand is zero if and only if D is on γ, and the '+' or '-' sign before the radicand holds according as D lies inside or outside γ, respectively.

THEOREM 6.2. If a, b, and c are positive numbers such that

$$a + b - c > 0, \quad a - b + c > 0, \quad -a + b + c > 0, \tag{4}$$

then, for any point D in the plane, there are two non-congruent equilateral triangles ABC such that $a = AD$, $b = BD$, $c = CD$. The sizes of the triangles are uniquely determined. D lies inside the circumcircle of the larger triangle and outside the circumcircle of the smaller one. If one of the inequalities (4) is replaced by an equality, then the size of the triangle ABC is uniquely determined, and D lies on the circumcircle of this triangle.

Here we shall also quote from the Editor's Notes about this problem:

"Eves gave the following formula for the side s of a convex regular n-gon, given the respective distances a, b, and c from a point P in the interior of the n-gon to three consecutive vertices of the n-gon:

$$s^2 = a^2 + 2b^2 \cos\gamma + c^2 \pm \Big(4(1 + \cos\gamma)(a^2b^2 + b^2c^2) + \\ + 2(1 + 2m\cos\gamma)c^2a^2 + (1 - 2m)(a^4 + c^4) + \\ + 4(\cos^2\gamma - 1)b^4\Big)^{1/2},$$

where $\gamma = ((n - 2)/n)180°$ and $m = \sqrt{1 - \cos\gamma}$."

T. C. Wales noted that the problem appears in L. A. Graham, 'Ingenious Mathematical Problems and Methods', Dover, NY (1959), p. 34 and pp. 189-190; and that the equations (3) are jointly equivalent to the equation

$$(5) \qquad 3(a^4 + b^4 + c^4 + s^4) = (a^2 + b^2 + c^2 + s^2)^2.$$

A. Wayne noted that he had himself proposed the same problem with the distances 3, 4 and 5 as Problem 3682, School Science and Mathematics, April 1977, pp. 353-354, and the solution appeared in the February 1978 issue, pp. 174-176, along with a note by C. W. Trigg that the problem had been discussed on page 450 of the April 1933 issue.

7. In [14], [21], [26], [51], [56], [59], [62], [63], [65] and [66] were given some solutions of the following problem and some of its analogues:

Let ABC be any triangle. Find the set of all points D in its plane (or in the space) such that the distances AD, BD, CD are the lengths of the sides of a triangle (or of a degenerate triangle).

8. In [70] and [72] were given some applications of the Möbius-Neuberg and Möbius-Pompeiu theorems.

G. Bercea [70] noted that Theorem 1.1 is an answer to the following problem of M. S. Klamkin:

If A, B, C denote the angles of an arbitrary triangle, then it is known (GI, p. 120) that the three triples (sin A, sin B, sin C), $(\cos\frac{A}{2}, \cos\frac{B}{2}, \cos\frac{C}{2})$, $(\cos^2\frac{A}{2}, \cos^2\frac{B}{2}, \cos^2\frac{C}{2})$ are sides of three triangles. Give a generalization which includes the latter three cases as special cases.

The above special cases follow upon putting in Theorem 1.1 D = O (the circumcentre), D = I (the incentre) and D = P where the point P is given by

$$PA/(s - a) = PB/(s - b) = PC/(s - c) = k, \quad a = BC,$$

$$b = CA, \quad c = AB$$

and s is semi-perimeter.

As we said, the proof of Theorem 1.1 is based on the identity (1) for complex numbers and the basic triangle inequality

(6) $|u + v| \leqslant |u| + |v|.$

M. S. Klamkin [72] exploited this simple method by applying (6) to a number of rather simple complex number identities to obtain simpler proofs of some known inequalities as well as obtaining some new inequalities. Here we shall give results similar to Theorem 1.1.

Let $A_1A_2A_3$ be a triangle with sides $a_1 = A_2A_3$, $a_2 = A_3A_1$, $a_3 = A_1A_2$, P a point in the plane of the triangle and $R_i = PA_i$ $(i = 1, 2, 3)$. Finally, M_1, etc. R, r, Δ denote the median from A_1, the circumradius, the inradius, the area, respectively, of $A_1A_2A_3$ and m_1, Δ_1, etc., denote the median from P and area of PA_2A_3. Then the following results are valid:

1° (a_1D_1, a_2D_2, a_3D_3) form a triangle, where

$$D_3^2 = \Sigma(R_1^2 + a_1^2) - 2(R_3^2 + a_3^2), \quad \text{etc.}$$

2° $(a_1E_1^2, a_2E_2^2, a_3E_3^2)$ form a triangle, where

$$E_1^4 = 3(R_2^4 + R_2^2R_3^2 + R_3^4) - 3a_1^2(R_2^2 + R_3^2) + a_1^4, \quad \text{etc.}$$

3° $(a_1^3M_1, a_2^3M_2, a_3^3M_3)$ form a triangle.

4° $(a_1M_1^3, a_2M_2^3, a_3M_3^3)$ form a triangle.

5° $(a_1m_1F_1^2, a_2m_2F_2^2, a_3m_3F_3^2)$ form a triangle, where

$$F_1^4 = (R_2^2 + R_3^2)^2 - 16\Delta_1^2, \quad \text{etc.}$$

6° $(a_1m_1R_1^2, a_2m_2R_2^2, a_3m_3R_3^2)$ form a triangle.

9. M. S. Klamkin and A. Meir [85] considered a real normed vector space which is called Ptolemaic if

$$\|x - y\|\|z\| + \|y - z\|\|x\| \geqslant \|z - x\|\|y\|$$

for all x, y, z in this space. They treated some results of I. J. Schoenberg [82] on Ptolemaic spaces and also gave related references [80], [81] and [84]. For an interesting application of Theorem 1.1 they noted [83].

10. Now, we shall give some comments on the above results:

1) As we noted in 2, Tweedie did not give a proof of Theorem 2.4. Here we shall give a simple proof of this theorem.

First we note the following result - a simple consequence of

Minkowski's inequality:

LEMMA 10.1. Let two triangles with sides a, b, c and a_1, b_1, c_1 be given. If $p \geqslant 1$, then $(a^p + a_1^p)^{1/p}$, $(b^p + b_1^p)^{1/p}$, $(c^p + c_1^p)^{1/p}$, are also sides of a triangle.

Now, we shall prove

THEOREM 10.2. Let a, b, c be the sides of a given triangle. If ax, by and cz $(x, y, z > 0)$ could be sides of a triangle, then the same holds for $a\sqrt{x^2 + d^2}$, $b\sqrt{y^2 + d^2}$, $c\sqrt{z^2 + d^2}$ $(d > 0)$.

Proof 1. Put in Lemma 1: $p = 2$, $a \to ax$, $b \to by$, $c \to cz$, $a_1 \to ad$, $b_1 \to bd$, $c_1 \to cd$, then Theorem 10.2 follows.

Proof 2. Here we shall give a direct proof. First, we shall prove the following inequality

$$\sqrt{(ax + cz)^2 + b^2d^2} \leqslant a\sqrt{x^2 + d^2} + c\sqrt{z^2 + d^2}. \tag{7}$$

By squaring, (7) becomes

$$2acxz - d^2(a^2 + c^2 - b^2) \leqslant 2ac\sqrt{(x^2 + d^2)(z^2 + d^2)}.$$

Squaring this inequality again we get

$$d^2(4acxz(a^2 + c^2 - b^2) + 4a^2c^2(x^2 + z^2) + \\ + d^2(4a^2c^2 - (a^2 + c^2 - b^2)^2)) \geqslant 0,$$

i.e.

$$d^2(4acxz((a + c)^2 - b^2) + 4a^2c^2(x - z)^2 + \\ + d^2(b^2 - (a - c)^2)((a + c)^2 - b^2)) \geqslant 0$$

which is evidently true (a, b, c are the sides of the triangle).

On the other hand by $\leqslant ax + cz$, i.e.

$$\sqrt{b^2y^2 + b^2d^2} \leqslant \sqrt{(ax + cz)^2 + b^2d^2},$$

which together with (7) gives

$$b\sqrt{y^2 + d^2} \leqslant a\sqrt{x^2 + d^2} + c\sqrt{z^2 + d^2}.$$

Proof of Theorem 2.4. Note that ABC and A'B'C' are two directly similar triangles in parallel planes. Let $A_1B_1C_1$ be the projection of triangle A'B'C' in the plane of the triangle ABC. This triangle is directly similar to triangle ABC, so using Theorem 2.2 we have that $BC \cdot AA_1$, $CA \cdot BB_1$, $AB \cdot CC_1$ could be the sides of a triangle. Now, using

Theorem 10.2, we find that the same holds for

$$BC\sqrt{AA_1^2 + d^2}, \quad CA\sqrt{BB_1^2 + d^2} \quad \text{and} \quad AB\sqrt{CC_1^2 + d^2},$$

i.e., for

$$BC \cdot AA', \quad CA \cdot BB' \quad \text{and} \quad AB \cdot CC',$$

where d is the distance of the two parallel planes.

Of course, Theorems 2.6, 2.5 and 2.3 are similar consequences of Theorem 10.2 and of Theorems 1.1, 1.2 and 2.1, respectively.

As we noted in 3, A. K. Humal [52] proved Möbius-Pompeiu's theorem for n-dimensional space. Note that using Theorem 10.2 we can similarly prove that Theorems 2.3, 2.4 and 2.6 are also valid in n-dimensional space. For example, we shall give:

Proof of Theorem 2.6 in n-dimensional space. Let a cartesian coordinate system be chosen such that points A, B, C are in the xy-plane, i.e. their coordinates are: $A(x_1, y_1, 0, \ldots, 0)$, $B(x_2, y_2, 0, \ldots, 0)$, $C(x_3, y_3, 0, \ldots, 0)$, and D is an arbitrary point in the space, i.e. $D(x_4, y_4, z_3, \ldots, z_n)$. Let AB = c, CA = b, BC = a, AD = x, BD = y, CD = z, $d^2 = z_3^2 + \ldots + z_n^2$, x' = AD', y' = BD', z' = CD', where D' is a projection of point D to the plane xy, i.e. $D'(x_4, y_4, 0, \ldots, 0)$. Then

$$x = \sqrt{(x_4 - x_1)^2 + (y_4 - y_1)^2 + z_3^2 + \ldots + z_n^2} = \sqrt{x'^2 + d^2}, \quad \text{etc.}$$

On the basis of Theorem 1.1, ax', by' and cz' could be sides of a triangle, so from Theorem 10.2 follows the same for

$$a\sqrt{x'^2 + d^2} = ax, \quad b\sqrt{y'^2 + d^2} = by, \quad c\sqrt{z'^2 + d^2} = cz.$$

2) Now, we shall prove the following theorem:

THEOREM 10.3. If P is a point in space, then (a_1D_1, a_3D_3), where a_1, D_1, etc., are defined as in 8.1°, form a triangle, too.

Proof. Let P' be a projection of P to the plane of the triangle, and let R_i', D_i' (i = 1, 2, 3) be defined as R_i, D_i but instead of point P we have P'. Using 8.1°, $(a_1D_1', a_2D_2', a_3D_3')$ form a triangle. Since

$$D_i^2 = \Sigma(R_1^2 + a_1^2) - 2(R_i^2 + a_i^2) = \Sigma(R_1'^2 + d^2 + a_1^2) -$$

$$- 2(R_i'^2 + d^2 + a_i^2) = \Sigma(R_1'^2 + a_1^2) - 2(R_i'^2 + a_i^2) + d^2 =$$

$$= D_i'^2 + d^2 \quad (i = 1, 2, 3),$$

where d is the distance of the point P to the plane of the triangle.

Now, using Theorem 10.2, we have that $a_i\sqrt{D_i'^2 + d^2} = a_iD_i$ (i = 1, 2, 3) could be sides of a triangle.

Remark. Theorem 10.3 holds in n-dimensional space, too.

3) Now, we shall give some simple extensions of Theorems 5.4, 5.5, 5.6 and 5.9.

The following result is a simple generalization of GI 13.3:

LEMMA 10.4. Let $f(x)$ be any non-negative, non-decreasing, subadditive function on the domain $x > 0$. If $a_1, a_2, \ldots, a_n$ form a polygon, then $f(a_1), f(a_2), \ldots, f(a_n)$ form a polygon, too.

For example, $f(x) = x^{1/p}$ $(p \geqslant 1)$ satisfies conditions of Lemma 10.4.

THEOREM 10.5. Let $A_1A_2 \ldots A_n$ be a regular polygon and M an arbitrary point in space. The expressions MA_i^p, i = 1, ..., n, where p is a real number from $[0, n-2]$ could be the sides of a convex polygon.

Proof. Theorem 5.6 says that MA_i^{n-2} (i = 1, ..., n) could be sides of a convex polygon. For $p \in [0, n-2]$, the function $f(x) = x^{p/(n-2)}$ satisfies the conditions of Lemma 10.4, and the theorem follows.

Similarly, using Theorem 5.9 and Lemma 10.4 we can prove:

THEOREM 10.6. Let $A_1A_2 \ldots A_n$ and $A_1'A_2' \ldots A_n'$ be any two regular polygons, which vertices are taken in the same sense of rotation, situated in parallel planes. The expressions $A_iA_i'^p$, i = 1, ..., n, where p is a real number from $[0, n-2]$ could be sides of a polygon (possibly degenerate).

Remark. Theorem 10.5 and 10.6 are also valid in n-dimensional space.

4) Now we give a simple proof of Lemma 5.11:

Let a triangle be given in a complex plane, and let the affixes of its vertices s_1, s_2, s_3 be the solutions of the equation $x^3 - R^3 = 0$. Then

$$\sum_{i=1}^{3} s_i(z - s_i)^2 = z^2\left(\sum_{i=1}^{3} s_i\right) - 2z\left(\sum_{i=1}^{3} s_i^2\right) + \left(\sum_{i=1}^{3} s_i^3\right) =$$

$$= \sum_{i=1}^{3} s_i^3 = 3R^3.$$

Therefore,

$$|s_j(z - s_j)^2| \leqslant \sum_{i \neq j} |s_i(z - s_i)|^2 + \sum_{i=1}^{3} |s_i|^3$$

and since $|s_i| = R$, $3R^2 = a^2$, we get

$$NA_j^2 \leqslant \sum_{i \neq j} NA_i^2 + a^2.$$

Similarly, we can prove the following generalization of Lemma 5.11:

LEMMA 10.7. Let $A_1A_2 \ldots A_n$ be a regular polygon, and R its circumradius. If N is a point in the plane of the polygon, then the following inequality is valid

$$NA_j^{n-1} - \sum_{i \neq j} NA_i^{n-1} \leqslant nR^{n-1} \quad (j = 1, \ldots, n).$$

Remark. Theorem 5.10 is also valid in n-dimensional space.

Comment by W. Janous. Theorem 5.10 can be generalized in the following (ultimate) way.

Let $A_1A_2A_3$ be an equilateral triangle in E^n, $n \geqslant 3$, with the sides a and P be a point of E^n. Let be $a_i = A_iP$ ($i = 1, 2, 3$). Furthermore, let M_1, M_2, M_3 be the points of intersection of the lines parallel to A_1A_2, A_2A_3, A_3A_1 and passing through A_3, A_1, A_2, respectively. Then it is necessary and sufficient for a_1^2, a_2^2, a_3^2 to form a triangle that P lies in the exterior of the three spheres with centres M_1, M_2, M_3 and radii a.

References

1. J. Neuberg, 'Sur les quadrangles complets', Mathesis (2) 1 (1891), 81-82.
2. D. Pompeiu, 'Une identité entre nombres complexes et un théorème de géométrie élémentaire', Bull. math. phys. Ecole polytechn. Bucarest 6 (1936), 6-7.
3. A. F. Möbius, 'Über eine Methode, um von Relationen, welche der Longimetrie angehören, zu entsprechenden Sätzen der Planimetrie zu gelangen', Ber. Verhandl. königl. Sächs. Ges. Wiss. Leipzig Math. Phys. Cl. 1852, 41-54.
4. J. Hadamard, Géométrie élémentaire, I, Paris 1898, p. 233.
5. C. Tweedie, 'Inequality Theorem Regarding the Lines Joining Corresponding Vertices of Two Equilateral, or Directly Similar, Triangles', Proc. Edinburgh Math. Soc. 22 (1903-1904), 22-26.
6. P. Pinkerton, 'Note on Mr. Tweedie's Theorem in Geometry', Ibid. 22 (1903-1904), 27.
7. F. G.-M. Exercises de géométrie, 5. éd., Tours-Paris 1912, pp. 522-523.
8. E. Rouché and Ch. de Comberousse, Traité de géométrie, I, 8. éd., Paris 1912, pp. 158-159.
9. De Lapierre, 'Relation entre deux problèmes classiques', Ens. sci. 4 (1930-1931), 25-26.
10. J. Lhermitte, 'Sur le lieu des point dont les distances aux trois sommets d'un triangle ABC sont proportionnelles aux nombres α, β, γ, Ibid. 4 (1930-1931), 150-152.
11. Huisman, 'Note', Ibid. 4 (1930-1931), 152-153.

12. D. Pompeiu, 'Remarques sur la note de M. Teodoriu', Bull. math. phys. Ecole polytechn. Bucarest 1 (1930), 161-163.
13. G. R. Veldkamp, 'Afleiding van eenige eigenschappen van koordenvierhoeken', Nieuw Tijdschr. Wisk. 18 (1931), 293-295.
14. D. Barbilian, 'Exkurs über die Dreiecke', Bull. math. Soc. Roumaine sci. 38,II (1936), 3-62.
15. G. Bratu, 'Un théorème de M. Pompeiu', Bull. math. phys. Ecole polytechn. Bucarest 7 (1937), 3.
16. N. Obrechkoff, 'Sur un théorème de M. Pompeiu', Ibid. 7 (1937), 4-5.
17. H. Hilton, 'Sur un théorème de M. D. Pompeiu', Ibid. 7 (1937), 5.
18. E. Abason, 'Sur une propriété du triangle équilatéral', Ibid. 7 (1937), 6-8.
19. E. Abason, 'Un théorème de géométrie', Ibid. 7 (1937), 8-13.
20. N. G. Botea, 'Une relation entre nombres complexes et la généralisation d'un théorème de géométrie élémentaire', Ibid. 7 (1937), 13-14.
21. D. Barbilian, 'Eine Verallgemeinung des Pompeiuschen Dreieckssatzes', Bull. math. Soc. Roumaine sci. 39 I (1937), 3-12.
22. E. Abason, 'Articol despre triunghiul echilateral', Gaz. Mat. 42 (1937), 352-360, 393-406.
23. G. Bratu, 'Sur le trapèze rectangle', Mathematica (Cluj) 13 (1937), 263-269.
24. P. Montel, 'Sur un problème de J. Bertrand', Bull. math. phys. École polytechn. Bucarest 9 (1937-1938), 1-7.
25. A. Angelesco, 'Sur un théorème de M. D. Pompeiu', Ibid. 9 (1937-1938), 7-8.
26. D. Germani, 'Sur un théorème de M. D. Pompeiu', Ibid. 9 (1937-1938), 12-17.
27. E. M. Abason, 'Sur la réciproque d'un théorème relatif au triangle équilatéral', Ibid. 9 (1937-1938), 27-37.
28. A. Hoborski, 'Elementarer Beweis eines planimetrischen Satzes des Herrn D. Pompeiu', Ibid. 9 (1937-1938), 9-12.
29. C. Postelnicesco, 'Sur un théorème de M. D. Pompeiu', Ibid. 9 (1937-1938), 38-39.
30. M. Petrovitch, 'A propos d'un théorème de M. Pompeiu', Bull. math. Soc. Roumaine sci. 40 (1938), 205-208.
31. V. Thébault, 'Sur un théorème de M. D. Pompeiu', Bull. math. phys. École polytechn. Bucarest 10 (1938-1939), 38-42.
32. D. Pompeiu, 'La géométrie et les imaginaires: démonstration de quelques théorèmes élémentaires', Ibid. 11 (1939), 29-34.
33. Ş. Gheorghiu, 'Sur un théorème de M. Pompeiu', Ibid. 12 (1941), 221-234.
34. T. Angheluţa, 'Une identité entre nombres complexes et un théorème de géométrie élémentaire', Bull. sci. Ecole polytechn. Timisoara 10 (1941), 249-252.
35. D. Pompeiu, 'Un théorème de géométrie', Bull. Sect. sci. Acad. Roumaine 24 (1941), 223-226.
36. M. Zacharias, 'Ein geometrischer Satz', Ibid. 25 (1943), 247-248.
37. I. Popa, 'Sur un théorème de géométrie élémentaire', Bull. École polytechn. Iaşi 2 (1947), 79-80.
38. G. H. Hardy, A Course of Pure Mathematics, III, Cambridge 1948, p. 105.
39. C. F. Manara, 'Un teorema sui triangoli equilateri', Period. Mat. (4) 31 (1953), 186-188.

40. S. V. Pavlović, 'Sur une démonstration géométrique d'un théorème de D. Pompeiu', Elem. Math. 8 (1953), 13-15.
41. J.-P. Sydler, 'Autre démonstration du théorème de Pompeiu', Ibid. 8 (1953), 15-16.
42. R. Lauffer, 'Ein Satz der elementaren Geometrie', Ibid. 8 (1953), 65, 136.
43. J.-P. Sydler, 'Généralisation d'un théorème de M. Pompeiu', Ibid. 8 (1953), 136-138.
44. R. Deaux, 'Théorème de Ptolémée et nombres complexes', Mathesis 65 (1956), 96-98.
45. S. V. Pavlović, 'Sur deux théorèmes de D. Pompeiu', Ibid. 65 (1956), 211-214.
46. W. Hansen, 'Ein Beweis der Ptolemäischen Ungleichung', Arch. Math. 7 (1956), 320-322.
47. G. R. Veldkamp, 'Een stelling uit de elementaire meetkunde van het platte vlak', Nieuw Tijdschr. Wisk. 44 (1956), 1-4.
48. O. Bottema, 'De stelling van Pompeiu', Ibid. 44 (1956), 183-184.
49. W. Boomstra, 'De stelling van Pompeiu nogmaals', Ibid. 44 (1956-1957), 285-288.
50. Z. A. Skopec, 'Nekotorie zamečanija k teoreme Pompeju', Mat. prosv. 1957, No. 2, 205-210.
51. S. Manolov and D. Kolarov, 'Opredeljane na oblastite v prostranstvoto, točkite na koito imat edno metrično svojstvo', Godišnik Univ. Sofija Mat. Fak. 52 (1957-1958), kn. 1, 165-181 (1959).
52. A. K. Humal, 'Dokazatel'stvo teoremi Pompeiju v n-mernom prostranstve, Izv. AN EstSSR 7 (1958), No. 2, 154-155.
53. Ju. M. Gajduk and A. N. Hovanskij, 'Kratkij obzor issledovanij po geometrii treugoljnika', Mat. v škole 1958, No. 5, 50-58.
54. D. Pompeiu, Opera Matematica, Bucarest 1959.
55. S. V. Pavlović, 'Sur un théorème de Neuberg et son inversion', Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 29-32 (1959), 5-9.
56. S. Manolov, 'Zadača 22', Fiz.-mat. spisanie 3 (36) (1960), 152-153.
57. S. V. Pavlović, 'O Pompeiu-Neubergovoj teoremi', Mat. biblioteka 19 (1961), 103-108.
58. S. V. Pavlović, 'Über die Erweiterung eines M. Zacharias-D. Pompeiu'schen Satzes', Publ. Inst. math. Beograd 1 (15) (1961), 93-100 (1962).
59. S. Manolov and D. Kolarov, 'Edin način na razdeljane prostranstvoto na oblasti', Fiz.-mat. spisanie 6 (39) (1963), 31-37.
60. O. Bottema, Euclides 38 (1963-1964), 129-137.
61. Z. A. Skopec, 'Ploščad treugoljnika Pompeju', Mat. v škole 1964, No. 2, 68-70.
62. S. Manolov and D. Kolarov, 'Edin način za razdeljane na pravite v ravninata na dve klasi', Fiz.-mat. spisanie 9 (1966), 272-278.
63. S. V. Pavlović, 'Über die Erweiterung eines elementargeometrischen Satzes von D. Pompeiu', Abh. Math. Sem. Hamburg 30 (1967), 54-60.
64. Z. A. Skopec, 'Priloženija kompleksnih čisel k zadačam elementarnoj geometrii', Mat. v škole 1967, No. 1, 63-71.
65. V. Kortenska, 'Vrhu edin način za razdeljane na pravite v ravninata na dve klasi', Godišnik Visš. tehn. učebn. zaved. Mat. 4 (1967), kn. 3, 139-147 (1972).
66. O. Bottema, 'Eine Erweiterung des Pompeiuschen Satzes', Abh. Math. Sem. Hamburg 33 (1969), 119-123.

67. GI: O. Bottema, R. Ž. Djordjević, R. R. Janić, D. S. Mitrinović, and P. M. Vasić, Geometric Inequalities, Groningen 1969.
68. D. Veljan, 'Zadatak 1018', Mat.-fiz. list 21 (1970-1971), 78-79.
69. 'Zadača 1330', Mat. v škole 1974, No. 1, 63 and No. 5, 87.
70. M. S. Klamkin and G. Bercea, 'Problem 729', Elem. Math. 29 (1974), 155 and 30 (1975), 133-134.
71. W. Mnich, 'O nierowności Ptolemeusza', Matematyka 31 (1978), 39-40.
72. M. S. Klamkin, 'Triangle Inequalities from the Triangle Inequality', Elem. Math. 34 (1979), 49-55.
73. M. Dincă, 'Extinderi ale teoremi lui Dimitrie Pompeiu', Gaz. Mat. 84 (1979), 368-370.
74. M. Dincă, 'In legătură cu nota matematică "Extinderi ale teoremei lui Dimitrie Pompeiu"', Gaz. Mat. 85 (1980), 198-199.
75. N. Schaumberger and H. Eves, 'Problem 187', College math. J. 1981, 155 and 1982, 278-282.
76. V. Bazon, 'In legătură cu teorema lui Pompeiu', Gaz. Mat. 88 (1983), 376-378.
77. M. Lascu, 'Asupra unor inegalităti într-un tetraedru', Gaz. Mat. 89 (1984), 397-399.
78. H. Zhu, 'A Tetrahedron Inequality', Math. Gaz. 68 (1984), 200-202.
79. I. Pop, 'In legătură cu teorema lui Dimitrie Pompeiu asupra triunghiului echilateral', Gaz. Mat. 90 (1985), 70-73.
80. T. Hayashi, 'Two Theorems on Complex Numbers', Tôhoku math. J. 4 (1913-1914), 68-70.
81. S. Barnard and J. M. Child, Higher Algebra, London 1949, p. 78.
82. I. J. Schoenberg, 'A Remark on M. M. Day's Characterization of Inner-product Spaces and a Conjecture of L. M. Blumenthal', Proc. Amer. Math. Soc. 3 (1952), 961-964.
83. D. Pedoe, 'A Geometric Proof of the Equivalence of Fermat's Principle and Snell's Law', Amer. Math. Monthly 71 (1964), 543.
84. T. M. Apostol, 'Ptolemy's Inequality and the Chordal Metric', Math. Mag. 40 (1967), 233-235.
85. M. S. Klamkin and A. Meir, 'Ptolemy's Inequality, Chordal Metric, Multiplicative Metric', Pacific J. Math. 101 (1982), 389-392.
86. N. Racliş, 'Théorème de fermeture', Bull. math. phys. École polytechn. Bucarest 12 (1941), 5-8.
87. T. Popoviciu, 'Quelques remarques sur un théorème de M. Pompeiu', Bull. math. Soc. Roumaine sci. 43 (1941), 27-43.
88. G. G. Constantinescu, 'On Some Geometrical Theorems' (Romanian), Pozitiva 1 (1941), 257-263.
89. J. Haantjes, 'De stelling van Ptolemeus', Simon Stevin 29 (1952), 25-31.

CHAPTER XIV

INEQUALITIES FOR QUADRILATERALS

1. Let ABCD be an inscribable quadrilateral. Then

$$p^{-2} + q^{-2} \leqslant 4^{-1}(a^{-2} + b^{-2} + c^{-2} + d^{-2}).$$

Ja. N. Sukonnik, 'Problem 823', Mat. v Škole 1970, No. 5, 72 and 1971, No. 3, 80.

2. Let ABCD be a given quadrilateral. Then

$$p^2q^2 \geqslant 4abcd \sin^2 \frac{A + C}{2}. \tag{1}$$

Proof. If we take the Brettschneider theorem in the form

$$p^2q^2 = (ac + bd)^2 - 4abcd \cos^2 \frac{A + C}{2},$$

then in view of $(ac + bd)^2 \geqslant 4abcd$ we obtain (1).

This result is due to M. Naydenov. For $A + C = \pi/2$, we get a result of W. Janous.

3. Let ABCD be a convex quadrilateral with $A + C > \pi$. Then,

$$\frac{p}{q} < \frac{ad + bc}{ab + cd}.$$

A. N. Danilov, 'Problem 593', Mat. v Škole 1969, No. 2, 73 and 1969, No. 6, 69.

4. In any convex quadrilateral the following inequalities are valid

$$2 < \sum \sin \frac{A}{2} \leqslant 2\sqrt{2} \quad \text{and} \quad 4(\sqrt{2} - 1) \leqslant \sum \tan \frac{A}{4} < 2.$$

This result is due to G. Mircea.

5. In any inscribable quadrilateral

$$\sum a^2 \geqslant pq + 2F/\sin \alpha \qquad \alpha \in \{A, B, C, D\},$$

with equality only if the quadrilateral is a square.

G. E. Müller, 'Problem 19700', Gaz. Mat. (Bucharest) 88 (1983), 220.

6. If a, b and p, q denote the lengths of sides and diagonals, respect-

ively, of a parallelogram, then

$$|a^2 - b^2| < pq.$$

T. P. Grigorjeva, 'Problem 1965', Mat. v Škole 1978, No. 2, 80 and 1979, No. 1, 70.

7. Let p and q be the lengths of diagonals of a trapezium that is circumscribed about a circle. If r denotes the inradius, then

$$p^2 + q^2 \geqslant 16r^2.$$

Remark. This result of Ja. N. Sukonnik is a generalization of a result of Ju. I. Gerasimov.

Ja. N. Sukonnik, 'Problem 1424', Mat. v Škole 1974, No. 5, 80 and 1975, No. 4, 82.

Ju. I. Gerasimov, 'Problem 639', Mat. v Škole 1969, No. 4, 73 and 1970, No. 3, 80.

8. If a quadrilateral has both an incircle and circumcircle, then

$$(1) \qquad (a + b + c + d)^2 \geqslant 8pq$$

with equality if and only if the quadrilateral is a square.

This inequality is due to M. S. Klamkin.

Comment by V. Mascioni. This inequality is equivalent to the trivial mean inequality $M_1(a, b, c, d) \leqslant M_2(a, b, c, d)$. In fact, the following are equivalent (for inscribable and circumscribable quadrilaterals, of course):

$$(1) \qquad (a + b + c + d)^2 \geqslant 8pq$$

$$4(a + c)^2 \geqslant 8pq$$

$$2(a + c)^2 + 2(b + d)^2 \geqslant 8pq$$

$$\Sigma a^2 \geqslant 2pq$$

$$4\Sigma a^2 \geqslant 2\Sigma a^2 + 4pq$$

$$4\Sigma a^2 \geqslant (\Sigma a)^2$$

$$M_2(a, b, c, d) \geqslant M_1(a, b, c, d).$$

A simple proof of (1) is given by

$$(\Sigma a)^2 = 4(a + c)^2 = 2(a + c)^2 + 2(b + d)^2 \geqslant 8ac + 8bd = 8pq$$

(note that for a circumscribable quadrilateral we have $a + c = b + d$, and for an inscribable quadrilateral $pq = ac + bd$).

Remark. (M. S. Klamkin) The stated inequality is also equivalent to $(\Sigma a)^2 \geqslant 8(ac + bd)$, for circumscribable quadrilaterals.

9. Let ABCD be any inscribable quadrilateral. Then the following inequality is valid

$$\frac{8\sqrt{abcd}}{(a + b)^2 + (c + d)^2} \leqslant \frac{\sin B}{\sin A} \leqslant \frac{(a + d)^2 + (b + c)^2}{8\sqrt{abcd}}.$$

Gh. Stoica, 'Problem 19466', Gaz. Mat. (Bucharest) **87** (1982), 423-424.

10. Let ABCD be any inscribable quadrilateral. Then the following inequalities are valid

(a) $\frac{1}{p^2} + \frac{1}{q^2} \leqslant (\frac{s}{2F})^2$; (b) $pq > \frac{1}{2}(\frac{2F}{s})^2$.

J. Brejcha and M. Kvašík, 'Problem 1', Rozhledy Mat. Fyz. **49** (1970-1971), 26-28.

11. If in a convex quadrilateral with $a \leqslant b \leqslant c \leqslant d$, then

$$F \leqslant \frac{3\sqrt{3}}{4} c^2.$$

E. Popa, 'Problem 8841', Gaz. Mat. (Bucharest) **B 20**, (1969), 100-102.

12. In a convex quadrilateral it is true that

(a) $2F \leqslant ab + cd$; (b) $2F \leqslant ac + bd$.

Equality holds exactly when the quadrilateral is inscribable and
(a) AC is a diameter of the circumcircle;
(b) BD is a diameter of the circumcircle.
Remark. The above equality conditions are given by M. Naydenov.
'Problem M 137', Kvant **1972**, No. 12, 37-38.

13. Let ABCD be a circumscribable quadrilateral with parallel sides AD and BC. Then

$$AB + CD \geqslant 2\sqrt{F}.$$

'Problem 1839', Mat. v Škole **1977**, No. 3, 75 and **1978**, No. 1, 76.

14. Let Q be a convex quadrilateral. Suppose that:
Length of any diagonal of Q $\geqslant$ length of any side of Q $\geqslant 1$.
Then

$$\frac{2}{\sqrt{3}} F - s + 1 \geqslant 0.$$

J. H. Folkman and R. L. Graham, 'A Packing Inequality for Compact Convex Subsets of the Plane', Canad. Math. Bull. 12 (1969), 747-752.

15. Let ABCD be a right circumscribable trapezium with parallel sides a and c ($a > c$) and inradius r. Then

$$c\sqrt{c/a} < 2r < \sqrt{ac}.$$

S. Horák and M. Zemek, 'Problem 3', Rozhledy Mat. Fyz. 59 (1980-1981), 82 and 60 (1981-1982), 79-81.

16. Let ABCD be a circumscribable quadrilateral with inradius r. Then

$$a + c \geqslant 4r.$$

T. A. Ivanova, 'Problem 1497', Mat. v Škole 1975, No. 2, 75 and 1976, No. 1, 73.

17. Let a, b, c, d be the sides (in that order) of a quadrilateral that is inscribed in one circle and circumscribed about another. Let R, r denote the circumradius and inradius, respectively. Then

(1) $$8R^2r^2 \geqslant abcd,$$

(2) $$8Rr\sqrt{2} \leqslant (a + c)^2,$$

(3) $$64(R^2 + 2r^2) \leqslant (a + c)^6/(abcd).$$

In each case there is equality if and only if the quadrilateral is regular.

Proof. Preliminary result. Let ABCD be the quadrilateral with AB = a, etc. Since opposite angles are supplementary, two adjacent angles are $\geqslant \pi/2$, say $A, B \geqslant \pi/2$, since $AC = 2R \sin B$, $BD = 2R \sin A$, Ptolemy's theorem gives

(4) $$ac + bd = 4R^2 \sin A \sin B.$$

Since $C = \pi - A$ and $D = \pi - B$, we have

$$a = r(\text{cotan}\,\frac{A}{2} + \text{cotan}\,\frac{B}{2}), \quad b = r(\tan\frac{A}{2} + \text{cotan}\,\frac{B}{2}),$$

$$c = r(\tan\frac{A}{2} + \tan\frac{B}{2}), \quad d = r(\text{cotan}\,\frac{A}{2} + \tan\frac{B}{2}).$$

Hence

(5) $$ac + bd = 4r^2(1 + \text{cosec}\,A\,\text{cosec}\,B)$$

and by (4) and (5),

(6) $$(ac + bd)^2 = 16R^2r^2(1 + \sin A + \sin B).$$

Note also that $a + c = b + d = 2r(\operatorname{cosec} A + \operatorname{cosec} B)$ and this attains its minimum value when ABCD is a square.

Proofs of (1), (2) and (3). By the arithmetic-geometric mean inequality

(7) $$ac + bd \geqslant 2(abcd)^{1/2},$$

(8) $$ac + bd \leqslant \frac{1}{4}(a + c)^2 + \frac{1}{4}(b + d)^2 = \frac{1}{2}(a + c)^2.$$

Inequality (1) follows immediately from (6) and (7).

By (6) and (8),

$$(a + c)^4 \geqslant 4(ac + bd)^2 = 64R^2r^2(1 + \sin A \sin B),$$

and since the minimum of $a + c$ occurs when $A = B = \pi/2$, it follows that $(a + c)^4 \geqslant 128R^2r^2$, i.e. (2) holds.

By (4), (5) and (8) we have

$$(a + c)^2 \geqslant 2(ac + bd) =$$

$$= 4R^2 \sin A \sin B + 4r^2(1 + \operatorname{cosec} A \operatorname{cosec} B).$$

Again, since the minimum of $a + c$ occurs when $A = B = \pi/2$, it follows that $(a + c)^2 \geqslant 4(R^2 + 2r^2)$. Inequality (3) follows from this and the fact that (by (7) and (8)) $(a + c)^4 \geqslant 16abcd$.

If the quadrilateral is a square, it is a trivial matter to show that equality holds in all cases. Conversely, if equality holds in (1), then there is equality in (8), i.e. $a = b = c = d$ and the quadrilateral being cyclic is a square; if equality holds in either (2) or (3), then $a + c$ is minimal, i.e. $A = B = \pi/2$ and again the quadrilateral is a square (of side 2r).

L. Carlitz and G. J. Griffith, 'Problem P186', Canad. Math. Bull. 15 (1972), 616-617.

Comments by V. Mascioni. Using (6), Klamkin's result in 8. becomes

$$8rR\sqrt{1 + \sin A \sin B} \leqslant (a + c)^2.$$

It follows that (2) is stronger than the inequality in 8.

(7) (true for inscribable quadrilaterals) may be written as $4abcd \leqslant p^2q^2$ and thus is a companion to Janous' result from 2.

18. Let ABCD be a quadrilateral inscribed in a circle of radius R and circumscribed about a circle of radius r. If s is the semiperimeter of the quadrilateral, then

(1) $s \leqslant \sqrt{4R^2 + r^2} + r$

with equality if and only if at least one pair of opposite angles are right angles;

(2) $s^2 \geqslant 8r(\sqrt{4R^2 + r^2} - r)$

with equality if and only if the quadrilateral is an isosceles trapezium;

(3) $s \leqslant 2R + (4 - 2\sqrt{2})r,$

(4) $s^2 \geqslant \frac{32\sqrt{2}}{3} Rr - \frac{16}{3} r^2,$

with equality in each case if and only if the quadrilateral is a square.

W. J. Blundon and R. H. Eddy, 'Problem 488', Nieuw Arch. Wisk. 26 (1978), 231 and 465-466.

19. Let U be the point of intersection of the diagonals of a convex quadrilateral ABCD. Let R_1, R_2, R_3, R_4 be the distances from U to the vertices of ABCD and r_1, r_2, r_3, r_4 the angle bisectors of the angles AUB = CAD = Ω, BUC = DUA. Then

(1) $R_1R_2R_3R_4 \sin^2 \Omega \geqslant 4r_1r_2r_3r_4$

with equality if and only if the quadrilateral is a rectangle; and

(2) $$2(r_1r_2 + r_1r_3 + r_1r_4 + r_2r_3 + r_2r_4 + r_3r_4) \leqslant (R_1R_3 + R_2R_4) + + (R_1R_2 + R_1R_4 + R_2R_3 + R_3R_4) \sin \Omega$$

with equality if and only if

$$R_1 = R_3, \quad R_2 = R_4, \quad R_1 : R_2 = R_4 : R_3.$$

S. Horák, 'Nerovnosti v geometrii', Rozhledy Mat. Fyz. 52 (1973-1974), 10-13.

20. Denote the midpoints of the diagonals AC and BD of a convex quadrilateral by E and F, respectively. Draw line segments AF, BE, CF and DE inside the quadrilateral and denote the new endpoints A_1, B_1, C_1 and D_1, respectively, in the order given. Then among the ratios

$$\frac{AF}{FA_1}, \frac{BE}{EB_1}, \frac{CF}{FC_1}, \frac{DE}{ED_1}$$

there is at least one not greater than 1.

I. MŐLLER, 'Problem Gy. 1495', Kőz. Mat. Lap. 47 (1973), 157.

21. If about a convex quadrilateral of area F is circumscribed a rectangle with maximal area F_{max}, then $F_{max} \geqslant 2F$.

V. I. Gridasov, 'Problem 669', Mat. v Škole 1969, No. 5, 75 and 1970, No. 4, 76.

22. A', B', C', D' are the midpoints of the sides BC, CD, DA, AB of a convex quadrilateral ABCD, the consecutive pairs of lines AA', BB', CC', DD' intersect each other at A_1, B_1, C_1, D_1. The area of the quadrilateral $A_1B_1C_1D_1$ is F_1, the quadrilateral ABCD has area F. Then

$$5F_1 \leqslant F \leqslant 6F_1.$$

This problem from Gaz. Mat. A can be found in Köz. Mat. Lap. 38 (1969), 174 and in Kvant 1976, No. 8, 16.

23. P, K, S, N are the midpoints of the sides AB, BC, CD, DA of a convex quadrilateral ABCD. Let (AK) ∩ (BN) = L, (KD) ∩ (NC) = M, (PC) ∩ (BS) = Q, (AS) ∩ (PD) = T. Then

(a) $\dfrac{F_{KMNL}}{F_{ABCD}} \leqslant \dfrac{1}{3}$;

(b) $\dfrac{1}{3} \leqslant \dfrac{F_{KMNL} + F_{PQST}}{F_{ABCD}} \leqslant \dfrac{1}{2}$.

V. Matizen, 'Problem 3', Kvant 1976, No. 8, 16.

24. A convex quadrilateral, for which the sum of all six distances between the vertices (i.e. the sum of lengths of the sides and diagonals) is S_1, contains another one, for which the sum is S_2.

(a) S_2 can be larger than S_1.

(b) $S_2 < \frac{4}{3} S_1$.

P. Gusjatnikov, 'Problem M 759', Kvant 1982, No. 8, 30 and 1983, No. 1, 42-43.

25. A triangle with area F contains a parallelogram with area F_1. Then

$$F_1 \leqslant \frac{1}{2} F.$$

This result of J. B. Tabov is a generalization of a result from: Zaeta, 'Problem 5', Ob po Matematika 1984, No. 3, 55 and 1985, No. 3, 60-61.

26. Let PQRS be inscribed in the convex quadrilateral ABCD with P on AB, Q on BC, etc., and let AP/BP = p, BQ/QC = q, etc. It is assumed no two vertices coincide. The area of a convex polygon with vertices P_1, P_2, ...,

P_n will be denoted $|P_1P_2 \ldots P_n|$. Let $|DAB| = A_1$, $|ABC| = A_2$, $|BCD| = A_3$, $|CDA| = A_4$. Then

$$|SAP| = \frac{p}{p+1}|SAB| = \frac{pA_1}{(1+p)(1+s)} .$$

Since $|ABCD| = \frac{1}{2}(A_1 + A_2 + A_3 + A_4)$ it follows,

$$F = |PQRS| = \Sigma\left(\frac{1}{2} - \frac{p}{(1+p)(1+s)}\right)A_1 =$$

$$= f_1A_1 + f_2A_2 + f_3A_3 + f_4A_4$$

where the coefficients f_1, f_2, f_3, f_4 depend only on the ratios p, q, r, s.

M. J. Pelling proved the following result:

$F = |PQRS|$ satisfies the following inequalities

(1) $$F \leqslant \left(1 + \frac{(1-pr)(1-qs)}{\Pi(1+p)}\right) \max(A_i) =$$

$$= (1 + (1 - p_1 - r_1)(1 - q_1 - s_1)) \max(a_i),$$

(2) $$F \geqslant \left(1 + \frac{(1-pr)(1-qs)}{\Pi(1+p)}\right) \min(A_i) =$$

$$= (1 + (1 - p_1 - r_1)(1 - q_1 - s_1)) \min(A_i)$$

where $p_1 = p/(1 + p) = AP/AB$ etc.

Proof. First, the coefficients f_i have the property that $f_i + f_j > 0$ for $i \in \{1, 3\}$ and $j \in \{2, 4\}$. For since $f_1 = \frac{1}{2} - p/(1 + p)(1 + s)$ and $f_2 = \frac{1}{2} - q/(1 + q)(1 + p)$ we have $f_1 + f_2 = (1 + s + ps + pqs)/$ $/(1 + p)(1 + q)(1 + s) > 0$ and the other cases follow similarly.

Suppose now that p, q, r, s are fixed, that $\max(A_i) = k$, and that F is maximized for A_i subject only to this condition. We show that then all the A_i must be equal to k. Obviously one must be, say A_4. If $A_2 < k$, then since geometrically $A_1 + A_3 = A_2 + A_4$ one of A_1, A_3 is less than k, say $A_1 < k$. But $f_1 + f_2 > 0$ so that F could be increased if A_1, A_2 were replaced by $A_1 + x$, $A_2 + x$, for a suitably small $x > 0$. So $A_2 = k$ which implies $A_1 = A_3 = k$.

Thus F is maximized only when $A_1 = A_2 = A_3 = A_4 = k$, so that

$$F = (f_1 + f_2 + f_3 + f_4)k.$$

A computation shows $\Sigma f_i = 1 + (1 - pr)(1 - qs)/\Pi(1 + p)$ whence (1) follows. (2) can be proved similarly by minimizing F subject to min $(A_i) = k$ and one finds again that $A_1 = A_2 = A_3 = A_4 = k$.

The following results are consequences of the above result:

(a) Equality holds in (1) or (2) if and only if all A_i are equal, i.e. if and only if ABCD is a parallelogram.

(b) From (1) it follows that if $pqrs = 1$ then $F \leqslant \max (A_i)$ with equality if and only if ABCD is a parallelogram and $pr = qs = 1$ and from (2) if $pr = qs$ then $F \geqslant \min (A_i)$ with equality if and only if ABCD is a parallelogram and $pr = qs = 1$. The condition $pqrs = 1$ can be expressed geometrically: a simple application of Menelaus' theorem shows it is equivalent to SP, RQ meeting on DB (or equally PQ, SR meeting on AC).

M. J. Pelling, 'Inequalities Involving the Area of a Quadrilateral Inscribed in a Convex Quadrilateral', Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 498-541 (1975), 188-190.

27. Let ABCD be a convex quadrilateral and let α be the angle between its diagonals. If P is an interior point and $W = \min (\Sigma AP)$, then

$$16F \leqslant 8F(1 + 1/\sin \alpha) \leqslant W^2(1 + \sin \alpha) \leqslant 4s^2.$$

H. Toepken, 'Aufgabe 301', Jahresb. d. Deutsch. Math.-Verein. 51, 2. Abt., Heft 2 (1941), 3.

28. Let $M = A_1A_2A_3A_4$ be a quadrangle in a Euclidean 2-space E^2 with the canonical coordinate system (x, y), and let P = (x, y) be an arbitrary point of E^2. We define a function $f(x, y) = f_M(P)$ corresponding to M by

$$f_M(P) = \sum_{i=1}^{4} |PA_iA_{i+1}|^2,$$

where $A_5 = A_1$ and $|ABP|$ denotes the area of the triangle determined by three points A, B and P. Then

$$F(M) = \int e^{-4r^2 f(x,y)} dE^2 \leqslant \frac{\pi}{2r^2 S(M)} e^{-r^2 S(M)^2},$$

where S(M) denotes the area of M and r a non-zero real number. The equality holds if and only if M is a parallelogram. Therefore, for any parallelogram M of the fixed area S = S(M), F(M) is independent of the shape of M.

S. Tanno, 'Some Functions on the Set of Triangles or Quadrangles', Kodai Math. J. 6 (1983), 110-115.

29. A normed linear space V is called quadrilateral if the inequality

(1) $$|a + b| + |b + c| + |c + a| \leqslant |a| + |b| + |c| + |a + b + c|$$

holds for all a, b, c ∈ V.

Such spaces were defined by M. F. Smiley and D. M. Smiley who conjectured that every two-dimensional normed linear space is quadrilateral. This conjecture was proved by A. Sudbery.

Of course inequality (1) is the well-known Hlawka inequality. For some other generalizations and related results of this inequality see AI, pp. 170-177, and a paper of J. E. Pečarić.

M. Chiriţă and R. Constantinescu noted the following interpretation of (1):

Let A_1, B_1, C_1 be the midpoints of the sides of a triangle ABC, and Let M be a point in its plane. Then

$$\Sigma MA + 3MG \geqslant 2\Sigma MA_1.$$

Remark. It is obvious that M can be an arbitrary point in space.

D. M. Smiley and M. F. Smiley, 'The Polygonal Inequalities', Amer. Math. Monthly 71 (1964), 755-760.

A. Sudbery, 'The Quadrilateral Inequality in Two Dimensions', Amer. Math. Monthly 82 (1975), 629-632.

J. E. Pečarić, 'Modified Version of a General Result of Vasić-Adamović-Kečkić and Some Remarks Concerning Inequalities for Convex Functions', Glasnik Matematički, 21 (41) (1986), 331-341.

M. Chiriţă and R. Constantinescu, 'Asupra unei inegalitati care caracterizeaza functiile convexe', Gaz. Mat. (Bucharest) 89 (1984), 241-242.

30. Let A_1, A_2, A_3, A_4 be any given non-collinear four points in a plane Π. If these points form a convex quadrilateral, then let P be the point of intersection of its diagonals. If there is p ∈ {1, 2, 3, 4} such that the point A_p lies inside (or on the boundary of) the triangle with vertices from the set $\{A_1, A_2, A_3, A_4\}\backslash\{A_p\}$, let then $P = A_p$. For any point Q of Π it is true that

$$\sum_{i=1}^{4} QA_i \geqslant \sum_{i=1}^{4} PA_i$$

with equality if and only if Q = P.

Peters, Briefwechsel zwischen K. F. Gauss und H. C. Schumacher, Bd. 3, Altona 1981, S. 13, 14, 24, 26-28, 30-32, 39-40, 44-45.

A. Engel, 'Geometrical Activities for the Upper Elementary School', Educ. Studies Math. 3 (1971), 372-376.

H. Schupp, 'Extremwertbestimmungen mit Hilfe der Dreiecksungleichung', Mathematikaunterricht 30 (1984), H. 6.

A. Fricke, 'Der Punkt kleinster gewichteter Entfernungssumme von gegebenen Punkten', Ibid. H. 6, 22-37.

31. Let x, y, z be three sides of a quadrilateral inscribed in a circle of radius r, and let the fourth side be the diameter of the circle. Then

$$xyz \leqslant r^3$$

with equality if and only if $x = y = z = r$.

E. Kraemer and S. Vaněček, 'Problem 4', Rozhledy Mat. Fyz. Praha 58 (1979-1980), 33-35.

32. Conjecture: If quadrilateral is convex, then

$$2 \max(a, b, c, d) \geq \sqrt{e^2 + f^2}.$$

When does the equality hold?

L. Tuţescu, 'Problem 20791*', Gaz. Mat. (Bucharest) 91 (1986), 220.

33. If a quadrilateral has a circumscribed circle with radius R and an inscribed circle with radius r, then the following interpolation of GI 15.15 is valid:

$$2R \geq \max(p, q) \geq 2\sqrt{2}\, r.$$

M. Lascu, 'Problem C:580', Gaz. Mat. (Bucharest) 91 (1986), 95.

Chapter XV

INEQUALITIES FOR POLYGONS

1. Let A_1, B_1, C_1, D_1, E_1 be the midpoints of sides of a convex pentagon ABCDE of area F. If F_1 is the area of the pentagon $A_1B_1C_1D_1E_1$, then

$$\frac{3}{4} F > F_1 > \frac{1}{2} F.$$

Remark. The second inequality is GI 16.2.

D. O. Škljarskij, N. N. Čencov, and I. M. Jaglom, Geometričeskie ocenki i zadači iz kombinatornoj geometrii, Moskva 1974, pp. 38 and 166-168.

2. Let ABCDE be a convex pentagon of area F_{ABCDE}. Then

$$F_{ABE} + F_{AED} + F_{EDC} + F_{BCD} \geqslant F_{ABCDE}$$

where F_{ABE} is the area of a triangle ABE, etc.

'Problem M 193', Kvant 1973, No. 11, 43-44.

3. Drawing in a triangle the three lines parallel to its sides and touching its incircle, we get a hexagon H having three pairs of parallel sides; then

$$\text{perimeter}(H) \leqslant \frac{4s}{3} \qquad \{E\}$$

where s denotes the semiperimeter of the triangle. This inequality is due to W. Janous.

4. Let $P_5(A_1, \ldots, A_5)$ be a convex pentagon and let $Q_5(B_1, \ldots, B_5)$ be a convex pentagon which is determined by the line segments A_iA_{i+2}, $i = 1, 2, \ldots, 5$ (e.g. $B_1 = A_1A_3 \cap A_5A_2$ and $B_5 = A_4A_1 \cap A_5A_2$). Let T be a point which is not in the exterior of Q_5. We denote by r_i the distances of T from the sides of P_5, d_i are the distances of T from the sides of Q_5 and $R_i = TA_i$, $i = 1, 2, \ldots, 5$. Then

$$3 \sum_{i=1}^{5} R_i > 2 \sum_{i=1}^{5} r_i + 4 \sum_{i=1}^{5} d_i.$$

A. Lupaş and S. B. Maurer, Math. Mag. 43 (1970), 279 and 44 (1971), 233.

5. If A_1, A_2, A_3, A_4, A_5 are five different points in a plane and $d_{ij} = d_{ji}$ stands for the distance of A_i and A_j, we have

$$(1) \qquad d_{14}d_{25} \leqslant d_{23}d_{45} + d_{12}d_{34} + d_{14}d_{23} + d_{12}d_{45} + d_{25}d_{34}.$$

Equality holds if and only if the points are colinear and in the order $A_1A_2A_3A_4A_5$.

Proof. If we apply GI 15.4 (or the Möbius-Neuberg theorem) to the quadrilaterals $A_2A_3A_4A_5$, $A_1A_2A_4A_5$, $A_1A_2A_3A_4$, and moreover the triangle inequality to $A_1A_3A_5$, we have respectively:

$$(2) \qquad d_{24}d_{35} \leqslant d_{23}d_{45} + d_{25}d_{34},$$

$$(3) \qquad d_{14}d_{25} \leqslant d_{12}d_{45} + d_{15}d_{24},$$

$$(4) \qquad d_{13}d_{24} \leqslant d_{12}d_{34} + d_{14}d_{23},$$

$$(5) \qquad d_{15}d_{24} \leqslant d_{13}d_{24} + d_{35}d_{24},$$

and by adding these inequalities we obtain (1). Equality holds if and only if it does for all relations (2), (3), (4) and (5). In this case it follows from (5) that A_1, A_3, A_5 are (in this order) on a line L. Furthermore, A_2, A_3, A_4, A_5 are on a circle and so are A_1, A_2, A_4, A_5 and A_1, A_2, A_3, A_4. Each pair of circles has three common points; hence the circles coincide and as L has three points of intersection with it, the points A_i are collinear. From (2) it follows the order has the possibilities $A_2A_3A_4A_5$ and $A_3A_4A_5A_2$; (3) implies $A_1A_2A_4A_5$ or $A_1A_5A_4A_2$; from (4) we have either $A_1A_2A_3A_4$ or $A_1A_4A_3A_2$. The only order satisfying all conditions is $A_1A_2A_3A_4A_5$. Conversely, if the points are collinear in this order, equality in (1) holds.

O. Bottema, Euclides 44 (1968-1969), 235-237.

6. If A_i $(i = 1, 2, \ldots, 6)$ are different points in a plane, d_{ij} being the distance A_iA_j, we have

$$(1) \qquad d_{14}d_{25}d_{36} \leqslant d_{12}d_{34}d_{56} + d_{23}d_{45}d_{61} + d_{14}d_{23}d_{56} + d_{25}d_{34}d_{61} + d_{36}d_{45}d_{12}.$$

Equality holds if and only if the points are on a circle in the order $A_1A_2A_3A_4A_5A_6$.

The proof goes along the same lines as that of 5.

Remark. That (1) with the equality sign holds for a convex hexagon inscribed in a circle, was proved by W. Fuhrmann, Synthetische Beweise planimetrischer Sätze (Berlin, 1890), 61; Cf. R. A. Johnson, Modern Geometry (1929), republished as Advanced Euclidean Geometry (Dover Publications, 1960), 65-66. The converse theorem and inequality (1) have been given by H. F. Sandham and V. W. Graham, Amer. Math. Monthly, 47 (1940), 642-643. For another proof of the converse theorem, see P. Bronkhorst, Euclides 44 (1968-1969), 55-56; O. Bottema (Id. 235-237) gave a proof of (1) by means of inversion making use of 5.

Comment by K. A. Post. 5. and 6. are consequences of the following identities

$$w_{14}w_{25} = w_{23}w_{45} + w_{12}w_{34} + w_{14}w_{23} + w_{12}w_{45} + w_{25}w_{34}$$

and

$$w_{14}w_{25}w_{36} = w_{12}w_{34}w_{56} + w_{23}w_{45}w_{16} + w_{14}w_{23}w_{56} + $$
$$+ w_{25}w_{34}w_{16} + w_{12}w_{45}w_{36}$$

where $w_{ij} = z_j - z_i$ $(i \neq j)$, z_i $(i = 1, \ldots, 6)$ are points in the complex plane.

7. Let $P_1P_2P_3P_4P_5P_6$ be a convex hexagon. Denote by F its area and by F_1 the area of the triangle $Q_1Q_2Q_3$, where Q_1, Q_2 and Q_3 are the midpoints of P_1P_4, P_2P_5, P_3P_6, respectively. Then

$$F_1 < \frac{1}{4} F.$$

D. O. Škljarskij, N. N. Čencov, and I. M. Jaglom, The same reference as in 1, pp. 38 and 169.

8. Let A, B, C be three points which are not in the exterior of a regular hexagon MNPQRS of side 1. Then

$$\min (AB, AC, BC) \leqslant \sqrt{3},$$

with equality if and only if the triangle ABC coincides with MPR or with NSQ.

This result is due to I. Tomescu.

9. Let on the elongations of the sides of a triangle ABC be marked the lengths $AK = AL = a$, $BM = BN = b$, $CP = CR = c$. Then

$$\text{area } (KLMNPR) \geqslant 13 \text{ area } (ABC).$$

Ju. I. Gerasimov, 'Problem 370', Mat. v Škole 1967, No. 3, 84 and 1968, No. 1, 72-73.

10. Let $(n - k)s$ be the perimeter of an n-gon with sides x_i, $1 \leqslant i \leqslant n$, $(n, k \in N)$ and let f be a strictly convex function for $x > 0$. If $x_i \leqslant s$,

$x_{n+i} = x_i$ $(1 \leq i \leq n)$, then

$$\sum_{i=1}^{n} f(x_i + x_{i+1} + \dots + x_{i+k-1}) \leq \sum_{i=1}^{n} f((n-k)(s-x_i)) \tag{1}$$

with equality if and only if $x_1 = x_2 = \dots = x_n$.

<u>Proof</u>. From Jensen's inequality

$$(n-k)f\left((n-k)s - \sum_{i=1}^{n-k} x_{j+i}\right) \leq \sum_{i=1}^{n-k} f((n-k)(s-x_{j+i})).$$

for $a_i = (n-k)(s-x_{j+i})$, $(1 \leq i \leq n-k)$, $j = 1, 2, \dots, n$; we obtain

$$(n-k)f\left((n-k)s - \sum_{i=1}^{n-k} x_{j+1}\right) \leq \sum_{i=1}^{n-k} f((n-k)(s-x_{j+1})).$$

Summing the latter set of inequalities for $j = 1, 2, \dots, n$, we obtain (1).

By letting $f(x) = \log x$ for $x > 0$ in (1), we obtain [1]:

$$\prod_{i=1}^{n} (x_i + x_{i+1} + \dots + x_{i+k-1}) \geq (n-k)^n \prod_{i=1}^{n} (s - x_i). \tag{2}$$

Of course, for $k = 1$ from (2) we get the known inequality of Mitrinović and Adamović (see AI, pp. 208-209), i.e. the following generalization of GI 1.3:

$$\prod_{i=1}^{n} x_i \geq \Pi(x_2 + x_3 + \dots + x_n - (n-2)x_1). \tag{3}$$

Another generalization of (3) was given by M. S. Klamkin. He proved the inequality

$$P^{1/n} \geq \sum_{i=1}^{n} w_i P_i^{1/n}, \tag{4}$$

where

$$P_i = \prod_{j=1}^{n} (T_i - \lambda a_{ij}), \quad T_i = \sum_{j=1}^{n} a_{ij},$$

$$P = \prod_{j=1}^{n} (T - \lambda a_j), \quad T = \sum_{j=1}^{n} a_j,$$

$$a_j = \sum_{i=1}^{m} w_i a_{ij}, \quad \sum_{i=1}^{m} w_i = 1, \quad a_{ij} > 0,$$

$$T_i - \lambda a_{ij} \geqslant 0 \quad \text{and} \quad w_i \geqslant 0.$$

There is equality if and only if the m vectors $(a_{i1}, a_{i2}, \ldots, a_{in})$, $i = 1, 2, \ldots, m$, are parallel.

Some results, similar to (3), were given by P. Sava, i.e. he proved in the case k = 1 the following two inequalities:

(5) $$\sum_{i=1}^{n} x_i \sqrt[n-1]{s - x_i} \geqslant n(n - 1) \prod_{i=1}^{n} \sqrt[n-1]{s - x_i},$$

(6) $$\sum_{i=1}^{n} \frac{x_i}{S - \sqrt{s - x_i}} \geqslant S \quad (S = \sum_{i=1}^{n} \sqrt{s - x_i}).$$

Remark. Here we note that from the fact that (1) is valid for every convex function defined on $(0, +\infty)$ and from the well-known theorem of majorization (AI, pp. 164-165) it follows that

$$(x_1 + x_2 + \ldots + x_k, \ldots, x_n + x_1 + \ldots + x_{k-1}) \prec$$

$$\prec (n - k)(s - x_1, \ldots, s - x_n).$$

This result is a generalization of a result from VIII, 2.1.5), so using the same Schur-concave functions as in Examples VIII, 2.1. 14°-16°, we can obtain further generalizations of these results.

J. E. Pečarić, R. R. Janić, and M. S. Klamkin, 'Some Cyclic Inequalities'. To appear.
M. S. Klamkin, 'Extensions of Some Geometric Inequalities', Math. Mag. 49 (1976), 28-30.
P. Sava, 'Asupra unor inegalităti într-un polygon', Gaz. Mat. (Bucharest) B 18 (1967), 56-57.

11. Consider any $(n + 1)$-gon, coplanar or not, simple or not, with sides a_i, $i = 1, 2, \ldots, n + 1$. If f is a non-decreasing convex function for $x > 0$, then

$$f(\frac{1}{n} a_{n+1}) \leqslant \frac{1}{n} \sum_{i=1}^{n} f(a_i),$$

with equality if and only if $a_1 = a_2 = \ldots = a_n = a_{n+1}/n$, i.e. the polygon is degenerate.

Remark. The case $f(x) = x^m$ $(m > 1)$ is given by M. S. Klamkin, as a generalization of a case m = 2 and n = 3, which was given as a problem in Crux Math. 2 (1976), 6 (Problem 106). The same problem was also given in Mat. v Škole as Problem 1478 by V. V. Malinin.

V. V. Malinin, 'Problem 1478', Mat. v Škole 1975, No. 1, 87 and

1975, No. 6, 84-85.
V. Linis, M. S. Klamkin, and F. G. B. Maskell, 'Problem 106', Crux Math. 2 (1976) 6, and 78-79.

12. Let $p = (n - 1)s$ be the perimeter of an n-gon with sides x_j, such that $x_j \leqslant s$ $(1 \leqslant j \leqslant n)$. If $f:(0, s] \to R^+$ and $g:(0, p] \to R^+$ are monotonic in the opposite sense and g is also strictly convex, then

$$(1) \qquad \sum_{i=1}^{n} f(x_i)g(x_i) \leqslant \sum_{i=1}^{n} f(x_i)g((n - 1)(s - x_i))$$

with equality if and only if $x_1 = \ldots = x_n$.

The inequality is reversed if f and g are monotone in the same sense and g is concave.

Proof. Let f and g be monotone in the opposite sense. Without loss of generality we can suppose that $x_1 \geqslant \ldots \geqslant x_n$. If $0 \leqslant y_1 \leqslant \ldots \leqslant y_n <$ s, then Čebyšev's inequality for monotone sequences gives

$$n \sum_{i=1}^{n} f(x_i)g(y_i) \geqslant \sum_{i=1}^{n} f(x_i) \sum_{i=1}^{n} g(y_i)$$

or

$$(n - 1) \sum_{i=1}^{n} f(x_i)g(y_i) \geqslant \sum_{i=1}^{n} f(x_i) \sum_{j \neq i} g(y_j).$$

Using Jensen's inequality for convex function g we get

$$\sum_{i=1}^{n} f(x_i)g(y_i) \geqslant \sum_{i=1}^{n} f(x_i)g(\sum_{j \neq i} y_j/(n - 1)).$$

If we put $y_j = (n - 1)(s - x_j)$, $(j = 1, \ldots, n)$, we have $\sum_{j \neq i} y_j = (n - 1)x_i$, i.e. (1) is valid.

Remark. For $f(t) = t^2$, $g(t) = 1/t$, we get an inequality of B. E. Lungulescu. For $n = 3$, $f(t) = t^x$, $g(t) = t^{-y}$ we get an inequality of V. Vâjâitu.

B. E. Lungulescu, 'Asupra unor poligoane', Gaz. Mat. (Bucharest) B 18 (1967), 352.
V. Vâjâitu, 'Asupra unei inegalitâti intr-un triunghi', Gaz. Mat. (Bucharest) 90 (1985), 191-192.
D. S. Mitrinović and J. E. Pečarić, 'An Inequality for a Polygon', Zbornik Fak. za Pomorstvo, Kotor 11-12 (1985-1986), 73-74.

13. If $a_1, a_2, \ldots, a_n$ are the sides of an n-gon, then

$$\frac{2n}{n-1}\sum_{i<j} a_i a_j \leqslant \left(\sum_{i=1}^{n} a_i\right)^2 \leqslant 4 \sum_{i<j} a_i a_j .$$

The first equality sign is valid if and only if the polygon is equilateral, and the second if and only if the polygon is degerate in which n - 2 of the sides are of length zero.

M. S. Klamkin, 'Problem Q 686', Math. Mag. 57 (1984), 244.

14. If $a_1, a_2, \ldots, a_n$ are the sides of a polygon, then for $k = 1, 2, \ldots, n$,

$$\frac{n+1}{S-a_k} \geqslant \sum_{\substack{i=1\\ i\neq k}}^{n} \frac{1}{S-a_i} \geqslant \frac{(n-1)^2}{(2n-3)(S-a_k)},$$

where $S = a_1 + a_2 + \ldots + a_n$.

M. S. Klamkin and G. Tsintsifas, 'Problem 506', Crux Math. 6 (1980), 16 and 7 (1981), 28-29.

15. If for all n-gons with sides $a_1, a_2, \ldots, a_n$ the following inequality

$$\sum \frac{1}{a_1} \leqslant \sum \frac{1}{-a_1 + a_2 + \ldots + a_n}$$

is valid, then $n = 3$.

This result is due to W. Janous.

16. Let $A_1A_2 \ldots A_n$ be a convex n-gon. We apply successively to each angle A_i weights $x_r \geqslant 0$ and $y_r \geqslant 0$, such that $\Sigma x_r = \Sigma y_r = 1$ (summations throughout are for $r = 1, 2, \ldots, n$) to form the weighted sums

$$\left.\begin{aligned} B_r &= x_r A_1 + x_{r+1} A_2 + \ldots + x_{r+n-1} A_n,\\ C_r &= y_r A_1 + y_{r+1} A_2 + \ldots + y_{r+n-1} A_n,\end{aligned}\right\} r = 1, 2, \ldots, n,$$

where all subscripts are reduced modulo n. Then

$$\left(n \operatorname{cotan} \frac{(n-2)\pi}{2n}\right)^{-1} \Sigma \operatorname{cotan} \frac{B_r}{2} \geqslant 1 \geqslant$$

$$\geqslant \left(n \cos \frac{(n-2)\pi}{2n}\right)^{-1} \Sigma \cos \frac{C_r}{2},$$

with equality if and only if the polygon is equilangular.

Remark. This is a generalization of a problem of J. Garfunkel given by M. S. Klamkin. The original problem is the special case for which $n = 4$, $(x_r) = (1, 0, 0, 0)$, and $(y_r) = (1/2, 1/2, 0, 0)$.

J. Garfunkel, E. C. Buissant des Amorie, and M. S. Klamkin, 'Problem 665', Crux Math. 7 (1981), 205 and 8 (1982), 221-222 and 280-281.

17. On a circle with the radius 1 we mark the points $A_1, A_2, \ldots, A_n$ counter-clockwise. If

$$A_1A_2 \cdot A_2A_3 \cdot \ldots \cdot A_nA_1 \geqslant 2^n \sin^n \frac{\pi}{n}, \quad n \geqslant 3,\ n \in N,$$

then the polygon $A_1A_2 \ldots A_n$ is regular.

This result is due to G. Mircea.

18. Let p be the perimeter and let S be the sum of the lengths of the diagonals of an n-gon. If the polygon is convex, then

$$S \geqslant \frac{n-3}{2} p.$$

This is a result of T. Popoviciu. The same result is given by V. F. Lev. Popoviciu also conjectured the following results:

$$S \leqslant \frac{n^2-8}{8} p \quad \text{(if n is even)},$$

$$S \leqslant \frac{n^2-9}{8} p \quad \text{(if n is odd)}.$$

Note that at the 25th International Math. Olympiad (1984) the following was posed as a problem:

For $n \geqslant 4$ there holds

$$\frac{n-3}{2} < \frac{S}{p} < \frac{1}{2}\left(\left[\frac{n}{2}\right] \cdot \left[\frac{n+1}{2}\right] - 2\right),$$

where $[x]$ denotes the greatest integer not exceeding x.

This is just the substance of Popoviciu's conjecture.

T. Popoviciu, 'Problem 4357', Gaz. Mat. (Bucharest) 39 (1933-1934), 258.

V. F. Lev, 'Problem M 846', Kvant 1984, No. 7, 53.

19. Let P be a convex n-gon with vertices $A_1, \ldots, A_n$, perimeter L and area F. Let $2\Theta_i$ be the measure of the interior angle at vertex A_i and set $C = \Sigma \cot\Theta_i$. Then

$$L^2 - 4FC \geqslant 0$$

and equality holds just when P has an inscribed circle.

Remark. (M. S. Klamkin) The proposed inequality is an immediate consequence of a theorem of S. Lhuilier (1750-1840) (see for example the book of L. A. Lyusternik, Convex Figures and Polyhedra, New York, 1963, pp. 118-119), which is itself a consequence of the Brunn-Minkowski inequality.

O. Bottema, Chapters on Elementary Geometry, The Hague, 1944, p. 43 (Dutch).

G. P. Henderson, O. Bottema, and M. S. Klamkin, 'Problem 615', Crux. Math. 7 (1981), 79 and 8 (1982), 57-58.

20. Let F, L, r, R be area, perimeter, radii of incircle and circumcircle, respectively, of a convex n-gon. Then

$$2n \tan \frac{\pi}{n} r \leqslant 2\sqrt{n \tan \frac{\pi}{n} F} \leqslant L \leqslant 2n \sin \frac{\pi}{n} R.$$

Equalities hold if and only if the n-gon is equilateral.

Remark. This is an interpolation of GI 16.23. Note that as a special case we have a generalization of Chapple-Euler's inequality, i.e. $R \geqslant r/\cos(\pi/n)$.

S. B. Gaškov, 'Neravenstva dlja ploščadi i perimetra vipuklogo mnogougol'nika', Kvant 1985, No. 10, 15-17.

21. Let $a_1, a_2, \ldots, a_n$ be the sides and r, R the radii of incircle and circumcircle, respectively, of a convex n-gon. Then

$$4nr^2 \tan^2 \frac{\pi}{n} \leqslant \Sigma a_1^2 \leqslant 9R^2$$

with equality if and only if the polygon is equilateral.

T. Kubota, 'Einige Ungleichheiten für das Dreieck und das konvexe Polygon', Tôhoku Math. J. 25 (1925), 122-126.

22. Let F be area of an n-gon inscribed in a circle of radius R. Then

$$R^2 \geqslant \frac{4\sqrt{3}\, F}{9(n - 2)}.$$

Proof. (W. Janous) This is a simple consequence of 20, i.e. of GI 16.23 since $\sin \frac{2\pi}{n} \leqslant \frac{3\sqrt{3}}{2} \frac{n - 2}{n}$.

Remark. For n = 3, this is the left-hand inequality from GI 7.9.

S. I. Majzus, 'Problem 898', Mat. v Škole 1971, No. 2, 74 and 1971, No. 6, 76.

23. For any convex n-gon, there are at least (n - 2) sides which are shorter than the longest diagonal and, furthermore, (n - 2) is best possible.

M. S. Klamkin, 'Polygonal Inequalities', Ontario Math. Bulletin 10 (1974), 18-21.

24. (i) Let $Q_1, \ldots, Q_{n+1}$ be n + 1 points in the space, O the midpoint of Q_1Q_{n+1}, and $E_i \in \{-1, 1\}$ ($i = 1, \ldots, n$, $E_{n+1} = E_1$). Then

$$(1) \qquad (\cos \frac{\pi}{n}) \sum_{i=1}^{n} \overline{OQ}_i^2 \geqslant \sum_{i=1}^{n} E_i E_{i+1}\, \overline{OQ}_i\, \overline{OQ}_{i+1} \cos(Q_i O Q_{i+1}).$$

For $E_i = 1$ ($i = 1, \ldots, n$) we get a result of N. Ozeki. (The proof of (1) is similar to Ozeki's proof.) Note that (1) is a generalization of the well-known asymmetric inequality of J. Wolstenholme (see Chapter VI). The same is valid for the following consequence of (i):

(ii) Let $x_1, \ldots, x_n$ ($x_{n+1} = x_1$) and $\delta_1, \ldots, \delta_n$ be real numbers and let $\sum_{i=1}^{n} \delta_i = (2m + 1)\pi$ ($m \in Z$). Then

$$(2) \qquad \left(\cos \frac{\pi}{n}\right) \sum_{i=1}^{n} x_i^2 \geqslant \sum_{i=1}^{n} x_i x_{i+1} \cos \delta_i .$$

(iii) Using the substitutions $\delta_i \to \delta_i + \pi$ ($i = 1, \ldots, n$) (2) becomes

$$(2') \qquad \left(\cos \frac{\pi}{n}\right) \sum_{i=1}^{n} x_i^2 \geqslant - \sum_{i=1}^{n} x_i x_{i+1} \cos \delta_i ,$$

where

$$\sum_{i=1}^{n} \delta_i = (2m + 1)\pi \quad \text{if } n \text{ is even,}$$

$$\sum_{i=1}^{n} \delta_i = 2m\pi \quad \text{if } n \text{ is odd.}$$

(iv) Let n be odd. Using the substitutions $\delta_i \to 2\delta_i$ ($i = 1, \ldots, n$), we get

$$\left(\cos \frac{\pi}{n}\right) \sum_{i=1}^{n} x_i^2 \geqslant - \sum_{i=1}^{n} x_i x_{i+1} \cos 2\delta_i ,$$

$$(3) \qquad 2 \sum_{i=1}^{n} x_i x_{i+1} \sin^2 \delta_i \leqslant \left(\cos \frac{\pi}{n}\right) \sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} x_i x_{i+1}$$

and

$$(4) \qquad 2 \sum_{i=1}^{n} x_i x_{i+1} \cos^2 \delta_i \geqslant \sum_{i=1}^{n} x_i x_{i+1} - \left(\cos \frac{\pi}{n}\right) \sum_{i=1}^{n} x_i^2 .$$

N. Ozeki, 'On P. Erdős' Inequality for the Triangle', J. College Arts Sci. Chiba Univ. 2 (1957), 247-250.

25. Let $A_1, \ldots, A_n$ be the vertices of a convex n-gon and P an internal point. Let $R_k = PA_k$, r_k the distance from P to the side $A_k A_{k+1}$, w_k the

segment of the bisector of the angle $A_kPA_{k+1} = 2\delta_i$ from P to its intersection with the side A_kA_{k+1}. It is known that (see GI 16.9) L. Fejes-Tóth has conjectured the following generalization of the well-known Erdős-Mordell inequality

$$(1) \qquad \sum_{k=1}^{n} R_k \geqslant \left(\sec \frac{\pi}{n}\right) \sum_{k=1}^{n} r_k .$$

One can find in GI 16.9 that inequality (1) for n = 4 was proved by A. Florian in 1958, and that in 1961, H. C. Lenhart proved (1) and the following generalization of GI 12.48:

$$(2) \qquad \sum_{k=1}^{n} R_k \geqslant \left(\sec \frac{\pi}{n}\right) \sum_{k=1}^{n} w_k .$$

But, inequality (2) (and therefore (1)) was firstly proved in 1958 by N. Ozeki, i.e. he proved the following generalization of (2):

(A) Put a point P inside a given polygon $A_1A_2 \ldots A_nA_1$, where the polygon is a star region to P. If $\lambda \geqslant 1$, then

$$(3) \qquad \sum_{i=1}^{n} R_i^{\lambda} \geqslant \left(\sec \frac{\pi}{n}\right) \sum_{i=1}^{n} w_i^{\lambda} .$$

where $\sec \frac{\pi}{n}$ is the best value.

In his proof, N. Ozeki used his result noted in 24 (i). Of course, it is possible to use inequality (2) from 24 (ii), i.e. the following special case of this result:

1° Let $x_1, \ldots, x_n$ $(x_{n+1} = x_1)$ be real numbers and $\delta_1, \ldots, \delta_n$ non-negative numbers such that $\sum_{i=1}^{n} \delta_i = \pi$. Then

$$(4) \qquad \left(\cos \frac{\pi}{n}\right) \sum_{i=1}^{n} x_i^2 \geqslant \sum_{i=1}^{n} x_i x_{i+1} \cos \delta_i .$$

Now, we shall show that using (4), i.e. (3), we can prove the following result:

2° Let the conditions of (A) be fulfilled and let $x_1, \ldots, x_n$ be non-negative numbers. If $\lambda \geqslant 1$, then

$$(5) \qquad \sum_{i=1}^{n} x_i^2 R_i^{\lambda} \geqslant \left(\sec \frac{\pi}{n}\right) \sum_{i=1}^{n} x_i x_{i+1} R_i^{*\lambda} ,$$

where $R_i^* = \sqrt{R_i R_{i+1}} \cos \delta_i$.

<u>Proof</u>. By substitution $x_i \to x_i R_i^{\lambda/2}$, (4) becomes

$$(\cos \frac{\pi}{n}) \sum_{i=1}^{n} x_i^2 R_i^\lambda \geq \sum_{i=1}^{n} x_i x_{i+1} (\sqrt{R_i R_{i+1}})^\lambda \cos \delta_i$$

and since $\lambda \geq 1$, i.e. $\cos \delta_i \geq \cos^\lambda \delta_i$, (5) follows.

Since

(6) $$w_i = \frac{2R_i R_{i+1}}{R_i + R_{i+1}} \cos \delta_i,$$

i.e.

$$w_i \leq \sqrt{R_i R_{i+1}} \cos \delta_i,$$

we have the well-known result

$$R_i^* \geq w_i \geq r_i.$$

So, the following results are also valid

(7) $$\sum_{i=1}^{n} x_i^2 R_i^\lambda \geq (\sec \frac{\pi}{n}) \sum_{i=1}^{n} x_i x_{i+1} w_i^\lambda$$

and

(8) $$\sum_{i=1}^{n} x_i^2 R_i^\lambda \geq (\sec \frac{\pi}{n}) \sum_{i=1}^{n} x_i x_{i+1} r_1^\lambda.$$

Remark. Inequality (5) for $x_i = \ldots = x_n = 1$ and $\lambda = 1$ was given by H. Vogler.

By using (6), (4) becomes

(9) $$2(\cos \frac{\pi}{n}) \sum_{i=1}^{n} x_i^2 \geq \sum_{i=1}^{n} x_i x_{i+1} w_i (R_i^{-1} + R_{i+1}^{-1}),$$

and in the case $x_1 = \ldots = x_n = 1$,

(10) $$2n \cos \frac{\pi}{n} \geq \sum_{i=1}^{n} R_i^{-1} (w_i + w_{i+1}).$$

Now, using the arithmetic-geometric means inequality, we get

(11) $$\prod_{i=1}^{n} (w_i + w_{i+1}) \leq 2^n \cos^n \frac{\pi}{n} \prod_{i=1}^{n} R_i$$

which is a generalization of GI 12.52.

Using again the arithmetic-geometric means inequality, we get the following inequality of N. Ozeki:

$$(12) \qquad \prod_{i=1}^{n} R_i \geq (\sec \frac{\pi}{n})^n \prod_{i=1}^{n} w_i$$

which is a generalization of GI 12.51. It is evident that (12) is better than GI 16.8.

Remark. The above results are generalizations of results from XI.5.

N. Ozeki, 'On P. Erdős' Inequality for the Triangle', J. College Arts Sci. Chiba Univ. 2 (1957), 247-250.

H. Vogler, 'Eine Bemerkung zum Erdős-Mordellschen Satz für Polygone', Anz. Österr. Akad. Wiss. Math.-naturwiss. Kl. 103, No. 1-14, (1966), 241-251.

D. S. Mitrinović and J. E. Pečarić, 'On the Erdős-Mordell Inequality for a Polygon', J. College Arts Sci. Chiba Univ. B 19 (1986), 3-6.

26. Let $A_1A_2 \ldots A_n$ be a convex polygon of area F and perimeter L, and M an internal point. Let $a_1, a_2, \ldots, a_n$ be the sides of the polygon and $r_1, r_2, \ldots, r_n$ the distances from M to the sides of the n-gon. Then

$$\sum_{k=1}^{n} \frac{a_k}{r_k} \geq \frac{L^2}{2F}.$$

This is a result of D. Buşneag. As a special case we get the following result of M. Bătineţu:

If a polygon is circumscribed about a circle of radius r, then

$$\sum_{k=1}^{n} \frac{a_k}{r_k} \geq \frac{L}{r} \qquad (L = 2rn \tan \frac{\pi}{n} > 2\pi r).$$

D. Buşneag, 'Problem 10876', Gaz. Mat. (Bucharest) B 22, (1971), 35.

D. M. Bătineţu, 'O inegalitate între medii ponderate si aplicaţii', Gaz. Mat. (Bucharest) 87 (1982), 250-252.

27. Let r and R be the radii of incircle and circumcircle, respectively, of a convex n-gon $A_1A_2 \ldots A_n$ and P an internal point. Let $R_k = PA_k$ and let $A(R_k)$ be the arithmetic mean of all R_k. Then

(a) $A(R_k) \geq \frac{4}{9} r(5 - \frac{r}{R})$ and $A(R_k) \geq \frac{2}{3} r \sqrt[4]{48 \frac{R}{r} - 15}$ for $n = 3$;

(b) $A(R_k) \geq 2r\sqrt{\tan \frac{\pi}{2n} \tan \frac{\pi}{n}}$ for $n \geq 3$.

J. Berkes, 'Aufgabe 620', Elem. Math. 25 (1970), 39.

28. Let P be a point in the plane of a regular n-gon $A_0A_1 \dots A_{n-1}$ and $R_k = PA_{k-1}$ $(k = 1, \dots, n)$. Let $M_r(R_k)$ be the mean of order r of all R_k. Then the following results are valid:

(1) $$M_1(R_k) \geqslant \frac{\sqrt{2}}{n} \cot\operatorname{an} \frac{\pi}{2n} M_2(R_k),$$

with equality if and only if P coincides with one of the vertices of the polygon.

(2) If $0 < r \leqslant 1$, then

$$M_r(R_k) \geqslant \frac{\sqrt{2}}{n^{1/r}} \left(\sum_{k=1}^{n-1} \sin^r \frac{k\pi}{n}\right)^{1/r} M_2(R_k),$$

with equality if and only if P coincides with one of the vertices of the polygon.

(3) Let m be an integer $(1 < m < n)$. Then

$$M_{2m}(R_k) \leqslant \frac{1}{\sqrt{2}} \binom{2m}{m}^{\frac{1}{2m}} M_2(R_k),$$

with equality if and only if P is the circumcentre of the polygon.

(4) Let m be an integer $(m > n)$. Then

$$\frac{M_{2m}(R_k)}{M_2(R_k)} \leqslant \frac{1}{\sqrt{2}} \left(\binom{2m}{m} + 2 \sum_{j=1}^{[m/n]} \binom{2m}{m + jn}\right)^{\frac{1}{2m}}$$

with equality in the following cases:

1° if n is even, P coincides with one of the vertices of the polygon;

2° if n is odd, P coincides with one of the midpoints of the arcs of circumcircle between two neighbouring vertices.

(5) $$\frac{1}{R_1^r} \sum_{i=1}^{n} R_i^r \geqslant \lambda_n^{(r)},$$

where

$$\lambda_n^{(r)} = \begin{cases} \sum_{k=0}^{n-1} \left|\cos \frac{k\pi}{n}\right|^r, & \text{if either n is even and } r > 0 \text{ or n odd and } r \geqslant 1; \\ \left(\sec^r \frac{\pi}{2n}\right) \sum_{k=0}^{n-1} \left|\cos \frac{(2k+1)\pi}{2n}\right|^r, & \text{if n is odd and } 0 < r < 1. \end{cases}$$

Equality occurs if and only if:

1° if either n is even and $r > 0$ or n is odd and $r > 1$, P coincides with A'_o (diametrically opposite point to A_o);

2° If n is odd and $0 < r < 1$, P coincides with one of the opposite vertices to A_o, i.e. $P = A_{(n-1)/2}$ or $P = A_{(n+1)/2}$;

3° if n is odd and $r = 1$, P coincides with any point of the arc $A_{\frac{n-1}{2}} A_{\frac{n+1}{2}}$ of circumcircle.

(6) $M_r(R_k) \geqslant C_n^{(r)} M_\infty(R_k)$,

where $C_n^{(r)} = (\frac{1}{n} \lambda_n^{(r)})^{1/r}$.

(7) $M_1(R_k) \geqslant C_n^{(1)} M_\infty(R_k)$

where

$$C_n^{(1)} = \begin{cases} \frac{1}{n} \operatorname{cotan} \frac{\pi}{2n}, & \text{if } n \text{ is even;} \\ \frac{1}{n} \operatorname{cosec} \frac{\pi}{2n}, & \text{if } n \text{ is odd.} \end{cases}$$

Independently of n it holds that

$$M_1(R_k) > \frac{2}{\pi} M_\infty(R_k)$$

where the constant $2/\pi$ cannot be improved.

(8) $M_{2m}(R_k) \geqslant \frac{1}{2}\binom{2m}{m}^{1/2m} M_\infty(R_k)$ $(n > m,\ m \in N)$.

T. Popoviciu, 'Quelques remarques sur un théorème de M. D. Pompeiu', Bull. Soc. Roumaine des Sciences **43** (1941), 27-43.
T. Popoviciu, 'Asupra poligoanelor regulate', Pozitiva II, **3-4** (1941), 3-8.

29. Let $A_1 \ldots A_n$ $(n > 4)$ be a polygon and let P be an internal point such that the angles A_iPA_{i+1} $(i = 1, \ldots, n;\ A_{n+1} = A_1)$ are acute. If $A_iA_{i+1} = a_i$, $PA_i = R_i$ $(i = 1, \ldots, n)$, then

$$\sum_{i=1}^{n} a_i^2/R_iR_{i+1} \geqslant 4n \sin^2 \frac{\pi}{n} \quad (R_{n+1} = R_1).$$

R. N. Gologan, 'Problem 11533', Gaz. Mat. (Bucharest) **B 23** (1972), 292-293.

30. F is the area of the polygon $A_1 \ldots A_n$, P is an arbitrary point on

the polygon $A_1 \ldots A_n$. Then, if $n \geqslant 4$,

$$(1) \qquad \sum_{i=1}^{n} R_i \geqslant 2\sqrt{2F}.$$

Equality holds if A_1PA_2 is a right isosceles triangle and $A_3 = \ldots = A_n = P$.

This is a result of N. Ozeki (see reference from 24). Note that (1) is also valid if P is an inner point of the polygon. If the polygon is circumscribable and inscribable, using GI 16.23 (i.e. 20), we get

$$(2) \qquad \sum_{i=1}^{n} R_i \geqslant 2r\sqrt{2n \tan \frac{\pi}{n}}.$$

For a triangle the analogous results are GI 12.18 and 12.14.

31. Let Π be a convex n-gon and let Π_1 be a new n-gon whose vertices are the midpoints of the sides of Π. If L and L_1 are perimeters and F and F_1 the areas of the polygons Π and Π_1, respectively, then

$$L_1 \geqslant \frac{1}{2} L \quad (n \geqslant 3) \quad \text{and} \quad F_1 \geqslant \frac{1}{2} F \quad (n \geqslant 4).$$

N. B. Vasiljev, 'Problem M 115', Kvant 1972, No. 8, 61-62.

32. Let Π be a convex polygon of area F and perimeter L and let a convex polygon Π_1 of area F_1 and perimeter L_1 lie in Π. Then

$$2F/L > F_1/L_1.$$

A. Kelarev, 'Problem M 690', Kvant 1981, No. 6, 31 and 1982, No. 2, 29-30.

33. Let a_n be the apothem of a regular n-gon ($n \geqslant 3$) inscribed in a circle of radius R. Then

$$(n + 1)a_{n+1} - na_n > R.$$

If we replace R by a number $R' > R$, the inequality is not true for arbitrary $n \geqslant 3$.

This is an extension of GI 16.20.

'Problem 9297', Gaz. Mat. (Bucharest) B 19 (1968).

M. G. Petkov, 'Problem 5', Mat.-Fiz. (Sofija), 1960, No. 4, 59.

34. Given a circle of radius R. Let o_n be the perimeter of the regular n-gon inscribed in the given circle, and O_n the perimeter of the circumscribed regular n-gon. Then

$$(o_n O_n)^{1/2} \geqslant 2R\pi.$$

Proof. Let $x \in (0, \pi/2)$. We then have

$$\sin x \tan x = \frac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} \frac{2 \tan \frac{x}{2}}{1 - \tan^2 \frac{x}{2}} > 4 \tan^2 \frac{x}{2} > x^2.$$

Therefore, if we take $x = \pi/n$ $(n \geqslant 3)$ it follows that

$$(\sin \frac{\pi}{n} \tan \frac{\pi}{n})^{1/2} > \frac{\pi}{n}, \text{ i.e.} (2nR \sin \frac{\pi}{n} \cdot 2nR \tan \frac{\pi}{n})^{1/2} > 2nR \frac{\pi}{n},$$

and the desired inequality is proved.

Remark. The above inequality of M. Jovanović is better than GI 16.22, but it is weaker than the well-known inequality of Ch. Huygens (1629-1695), i.e. GI 16.21: $\frac{2}{3} o_n + \frac{1}{3} O_n > 2\pi R$.

Recently, E. Braune gave the following inequality:

$$o_n + O_n > 4\pi R/(\cos \frac{\pi}{n})^{1/6}.$$

M. S. Jovanović, 'Some Inequalities Involving Elements of a Triangle and Polygon', Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 357-380 (1971), 81-85.

E. Braune, 'Problem 1137', Math. Mag. 56 (1983), 53-54.

35. If L, F and L_1, F_1, denote the perimeters and areas, respectively, of two polygons Π and Π_1 circumscribed about a given circle such that the greatest side of Π_1 is less than the smallest side of Π, then

$$F > F_1 \quad \text{and} \quad L > L_1.$$

Remark. This result is Klamkin's extension of a result of J. V. Uspensky (the part for perimeters is due to Uspensky).

J. V. Uspensky, 'A Curious Case of Mathematical Induction in Geometry', Amer. Math. Monthly 34 (1927), 247-250.

M. S. Klamkin, 'Inequalities for Inscribed and Circumscribed Polygons', Amer. Math. Monthly 87 (1980), 469-473.

36. Let the complex numbers a_o, a_1, ..., a_{n-1} represent the vertices of a polygon Π. Then the following inequality of Weitzenböck is valid

$$(\sum_k |a_k - a_o|^2)^2 \geqslant 16 \sum_{i<j} \phi_{i,j}^2$$

where $\phi_{i,j}$ is the area of the triangle $a_o a_i a_j$.

B. H. Neumann proved the slightly more general inequality

$$(\sum_k |a_k|^2)^2 \geqslant 16 \sum_{i<j} f_{i,j}^2$$

where $f_{i,j}$ denotes the area of the triangle Oa_ia_j.

B. H. Neumann, 'Some Remarks on Polygons', J. London Math. Soc. 16 (1941), 230-245.

37. Let H denote the regular n-gon with unit side length, $n \geqslant 4$. If K is a convex n-gon inscribed in H with side lengths x_i $(i = 1, \ldots, n)$, then

$$\frac{n(1 - \cos\Theta)}{2} \leqslant \sum_{i=1}^{n} x_i^2 \leqslant n(1 - \cos\Theta),$$

where Θ denotes the internal angle of H.

M. Guan, E. T. H. Wang, W. Janous, and H. O. Kim, 'Problem E 3059', Amer. Math. Monthly 91 (1984), 580.

38. Let $A_1 \ldots A_n$ be a convex n-gon of area F and M an internal point. If r_k and R_k are inradius and circumradius of the triangle A_kMA_{k+1}, respectively, then

(1) $$\sum_{i=1}^{n} r_i \leqslant \sqrt{nF/3\sqrt{3}}$$

and

(2) $$\sum_{i=1}^{n} R_i > 2\sqrt{F/3\sqrt{3}}.$$

This result is due to Hr. Karanikolov.

A conversion of (1) is given as Problem M 126, Kvant 1972, No. 10, 40:

If an n-gon is circumscribed about a circle of radius r and divided into n arbitrary triangles, then

(3) $$\sum_{i=1}^{n} r_i > r.$$

K. Kr'steva-Prevalska gave the following generalization of (1) and (2):

The largest circle lying in a polygon shall be denoted its incircle, and the smallest circle containing a polygon shall be denotes its circumcircle.

Let a convex n-gon be divided into p arbitrary convex m-gons with inradii and circumradii r_k, R_k $(k = 1, \ldots, p)$. Then

$$\sum_{i=1}^{p} r_i \leqslant \sqrt{pF/m} \tan \frac{\pi}{m} \quad \text{and} \quad \sum_{i=1}^{p} R_i > \sqrt{2F/m} \sin \frac{2\pi}{m} .$$

Hr. Karanikolov, 'Njakoi zavisimosti za edin izp'knal n-'g'lnik', God. na VTUZ - Matematika, t. 1, kn. 3, 1964.
K. Kr'steva-Prevalska, 'V'rhu njakoi neravenstva za izp'knal n-'g'lnik', God. na VTUZ - Matematika, t. 3, kn. 1, 1967, 123-125.

39. Let $P_1, P_2, \ldots, P_n$ $(n \geqslant 2)$ be arbitrary points in the plane. Then

$$\max_{1\leqslant i<j\leqslant n} P_iP_j > \frac{\sqrt{3}}{2}(\sqrt{n} - 1) \min_{1\leqslant i<j\leqslant n} P_iP_j .$$

'Problem 3', Matematika (Sofija) 1983, No. 10, 24 and 1984, No. 5, 39.

40. Let $A_1A_2 \ldots A_{n+1}$ be a polygonal line and $B_i \in A_iA_{i+1}$, $i = 1, \ldots, n$. Then

$$\min\left(\frac{\text{area } B_nA_1B_1}{\text{area } A_nA_1A_2}, \frac{\text{area } B_1A_2B_2}{\text{area } A_1A_2A_3}, \ldots, \frac{\text{area } B_{n-1}A_nB_n}{\text{area } A_{n-1}A_nA_1}\right) \leqslant \frac{1}{4} .$$

F. S. Pîrvănescu, 'Problem O:273', Gaz. Mat. (Bucharest) 87 (1982), 45.

41. The plane polygon of n sides of which n - 1 sides are of given lengths a, b, c, ..., has the maximum area F_m when the polygon is inscribed in the circle having the n-th side of unknown length as its diameter. Then the following inequalities are valid

$$\frac{1}{4} \Sigma a\sqrt{b^2 + c^2 + d^2 + \ldots} \leqslant F_m \leqslant \frac{1}{4} \Sigma a(b + c + d + \ldots).$$

T. Hayashi, 'An Inequality in Polygon-Geometry', Tôhoku Math. J. 23 (1924), 90-93.

42. If we denote the diameter, width, perimeter and area of a convex n-gon by d, w, p and F, respectively, then the following inequalities are valid:

(1) $$2n(\tan \frac{\pi}{2n})w \leqslant p \leqslant 2n(\sin \frac{\pi}{2n})d$$

with equalities for regular n-gons, and

(2) $$w^2/\sqrt{3} \leqslant F \leqslant \frac{n}{2}(\cos \frac{\pi}{n} \tan \frac{\pi}{2n})d^2$$

with the first equality only for equilateral triangles, and the second one for regular n-gons.

Remark. This is a result of K. Reinhardt from 1922 (see Kvant 1985, No. 10, 18-19).

43. Let $g(s) = (\cot \frac{\pi}{s})/(4s)$ and let $z_i = 1/(g(i+1) - g(i))$, $i = 3, 4, \ldots$ Let a convex polygon of at most six sides be decomposed into N parts of perimeters $p_1, \ldots, p_N$. If $\min p_i/\max p_i \geqslant \sqrt{z_5/z_6} \approx 0.7713$, then the area F of the polygon satisfies the inequality

$$F \leqslant (\sqrt{3} \sum_{i=1}^{N} p_i^2)/24.$$

L. Fejes Tóth, 'Isoperimetric Problems for Tilings', Discrete Geometrie, 3, Kolloq. Salzburg 1985, 101-110.

44. Given a convex polygon P with m $(m > 4)$ sides and perimeter L(p). We define $T_i(P)$ to be the (possibly infinite) triangle outside P bounded by a side L_i of P and the prolongations of two adjacent sides of P. Let

$$G(P) = \min_{1 \leqslant i \leqslant m} t_i(P)/\text{area}\,(P),$$

where $t_i(P)$ is the area of $T_i(P)$. Then the following results are valid:

$$G(P)\ \text{area}\ (P) \leqslant L^2(P)(2m)^{-2} \tan(2\pi/m) \quad (m > 4),$$

$$G(P) \leqslant \sqrt{5}/5 \quad (m = 5),$$

$$G(P) \leqslant 1/6 \quad (m = 6).$$

M. Longinetti, 'An Isoperimetric Inequality for Convex Polygons and for Equichordal Convex Sets', Pubbl. dell'Instit. Anal. Glob. Appl. (Firenze) 18 (1984), 1-17.
M. Longinetti, 'Una proprietà di massimo dei poligoni affinemente regolari', Ibid. 20 (1984), 1-11.
M. Longinetti, 'Una proprietà di massimo dei poligoni affinemente regolari', Rend. Circ. Mat. Palermo (2) 34 (1985), 448-459.

45. Let $A_1A_2 \ldots A_n$ be a polygon inscribed in a circle of centre O, and let O be in the interior of the polygon. If the normals from O to the sides of the polygon meet the circle in $B_1, B_2, \ldots, B_n$, respectively, then

$$\text{area}\ (B_1B_2 \ldots B_n) \geqslant \text{area}\ (A_1A_2 \ldots A_n),$$

with equality for a regular polygon.
'Problem 2799', Mat. v Škole 1985, No. 5, 60-61.

46. Let F_i be the area of a convex polygon inscribed in a circle of

radius 1, and let F_c be the area of a circumscribable polygon whose points of tangency to the circle are vertices of the first polygon. Then

$$F_i + F_c \geqslant 6,$$

with equality only for a square.

L. Moisotte, 1850 exercises de mathématiques, Paris 1978, p. 95.

Chapter XVI

INEQUALITIES FOR A CIRCLE

1. Let $|p - q|$ denote the Euclidean distances between p and q. K. R. Stolarsky proved the following two results:

1° If p_1, p_2, and p_3 are points on the unit circle U, and $0 \leqslant \lambda \leqslant 2$, then there is a $p \in U$ such that

$$(1) \qquad \sum_{i=1}^{3} |p - p_i|^{\lambda} \geqslant 2 + 2^{\lambda},$$

and this is best possible.

2° Let $e_1, \ldots, e_n$ be the vertices of a regular n-gon ($e_i \in U$, $1 \leqslant i \leqslant n$). Let $p \in U$, and for $0 \leqslant \lambda < 2n$ let

$$(2) \qquad T_{\lambda}(p) = \sum_{i=1}^{n} |p - e_i|^{\lambda}.$$

If λ is an even integer, then $T_{\lambda}(p)$ is constant. Otherwise, let m be the integer such that

$$2m < \lambda < 2(m + 1).$$

If m is even (odd), then $T_{\lambda}(p)$ is maximal (minimal) if and only if p bisects the arc between two consecutive e_i's. Moreover, $T_{\lambda}(p)$ is minimal (maximal) if and only if $p = e_i$ for some i.

K. B. Stolarsky, 'The Sum of the Distances to Certain Pointsets on the Unit Circle', Pacific J. Math. 59 (1975), 241-251.

2. Let $X_1, \ldots, X_n$ be points on the unit circle (hypersphere). Then

$$s = \sum_{1 \leqslant i < j \leqslant n} |X_i X_j|^2 \leqslant n^2.$$

Proof. Let O' be the centre of the unit circle (hypersphere). If $\vec{x}_i = \vec{OX}_i$ ($i = 1, 2, \ldots, n$), then $|X_i X_j|^2 = (\vec{x}_j - \vec{x}_i)^2$, and therefore

$$2s = \sum_{i=1}^{n} \sum_{j=1}^{n} (\vec{x}_i - \vec{x}_j)^2 = 2n \sum_{i=1}^{n} \vec{x}_i^2 - 2 \sum_{i=1}^{n} \sum_{j=1}^{n} \vec{x}_i \vec{x}_j =$$

$$= 2n \sum_{i=1}^{n} \vec{x}_i^2 - 2\vec{x} \cdot \vec{x},$$

where $\vec{x} = \vec{x}_1 + \vec{x}_2 + \ldots + \vec{x}_n$. Hence $s = n^2 - |\vec{x}|^2 \leqslant n^2$, with equality if and only if $\vec{x}_1 + \vec{x}_2 + \ldots + \vec{x}_n = \vec{0}$.

V. Prasolov, 'Problem M 742', Kvant 1982, No. 10, 32.

3. Let A, B, C, D be points on a circle of radius R. Then

$$AB^2 + BC^2 + CA^2 - DA^2 - DB^2 - DC^2 \leqslant 4R^2.$$

'Problem 2079', Mat. v Škole 1978, No. 6, 57 and 1979, No. 5, 65.

4. $P_1, P_2, \ldots, P_n$ are fixed points inside a circle of radius r and point P moves on the circumference. Then

$$\max \sqrt[n]{PP_1 \ldots PP_n} \geqslant r \quad \text{and} \quad \min \sqrt[n]{PP_1 \ldots PP_n} \leqslant r.$$

'Problem P. 248', Köz. Mat. Lapok 50 (1975), 174.

5. Let a radius be drawn in a semi-circle of diameter AB ($|AB| = 2R$) and let circles be inscribed in each sector which touch AB in points M and N, respectively. Then

$$|MN| \leqslant \frac{8 - \sqrt{2}}{8} R.$$

K. V. Vetrov, 'Problem 1696', Mat. v Škole 1976, No. 3, 78 and 1977, No. 1, 85-86.

6. Let the point C be given on the diameter AB of a semi-circle of radius r such that AC = a, and let M be a point on the semi-circle. If the parallels to AM, BM through C cut BM, AM in P, Q, respectively, then

$$\text{per } (MPCQ) \leqslant 2\sqrt{(2r - a)^2 + a^2}$$

with equality if and only if

$$\frac{CP}{CQ} = \left(\frac{2r - a}{a}\right)^2.$$

Zeitschr. math. naturwiss. Unterr. 28 (1897), 344-345.

7. Let F and L be the area and the length of an arc of the segment of a circle, respectively. Then

$$L^2 \geqslant 2\pi F$$

with equality if and only if the segment is a semi-circle.

H. Dörrie, Zeitschr. math. naturwiss. Unterr. 69 (1938), 166-168.

8. Given a circle c of circumference C and area S. Let p_n be the perimeter of the regular n-gon inscribed in c and P_n the perimeter of the circumscribed regular n-gon. In 1654, Ch. Huygens published the following inequalities:

(1) $$\frac{4}{3} p_{2n} - \frac{1}{3} p_n \leqslant C < \frac{2}{3} p_{2n} + \frac{1}{3} P_{2n},$$

(2) $$p_{2n} + \frac{1}{3}(p_{2n} - p_n) < C <$$

$$< p_{2n} + \frac{1}{3}(p_{2n} - p_n)(4p_{2n} + p_n)/(2p_{2n} + 3p_n).$$

If S_n is the area of regular n-gon inscribed in c, then the first inequality in (1) is equivalent to

(3) $$S \geqslant \frac{4}{3} S_{2n} - \frac{1}{3} S_n.$$

Note that Lin Hui proved the following inequalities (about 260 A.D.):

(4) $$S_{2n} < S < 2S_{2n} - S_n.$$

It is known (see for example GI 16.21) that instead of the second inequality in (1) the following inequality can be proved:

(5) $$C < \frac{2}{3} p_n + \frac{1}{3} P_n.$$

Remark. In the proof of (5), M. B. Balk and N. A. Paravjan used the following inequality

$$A < \frac{2}{3} \sin A + \frac{1}{3} \tan A.$$

For a result similar to (5) see also XV.34. M. B. Balk also gave a generalization of (2).

Now, we shall give some other results from the paper of Balk and Paravjan:

(a) Let the isosceles triangle ABC be inscribed in the segment of chord AB. Further, let isosceles triangles ACM and CBN be inscribed in the segments of chords AC and BC. Then

$$F_{ACB} < 4(F_{AMC} + F_{CNB}) \quad \text{and} \quad F_{AMC} > \frac{1}{8} F_{ACB}.$$

If F_C is the area of the segment of chord AB, then

$$\frac{F_C}{F_{ABC}} \geqslant \frac{4}{3} .$$

(b) If the tangents in the points A, B and C determine a new triangle DEF (D is the point of intersection of tangents in A and B), then

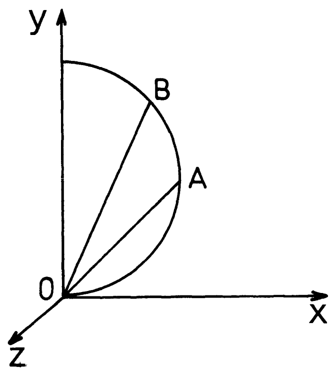
$$F_{ACB} < 2F_{EDF} \quad \text{and} \quad F_C \leqslant \frac{2}{3} F_{ABD}.$$

M. B. Balk and N. A. Paravjan, 'Neravenstva Gjujgensa i ih primenenie', Mat. v škole 1974, No. 2, 70-73.
M. B. Balk, 'Primenenie proizvodnoj k rijasneniju istinitosti neravenstv', Mat. v škole 1975, No. 6, 47-53.

9. J. V. Uspensky proved the following result:
Two arcs AB and AC, neither exceeding a semi-circumference, being taken on the same circle and the latter being the greater of the two, the following inequality holds:

(1) $$\frac{AB}{AC} > \frac{\text{arc } AB}{\text{arc } AC} .$$

H. A. Simmons noted that $(AB/AC)^2$ does not exceed arc AB/arc AC, and he noted that an inequality $p^n > q$ can be called STRONG if it is true for all pairs of values of p, q on the interval in which they are considered while the inequality $p^{n+1} > q$ is not true for some choice of p, q on this interval, n being a positive integer. He proved several such inequalities. Here we shall give his results.



Let a semi-circle in the xy-plane be represented by equation

(2) $$r = 2a \sin \Theta, \quad 0 \leqslant \Theta \leqslant \pi/2,$$

and denote the coordinates of A and B, two points on the circumference, by (r_1, Θ_1) and (r_2, Θ_2), respectively, where $0 < \Theta_1 < \Theta_2 \leqslant \pi/2$, and denote by a_1, a_2 the lengths of the arcs OA and OB, respectively. Let k_1 and k_2 denote the areas of the segments of OA and OB, respectively, of the circle. Let v_1 and v_2 denote the volumes of the segments of the sphere (obtained by rotating the semi-circle about the y-axis) which lie below the planes ZOA and ZOB, respectively, OZ being perpendicular to the xy-plane. Let s_1 and s_2 denote the areas of the spherical portions of the former and latter segments, respectively. Then the following inequalities are valid:

(3) $(r_1/r_2)^3 > k_1/k_2$, (4) $(a_1/a_2)^2 > k_1/k_2$,

(5) $(r_1/r_2)^2 > s_1/s_2$, (6) $a_1/a_2 > s_1/s_2$,

(7) $(r_1/r_2)^4 > v_1/v_2$, (8) $(a_1/a_2)^2 > v_1/v_2$.

All the above inequalities are strong.

Proofs of (3) and (4). To prove (3), we first observe that $(r_1/r_2)^3 = \sin^3 \Theta_1/\sin^3 \Theta_2$, by (2), and that the areas k_1, k_2 have the values

$$k_1 = \frac{1}{2}\int_0^{\Theta_1} 4a^2 \sin^2 \Theta \, d\Theta = a^2(\Theta_1 - \sin \Theta_1 \cos \Theta_1),$$

$$k_2 = a^2(\Theta_2 - \sin \Theta_2 \cos \Theta_2).$$

We need to prove, then, that

$$(9) \qquad \frac{\sin^3 \Theta_1}{\sin^3 \Theta_2} > \frac{\Theta_1 - \sin \Theta_1 \cos \Theta_1}{\Theta_2 - \sin \Theta_2 \cos \Theta_2}, \quad \text{or}$$

$$\frac{\sin^3 \Theta_1}{\Theta_1 - \sin \Theta_1 \cos \Theta_1} > \frac{\sin^3 \Theta_2}{\Theta_2 - \sin \Theta_2 \cos \Theta_2}.$$

Since $0 < \Theta_1 < \Theta_2 \leqslant \pi/2$, the inequality (9) will be established if it is shown that $y = \sin^3 x/(x - \sin x \cos x)$ decreases as x increases from 0 to $\pi/2$, the value $x = 0$ itself being excluded. Differentiating y with respect to x and simplifying the result by means of the identity $\sin^2 x + \cos^2 x = 1$, we find

$$y' = \frac{\sin^2 x(3x \cos x - 3 \sin x + \sin^3 x)}{(x - \sin x \cos x)^2}.$$

In this equation, $y' < 0$ if $3x \cos x - 3 \sin x + \sin^3 x < 0$, or if

(10) $$x < \frac{2}{3} \tan x + \frac{1}{3} \sin x \cos x, \quad 0 < x \leqslant \pi/2.$$

To prove (10), consider the curves

$$y = x, \quad y = \frac{2}{3} \tan x + \frac{1}{3} \sin x \cos x.$$

Each of these curves passes through the origin with slope 1, but the latter curve has, at every point, the slope $\frac{2}{3} \sec^2 x + \frac{1}{3} \cos 2x$, which is greater than 1 throughout the interval $0 < x \leqslant \pi/2$; for

$$\frac{2 \sec^2 x}{3} + \frac{\cos 2x}{3} = \frac{2 + 2 \cos^4 x - \cos^2 x}{3 \cos^2 x} > 1$$

since $(\cos^2 x - 1)^2 > 0$ on the interval $0 < x \leqslant \pi/2$. Consequently, on this interval the latter curve lies above the former. Hence (10) holds, $y' < 0$, and the function $\sin^3 x/(x - \sin x \cos x)$ decreases, as was to be proved. This proves (3). Take $\Theta_1 = \pi/6$ and $\Theta_2 = \pi/3$, then $(r_1/r_2)^4 < k_1/k_2$, since

$$\frac{\sin^4 \pi/6}{\sin^4 \pi/3} < \frac{(\pi/6) - \sin(\pi/6) \cos(\pi/6)}{(\pi/3) - \sin(\pi/3) \cos(\pi/3)},$$

and, therefore, (3) is strong.

Remark. For inequality (10) see also Remark in 8.

To prove (4), we first observe that

$$\frac{a_1^2}{a_2^2} = \frac{\Theta_1^2}{\Theta_2^2}, \quad \frac{k_1}{k_2} = \frac{\Theta_1 - \sin \Theta_1 \cos \Theta_1}{\Theta_2 - \sin \Theta_2 \cos \Theta_2}.$$

We need to prove, then, that

(11) $$\frac{\Theta_1^2}{\Theta_2^2} > \frac{\Theta_1 - \sin \Theta_1 \cos \Theta_1}{\Theta_2 - \sin \Theta_2 \cos \Theta_2}, \quad \text{or}$$

$$\frac{\Theta_1^2}{\Theta_1 - \sin \Theta_1 \cos \Theta_1} > \frac{\Theta_2^2}{\Theta_2 - \sin \Theta_2 \cos \Theta_2}.$$

To establish (11), we need only to show that $y = x^2/(x - \sin x \cos x)$

decreases as x increases from 0 to $\pi/2$, the value of $x = 0$ itself being excluded, since $0 < \Theta_1 < \Theta_2 \leqslant \pi/2$. Differentiating y with respect to x and simplifying the result by means of the identity $\sin^2 x + \cos^2 x = 1$, we find

$$y' = \frac{2x \cos x(x \cos x - \sin x)}{(x - \sin x \cos x)^2} < 0,$$

since $x < \tan x$ on the interval $0 < x \leqslant \pi/2$. Hence (11) is true. This proves (4). That this inequality is strong follows from the following example:

$\Theta_1 = \pi/6$, $\Theta_2 = \pi/3$, then $(a_1/a_2)^3 < k_1/k_2$ since

$$\frac{(\pi/6)^3}{(\pi/3)^3} < \frac{(\pi/6) - \sin(\pi/6)\cos(\pi/6)}{(\pi/3) - \sin(\pi/3)\cos(\pi/3)} .$$

In his paper, H. A. Simmons also proved the following result:

Let P and Q be two polygons inscribed in the same circle of radius a. Denote by $p_1, p_2, \ldots, p_n$ the lengths of the sides of P in increasing order of magnitude and by $u_1, u_2, \ldots, u_n$ their respective intercepted arcs (each $\leqslant \pi a$) on the circumference, which will be taken as the unit of arc. Denote by $q_1, q_2, \ldots, q_m$ the lengths of the sides of Q in decreasing order of magnitude, and by $v_1, v_2, \ldots, v_m$ the lengths of their respective intercepted arcs, and suppose $q_1 < p_1$. Denote by j_i ($i = 1, 2, \ldots, n$) the area of the segment of the circle which is bounded by the i-th side of P and its intercepted arc, and by k_t ($t = 1, 2, \ldots, m$) the area which is bounded by the t-th side of Q and its intercepted arc. If $J = j_1 + j_2 + \ldots + j_n$ and $K = k_1 + k_2 + \ldots + k_m$, then

$$(12) \qquad \frac{J}{K} = \frac{\sum_{i=1}^{n} j_i}{\sum_{i=1}^{m} k_i} > \frac{\sum_{i=1}^{n} p_i^3}{\sum_{i=1}^{m} q_i^3} > \frac{\sum_{i=1}^{n} p_i^2}{\sum_{i=1}^{m} q_i^2} > 1 > \frac{\sum_{i=1}^{n} p_i}{\sum_{i=1}^{m} q_i} ,$$

and

$$(13) \qquad \frac{J}{K} > \frac{\sum_{i=1}^{n} u_i^2}{\sum_{i=1}^{m} v_i^2} > \frac{\sum_{i=1}^{n} u_i}{\sum_{i=1}^{m} v_i} = 1$$

J. V. Uspensky, 'A Curious Case of Mathematical Induction in Geometry', Amer. Math. Monthly 34 (1927), 247-250.

H. A. Simmons, 'Strong and Weak Inequalities Involving the Ratio of Two Chords or Two Arcs of a Circle; Chains of Inequalities', Amer. Math. Monthly 35 (1928), 122-130.

10. Let C_1 and C_2 be two distinct concentric circles with centre O of radii R_1 and R_2, respectively; assume $R_2 < R_1$. Let P, $(P \neq O)$, be any point inside C_2 and let the line L pass through OP. Construct a circle C_4 of radius R_4, where $OP + R_4 > R_2 > R_2 - OP$ so that C_4 cuts C_2. Let Q be one of the points of intersection of C_4 and C_2. Let the line PQ intersect C_1 at R. Construct the circle C_3 with centre P passing through R. Then

(i) $\dfrac{OR}{OQ} < \dfrac{OP + PR}{OP + PQ}$ if and only if $PQ \cdot PR < OP$;

(ii) $\dfrac{OR}{OQ} > \dfrac{OP + PR}{OP + PQ}$ if and only if $PQ \cdot PR > OP$;

(iii) $\dfrac{OR}{OQ} = \dfrac{OP + PR}{OP + PQ}$ if and only if Q and R are inverse points with respect to the circle centered at P and passing through O.

J. M. Sloss, 'A Four Circle Inequality', Math. Mag. 37 (1964), 327-330.

11. Conjectures: Let in the circle C(O, R) be inscribed three circles $C(O_k, R_k)$, $1 \leqslant k \leqslant 3$, so that they touch the first circle from the interior and one touches the other two from the exterior. Further, let the circle $C(\omega, \rho)$ touch these three circles in the exterior. Then

(1) $R_1^{-1/2}$, $R_2^{-1/2}$, $R_3^{-1/2}$ can be the sides of a triangle;

(2) $$\frac{1}{R} + \frac{1}{\rho} = 4\left(\frac{R_1 + R_2 + R_3}{R_1R_2R_3}\right)^{1/2} \geqslant \frac{4\sqrt{3}}{\sqrt[3]{R_1R_2R_3}} \geqslant \frac{12\sqrt{3}}{R_1 + R_2 + R_3}.$$

Remark. The result given in (2) is an interpolating inequality of an inequality of V. Nikula.

V. Nikula, 'Problem C:177', Gaz. Mat. (Bucharest) 87 (1982), 46.

12. Let on the diameter AB of a semi-circle of radius R be given the point C, and let D be a point on the semi-circle such that $CD \perp AB$. Further, let two circles of radii r_1 and r_2 be inscribed in figures ACD and CBD, i.e. they touch the diameter AB, the normal CD and the semi-circle. Then

$$r_1 + r_2 \leqslant 2R(\sqrt{2} - 1).$$

'Problem 1323', Mat. v Škole 1974, No. 5, 84.

13. Let a, b be two coplanar non-intersecting circles of radii r_a and r_b, and let the distance between their centres be d. Then the inversive distance (ab), between the circles is defined as

$$(ab) = \cosh^{-1}\left(|r_a^2 + r_b^2 - d^2|/(2r_a r_b)\right).$$

(ab) has the property that it is invariant under the group of inversive transformations of the plane; further, if a, b, c are three circles belonging to the same non-intersecting co-axial system, then

$$(ab) + (bc) = (ca).$$

H. S. M. Coxeter gave an interesting 'non-triangle' inequality, namely that if a, b, c are three nested circles with $r_a < r_b < r_c$ which do not belong to the same co-axial system, then

$$(ab) + (bc) < (ca).$$

H. S. M. Coxeter, 'Inversive Geometry', Educ. Stud. in Math. 3 (1971), 310-321.
R. Shail, 'An Electrostatic Proof of the Inversive Distance Inequality', Math. Gaz. 56 (1972), 328-329.

14. Let γ_1 and γ_2 be two circular arcs through the points A and B, and let r_1 and r_2 be their radii. Then we have

$$r_1 < r_2 \Rightarrow \gamma_1 > \gamma_2.$$

K. Mahler, 'On a Question in Elementary Geometry', Simon Stevin 28 (1951), 90-97.

15. Let r be the radius of a circle which can be covered by three circular disks of given radii a, b, c. If these radii are equal to sides of an acute triangle with the circumradius R, then it holds $r \leqslant R$, and otherwise $r \leqslant \max(a, b, c)$..

Let r be the radius of a circle which can be covered by four circles of given radii a, b, c, d. If these radii are equal to sides of a quadrilateral inscribed in a circle with the radius R, and if the centre of this circle lies inside the quadrilateral, then it holds $r \leqslant R$, and otherwise $r \leqslant \max(a, b, c, d)$.

For five or more circles the problem is unsolved.

J. Molnár, 'Über eine elementargeometrische Extremalaufgabe', Mat. Fiz. Lapok 49 (1942), 238-248 (Hungarian).

16. Let two regular polygons of n and n + 1 sides be inscribed in a circle of radius 1. Then there exist two vertices of these polygons, for example A is the vertex of the first polygon and B of the second one, such that

$$\text{arc } AB < 2\pi/n(n + 1).$$

L. Moisotte, 1850 Exercises de Mathématiques, Paris 1978, p. 95.

17. Let Ω be a semi-circle of unit radius, with diameter AA_0. Consider a sequence of circles δ_i, all interior to Ω, such that δ_1 is tangent to Ω and to AA_0, δ_2 is tangent to Ω and to the chord AA_1 tangent to δ_1, δ_3 is tangent to Ω and to the chord AA_2 tangent to δ_2, etc. Of course, $A_{i+1} \neq A_i$ and δ_i and δ_{i+1} are on opposite sides of AA_i for each i. Then

$$r_1 + r_2 + r_3 + \ldots < 1.$$

J. Dou, 'Problem 1057', Crux Math. 11 (1985), 189 and 12 (1986), 288-289.

Chapter XVII

PARTICULAR INEQUALITIES IN PLANE GEOMETRY

1. Some Isoperimetric Inequalities

1.1. A classical isoperimetric inequality states that, for a simple closed curve C of length L in the plane, the area F enclosed by C satisfies

(1) $$L^2 - 4\pi F \geqslant 0.$$

Since equality holds if C is a circle, it follows that the circle encloses a maximum area among all curves of the same length. It does not follow that the circle is the only curve enclosing the maximum area. That statement requires a separate proof.

From 1921 to 1929, Bonnesen proved a series of inequalities of the form

(2) $$L^2 - 4\pi F \geqslant B,$$

where B is an expression having the following properties:

1° B is non-negative;
2° B can vanish only if C is a circle;
3° B has geometric significance.

BONNESEN'S THEOREM. Let $\mathcal{D}$ be a plane domain bounded by a rectifiable Jordan curve C of length L. Let F, ρ, R denote the area, inradius, and circumradius of $\mathcal{D}$, respectively. Then if r satisfies $\rho < r < R$, the following inequalities are valid:

(3) $$L^2 - 4\pi F > (L - 2\pi r)^2,$$

(4) $$L^2 - 4\pi F > \left(\frac{F}{r} - \pi r\right)^2,$$

(5) $$L^2 - 4\pi F > \left(L - \frac{2F}{r}\right)^2,$$

(6) $$L - 2\pi r > \frac{F}{r} - \pi r,$$

(7) $$L - \frac{2F}{r} > \pi r - \frac{F}{r},$$

(8) $$\frac{(L - 2\pi r)^2 - \left(L - \frac{2F}{r}\right)^2}{\frac{F}{r} - \pi r} > 0 \qquad (F \neq \pi r^2),$$

(9) $$rL > F + \pi r^2,$$

(10) $$\frac{L - \sqrt{L^2 - 4\pi F}}{2\pi} < r < \frac{L + \sqrt{L^2 - 4\pi F}}{2\pi} \quad (L^2 > 4\pi F).$$

This is a main result from which various Bonnesen inequalities may be derived.

Remark. Each domain has a unique circle of smallest radius which encloses it; that is the circumscribed circle of radius R. There may not be a unique inscribed circle - that is, a unique circle of maximum radius lying in $\mathcal{D} \cup C$ - but the maximum radius of all such circles if well-defined, and it is the inradius ρ of $\mathcal{D}$.

The above text is written according to a nice review of R. Osserman. See also the 77 references in the paper of Osserman.

R. Osserman, 'Bonnesen-style Isoperimetric Inequalities', Amer. Math. Monthly 86 (1979), 1-29.

1.2. If L is the length of a convex polygon P, F its area, r the radius of the largest inscribed circle, and s the length of any chord through the centre of a largest inscribed circle, then

$$L^2 - 4\pi F \geqslant \frac{\pi^2}{4}(s - 2r)^2.$$

This is a sharpened isoperimetric inequality for convex polygons.

H. Hadwiger, 'Eine elementare Ableitung der isoperimetrischen Ungleichung für Polygone', Comment. Math. Helv. 16 (1944), 305-309.

1.3. Let F_n (respectively f_n) be the area of an n-gon with the maximal (respectively minimal) area inscribed in (respectively circumscribed about) a given convex figure. Then

$$F_{n-1} + F_{n+1} \leqslant 2F_n \quad (n \geqslant 4),$$

$$f_{n-1} + f_{n+1} \geqslant 2f_n \quad (n \geqslant 4).$$

Remark. These inequalities play a central part in the problems of the densest packing and the thinnest covering with convex discs.

C. H. Dowker, 'On Minimum Circumscribed Polygons', Bull. Amer. Math. Soc. 50 (1944), 120-122.

L. Fejes-Tóth, 'Megjegyzések Dowker sokszögtételeihaz', Mat. Lapok 6 (1955), 176-179.

14. Let P_n (respectively p_n) be the perimeter of an n-gon with the maximal (respectively minimal) perimeter inscribed in (respectively circumscribed about) a given convex figure. Then

$$P_{n-1} + P_{n+1} \leqslant 2P_n \quad (n \geqslant 4),$$

$$p_{n-1} + p_{n+1} \geqslant 2p_n \quad (n \geqslant 4).$$

L. Fejes-Tóth, 'Megjegyzések Dowker sokszögtételeihaz', Mat. Lapok 6 (1955), 176-179.
J. Molnár, 'Konvex tartományok beírt és körülirt poligonjairól', Mat. Lapok 6 (1955), 210-218.

1.5. Let $A_1, A_2, \ldots, A_n$ be n ($n \geqslant 3$) given points. If $\alpha_n = \min \measuredangle A_iA_jA_k$ for all $i, j, k \in \{1, 2, \ldots, n\}$, then

$$\alpha_n \leqslant \pi/n$$

with equality only if $A_1, A_2, \ldots, A_n$ are the vertices of a regular n-gon.

H. T. Croft, 'Some Geometrical Thoughts II', Math. Gaz. 51 (1967), 125-129.

1.6. Let $r \in R^+$ and $4r < L < 2r\pi$. There is one and only one (modulo a rotation) polygon P_0 with the perimeter L inscribed in a given circle with radius r such that P_0 has all sides equal except one and this one is not greater than any of them. If F_0 is the area of this polygon and F the area of any convex figure K with the perimeter L contained in a circle of radius r, then

$$F \geqslant F_0$$

with equality if and only if $K = P_0$.

S. J. Taylor, 'Some Simple Geometrical Extremal Problems', Math. Gaz. 37 (1953), 188-198.

1.7. Let C be a closed rectifiable curve of length L, which bounds a figure K of area F and let $r \in R^+$. If K_r is the set of points whose distances from K are not greater than r (K_r is generally called parallel domain of K at distance r) and if F_r is the area of K_r, then

$$F_r \leqslant F + Lr + r^2\pi$$

with equality if and only if C is a convex curve.

1.8. Let C be a given rectifiable curve of length L and $r \in R^+$. If C_r is the set of points whose distances from C are not greater than r and if F_r is the area of C_r, then

$$F_r \leqslant 2Lr + r^2\pi.$$

Equality holds if and only if C is a smooth curve and $r \leqslant \min\,(r_1, r_2/2)$, where r_1 is the limit inferior of radii of circles through all triples of different points on C and r_2 is the distance of two end points of C.

H. Hornich, 'Eine allgemeine Ungleichung für Kurven', Monatsh. Math. 47 (1939), 432-438.

1.9. Let F be the area of a rectangle with the minimal area, which contains a given arc of a curve of length L. Then

$$F \leqslant \frac{1}{4} L^2$$

with equality if and only if the given arc is formed by two adjacent sides of a square.

J. H. McKay, 'The William Lowell Putnam Mathematical Competition. Problem B-4', Amer. Math. Monthly 77 (1970), 724 and 727.

1.10. Inside a quadratic basin with the side length 2a + b there is a concentric and homothetic quadratic island with side of length b. By means of two boards of length x (the width can be neglected) one can cross from the bank to the island. Then

$$x \geqslant \frac{2\sqrt{2}}{3} a.$$

S. Newman and H. Marston, 'Problem 605', Math. Mag. 39 (1966), 194.

1.11. Let $0 < L < 4r\pi$ and let ϕ be the unique solution of the equation $\phi + \sin\phi = \frac{L}{2r}$. If F is the area of a convex figure K of perimeter L, which is bounded by arcs and chords of a circle with radius r, then

$$F \leqslant r^2\left(\phi + \left(\phi - \frac{L}{2r}\right)\sqrt{1 - \left(\phi - \frac{L}{2r}\right)^2}\right)$$

with equality if and only if K is a segment of a circle with radius r.

S. J. Taylor, 'Some Simple Geometrical Extremal Problems', Math. Gaz. 37 (1953), 188-198.

1.12. Let F be the area of a figure K of constant width d. Then

$$F \geqslant \frac{1}{2}(\pi - 3)d^2.$$

Equality holds if and only if K is the triangle of Reuleaux (i.e. the common part of three equal circular discs with radius d and with centres in vertices of an equilateral triangle of side length d).

S. J. Taylor, 'Some Simple Geometrical Extremal Problems', Math. Gaz. 37 (1953), 188-198.

1.13. Let F be the area of a convex figure K of perimeter L and diameter not greater than d, where $2d \leqslant L \leqslant 3d$. Then

$$F \geqslant \frac{1}{4}(L - 2d)\sqrt{4Ld - L^2}$$

with equality if and only if K is a triangle with side lengths d, d, L - 2d.

S. J. Taylor, 'Some Simple Geometrical Extremal Problems', Math. Gaz. 37 (1953), 188-198.

1.14. Let K be a bounded, closed convex set in the Euclidean plane. If we denote the diameter, width, perimeter, area, inradius, and circumradius of K by d, w, p, A, r, and R, respectively, then the following inequalities are valid:

(1) $(w - 2r)d < w^2/2$ (I),

(2) $(w - 2r)d \leqslant 2wr/\sqrt{3}$ (E),

(3) $(w - 2r)R < w^2/4$ (I),

(4) $(w - 2r)R \leqslant 2wr/3$ (E),

(5) $(w - 2r)p \leqslant 2w^2/\sqrt{3}$ (E),

(6) $(w - 2r)A < w^3/4$ (I),

(7) $(w - 2r)A \leqslant w^2r/\sqrt{3}$ (E),

where we use the following notations:

(E) - equality is valid when K is an equilateral triangle,

(I) - the upper bound is the limit as K approaches an 'infinite' isosceles triangle' of fixed base and unbounded altitude.

The inequalities are the best possible.

By Blaschke's Theorem (W. Blaschke, Kreis und Kugel, 2te Aufl., Berlin, 1956), every bounded convex figure of width w contains a circle of radius w/3. It follows that $w \leqslant 3r$; equality holds here if and only if the figure is an equilateral triangle. Therefore, the following results are also valid:

(8) $(w - 2r)d \leqslant 2\sqrt{3}\, r^2$ (E),

(9) $(w - 2r)R \leqslant 2r^2$ (E),

(10) $(w - 2r)p \leqslant 2\sqrt{3}\, wr \leqslant 6\sqrt{3}\, r^2$ (E),

(11) $(w - 2r)A \leqslant \sqrt{3}\, wr^2 \leqslant 3\sqrt{3}\, r^3$ (E).

P. R. Scott, 'A Family of Inequalities for Convex Sets', Bull. Austral. Math. Soc. 20 (1979), 237-245.

1.15. If P is an inscribed n-gon of maximum area in a convex region K, then

$$\text{area (P)} \geq \frac{n}{2\pi} \sin \frac{2\pi}{n} \text{ area (K)},$$

and equality holds only if K is an ellipse.

G. D. Chakerian and L. H. Lange, 'Geometric Extremum Problems', Math. Mag. 44 (1971), 57-69.

1.16. Let E be an ellipse inscribed in a square of side a. Then

$$\text{per(E)} \leq a\pi$$

with equality if and only if E is a circle.

'Problem A.4', Fiz. Mat. Spisanie (Sofija) 18 (51) (1975), 114-145.

1.17. Let F be a compact planar set, not contained in a line. Let

$I = \{\sup \text{ area of triangle ABC}: \{A, B, C\} \subset F\}$,

$J = \{\inf \text{ area of triangle ABC}: F \subset \text{closed triangle ABC}\}$.

Then

$$4I \geq J.$$

Equality occurs when F is a circle or when F comprises the vertices of a square.

I. J. Schoenberg, 'Problem 6466', Amer. Math. Monthly 91 (1984), 441 and 92 (1985), 742.

2. Various Particular Inequalities

2.1. Let $P_i = (x_i, y_i)$, $i = 1, 2, 3$, $x_1 < x_2 < x_3$, be points in the Cartesian (x, y)-plane and let R denote the radius of the circumcircle of the triangle $P_1P_2P_3$. Then

$$\frac{1}{R} < 2\left|\frac{y_1}{(x_1 - x_2)(x_1 - x_3)} + \frac{y_2}{(x_2 - x_3)(x_2 - x_1)} + \frac{y_3}{(x_3 - x_1)(x_3 - x_2)}\right|$$

unless both sides vanish. The constant 2 is the best.

I. J. Schoenberg and W. J. Blundon, 'Problem 77-9', SIAM Review 19 (1977), 329 and 20 (1978), 399-400.

2.2. Let M, N, P be the midpoints of the sides BC, CA, AB of a triangle, respectively. If $A' \in BM$, $B' \in CN$, $C' \in AP$ and $BC > CA > AB$, then

$$\Sigma AA' < \Sigma AB.$$

Ş. Barcanescu, Gaz. Mat. (Bucharest) B 18 (1967), 546 and B 19 (1968), 665-666.

2.3. Let u, v, w be the distances of the feet of the angle-bisectors of the angles A, B, C to the remaining sides, respectively. Then

$$\Sigma u/h_a \geqslant 3/2.$$

E. A. Bokov, 'Problem 1589', Mat. v Škole 1975, No. 5, 65 and 1976, No. 3, 79-80.

2.4. Let ϕ be the angle between the side a and the median m_b of a triangle. Then

$$\cot A \geqslant \frac{2\sqrt{2} - 3\cos\phi}{\sin\phi}.$$

E. G. Gotman, 'Problem 1472', Mat. v Škole 1975, No. 1, 86 and 1975, No. 6, 82-83.

2.5. Let ABC be a given triangle, neither equilateral nor isosceles, with base AC. Let D be the foot of angle-bisector w_b and M the foot of the normal from C to BD. Then

(a) The parallel to AB through M passes through the midpoints B' and A' of sides AC and BC, respectively.

(b) If AA' ⊥ BD, then by joining A and M we get two parallelograms ABA'M and AA'CM, one of which is a rhomb; if d_i $(i = 1, \ldots, 4)$ are the diagonals of these parallelograms and h is the distance between the centres of these parallelograms, then

$$h < \frac{1}{4}\Sigma d_i.$$

(c) If the above parallelograms are rhombs, and the angle ABC = 60°, then they are equal and

$$h < \frac{d + D}{2} \quad \text{and} \quad \frac{D}{2d} + \frac{d}{D} > \frac{1}{6}$$

where d and D are diagonals of one of the rhombs.

I. Ionescu, 'Problem 8772', Gaz. Mat. (Bucharest) B 20 (1969), 27-29.

2.6. Take cevians AA', BB', CC' which concur in an interior point L of the medial triangle. Then

$$\left|\frac{A'C}{A'B} - \frac{B'C}{B'A}\right| < 1 < \frac{A'C}{A'B} + \frac{B'C}{B'A}.$$

This result is due to V. Cîrtoaje. An equivalent result is given by E. Kraemer.

V. Cîrtoaje, 'Problem 11420', Gaz. Mat. (Bucharest) B 22 (1971), 540.

E. Kraemer and R. Urban, 'Problem 6', Rozhledy Mat. Fyz. 55 (1976-1977), 30 and 56 (1977-1978), 82-87.

2.7. Let AT_1, BT_2, CT_3 be the extended angle-bisectors to the circumcircle of a triangle ABC. Perpendiculars T_1H_1, T_2H_2, T_3H_3 are drawn to sides AC, BA, CB, respectively. Then

$$(1) \qquad \Sigma x T_1H_1 \leqslant R\Sigma \frac{yz}{x},$$

where x, y, z are non-negative numbers.

Proof. From $T_1H_1 = AT_1 \sin\frac{A}{2} \leqslant 2R\sin\frac{A}{2}$ and two similar results, we get

$$\Sigma x T_1H_1 \leqslant 2R\Sigma x \sin\frac{A}{2}.$$

Using the well-known inequality (see VI.1.2. with $n = 0$, $p = 1$, $x^2 \to yz/x$, etc.)

$$\Sigma x \sin\frac{A}{2} \leqslant \frac{1}{2}\Sigma \frac{yz}{x},$$

we get (1).

Remark. For $x = y = z = 1$ we get a statement of J. Garfunkel.

J. Garfunkel and C. W. Dodge, 'Problem 374', Pi Mu Epsilon J. 6 (1976), 306 and 6 (1978), 557-558.

2.8. Let ABC be a triangle with sides a, b, c in the usual order, and let l_a, l_b, l_c and l'_a, l'_b, l'_c be two sets of concurrent cevians, with l_a, l_b, l_c intersecting a, b, c in L, M, N, respectively. If

$$l_a \cap l'_b = P, \quad l_b \cap l'_c = Q, \quad l_c \cap l'_a = R,$$

then, independently of the choice of concurrent cevians l'_a, l'_b, l'_c, we have

$$(1) \qquad \frac{AP}{PL}\cdot\frac{BQ}{QM}\cdot\frac{CR}{RN} = \frac{abc}{BL\cdot CM\cdot AN} \geqslant 8,$$

with equality occurring just when l_a, l_b, l_c are the medians of the triangle.

Remark. The above result is given as Problem 790 in Crux Math., and extends Problems 588 and 685 from the same journal.

R. H. Eddy and J. Dou, 'Problem 790', Crux Math. 8 (1982), 278 and 9 (1983), 60-62.
J. Garfunkel and R. H. Eddy, 'Problem 588', Crux Math. 6 (1980), 317 and 7 (1981), 306-307.
J. T. Groenman and K. Satyanarayana, 'Problem 685', Crux Math. 7 (1981), 275 and 8 (1982), 292-293.

2.9. Let ABC be a triangle with area F, sides a, b, c, medians m_a, m_b, m_c, and interior angle-bisectors w_a, w_b, w_c. If

$$w_a \cap m_b = P, \quad w_b \cap m_c = Q, \quad w_c \cap m_a = R,$$

and if δ denotes the area of triangle PQR, then

$$\delta/F < 1/6.$$

Proof. Using barycentric coordinates and vectors with an origin outside the plane of triangle ABC (assumed for now to be non-degenerate), the incenter I and centroid G of the triangle are given by

$$\vec{I} = (\Sigma a\vec{A})/(\Sigma a) \quad \text{and} \quad \vec{G} = \frac{1}{3}(\Sigma\vec{A}).$$

Then

$$\vec{P} = \vec{A} + u(\vec{I} - \vec{A}) = \vec{B} + v(\vec{G} - \vec{B}),$$

and equating the coefficients of $\vec{A}$ and $\vec{B}$ on both sides gives $v = 3c/(b + 2c)$. Thus

$$\vec{P} = (c\vec{A} + b\vec{B} + c\vec{C})/(b + 2c),$$

and similarly,

$$\vec{Q} = (a\vec{A} + a\vec{B} + c\vec{C})/(c + 2a) \quad \text{and} \quad \vec{R} = (a\vec{A} + b\vec{B} + b\vec{C})/(a + 2b).$$

Now we find

$$2\delta = |\vec{Q}\times\vec{R} + \vec{R}\times\vec{P} + \vec{P}\times\vec{Q}| = f(a, b, c)|\vec{B}\times\vec{C} + \vec{C}\times\vec{A} + \vec{A}\times\vec{B}| =$$

$$= f(a, b, c) \cdot 2F,$$

where

$$f(a, b, c) = \frac{\Sigma bc^2 - 2abc}{\Pi(b + 2c)}.$$

Thus $\delta/F = f(a, b, c)$ for non-degenerate triangles ABC. We will show that $f(a, b, c) \leqslant 1/6$ holds for all triangles, even for degenerate ones (for which we assume that δ/F is defined by $f(a, b, c)$).

The inequality $f(a, b, c) \leqslant 1/6$ is easily shown to be equivalent to

$$2\Sigma bc(a + b - c) + 21abc \geqslant 0,$$

and this is clearly true since $a + b - c \geqslant 0$, etc. Equality holds just when $abc = 0$, that is, just when ABC is a degenerate triangle with one side of length zero.

J. Tabov, S. Troyanski, and M. S. Klamkin, 'Problem 883', Crux Math. **9** (1983), 275 and **10** (1984), 28-29.

2.10. Let r_1, r_2, r_3 be the radii of circles which touch two of the extended sides of a triangle and its circumcircle. Then

$$r_1 + r_2 + r_3 \geqslant 6R.$$

Ju. I. Gerasimov, 'Problem 420', Mat. v Škole 1967, No. 6, 77 and 1968, No. 4, 82.

2.11. Let three circles of radii r_1, r_2, r_3 be given in a triangle ABC, such that they touch two sides and its incircle. Then

$$\Pi \frac{r - r_1}{r + r_1} \leqslant \frac{1}{8}.$$

M. Stanković and Z. M. Stojaković, 'Problem 175', Mat. Vesnik 7 (22) (1970), 277.

2.12. Conjecture: let r be radius of the incircle of an arbitrary triangle lying in the closed unit square. Then

$$r \leqslant \frac{1}{4}(\sqrt{5} - 1).$$

L. Funar, 'Problem 6477', Amer. Math. Monthly 81 (1984), 588.

2.13. The Wallace point W of any four points A_1, A_2, A_3, A_4 on a circle with centre O may be defined by the vector equation (Bottema and Groenman):

$$\vec{OW} = \frac{1}{2} \Sigma \vec{OA}_1.$$

Let δ be a cyclic quadrilateral the Wallace point of whose vertices lies inside δ. Let a_i (i = 1, 2, 3, 4) be the sides of δ, and let G_i be the midpoint of the side opposite to a_i. Then

$$\Sigma a_1 \cdot G_1X \geqslant 2S,$$

where X ranges over all the points of the plane of δ, S is the area of δ, with equality if X is the Wallace point.

J. B. Tabov, 'Problem 1046', Crux Math. 11 (1985), 147.

2.14. Let an equilateral triangle and a square be inscribed in a circle of radius r. If F is the area of their joint part and if

$$F_m = \frac{r^2}{6}(9\sqrt{2} + 2\sqrt{6} - 6\sqrt{3}) \sim 1{,}20577r^2,$$

$$F_M = \frac{r^2}{4}(8\sqrt{3} - 9) \sim 1,21410r^2,$$

then

$$F_m \leqslant F \leqslant F_M.$$

The first equality is valid if and only if the square and the triangle have two parallel sides, and the second equality is valid if and only if they have one joint vertex.

Remark. Note that $(F_M - F_m)/F_M \sim 0,00687 < 7 \cdot 10^{-3}$.

H. Demir and M. Goldberg, 'Problem 537', Math. Mag. 37 (1964), 277-278.

2.15. Let an equilateral triangle and a square be circumscribed about a circle of radius r. If F is area of their joint part and if

$$F_m = r^2(4\sqrt{6} + 2\sqrt{2} - 9) \sim 3,62639r^2,$$

$$F_M = \frac{r^2}{3}(18 - 4\sqrt{3}) \sim 3,69060r^2,$$

then

$$F_m \leqslant F \leqslant F_M.$$

The first equality is valid if and only if one vertex of the triangle and one vertex of the square lie on the same semi-line from the centre of the circle, and the second equality is valid if and only if the square and the triangle have a joint side.

Remark. Note that $(F_m - F_M)/F_M \sim 0,0174 < 2 \cdot 10^{-2}$.

2.16. If $L(a, b)$ denotes the perimeter of the ellipse $x^2/a^2 + y^2/b^2 = 1$, and $a \geqslant a' \geqslant b$, then

$$L(a, b)^2 - L(a', b)^2 \geqslant 16(a^2 - a'^2).$$

M. S. Klamkin, 'Problem 5754', Amer. Math. Monthly 78 (1971), 202.

2.17. Being given the ellipse (E): $x^2/a^2 + y^2/b^2 = 1$, $a, b > 0$ with the property that $\pi ab = 2$ and n $(n \geqslant 1)$ different points $M_i(x_i, y_i)$, $i = 1, \ldots, n$, of the boundary or interior, but different from the vertices of (E). Then

$$\left(\sum_{i=1}^{n} |x_i|^{-1}\right)\left(\sum_{i=1}^{n} |y_i|^{-1}\right) \geqslant \pi n^2. \tag{1}$$

Equality in (1) occurs for n = 1, 2, 3 or 4 if and only if the n points belong to the set

$$\{M_1(-\frac{\sqrt{2}}{2}a, -\frac{\sqrt{2}}{2}b), M_2(-\frac{\sqrt{2}}{2}a, \frac{\sqrt{2}}{2}b), M_3(\frac{\sqrt{2}}{2}a, -\frac{\sqrt{2}}{2}b),$$

$$M_4(\frac{\sqrt{2}}{2}a, \frac{\sqrt{2}}{2}b)\}.$$

For $n \geqslant 5$, there never occurs equality in (1).
This is a result of G. Mircea.

2.18. Let triangle ABC be inscribed in an ellipse. Erect the normals from the centre of circumcircle to the sides of triangle. Let the normal to the side BC cut this side in A_1 and the ellipse in A_2 and let $A_1A_2 = t_1$. Similarly, we define t_2, t_3. Then

$$\Sigma t_1 \geqslant 3r.$$

V. Gh. Voda, 'Problem O:27', Gaz. Mat. (Bucharest) **84** (1979), 168 and 486-487.

2.19. Let triangle ABC be inscribed in an ellipse (E), and let $A_1 \in BC$, $B_1 \in CA$, $C_1 \in AB$. If $A_2 = AA_1 \cap (E)$, $B_2 = BB_1 \cap (E)$, $C_2 = CC_1 \cap (E)$, then

$$\Sigma \frac{|AA_1|}{|A_1A_2|} \geqslant 9.$$

V. Marinescu, 'O extindere a problemei 12357', Gaz. Mat. (Bucharest) **85** (1980), 250-253.

2.20. By a Jordan diamond ABCD we shall mean a configuration of two orthogonally intersecting line segments AC and BD, together with a rectifiable Jordan curve ABCD on which the endpoints of these segments lie. Then

$$(1) \qquad 4(AC^2 + BD^2) \leqslant L^2,$$

where L is the length of the Jordan curve ABCD.

Remark. The proof of (1) follows as an immediate consequence of the first part of the following quadrilateral inequality

$$2s \geqslant 2(p^2 + q^2)^{1/2} \geqslant (p^2 + q^2 - 2pq\cos\Theta)^{1/2} +$$

$$+ (p^2 + q^2 + 2pq\cos\Theta)^{1/2} \geqslant p + q \geqslant s,$$

where s, p, q denote the semi-perimeter and lengths of the diagonals, respectively, of a plane convex quadrilateral, and Θ is the angle between the diagonals.

M. S. Klamkin and E. C. Schlesinger, 'Diamond Inequalities', Math. Mag. 50 (1977), 96-98.

2.21. Let a_1, b_1 and a_2, b_2 be the half-axes ($a_1 > a_2$) of two confocal ellipses and s the length of the segment between these ellipses on any half-line from a focus. Then

$$s \leqslant \frac{(b_1 - b_2)^2}{a_1 - a_2}.$$

'Aufgabe 2', Elem. Math. 21 (1966), 42.

2.22. Let an arc p of any conic section with chord AB be given, and let C be the point on p where the tangent line is parallel to AB. If T is the area of triangle ABC and S is the area of the segment bounded by the arc p and the chord AB, then

$T/S < 3/4$ if p is on an ellipse,

$T/S = 3/4$ if p is on a parabola,

$T/S > 3/4$ if p is on a branch of a hyperbola.

Remark. The equality case is given by Archimedes. The above generalization of this classic result was given by M. Golomb and H. Haruki. The new proofs were given by H. Haruki and O. Bottema.

M. Golomb and H. Haruki, 'An Inequality for Elliptic and Hyperbolic Segments', Math. Mag. 46 (1973), 152-155.

H. Haruki, 'An Application of Conformal Mapping to an Inequality for Elliptic and Hyperbolic Segments', Mathematicae Notae 27 (1979-1980), 15-22.

O. Bottema, 'Archimedes Revisited', Math. Mag. 57 (1984), 224-225.

2.23. Let r be the radius of the incircle of an arbitrary triangle lying in a closed convex figure F of width w, and let R be the radius of the incircle of F. Then

(1) $1/4 \leqslant (\sup r)/w \leqslant 1/2$.

(2) Conjecture: $1/2 \leqslant \sup r/R \leqslant 1$.

L. Funar, 'Problem 6478', Amer. Math. Monthly 91 (1984), 588.

2.24. Let p, h_a, t, d, T, r and q be the perimeter, altitude to side a, thickness (minimal altitude), diameter (maximal side), area, inradius, and the perimeter of the orthic (pedal) triangle, respectively, of a triangle ABC. The following results are valid:

(1) Let ABC be a triangle circumscribed about a closed curve Γ of length L, and suppose the notation is arranged so that A is a maximal angle. Then

$$p \leqslant \frac{\Sigma \sin A}{\Pi\, \text{Sin } A}\,,\,, \quad t \leqslant \frac{L}{2}\frac{1}{\text{Sin } A}\,, \quad d \leqslant \frac{L}{2}\,\frac{1}{\text{Sin } A}\,\frac{\sin A}{\sin B \sin C}\,.$$

$$T \leqslant \frac{L^2}{8}\,\frac{1}{\text{Sin}^2 A}\,\frac{\sin A}{\sin B \sin C}\,,$$

where

$$\text{Sin } x = \begin{cases} \sin x, & \text{for } 0 \leqslant x \leqslant \pi/2 \\ 1, & \text{for } \pi/2 \leqslant x \leqslant \pi. \end{cases}$$

(2) Let Γ be a closed curve of length L and A'B'C' a given triangle with angles A, B and C, the notation being arranged so that A is a maximal angle. Then there exists a triangle ABC similar to A'B'C' circumscribed about Γ so that all the following inequalities hold:

$$p \leqslant \frac{L}{2\pi}\,\frac{(\Sigma \sin A)^2}{\Pi \sin A}\,, \quad t \leqslant \frac{L}{2\pi}\,\frac{\Sigma \sin A}{\sin A}\,,$$

$$d \leqslant \frac{L}{2\pi}\,\frac{\Sigma \sin A}{\sin B \sin C}\,, \quad T \leqslant \frac{L^2}{8\pi^2}\,\frac{(\Sigma \sin A)^2}{\Pi \sin A}\,.$$

J. E. Wetzel, 'Triangular Covers for Closed Curves of Constant Length', Elem. Math. 25 (1970), 78-82.

2.25. Let P be the perimeter of a convex quadrilateral of diameter d. Then

$$P \leqslant (2 + \sqrt{6} - \sqrt{2})d$$

with equality if and only if the given quadrilateral is of the form ABCD with AB = AC = AD = BD = d, BC = CD = $\frac{1}{2}(\sqrt{6} - \sqrt{2})d$.

S. J. Taylor, 'Some Simple Geometrical Extremal Problems', Math. Gaz. 37 (1953), 188-198.

2.26. Let $3d \leqslant L \leqslant (2 + \sqrt{6} - \sqrt{2})d$. There is one and only one (modulo an isometry) convex quadrilateral Q_o = ABCD of perimeter L such that AB = AC = AD = BD = d. Let F_o be the area of Q_o and F the area of any convex quadrilateral Q at perimeter L and the diameter not greater than d. Then

$$F \geqslant F_o$$

with equality if and only if $Q = Q_o$.

S. J. Taylor, 'Some Simple Geometrical Extremal Problems', Gaz. Math. 37 (1953), 188-198.

2.27. On the same line two lengths OA = a and AB = b are given (A between

O and B) and p is a line through O with $\measuredangle (p, AB) = \alpha$. For any point P on p it holds

$$\measuredangle APB \leqslant \arctan \frac{b \sin \alpha}{2\sqrt{a^2 + ab} - (2a + b) \cos \alpha}.$$

Equality holds if and only if P is the point of contact of p with a circle through A and B.

F. Enriques and A. Nordio, 'Questione 93', Period. Mat. (4) 5 (1925), 46, 123-127.

Kohrs and Jaquet, 'Bemerkungen zur Hochwertaufgabe von Regiomontanus', Zeitschr. math. naturwiss. Unterr. 68 (1937), 53-54.

C. J. Coe, 'Problems on Maxima and Minima', Amer. Math. Monthly 49 (1942), 33-37.

B. Brady and P. D. Thomas, 'Problem 363', Math. Mag. 33 (1959-1960), 53-54.

Remark. For $\alpha = \pi/2$ we have a problem of Regiomontanus:

Niebel, 'Zur Behandlung von Hoch- und Tiefwertaufgaben', Zeitschr. math. naturwiss. Unterr. 67 (1936), 191-192.

R. Baldus, 'Elementare Lösung einiger einfacher Fragen über Maxima und Minima', Ibid. 71 (1940), 30-36, 51-60.

H. Rauter, 'Zur Extremwertaufgabe Regiomontans', Ibid. 72 (1941), 162-164.

J. H. Butchart and L. Moser, 'No calculus, please', Scripta Math. 18 (1952), 221-236.

J. A. Tierney, 'Elementary Techniques in Maxima and Minima', Math. Teacher 46 (1953), 484-486.

C. S. Ogilvy, 'Problem E 1128', Amer. Math. Monthly 62 (1955), 184.

B. Djerasimović, 'O elementarnim metodama odredivanja ekstremnih vrednosti funkcija', Nastava mat. fiz. 6 (1957), 275-280.

2.28. Two corridors of widths a and b form a right angle in a horizontal plane. Let L be the length of a horizontal ladder, which can be carried from one corridor to the second one. Then

$$L \leqslant (a^{2/3} + b^{2/3})^{3/2}.$$

C. J. Coe, 'Problems on Maxima and Minima', Amer. Math. Monthly 49 (1942), 33-37.

T. J. Fletcher, 'Easy Ways of Going Round the Bend', Math. Gaz. 57 (1973), 16-22.

2.29. On the same line two lengths AO = a and OB = b are given (O between A and B). If $c < \min (a, b)$, then for any point P with $OP \leqslant c$ it holds

$$\frac{a^2}{AP} + \frac{b^2}{BP} \geqslant \frac{(a + b)\sqrt{ab}}{\sqrt{ab + c^2}}$$

with equality if and only if P is a point such that OP = c and AP : BP = a : b.

B. B. Misra and I. A. Dodes, 'Problem E 1018', Amer. Math. Monthly 60 (1953), 116-117.

2.30. At the ends A and B of the line segment AB = d the normals a and b are erected and C is a point on a such that AC = c. For any point X of the segment AB the line CX intersects b at the point D. Then

$$\text{area }(ACX) + \text{area }(BDX) \leqslant cd(\sqrt{2} - 1)$$

with equality if and only if $AX = d\sqrt{2}/2$.

A. Agostini and G. Gobesso, 'Questione 84', Period. Mat. (4) 4 (1924), 257, 453-455.

2.31. Three segments AB = a, BC = b and CD = c are given successively on a line. Let P be any point such that $\measuredangle APB = \measuredangle CPD = \phi$. Then

$$\tan \phi \leqslant \sqrt{\frac{ac}{(a+b+c)b}}\,.$$

Equality holds if and only if P has the coordinates

$$x = \frac{a(a+b+c)}{a+c}, \quad y = \frac{\sqrt{abc(a+b+c)}}{a+c}$$

with respect to the Cartesian coordinate system with the origin A and the half-line AB as the positive x-half-axis. In this case, the circles PAB and PCD have P as the point of contact.

Beilis, 'Aufgabe 652', Zeitschr. math. naturwiss. Unterr. 51 (1920), 277-278.

2.32. Let R be a regular n-gon with the incircle k and the circumcircle K. Let s be an n-gon circumscribed about k and S an n-gon inscribed in K. If s* is the intersection $s \cap K$ and S* the convex closure of $s \cup k$, then

$$\text{area }(s^*) \geqslant \text{area }(R),$$

$$\text{per }(S^*) \leqslant \text{per }(R).$$

L. Fejes-Tóth, 'Körbe és kör köré írt sokszögekről', Mat. Lapok 10 (1959), 23-25.

2.33. Let r be the inradius of a triangle and r_1, r_2, r_3 the radii of its three Malfatti circles (inscribed circles which touch two sides and two other circles). Then

$$(3 + \sqrt{3})\Sigma \frac{1}{r_1} \leqslant r \leqslant \frac{3 + \sqrt{3}}{9} \Sigma r_1.$$

J. Garfunkel, 'Problem 1067', Crux Math. 11 (1985), 221.

W. Janous (private communication).

2.34. Let $n \in N$. There shall be given n segments, each of length $\geqslant 1$ and $< \frac{1}{2^n\sqrt{5}}((1+\sqrt{5})^n - (1-\sqrt{5})^n)$. Then at least one triangle may be formed with three of these segments.

This result is due to W. Janous.

2.35. Let T be a given triangle ABC, and let P be any point in the plane of T. It is known that there exists a triangle T_o, possibly degenerate, with sides a • PA, b • PB, and c • PC (Möbius-Neuberg's theorem). Let R_o be the circumradius of T_o. The locus of all the points P for which

(1) $$PA \cdot PB \cdot PC \leqslant RR_o$$

consists of the closed circular disk δ with centre O and radius $R\sqrt{2}$. Equality holds in (1) when P = O and when OP = $R\sqrt{2}$, and the inequality is reversed when P lies outside δ.

G. Tsintsifas, 'Problem 872', Crux Math. 9 (1983), 24 and 10 (1984), 18 and 10 (1984), 334-335.

2.36. The straight line L splits a convex figure into two parts the areas of which are in ratio 1 : t ($t \geqslant 1$). The parts are projected perpendicularily on a line n orthogonal to L. The maximal ratio of the lengths of the two projections is $\sqrt{t}(\sqrt{t} + \sqrt{t+1})$.

W. Janous, 'Problem 1145', Crux Math. 12 (1986), 107.

2.37. To every closed convex curve there exists an inscribed n-gon of maximal area t_n and a circumscribed n-gon of minimal area T_n so that

$$\frac{T_n - t_n}{T_n} \leqslant \sin^2 \frac{\pi}{n} .$$

D. Lazár, 'Sur l'approximation des courbes convexes par des polygones', Acta Univ. Szeged. Sect. Sci. Math. 11 (1947), 129-132.

Comment by L. Fejes-Tóth. This inequality is very important. The following analogues are also valid:

There exists an inscribed n-gon of maximal perimeter p_n and a circumscribed n-gon of minimal perimeter P_n so that

(a) $$\frac{P_n - p_n}{P_n} \leqslant 2 \sin^2 \frac{\pi}{2n}.$$

L. Fejes-Tóth, 'Über die Approximation konvexer Kurven durch Polygonfolgen', Compositio Math. 6 (1939), 456-467.

(b) $$t_n \geqslant \frac{n}{2\pi} \sin \frac{2\pi}{n} T,$$

where T is the area of the given curve.

E. Sas, 'Über eine Extremaleigenschaft der Ellipsen', Compositio Math. 6 (1939), 468-471.

(c) $$P_n \leqslant \frac{n}{\pi} \tan \frac{\pi}{n} P, \quad p_n \geqslant \frac{n}{\pi} \sin \frac{\pi}{n} P,$$

where P is the perimeter of the given curve.

R. Schneider, 'Zwei Extremalaufgaben für konvexe Bereiche', Acta Math. Sci. Hungar. 22 (1971), 379-383.

2.38. Let N denote the set of the first n natural numbers. For k = 1, ..., n - 1, let S be a subset $(s_1, s_2, \ldots, s_k)$ of N such that $1 \leqslant s_1 <$

$\ldots < s_k \leqslant n$. We call S a k-shuffle of N. The $(n - k)$-shuffle $(t_1, \ldots, t_{n-k})$ is called the complementary shuffle of S with respect to N and is called the complementary shuffle of S with respect to N and is denoted by $\bar{S}$ if $S \cup \bar{S} = N$. If $|P_{s_j} P_{t_i}|$ denotes the distance between points P_{s_j} and P_{t_i}, define

$$d(S) = \prod_{j=1}^{k} \prod_{i=1}^{n-k} |P_{s_j} P_{t_i}|.$$

Then the following results are valid:

(1) Let $P_1, \ldots, P_n$ be n distinct points in the Euclidean plane and let P be some point in the same plane. Then, for $k = 1, \ldots, n - 1$, we have

$$\sum_S \frac{(|PP_{t_1}| \ldots |PP_{t_{n-k}}|)^k}{d(S)} \geqslant 1,$$

where the sum is extended over all k-shuffles S.

(2) Let $P_1, \ldots, P_n$ denote n distinct points on a circle of unit radius. Then

$$\sum_S 1/d(S) \geqslant 1.$$

Equality is attained if and only if the points form a regular n-gon.

G. Z. Chang, 'Planar Metric Inequalities Derived from the Vandermonde Determinant', Amer. Math. Monthly 92 (1985), 495-499.

2.39. Conjecture: let O_1, O_2, O_3 be the centres of three Malfatti circles of the triangle $M_1M_2M_3$. Circle O is externally tangent to these three circles and the sides of triangle $G_1G_2G_3$ are each tangent to O and one of the smaller circles. Then

$$P(\Delta G_1G_2G_3) \geqslant P(\Delta M_1M_2M_3) + P(\Delta O_1O_2O_3),$$

where P stands for perimeter. Equality is attained when $\Delta O_1O_2O_3$ is equilateral.

J. Garfunkel, 'Problem 1150*', Crux Math. 12 (1986), 108.

2.40. At any point $P(a \cos \Theta, b \sin \Theta)$ of an ellipse of semi-axes a and b $(a > b)$, draw a normal line and let Q be the other meeting point. Then

$$\sqrt{27}a^2b^2/(\sqrt{a^2 + b^2})^3 \leqslant PQ \leqslant 2a \quad \text{in case of } a \geqslant \sqrt{2}\, b.$$

Equalities hold when $\sin \Theta = b\sqrt{2a^2 - b^2}/\sqrt{a^4 + b^4}$ and $\Theta = 180°$. If $a < \sqrt{2}b$, then $2b \leqslant PQ \leqslant 2a$.

This result is due to H. Fukagawa.

2.41. Let $P(a\cos\Theta, b\sin\Theta)$ be any point of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, and let $A(a, 0)$ and $B(0, b)$ be the ends of semi-axes. In case of $0 \leqslant \Theta < 90°$, the angle APB is maximum if $\Theta = 45°$. In case of $90° < \Theta \leqslant 360°$, the angle APB is maximum if $\Theta = 315°$.

This result is due to H. Fukagawa and M. Kinosita.

2.42. J. Aczél, W. Gilbert and C. T. Ng established the following result: Given two lines L_1 and L_2 intersecting at A, a point D on the angle-bisector of A and the line p through D perpendicular to AD intersecting L_1, L_2 in B, C, respectively. Let q be any line through D intersecting L_1, L_2 in E, F, respectively. Then if E'F' is the orthogonal projection of EF onto p,

$$\overline{E'F'} \geqslant \overline{BC}.$$

An interesting generalization in form of a functional inequality is given by M. S. Klamkin and A. Meir. They also considered some similar functional inequalities and some extensions to higher dimensions.

J. Aczél, W. Gilbert, and C. T. Ng, 'An Elementary Solution to Question 3 of the Twelfth Canadian Mathematics Olympiad', Can. Math. Soc. Notes 13 (1981), 16-17.

M. S. Klamkin and A. Meir, 'Extensions of an Elementary Geometric Inequality', Aequations Mathematicae 26 (1983), 197-207.

2.43. M. Lascu communicated to us several very interesting inequalities involving symmedians s_a, s_b, s_c:

(1) $\Sigma m_a + 2\min(a, b, c) \leqslant \Sigma s_a + 2\max(a, b, c)$, {E}

(2) $w_a \leqslant 2(m_a^{-1} + s_a^{-1})^{-1}$, {E}

(3) $3 \leqslant \Sigma \frac{s_a}{h_a} \leqslant \frac{9\sqrt{3}}{2}\frac{R}{s}$, {E}

(4) $\Sigma w_a \geqslant \Sigma s_a \geqslant \Sigma g_a$, {E}

(5) $\Sigma a^2 s_b s_c \leqslant \frac{1}{4}(\Sigma a^2)^2$, {E}

(6) $\Sigma s_a \leqslant 3(R + r)$, {E}

(7) $\Sigma m_a s_a \leqslant 4s^2$, (Δ_a)

(8) If A', B', C' denote the second points of intersection of the symmedians and the circumcircle of the triangle ABC, then

$$\text{(i)} \quad 6R \geqslant \sum AA' \geqslant \frac{8\sqrt{3}}{3}\frac{F}{R}, \qquad \text{(ii)} \quad \sum \frac{1}{AA' \cdot BB'} \geqslant \frac{8\sqrt{3}}{3}\frac{F}{R}.$$

2.44. Let UV = 2a be any axis of an ellipse, A, B two given different points on the ellipse, A', B' the points symmetric to A, B with respect to the line UV and T any point of the ellipse different from A, B, A', B'. If M, N are the intersection points of the line UV with the axes of symmetry of segments $\overline{AT}$, $\overline{BT}$, then MN = const. and

$$0 \leqslant MN \leqslant |a - r|,$$

where r is the radius of curvature in the vertices U and V. The first equality holds if and only if B = A', and the second one if and only if {A, B} = {U, V}.

E. Kraemer and A. Toufar, 'Problem 5', Rozhledy Mat.-Fyz. 58 (1979-1980), 35-37.

2.45. Let C be a circle about O with radius R and let F be an ellipse contained in C with semi-minor axis b and foci F_1, F_2. Set $d_1 = \overline{OF}_1$, $d_2 = \overline{OF}_2$.

(1) There exists a triangle inscribed in C and circumscribed about F, if and only if

$$(R^2 - d_1^2)(R^2 - d_2^2) = 4b^2R^2.$$

(2) A necessary and sufficient condition for the existence of a triangle which includes F and is included in C is

$$(R^2 - d_1^2)(R^2 - d_2^2) \geqslant 4b^2R^2.$$

M. Goldberg and G. Zwas, 'On Inscribed Circumscribed Conics', Elem. Math. 31 (1976), 36-38.

Chapter XVIII

INEQUALITIES FOR SIMPLEXES IN E^n $(n \geqslant 2)$

1. Inequalities for r, ρ_i, h_i, F_i $(i = 1, 2, \ldots, n + 1)$ in a Simplex

Let V be the volume and r inradius of a given n-simplex $A = A_1A_2 \ldots A_{n+1}$ in n-dimensional Euclidean space E^n $(n \geqslant 2)$. For any $i \in \{1, 2, \ldots, n\}$ let F_i be the (n-1)-dimensional content of (n-1)-simplex $A_i = A_1 \ldots A_{i-1}A_{i+1} \ldots A_{n+1}$, let $h_i = A_iA_i'$ be the altitude of A from vertex A_i, i.e. the distance from A_i to the hyperplane $a_i = A_1 \ldots A_{i-1}A_{i+1} \ldots A_{n+1}$, and let ρ_i be the radius of i-th escribed hypersphere of A. Let

$$F = \sum_{i=1}^{n+1} F_i \tag{1}$$

be the 'total area' of A. For every $i \in \{1, 2, \ldots, n + 1\}$ we have

$$h_i = \frac{nV}{F_i} \quad (i = 1, 2, \ldots, n + 1). \tag{2}$$

Equalities

$$nV = \sum_{i=1}^{n+1} F_i r; \qquad nV = \sum_{\substack{j=1\\ j\neq i}}^{n+1} F_j\rho_i - F_i\rho_i \quad (i = 1, 2, \ldots, n + 1)$$

imply by (1)

$$r = \frac{nV}{F}, \tag{3}$$

$$\rho_i = \frac{nV}{F - 2F_i} \quad (i = 1, 2, \ldots, n + 1). \tag{4}$$

From (1), (2) and (3), and from (1), (3) and (4) respectively, we obtain

$$H_{n+1}(h_i) = (n + 1)r, \tag{5}$$

$$H_{n+1}(\rho_i) = \frac{n + 1}{n - 1} r. \tag{6}$$

In every inequality in this section the equality sign appears iff A is an equifacial simplex, i.e.iff $F_1 = F_2 = \dots = F_{n+1}$ or equivalently $h_1 = h_2 = \dots = h_{n+1}$.

1.1. If $m \in (-2, 0) \cup R^+$, then

$$M_{n+1}^{[k]}(F + mF_i) \geq \frac{n+m+1}{n+1} F \qquad (k \in (1, +\infty])$$

and for every $k \in [-\infty, 1)$ we have the opposite inequality.

Proof. For every $i \in \{1, 2, \dots, n+1\}$ we have $F > 2F_i$ and so $F + mF_i > 0$. Now,

$$M_{n+1}^{[k]}(F + mF_i) \gtreqless A_{n+1}(F + mF_i) = \frac{1}{n+1} \sum_{i=1}^{n+1} (F + mF_i) =$$

$$= \frac{n+m+1}{n+1} F.$$

GI 1.15, 1.20 ($n = 2$, $m = -2$, $k \in \{-1, \frac{1}{2}\}$).

1.2. $$M_{n+1}^{[k]}\left(\frac{F_i}{F + mF_i}\right) \leq \frac{1}{n+m+1} \qquad (k \in [-\infty, 1],\ m \in R^+).$$

The opposite inequality is valid for $k \in [1, +\infty]$, $m \in [-2, 0)$.

Proof. By 1.1 for every $m \in [-2, +\infty)$

$$H_{n+1}(F + mF_i) \lesseqgtr \frac{n+m+1}{n+1} F.$$

Therefore, we have in our two cases

$$M_{n+1}^{[k]}\left(\frac{F_i}{F + mF_i}\right) \lesseqgtr A_{n+1}\left(\frac{F_i}{F + mF_i}\right) = \frac{1}{m} - \frac{F}{m} \cdot A_{n+1}\left(\frac{1}{F + mF_i}\right) =$$

$$= \frac{1}{m} - \frac{F}{m} \cdot \frac{1}{H_{n+1}(F + mF_i)} \lesseqgtr \frac{1}{m} - \frac{F}{m} \cdot \frac{n+1}{(n+m+1)F} =$$

$$= \frac{1}{n+m+1}.$$

GI 1.16 ($n = 2$, $m = -1$, $k = 1$).

Remark. Inequality 1.2 (with the sign $\leq$) is valid also if $k \in [-\infty, -1]$, $m \in [-2, 0)$. Indeed, from $-k \geq 1$ we obtain

$$M_{n+1}^{[k]}\left(\frac{F_i}{F + mF_i}\right) = \frac{1}{M_{n+1}^{[-k]}\left(\frac{F + mF_i}{F_i}\right)} \leq \frac{1}{A_{n+1}\left(\frac{F + mF_i}{F_i}\right)} =$$

$$= \frac{1}{m + F \cdot A_{n+1}(\frac{1}{F_i})} \leqslant \frac{1}{m + F \cdot H_{n+1}(\frac{1}{F_i})} = \frac{1}{n + m + 1} .$$

1.3. $$M_{n+1}^{[k]}\left(\frac{F_i}{F - F_i}\right) < \frac{2}{n + 1} \quad (k \in [-\infty, 1]).$$

Proof. For every $i \in \{1, 2, \ldots, n + 1\}$ we have $F > 2F_i$, from which it follows

$$\frac{F_i}{F - F_i} < \frac{2F_i}{F} .$$

Therefore,

$$M_{n+1}^{[k]}\left(\frac{F_i}{F - F_i}\right) \leqslant A_{n+1}\left(\frac{F_i}{F - F_i}\right) < A_{n+1}\left(\frac{2F_i}{F}\right) = \frac{2}{n + 1} .$$

GI 1.16 ($n = 2$, $k = 1$).

1.4. $$M_{n+1}^{[k]}\left(\frac{F + pF_i}{F + mF_i}\right) \leqslant \frac{n + p + 1}{n + m + 1} \quad (k \in [-\infty, 1];\ 0 < m < p \text{ or } -2 \leqslant p < m < 0).$$

The opposite inequality is valid if $k \in [1, +\infty]$; $-2 \leqslant p < m$, $m > 0$ or $-2 \leqslant m < p$, $m < 0$.

Proof. If $k \in [-\infty, 1]$ respectively $k \in [1, +\infty]$, then

$$M_{n+1}^{[k]}\left(\frac{F + pF_i}{F + mF_i}\right) \lesseqgtr A_{n+1}\left(\frac{F + pF_i}{F + mF_i}\right).$$

On the other hand, by 1.2 for $m > 0$ respectively $m < 0$ we have

$$A_{n+1}\left(\frac{F_i}{F + mF_i}\right) \lesseqgtr \frac{1}{n + m + 1} .$$

Therefore, for $0 < m < p$ or $p < m < 0$ respectively for $m > 0$, $m > p$ or $m < 0$, $m < p$ we get

$$(p - m) A_{n+1}\left(\frac{F_i}{F + mF_i}\right) \lesseqgtr \frac{p - m}{n + m + 1} .$$

Now, 1.4 is a consequence of the equality

(7) $$A_{n+1}\left(\frac{F + pF_i}{F + mF_i}\right) = 1 + (p - m) A_{n+1}\left(\frac{F_i}{F + mF_i}\right).$$

GI 1.17 ($n = 2$, $k = -1$, $m = -1$, $p = 2$).

1.5. $$M_{n+1}^{[k]}\left(\frac{F + pF_i}{F - F_i}\right) < \frac{n + 2p + 3}{n + 1} \quad (k \in [-\infty, 1],\ p > -1).$$

The opposite inequality is valid if $k \in [1, +\infty]$, $p < -1$.

Proof. For $k \in [-\infty, 1]$ respectively $k \in [1, +\infty]$ we have

$$M_{n+1}^{[k]}\left(\frac{F + pF_i}{F - F_i}\right) \lesseqgtr A_{n+1}\left(\frac{F + pF_i}{F - F_i}\right)$$

and by 1.3

$$A_{n+1}\left(\frac{F}{F - F_i}\right) < \frac{2}{n + 1}.$$

Therefore, for $p > -1$ respectively $p < -1$ it follows

$$(p + 1)A_{n+1}\left(\frac{F}{F - F_i}\right) \lessgtr \frac{2p + 2}{n + 1}.$$

Now, inequality 1.5 is a consequence of the equality (7) (with $m = -1$).

GI 1.17 ($n = 2$, $k = 1$, $p = 2$).

1.6. $$M_{n+1}^{[k]}(\rho_i) \gtreqless \frac{n + 1}{n - 1} r \quad (k \in (-1, +\infty] \text{ respectively } k \in [-\infty, -1)).$$

Proof. Follows from (6).

M. Stanković [106] ($k = 0$); V. Thébault [119] ($n = 3$, $k = 0$); F. Leuenberger [67] ($n = 3$, $k = 1$); G. Kalajdžić [43] ($n = 3$, $k = 1$); S. Horák [40] ($n = 3$, $k = 1$); GI 5.41 ($n = 2$, $k = 1$).

1.7. $$M_{n+1}^{[k]}\left(\frac{F_i}{\rho_i}\right) \leqslant \frac{n - 1}{(n + 1)^2} \cdot \frac{F}{r} \quad (k \in [-\infty, 1]).$$

Proof. First we have

(8) $$\sum_{i=1}^{n+1} F_i^2 \geqslant \frac{1}{n + 1}\left(\sum_{i=1}^{n+1} F_i\right)^2 = \frac{F^2}{n + 1},$$

(9) $$\sum_{i=1}^{n+1} F_i(F - 2F_i) = F^2 - 2\sum_{i=1}^{n+1} F_i^2 \leqslant$$

$$\leqslant F^2 - 2 \cdot \frac{F^2}{n + 1} = \frac{n - 1}{n + 1} F^2.$$

By (4), (9) and (3) we obtain successively

$$M_{n+1}^{[k]}\left(\frac{F_i}{\rho_i}\right) \leqslant A_{n+1}\left(\frac{F_i(F - 2F_i)}{nV}\right) = \frac{1}{(n+1)nV}\sum_{i=1}^{n+1} F_i(F - 2F_i) \leqslant$$

$$\leqslant \frac{(n-1)F^2}{(n+1)^2 nV} = \frac{n-1}{(n+1)^2}\cdot\frac{F}{r}.$$

Remark. For $n = 2$ the right side of 1.7 is $\frac{1}{9}\cdot\frac{2s}{r}$, which is a better result than $\frac{1}{8}\cdot\frac{2s}{r}$ in GI 5.47 (for $k = \frac{1}{2}$).

1.8. $$M_{n+1}^{[k]}\left(\frac{\rho_i - r}{r}\right) \leqslant \frac{2}{n-1} \quad (k \in [-\infty, 1]).$$

Proof. Follows from 1.2, because by (3) and (4) we have equalities

$$\frac{\rho_i - r}{r} = \frac{2F_i}{F - 2F_i} \quad (i = 1, 2, \ldots, n + 1).$$

M. Stanković [105] ($n = 3$, $k = -1$).

1.9. $$M_{n+1}^{[k]}\left(\frac{\rho_i - r}{\rho_i + r}\right) \gtreqless \frac{1}{n} \quad (k \in [1, +\infty] \text{ respectively } k \in [-\infty, -1]).$$

Proof. Follows from 1.2 (see also Remark), because by (3) and (4)

(10) $$\frac{\rho_i - r}{\rho_i + r} = \frac{F_i}{F - F_i} \quad (i = 1, 2, \ldots, n + 1).$$

G. Păun [77] ($n = 3$, $k = -1$).

1.10. $$M_{n+1}^{[k]}\left(\frac{\rho_i - r}{\rho_i + r}\right) < \frac{2}{n+1} \quad (k \in [-\infty, 1]).$$

Proof. Follows from 1.3 according to (10).
Remark. Inequality 1.10 is interesting only for $k \in (-1, 1]$, because for $k \in [-\infty, -1]$ the inequality 1.9 is better.

1.11. $$M_{n+1}^{[k]}(h_i) \gtreqless (n + 1)r \quad (k \in (-1, +\infty] \text{ respectively } k \in [-\infty, -1)).$$

Proof. Follows immediately from (5).
V. Volenec [139]; [151] ($n = 3$, $k = 4$); [156] ($n = 3$, $k = 1$); GI 6.8, 6.10 ($n = 2$, $k = 1$); GI 6.10, 6.16 ($n = 2$, $k = 3$).

1.12. $$M_{n+1}^{[k]}\left(\frac{1}{h_i - 2r}\right) \gtreqless \frac{1}{(n-1)r} \quad (k \in [1, +\infty] \text{ respectively } k \in [-\infty, -1]).$$

Proof. Follows from 1.2, because by (2) and (3) we have

$$\frac{1}{h_i - 2r} = \frac{1}{r} \cdot \frac{F_i}{F - 2F_i} \quad (i = 1, 2, \ldots, n + 1).$$

GI 6.21 (n = 2, k = 1).

1.13. $$M_{n+1}^{[k]}\left(\frac{h_i + r}{h_i - r}\right) \gtrless \frac{n + 2}{n} \quad (k \in [1, +\infty] \text{ respectively } k \in [-\infty, -1]),$$

Proof. By (2) and (3) we have

$$(11) \qquad \frac{h_i + r}{h_i - r} = \frac{F + F_i}{F - F_i} \quad (i = 1, 2, \ldots, n + 1).$$

Therefore, for $k \in [1, +\infty]$ inequality 1.13 follows by 1.4 with $p = 1$, $m = -1$. On the other hand, because of equalities

$$\frac{h_i - r}{h_i + r} = \frac{F - F_i}{F + F_i} \quad (i = 1, 2, \ldots, n + 1)$$

by 1.4 (with $p = -1$, $m = 1$) it follows for $k \in [-\infty, -1]$, i.e. for $-k \in [1, +\infty]$

$$M_{n+1}^{[-k]}\left(\frac{h_i - r}{h_i + r}\right) \geq \frac{n}{n + 2} .$$

Therefore,

$$M_{n+1}^{[k]}\left(\frac{h_i + r}{h_i - r}\right) = \left[M_{n+1}^{[-k]}\left(\frac{h_i - r}{h_i + r}\right)\right]^{-1} \leq \frac{n + 2}{n} .$$

GI 6.22 (n = 2, k = 1).

1.14. $$M_{n+1}^{[k]}\left(\frac{h_i + r}{h_i - r}\right) < \frac{n + 5}{n + 1} \quad (k \in [-\infty, 1]).$$

Proof. Follows from 1.5 by (11).

Remark. Inequality 1.14 is interesting only for $k \in (-1, 1]$, because for $k \in [-\infty, -1]$ it is weaker than 1.13.

1.15. $$M_{n+1}^{[k]}\left(\frac{\rho_i}{h_i}\right) \gtrless \frac{1}{n - 1} \quad (k \in [1, +\infty] \text{ respectively } k \in [-\infty, -1]).$$

Proof. Follows from 1.2, because by (2) and (4) we have

$$\frac{\rho_i}{h_i} = \frac{F_i}{F - 2F_i} \quad (i = 1, 2, \ldots, n + 1).$$

V. Volenec [138] ($k = -1$); G. Kalajdžić [43] ($n = 3$, $k = -1$); S. Horák [37] ($n = 3$, $k = -1$).

1.16. $$M_{n+1}^{[k]}\left(\frac{1}{\rho_i h_i}\right) \leqslant \frac{n - 1}{(n + 1)^2} \cdot \frac{1}{r^2} \quad (k \in [-\infty, 1]).$$

Proof. By (2), (3), (4) and (9) it follows successively

$$M_{n+1}^{[k]}\left(\frac{1}{\rho_i h_i}\right) \leqslant A_{n+1}\left(\frac{1}{\rho_i h_i}\right) = \frac{1}{n + 1} \sum_{i=1}^{n+1} \frac{F_i(F - 2F_i)}{r^2F^2} \leqslant$$

$$\leqslant \frac{1}{(n + 1)r^2F^2} \cdot \frac{n - 1}{n + 1} F^2 = \frac{n - 1}{(n + 1)^2} \cdot \frac{1}{r^2} .$$

1.17. $$\frac{n - 3}{n + 1} < A_{n+1}\left(\frac{h_i - \rho_i}{h_i + \rho_i}\right) \leqslant \frac{n - 2}{n} .$$

Proof. Follows from 1.4 and 1.5 ($p = -3$, $m = -1$), because by (2) and (4)

$$\frac{h_i - \rho_i}{h_i + \rho_i} = \frac{F - 3F_i}{F - F_i} \quad (i = 1, 2, \ldots, n + 1).$$

GI 6.24 ($n = 2$).

1.18. $$M_{n+1}^{[k]}\left(\frac{F_i}{\sum\limits_{\substack{j=1\\ j\neq i}}^{n+1} h_j}\right) \leqslant \frac{1}{n(n + 1)^2} \cdot \frac{F}{r} \quad (k \in [-\infty, 1]).$$

Proof. By (2), (8) and (3) we obtain

$$M_{n+1}^{[k]}\left(\frac{F_i}{\sum\limits_{\substack{j=1\\ j\neq i}}^{n+1} h_j}\right) \leqslant A_{n+1}\left(\frac{F_i}{\sum\limits_{\substack{j=1\\ j\neq 1}}^{n+1} h_j}\right) = \frac{1}{n(n + 1)V} \sum_{i=1}^{n+1}\left(\frac{F_i}{\sum\limits_{\substack{j=1\\ j\neq i}}^{n+1} \frac{1}{F_j}}\right) =$$

$$= \frac{1}{n(n + 1)V} \sum_{i=1}^{n+1} \left[\frac{F_i}{n} \cdot\right.$$

$$\left. \cdot\ H_n(F_1, \ldots, F_{i-1}, F_{i+1}, \ldots, F_{n+1})\right] \leqslant$$

$$\leqslant \frac{1}{n^2(n+1)V}\sum_{i=1}^{n+1}[F_i \cdot A_n(F_1, \ldots, F_{i-1}, F_{i+1}, \ldots, F_{n+1})] =$$

$$= \frac{1}{n^3(n+1)V}\sum_{i=1}^{n+1} F_i(F - F_i) = \frac{1}{n^3(n+1)V}(F^2 - \sum_{i=1}^{n+1} F_i^2) \leqslant$$

$$\leqslant \frac{1}{n^3(n+1)V}(F^2 - \frac{F^2}{n+1}) = \frac{1}{n(n+1)^2} \cdot \frac{F}{r}.$$

Symmetric means of $x_1, x_2, \ldots, x_{n+1} \in R^+$ can be defined by

$$S_{n+1}^{[k]}(x_i) = \left[\frac{1}{\binom{n+1}{k}} \sum_{\substack{i_1,\ldots,i_k=1 \\ i_1<\ldots<i_k}}^{n+1} \left(\prod_{j=1}^{k} x_{i_j}\right)\right]^{1/k} \quad (k \in \{1, 2, \ldots, n+1\}),$$

(12) $$S_{n+1}^{[k]}(x_i) = \left[S_{n+1}^{[-k]}(x_i)\right]^{-1} \quad (k \in \{-1, -2, \ldots, -(n+1)\}).$$

Obviously,

$$S_{n+1}^{[1]}(x_i) = A_{n+1}(x_i), \quad S_{n+1}^{[-1]}(x_i) = H_{n+1}(x_i),$$

$$S_{n+1}^{[n+1]}(x_i) = S_{n+1}^{[-n-1]}(x_i) = G_{n+1}(x_i).$$

According to AI 2.15.1 (Theorem 4) it holds:

$$H_{n+1} = S_{n+1}^{[-1]} \leqslant S_{n+1}^{[-2]} \leqslant \ldots \leqslant S_{n+1}^{[-n-1]} = G_{n+1} =$$

$$= S_{n+1}^{[n+1]} \leqslant S_{n+1}^{[n]} \leqslant \ldots \leqslant S_{n+1}^{[1]} = A_{n+1}$$

with equalities iff $x_1 = x_2 = \ldots = x_{n+1}$.

Now, if $H_{n+1}(x_i) \geqslant a$, then

$$S_{n+1}^{[k]}(x_i) \geqslant a \quad (k \in \{\pm 1, \pm 2, \ldots, \pm(n+1)\}).$$

and if $A_{n+1}(x_i) \leqslant a$, then

$$S_{n+1}^{[k]}(x_i) \leqslant a \quad (k \in \{\pm 1, \pm 2, \ldots, \pm(n+1)\}).$$

Applying the results from 1.1 - 1.8, 1.10, 1.11, 1.14, 1.16, 1.18 we obtain the following inequalities 1.19-1.31.

1.19. $S_{n+1}^{[k]}(F + mF_i) \leqslant \frac{n+m+1}{n+1} F \quad (k \in \{-1, \pm 2, \ldots, \pm(n+1)\};\ m \geqslant -2).$

1.20. $S_{n+1}^{[k]}\left(\frac{F_i}{F + mF_i}\right) \leqslant \frac{1}{n+m+1} \quad (k \in \{\pm 1, \pm 2, \ldots, \pm(n+1)\};\ m > 0).$

1.21. $S_{n+1}^{[k]}\left(\frac{F_i}{F - F_i}\right) < \frac{2}{n+1} \quad (k \in \{\pm 1, \pm 2, \ldots, \pm(n+1)\}).$

1.22. $S_{n+1}^{[k]}\left(\frac{F + pF_i}{F + mF_i}\right) \leqslant \frac{n+p+1}{n+m+1} \quad (k \in \{\pm 1, \pm 2, \ldots, \pm(n+1)\};\ 0 < m < p \text{ or } -2 \leqslant p < m < 0).$

1.23. $S_{n+1}^{[k]}\left(\frac{F + pF_i}{F - F_i}\right) < \frac{n+2p+3}{n+1} \quad (k \in \{\pm 1, \pm 2, \ldots, \pm(n+1)\};\ p > -1).$

1.24. $S_{n+1}^{[k]}(\rho_i) \geqslant \frac{n+1}{n-1} r \quad (k \in \{1, \pm 2, \ldots, \pm(n+1)\}).$

1.25. $S_{n+1}^{[k]}\left(\frac{F_i}{\rho_i}\right) \leqslant \frac{n-1}{(n+1)^2} \cdot \frac{F}{r} \quad (k \in \{\pm 1, \pm 2, \ldots, \pm(n+1)\}).$

1.26. $S_{n+1}^{[k]}\left(\frac{\rho_i - r}{r}\right) \leqslant \frac{2}{n-1} \quad (k \in \{\pm 1, \pm 2, \ldots, \pm(n+1)\}).$

1.27. $S_{n+1}^{[k]}\left(\frac{\rho_i - r}{\rho_i + r}\right) < \frac{2}{n+1} \quad (k \in \{1, \pm 2, \ldots, \pm(n+1)\}).$

1.28. $S_{n+1}^{[k]}(h_i) \geqslant (n+1)r \quad (k \in \{1, \pm 2, \ldots, \pm(n+1)\}).$

1.29. $S_{n+1}^{[k]}\left(\frac{h_i + r}{h_i - r}\right) < \frac{n+5}{n+1} \quad (k \in \{1, \pm 2, \ldots, \pm(n+1)\}).$

1.30. $S_{n+1}^{[k]}\left(\frac{1}{\rho_i h_i}\right) \leqslant \frac{n-1}{(n+1)^2} \cdot \frac{1}{r^2} \quad (k \in \{\pm 1, \pm 2, \ldots, \pm(n+1)\}).$

1.31. $S_{n+1}^{[k]}\left(\frac{F_i}{\sum_{\substack{j=1 \\ j \neq i}}^{n+1} h_j}\right) \leqslant \frac{1}{n(n+1)^2} \cdot \frac{F}{r} \quad (k \in \{\pm 1, \pm 2, \ldots, \pm(n+1)\}).$

Now, let $x_1, x_2, \ldots, x_{n+1} \in R^+$, $k \in \{\pm 1, \pm 2, \ldots, \pm(n+1)\}$, $a \in R$. If $k > 0$ and $G_{n+1}(x_i) \geqslant g$, $a > 0$ or $G_{n+1}(x_i) \leqslant g$, $a < 0$, then we obtain

$$S_{n+1}^{[k]}(x_i^a) \geqslant G_{n+1}(x_i^a) = [G_{n+1}(x_i)]^a \geqslant g^a$$

and the opposite inequalities hold if $k < 0$ and $G_{n+1}(x_i) \geqslant g$, $a < 0$ or $G_{n+1}(x_i) \leqslant g$, $a > 0$. But, by (12), these two statements are mutually equivalent. Applying these results on 1.19-1.31 we obtain in some cases more general inequalities 1.32-1.44.

1.32. $$S_{n+1}^{[k]}((F + mF_i)^a) \geqslant \left(\frac{n+m+1}{n+1} F\right)^a \quad (k \in \{2, \ldots, n+1\},\ m \geqslant -2,\ a < 0).$$

1.33. $$S_{n+1}^{[k]}\left(\left(\frac{F_i}{F + mF_i}\right)^a\right) \geqslant \frac{1}{(n+m+1)^a} \quad (k \in \{1, 2, \ldots, n+1\},\ m > 0,\ a < 0).$$

1.34. $$S_{n+1}^{[k]}\left(\left(\frac{F_i}{F - F_i}\right)^a\right) > \left(\frac{2}{n+1}\right)^a \quad (k \in \{1, 2, \ldots, n+1\},\ a < 0).$$

1.35. $$S_{n+1}^{[k]}\left(\left(\frac{F + pF_i}{F + mF_i}\right)^a\right) \geqslant \left(\frac{n+p+1}{n+m+1}\right)^a \quad (k \in \{1, 2, \ldots, n+1\},\ a < 0;\ 0 < m < p \text{ or } -2 \leqslant p < m < 0).$$

1.36. $$S_{n+1}^{[k]}\left(\left(\frac{F + pF_i}{F - F_i}\right)^a\right) > \left(\frac{n+2p+3}{n+1}\right)^a \quad (k \in \{1, 2, \ldots, n+1\},\ p > -1,\ a < 0).$$

1.37. $$S_{n+1}^{[k]}(\rho_i^a) \geqslant \left(\frac{n+1}{n-1} r\right)^a \quad (k \in \{1, 2, \ldots, n+1\},\ a < 0).$$

1.38. $$S_{n+1}^{[k]}\left(\left(\frac{F_i}{\rho_i}\right)^a\right) \geqslant \left[\frac{n-1}{(n+1)^2} \cdot \frac{F}{r}\right]^a \quad (k \in \{1, 2, \ldots, n+1\},\ a < 0).$$

1.39. $$S_{n+1}^{[k]}\left(\left(\frac{\rho_i - r}{r}\right)^a\right) < \left(\frac{2}{n-1}\right)^a \quad (k \in \{1, 2, \ldots, n+1\},\ a < 0).$$

1.40. $$S_{n+1}^{[k]}\left(\left(\frac{\rho_i - r}{\rho_i + r}\right)^a\right) > \left(\frac{2}{n+1}\right)^a \quad (k \in \{1, 2, \ldots, n+1\},\ a < 0).$$

1.41. $$S_{n+1}^{[k]}(h_i^a) \geqslant [(n+1)r]^a \quad (k \in \{1, 2, \ldots, n+1\},\ a > 0).$$

V. Volenec [139].

1.42. $$S_{n+1}^{[k]}\left(\left(\frac{h_i + r}{h_i - r}\right)^a\right) > \left(\frac{n+5}{n+1}\right)^a \quad (k \in \{1, 2, \ldots, n+1\},\ a < 0).$$

1.43. $$S_{n+1}^{[k]}\left(\left(\frac{1}{\rho_i h_i}\right)^a\right) \geqslant \left[\frac{n-1}{(n+1)^2} \cdot \frac{1}{r^2}\right]^a \quad (k \in \{1, 2, \ldots, n+1\},\ a < 0).$$

1.44. $$S_{n+1}^{[k]}\left(\left(\frac{F_i}{\sum_{\substack{j=1 \\ j\neq i}}^{n+1} h_i}\right)^a\right) \geqslant \left[\frac{F}{n(n+1)^2 r}\right]^a \quad (k \in \{1, 2, \ldots, n+1\},\ a < 0).$$

Finally, we give only three of many further possible generalizations.

1.45. For every convex function $f : R^+ \to R$ we have

$$A_{n+1}(f(F + mF_i)) \gtreqless f(A_{n+1}(F + mF_i)) = f\left(\frac{n+m+1}{n+1} F\right).$$

and the opposite inequality is valid for every concave function f.

Proof. By (1) we obtain

$$A_{n+1}(f(F + mF_i)) \gtreqless f(A_{n+1}(F + mF_i)) = f\left(\frac{n+m+1}{n+1} F\right).$$

1.46. For every convex respectively concave function $f : R^+ \to R$ we have

$$A_{n+1}\left(f\left(\frac{1}{\rho_i}\right)\right) \gtreqless f\left(\frac{n-1}{n+1} \cdot \frac{1}{r}\right).$$

Proof. By (6) it follows

$$A_{n+1}\left(f\left(\frac{1}{\rho_i}\right)\right) \gtreqless f\left(A_{n+1}\left(\frac{1}{\rho_i}\right)\right) = f\left(\frac{1}{H_{n+1}(\rho_i)}\right) = f\left(\frac{n-1}{n+1} \cdot \frac{1}{r}\right).$$

1.47. For every convex respectively concave function $f : R^+ \to R$ we have

$$A_{n+1}\left(f\left(\frac{1}{h_i}\right)\right) \gtreqless f\left(\frac{1}{(n+1)r}\right).$$

Proof. We work similarly as for 1.46, but according to (5).

2. Inequalities for the Simplex and a Point

Let $A = A_1 A_2 \ldots A_{n+1}$ be the given simplex in E^n $(n \geqslant 2)$, P any point inside A (when P is any point in E^n, then it will be emphasized separately). For every $i \in \{1, 2, \ldots, n+1\}$ let B_i be the intersection of the line A_iP with the hyperplane $a_i = A_1 \dots A_{i-1}A_{i+1} \dots A_{n+1}$, further let $R_i = A_iP$ and let r_i and h_i be the distances of the points P and A_i to the hyperplane a_i. Obviously,

(1) $$A_iB_i \geqslant h_i \quad (i = 1, 2, \dots, n + 1),$$

(2) $$r_i' = PB_i \geqslant r_i \quad (i = 1, 2, \dots, n + 1)$$

with equalities iff $A_iB_i \perp a_i$. Therefore, all equalities in (1) and (2) hold iff A is an orthocentric simplex and P is its orthocentre H, what will be designed by $\{A_O, P = H\}$. From similar triangles we obtain equalities

(3) $$\lambda_i = \frac{PB_i}{A_iB_i} = \frac{r_i}{h_i} \quad (i = 1, 2, \dots, n + 1),$$

where $\lambda_1, \lambda_2, \dots, \lambda_{n+1}$ are the barycentric coordinates of P in A. We have

(4) $$\sum_{i=1}^{n+1} \lambda_i = 1,$$

(5) $$\sum_{i=1}^{n+1} (1 - \lambda_i) = n,$$

(6) $$\underline{P} = \sum_{i=1}^{n+1} \lambda_i \underline{A}_i,$$

and by applying of A-G-inequality for means from (4) it follows

(7) $$\prod_{i=1}^{n+1} \lambda_i \leqslant \frac{1}{(n + 1)^{n+1}} \quad \{P = G\}.$$

The sign $\{P = G\}$ beyond an inequality means that the equality sign holds iff $P = G$ and $\{A_R\}$ means that equality holds iff A is a regular simplex. Let O, I be centres and R, r radii of circumscribed and inscribed hyperspheres and G the centroid of A. For every $i, j \in \{1, 2, \dots, n + 1\}$ $(i < j)$ let $a_{ij} = A_iA_j$ and for every $i \in \{1, 2, \dots, n + 1\}$ let F_i be the $(n - 1)$-dimensional content of $(n - 1)$-simplex $A_i = A_1 \dots A_{i-1}A_{i+1} \dots A_{n+1}$. Moreover, let V be the n-dimensional content and $m_1, m_2, \dots, m_{n+1}$ the lengths of medians of A.

2.1. If $f : R^+ \to R$ is a convex function, then for every $k \in \{1, 2, \dots, n + 1\}$

$$\sum_{\substack{i_1,\ldots,i_k=1\\ i_1<\ldots<i_k}}^{n+1} f\left(\sum_{j=1}^{k} \frac{PB_{i_j}}{A_{i_j}B_{i_j}}\right) \geq \binom{n+1}{k} f\left(\frac{k}{n+1}\right) \quad \{P = G\},$$

and if f is a concave function, then we have the opposite inequality.

Proof. By (3) and (4) we have successively

$$\sum_{\substack{i_1,\ldots,i_k=1\\ i_1<\ldots<i_k}}^{n+1} f\left(\sum_{j=1}^{k} \lambda_{i_j}\right) \gtreqless \binom{n+1}{k} f\left[\frac{1}{\binom{n+1}{k}} \sum_{\substack{i_1,\ldots,i_k=1\\ i_1<\ldots<i_k}}^{n+1} \left(\sum_{j=1}^{k} \lambda_{i_j}\right)\right] =$$

$$= \binom{n+1}{k} f\left(\frac{k}{n+1} \sum_{j=1}^{n+1} \lambda_i\right) = \binom{n+1}{k} f\left(\frac{k}{n+1}\right).$$

V. Volenec [139] ($f(x) = \ln x$); M. S. Klamkin and G. A. Tsintsifas (private communication) ($f(x) = x^a$, $a \in (-\infty, 0) \cup (1, +\infty)$, $k = 1$; $f(x) = (1 - x)^{-a}$, $a \in (-\infty, -1) \cup (0, +\infty)$, $k = 1$).

Remark. According to (3) we can write an equivalent inequality

2.1' $$\sum_{\substack{i_1,\ldots,i_k=1\\ i_1<\ldots<i_k}}^{n+1} f\left(\sum_{j=1}^{k} \frac{r_{i_j}}{h_{i_j}}\right) \gtreqless \binom{n+1}{k} f\left(\frac{k}{n+1}\right) \quad \{P = G\}.$$

2.2. If $f : R^+ \to R$ is a non-increasing function such that $f(\exp) : R \to R$ is a convex function, thenfor every $k \in \{1, 2, \ldots, n + 1\}$

$$\sum_{\substack{i_1,\ldots,i_k=1\\ i_1<\ldots<i_k}}^{n+1} f\left(\prod_{j=1}^{k} \frac{PB_{i_j}}{A_{i_j}B_{i_j}}\right) \geq \binom{n+1}{k} f\left(\frac{1}{(n+1)^k}\right) \quad \{P = G\}$$

and if f is a non-decreasing and f(exp) a concave function, then we have the opposite inequality.

Proof. According to (3) and (7) working as in 2.1 we obtain

$$\sum_{\substack{i_1,\ldots,i_k=1\\ i_1<\ldots<i_k}}^{n+1} f\left(\prod_{j=1}^{k} \lambda_{i_j}\right) = \sum_{\substack{i_1,\ldots,i_k=1\\ i_1<\ldots<i_k}}^{n+1} f\left(\exp\left(\sum_{j=1}^{k} \ln \lambda_{i_j}\right)\right) \gtreqless$$

$$\gtrless \binom{n+1}{k} f\left(\exp\left(\frac{k}{n+1}\sum_{i=1}^{n+1}\ln \lambda_i\right)\right) = \binom{n+1}{k} f\left(\left(\prod_{i=1}^{n+1}\lambda_i\right)^{\frac{k}{n+1}}\right) \gtrless$$

$$\gtrless \binom{n+1}{k} f\left(\left(\frac{1}{(n+1)^{n+1}}\right)^{\frac{k}{n+1}}\right) = \binom{n+1}{k} f\left(\frac{1}{(n+1)^k}\right).$$

D. M. Bătinețu [7] ($n = 3$).

Remark. The same as in 2.1.

2.3. $$\sum_{\substack{i_1,\ldots,i_k=1\\ i_1<\ldots<i_k}}^{n+1} \prod_{j=1}^{k} \left(\frac{PB_{i_j}}{A_{i_j}B_{i_j}}\right)^a \leqslant \binom{n+1}{k}\frac{1}{(n+1)^{ka}} \quad (k \in \{1, 2, \ldots, n+1\},\ 0 < a \leqslant 1)\ \{P = G\}.$$

Proof. According to inequalities for means (AI 2.15.1, Theorem 4 and 2.14.2, Theorem 1) from (4) it follows

$$\sum_{\substack{i_1,\ldots,i_k=1\\ i_1<\ldots<i_k}}^{n+1} \prod_{j=1}^{k} \lambda_{i_j}^a \leqslant \binom{n+1}{k}\left(\frac{1}{n+1}\sum_{i=1}^{n+1}\lambda_i^a\right)^k \leqslant$$

$$\leqslant \binom{n+1}{k}\left(\frac{1}{n+1}\sum_{i=1}^{n+1}\lambda_i\right)^{ka} = \binom{n+1}{k}\frac{1}{(n+1)^{ka}}.$$

M. S. Klamkin and G. A. Tsintsifas (private communication) ($k = 1$).

Remark. Similar as in 2.1.

2.4. If $f : R^+ \to R$ is a non-decreasing function such that $f(\exp) : R \to R$ is a convex function, then

$$\sum_{\substack{i_1,\ldots,i_k=1\\ i_1<\ldots<i_k}}^{n+1} f\left(\prod_{j=1}^{k}\frac{A_{i_j}B_{i_j}}{PB_{i_j}}\right) \geqslant \binom{n+1}{k} f((n+1)^k) \quad (k \in \{1, 2, \ldots, n+1\})\ \{P = G\}$$

and if f is a non-increasing and $f(\exp)$ a concave function, then we have the opposite inequality.

Proof. According to (3)

$$\frac{A_iB_i}{PB_i} = \frac{1}{\lambda_i} \quad (i = 1, 2, \ldots, n+1)$$

and by (7) analogously as in 2.2 we obtain

$$\sum_{\substack{i_1,\ldots,i_k=1 \\ i_1<\ldots<i_k}}^{n+1} f\left(\prod_{j=1}^{k} \frac{1}{\lambda_{i_j}}\right) \gtreqless \binom{n+1}{k} f\left(\left(\prod_{j=1}^{n+1} \frac{1}{\lambda_i}\right)^{\frac{1}{n+1}}\right) \gtreqless$$

$$\gtreqless \binom{n+1}{k} f\left(\left((n+1)^{n+1}\right)^{\frac{k}{n+1}}\right) =$$

$$= \binom{n+1}{k} f((n+1)^k).$$

V. Volenec [139] ($f(x) = x^a$, $a > 0$); H. Gabai [25] ($f(x) = x$, $k = n + 1$).

Remark 1. Because of (3) we can write an equivalent inequality

2.4' $$\sum_{\substack{i_1,\ldots,i_k=1 \\ i_1<\ldots<i_k}}^{n+1} f\left(\prod_{j=1}^{k} \frac{h_{i_j}}{r_{i_j}}\right) \gtreqless \binom{n+1}{k} f((n+1)^k) \quad (k \in \{1, 2, \ldots, n+1\}) \ \{P = G\}.$$

Ž. Živanović [149] ($f(x) = x$, $k = 1$); GI 12.11, 12.12 ($f(x) = x$, $n = 2$, $k \in \{1, 3\}$).

Remark 2. Because of $A_iP = R_i$ and (2) for any $i \in \{1, 2, \ldots, n+1\}$ we have

$$\frac{A_iB_i}{PB_i} = 1 + \frac{A_iP}{BP_i} \leqslant 1 + \frac{R_i}{r_i},$$

and 2.4 implies the inequality

2.4'' $$\sum_{\substack{i_1,\ldots,i_k=1 \\ i_1<\ldots<i_k}}^{n+1} f\left(\prod_{j=1}^{k} \left(1 + \frac{R_{i_j}}{r_{i_j}}\right)\right) \gtreqless \binom{n+1}{k} f((n+1)^k) \quad (k \in \{1, 2, \ldots, n+1\}) \ \{A_R, P = O\}.$$

2.5. With the same conditions as in 2.1 we have

$$\sum_{\substack{i_1,\ldots,i_k=1 \\ i_1<\ldots<i_k}}^{n+1} f\left(\sum_{j=1}^{k} \frac{A_{i_j}P}{A_{i_j}B_{i_j}}\right) \gtreqless \binom{n+1}{k} f\left(\frac{kn}{n+1}\right) \quad \{P = G\}.$$

Proof. By (3)

$$\frac{A_iP}{A_iB_i} = 1 - \lambda_i \quad (i = 1, 2, \ldots, n + 1),$$

and analogously as in 2.1 it follows by (5)

$$\sum_{\substack{i_1,\ldots,i_k=1 \\ i_1<\ldots<i_k}}^{n+1} f\left(\sum_{j=1}^{k} (1 - \lambda_{i_j})\right) \gtrless \binom{n+1}{k} f\left(\frac{k}{n+1} \sum_{i=1}^{n+1} (1 - \lambda_i)\right) =$$

$$= \binom{n+1}{k} f\left(\frac{kn}{n+1}\right).$$

V. Volenec [139] ($f(x) = \ln x$).

Remark 1. If $f : R^+ \to R$ is a convex non-decreasing function, then

2.5' $$\sum_{\substack{i_1,\ldots,i_k=1 \\ i_1<\ldots<i_k}}^{n+1} f\left(\sum_{j=1}^{k} \frac{R_{i_j}}{h_{i_j}}\right) \geqslant \binom{n+1}{k} f\left(\frac{kn}{n+1}\right) \quad (k \in \{1, 2, \ldots, n + 1\}) \; \{A_R, P = O\}.$$

This follows from 2.5 because of $A_iP = R_i$ and (1). According to (5) the inequality 2.5 holds also (as an equality) for $k = n + 1$ and therefore 2.5' holds also for $k = n + 1$ with $\{A_O, P = H\}$.

G. Kalajdžić [43] ($f(x) = x$, $n = 3$, $k \in \{1, n + 1\}$); S. Horák [40] ($f(x) = x$, $n = 3$, $k \in \{1, n + 1\}$).

Remark 2. With the same conditions as in 2.5' we have

2.5'' $$\sum_{\substack{i_1,\ldots,i_k=1 \\ i_1<\ldots<i_k}}^{n+1} f\left(\sum_{j=1}^{k} \frac{R_{i_j}}{R_{i_j} + r_{i_j}}\right) \geqslant \binom{n+1}{k} f\left(\frac{kn}{n+1}\right) \quad (k \in \{1, 2, \ldots, n + 1\}) \; \{A_R, P = O\}.$$

which is a consequence of 2.5 because of $A_iP = R_i$ and (2), i.e. of $A_iB_i \geqslant R_i + r_i$.

J. Berkes [8] ($f(x) = x$, $n = 3$; any $k \in \{1, 2, \ldots, n + 1\}$ gives the same result!); GI 12.36 ($f(x) = x$, $n = 2$).

2.6. With the same conditions as in 2.2 we have

$$\sum_{\substack{i_1,\ldots,i_k=1 \\ i_1<\ldots<i_k}}^{n+1} f\left(\prod_{j=1}^{k} \frac{A_{i_j}P}{A_{i_j}B_{i_j}}\right) \gtrless \binom{n+1}{k} f\left(\left(\frac{n}{n+1}\right)^k\right) \quad (k \in \{1, 2, \ldots, n + 1\}) \; \{P = G\}.$$

Proof. On the basis of the A-G-inequality for means, from (5) we get

$$\prod_{i=1}^{n+1} (1 - \lambda_i) \leqslant \left(\frac{n}{n+1}\right)^{n+1}$$

and analogously as in 2.2 it follows

$$\sum_{\substack{i_1,\ldots,i_k=1 \\ i_1<\ldots<i_k}}^{n+1} f\left(\prod_{j=1}^{k} (1 - \lambda_{i_j})\right) \gtreqless \binom{n+1}{k} f\left(\left(\prod_{i=1}^{n+1} (1 - \lambda_i)\right)^{\frac{k}{n+1}}\right) \gtreqless$$

$$\gtreqless \binom{n+1}{k} f\left(\left(\left(\frac{n}{n+1}\right)^{n+1}\right)^{\frac{k}{n+1}}\right) =$$

$$= \binom{n+1}{k} f\left(\left(\frac{n}{n+1}\right)^{k}\right).$$

D. M. Bătineţu [7] (n = 3).

2.7. With the same conditions as in 2.3 we have

$$\sum_{\substack{i_1,\ldots,i_k=1 \\ i_1<\ldots<i_k}}^{n+1} \prod_{j=1}^{k} \left(\frac{A_{i_j}P}{A_{i_j}B_{i_j}}\right)^{a} \leqslant \binom{n+1}{k} \left(\frac{n}{n+1}\right)^{ka} \quad (k \in \{1, 2, \ldots, n+1\},\ 0 < a \leqslant 1)\ \{P = G\}.$$

Proof. Analogously as in 2.3 from (5) it follows

$$\sum_{\substack{i_1,\ldots,i_k=1 \\ i_1<\ldots<i_k}}^{n+1} \prod_{j=1}^{k} (1 - \lambda_{i_j})^{a} \leqslant \binom{n+1}{k} \left(\frac{1}{n+1} \sum_{i=1}^{n+1} (1 - \lambda_i)\right)^{ka} =$$

$$= \binom{n+1}{k} \left(\frac{n}{n+1}\right)^{ka}.$$

2.8. With the same conditions as in 2.4 we have

$$\sum_{\substack{i_1,\ldots,i_k=1 \\ i_1<\ldots<i_k}}^{n+1} f\left(\prod_{j=1}^{k} \frac{A_{i_j}B_{i_j}}{A_{i_j}P}\right) \gtreqless \binom{n+1}{k} f\left(\left(\frac{n+1}{k}\right)^{k}\right) \quad \{P = G\}.$$

Proof. Since

$$\frac{A_iB_i}{A_iP} = \frac{1}{1 - \lambda_i} \quad (i = 1, 2, \ldots, n + 1),$$

from (5) on the basis of the A-G-inequality for means we obtain

$$\prod_{i=1}^{n+1} \frac{1}{1-\lambda_i} = \prod_{i=1}^{n+1} \frac{1}{n(1-\lambda_i)} \sum_{j=1}^{n+1} (1-\lambda_j) \geqslant$$

$$\geqslant \prod_{i=1}^{n+1} \left[\frac{n+1}{n(1-\lambda_i)} \left(\prod_{j=1}^{n+1} (1-\lambda_j)\right)^{\frac{1}{n+1}}\right] =$$

$$= \left(\frac{n+1}{n}\right)^{n+1}.$$

Therefore, it follows analogously as in 2.2

$$\sum_{\substack{i_1,\ldots,i_k=1\\ i_1<\ldots<i_k}}^{n+1} f\left(\prod_{j=1}^{k} \frac{1}{1-\lambda_{i_j}}\right) \gtrless \binom{n+1}{k} f\left(\left(\prod_{i=1}^{n+1} \frac{1}{1-\lambda_i}\right)^{\frac{k}{n+1}}\right) \gtrless$$

$$\gtrless \binom{n+1}{k} f\left(\left(\left(\frac{n+1}{n}\right)^{n+1}\right)^{\frac{k}{n+1}}\right) =$$

$$= \binom{n+1}{k} f\left(\left(\frac{n+1}{n}\right)^{k}\right).$$

V. Volenec [139] ($f(x) = x^a$, $a > 0$); Ž. Živanović [149] ($f(x) = x$, $k = 1$); F. Abeles [1] ($f(x) = x$, $k = n + 1$); G. Georgescu [27] ($f(x) = x$, $n = 3$, $k = 1$); M. Stan [103] ($f(x) = x$, $n = 3$, $k = 1$); I. A. Kušnir [62] ($f(x) = x$, $n = 3$, $k = 1$, $P = O$); GI 12.40 = 9.14 ($f(x) = x$, $n = 2$, $k = 1$).

2.9. With the same conditions as in 2.4 we have

$$\sum_{\substack{i_1,\ldots,i_k=1\\ i_1<\ldots<i_k}}^{n+1} f\left(\prod_{j=1}^{k} \frac{A_{i_j}P}{PB_{i_j}}\right) \gtrless \binom{n+1}{k} f(n^k) \qquad (k \in \{1, 2, \ldots, n+1\}) \quad \{P = G\}.$$

Proof. Now,

$$\frac{A_iP}{PB_i} = \frac{1-\lambda_i}{\lambda_i} \qquad (i = 1, 2, \ldots, n+1)$$

and according to the A-G-inequality for means it follows by (4)

$$\prod_{i=1}^{n+1} \frac{1-\lambda_i}{\lambda_i} = \prod_{i=1}^{n+1} \left(\frac{1}{\lambda_i} \sum_{\substack{j=1\\ j\neq i}}^{n+1} \lambda_j\right) \geqslant \prod_{i=1}^{n+1} \left(\frac{n}{\lambda_i}\left(\prod_{\substack{j=1\\ j\neq i}}^{n+1} \lambda_j\right)^{\frac{1}{n}}\right) = n^{n+1}.$$

Therefore, analogously as in 2.2 we get

$$\sum_{\substack{i_1,\dots,i_k=1\\ i_1<\dots<i_k}}^{n+1} f\left(\prod_{j=1}^{k} \frac{1-\lambda_{i_j}}{\lambda_{i_j}}\right) \gtreqless \binom{n+1}{k} f\left(\left(\prod_{i=1}^{n+1} \frac{1-\lambda_i}{\lambda_i}\right)^{\frac{k}{n+1}}\right) \gtreqless$$

$$\gtreqless \binom{n+1}{k} f\left(\left(n^{n+1}\right)^{\frac{k}{n+1}}\right) = \binom{n+1}{k} f(n^k).$$

V. Volenec [139] ($f(x) = x^a$, $a > 0$); J. Schopp [95] ($f(x) = x$); A. Oppenheim [76] ($f(x) = x^a$, $a > 0$, $k = 1$); J. Schopp [91] ($f(x) = x$, $k = n + 1$); O. Reutter [86] ($f(x) = x$, $k = 1$); H. Gabai [25] ($f(x) = x$, $k \in \{1, n + 1\}$); V. L. Rabinovič and I. M. Jaglom [84] ($f(x) = x$, $k = n + 1$); G. A. Tsintsifas (private communication) ($f(x) = x$, $k \in \{1, 2, n + 1\}$); D. M. Bătineţu [7] ($f(x) = x^a$, $a > 0$, $n = 3$); D. M. Stan [104] ($f(x) = x^a$, $a > 0$, $n = 3$, $k = 1$); G. P. Bevz [11] ($f(x) = x^a$, $a > 0$, $n = 3$, $k = 1$); V. Thébault [120] ($f(x) = x$, $n = 3$, $k \in \{1, 4\}$), [123] ($f(x) = x$, $n = 3$, $k \in \{1, 4\}$); M. Stanković [105] ($f(x) = x$, $n = 3$, $k \in \{1, 4\}$); D. O. Škljarskij, N. N. Čencov, and I. M. Jaglom [101, p. 308-309] ($f(x) = x$, $n = 3$, $k = 4$); M. Erdmann [19] ($f(x) = x$, $n = 3$, $k = 1$); GI 12.38, 12.47, 12.39 ($f(x) = x$, $n = 2$, $k \in \{1, 2, 3\}$); J. T. Groenman [33] ($f(x) = x$, $n = 2$, $k \in \{1, 3\}$, $P = O$).

Remark. According to $A_iP = R_i$ and (2) it follows from 2.9 (with the same conditions):

2.9' $$\sum_{\substack{i_1,\dots,i_k=1\\ i_1<\dots<i_k}}^{n+1} f\left(\prod_{j=1}^{k} \frac{R_{i_j}}{r_{i_j}}\right) \gtreqless \binom{n+1}{k} f(n^k) \qquad (k \in \{1, 2, \dots, n + 1\}) \ \{A_R, P = O\}.$$

V. Volenec [139] ($f(x) = x^a$, $a > 0$); J. Schopp [95] ($f(x) = x$); A. Oppenheim [76] ($f(x) = x^a$, $a > 0$, $k = 1$); J. Schopp [91] ($f(x) = x$, $k = n + 1$), [92] ($f(x) = x$, $k \in \{1, n + 1\}$, $P = I$); O. Reutter [86] ($f(x) = x$, $k = 1$); J. Berkes [8] ($f(x) = x$, $n = 3$, $k = 4$); V. Thébault [123] ($f(x) = x$, $n = 3$, $k \in \{1, 4\}$), [124] ($f(x) = x$, $n = 3$, $k = 1$, $P = I$); M. Stanković [105] ($f(x) = x$, $n = 3$, $k = 4$); D. O. Škljarskij, N. N. Čencov, and I. M. Jaglom [101, p. 59, 307-310] ($f(x) = x$, $n = 3$, $k = 4$; $f(x) = x$, $n = 3$, $k = 1$, $P = I$); GI 12.25, 12.1 ($f(x) = x$, $n = 2$, $k = 3$; $f(x) = x$, $n = 2$, $k = 1$, $P = I$).

2.10. With the same conditions as in 2.2 we have

$$\sum_{\substack{i_1,\ldots,i_k=1\\ i_1<\ldots<i_k}}^{n+1} f\left(\prod_{j=1}^{k} \frac{PB_{i_j}}{A_{i_j}P}\right) = \binom{n+1}{k} f\left(\frac{1}{n^k}\right) \qquad (k \in \{1, 2, \ldots, n+1\}) \quad \{P = G\}.$$

Proof. Since

$$\frac{PB_i}{A_iP} = \frac{\lambda_i}{1 - \lambda_i} \qquad (i = 1, 2, \ldots, n+1)$$

and according to results from the proof of 2.9, we have

$$\prod_{i=1}^{n+1} \frac{\lambda_i}{1 - \lambda_i} \leqslant \frac{1}{n^{n+1}},$$

so analogously as in 2.2 we get

$$\sum_{\substack{i_1,\ldots,i_k=1\\ i_1<\ldots<i_k}}^{n+1} f\left(\prod_{j=1}^{k} \frac{\lambda_{i_j}}{1 - \lambda_{i_j}}\right) \gtreqless \binom{n+1}{k} f\left(\left(\prod_{i=1}^{n+1} \frac{\lambda_i}{1 - \lambda_i}\right)^{\frac{1}{n+1}}\right) \gtreqless$$

$$\gtreqless \binom{n+1}{k} f\left(\left(\frac{1}{n^{n+1}}\right)^{\frac{k}{n+1}}\right) = \binom{n+1}{k} f\left(\frac{1}{n^k}\right).$$

M. S. Klamkin and G. A. Tsintsifas (private communication) ($f(x) = x^a$, $a > 0$, $k = 1$).

2.11. $$\sum_{i=1}^{n+1} \left(\frac{PB_i}{A_iP}\right)^a \geqslant \frac{n+1}{n^a} \qquad (a \geqslant 1) \quad \{P = G\}.$$

Proof. From 2.8 with $f(x) = x$ and $k = 1$ we obtain

$$\sum_{i=1}^{n+1} \frac{A_iB_i}{A_iP} \geqslant \frac{(n+1)^2}{n},$$

and so it follows successively

$$\sum_{i=1}^{n+1} \frac{PB_i}{A_iP} = \sum_{i=1}^{n+1} \left(\frac{A_iB_i}{A_iP} - 1\right) \geqslant \frac{(n+1)^2}{n} - (n+1) = \frac{n+1}{n},$$

$$\sum_{i=1}^{n+1} \left(\frac{PB_i}{A_iP}\right)^a \geqslant (n+1)\left(\frac{1}{n+1} \sum_{i=1}^{n+1} \frac{PB_i}{A_iP}\right)^a \geqslant$$

$$\geqslant (n+1)\left(\frac{1}{n+1}\cdot\frac{n+1}{n}\right)^a = \frac{n+1}{n^a}.$$

A. Oppenheim [76]; V. Volenec [139] (a = 1); M. S. Klamkin [55] (a = 1, P = O); G. Tsintsifas and S. Rabinowitz [132] (a = 1, n = 2, P = O).

2.12. $$\sum_{i=1}^{n+1} \varepsilon_i^2 \cdot A_iP \geqslant 2 \sum_{\substack{i,j=1\\ i<j}}^{n+1} \varepsilon_i\varepsilon_j\sqrt{PB_i \cdot PB_j} \quad (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n+1} \in \mathbb{R}^+)$$

with equality iff all numbers $(\varepsilon_i/\lambda_i)\sqrt{PB_i}$ (i = 1, 2, ..., n + 1) are equal.

Proof. As in 2.9, because of (4) it follows

$$A_iP = \frac{1-\lambda_i}{\lambda_i} PB_i = \sum_{\substack{j=1\\ j\neq i}}^{n+1} \frac{\lambda_j}{\lambda_i} PB_i,$$

and hence

$$\sum_{i=1}^{n+1} \varepsilon_i^2 \cdot A_iP = \sum_{\substack{i,j=1\\ i<j}}^{n+1} \left(\frac{\lambda_j}{\lambda_i}\varepsilon_i^2 \cdot PB_i + \frac{\lambda_i}{\lambda_j}\varepsilon_j^2 \cdot PB_j\right) \geqslant$$

$$\geqslant \sum_{\substack{i,j=1\\ i<j}}^{2n+1} \left(\sqrt{\frac{\lambda_j}{\lambda_i}}\,\varepsilon_i\sqrt{PB_i}\sqrt{\frac{\lambda_i}{\lambda_j}}\,\varepsilon_j\sqrt{PB_j}\right) =$$

$$= 2\sum_{\substack{i,j=1\\ i<j}}^{n+1} \varepsilon_i\varepsilon_j\sqrt{PB_i \cdot PB_j}.$$

A. Oppenheim [76]; G. A. Tsintsifas 1967 (private communication) ($\varepsilon_1 = \varepsilon_2 = \ldots = \varepsilon_{n+1} = 1$); GI 12.43 (n = 2, $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 1$).

Remark. According to $A_iP = R_i$ and (2) it follows from 2.12:

2.12' $$\sum_{i=1}^{n+1} \varepsilon_i^2 R_i \geqslant 2 \sum_{\substack{i,j=1\\ i<j}}^{n+1} \varepsilon_i\varepsilon_j\sqrt{PB_i \cdot PB_j} \quad (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n+1} \in R^+)$$

with equality iff A is an orthocentric simplex, P its orthocentre and all numbers $(\varepsilon_i/\lambda_i)\sqrt{PB_i}$ (i = 1, 2, ..., n + 1) are equal.

A. Oppenheim [76]; G. Kalajdžić [43] (n = 3, $\varepsilon_i = 1$); G. Kalajdžić and S. Srećković [44] (n = 3, $\varepsilon_i = 1$); D. O. Škljarskij, N. N. Čencov and I. M. Jaglom [101, p. 59, 306-307] ($n = 3$, $\varepsilon_i = \sqrt{r_i}$); GI 12.31, 12.34 ($n = 2$, $\varepsilon_i \in \{\sqrt{r_i}, \frac{1}{R_i\sqrt{r_i}}\}$.

2.13. $$\sum_{i=1}^{n+1} \varepsilon_i^2(n \cdot A_iB_i - A_iP) \geqslant 2 \sum_{\substack{i,j=1\\ i<j}}^{n+1} \varepsilon_i\varepsilon_j\sqrt{A_iP \cdot A_jP} \quad (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n+1} \in R^+)$$

with equality iff all numbers $\varepsilon_i\sqrt{A_iP}/((1 - \lambda_i)$ ($i = 1, 2, \ldots, n + 1$) are equal.

Proof. Because of (4) and (6) the point with radius vector

$$\frac{1}{1 - \lambda_i}(\underline{P} - \lambda_i\underline{A}_i) = \frac{1}{\sum\limits_{\substack{j=1\\ j\neq i}}^{n+1} \lambda_j} \sum_{\substack{j=1\\ j\neq i}}^{n+1} \lambda_j\underline{A}_j$$

is simultaneously centroid of the points P, A_i with masses 1, $-\lambda_i$ and centroid of the points $A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_{n+1}$ with masses $\lambda_1, \ldots, \lambda_{i-1}, \lambda_{i+1}, \ldots, \lambda_{n+1}$. Therefore, this point is the intersection B_i of line A_iP with hyperplane a_i. Further, it follows successively

$$\sum_{i=1}^{n+1} \frac{1}{n}(1 - \lambda_i)\underline{B}_i = \frac{1}{n}\sum_{i=1}^{n+1} (\underline{P} - \lambda_i\underline{A}_i) =$$

$$= \frac{1}{n}\left[(n + 1)\underline{P} - \sum_{i=1}^{n+1} \lambda_i\underline{A}_i\right] = \underline{P},$$

and P is the centroid of points $B_1, B_2, \ldots, B_{n+1}$ with masses

(8) $$\frac{1}{n}(1 - \lambda_1),\ \frac{1}{n}(1 - \lambda_2),\ \ldots,\ \frac{1}{n}(1 - \lambda_{n+1}).$$

According to (5), the numbers (8) are barycentric coordinates of P in the simplex $\mathcal{B} = B_1B_2 \ldots B_{n+1}$. When going from the pair $(\mathcal{A}, P)$ to the pair $(\mathcal{B}, P)$, it is necessary for every $i \in \{1, 2, \ldots, n + 1\}$ to make the substitution

$$\lambda_i \to \frac{1}{n}(1 - \lambda_i).$$

Therefore, the ratio

$$\frac{A_iP}{PB_i} = \frac{1 - \lambda_i}{\lambda_i}$$

must be substituted by

$$\frac{1-\frac{1}{n}(1-\lambda_i)}{\frac{1}{n}(1-\lambda_i)} = \frac{n-1+\lambda_i}{1-\lambda_i} = \frac{n-1+\frac{PB_i}{A_iB_i}}{1-\frac{PB_i}{A_iB_i}} =$$

$$= \frac{n \cdot A_iB_i - A_iP}{A_iP},$$

i.e. we must make the substitutions

$$(9) \qquad A_iP \to n \cdot A_iB_i - A_iP, \quad PB_i \to A_iP \quad (i = 1, 2, \ldots, n+1).$$

Applying these substitutions, from 2.12 it follows 2.13.

A. Oppenheim [76]; G. A. Tsintsifas 1967 (private communication) ($\varepsilon_1 = \varepsilon_2 = \ldots = \varepsilon_{n+1} = 1$); GI 12.46 ($n = 2$, $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 1$).

2.14. $$\prod_{\substack{i_1,\ldots,i_k=1\\ i_1<\ldots<i_k}}^{n+1} \sum_{j=1}^{k} \left(n - \frac{A_{i_j}P}{A_{i_j}B_{i_j}}\right) \leqslant \left(\frac{kn^2}{n+1}\right)^{\binom{n+1}{k}}$$ ($k \in \{1, 2, \ldots, n+1\}$) $\{P = G\}$.

Proof. From 2.5 with $f(x) = \ln x$ it follows the inequality

$$\prod_{\substack{i_1,\ldots,i_k=1\\ i_1<\ldots<i_k}}^{n+1} \sum_{j=1}^{k} \frac{A_{i_j}P}{A_{i_j}B_{i_j}} \leqslant \left(\frac{kn}{n+1}\right)^{\binom{n+1}{k}}$$

from which it follows 2.14 by (9). At the same time we must make also the substitutions

$$(10) \qquad A_iB_i \to n \cdot A_iB_i \quad (i = 1, 2, \ldots, n+1),$$

because $A_iB_i = A_iP + PB_i$ must be substituted by $(n \cdot A_iB_i - A_iP) + A_iP = n \cdot A_iB_i$.

2.15. $$\sum_{\substack{i_1,\ldots,i_k=1\\ i_1<\ldots<i_k}}^{n+1} \prod_{j=1}^{k} \left(n - \frac{A_{i_j}P}{A_{i_j}B_{i_j}}\right)^a \geqslant \binom{n+1}{k} \left(\frac{n^2}{n+1}\right)^{ka}$$ ($k \in \{1, 2, \ldots, n+1\}$, $0 < a \leqslant 1$) $\{P = G\}$.

Proof. Follows from 2.7 by the substitutions (9) and (10).

2.16. $$\sum_{\substack{i_1,\ldots,i_k=1\\ i_1<\ldots<i_k}}^{n+1} \prod_{j=1}^{k} \left(n \cdot \frac{A_{i_j} B_{i_j}}{A_{i_j} P} - 1\right)^a \geqslant \binom{n+1}{k} n^{ka}$$ $(k \in \{1, 2, \ldots, n+1\},\ a > 0)$ $\{P = G\}$.

Proof. We must put $f(x) = x$ in 2.9 and then apply the substitutions (9).

A. Oppenheim [76] $(k = 1)$.

2.17. $$\sum_{i=1}^{n+1} A_i P^2 \geqslant 2 \sum_{\substack{i,j=1\\ i<j}}^{n+1} (PB_i \cdot PB_j) + n(n^2-1) \prod_{i=1}^{n+1} PB_i^{\frac{2}{n+1}}$$ $\{m_1 = m_2 = \ldots = m_{n+1},\ P = G\}$.

Proof. As in 2.9 we have

$$A_i P = \frac{1-\lambda_i}{\lambda_i} PB_i = \sum_{\substack{j=1\\ j\neq i}}^{n+1} \frac{\lambda_j}{\lambda_i} PB_i,$$

and it follows

$$\sum_{i=1}^{n+1} A_i P^2 = \sum_{i=1}^{n+1} \left(\sum_{j=1}^{n+1} \frac{\lambda_j}{\lambda_i} PB_i\right)^2 =$$

$$= \sum_{\substack{i,j=1\\ i<j}}^{n+1} \left(\frac{\lambda_j^2}{\lambda_i^2} PB_i^2 + \frac{\lambda_j^2}{\lambda_i^2} PB_j^2\right) + 2 \sum_{i=1}^{n+1} \left(\sum_{\substack{j,k=1\\ j,k\neq i\\ j<k}}^{n+1} \frac{\lambda_j \lambda_k}{\lambda_i^2} PB_i^2\right) \geqslant$$

$$\geqslant 2 \sum_{\substack{i,j=1\\ i<j}}^{n+1} (PB_i \cdot PB_j) + n(n^2-1) \prod_{i=1}^{n+1} PB_i^{\frac{2}{n+1}}.$$

Equality occurs iff for every $i, j, k \in \{1, 2, \ldots, n+1\}$ $(i \neq j \neq k \neq i)$.

(11) $$\frac{\lambda_j \lambda_k}{\lambda_i^2} PB_i^2 = \alpha = \text{const.},$$

and for every $i, j \in \{1, 2, \ldots, n+1\}$ $(i < j)$

$$\frac{\lambda_j}{\lambda_i} PB_i = \frac{\lambda_i}{\lambda_j} PB_j,$$

i.e. for every $i \in \{1, 2, \ldots, n+1\}$

$$\frac{\lambda_i^2}{PB_i} = \beta = \text{const.} \tag{12}$$

But, from (11) and (12) it follows $\lambda_i^2\lambda_j\lambda_k = \alpha\beta^2$, what together with $\lambda_i\lambda_j^2\lambda_k = \alpha\beta^2$ implies $\lambda_i = \lambda_j$. Therefore, $P = G$, and from (12) it follows $PB_i = \text{const.}$, i.e. $A_iB_i = (n+1)PB_i = \text{const.}$

G. A. Tsintsifas 1967 (private communication); GI 12.44 ($n = 2$).

2.18. $$\sum_{\substack{i,j=1\\ i<j}}^{n+1} (A_iP \cdot A_jP) \geq \sum_{\substack{i,j=1\\ i<j}}^{n+1} (PB_i \cdot PB_j) + \frac{1}{2} n(n^2-1)(n+1) \prod_{i=1}^{n+1} PB_i^{\frac{2}{n+1}} \quad \{m_i = \text{const.}, P = G\}.$$

Proof. As in 2.9 we have

$$A_iP = \frac{1-\lambda_i}{\lambda_i} PB_i,$$

and so it follows by (4) and (7)

$$\sum_{\substack{i,j=1\\ i<j}}^{n+1} A_iP \cdot A_jP = \sum_{\substack{i,j=1\\ i<j}}^{n+1} \left[\frac{(1-\lambda_i)(1-\lambda_j)}{\lambda_i\lambda_j} PB_i \cdot PB_j\right] =$$

$$= \sum_{\substack{i,j=1\\ i<j}}^{n+1} \left[\left(\frac{1-\lambda_i-\lambda_j}{\lambda_i\lambda_j} + 1\right) PB_i \cdot PB_j\right] =$$

$$= \sum_{\substack{i,j=1\\ i<j}}^{n+1} \left[\frac{PB_i \cdot PB_j}{\lambda_i\lambda_j}\left(\sum_{\substack{k=1\\ k\neq i,j}}^{n+1} \lambda_k\right) + PB_i \cdot PB_j\right] \geq$$

$$\geq \sum_{\substack{i,j=1\\ i<j}}^{n+1} \left[\frac{PB_i \cdot PB_j}{\lambda_i\lambda_j} (n-1)\left(\prod_{\substack{k=1\\ k\neq i,j}}^{n+1} \lambda_k^{\frac{1}{n-1}}\right) + \right.$$

$$+ PB_i \cdot PB_j\Big] = (n-1)\left(\prod_{k=1}^{n+1} \lambda_k^{\frac{1}{n-1}}\right) \sum_{\substack{i,j=1\\ i<j}}^{n+1} \frac{PB_i \cdot PB_j}{\lambda_i^{\frac{n}{n-1}} \lambda_j^{\frac{n}{n-1}}} +$$

$$+ \sum_{\substack{i,j=1\\ i<j}}^{n+1} PB_i \cdot PB_j \geqslant$$

$$\geqslant (n-1)\left(\prod_{i=1}^{n+1} \lambda_i^{\frac{1}{n-1}}\right) \frac{n(n+1)}{2} \frac{\prod_{i=1}^{n+1} PB_i^{\frac{2}{n+1}}}{\prod_{i=1}^{n+1} \lambda_i^{\frac{2n}{(n+1)(n-1)}}} +$$

$$+ \sum_{\substack{i,j=1\\ i<j}}^{n+1} PB_i \cdot PB_j = \sum_{\substack{i,j=1\\ i<j}}^{n+1} PB_i \cdot PB_j +$$

$$+ \frac{1}{2} n(n^2-1) \frac{\prod_{i=1}^{n+1} PB_i^{\frac{2}{n+1}}}{\prod_{i=1}^{n+1} \lambda_i^{\frac{1}{n+1}}} \geqslant \sum_{\substack{i,j=1\\ i<j}}^{n+1} PB_i \cdot PB_j +$$

$$+ \frac{1}{2} n(n^2-1)(n+1) \prod_{i=1}^{n+1} PB_i^{\frac{2}{n+1}}.$$

Equalities hold iff λ_i = const. and PB_i = const.

G. A. Tsintsifas 1967 (private communication); GI 12.45 ($n = 2$).

2.19. $$\left(\sum_{i=1}^{n+1} A_iP^a\right)\left(\sum_{i=1}^{n+1} \frac{1}{PB_i^a}\right) \geqslant (n+1)^2 n^a \quad (a \in R^+)\ \{m_1 = m_2 = \dots = m_{n+1},\ P = G\}.$$

Proof. According to Cauchy's inequality we have

$$\left(\sum_{i=1}^{n+1} A_iP^a\right)\left(\sum_{i=1}^{n+1} \frac{1}{PB_i^a}\right) \geqslant \left[\sum_{i=1}^{n+1} \left(\frac{A_iP}{PB_i}\right)^{\frac{a}{2}}\right]^2.$$

Equality occurs iff $A_iP \cdot PB_i$ = const. From 2.9 setting $f(x) = x^a$, $k = 1$, we get

$$\sum_{i=1}^{n+1} \left(\frac{A_iP}{PB_i}\right)^{\frac{a}{2}} \geqslant (n+1)n^{\frac{a}{2}}$$

with equality only for $P = G$, i.e. for $A_iP/PB_i = n$. Both equalities appear iff $P = G$ and A_iP = const., i.e. $nm_i = (n+1)A_iP$ = const.

V. L. Rabinovič and I. M. Jaglom [84]; D. O. Škljarskij, N. N. Čencov, and I. M. Jaglom [101, p. 312] ($n = 3$, $a \in \{1, 2\}$).

Remark. Because of $R_i = A_iP$ and (2) from 2.19 it follows

2.19' $$\left(\sum_{i=1}^{n+1} R_i^a\right)\left(\sum_{i=1}^{n+1} \frac{1}{r_i^a}\right) \geqslant (n+1)^2 n^a \qquad (a \in R^+)\ \{A_R,\ P = O\}.$$

V. L. Rabinovič and I. M. Jaglom [84]; D. O. Škljarskij, N. N. Čencov, and I. M. Jaglom [101, p. 62, 310-312] ($n = 3$, $a \in \{1, 2\}$).

2.20. Let $k > 0$. For any $i \in \{1, 2, \ldots, n+1\}$ let C_i be the point of segment $\overline{PB_i}$ such that $PC_i : C_iB_i = k$. Then

$$\sum_{i=1}^{n+1} \frac{A_iC_i}{C_iB_i} \geqslant (n+1)[(n+1)k + n] \qquad \{P = G\}.$$

Proof. As

$$\frac{A_iC_i}{C_iB_i} = \frac{A_iP + PC_i}{C_iB_i} = \frac{A_iP + \frac{k}{k+1}PB_i}{\frac{1}{k+1}PB_i} = (k+1)\frac{A_iP}{PB_i} + k,$$

so by 2.9 (with $f(x) = x$, $k = 1$) it follows

$$\sum_{i=1}^{n+1} \frac{A_iC_i}{C_iB_i} = (k+1)\sum_{i=1}^{n+1} \frac{A_iP}{PB_i} + (n+1)k \geqslant$$

$$\geqslant (k+1)n(n+1) + (n+1)k = (n+1)[(n+1)k + n].$$

J. Weinstein and I. Dăncilă [141] ($n = 3$).

2.21. With the same conditions as in 2.20 we have

$$\sum_{i=1}^{n+1} \frac{C_iB_i}{A_iC_i} \geqslant \frac{n+1}{(n+1)k + n} \qquad \{P = G\}.$$

Proof. As by (3)

$$\frac{C_iB_i}{A_iC_i} = \frac{C_iB_i}{A_iB_i - C_iB_i} = \frac{\dfrac{PB_i}{A_iB_i}}{\dfrac{PB_i}{C_iB_i} - \dfrac{PB_i}{A_iB_i}} = \frac{\lambda_i}{k + 1 - \lambda_i} = \frac{k + 1}{k + 1 - \lambda_i} - 1,$$

so by (4) follows

$$\sum_{i=1}^{n+1} \frac{C_iB_i}{A_iB_i} = (k + 1) \sum_{i=1}^{n+1} \frac{1}{k + 1 - \lambda_i} - (n + 1) =$$

$$= \frac{k + 1}{(n + 1)(k + 1) - 1} \sum_{i=1}^{n+1} (k + 1 - \lambda_i) \sum_{i=1}^{n+1} \frac{1}{k + 1 - \lambda_i} -$$

$$- (n + 1) \geqslant \frac{k + 1}{(n + 1)(k + 1) - 1}(n + 1)^2 - (n + 1) =$$

$$= \frac{n + 1}{(n + 1)k + n} .$$

J. Weinstein and I. Dăncilă [141] (n = 3).

2.22. For any points P and Q

$$\sum_{\substack{i,j=1\\ i<j}}^{n+1} \lambda_i\lambda_j a_{ij}^2 \leqslant \sum_{i=1}^{n+1} \lambda_i \cdot QA_i^2 \qquad \{Q = P\}.$$

<u>Proof</u>. Let Q be the origin. Because of (6) and (4) it follows

$$0 \leqslant QP^2 = \left(\sum_{i=1}^{n+1} \lambda_i \underline{A}_i\right)^2 = \sum_{i=1}^{n+1} \lambda_i^2 \cdot QA_i^2 + 2 \sum_{\substack{i,j=1\\ i<j}}^{n+1} \lambda_i\lambda_j \underline{A}_i \cdot \underline{A}_j =$$

$$= \sum_{i=1}^{n+1} \lambda_i^2 \cdot QA_i^2 + \sum_{\substack{i,j=1\\ i<j}}^{n+1} \lambda_i\lambda_j (QA_i^2 + QA_j^2 - a_{ij}^2) =$$

$$= \sum_{i=1}^{n+1} \lambda_i^2 \cdot QA_i^2 + \sum_{\substack{i,j=1\\ i\neq j}}^{n+1} \lambda_i\lambda_j \cdot QA_i^2 - \sum_{\substack{i,j=1\\ i<j}}^{n+1} \lambda_i\lambda_j a_{ij}^2 =$$

$$= \left(\sum_{i=1}^{n+1} \lambda_i\right)\left(\sum_{i=1}^{n+1} \lambda_i \cdot QA_i^2\right) - \sum_{\substack{i,j=1\\ i<j}}^{n+1} \lambda_i\lambda_j a_{ij}^2 =$$

$$= \sum_{i=1}^{n+1} \lambda_i \cdot QA_i^2 - \sum_{\substack{i,j=1\\ i<j}}^{n+1} \lambda_i\lambda_j a_{ij}^2 .$$

G. A. Tsintsifas 1967 (private communication); M. S. Klamkin [48] (Q = O); V. Devidé [16] (P = G, (Q = O); A. Simeonov [100] (n = 3, P = I); C. Băluţă [5] (n = 3, A regular tetrahedron, P = O); GI 14.1 (n = 2, Q = O).

2.23. If P and Q are any points, then

$$\sum_{i=1}^{n+1} \lambda_i \cdot PA_i^2 \leqslant \sum_{i=1}^{n+1} \lambda_i \cdot QA_i^2 \quad \{Q = P\}.$$

Proof. If Q is the origin, then from (6) and from the equality

$$\sum_{i=1}^{n+1} \lambda_i \overrightarrow{PA_i} = \vec{0}$$

it follows

$$0 \leqslant QP^2 = \underline{P} \cdot \left(\sum_{i=1}^{n+1} \lambda_i \underline{A}_i\right) = \underline{P} \cdot \left[\sum_{i=1}^{n+1} (\lambda_i \underline{A}_i + \lambda_i \overrightarrow{PA_i})\right] =$$

$$= \sum_{i=1}^{n+1} [\lambda_i \underline{P} \cdot (\underline{A}_i + \overrightarrow{PA_i})] = \sum_{i=1}^{n+1} \lambda_i (\underline{A}_i - \overrightarrow{PA_i})(\underline{A}_i + \overrightarrow{PA_i}) =$$

$$= \sum_{i=1}^{n+1} \lambda_i (QA_i^2 - PA_i^2).$$

G. A. Tsintsifas 1967 (private communication); E. Egerváry [17] (n ∈ {2, 3}); R. Estève [20] (n = 3); D. O. Škljarskij, N. N. Čencov, and I. M. Jaglom [101, p. 37, 221-224, 225-228] (n ∈ {2, 3}, P = O); C. Băluţă [5] (n = 3, A regular tetrahedron, P = O).

2.24. Let $\lambda_1, \lambda_2, \ldots, \lambda_{n+1} \in R$, $\lambda = \lambda_1 + \lambda_2 + \ldots + \lambda_{n+1} \neq 0$. For any point Q we have

$$\sum_{i=1}^{n+1} \lambda_i \cdot QA_i^2 \geqslant \frac{1}{\lambda^2}\left[\sum_{\substack{i,j=1\\ i<j}}^{n+1} \lambda_i\lambda_j(\lambda_i + \lambda_j)a_{ij}^2 + \right.$$

$$\left. + \sum_{\substack{i,j,k=1\\ i<j<k}}^{n+1} \lambda_i\lambda_j\lambda_k(a_{ij}^2 + a_{ik}^2 + a_{jk}^2)\right]$$

with equality iff Q is the point with barycentric coordinates

$$\frac{\lambda_1}{\lambda}, \frac{\lambda_2}{\lambda}, \ldots, \frac{\lambda_{n+1}}{\lambda} \tag{13}$$

in the simplex A.

Proof. Let P be the point with barycentric coordinates (13) in the simplex A, and let Q be the origin. Then

$$\underline{P} = \frac{1}{\lambda}\sum_{j=1}^{n+1} \lambda_j \underline{A}_j,$$

and according to 2.23 it follows successively

$$\sum_{i=1}^{n+1} \lambda_i \cdot QA_i^2 \geqslant \sum_{i=1}^{n+1} \lambda_i \cdot PA_i^2 = \sum_{i=1}^{n+1} \lambda_i (\underline{A}_i - \underline{P})^2 =$$

$$= \sum_{i=1}^{n+1} \lambda_i \left[\frac{1}{\lambda}\left(\sum_{j=1}^{n+1} \lambda_j\right)\underline{A}_i - \frac{1}{\lambda}\sum_{j=1}^{n+1} \lambda_j \underline{A}_j\right]^2 =$$

$$= \frac{1}{\lambda^2}\sum_{i=1}^{n+1} \lambda_i \left(\sum_{\substack{j=1\\ j\neq i}}^{n+1} \lambda_j \cdot \overrightarrow{A_iA_j}\right)^2 =$$

$$= \frac{1}{\lambda^2}\sum_{i=1}^{n+1} \lambda_i \left(\sum_{\substack{j=1\\ j\neq i}}^{n+1} \lambda_j \cdot A_iA_j^2 + \sum_{\substack{j,k=1\\ j,k\neq i\\ j<k}}^{n+1} 2\lambda_j\lambda_k \cdot \overrightarrow{A_iA_j} \cdot\right.$$

$$\left. \cdot \overrightarrow{A_iA_k}\right) = \frac{1}{\lambda^2}\left[\sum_{i=1}^{n+1}\sum_{\substack{j=1\\ j\neq i}}^{n+1} \lambda_i\lambda_j^2 a_{ij}^2 + \right.$$

$$\left. + \sum_{i=1}^{n+1}\sum_{\substack{j,k=1\\ j,k\neq i\\ j<k}}^{n+1} \lambda_i\lambda_j\lambda_k (a_{ij}^2 + a_{ik}^2 - a_{jk}^2)\right] =$$

$$= \frac{1}{\lambda^2}\left[\sum_{\substack{i,j=1\\ i<j}}^{n+1} \lambda_i\lambda_j(\lambda_i + \lambda_j)a_{ij}^2 + \right.$$

$$\left. + \sum_{\substack{i,j,k=1\\ i<j<k}}^{n+1} \lambda_i\lambda_j\lambda_k (a_{ij}^2 + a_{ik}^2 + a_{jk}^2)\right].$$

2.25. $$\sum_{i=1}^{n+1} \lambda_i R_i \leqslant R \quad \{P = O\}.$$

Proof. From 2.23 (with $Q = O$) it follows by (4)

$$\sum_{i=1}^{n+1} \lambda_i R_i^2 \leqslant R^2,$$

and applying Cauchy's inequality, we get

$$R^2 \geqslant \left(\sum_{i=1}^{n+1} \lambda_i\right)\left(\sum_{i=1}^{n+1} \lambda_i R_i^2\right) \geqslant \left(\sum_{i=1}^{n+1} \lambda_i R_i\right)^2.$$

G. A. Tsintsifas (private communication).

2.26. $$\sum_{i=1}^{n+1} GA_i \leqslant (n + 1)\sqrt{R^2 - OG^2} \quad \{m_1 = m_2 = \ldots = m_{n+1}\}.$$

Proof. According to a result from the proof of 2.23 we have with $P = G$, $Q = O$

$$OG^2 = \frac{1}{n + 1} \sum_{i=1}^{n+1} OA_i^2 - \frac{1}{n + 1} \sum_{i=1}^{n+1} GA_i^2,$$

and hence because of $OA_i = R$ $(i = 1, 2, \ldots, n + 1)$

$$\frac{1}{n + 1} \sum_{i=1}^{n+1} GA_i^2 = R^2 - OG^2.$$

Therefore, it follows

$$\frac{1}{n + 1} \sum_{i=1}^{n+1} GA_i \leqslant \sqrt{\frac{1}{n + 1} \sum_{i=1}^{n+1} GA_i^2} = \sqrt{R^2 - OG^2}.$$

G. Kalajdžić [43] (n = 3).

2.27. $$\sum_{\substack{i,j=1 \\ i<j}}^{n+1} \lambda_i \lambda_j a_{ij} \leqslant \sqrt{\frac{n}{2(n + 1)}}\, R \quad \{m_1 = m_2 = \ldots = m_{n+1},\ P = G\}.$$

Proof. From 2.22 with $Q = O$ it follows by (4)

$$\sum_{\substack{i,j=1 \\ i<j}}^{n+1} \lambda_i \lambda_j a_{ij}^2 \leqslant R^2 \quad \{P = O\}.$$

By applying Cauchy's inequality we obtain

$$\left(\sum_{\substack{i,j=1\\ i<j}}^{n+1} \lambda_i\lambda_j\right)R^2 \geqslant \left(\sum_{\substack{i,j=1\\ i<j}}^{n+1} \lambda_i\lambda_j\right)\left(\sum_{\substack{i,j=1\\ i<j}}^{n+1} \lambda_i\lambda_j a_{ij}^2\right) \geqslant$$

$$\geqslant \left(\sum_{\substack{i,j=1\\ i<j}}^{n+1} \lambda_i\lambda_j a_{ij}\right)^2.$$

But, from (4) by an inequality for symmetric means (AI 2.15.1, Theorem 4) we get

$$\sum_{\substack{i,j=1\\ i<j}}^{n+1} \lambda_i\lambda_j \leqslant \binom{n+1}{2}\left(\frac{1}{n+1}\sum_{i=1}^{n+1} \lambda_i\right)^2 = \frac{n}{2(n+1)} \quad \{P = G\}.$$

Therefore, it follows

$$\frac{n}{2(n+1)} R^2 \geqslant \left(\sum_{\substack{i,j=1\\ i<j}}^{n+1} \lambda_i\lambda_j a_{ij}\right)^2 \quad \{P = G = O\}.$$

But, $\{P = G = O\}$ is equivalent to $\{m_1 = m_2 = \dots = m_{n+1}, P = G\}$.

G.A. Tsintsifas (private communication).

2.28. $$\sum_{i=1}^{n+1} \frac{R_i}{xR_i + r_i} \geqslant \sum_{i=1}^{n+1} \frac{R_i}{xR_i + r_i'} \geqslant \frac{n(n+1)}{nx+1} \quad (0 \leqslant x \leqslant 1)$$

with $\{A_R, P = O\}$ for $x \in [0, 1)$ and $\{A_O, P = H\}$ for $x = 1$.

O. Reutter [86]; F. Leuenberger [68] ($n = 3$, $x \in \{0, \frac{1}{3}, 1\}$); GI 12.38, 12.36 ($n = 2$, $x \in \{0, 1\}$).

2.29. $$\sum_{i=1}^{n+1} F_iR_i \geqslant n \sum_{i=1}^{n+1} F_ir_i \quad \{A_O, P = H\}.$$

Proof. As we have the equalities $F_ih_i = nV$ ($i = 1, 2, \dots, n+1$),

(14) $$\sum_{i=1}^{n+1} F_ir_i = nV,$$

from the inequalities

$$R_i + r_i \geqslant h_i \quad (i = 1, 2, \dots, n+1) \quad \{A_iB_i \perp a_i\}$$

follows successively

$$\sum_{i=1}^{n+1} F_iR_i \geqslant \sum_{i=1}^{n+1} F_ih_i - \sum_{i=1}^{n+1} F_ir_i = (n + 1)nV - nV =$$

$$= n^2V = n \sum_{i=1}^{n+1} F_ir_i .$$

J. Schopp [94]; V. Thébault [121] (n = 3), [122] (n = 3); D. O. Škljarskij, N. N. Čencov, and I. M. Jaglom [101, p. 59, 306] (n = 3); [153] (n = 3); J. B. Tabov [115] (n = 3); GI 12.19 (n = 2).

Remark 1. In the case of the special simplex A with $F_1 = F_2 = \ldots = F_{n+1}$ we obtain an analogous of the Erdős-Mordell inequality (GI 12.13) in the form

2.29' $$\sum_{i=1}^{n+1} R_i \geqslant n \sum_{i=1}^{n+1} r_i \quad \{A_O, P = H\}.$$

Remark 2. With

$$F_1 = F_2 = \ldots = F_{n+1} = \frac{F}{n + 1} = \frac{nV}{(n + 1)r}$$

it follows by (14)

$$\sum_{i=1}^{n+1} r_i = (n + 1)r$$

and in the case of a regular simplex A with nr = R we get from 2.29

2.29'' $$\sum_{i=1}^{n+1} R_i \geqslant (n + 1)R \quad \{A_R, P = O\}.$$

E. A. Morozova and I. S. Petrakov [75] (n = 3); D. M. Bătineţu [6] (n = 3).

2.30. $$\prod_{i=1}^{n+1} F_ir_i \leqslant \left(\frac{n}{n + 1}\right)^{n+1} V^{n+1} \quad \{P = G\}.$$

Proof. It follows from (14) by applying the A-G-inequality for means.

D. O. Škljarskij, N. N. Čencov, and I. M. Jaglom [101, p. 50, 281, 282-283] (n ∈ {2, 3}); GI 12.29 (n = 2).

2.31. $$\sum_{i=1}^{n+1} \frac{F_i}{r_i} \geqslant \frac{F^2}{nV} \quad \{P = I\}.$$

Proof. We have by (14)

$$nV \sum_{i=1}^{n+1} \frac{F_i}{r_i} = \left(\sum_{i=1}^{n+1} F_i r_i\right)\left(\sum_{i=1}^{n+1} \frac{F_i}{r_i}\right) =$$

$$= \sum_{i=1}^{n+1} F_i^2 + \sum_{\substack{i,j=1\\ i<j}}^{n+1} F_i F_j \left(\frac{r_i}{r_j} + \frac{r_j}{r_i}\right) \geqslant$$

$$\geqslant \sum_{i=1}^{n+1} F_i^2 + 2 \sum_{\substack{i,j=1\\ i<j}}^{n+1} F_i F_j = \left(\sum_{i=1}^{n+1} F_i\right)^2 = F^2.$$

[156].

2.32. $$\sum_{i=1}^{n+1} \frac{x_i}{r_i^p} \geqslant \frac{1}{(nV)^p}\left[\sum_{i=1}^{n+1} (x_i F_i^p)^{\frac{1}{p+1}}\right]^{p+1}$$ $(x_1, x_2, \ldots, x_{n+1} \in R^+ \cup \{0\},\ p > 0)$

with equality only in the case of proportionality of sequences

$$r_i, \left(\frac{x_i}{F_i}\right)^{\frac{1}{p+1}} \quad (i = 1, 2, \ldots, n + 1).$$

Proof. Applying Hölder's inequality to the sequences

$$(r_i F_i)^{\frac{1}{p+1}}, \frac{x_i^{\frac{1}{p+1}}}{r_i^{\frac{p}{p+1}}} \quad (i = 1, 2, \ldots, n + 1)$$

it follows

$$\left(\sum_{i=1}^{n+1} r_i F_i\right)^{\frac{p}{p+1}}\left(\sum_{i=1}^{n+1} \frac{x_i}{r_i^p}\right)^{\frac{1}{p+1}} \geqslant \sum_{i=1}^{n+1} \left(x_i F_i^p\right)^{\frac{1}{p+1}}$$

and by (14) we obtain 2.32.

M. S. Klamkin [52]; T. Andreescu and I. V. Maftei [4] ($n = 3$, $p = 1$, $x_i = F_i$).

2.33. For every $p \in (\frac{3}{2}, +\infty)$ and every $x_1, x_2, \ldots, x_{n+1} \in R^+ \cup \{0\}$ we have

$$\sum_{i=1}^{n+1} x_i r_i^{\frac{1}{p}} \leqslant \left(\sum_{i=1}^{n+1} x_i^{\frac{2p}{2p-3}}\right)^{\frac{2p-3}{2p}} \left[\frac{(n+1)^3}{n^2} R^2\right]^{\frac{1}{2p}} \quad \{A_R,\ P = O,\ x_1 = x_2 = \dots = x_{n+1}\}.$$

Proof. Applying of Hölder's inequality to the sequences

$$\left(\frac{r_i}{h_i}\right)^{\frac{1}{p}},\ x_i h_i^{\frac{1}{p}} \quad (i = 1, 2, \dots, n+1)$$

respectively,

$$x_i^q,\ h_i^{\frac{q}{p}} \quad (i = 1, 2, \dots, n+1)$$

we obtain

$$(15) \qquad \left(\sum_{i=1}^{n+1} \frac{r_i}{h_i}\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n+1} x_i^q h_i^{\frac{q}{p}}\right)^{\frac{1}{q}} \geqslant \sum_{i=1}^{n+1} x_i r_i^{\frac{1}{p}},$$

$$(16) \qquad \left(\sum_{i=1}^{n+1} x_i^{\frac{2p}{2p-3}}\right)^{\frac{(2p-3)q}{2p}} \left(\sum_{i=1}^{n+1} h_i^2\right)^{\frac{q}{2p}} \geqslant \sum_{i=1}^{n+1} x_i^q h_i^{\frac{q}{p}},$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Equalities occur iff the sequences

$$\frac{r_i}{h_i},\ x_i^q h_i^{\frac{q}{p}} \quad (i = 1, 2, \dots, n+1)$$

respectively the sequences

$$x_i^{\frac{2p}{2p-3}},\ h_i^2 \quad (i = 1, 2, \dots, n+1)$$

are proportional. By (3) and (4) we have

$$\sum_{i=1}^{n+1} \frac{r_i}{h_i} = 1$$

and from (15) and (16) it follows

$$(17) \qquad \sum_{i=1}^{n+1} x_i r_i^{\frac{1}{p}} \leqslant \left(\sum_{i=1}^{n+1} x_i^{\frac{2p}{2p-3}}\right)^{\frac{2p-3}{2p}} \left(\sum_{i=1}^{n+1} h_i^2\right)^{\frac{1}{2p}}$$

with equality iff the sequences

$$r_i,\ x_i^{\frac{3p}{2p-3}},\ h_i^3 \quad (i = 1, 2, \ldots, n + 1)$$

are proportional. Indeed, because of $q = \frac{p}{p-1}$ from

$$\frac{r_i}{h_i} = \alpha x_i^q h_i^{\frac{q}{p}},\ x_i^{\frac{2p}{2p-3}} = \beta h_i^2 \quad (i = 1, 2, \ldots, n + 1)$$

it follows

$$r_i = \alpha\beta^{-\frac{q}{2}} x_i^{\frac{3p}{2p-3}} \quad \alpha\beta^{\frac{2p-3}{2(p-1)}} h_i^3 \quad (i = 1, 2, \ldots, n + 1).$$

According to 2.50 we have

$$\sum_{i=1}^{n+1} h_i^2 \leqslant \frac{(n+1)^3}{n^2} R^2 \quad \{A_R\}.$$

Therefore, from (17) it follows 2.33 with equality iff A is a regular simplex and further h_i = const., r_i = const., x_i = const.

M. S. Klamkin [51].

2.34. $$\prod_{i=1}^{n+1} r_i \leqslant (\frac{R}{n})^{n+1} \quad \{A_R,\ P = O\}.$$

Proof. Follows from 2.33 with $x_1 = x_2 = \ldots = x_{n+1} = 1$, $p = 2$ according to the A-G-inequality for the numbers $\sqrt{r_i}$ $(i = 1, 2, \ldots, n + 1)$.

L. Gerber [29].

2.35. $$\min(h_1, h_2, \ldots, h_{n+1}) \leqslant \sum_{i=1}^{n+1} r_i \leqslant$$
$$\leqslant \max(h_1, h_2, \ldots, h_{n+1}) \quad \{h_1 = h_2 = \ldots = h_{n+1}\}.$$

Proof. For every $i \in \{1, 2, \ldots, n + 1\}$ let V_i be the volume of the simplex $A_1 \ldots A_{i-1}PA_{i+1} \ldots A_{n+1}$. Then we have $V_1 + V_2 + \ldots + V_{n+1} = V$ and $h_iV_i = r_iV$ $(i = 1, 2, \ldots, n + 1)$. Therefore, 2.35 follows from

$$\min(h_1, h_2, \ldots, h_{n+1}) \sum_{i=1}^{n+1} V_i \leqslant \sum_{i=1}^{n+1} h_i V_i \leqslant$$

$$\leqslant \max(h_1, h_2, \ldots, h_{n+1}) \sum_{i=1}^{n+1} V_i .$$

M. Stanković [106]; L. M. Lopovok and V. Crişan [70] (n = 3); M. Stanković [105] (n = 3); D. O. Škljarskij, N. N. Čencov, and I. M. Jaglom [101, p. 281-282] (n = 3); F. S. Pirvanescu [82] (n = 3, an affine generalization); GI 12.9 (n = 2).

Remark. V. Crişan in [70] also has an affine generalization in the case n = 3.

2.36. For any i, j ∈ {1, 2, ..., n + 1} (i < j) let P_{ij} be the point symmetric to the point P with respect to the inner hyperplane of symmetry of hyperplanes a_i and a_j. For any i, j, k ∈ {1, 2, ..., n + 1} (i < j) let $r_{ij,k}$ be the distance of the point P_{ij} to the hyperplane a_k (especially, we have $r_{ij,i} = r_j$, $r_{ij,j} = r_i$). Then we have

$$\sum_{i=1}^{n+1} R_k \geqslant 2 \sum_{k=1}^{n+1} r_k + \frac{2}{n(n-1)} \sum_{\substack{i,j,k,h=1 \\ i<j \\ k,h\neq i,j \\ h\neq k}}^{n+1} \frac{F_h}{F_k} r_{ij,h}$$

with equality iff $F_1 = F_2 = \ldots = F_{n+1}$ and if for every i, j, k ∈ {1, 2, ..., n + 1} (i < j, k ≠ i, j) we have $P_{ij}A_k \perp a_k$.

Proof. For any i, j ∈ {1, 2, ..., n + 1} (i < j)

$$\sum_{h=1}^{n+1} F_h r_{ij,h} = nV,$$

i.e.

$$\frac{F_i}{F_k} r_j + \frac{F_j}{F_k} r_i + r_{ij,k} + \sum_{\substack{h=1 \\ h\neq i,j,k}}^{n+1} \frac{F_h}{F_k} r_{ij,h} = h_k \quad (i, j, k \in \{1, 2, \ldots, n+1\};\ i < j;\ k \neq i, j).$$

On the other hand, we have obviously

$$R_k + r_{ij,k} \geqslant h_k \quad (i, j \in \{1, 2, \ldots, n+1\};\ i < j;\ k \neq i, j)$$

with equality iff $P_{ij}A_k \perp a_k$. Hence

$$R_k \geqslant \frac{F_i}{F_k} r_j + \frac{F_j}{F_k} r_i + \sum_{\substack{h=1 \\ h\neq i,j,k}}^{n+1} \frac{F_h}{F_k} r_{ij,h} \quad (i, j, k \in \{1, 2, \ldots, n+1\};\ i < j;\ k \neq i, j)$$

with the same conditions for equality. By addition of these $\frac{1}{2}(n + 1)n(n - 1)$ inequalities, application of inequalities

$$\frac{F_p}{F_q} + \frac{F_q}{F_p} \geqslant 2 \quad (p, q \in \{1, 2, \ldots, n + 1\}, p < q),$$

and division of the obtained inequality by $n(n - 1)$, it follows finally 2.33.

V. Thébault [123] ($n = 3$); GI 12.13 ($n = 2$).

2.37. If a is (the unique) positive solution of the equation

$$x \sum_{i=1}^{n+1} \frac{1}{x + R_i^2} = 1,$$

then

$$V \leqslant \frac{1}{n!\sqrt{a}} \sum_{i=1}^{n+1} \sqrt{a + R_i^2} \quad \{A_O, P = H\}.$$

D. Slepian [102]; M. M. Ali [3]; L. Gerber [29].

2.38. $$V \leqslant \frac{1}{n!} R^n \sqrt{\frac{(n + 1)^{n+1}}{n^n}} \quad \{A_R\}.$$

Proof. Follows from 3.2 and 3.7, or from 2.37 ($P = O$), or from 2.44 ($P = O$).

L. Frejes Tóth [23, p. 313]; M. M. Ali [3]; L. Gerber [29]; G. Krammer [61] ($n = 3$); A. Heppes [37] ($n = 3$); G. Kalajdžić [43] ($n = 3$); D. O. Škljarskij, N. N. Čencov, and I. M. Jaglom [101, p. 46] ($n = 3$); GI 7.9 ($n = 2$).

2.39. If a is (the unique) positive solution of the equation

$$x \sum_{i=1}^{n+1} \frac{1}{x + r_i^2} = 1,$$

then

$$V \geqslant \frac{n^n}{n!\sqrt{a}} \sum_{i=1}^{n+1} \sqrt{a + r_i^2} \quad \{A_O, P = H'\}.$$

where H' is the image of H in the homothety with centre G and coefficient $-\frac{1}{n}$.

D. Slepian [102]; M. M. Ali [3]; L. Gerber [29].

2.40. $$V \geqslant \frac{1}{n!} r^n \sqrt{n^n (n+1)^{n+1}} \quad \{A_R\}.$$

Proof. Follows from 2.39 with P = I, or from 3.10 because of $Fr = nV$.

L. Fejes Tóth [23, p. 291]; M. M. Ali [3]; L. Gerber [29]; J. Zh. Zhang and L. Yang [148]; D. O. Škljarskij, N. N. Čencov, and I. M. Jaglom [101, p. 46] (n = 3); GI 7.9 (n = 2).

2.41. $$V \leqslant \frac{1}{n!} \sqrt{\frac{(n+1)^{n+1}}{n^n}} \left[M_{n+1}^{[p]}(R_i)\right]^n \quad (p > 0) \ \{A_R,\ P = O,\ p \geqslant p_n\},$$

where

$$p_n = 2n \frac{\ln(n+1) - \ln n}{(n+1)\ln(n+1) - n \ln n}.$$

L. Gerber [29]; C. M. Petty and D. Waterman [79] (p = 1); GI 12.18 (n = 2, p = 1).

Remark. For $n \geqslant 2$ we have $p_n < 1$.

2.42. $$M_{n+1}^{[p]}(R_i) \geqslant nr \quad (p > 0) \ \{A_R,\ P = O,\ p \geqslant p_n\},$$

where p_n is defined as in 2.41.

Proof. Follows by 2.40 and 2.41.

L. Gerber [29]; J. Berkes [9] (p = 1); E. A. Morozova and I. S. Petrakov [75] (n = 3, p = 1, A regular tetrahedron); D. M. Bătineţu [6] (n = 3, p = 1, A regular tetrahedron); GI 12.14 (n = 2, p = 1).

2.43. If P is an internal or boundary point of A, then

$$V \leqslant \frac{1}{n!}\left(\frac{n+1}{n}\right)^{\frac{n}{p}} \left[M_{n+1}^{[p]}(R_i)\right]^n \quad (p > 0).$$

Equality occurs if P is a vertex of A and if the edges from this vertex are mutually orthogonal and equal. If $p < p_n$ (p_n is defined in 2.41), then equality occurs only in such a case.

L. Gerber [29].

Remark. If $p = p_n$, then 2.41 and 2.43 give the same upper bound for V.

2.44. For any $s \in \{2, \ldots, n+1\}$ and any $i_1, i_2, \ldots, i_s \in \{1, 2, \ldots, n+1\}$ ($i_1 < i_2 < \ldots < i_s$) let $V_{i_1,\ldots,i_s}$ be the content of the (s - 1)-

simplex $A_{1_i} A_{i_2} \dots A_{i_s}$. Then

$$\sum_{\substack{i_1,\dots,i_s=1 \\ i_1<\dots<i_s}}^{n+1} V^{\lambda}_{i_1,\dots,i_s} \leqslant$$

$$\leqslant \binom{n+1}{s}\left[\frac{\sqrt{s}}{(s-1)!}\left(\frac{1}{n}\sum_{i=1}^{n+1} R_i^2\right)^{\frac{s-1}{2}}\right]^{\lambda} \quad (0 < \lambda \leqslant 2)\ \{A_R,\ P = O\}.$$

Proof. Follows from 3.8, because by 2.22 (with $Q = P = G$)

$$\sum_{\substack{i,j=1 \\ i<j}}^{n+1} a_{ij}^2 = (n+1)\sum_{i=1}^{n+1} GA_i^2,$$

and according to 2.23 (with $P = G$ and with substitution $Q \to P$) we have

$$\sum_{i=1}^{n+1} GA_i^2 \leqslant \sum_{i=1}^{n+1} R_i^2 \quad \{P = G\}.$$

D. O. Škljarskij, N. N. Čencov, and I. M. Jaglom [101, p. 49, 280] ($n = 3$, $\lambda = 1$, $s \in \{2, 3\}$); O. Chisini [14] ($n = 3$, $\lambda = 1$, $s \in \{2, 3\}$, $P = O$); B. R. Venkataraman [135] ($n = 3$, $\lambda = 1$, $s = 2$, $P = O$); GI 12.53 ($n = s = \lambda = 2$).

2.45. $$V \geqslant \frac{1}{n!}\sqrt{n^n(n+1)^{n+1}}\left[M_{n+1}^{[a]}(r_i)\right]^n \quad (a \leqslant 0)\ \{A_R,\ P = O\}.$$

Proof. Follows from 2.30 and 3.14 using the inequality

$$M_{n+1}^{[0]}(r_i) \geqslant M_{n+1}^{[a]}(r_i).$$

L. Gerber [29].

2.46. If V' is the volume of simplex $\mathcal{B} = B_1B_2 \dots B_{n+1}$, then

$$V' \leqslant \frac{1}{n^n} V \quad \{P = G\}.$$

Proof. For any $i \in \{1, 2, \dots, n+1\}$ let V_i and V_i' be the volumes of the simplexes $A_1 \dots A_{i-1}PA_{i+1} \dots A_{n+1}$ and $B_1 \dots B_{i-1}PB_{i+1} \dots B_{n+1}$. We have

$$\frac{V_i'}{V_i} = \prod_{\substack{j=1\\ j\neq i}}^{n+1} \frac{PB_j}{A_jP}, \quad \frac{V_i}{V} = \frac{PB_i}{A_iB_i} = \lambda_i \quad (i = 1, 2, \ldots, n+1),$$

and by (5) it follows

$$\frac{V'}{V} = \sum_{i=1}^{n+1} \frac{V_i'}{V} = \sum_{i=1}^{n+1}\left[\left(\prod_{j=1}^{n+1} \frac{PB_j}{A_jP}\right) \cdot \frac{A_iP}{A_iB_i}\right] =$$

$$= \left(\prod_{j=1}^{n+1} \frac{PB_j}{A_jP}\right)\left[\sum_{i=1}^{n+1} (1 - \lambda_i)\right] = n\left(\sum_{i=1}^{n+1} \frac{PB_i}{A_iP}\right).$$

But, according to 2.9 we have

$$\prod_{i=1}^{n+1} \frac{PB_i}{A_iP} \leqslant \frac{1}{n^{n+1}} .$$

O. Reutter and F. Leuenberger [87]; G. Tsintsifas 1967 (private communication); M. S. Klamkin [47]; L. Stojanov [110] (n = 3).

2.47. For any $i \in \{1, 2, \ldots, n+1\}$ let P_i be the point of the half-line A_iP such that $A_iP_i : A_iP = 1:\alpha_i$, where $\alpha_1, \alpha_2, \ldots, \alpha_{n+1} \in R^+$ are given numbers. The hyperplane through P_i parallel to hyperplane a_i intersects the line A_iA_j in a point P_{ij}, where $j \in \{1, 2, \ldots, n+1\}\backslash \{i\}$. For any $i \in \{1, 2, \ldots, n+1\}$ let V_i be the volume of the simplex $P_{i1} \ldots P_{i,i-1}A_iP_{i,i+1} \ldots P_{i,n+1}$. Then

$$\sum_{i=1}^{n+1} V_i \geqslant \frac{n^n}{\left(\sum_{i=1}^{n+1} \alpha_i^{\frac{n}{n-1}}\right)^{n-1}} V$$

with equality iff

$$(18) \qquad \lambda_i = 1 - \frac{n\alpha_i^{\frac{n}{n-1}}}{\sum_{j=1}^{n+1} \alpha_j^{\frac{n}{n-1}}} \qquad (i = 1, 2, \ldots, n+1).$$

(If $\alpha_1 = \alpha_2 = \ldots = \alpha_{n+1}$, then $P = G$.)

Proof. From

$$\frac{A_iP}{A_iB_i} = 1 - \lambda_i \quad (i = 1, 2, \ldots, n+1)$$

it follows

$$\frac{A_iP_i}{A_iB_i} = \frac{1 - \lambda_i}{\alpha_i} = \frac{A_iP_{ij}}{A_iA_j} \quad (i, j = 1, 2, \ldots, n + 1; i \neq j).$$

Therefore, we obtain

$$\frac{V_i}{V} = \prod_{\substack{j=1 \\ j\neq i}}^{n+1} \frac{A_iP_{ij}}{A_iA_j} = \left(\frac{1 - \lambda_i}{\alpha_i}\right)^n \quad (i = 1, 2, \ldots, n + 1).$$

But, by Hölder's inequality we have

$$\left[\sum_{i=1}^{n+1} \left(\frac{1 - \lambda_i}{\alpha_i}\right)^n\right]^{\frac{1}{n}} \left(\sum_{i=1}^{n+1} \alpha_i^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}} \geqslant \sum_{i=1}^{n+1} (1 - \lambda_i),$$

and by (5) it follows 2.47. Equality occurs iff

$$\left(\frac{1 - \lambda_i}{\alpha_i}\right)^n = k^n \alpha_i^{\frac{n}{n-1}} \quad (i = 1, 2, \ldots, n + 1),$$

i.e.

$$1 - \lambda_i = k\alpha_i^{\frac{n}{n-1}} \quad (i = 1, 2, \ldots, n + 1),$$

where k = const. Because of (5) we get further

$$k \sum_{i=1}^{n+1} \alpha_i^{\frac{n}{n-1}} = n,$$

and from the last two equalities it follows (18).

M. S. Klamkin [50]; G. A. Tsintsifas 1967 (private communication) ($\alpha_1 = \alpha_2 = \ldots = \alpha_{n+1} = 2$); GI 14.30 ($n = 2$, $\alpha_1 = \alpha_2 = \alpha_3 = 2$).

2.48. Let P be a point inside the circumscribed hypersphere of A. For any $i \in \{1, 2, \ldots, n + 1\}$ let A_i' be the second point of intersection of the line A_iP with this hypersphere. If

$$K = \sum_{i=1}^{n+1} \frac{A_iP}{PA_i'},$$

then

$$K \lesseqgtr n + 1$$

when P is a point inside, on or outside the hypersphere with diameter $\overline{OG}$.
G. A. Tsintsifas [130].

2.49. $$\frac{n+1}{n^2} \sum_{\substack{i,j=1\\i<j}}^{n+1} a_{ij} < \sum_{i=1}^{n+1} m_i < \frac{2}{n} \sum_{\substack{i,j=1\\i<j}}^{n+1} a_{ij}.$$

Proof. We have

$$\sum_{i=1}^{n+1} m_i = \frac{1}{n} \sum_{\substack{i,j=1\\i<j}}^{n+1} (m_i + m_j) = \frac{1}{n} \sum_{\substack{i,j=1\\i<j}}^{n+1} \left(\frac{n+1}{n} GA_i + \frac{n+1}{n} GA_j\right) =$$

$$= \frac{n+1}{n^2} \sum_{\substack{i,j=1\\i<j}}^{n+1} (GA_i + GA_j) > \frac{n+1}{n^2} \sum_{\substack{i,j=1\\i<j}}^{n+1} a_{ij}.$$

Because of Cauchy's inequality we obtain for any $i \in \{1, 2, \ldots, n+1\}$

$$\left(\sum_{\substack{j=1\\j\neq i}}^{n+1} a_{ij}\right)^2 - n^2 m_i^2 = \left(\sum_{\substack{j=1\\j\neq i}}^{n+1} a_{ij}\right)^2 - \left(\sum_{\substack{j=1\\j\neq i}}^{n+1} \overrightarrow{A_iA_j}\right)^2 =$$

$$= \left(\sum_{\substack{j,k=1\\j,k\neq i}}^{n+1} a_{ij}a_{ik}\right) - \left(\sum_{\substack{j,k=1\\j,k\neq i}}^{n+1} \overrightarrow{A_iA_j} \cdot \overrightarrow{A_iA_k}\right) =$$

$$= \sum_{\substack{j,k=1\\j,k\neq i}}^{n+1} \left(\sqrt{A_iA_j^2}\sqrt{A_iA_k^2} - \overrightarrow{A_iA_j} \cdot \overrightarrow{A_iA_k}\right) > 0,$$

i.e.

$$m_i < \frac{1}{n} \sum_{\substack{j=1\\j\neq i}}^{n+1} a_{ij} \quad (i = 1, 2, \ldots, n+1).$$

Therefore, it follows

$$\sum_{i=1}^{n+1} m_i < \frac{1}{n} \sum_{i=1}^{n+1} \sum_{\substack{j=1\\j\neq i}}^{n+1} a_{ij} = \frac{2}{n} \sum_{\substack{i,j=1\\i<j}}^{n+1} a_{ij}.$$

R. Robinson and J. K. Peterson [88]; E. Piccioli [81] ; M. S. Klamkin (private communication); V. Thébault [118] (n = 3); GI 8.1 (n = 2).

2.50. $$(n + 1)r \leqslant M_{n+1}^{[-1]}(t_i) \leqslant M_{n+1}^{[k]}(t_i) \leqslant M_{n+1}^{[2]}(t_i) \leqslant$$

$$\leqslant \frac{n + 1}{n} R \quad (-1 < k < 2,\ t \in \{h, m\}).$$

For t = h the first equality occurs always and the last equality occurs iff A is a regular simplex. For t = m the first equality occurs iff A is a regular simplex. All other equalities occur iff $t_1 = t_2 = \ldots = t_{n+1}$.

Proof. Obviously, $h_i \leqslant m_i$ (i = 1, 2, ..., n + 1) with equalities iff A is an orthocentric simplex and G = H. But, in this case it can be easily proved (by induction on n) that A is a regular simplex. Hence, it suffices to prove

$$(n + 1)r = M_{n+1}^{[-1]}(h_i), \qquad M_{n+1}^{[2]}(m_i) \leqslant \frac{n + 1}{n} R.$$

But, the first equality is the equality (5) in 1. On the other hand, from

$$\sum_{i=1}^{n+1} \vec{GA_i} = \vec{0},$$

(19) $$GA_i = \frac{n}{n + 1} m_i \qquad (i = 1, 2, \ldots, n + 1)$$

it follows

$$(n + 1)R^2 = \sum_{i=1}^{n+1} OA_i^2 = \sum_{i=1}^{n+1} (\vec{OG} + \vec{GA_i})^2 =$$

$$= (n + 1)OG^2 + 2\,\vec{OG} \cdot \sum_{i=1}^{n+1} \vec{GA_i} + \sum_{i=1}^{n+1} GA_i^2 =$$

$$= (n + 1)OG^2 + \left(\frac{n}{n + 1}\right)^2 \sum_{i=1}^{n+1} m_i^2,$$

i.e.

$$\left[M_{n+1}^{[2]}(m_i)\right]^2 = \frac{1}{n + 1} \sum_{i=1}^{n+1} m_i^2 \leqslant \left(\frac{n + 1}{n} R\right)^2.$$

Equality appears iff O = G, i.e. iff $m_1 = m_2 = \ldots = m_{n+1}$.

F. Leuenberger [66]; J. Schopp [97]; J. Berkes and F. Leuenberger [10]; G. Kalajdžić [43] ($n = 3$); [156] ($n = 3$); GI 6.8, 6.16, 8.3 ($n = 2$).

2.51. $R \geqslant nr \quad \{A_R\}$.

Proof. Included in 2.50.

P. K. Kashikar [45]; L. Fejes Tóth [21, p. 188], [22], [23, p. 313]; F. Leuenberger [66]; M. S. Klamkin and G. A. Tsintsifas [58] (the proof is valid only in the case when O lies inside A); Moret-Blanc [74] ($n = 3$); H. Laurent [65] ($n = 3$); B. R. Venkataraman [135] ($n = 3$); D. O. Škljarskij, N. N. Čencov, and I. M. Jaglom [101, p. 42, 244-245] ($n = 3$); Ja. N. Sukonnik [113] ($n = 3$, A is a triangular regular pyramid); GI 5.1 ($n = 2$).

Remark. Inequality 2.51 follows from 2.52.

2.52. $R^2 \geqslant n^2r^2 + OI^2 \quad \{A_R\}$.

M. S. Klamkin [57].

2.53. If $f:R^+ \to R$ is a non-decreasing concave function and $t \in \{h, m\}$, then

$$\sum_{i=1}^{n+1} f(t_i) \leqslant (n + 1)f\left(\frac{n + 1}{n} R\right),$$

and if f is a non-increasing convex function, then we have the opposite inequality. For $t = h$ the equality occurs iff A is a regular simplex, and for $t = m$ equality occurs iff $m_1 = m_2 = \dots = m_{n+1}$.

Proof. According to 2.50 it follows successively

$$\sum_{i=1}^{n+1} f(t_i) = (n + 1)M_{n+1}^{[1]}(f(t_i)) \lesseqgtr (n + 1)(f(M_{n+1}^{[1]}(t_i)) \lesseqgtr$$

$$\lesseqgtr (n + 1)f\left(\frac{n + 1}{n} R\right).$$

2.54. If $f:R^+ \to R$ is a non-decreasing convex respectively a non-increasing concave function and if $t \in \{h, m\}$, then

$$\sum_{i=1}^{n+1} f(t_i) \gtreqless (n + 1)f((n + 1)r).$$

For $t = h$ equality occurs iff $h_1 = h_2 = \dots = h_{n+1}$, and for $t = m$ equality occurs iff A is a regular simplex.

Proof. Analogously as in 2.53.

2.55. $$\left(\sum_{i=1}^{n+1} h_i\right)\left(\sum_{i=1}^{n+1} m_i^{n-1}\right) \geqslant n!(n+1)^{\frac{n+1}{2}} n^{-\frac{n}{2}} V \qquad \{A_R\}.$$

M. S. Klamkin [56]; D. M. Milošević and P. Bundschuh [73] ($n = 3$).

2.56. Let h be any hyperplane through G which intersects every edge $A_{n+1}A_i$ ($i \in \{1, 2, \ldots, n\}$) in an internal point G_i. If V_{n+1} is the volume of simplex $G_1 \ldots G_n A_{n+1}$, then

$$\frac{V}{V_{n+1}} \leqslant \left(\frac{n+1}{n}\right)^n \qquad \{h \parallel a_{n+1}\}.$$

Proof. If $\mu_1, \mu_2, \ldots, \mu_n$ are the barycentric coordinates of G in the $(n - 1)$-simplex $G_1G_2 \ldots G_n$, then

$$\sum_{i=1}^{n} \mu_i = 1, \quad \overrightarrow{A_{n+1}G} = \sum_{i=1}^{n} \mu_i \cdot \overrightarrow{A_{n+1}G_i}.$$

For any $i \in \{1, 2, \ldots, n\}$ let

$$\overrightarrow{A_{n+1}G_i} = \frac{1}{\nu_i} \overrightarrow{A_{n+1}A_i}.$$

Then it follows

$$\frac{1}{n+1} \sum_{i=1}^{n} \overrightarrow{A_{n+1}A_i} = \overrightarrow{A_{n+1}G} = \sum_{i=1}^{n} \frac{\mu_i}{\nu_i} \overrightarrow{A_{n+1}A_i}$$

and therefore

$$\frac{\mu_i}{\nu_i} = \frac{1}{n+1} \qquad (1 = 1, 2, \ldots, n).$$

Now, we obtain

$$\frac{V}{V_{n+1}} = \prod_{i=1}^{n} \nu_i = (n+1)^n \prod_{i=1}^{n} \mu_i \leqslant$$

$$= (n+1)^n \left(\frac{1}{n} \sum_{i=1}^{n} \mu_i\right)^n = \left(\frac{n+1}{n}\right)^n.$$

Equality occurs iff $\mu_1 = \mu_2 = \ldots = \mu_n = \frac{1}{n}$, i.e. iff $\nu_1 = \nu_2 = \ldots = \nu_n = \frac{n+1}{n}$.

F. Abeles [1].

2.57. For any $i \in \{1, 2, \ldots, n+1\}$ let A_i' be the second point of intersection of the line A_iG with the circumscribed hypersphere Ω of A. If $0 < \alpha \leqslant 2$, then

$$\sum_{i=1}^{n+1} GA_i'^{\alpha} \geqslant \sum_{i=1}^{n+1} GA_i^{\alpha} \quad \{m_1 = m_2 = \ldots = m_{n+1}\}.$$

Proof. The value opposite to the potency of G with respect to Ω is

$$(20) \qquad R^2 - OG^2 = GA_i \cdot GA_i' \quad (i = 1, 2, \ldots, n+1),$$

and in the proof of 2.50 it is proved that

$$(21) \qquad R^2 - OG^2 = \frac{n^2}{(n+1)^3} \sum_{i=1}^{n+1} m_i^2.$$

By (20), (21) and (19) it follows

$$\sum_{i=1}^{n+1} GA_i'^{\alpha} = (R^2 - OG^2)^{\alpha} \sum_{i=1}^{n+1} \frac{1}{GA_i^{\alpha}} =$$

$$= \frac{n^{\alpha}}{(n+1)^{\alpha}}\left(\frac{1}{n+1}\sum_{i=1}^{n+1} m_i^2\right)^{\alpha}\left(\sum_{i=1}^{n+1}\frac{1}{m_i^{\alpha}}\right) \geqslant$$

$$\geqslant \frac{n^{\alpha}}{(n+1)^{\alpha}}\left(\frac{1}{n+1}\sum_{i=1}^{n+1} m_i^{\alpha}\right)^{2}\left(\sum_{i=1}^{n+1}\frac{1}{m_i^{\alpha}}\right) =$$

$$= \frac{n^{\alpha}}{(n+1)^{\alpha+2}}\left(\sum_{i=1}^{n+1} m_i^{\alpha}\right)\left(\sum_{i=1}^{n+1} m_i^{\alpha}\right)\left(\sum_{i=1}^{n+1}\frac{1}{m_i^{\alpha}}\right) \geqslant$$

$$\geqslant \frac{n^{\alpha}}{(n+1)^{\alpha+2}}\left(\sum_{i=1}^{n+1} m_i^{\alpha}\right)(n+1)^2 = \sum_{i=1}^{n+1}\left(\frac{n}{n+1} m_i\right)^{\alpha} =$$

$$= \sum_{i=1}^{n+1} GA_i^{\alpha}.$$

W. Janous (private communication); M. S. Klamkin [54] ($\alpha = 1$).

2.58. With the same conditions as in 2.57 we have

$$\prod_{i=1}^{n+1} GA_i' \geq \prod_{i=1}^{n+1} GA_i \quad \{m_1 = m_2 = \dots = m_{n+1}\}.$$

Proof. By (20), (21) and (19) it follows

$$\prod_{i=1}^{n+1} GA_i' = (R^2 - OG^2)^{n+1} \prod_{i=1}^{n+1} \frac{1}{GA_i} =$$

$$= (\frac{n}{n+1})^{n+1}\left(\frac{1}{n+1}\sum_{i=1}^{n+1} m_i^2\right)^{n+1}\left(\prod_{i=1}^{n+1}\frac{1}{m_i}\right) \geq$$

$$\geq (\frac{n}{n+1})^{n+1}\left(\prod_{i=1}^{n+1} m_i^2\right)\left(\prod_{i=1}^{n+1}\frac{1}{m_i}\right) = \prod_{i=1}^{n+1}(\frac{n}{n+1} m_i) =$$

$$= \prod_{i=1}^{n+1} GA_i.$$

M. S. Klamkin [54].

2.59. With the same conditions as in 2.57 let V' be the volume of the simplex $A_1'A_2' \dots A_{n+1}'$. Then

$$V' \geq V \quad \{m_1 = m_2 = \dots = m_{n+1}\}.$$

Proof. For any $i \in \{1, 2, \dots, n+1\}$ let V_i and V_i' be the volumes of simplexes $A_1 \dots A_{i-1}GA_{i+1} \dots A_{n+1}$ and $A_1' \dots A_{i-1}'GA_{i+1}' \dots A_{n+1}'$. Then $V' = V_1' + V_2' + \dots + V_{n+1}'$,

$$\frac{V_i'}{V_i} = \prod_{\substack{j=1\\ j\neq i}}^{n+1} \frac{GA_j'}{GA_j}, \quad V_i = \frac{V}{n+1} \qquad (i = 1, 2, \dots, n+1),$$

and by (20), (21) and (19) it follows successively

$$\frac{V'}{V} = \frac{1}{n+1}\sum_{i=1}^{n+1}\frac{V_i'}{V_i} = \frac{1}{n+1}\sum_{i=1}^{n+1}\left(\prod_{\substack{j=1\\ j\neq i}}^{n+1}\frac{GA_j'}{GA_j}\right) =$$

$$= \frac{1}{n+1}\sum_{i=1}^{n+1}\left(\prod_{\substack{j=1\\ j\neq i}}^{n+1}\frac{R^2 - OG^2}{GA_j^2}\right) =$$

$$= \frac{1}{n+1} \sum_{i=1}^{n+1} \left[(R^2 - OG^2)^n \cdot GA_i^2 \cdot \prod_{j=1}^{n+1} \frac{1}{GA_j^2} \right] =$$

$$= \frac{(R^2 - OG^2)^n}{n+1} \left(\prod_{j=1}^{n+1} \frac{1}{GA_j^2} \right) \left(\sum_{i=1}^{n+1} GA_i^2 \right) =$$

$$= \left(\prod_{i=1}^{n+1} \frac{1}{m_i^2} \right) \left(\frac{1}{n+1} \sum_{i=1}^{n+1} m_i^2 \right)^{n+1} \geqslant \left(\prod_{i=1}^{n+1} \frac{1}{m_i^2} \right) \left(\prod_{i=1}^{n+1} m_i^2 \right) = 1.$$

M. S. Klamkin [53]; GI 10.2 (n = 2).

2.60. Let A be such a simplex that O is an internal point of it. For any $i \in \{1, 2, \ldots, n+1\}$ let ϕ_i be the angle between the circumscribed hypersphere and the hyperplane a_i. With the same conditions as in 2.4 we have

$$\sum_{\substack{i_1=\ldots=i_k=1 \\ i_1<\ldots<i_k}}^{n+1} f\left(\prod_{j=1}^{k} \frac{1}{\cos \phi_{i_j}} \right) \gtreqless \binom{n+1}{k} f(n^k) \qquad \{A_R\}.$$

Proof. Follows from 2.9' (with P = O) because of equalities

$$(22) \qquad R_i = R, \quad \frac{R}{r_i} = \frac{1}{\cos \phi_i} \quad (i = 1, 2, \ldots, n+1).$$

V. Volenec [139] ($f(x) = x^a$, $a > 0$); J. Schopp [93] ($f(x) = x$, $k = n + 1$); V. Thébault [123] ($f(x) = x$, $n = 3$, $k \in \{1, 4\}$), [125] ($f(x) = x$, $n = 3$, $k = 4$); GI 2.45, 2.47, 2.23 ($f(x) = x$, $n = 2$, $k \in \{1, 2, 3\}$).

2.61. With the same conditions as in 2.4 and 2.60 we have

$$\sum_{\substack{i_1=\ldots=i_k=1 \\ i_1<\ldots<i_k}}^{n+1} f\left(\prod_{j=1}^{k} \frac{1 + \cos \phi_{i_j}}{\cos \phi_{i_j}} \right) \gtreqless \binom{n+1}{k} f((n+1)^k) \qquad \{A_R\}.$$

Proof. Follows from 2.4'' (with P = O) by (22).

2.62. If $h_1 = h_2 = \ldots = h_{n+1} = h$, then

$$\sum_{i=1}^{n+1} \frac{1}{r_i} \geqslant \frac{(n+1)^2}{h} \qquad \{P = I\}.$$

Proof. From (3) and (4) we obtain

$$\sum_{i=1}^{n+1} r_i = h, \tag{23}$$

and therefore it follows

$$h \sum_{i=1}^{n+1} \frac{1}{r_i} = \left(\sum_{i=1}^{n+1} r_i\right)\left(\sum_{i=1}^{n+1} \frac{1}{r_i}\right) \geqslant (n + 1)^2$$

with equality iff $r_1 = r_2 = \ldots = r_{n+1}$.

C. Tănase [116] ($n = 3$, A is a regular tetrahedron).

2.63. Let a be the edge length of a regular simplex A. For any point P we have

$$\sum_{\substack{i,j=1\\ i\leqslant j}}^{n+1} R_iR_j \geqslant \frac{n(n + 2)}{4} a^2 \quad \{P = O\}.$$

Proof. From 2.23 (with $P = O$ and with substitution $Q \to P$) we obtain

$$\sum_{i=1}^{n+1} R_i^2 \geqslant (n + 1)R^2 \quad \{P = O\},$$

and from 2.22 (with $P = Q = O$ for the regular simplex A) it follows

$$R^2 = \frac{n}{2(n + 1)} a^2. \tag{24}$$

Therefore, by 2.29'' we have successively

$$\sum_{\substack{i,j=1\\ i<j}}^{n+1} R_iR_j = \frac{1}{2}\left[\left(\sum_{i=1}^{n+1} R_i\right)^2 + \sum_{i=1}^{n+1} R_i^2\right] \geqslant \frac{1}{2}[(n + 1)^2R^2 +$$

$$+ (n + 1)R^2] = \frac{1}{2}(n + 1)(n + 2)R^2 = \frac{n(n + 2)}{4} a^2.$$

C. Băluţă [5] ($n = 3$).

2.64. If V' is the volume of the pedal simplex $P = P_1P_2 \ldots P_{n+1}$ of P with respect to a regular simplex A, then

$$V' \leqslant \frac{1}{n^n} V \quad \{P = O\}.$$

Proof. For any $i \in \{1, 2, \ldots, n+1\}$ let V_i' be the volume of the simplex $P_1 \ldots P_{i-1}PP_{i+1} \ldots P_{n+1}$. On the other hand, the simplex $A_1 \ldots A_{i-1}OA_{i+1} \ldots A_{n+1}$ has the volume $\frac{V}{n+1}$. If h is the length of altitude of A, then (23) is valid. Moreover, we have

$$\frac{V_i'}{\frac{1}{n+1}V} = \prod_{\substack{j=1\\ j\neq i}}^{n+1} \frac{PP_j}{OA_j} = \left(\frac{n+1}{n}\right)^n \frac{1}{h^n r_i} \prod_{j=1}^{n+1} r_j \quad (i = 1, 2, \ldots, n+1).$$

Therefore, it follows successively

$$\frac{V'}{V} = \sum_{i=1}^{n+1} \frac{V_i'}{V} = \left(\frac{n+1}{nh}\right)^n \left(\prod_{j=1}^{n+1} r_j\right)\left(\frac{1}{n+1} \sum_{i=1}^{n+1} \frac{1}{r_i}\right) \leqslant$$

$$\leqslant \left(\frac{n+1}{nh}\right)^n \left(\prod_{i=1}^{n+1} r_i\right)\left(\prod_{i=1}^{n+1} \frac{1}{r_i}\right)^{\frac{1}{n+1}} = \left(\frac{n+1}{nh}\right)^n \left(\prod_{i=1}^{n+1} r_i\right)^{\frac{n}{n+1}} \leqslant$$

$$\leqslant \left(\frac{n+1}{nh}\right)^n \left(\frac{1}{n+1} \sum_{i=1}^{n+1} r_i\right)^n = \left(\frac{n+1}{nh}\right)^n \left(\frac{h}{n+1}\right)^n = \frac{1}{n^n}.$$

Remark. If P' is the image of P under the homothetic transformation with centre P and the coefficient $\lambda \in R^+$ and if V'' is the volume of P', then we have the more general inequality

$$V'' \leqslant \left(\frac{\lambda}{n}\right)^n V \quad \{P = O\}.$$

G. Kalajdžić [43] ($n = 3$); M. S. Klamkin [46] ($n = 3$, $\lambda \in \{2, 3\}$).

2.65. If E and E' are total edge lengths of simplexes A and P as in 2.64, then

$$\frac{1}{n} E \leqslant E' < \sqrt{\frac{2}{n(n+1)}}\, E \quad \{P = O\}.$$

Proof. Let $\vec{r}_i = \overrightarrow{PP_i}$ ($i = 1, 2, \ldots, n+1$). From $P_iP_j < r_i + r_j$ ($i, j = 1, 2, \ldots, n+1$; $i < j$) it follows by adding

$$(25) \qquad E' < n \sum_{i=1}^{n+1} r_i.$$

All angles $\sphericalangle(\vec{r}_i, \vec{r}_j)$ and all angles $\sphericalangle(\overrightarrow{OA_i}, \overrightarrow{OA_j})$ are equal to an angle ϕ. From

$$\sum_{i=1}^{n+1} \vec{OA_i} = \vec{0}$$

by squaring it follows

$$(n + 1)R^2 + \frac{n(n + 1)}{2} \cdot 2R^2 \cos \phi = 0$$

i.e.

$$\cos \phi = - \frac{1}{n} .$$

Therefore, for any $i, j \in \{1, 2, \ldots, n + 1\}$ $(i < j)$ we have

$$P_iP_j = |\vec{r}_i - \vec{r}_j| = \sqrt{r_i^2 + r_j^2 - 2\vec{r}_i\vec{r}_j} = \sqrt{r_i^2 + r_j^2 + \frac{2}{n} r_ir_j} =$$

$$= \sqrt{\frac{n + 1}{2n}(r_i + r_j)^2 + \frac{n - 1}{2n}(r_i - r_j)^2} \geqslant \sqrt{\frac{n + 1}{2n}}(r_i + r_j)$$

with equality iff $r_i = r_j$. By adding we obtain

$$\text{(26)} \qquad E' \geqslant n\sqrt{\frac{n + 1}{2n}} \sum_{i=1}^{n+1} r_i \qquad \{P = O\}.$$

According to (24) we have

$$E = (n + 1)\sqrt{\frac{n(n + 1)}{2}} R$$

and together with (23) and with the equality

$$R = \frac{n}{n + 1} h$$

it follows that

$$\sum_{i=1}^{n+1} r_i = \frac{1}{n} \sqrt{\frac{2}{n(n + 1)}} E.$$

Therefore, from (25) and (26), the inequalities 2.65 follow.

Remark 1. According to (23) and (25) the inequality $E' < nh$ follows, which cannot be improved. Indeed, it is sufficient to take the point P very close to a vertex of $\mathcal{A}$.

Remark 2. If $\mathcal{P}'$ is the image of $\mathcal{P}$ under the homothetic transformation with centre P and the coefficient $\lambda \in R^+$ and if E'' is the total edge length of $\mathcal{P}'$, then we have the more general inequality

$$\frac{\lambda}{n} E \leqslant E'' < \lambda\sqrt{\frac{2}{n(n+1)}}\, E.$$

M. S. Klamkin [46] ($n = \lambda = 3$).

3. Other Inequalities for a Simplex

We use the notations and formulas from 1 and 2.

3.1. $$\sum_{\substack{i,j=1\\ i<j}}^{n+1} a_{ij}^4 + \sum_{\substack{i,j,k,h=1\\ k,h\neq i,j\\ i<j;k<h}}^{n+1} a_{ij}^2 a_{kh}^2 - 2\sum_{\substack{i,j,k=1\\ j,k\neq i\\ j<k}}^{n+1} a_{ij}^2 a_{ik}^2 \geqslant 0 \quad \{A_0\}.$$

Proof. Inequality 3.1 is equivalent with the obvious inequality

$$\sum_{\substack{i,j,k,h=1\\ i<j<k<h}}^{n+1} [(a_{ij}^2 + a_{kh}^2 - a_{ik}^2 - a_{jh}^2)^2 + (a_{ij}^2 + a_{kh}^2 - a_{ih}^2 - a_{jk}^2)^2 +$$
$$+ (a_{ik}^2 + a_{jh}^2 - a_{ih}^2 - a_{jk}^2)^2] \geqslant 0.$$

Equality occurs iff for any $i, j, k, h \in \{1, 2, ..., n+1\}$ ($i < j < k < h$) we have

$$a_{ij}^2 + a_{kh}^2 = a_{ik}^2 + a_{jh}^2 = a_{ih}^2 + a_{jk}^2,$$

i.e. if $A_iA_jA_kA_h$ is an orthocentric tetrahedron.

O. Bottema [12].

3.2. $$M_{\frac{n(n+1)}{2}}^{[k]}(a_{ij}) \leqslant M_{\frac{n(n+1)}{2}}^{[2]}(a_{ij}) \leqslant R\sqrt{\frac{2(n+1)}{n}} \quad (k \in [-\infty, 2)).$$

The first equality occurs iff A is a regular simplex and the second equality occurs iff $m_1 = m_2 = ... = m_{n+1}$.

Proof. The second inequality follows from 2.22 with $P = G$ and $Q = O$.

V. Devidé [16]; J. Schopp [97]; E. Hille [158]; G. D. Chakerian and M. S. Klamkin [13]; G. Kalajdžić [43] ($n = 3$); D. O. Škljarskij, N. N. Čencov, and I. M. Jaglom [101, p. 50, 280] ($n = 3$); I. Tomescu [129] ($n = 3$); M. Lascu [64]; GI 5.3, 5.13, 5.22, 5.27 ($n = 2$).

3.3. If $f: R^+ \to R$ is a non-decreasing convex function, then

$$\sum_{\substack{i,j=1\\ i<j}}^{n+1} f(a_{ij}) \leqslant \frac{n(n+1)}{2}\, f\left(R\sqrt{\frac{2(n+1)}{n}}\right) \qquad \{A_R\}$$

and if f is a non-increasing concave function, then we have the opposite inequality.

Proof. By 3.2 we obtain

$$\sum_{\substack{i,j=1\\ i<j}}^{n+1} f(a_{ij}) \lesseqgtr \frac{n(n+1)}{2}\, f\left(\frac{2}{n(n+1)} \sum_{\substack{i,j=1\\ i<j}}^{n+1} a_{ij}\right) \lesseqgtr$$

$$\lesseqgtr \frac{n(n+1)}{2}\, f\left(R\sqrt{\frac{2(n+1)}{n}}\right).$$

3.4. If O is an internal point of A and $k \in (1, +\infty]$, then

$$M^{[k]}_{\frac{n(n+1)}{2}}(a_{ij}) > \frac{4}{n+1} R.$$

Proof. Follows from XX.9.12 (with $m = n + 1$).
G. D. Chakerian and M. S. Klamkin [13].

3.5. If $d = \max a_{ij}$ $(i, j \in \{1, 2, \ldots, n+1\},\ i < j)$ is the diameter of A, then

$$d \geqslant R\, \frac{2(n+1)}{n} \qquad \{A_R\}.$$

R. Alexander [2].

3.6. $$\sum_{\substack{i,j=1\\ i<j}}^{n} a_{ij}^2 \leqslant (n-1)^2 R^2 + \sum_{i=1}^{n} a_{i,n+1}^2$$

with equality iff $\overrightarrow{OA}_{n+1} = n \cdot \overrightarrow{OG}_{n+1}$, where G_{n+1} is the centroid of $(n-1)$-simplex $A_{n+1} = A_1A_2 \ldots A_n$.

Proof. For any $i, j \in \{1, 2, \ldots, n+1\}$ $(i < j)$ from

$$a_{ij}^2 = (\overrightarrow{OA}_j - \overrightarrow{OA}_i)^2 = 2R^2 - 2\, \overrightarrow{OA}_i \cdot \overrightarrow{OA}_j$$

it follows

$$2\, \overrightarrow{OA}_i \cdot \overrightarrow{OA}_j = 2R^2 - a_{ij}^2.$$

Therefore, we obtain successively

$$0 \leqslant (\overrightarrow{OA}_{n+1} - \sum_{i=1}^{n} \overrightarrow{OA}_i)^2 =$$

$$= (n+1)R^2 + \sum_{\substack{i,j=1 \\ i<j}}^{n} 2\,\overrightarrow{OA}_i \cdot \overrightarrow{OA}_j - \sum_{i=1}^{n} 2\,\overrightarrow{OA}_i \cdot \overrightarrow{OA}_{n+1} =$$

$$= (n+1)R^2 + \sum_{\substack{i,j=1 \\ i<j}}^{n} (2R^2 - a_{ij}^2) - \sum_{i=1}^{n} (2R^2 - a_{i,n+1}^2) =$$

$$= (n-1)^2R^2 + \sum_{i=1}^{n} a_{i,n+1}^2 - \sum_{\substack{i,j=1 \\ i<j}}^{n} a_{ij}^2.$$

Equality occurs iff

$$\overrightarrow{OA}_{n+1} = \sum_{i=1}^{n} \overrightarrow{OA}_i = n \cdot \overrightarrow{OG}_{n+1}.$$

E. G. Gotman [31] ($n = 3$).

3.7. $$V \leqslant \frac{1}{n!}\sqrt{\frac{n+1}{2^n}}\left[M^{[k]}_{\frac{n(n+1)}{2}}(a_{ij})\right]^n \quad (k \in [0, +\infty]) \quad \{A_R\}.$$

Proof. Follows from 3.14 by induction on n.

D. Veljan and G. Korchmáros [134] ($k = 0$); J. Zh. Zhang and L. Yang [148] ($k = 0$); G. Korchmáros [59] ($k = 0$); G. Sansone [90] ($n = 3$, $k = 1$); Z. E. Melzak [72] ($n = 3$, $k = 1$); D. Voiculescu and F. Leuenberger [137] ($n = 3$, $k = 0$); D. O. Škljarskij, N. N. Čencov, and I. M. Jaglom [101, p. 46, 260-261, 32, 166] ($n = 3$, $k \in \{1, +\infty\}$); G. Kalajdžić [43] ($n = 3$, $k = 2$); GI 4.14, 4.2, 4.4 ($n = 2$, $k \in \{0, 1, 2\}$).

Remark. From 3.7 and 2.40 it follows for any $k \in [0, +\infty]$ the inequality

3.7' $$M^{[k]}_{\frac{n(n+1)}{2}}(a_{ij}) \geqslant r\sqrt{2n(n+1)} \quad \{A_R\},$$

which completes the inequality 3.2.

J. Fickett [24] ($n = 3$, $k = +\infty$).

3.8. With the same conditions as in 2.44 we have for any $\lambda \in (0, 2]$

$$\sum_{\substack{i_1,\ldots,i_s=1\\ i_1<\ldots<i_s}}^{n+1} V^{\lambda}_{i_1,\ldots,i_s} \leqslant$$

$$\leqslant \binom{n+1}{s}\left\{\frac{1}{(s-1)!}\sqrt{\frac{s}{2^{s-1}}}\left[M^{[2]}_{\frac{n(n+1)}{2}}(a_{ij})\right]^{s-1}\right\}^{\lambda} \quad \{A_R\}.$$

R. M. Tanner [117]; G. Kalajdžić [43] ($n = 3$, $s = 4$); D. O. Škljarskij, N. N. Čencov, and I. M. Jaglom [101, p. 48, 275-277] ($n = 3$, $s \in \{2, 3\}$, $\lambda = 1$); N. Şaganai [89] ($n = 3$, $s = 3$, $\lambda = 1$); GI 4.4 ($n = 2$, $s = 3$).

3.9. For every p such that

$$p \geqslant p_n = 2n\,\frac{\ln(n+1) - \ln n}{(n+1)\ln(n+1) - n\ln n}$$

we have (with $a_{n+1,n+2} = a_{1,n+1}$)

$$V \leqslant \frac{1}{n!}\sqrt{\frac{n+1}{2^n}}\left[M^{[p]}_{n+1}(a_{i,i+1})\right]^n \quad \{A_R\}.$$

L. Gerber [29].

3.10. $$\frac{F^n}{V^{n-1}} \geqslant \frac{1}{n!}\sqrt{n^{3n}(n+1)^{n+1}} \quad \{A_R\}.$$

Proof. Follows from 3.14 according to the A-G-inequality for means.

H. Hadwiger [36]; J. Steiner [107] ($n \in \{2, 3\}$), [108] ($n \in \{2, 3\}$); R. Sturm [111] ($n = 3$), [112, p. 115-116] ($n = 3$); G. Sansone [90] ($n = 3$); B. R. Venkataraman [135] ($n = 3$); K. S. K. Iyengar and K. V. Iyengar [41] ($n = 3$); D. O. Škljarskij, N. N. Čencov, and I. M. Jaglom [101, p. 46, 257-262] ($n = 3$); GI 4.2 ($n = 2$).

3.11. For any $s \in \{1, 2, \ldots, n\}$ let Δ_s be the sum of squares of contents of all s-simplexes $A_{i_1}A_{i_2}\ldots A_{i_{s+1}}$ ($1 \leqslant i_1 < i_2 < \ldots < i_{s+1} \leqslant n+1$) and

$$v_s = \frac{(s!)^2\Delta_s}{(n+1)\binom{n}{s}}.$$

Then we have

$$\sqrt[s]{v_s} \geqslant \sqrt[s+1]{v_{s+1}} \quad (s = 1, 2, \ldots, n+1) \quad \{A_R\}.$$

Proof. Follows from XX.9.17 with $m = n + 1$, $a_1 = a_2 = \ldots = a_{n+1} = 1$.

E. M. Goljberg [30]; J. Zh. Zhang and L. Yang [161], [148]; W. Jänichen [42] ($n = 3$); GI 4.4 ($n = 2$).

Remark. Inequality $\sqrt[n]{v_n} \leqslant v_1$ follows from 3.7 and the inequalities $\sqrt[s]{v_s} \leqslant v_1$ are the inequalities 3.8 with $\lambda = 2$.

3.12. With the same conditions as in 3.11 we have

$$v_{s-1}v_{s+1} \leqslant (v_s)^2 \quad (s = 2, \ldots, n - 1) \quad \{A_R\}.$$

Proof. Follows from XX.9.18 with $m = n + 1$, $a_1 = a_2 = \ldots = a_{n+1} = 1$.

E. M. Goljberg [30]; J. Zh. Zhang and L. Yang [161], [148]; W. Jänichen [42] ($n = 3$).

3.13. With the same conditions as in 3.11 we have

$$\sqrt[n-s]{\frac{v_n}{v_s}} \geqslant \sqrt[n-s-1]{\frac{v_n}{v_{s+1}}} \quad (s = 1, 2, \ldots, n - 2) \quad \{A_R\}.$$

E. M. Goljberg [30].

3.14. $$V^{n-1} \leqslant n!\sqrt{\frac{(n+1)^{n-1}}{n^{3n}}}\left(\prod_{i=1}^{n+1} F_i\right)^{\frac{n}{n+1}} \quad \{A_R\}.$$

Proof. Follows from XX.9.17 with $m = n + 1$, $k = n + 1$, $l = n$, $a_i = F_i^2$ ($i = 1, 2, \ldots, n + 1$) applying the A-G-inequality

$$\sum_{i=1}^{n+1} F_i^2 \geqslant (n + 1)\left(\prod_{i=1}^{n+1} F_i\right)^{\frac{2}{n+1}}.$$

G. Korchmáros [134]; J. Zh. Zhang and L. Yang [148]; GI 4.14 ($n = 2$).

Remark. 3.14 implies

3.14' $$V^{n-1} \leqslant n!\sqrt{\frac{(n+1)^{n-1}}{n^{3n}}}\left[M_{n+1}^{[k]}(F_i)\right]^n \quad (k \in [0, +\infty]) \quad \{A_R\}.$$

L. Yang and J. Zh. Zhang [147] ($k = 2$).

3.15. $$V \geqslant \frac{1}{n!}\sqrt{\frac{n^n}{(n+1)^{n-1}}}\left(\prod_{i=1}^{n+1} h_i\right)^{\frac{n}{n+1}} \quad \{A_R\}.$$

Proof. Follows from 3.14 because of the equalities $F_iR_i = nV$ ($i = 1, 2, \ldots, n + 1$).

J. Zh. Zhang and L. Yang [148].

3.16. Let $k \in \{1, 2, \dots, [\frac{n+1}{2}]\}$ and let $(i_1, i_2, \dots, i_{n+1})$ be any permutation of the set $\{1, 2, \dots, n+1\}$. If $h_{i_1 \dots i_k}$ is the distance of the $(k-1)$-plane $A_{i_1} \dots A_{i_k}$ and the $(n-k)$-plane $A_{i_{k+1}} \dots A_{i_{n+1}}$ and if h_k is the minimum of all distances $h_{i_1 \dots i_k}$, then

$$h_k \leqslant \frac{n+1}{\sqrt{nk(n-k+1)}} R \qquad \{A_R\}.$$

R. Alexander [2].

3.17. With the same conditions as in 3.16, let $w = \min h_k$ $(k \in \{1, 2, \dots, [\frac{n+1}{2}]\})$, i.e. $w = h_{[\frac{n+1}{2}]}$, be the width of A. Then

$$w \leqslant \frac{n+1}{\sqrt{n[\frac{n+1}{2}](n+1-[\frac{n+1}{2}])}} R \qquad \{A_R\}.$$

R. Alexander [2].
Remark. Follows from 3.16 or from 3.19.

3.18. With the same conditions as in 3.17 we have

$$w \leqslant \sqrt{\frac{n^3}{[\frac{n+1}{2}](n+1-[\frac{n+1}{2}])}} \cdot \frac{V}{M_{n+1}^{[2]}(F_i)} \qquad \{A_R\}.$$

L. Yang and J. Zh. Zhang [147].

3.19. With the same conditions as in 3.17 we have

$$w \leqslant \frac{(n!)^{\frac{1}{n}}(n+1)^{\frac{n-1}{2n}}}{\sqrt{[\frac{n+1}{2}](n+1-[\frac{n+1}{2}])}} V^{\frac{1}{n}} \qquad \{A_R\}.$$

L. Yang and J. Zh. Zhang [147].
Remark. Follows from 3.18 and 3.14' (with $k = 2$).

3.20. $$V \leqslant \frac{1}{n!} \prod_{i=1}^{n} a_{i,n+1}.$$

Equality holds iff the edges $A_1A_{n+1}, A_2A_{n+1}, \dots, A_nA_{n+1}$ are mutually orthogonal.
J. Hadamard [35]; R. Sturm [111] ($n = 3$).
Remark. Inequality 3.20 implies

3.20' $$V \leqslant \frac{1}{n!}\left[M_n^{[k]}(a_{i,n+1})\right]^n \quad (k \in [0, +\infty]).$$

Equality holds iff the edges A_1A_{n+1}, A_2A_{n+1}, ..., A_nA_{n+1} are equal and mutually orthogonal.

3.21. $$V \leqslant \frac{n!}{n^{2n}}\left(\sum_{i=1}^{n} F_i\right)^{\frac{n}{n-1}}.$$

Equality holds iff the edges A_1A_{n+1}, A_2A_{n+1}, ..., A_nA_{n+1} are equal and mutually orthogonal.

J. Steiner [108]* (n = 3); R. Sturm [112, p. 111-112]* (n = 3).

Remark. Inequality 3.21 implies

3.21' $$V \leqslant \frac{n!}{n^n}\left[M_n^{[k]}(F_i)\right]^{\frac{n}{n-1}} \quad (k \in [1, +\infty])$$

with the same conditions for equality as in 3.21.

3.22. Let O be in interior of A and for any $i \in \{1, 2, ..., n + 1\}$ let $\bar{R}_i$ be the circumradius of A_i. Then

$$\min_i \bar{R}_i^2 \leqslant R^2 - r^2 \leqslant \max_i \bar{R}_i^2 \quad \{O = I\}.$$

Proof. We use the notation from 2 with P = O. Let $\min_i \bar{R}_i = \bar{R}_j$, $\max_i \bar{R}_i = \bar{R}_k$. Then from

(1) $$R^2 = r_i^2 + \bar{R}_i^2 \quad (i = 1, 2, ..., n + 1)$$

we obtain $r_j = \max_i r_i$, $r_k = \min_i r_i$. The inequalities

$$\sum_{i=1}^{n+1} F_i r_i = nV = \sum_{i=1}^{n+1} F_i r$$

imply

$$r = \frac{1}{F}\sum_{i=1}^{n+1} F_i r_i$$

and so

$$r_k = \min_i r_i \leqslant r \leqslant \max_i r_i = r_j \quad \{r_1 = r_2 = ... = r_{n+1}\}.$$

Therefore, we get by (1)

$$\min_i \bar{R}_i^2 = \bar{R}_j^2 = R^2 - r_j^2 \leqslant R^2 - r^2 \leqslant R^2 - r_k^2 = \bar{R}_k^2 = \max_i \bar{R}_i^2 \qquad \{O = I\}.$$

Gh. Stoica [109].

3.23. Let M be any point of the edge $\overline{A_1A_2}$ of A. The hyperplane through M parallel to the hyperplane a_1 intersects the edges $\overline{A_1A_3}, \ldots, \overline{A_1A_{n+1}}$ in the points $A_{13}, \ldots, A_{1,n+1}$ and the hyperplane through M parallel to a_2 intersects the edges $\overline{A_2A_3}, \ldots, \overline{A_2A_{n+1}}$ in the points $A_{23}, \ldots, A_{2,n+1}$. If V, V_1 and V_2 are the volumes of simplexes A, $A_1MA_{13} \ldots A_{1,n+1}$ and $A_2MA_{23} \ldots A_{2,n+1}$, then

$$V_1^2 + V_2^2 \geqslant \frac{1}{2^{2n-1}} V^2 \qquad \{A_1M = MA_2\}.$$

Proof. Let

$$\frac{A_1M}{A_1A_2} = \lambda, \quad \frac{A_2M}{A_2A_1} = 1 - \lambda.$$

Then, we have successively

$$\frac{V_1}{V} = \lambda^n, \quad \frac{V_2}{V} = (1 - \lambda)^n,$$

$$\frac{1}{2}(V_1^2 + V_2^2) \geqslant \left(\frac{V_1 + V_2}{2}\right)^2 = \left[\frac{\lambda^n + (1 - \lambda)^n}{2}\right]^2 V^2 \geqslant$$

$$\geqslant \left[\left(\frac{\lambda + 1 - \lambda}{2}\right)^n\right]^2 V^2 = \frac{1}{2^{2n}} V^2.$$

G. Pirvănescu [83].

3.24. For every i, j $\in \{1, 2, \ldots, n + 1\}$ $(i < j)$ the point $A_{ij} = A_{ji}$ divides the edge $\overline{A_iA_j}$ of A in the ratio $1:\lambda_{ij} = \lambda_{ji}:1$, where $\lambda_{ij} \in R^+$ is given. If V' is the volume of the polytope with vertices A_{ij} $(i, j = 1, 2, \ldots, n + 1;\ i < j)$, then

$$V' \leqslant \left\{1 - (n + 1)\left[\prod_{\substack{i,j=1 \\ i<j}}^{n+1} \frac{\lambda_{ij}}{(1 + \lambda_{ij})^2}\right]^{\frac{1}{n+1}}\right\} V$$

with equality iff we have the equality of all the products

$$\prod_{\substack{j=1\\ j\neq i}}^{n+1} (1+\lambda_{ij}) \quad (i = 1, 2, \dots, n+1).$$

Proof. For any i, j $\in \{1, 2, \dots, n+1\}$ $(i < j)$ let $a_{ij} = A_iA_j$. Then

$$A_iA_{ij} = \frac{a_{ij}}{1+\lambda_{ij}}.$$

If V_i' is the volume of simplex $A_{i,1} \dots A_{i,i-1}A_iA_{i,i+1} \dots A_{i,n+1}$, then

$$\frac{V_i'}{V} = \prod_{\substack{j=1\\ j\neq i}}^{n+1} \frac{1}{1+\lambda_{ij}},$$

$$V' = V - \sum_{i=1}^{n+1} V_i' = \left[1 - \sum_{i=1}^{n+1} \left(\prod_{\substack{j=1\\ j\neq i}}^{n+1} \frac{1}{1+\lambda_{ij}}\right)\right] V \leqslant$$

$$\leqslant \left\{1 - (n+1)\left[\prod_{i=1}^{n+1}\left(\prod_{\substack{j=1\\ j\neq i}}^{n+1} \frac{1}{1+\lambda_{ij}}\right)\right]^{\frac{1}{n+1}}\right\} V =$$

$$= \left\{1 - (n+1)\left[\prod_{\substack{i,j=1\\ i<j}}^{n+1} \frac{1}{(1+\lambda_{ij})(1+\lambda_{ji})}\right]^{\frac{1}{n+1}}\right\} V =$$

$$= \left\{1 - (n+1)\left[\prod_{\substack{i,j=1\\ i<j}}^{n+1} \frac{\lambda_{ij}}{(1+\lambda_{ij})^2}\right]^{\frac{1}{n+1}}\right\} V.$$

G. Kalajdžić [43] $(n = 3;\ \lambda_{ij} = \lambda$ or $\lambda_{ij} = 1/\lambda$ for $i, j = 1, 2, \dots, n+1;\ i \neq j)$.

3.25. Let P be a parallelotope inscribed in A such that A_{n+1} is one vertex of P, the opposite vertex P of P is an internal point of A_{n+1} and n vertices of P are points of the edges $\overline{A_{n+1}A_1}, \overline{A_{n+1}A_2}, \dots, \overline{A_{n+1}A_n}$. If V' is the volume of P, then

$$\frac{V'}{V} \leqslant \frac{n!}{n^n} \quad \{P = G_{n+1}\}.$$

Proof. Applying an affinity we can obtain the edges $A_{n+1}A_1$, $A_{n+1}A_2$, ..., $A_{n+1}A_n$ to be orthogonal and to have unit lengths. Let A_{n+1} be the origin and $A_{n+1}A_1$, $A_{n+1}A_2$, ..., $A_{n+1}A_n$ the coordinate axes. Then the hyperplane a_{n+1} has the equation $x_1 + x_2 + \dots + x_n = 1$. If $P = (p_1, p_2, \dots, p_n)$, then

$$\frac{V'}{V} = \frac{1}{n!} \prod_{i=1}^{n} p_i \leqslant \left(\frac{1}{n} \sum_{i=1}^{n} p_i\right)^n = \frac{1}{n^n} \quad \{p_1 = p_2 = \dots = p_n\}.$$

F. G.-M. [150, p. 198-199] (n = 3).

3.26. If F' is the (n - 1)-content of the section of A with any hyperplane, then

$$F' \leqslant \max (F_1, F_2, \dots, F_{n+1}) \quad (n \in \{2, 3, 4\}),$$

$$\max F' \geqslant \max (F_1, F_2, \dots, F_{n+1}) \quad (n \in \{5, 6, \dots\}).$$

J. Philip [80]; E. Ehrhart, J. J. A. M. Brands, G. Laman, and H. G. Eggleston [18] (n = 3); D. W. Walkup [140] (n = 5); M. J. Pelling [78] (n = 3); V. Senderov [99] (n = 3).

3.27. For any $i \in \{1, 2, \dots, n\}$ let $F_{i,n+1}$ be the (n - 2)-content of (n - 2)-simplex $A_{i,n+1} = A_1 \dots A_{i-1}A_{i+1} \dots A_n$ and let

$$F'_{n+1} = \sum_{i=1}^{n} F_{i,n+1}.$$

Then

$$F \geqslant F_{n+1} + \sqrt{F_{n+1}^2 + \frac{1}{(n-1)^2} h_{n+1}^2 F'^2_{n+1}}$$

with equality iff the foot A'_{n+1} of the altitude from A_{n+1} is simultaneously the circumcentre of the (n - 1)-simplex $A_{n+1} = A_1A_2 \dots A_n$.

Proof. For any $i \in \{1, 2, \dots, n\}$ let $r_{i,n+1}$ be the distance of the point A'_{n+1} from the (n - 2)-plane $A_1 \dots A_{i-1}A_{i+1} \dots A_n$ and let $h_{i,n+1}$ be the altitude of the (n - 1)-simplex $A_i = A_1 \dots A_{i-1}A_{i+1} \dots A_{n+1}$ from the vertex A_{n+1}. Then the equalities

$$(n - 1)F_i = F_{i,n+1}h_{i,n+1},$$

$$h_{i,n+1} = \sqrt{r_{i,n+1}^2 + h_{n+1}^2} \quad (i = 1, 2, \dots, n)$$

imply

$$(2) \qquad (n-1)\sum_{i=1}^{n} F_i = \sum_{i=1}^{n} \sqrt{F_{i,n+1}^2 r_{i,n+1}^2 + F_{i,n+1}^2 h_{n+1}^2}.$$

On the other hand

$$(3) \qquad (n-1)F_{n+1} = \sum_{i=1}^{n} F_{i,n+1} r_{i,n+1}.$$

For any n vectors $\underline{v}_i$ (i = 1, 2, ..., n) with Cartesian coordinates x_i, y_i in a plane we have the inequality

$$\sum_{i=1}^{n} \sqrt{x_i^2 + y_i^2} = \sum_{i=1}^{n} |\underline{v}_i| \geq \left|\sum_{i=1}^{n} \underline{v}_i\right| = \sqrt{\left(\sum_{i=1}^{n} x_i\right)^2 + \left(\sum_{i=1}^{n} y_i\right)^2}$$

with equality iff the considered vectors are parallel, i.e. iff the numbers x_i and y_i (i = 1, 2, ..., n) are proportional. Therefore, with $x_i = F_{i,n+1} r_{i,n+1}$, $y_i = F_{i,n+1} h_{n+1}$ (i = 1, 2, ..., n) from (2) and (3) we obtain

$$(n-1)(F - F_{n+1}) = (n-1)\sum_{i=1}^{n} F_i \geq$$

$$\geq \sqrt{\left(\sum_{i=1}^{n} F_{i,n+1} r_{i,n+1}\right)^2 + \left(\sum_{i=1}^{n} F_{i,n+1} h_{n+1}\right)^2} =$$

$$= \sqrt{(n-1)^2 F_{n+1}^2 + h_{n+1}^2 F'^2_{n+1}}$$

with equality iff $r_{1,n+1} = r_{2,n+1} = \dots = r_{n,n+1}$.

E. G. Gotman [32] (n = 3).

3.28. Let $I_{o,n+1}$ be the incentre and $r_{0,n+1}$ the inradius of the (n - 1)-simplex $A_{n+1} = A_1 A_2 \dots A_n$ and let $h_{n+1} = A_{n+1}A'_{n+1}$ be the altitude of A. Then

$$r \leq \frac{h_{n+1} r_{0,n+1}}{r_{0,n+1} + \sqrt{h_{n+1}^2 + r_{0,n+1}^2}}. \qquad \{A'_{n+1} = I_{0,n+1}\}.$$

<u>Proof</u>. With the notation from 3.27 we have the equalities

$$F = \frac{nV}{r} = \frac{h_{n+1} F_{n+1}}{r}, \qquad F'_{n+1} = \frac{(n-1)F_{n+1}}{r_{0,n+1}}$$

and the inequality 3.27 takes the following form (equivalent to 3.28):

$$\frac{h_{n+1}}{r} \geqslant 1 + \sqrt{1 + \frac{h_{n+1}^2}{r_{0,n+1}^2}} .$$

P. K. Kashikar [45].

3.29. For any $i \in \{1, 2, \ldots, n + 1\}$ let V_i' be the volume of the simplex A_i' which is cut off from A by the second tangential hyperplane of the inscribed hypersphere of A parallel to the hyperplane a_i. Then

$$\sum_{i=1}^{n+1} V_i' \geqslant \frac{(n - 1)^n}{(n + 1)^{n-1}} V\{h_1 = h_2 = \ldots = h_{n+1}\}.$$

Proof. We have the equality

$$(4) \qquad \sum_{i=1}^{n+1} \frac{1}{h_i} = \frac{1}{r}$$

and because of the similarity of simplexes A_i' and A we have

$$V_i' = \left(\frac{h_i - 2r}{h_i}\right)^n V \quad (i = 1, 2, \ldots, n + 1).$$

Therefore,

$$\sum_{i=1}^{n+1} V_i' = V \sum_{i=1}^{n+1} (1 - \frac{2r}{h_i})^n \geqslant \frac{V}{(n + 1)^{n-1}} \left[\sum_{i=1}^{n+1} (1 - \frac{2r}{h_i})\right]^n =$$

$$= \frac{V}{(n + 1)^{n-1}} \left(n + 1 - 2r \sum_{i=1}^{n+1} \frac{1}{h_i}\right)^n = \frac{(n - 1)^n}{(n + 1)^{n-1}} V.$$

T. Rajkov [85] (n = 3).

3.30. If r_i' ($i \in \{1, 2, \ldots, n + 1\}$) is the inradius of the simplex A_i' from 3.29, then

$$\sum_{i=1}^{n+1} h_i \geqslant \frac{(n + 1)^2}{n - 1} \sum_{i=1}^{n+1} r_i' \quad \{h_1 = h_2 = \ldots = h_{n+1}\}.$$

Proof. By (4) and the equalities

$$\frac{r_i'}{r} = \frac{h_i - 2r}{h_i} \quad (i = 1, 2, \ldots, n + 1)$$

we obtain successively

$$\sum_{i=1}^{n+1} r_i' = \left(n + 1 - 2r \sum_{i=1}^{n+1} \frac{1}{h_i}\right) r = (n - 1)r,$$

$$\frac{1}{n + 1} \sum_{i=1}^{n+1} h_i \geqslant \frac{n + 1}{\sum_{i=1}^{n+1} \frac{1}{h_i}} = (n + 1)r = \frac{n + 1}{n - 1} \sum_{i=1}^{n+1} r_i'.$$

[154].

3.31. For any $i, j \in \{1, 2, \ldots, n + 1\}$ $(i \neq j)$ let α_{ij} be the angle between the hyperplanes a_i and a_j. Then

$$\frac{n^2}{4} \pi \leqslant \sum_{\substack{i,j=1 \\ i<j}}^{n+1} \alpha_{ij} \leqslant \binom{n}{2} \pi \quad (n \text{ even}),$$

$$\frac{n^2 - 1}{4} \pi \leqslant \sum_{\substack{i,j=1 \\ i<j}}^{n+1} \alpha_{ij} \leqslant \binom{n}{2} \pi \quad (n \text{ odd}).$$

J. W. Gaddum [26].

3.32. With the same conditions as in 3.31 we have

$$\sum_{\substack{i,j=1 \\ i<j}}^{n+1} \cos \alpha_{ij} \leqslant \frac{n + 1}{2} \quad \{h_1 = h_2 = \ldots = h_{n+1}\}.$$

Proof. The equalities

$$\sum_{\substack{j=1 \\ j \neq i}}^{n+1} F_j \cos \alpha_{ij} = F_i \quad (i = 1, 2, \ldots, n + 1)$$

imply successively

$$(5) \qquad \sum_{\substack{j=1 \\ j \neq i}}^{n+1} \frac{F_j}{F_i} \cos \alpha_{ij} = 1 \quad (i = 1, 2, \ldots, n + 1),$$

$$n + 1 = \sum_{i=1}^{n+1} \sum_{\substack{j=1 \\ j\neq i}}^{n+1} \left(\frac{F_j}{F_i} \cos \alpha_{ij}\right) = \sum_{\substack{i,j=1 \\ i<j}}^{n+1} \left(\frac{F_i}{F_j} + \frac{F_j}{F_i}\right) \cos \alpha_{ij} \geqslant$$

$$\geqslant 2 \sum_{\substack{i,j=1 \\ i<j}}^{n+1} \cos \alpha_{ij}$$

with equality iff $F_1 = F_2 = \ldots = F_{n+1}$.

R. Hoppe [159] (n = 3); A. Marmion [71] (n = 3); T. M. Korikova [60] (n = 3); S.-N. Deaconu [15] (n = 3); GI 2.16 (n = 2).

3.33. With the same notation as in 3.31 let $c_{ii} = 1$ and $c_{ij} = -\cos \alpha_{ij}$ for any $i, j \in \{1, 2, \ldots, n + 1\}$ $(i \neq j)$. If C_{ij} is the cofactor of c_{ij} of the determinant $\det(c_{ij})$ $(i, j = 1, 2, \ldots, n + 1)$, then

$$C_{ij} > 0 \quad (i, j \in \{1, 2, \ldots, n + 1\}).$$

L. Yang and J. Zh. Zhang [146].

Remark. The equality $\det(c_{ij}) = 0$ $(i, j = 1, 2, \ldots, n + 1)$ and the inequalities 3.33 are necessary and sufficient conditions for the existence of a simplex with dihedral angles α_{ij} $(i, j = 1, 2, \ldots, n + 1; i < j)$.

L. Yang and J. Zh. Zhang [146].

3.34. The orthocentre H of an orthocentric simplex A lies inside, on or outside the circumhypersphere of this simplex iff

$$\sum_{\substack{i,j=1 \\ i<j}}^{n+1} a_{ij}^2 \gtreqless 4nR^2.$$

Proof. In the proof of 2.22 we proved the equality

$$PQ^2 = \sum_{i=1}^{n+1} \lambda_i QA_i^2 - \sum_{\substack{i,j=1 \\ i<j}}^{n+1} \lambda_i \lambda_j a_{ij}^2,$$

where $\lambda_1, \lambda_2, \ldots, \lambda_{n+1}$ are barycentric coordinates of P in A. With P = G and Q = O we have

$$OG^2 = R^2 - \frac{1}{(n + 1)^2} \sum_{\substack{i,j=1 \\ i<j}}^{n+1} a_{ij}^2.$$

Since OG:OH = (n - 1):(n + 1), we obtain

$$(n-1)^2 OH^2 = (n+1)^2 R^2 - \sum_{\substack{i,j=1\\ i<j}}^{n+1} a_{ij}^2 .$$

Therefore, $OH \lesseqgtr R$ is equivalent to 3.34.

G. A. Tsintsifas (private communication).

3.35. In a simplex A with $F_1 = F_2 = \dots = F_{n+1}$ (with the same notation as in 3.31) for any $i \in \{1, 2, \dots, n+1\}$ and any $k \in (1, +\infty]$ respectively $k \in [-\infty, 1)$, we have

$$M_n^{[k]}(\cos \alpha_{ij}) \gtreqless \frac{1}{n}$$

with equality iff $\cos \alpha_{i1} = \dots = \cos \alpha_{i,i-1} = \cos \alpha_{i,i+1} = \dots = \cos \alpha_{i,n+1}$.

Proof. From (5) we obtain now the equality

$$\sum_{\substack{j=1\\ j\neq i}}^{n+1} \cos \alpha_{ij} = 1$$

and 3.35 is a consequence of the inequality for means.

V. Gh. Vodă [136] ($n = 3$, $k \in \{0, 2\}$).

3.36. In a simplex A with mutually orthogonal edges from the vertex A_{n+1} (with the same notation as in 3.31) for any $k \in [-\infty, 2)$ respectively $k \in (2, +\infty]$ we have

$$M_n^{[k]}(\cos \alpha_{i,n+1}) \lesseqgtr \frac{1}{\sqrt{n}} \quad \{a_{1,n+1} = a_{2,n+1} = \dots = a_{n,n+1}\}.$$

Proof. If A_{n+1} is the origin and $A_{n+1}A_i$ ($i = 1, 2, \dots, n$) are the coordinate axes, then the hyperplane a_{n+1} has the equation

$$\sum_{i=1}^{n} \frac{x_i}{a_{i,n+1}} - 1 = 0. \tag{6}$$

The distance $h_{n+1} = A_{n+1}A'_{n+1}$ of A_{n+1} from this hyperplane is given by

$$h_{n+1} = \frac{1}{\sum_{i=1}^{n} \frac{1}{a_{i,n+1}^2}},$$

i.e. by

(7) $$\frac{1}{h_{n+1}^2} = \sum_{i=1}^{n} \frac{1}{a_{i,n+1}^2} .$$

For any $i \in \{1, 2, \ldots, n\}$ we have $\alpha_{i,n+1} = \measuredangle A'_{n+1}A_{n+1}A_i$ and by (6) we obtain

$$\cos \alpha_{i,n+1} = \frac{h_{n+1}}{a_{i,n+1}} \quad (i = 1, 2, \ldots, n),$$

$$M_n^{[2]}(\cos \alpha_{i,n+1}) = \sqrt{\frac{1}{n} \sum_{i=1}^{n} \cos^2 \alpha_{i,n+1}} =$$

$$= \sqrt{\frac{1}{n} h_{n+1}^2 \sum_{i=1}^{n} \frac{1}{a_{i,n+1}^2}} = \frac{1}{\sqrt{n}} .$$

[162] (n = 3, k = 0); Ju. I. Gerasimov [28] (n = 3, k = 1).

3.37. In the simplex A from 3.36 we have

$$\sum_{i=1}^{n} F_i \geqslant \frac{1}{(n-1)!} \sqrt{n}^{n+1} h_{n+1}^{n-1} \quad \{a_{1,n+1} = a_{2,n+1} = \ldots = a_{n,n+1}\}.$$

Proof. The equality

$$F_i = \frac{1}{(n-1)!} \cdot \frac{1}{a_{i,n+1}} \prod_{j=1}^{n} a_{j,n+1} \quad (i = 1, 2, \ldots, n)$$

implies by (7) successively

$$\sum_{i=1}^{n} F_i = \frac{1}{(n-1)!} \sum_{i=1}^{n} \left(\frac{1}{a_{i,n+1}} \prod_{j=1}^{n} a_{j,n+1} \right) =$$

$$= \frac{n}{(n-1)!} \left(\prod_{j=1}^{n} a_{j,n+1} \right) \left(\frac{1}{n} \sum_{i=1}^{n} \frac{1}{a_{i,n+1}} \right) \geqslant$$

$$\geqslant \frac{n}{(n-1)!} \left(\prod_{j=1}^{n} a_{j,n+1} \right) \left(\prod_{i=1}^{n} \frac{1}{a_{i,n+1}} \right)^{\frac{1}{n}} =$$

$$= \frac{n}{(n-1)!} \left[\left(\prod_{i=1}^{n} a_{i,n+1}^2 \right)^{\frac{1}{n}} \right]^{\frac{n-1}{2}} \geqslant$$

$$\geqslant \frac{n}{(n-1)!}\left(\frac{n}{\sum_{i=1}^{n} \frac{1}{a_{i,n+1}^2}}\right)^{\frac{n-1}{2}} = \frac{n}{(n-1)!}\left(nh_{n+1}^2\right)^{\frac{n-1}{2}} =$$

$$= \frac{1}{(n-1)!}\sqrt{n^{n+1}}\; h_{n+1}^{n-1}.$$

[155] (n = 3); W. Janous (private communication) (n = 3).

3.38. In the simplex A from 3.36 we have

$$F \geqslant \frac{n+\sqrt{n}}{n!}\sqrt{n^{n+1}}\; h_{n+1}^{n-1} \qquad \{a_{1,n+1} = a_{2,n+1} = \dots = a_{n,n+1}\}.$$

Proof. The equalities $F_i a_{i,n+1} = nV$ $(i = 1, 2, \dots, n)$, $F_{n+1}h_{n+1} = nV$ and the equality (7) imply a generalized Pythagorean formula

$$\sum_{i=1}^{n} F_i^2 = F_{n+1}^2$$

and therefore the inequality

$$F_{n+1} = \sqrt{n}Q_n(F_i) \geqslant \sqrt{n}A_n(F_i) = \frac{\sqrt{n}}{n}\sum_{i=1}^{n} F_i \qquad \{F_1 = F_2 = \dots = F_n\}.$$

According to 3.37 we obtain

$$F = \sum_{i=1}^{n} F_i + F_{n+1} \geqslant \left(1 + \frac{\sqrt{n}}{n}\right)\sum_{i=1}^{n} F_i \geqslant$$

$$\geqslant \frac{n+\sqrt{n}}{n} \cdot \frac{1}{(n-1)!}\sqrt{n^{n+1}}\; h_{n+1}^{n-1} = \frac{n+\sqrt{n}}{n!}\sqrt{n^{n+1}}\; h_{n+1}^{n-1}$$

with equality iff $a_{1,n+1} = a_{2,n+1} = \dots = a_{n,n+1}$.

3.39. In the simplex A from 3.36 we have

$$V \geqslant \frac{\sqrt{n^n}}{n!} h_{n+1}^{n} \qquad \{a_{1,n+1} = a_{2,n+1} = \dots = a_{n,n+1}\}.$$

Proof. The equality $nV = a_{1,n+1}a_{2,n+1} \dots a_{n,n+1}$ implies by (7)

$$\frac{V}{h_{n+1}^{n}} = \frac{1}{n!}\left(\prod_{i=1}^{n} a_{i,n+1}\right)\left(\sum_{i=1}^{n} \frac{1}{a_{i,n+1}^2}\right)^{\frac{n}{2}} \geqslant$$

$$\geq \frac{1}{n!}\left(\prod_{i=1}^{n} a_{i,n+1}\right)\left[n\left(\prod_{i=1}^{n} \frac{1}{a_{i,n+1}^2}\right)^{\frac{1}{n}}\right]^{\frac{n}{2}} = \frac{n^n}{n!}$$

with equalityiff $a_{1,n+1} = a_{2,n+1} = \ldots = a_{n,n+1}$.

W. Janous (private communication) ($n = 3$).

3.40. In the simplex A from 3.36 let P be any point in the hyperplane a_{n+1} and for any $i \in \{1, 2, \ldots, n\}$ let

$$q_i = \frac{A_i P}{A_i A_{n+1}} .$$

Then

$$\sum_{i=1}^{n} q_i^2 \geq n - 1 \quad \{A_{n+1}P \perp a_{n+1}\}.$$

Proof. With the notation from the proof of 3.36 we have the equalities (6) and (7) and the equalities

$$A_{n+1}P^2 = \sum_{i=1}^{n} p_i^2,$$

$$A_iP^2 = \sum_{j=1}^{n} p_j^2 - p_i^2 + (p_i - a_{i,n+1})^2 =$$

$$= A_{n+1}P^2 - 2p_i a_{i,n+1} + a_{i,n+1}^2 \qquad (i = 1, 2, \ldots, n),$$

where $P = (p_1, p_2, \ldots, p_n)$. Therefore,

$$\sum_{i=1}^{n} q_i^2 = \sum_{i=1}^{n} \frac{A_iP^2}{a_{i,n+1}^2} = A_{n+1}P^2 \sum_{i=1}^{n} \frac{1}{a_{i,n+1}^2} - 2\sum_{i=1}^{n} \frac{p_i}{a_{i,n+1}} + n =$$

$$= n - 2 + \frac{A_{n+1}P^2}{h_{n+1}^2} .$$

But we have

$$A_{n+1}P \geq h_{n+1} \quad \{A_{n+1}P \perp a_{n+1}\}.$$

J. Schopp [98]; G. Szász [114] ($n = 3$).

Remark 1. The ratios q_i $(i = 1, 2, \dots, n)$ are ratios of shortening of an axonometry in E^n and this axonometry is orthogonal iff $A_{n+1}P \perp a_{n+1}$.

Remark 2. As $q_i < 1$ $(i = 1, 2, \dots, n)$, so $q_i > q_i^2$ and from 3.40 we obtain

3.40' $$\sum_{i=1}^{n} q_i > n - 1.$$

3.41. With the same notation as in 3.40 let $A_{n+1}P \perp a_{n+1}$. Then

$$\sum_{i=1}^{n} q_i \leq \sqrt{n(n-1)} \qquad \{a_{1,n+1} = a_{2,n+1} = \dots = a_{n,n+1}\}.$$

Proof. Follows from the equality 3.40

$$\sum_{i=1}^{n} q_i^2 = n - 1$$

by the use of the A-Q-inequality for means.

J. Schopp [98]; G. Szász [114] $(n = 3)$.

3.42. Let A be a regular simplex with the edge length a and p a hyperplane. If $\bar{A}$ is the orthogonal projection of A onto p and $\bar{R}$ the circumradius of $\bar{A}$, then

$$\bar{R} \geq \frac{n-1}{2n+2}\, a \quad (n \text{ odd}), \qquad \bar{R} \geq \frac{2n-1}{8n(n+1)}\, a \quad (n \text{ even}).$$

B. Weissbach [142].

4. Inequalities for Two (or More) Simplexes

Let $A = A_1A_2 \dots A_{n+1}$ and $A' = A_1'A_2' \dots A_{n+1}'$ be two given simplexes. For the first simplex A we use the notation from 1, 2 and 3 and an analogous notation for the second simplex A'. For example, if V is the volume of A, then V' is the volume of A', etc. The sign $\{A \sim A'\}$ means that equality holds iff the simplexes A and A' are similar.

4.1. For any points P and P' we have

$$RR' \geq OP \cdot O'P' + \sum_{\substack{i,j=1 \\ i<j}}^{n+1} \sqrt{\lambda_i \lambda_j \lambda_i' \lambda_j'}\, a_{ij} a_{ij}' \qquad \{A \sim A'\}.$$

Proof. In the proof of 2.22 we proved an equality, which takes the form

$$R^2 = OP^2 + \sum_{\substack{i,j=1\\ i<j}}^{n+1} \lambda_i\lambda_j a_{ij}^2,$$

if $Q = O$. Similarly,

$$R'^2 = O'P'^2 + \sum_{\substack{i,j=1\\ i<j}}^{n+1} \lambda_i'\lambda_j' a_{ij}'^2.$$

Using Cauchy's inequality, we obtain

$$RR' = \sqrt{OP^2 + \sum_{\substack{i,j=1\\ i<j}}^{n+1} \lambda_i\lambda_j a_{ij}^2}\sqrt{O'P'^2 + \sum_{\substack{i,j=1\\ i<j}}^{n+1} \lambda_i'\lambda_j' a_{ij}'^2} \geq$$

$$\geq OP \cdot O'P' + \sum_{\substack{i,j=1\\ i<j}}^{n+1} \sqrt{\lambda_i\lambda_j\lambda_i'\lambda_j'}\, a_{ij} a_{ij}'.$$

M. S. Klamkin [49].

Remark. With $P = G$ and $P' = G'$, we obtain

4.1' $$(n + 1)^2 RR' \geq \sum_{\substack{i,j=1\\ i<j}}^{n+1} a_{ij} a_{ij}' \qquad \{A \sim A',\ m_1 = m_2 = \dots = m_{n+1}\}.$$

G. Tsintsifas (private communication).

4.2. For any points P and P' we have

$$RR' \geq OP \cdot O'P' + \sum_{i=1}^{n+1} \sqrt{\lambda_i\lambda_i'}\, R_i R_i'$$

with equality iff $OP : \sqrt{\lambda_1}R_1 : \sqrt{\lambda_2}R_2 : \dots : \sqrt{\lambda_{n+1}}R_{n+1} = O'P' : \sqrt{\lambda_1'}R_1' : \sqrt{\lambda_2'}R_2' : \dots : \sqrt{\lambda_{n+1}'}R_{n+1}'$.

Proof. In the proof of 2.23 we proved an equality, which takes the form

$$R^2 = OP^2 + \sum_{i=1}^{n+1} \lambda_i R_i^2,$$

if $Q = O$. The further procedure is analogous as in the proof of 4.1.

G. A. Tsintsifas (private communication).

4.4. $$\sum_{\substack{i,j=1\\i\neq j}}^{n+1} a_{ij}^2 B_{ij} \geqslant 2n(n!)^{\frac{2}{n}} V^{\frac{2}{n}} V'^{2-\frac{2}{n}},$$

where $\mathcal{A}$ and $\mathcal{B}$ are any two simplexes and for any $i, j \in \{1, 2, \ldots, n+1\}$ $(i \neq j)$ B_{ij} is the cofactor of the element $-\frac{1}{2} b_{ij}^2$ of the determinant

$$\begin{vmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & -\frac{1}{2} b_{12}^2 & \cdots & -\frac{1}{2} b_{1,n+1}^2 \\ 1 & -\frac{1}{2} b_{21}^2 & 0 & \cdots & -\frac{1}{2} b_{2,n+1}^2 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 1 & -\frac{1}{2} b_{n+1,1}^2 & -\frac{1}{2} b_{n+1,2}^2 & \cdots & 0 \end{vmatrix}.$$

L. Yang and J. Zh. Zhang [145]; GI 10.8 $(n = 2)$.

4.5. If $\mathcal{A}$ is any simplex and $\mathcal{A}'$ such a simplex that for any $i, j \in \{1, 2, \ldots, n+1\}$ $(i < j)$ the angle between the hyperplanes a_i' and a_j' is acute or right, then

$$\frac{V}{V'} \leqslant \left(\max_{i,j} \frac{a_{ij}}{a_{ij}'}\right)^n \quad \{\mathcal{A} \sim \mathcal{A}'\}.$$

L. Yang and J. Zh. Zhang [145].

4.6. Through a point P_{n+1} inside the $(n-1)$-simplex $\mathcal{A}_{n+1}$ draw for any $i \in \{1, 2, \ldots, n\}$ the parallel to the edge $A_{n+1}A_i$ which intersects the $(n-1)$-simplex $\mathcal{A}_i$ in a point P_i. If V' is the volume of the simplex $\mathcal{P} = P_1P_2 \ldots P_{n+1}$, then

$$\frac{V'}{V} \leqslant \frac{1}{n^n} \quad \{P_{n+1} = G_{n+1}\}.$$

<u>Proof</u>. For any $i \in \{1, 2, \ldots, n\}$ the lines A_iP_{n+1} and $A_{n+1}P_i$ intersect the $(n-2)$-simplex $A_1 \ldots A_{i-1}A_{i+1} \ldots A_n$ in the same point $B_{i,n+1}$ and two similar triangles $P_{n+1}P_iB_{i,i+1}$ and $A_iA_{n+1}B_{i,i+1}$ imply

$$\frac{P_{n+1}P_i}{A_{n+1}A_i} = \frac{P_{n+1}B_{i,n+1}}{A_iB_{i,n+1}}.$$

From 2.1 (with $f(x) = \ln x$, $k = 1$ and the substitution $h \to n - 1$) we get

$$\prod_{i=1}^{n} \frac{P_{n+1}B_{i,n+1}}{A_iB_{i,n+1}} \leqslant \frac{1}{n^n} .$$

Therefore,

$$\frac{V'}{V} = \prod_{i=1}^{n} \frac{P_{n+1}P_i}{A_{n+1}A_i} \leqslant \frac{1}{n^n} .$$

V. Thébault and L. M. Kelly [160] ($n = 3$).

Remark. In the extremal case we also have $P_i = G_i$ ($i = 1, 2, \dots, n$) and $\mathcal{P}$ is the medial simplex of $\mathcal{A}$.

4.7. Let O be the origin, $E_1, E_2, \dots, E_n$ the unit points of affine coordinate axes and $P = (p_1, p_2, \dots, p_n)$ a point such that $p_1, p_2, \dots, p_n \in R^+$. Any hyperplane through P intersects the positive coordinate half-axes in n points $A_1, A_2, \dots, A_n$. If V and V' are the volumes of the simplexes $OA_1A_2 \dots A_n$ and $OE_1E_2 \dots E_n$, then

$$V \geqslant n^n\left(\prod_{i=1}^{n} p_i\right)V' \qquad \{OA_1 = np_1,\ OA_2 = np_2,\ \dots,\ OA_n = np_n\}.$$

Proof. Let $a_i = OA_i$ ($i = 1, 2, \dots, n$). The hyperplane $A_1A_2 \dots A_n$ has the equation

$$\sum_{i=1}^{n} \frac{x_i}{a_i} = 1$$

and therefore we obtain

$$\frac{V'}{V} = \prod_{i=1}^{n} \frac{1}{a_i} = \left(\prod_{i=1}^{n} \frac{1}{p_i}\right)\left(\prod_{i=1}^{n} \frac{p_i}{a_i}\right) \leqslant \left(\prod_{i=1}^{n} \frac{1}{p_i}\right)\left(\frac{1}{n} \sum_{i=1}^{n} \frac{p_i}{a_i}\right)^n =$$

$$= \frac{1}{n^n} \prod_{i=1}^{n} \frac{1}{p_i}$$

with equality iff $p_1:a_1 = p_2:a_2 = \dots = p_n:a_n = 1:n$.

D. Veljan [133]; J. Steiner [108] ($n \in \{2, 3\}$); V. Thébault [126] ($n = 3$); [151] ($n = 3$); L. H. Lange [63] ($n \in \{2, 3\}$); J. A. Tierney [127] ($n = 2$).

4.8. If $\mathcal{A}$, $\mathcal{A}'$, $\mathcal{A}''$ are simplexes such that for any $i, j \in \{1, 2, \dots,$

$n + 1\}$ $(i < j)$ we have $a_{ij}^2 + a_{ij}'^2 = a_{ij}''^2$, then

$$V^{\frac{2}{n}} + V'^{\frac{2}{n}} \leqslant V''^{\frac{2}{n}} \quad \{A \sim A' \sim A''\}.$$

L. Yang and J. Zh. Zhang [144].

4.9. With the same conditions as in 4.8 we have

$$h_i^2 + h_i'^2 \leqslant h_i''^2 \quad (i = 1, 2, \ldots, n + 1) \quad \{A \sim A' \sim A''\}.$$

L. Yang and J. Zh. Zhang [143].

4.10. With the same conditions as in 4.8 we have

$$R^2 + R'^2 \geqslant R''^2 \quad \{A \sim A' \sim A''\}.$$

L. Yang and J. Zh. Zhang [143].

Remark. Analogous inequalities as in 4.8-4.10 hold also in the case of four or more simplexes.

4.11. Let A' be a simplex similar to A and inscribed into A so that for any $i \in \{1, 2, \ldots, n + 1\}$ the point A_i' lies in the hyperplane a_i. If

$$\lambda = \frac{A_i'A_j'}{A_iA_j} \quad (i, j \in \{1, 2, \ldots, n + 1\}, (i < j),$$

then

$$\lambda \leqslant \frac{1}{n}.$$

G. Tsintsifas and G. P. Henderson [131].

4.12. Let $m \in \{1, 2, \ldots, n\}$ and let M_m respectively M_m' be the total m-dimensional content of all m-dimensional faces of a simplex A respectively A'. Let q and r be the quotient and the remainder on division of $n + 1$ by $m + 1$, i.e. q and r are integers such that $n + 1 = q(m + 1) + r$, $0 \leqslant r < m + 1$. If $A' \subset A$, then

$$M_m' < \frac{q^{m+1-r}(q+1)^r}{n + 1 - m} M_m$$

and the fraction on the right-hand side is the best possible constant.

C. Linderholm [69]; W. Holsztyński and W. Kuperberg [38] (n = 3, m = 1), [39] (n = 3, m = 1); V. M. Tihomirov [128] (n = 3, m = 1); P. B. Gusjatnikov [34] (n = 3, m = 1).

4.13. If $\mathcal{A}$ and $\mathcal{A}'$ are two simplexes inscribed in the same hypersphere Σ with the radius R and if $u = \min A_iA_j'$ for $i, j \in \{1, 2, \ldots, n + 1\}$, then

$$GG' < \frac{2nR + u}{n + 1} .$$

Proof. For any $i, j \in \{1, 2, \ldots, n + 1\}$ let G_i and G_j' be the centroids of the $(n - 1)$-simplexes $\mathcal{A}_i = A_1 \ldots A_{i-1}A_{i+1} \ldots A_{n+1}$ and $\mathcal{A}_j' = A_1' \ldots A_{j-1}'A_{j+1}' \ldots A_{n+1}'$. The points G_i, G_j' lie inside the hypersphere Σ with the centre O. We have

$$\frac{A_iG}{A_iG_i} = \frac{n}{n + 1} ,$$

so the homothetic transformation with the centre A_i and the coefficient $\frac{n}{n + 1}$ maps the hypersphere Σ onto a hypersphere Σ_i with the centre O_i and radius

$$R_i = \frac{n}{n + 1} R,$$

and the point G_i into the point G inside the hypersphere Σ_i. Here we have

$$\frac{A_iO_i}{A_iO} = \frac{n}{n + 1} , \quad \text{i.e.} \quad \frac{OO_i}{OA_i} = \frac{1}{n + 1} .$$

Analogously, the homothetic transformation with the centre A_j' and the coefficient $\frac{n}{n + 1}$ maps the hypersphere Σ onto a hypersphere Σ_j' with centre O_j' and radius

$$R_j' = \frac{n}{n + 1} R,$$

and the point G_j' into the point G' inside the hypersphere Σ_j'. Here also

$$\frac{OO_j'}{OA_j'} = \frac{1}{n + 1} .$$

Therefore, we have

$$\frac{O_iO_j'}{A_iA_j'} = \frac{1}{n + 1} .$$

The hyperspheres Σ_i and Σ'_j are contained inside the hypersphere with a diameter $O_iO'_j + R_i + R'_j$ and so we obtain

$$GG' < O_iO'_j + R_i + R'_j = \frac{2nR + A_iA'_j}{n + 1}.$$

But this inequality holds for every i, j ∈ {1, 2, ..., n + 1}.
J. Schopp [96]; GI 10.6 (n = 2).

4.14. If H and H' are the orthocentres of two orthocentric simplexes A and A' inscribed in the same hypersphere Σ with the radius R and if $u = \min A_iA'_j$ for i, j ∈ {1, 2, ..., n + 1}, then

$$HH' < \frac{2nR + u}{n - 1}.$$

Proof. Let G and G' be the centroids of A and A' and let O be the centre of Σ. The points O, G, H respectively O, G', H' lie on the Euler-lines of A respectively of A' and we have

$$\frac{OG}{OH} = \frac{OG'}{OH'} = \frac{n-1}{n+1}, \quad GG' = \frac{n-1}{n+1} HH'$$

and 4.13 implies 4.14.
J. Schopp [96]; GI 10.7 (n = 2).

References

1. F. Abeles, 'Inequalities for a Simplex and the Number e', J. Geom. 15 (1980), 149-152.
2. R. Alexander, 'The Width and Diameter of a Simplex', Geom. Ded. 6 (1977), 87-94.
3. M. M. Ali, 'On Some Extremal Simplexes', Pacific J. Math. 33 (1970), 1-14.
4. T. Andreescu and I. V. Maftei, 'Problema 3', Gaz. Mat. 87 (1982), 262.
5. C. Băluţă, 'Problema 8605', Gaz. Mat. B 19 (1968), 661.
6. D. M. Bătineţu, 'Asupra problemei 7764 dată la O.I.M. 1966', Gaz. Mat. B 19 (1968), 652-653.
7. D. M. Bătineţu, 'Unele inegalităţi în triunghiuri şi tetraedre', Gaz. Mat. B 21 (1970), 131-135.
8. J. Berkes, 'Einfacher Beweis und Verallgemeinerung einer Dreiecksungleichung', Elem. Math. 12 (1957), 121-123.
9. J. Berkes, 'Einfacher Beweis und Verallgemeinerung einer Dreiecksungleichung', Elem. Math. 22 (1967), 135-136.
10. J. Berkes and F. Leuenberger, 'Aufgabe 372', Elem. Math. 16 (1961), 40-41.
11. G. P. Bevz, 'Zadača 700', Mat. v Škole 1970, No. 4, 85-86.
12. O. Bottema, 'Problem 711', Nieuw Arch. Wisk. (4) 4 (1986), 82-83.
13. G. D. Chakerian and M. S. Klamkin, 'Inequalities for Means of Distances', Amer. Math. Monthly 80 (1973), 1009-1017.

14. O. Chisini, 'Questioni 98-99', Period. Mat. (4) 5 (1925), 123, 294-299, 371.
15. S.-N. Deaconu, 'Problema 20125', Gaz. Mat. 90 (1985), 155.
16. V. Devidé, 'Zadatak 150', Glasnik Mat.-Fiz. Astr. 7 (1952), 190-191.
17. E. Egerváry, 'Über ein Minimumproblem der Elementargeometrie', J. reine angew. Math. 178 (1938), 174-176.
18. E. Ehrhart, J. J. A. M. Brands, G. Laman, and H. G. Eggleston, 'Problem 5006', Amer. Math. Monthly 69 (1962), 63 and 70 (1963), 338-339, 1108.
19. M. Erdmann, 'Zadanie 8', Matematyka 1974, 183.
20. R. Estève, 'Sur une étude du tétraèdre', Bull. Assoc. Prof. Math. Ens. Public 41 (1961-1962), 165-173.
21. L. Fejes Tóth, Lagerungen in der Ebene, auf der Kugel und im Raum, Berlin-Göttingen-Heidelberg 1953.
22. L. Fejes Tóth, 'Extremum Properties of the Regular Polytopes', Acta Math. Acad. Sci. Hungar 6 (1955), 143-146.
23. L. Fejes Tóth, Regular Figures, Oxford-London-Edinburgh-New York-Paris-Frankfurt 1964.
24. J. Fickett, 'Problem E 2848', Amer. Math. Monthly 89 (1982), 335-336.
25. H. Gabai, 'Inequalities for Simplexes', Ibid. 73 (1966), 1083-1087.
26. J. W. Gaddum, 'Distance Sums on a Sphere and Angle Sums in a Simplex', Ibid. 63 (1956), 91-96.
27. G. Georgescu, 'Problema 6820', Gaz. Mat. B 16 (1965), 174.
28. Ju. I. Gerasimov, 'Zadača 550', Mat. v Škole 1969, No. 4, 77.
29. L. Gerber, 'The Orthocentric Simplex as an Extreme Simplex', Pacific J. Math. 56 (1975), 97-111.
30. E. M. Goljberg, 'Ob ocenkah objoma n-simpleksa čerez objoma ego granej', Vestnik Leningrad. Univ. 16 (1961), No. 13, 5-10.
31. E. G. Gotman, 'Zadača 1579', Mat. v Škole 1976, No. 2, 79-80.
32. E. G. Gotman, 'Neravenstva v geometričeskih zadačah', Ibid. 1985, No. 3, 46-48.
33. J. T. Groenman, 'Comment on Problem 718', Crux Math. 9 (1983), 82-83.
34. P. B. Gusjatnikov, 'Zadača M 759', Kvant 1983, No. 1, 42-43.
35. J. Hadamard, 'Resolution d'une question relative aux déterminants', Bull. Sci. Math. 2 (1893), 240-248.
36. H. Hadwiger, Vorlesungen über Inhalt, Oberfläche und Isoperimetrie, Berlin-Göttingen-Heidelberg 1957.
37. A. Heppes, 'Egy tetraéder felszínére vonatkozó szélsőértékfeladat', Mat. Lapok 12 (1961), 59-61.
38. W. Holsztyński and W. Kuperberg, 'O pewnej wlasności czworościanów', Wiadom. Mat. 6 (1962-1963), 13-16.
39. W. Holsztyński and W. Kuperberg, 'On a Property of Tetrahedra', Alabama J. Math. 1 (1977), 40-42.
40. S. Horák, 'Nerovnosti v geometrii. III', Rozhledy Mat. Fyz. 52 (1973-1974), 448-452.
41. M. S. K. Iyengar and K. V. Iyengar, 'On a Problem Relating to a Tetrahedron', Proc. Indian Acad. Sci. A 7 (1938), 305-311.
42. W. Jänichen, 'Aufgabe 623', Elem. Math. 26 (1971), 65-67.
43. G. Kalajdžić, 'Some Inequalities Concerning a Tetrahedron', Publ. Elektrotehn. Fak. Univ. Beograd. No. 320-328 (1970), 51-58.
44. G. Kalajdžić and S. Srećković, 'Problem 8', Matematika 4 (1975), No. 2, 99.
45. P. K. Kashikar, 'A Note on the Ratio of the In- and Circumradii of

a Tetrahedron', Math. Student 7 (1939), 61-63.
46. M. S. Klamkin, 'Notes on Inequalities Involving Triangles or Tetrahedrons', Publ. Elektrotehn. Fak. Univ. Beograd. No. 330-337 (1970), 1-15.
47. M. S. Klamkin, 'A Volume Inequality for Simplexes', Ibid. No. 357-380, (1971), 3-4.
48. M. S. Klamkin, 'An Identity for Simplexes and Related Inequalities', Simon Stevin 48 (1974-1975), 57-64.
49. M. S. Klamkin, 'Geometric Inequalities via the Polar Moment of Inertia', Math. Mag. 48 (1975), 44-46.
50. M. S. Klamkin, 'Problem 224', Crux Math. 3 (1977), 203-204.
51. M. S. Klamkin, 'Comment to Problem 959', Math. Mag. 50 (1977), 212-213.
52. M. S. Klamkin, 'Extensions of a Triangle Inequality of Carlitz', Publ. Elektrotehn. Fak. Univ. Beograd. No. 602-633 (1978), 147-149.
53. M. S. Klamkin, 'Problem 78-20', SIAM Rev. 21 (1979), 569-570.
54. M. S. Klamkin, 'Generalization of Problem 723', Crux Math. 8 (1982), 147-149.
55. M. S. Klamkin, 'Generalization of Problem 718', Ibid. 9 (1983), 83.
56. M. S. Klamkin, 'Aufgabe 902', Elem. Math. 39 (1984), 154-155.
57. M. S. Klamkin, 'Problem 85-26', SIAM Rev. 27 (1985), 576.
58. M. S. Klamkin and G. A. Tsintsifas, Math. Mag. 52 (1979), 20-22.
59. G. Korchmáros, 'Una limitazione per il volume di un simplesso n-dimensionale avente spigoli di date lunghezze', Atti Accad. Lincei, Rend. (8) 56 (1974), 876-879.
60. T. M. Korikova, 'Nekotorie geometričeskie neravenstva i ih vektornoe rešenie', Mat. v škole 1977, No. 3, 64-67.
61. G. Krammer, 'A szabályos tetraéder egy szélsőértéktulajdonsága', Mat. Lapok 12 (1961), 54-58.
62. I. A. Kušnir, 'Zadača 2775', Mat. v škole 1985, No. 4, 71.
63. L. H. Lange, 'Cutting Certain Minimum Corners', Amer. Math. Monthly 83 (1976), 361-365.
64. M. Lascu, 'Asupra unor inegalităţi într-un tetraedru', Gaz. Mat. 89 (1984), 397-399.
65. H. Laurent, Interméd. Math. 7 (1900), 60.
66. F. Leuenberger, 'Extremaleigenschaften der Summe der wichtigsten Ecktransversalen des n-dimensionalen Simplex', Elem. Math. 15 (1960), 81-82.
67. F. Leuenberger, 'Problem E 1402', Amer. Math. Monthly 67 (1960), 803-804.
68. F. Leuenberger, 'Aufgabe 367', Elem. Math. 16 (1961), 16.
69. C. Linderholm, 'An Inequality for Simplices', Geom. Ded. 21 (1986), 67-73.
70. L. M. Lopovok and V. Crişan, 'Problema 6156', Gaz. Mat. 16 (1965), 26-27.
71. A. Marmion, 'Sur quelques inégalités concernant les angles dièdres d'un tétraèdre', Mathesis 57 (1948), 343-345.
72. Z. E. Melzak, 'An Isoperimetric Inequality for Tetrahedra', Canad. Math. Bull. 9 (1966), 667-669.
73. D. M. Miloševič and P. Bundschuh, 'Aufgabe 890', Elem. Math. 38 (1983), 162-163.
74. Moret-Blanc, Nouv. Ann. Math. (3) 13 (1894), 25*-27*.
75. E. A. Morozova and I. S. Petrakov, 'VIII meždunarodnaja', Mat. v škole 1966, No. 6, 63, 65.

76. A. Oppenheim, 'Inequalities for a Simplex and an Internal Point', Publ. Elektrotehn. Fak. Univ. Beograd. No. 200-209 (1967), 17-20.
77. G. Păun, 'Problema 9613', Gaz. Mat. B 20 (1969), 308.
78. M. J. Pelling, 'Inequalities Involving the Area of a Quadrilateral Inscribed in a Convex Quadrilateral', Publ. Elektrotehn. Fak. Univ. Beograd. No. 498-541 (1975), 188-190.
79. C. M. Petty and D. Waterman, 'An Extremal Theorem for N-simplexes', Monatsh. Math. 59 (1955), 320-322.
80. J. Philip, 'Plane Sections of Simplices', Math. Programming 3 (1972), 312-325.
81. E. Piccioli, 'Sulle mediani dell'n-edro nello spazio lineare S_{n-1}', Boll. Mat. (3) 1 (1939), 29-30.
82. F. S. Pîrvănescu, 'Asupra unor inegalităţi într-un tetraedru', Gaz. Mat. 87 (1982), 57-58.
83. G. Pîrvănescu, 'Problema 17873', Ibid. 85 (1980), 112-113.
84. V. L. Rabinovič and I. M. Jaglom, 'O neravenstvah, rodstvennih neravenstvu Erdöša-Mordella dlja treugoljnika', Uč. Zap. Moskov. Gos. Ped. Inst. 401 (1971), t. 2, 123-128.
85. T. Rajkov, 'Zadača 4', Matematika (Sofija) 1975, No. 4, 37.
86. O. Reutter, 'Verallgemeinerung von Aufgabe 367', Elem. Math. 16 (1961), 16-18.
87. O. Reutter and F. Leuenberger, 'Aufgabe 454', Ibid. 19 (1964), 63-64.
88. R. Robinson and J. K. Peterson, 'Problem 3683', Amer. Math. Monthly 42 (1935), 632-635.
89. N. Şaganai, 'Problem E:7464', Gaz. Mat. 86 (1981), 466.
90. G. Sansone, 'Sulle espressioni del volume del tetraedro e su qualche problema di massimo'. Period. Mat. (4) 3 (1923), 20-50.
91. J. Schopp, 'Über eine Extremaleigenschaft des Simplex im n-dimensionalen Raum', Elem. Math. 13 (1958), 106-107.
92. J. Schopp, 'Généralisation de la Question 3846', Mathesis 68 (1959), 95-96.
93. J. Schopp, 'Généralisation de la Question 3853', Ibid. 68 (1959), 304.
94. J. Schopp, 'The Inequality of Steensholt for an n-dimensional Simplex', Amer. Math. Monthly 66 (1959), 896-897.
95. J. Schopp, 'Extremaleigenschaften der Ecktransversalen des n-dimensionalen Simplex', Elem. Math. 14 (1959), 61-62.
96. J. Schopp, 'Verallgemeinerung von Aufgabe 353', Ibid. 15 (1960), 85-86.
97. J. Schopp, 'Simplexungleichungen', Ibid. 16 (1961), 13-16.
98. J. Schopp, 'Über die n-dimensionalen Axonometrien', Ibid. 19 (1964), 108-110.
99. V. Senderov, 'Zadača M 615', Kvant 1981, No. 1, 30.
100. A. Simeonov, 'Zadača 3', Obzornik Mat. (Sofija) 1984, No. 1, 60-61.
101. D. O. Škljarskij, N. N. Čencov, and I. M. Jaglom, Geometričeskie neravenstva i zadači na maksimum i minimum, Moskva 1970.
102. D. Slepian, 'The Content of Some Extreme Simplexes', Pacific J. Math. 31 (1969), 795-808.
103. M. Stan, 'Asupra unor relaţii în triunghiuri şi tetraedre', Gaz. Mat. B 19 (1968), 334-335.
104. D. M. Stan, 'Problem 8964', Ibid. B 20 (1969), 227.
105. M. Stanković, 'Neke nejednakosti za tetraedar', Mat. Biblioteka 41 (1969), 200-202.

106. M. Stanković, 'Some Inequalities for a Simplex', Publ. Elektrotehn. Fak. Univ. Beograd. No. 330-337 (1970), 53-54.
107. J. Steiner, 'Einfache Beweise der isoperimetrischen Hauptsätze', J. reine angew. Math. 18 (1838), 281-296; Ges. Werke II, 75-91.
108. J. Steiner, 'Über Maximum und Minimum bei den Figuren in der Ebene, auf der Kugelfläche und im Raume überhaupt', Ibid. 24 (1842), 93-152, 189-250; Ges. Werke II, 177-308.
109. Gh. Stoica, 'Problema C:315', Gaz. Mat. 88 (1983), 256.
110. L. Stojanov, 'Zadača 3', Matematika (Sofija) 1982, No. 10, 35-36.
111. R. Sturm, 'Würfel und reguläres Tetraeder als Maximum und Minimum', J. reine angew. Math. 97 (1884), 1-12.
112. R. Sturm, Maxima und Minima in der elementaren Geometrie, Leipzig-Berlin 1910.
113. Ja. N. Sukonnik, 'Zadača 1772', Mat. v škole 1977, No. 4, 72.
114. G. Szász, 'Über die normale Axonometrie', Elem. Math. 18 (1963), 58-60.
115. J. B. Tabov, 'Problem 1065', Crux Math. 11 (1985), 220.
116. C. Tănase, 'Problema 18253', Gaz. Mat. 87 (1982), 141.
117. R. M. Tanner, 'Some Content Maximizing Properties of the Regular Simplex', Pacific J. Math. 52 (1974), 611-616.
118. V. Thébault, 'Sur deux propriétiés des médianes du tétraèdre', Boll. Mat. (3) 1 (1939), 13-14.
119. V. Thébault, 'Question 12', Mathesis 64 (1955), 220.
120. V. Thébault, 'Question 3595', Ibid. 65 (1956), 554-555.
121. V. Thébault, 'Question 3', Ibid. 66 (1957), 123.
122. V. Thébault, 'Problem E 1264', Amer. Math. Monthly 64 (1957), 744-745.
123. V. Thébault, 'Notes de géométrie élémentaire', Mathesis 67 (1958), suppl., 24 p.
124. V. Thébault, 'Question 3846', Ibid. 68 (1959), 95.
125. V. Thébault, 'Question 3853', Ibid. 68 (1959), 304.
126. V. Thébault, 'Problem E 1394', Amer. Math. Monthly 67 (1960), 595-596.
127. J. A. Tierney, 'Elementary Techniques in Maxima and Minima', Math. Teacher 46 (1953), 484-486.
128. V. M. Tihomirov, 'Ob odnoj olimpiadnoj zadače', Kvant 1983, No. 1, 22-25.
129. I. Tomescu, 'Problema O:153', Gaz. Mat. 85 (1980), 325.
130. G. Tsintsifas,'Problem E 2987', Amer. Math. Monthly 92 (1985), 669.
131. G. Tsintsifas and G. P. Henderson, 'Problem 606', Crux Math. 8 (1982), 108-109.
132. G. Tsintsifas and S. Rabinowitz, 'Problem 718', Ibid. 9 (1983), 82-84.
133. D. Veljan, 'Problem 14', Glasnik Mat. 5 (25) (1970), 354-355.
134. D. Veljan and G. Korchmáros, 'Aufgabe 629 A', Elem. Math. 27 (1972), 14-15.
135. B. R. Venkataraman, 'Inequalities Connected with a Tetrahedron', Math. Student 6 (1938), 78-79.
136. V. Gh. Vodă, 'Zadača 5', Mat. v škole 1963, No. 5, 89.
137. D. Voiculescu and F. Leuenberger, 'Aufgabe 583', Elem. Math. 24 (1969), 116-117.
138. V. Volenec, 'Einige Sätze über die Halbmesser von Hyperkugeln, die alle Seiten eines Simplexes in E^n berühren', Glasnik Mat. 4 (24) (1969), 131-137.

139. V. Volenec, 'Über einigen algebraische und geometrische Ungleichungen', Publ. Elektrotehn. Fak. Univ. Beograd. No. 274-301 (1969), 103-113.
140. D. W. Walkup, 'A Simplex with a Large Cross Section', Amer. Math. Monthly 75 (1968), 34-36.
141. J. Weinstein and I. Dăncilă, 'Problema 7248', Gaz. Mat. 17 (1966), 455-456.
142. B. Weissbach, 'Über Umkugeln von Projektionen regulärer Simplexe', Beitr. Algebra Geom. 16 (1983), 127-137.
143. L. Yang and J. Zh. Zhang, 'Two Inequalities of Higher Dimensional Metric Geometry' (Chinese), J. Chengdu Univ. Sci. Technol. 1981, 63-70.
144. L. Yang and J. Zh. Zhang, 'On a Conjecture of R. Alexander', Kexue Tongbao 27 (1982), 699-703.
145. L. Yang and J. Zh. Zhang, 'A Generalization to Several Dimensions of the Neuberg-Pedoe Inequality, with Applications', Bull. Austral. Math. Soc. 27 (1983), 203-214.
146. L. Yang and J. Zh. Zhang, 'A Sufficient and Necessary Condition for Embedding a Simplex with Prescribed Dihedral Angles in E^n' (Chinese), Acta Math. Sinica 26 (1983), 250-256.
147. L. Yang and J. Zh. Zhang, 'Application of the Metric Equation to a Conjecture of Sallee' (Chinese), Ibid. 26 (1983), 488-493.
148. J. Zh. Zhang and L. Yang, 'A Class of Geometric Inequalities Concerning the Mass-point System' (Chinese), J. China Univ. Sci. Technol. 11 (1981), 1-8.
149. Ž. Živanović, 'Inequalities for Simplexes', Publ. Elektrotehn. Fak. Univ. Beograd. No. 230-241 (1968), 73-75.
150. F. G.-M., Exercices de géométrie, 4. éd., Tours-Paris 1912.
151. 'Zadača 32', Mat. v škole 1956, No. 1, 91-92.
152. 'Zadača 50', Ibid. 1964, No. 3, 78-79.
153. 'Feladat P.205', Közöp. Mat. Lapok 48 (1974), 126 and 51 (1975), 73-75.
154. 'Zadača 4', Matematika (Sofija) 1975, No. 2, 38-39.
155. 'Zadača 1599', Mat. v škole 1976, No. 3, 82.
156. 'Zadača 1955', Ibid. 1978, No. 6, 62-63.
157. 'Feladat 2425', Közöp. Mat. Lapok 1983, 202-203.
158. E. Hille, 'Some Geometric Extremal Problems', J. Austral. Math. Soc. 6 (1966), 122-128.
159. R. Hoppe, 'Maximum der Ecken eines Tetraeders für den Fall ihrer Gleichheit', Arch. Math. Phys. (2) 10 (1891), 111-112.
160. V. Thébault and L. M. Kelly, 'Problem 31', Math. Mag. 23 (1949-1950), 108.
161. J. Zh. Zhang and L. Yang, 'A Class of Geometric Inequalities on Finite Points (Chinese)', Acta Math. Sinica 23 (1980), 740-749.
162. 'Zadača 15', Mat. v škole 1955, No. 5, 91-93.

Chapter XIX

INEQUALITIES FOR TETRAHEDRA

1. Inequalities for the Edge Lengths and Face Areas of a Tetrahedron

Let $a = BC$, $b = CA$, $c = AB$, $a' = AD$, $b' = BD$, $c' = CD$ be the edge lengths of a tetrahedron $T = ABCD$ and F_A, F_B, F_C, F_D the areas of its faces BCD, CDA, DAB, ABC. We shall use the expressions

$$E = a + b + c + a' + b' + c', \quad F = F_A + F_B + F_C + F_D,$$

$$u = a^2 + a'^2, \quad v = b^2 + b'^2, \quad w = c^2 + c'^2,$$

$$p_A = b'c' + c'a + ab', \quad p_B = bc' + c'a' + a'b,$$

$$p_C = b'c + ca' + a'b', \quad p_D = bc + ca + ab,$$

$$P = p_A + p_B + p_C + p_D, \quad Q = a^2 + b^2 + c^2 + a'^2 + b'^2 + c'^2.$$

The signs $\{T_R\}$, $\{T_{ID}\}$ and $\{T_O\}$ mean that equality holds iff T is a regular tetrahedron, an isodynamic tetrahedron (tetrahedron with $aa' = bb' = cc'$) and an orthocentric tetrahedron (tetrahedron with $u = v = w$) respectively.

1.1. Let $a, b, c, a', b', c' \in R^+$. The inequalities

(1) $$a + b > c, \quad a + c > b, \quad b + c > a,$$

(2) $$\begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & c^2 & b^2 & a'^2 \\ 1 & c^2 & 0 & a^2 & b'^2 \\ 1 & b^2 & a^2 & 0 & c'^2 \\ 1 & a'^2 & b'^2 & c'^2 & 0 \end{vmatrix} > 0$$

are necessary and sufficient conditions for the existence of a tetrahedron $T = ABCD$ with edge lengths $a = BC$, $b = CA$, $c = AB$, $a' = AD$, $b' = BD$, $c' = CD$.

Proof. The necessity of conditions (1) and (2) is obvious, because the determinant in (2) equals $288V^2$, where V is the volume of T. Let us prove the sufficiency of our conditions. The inequality (2) can be written in the form

$$(3)\qquad (a^4 - a^2b^2 - a^2c^2 + 2a^2a'^2 - a^2b'^2 - a^2c'^2 - b^2b'^2 + b^2c'^2 + b'^2c^2 - c^2c'^2)^2 < (2a^2b^2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4)(2a^2b'^2 + 2a^2c'^2 + 2b'^2c'^2 - a^4 - b'^4 - c'^4).$$

According to (1) there is a triangle ABC with BC = a, CA = b, AB = c, and then

$$2a^2b^2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4 > 0$$

(by Heron's formula). Therefore, from (3) we obtain

$$2a^2b'^2 + 2a^2c'^2 + 2b'^2c'^2 - a^4 - b'^4 - c'^4 > 0,$$

and there is a triangle with sides a, b', c'. Let D' and D'' be the points in the plane ABC (D' on the same side of the line BC as the point A, and D'' on the opposite side of this line) such that BD' = BD'' = b', CD' = CD'' = c'. The tetrahedron T exists if AD' < a' < AD''. But, we have

$$AD'^2 = (m - n)^2 + (h - k)^2 = (m - n)^2 + h^2 + k^2 - 2hk,$$

$$AD''^2 = (m - n)^2 + (h + k)^2 = (m - n)^2 + h^2 + k^2 + 2hk,$$

where h = AH, k = D'K = D''K are the altitudes of triangles ABC, D'BC and D''BC and m = BH, n = BK. Therefore, the tetrahedron T exists if

$$(m - n)^2 + h^2 + k^2 - 2hk < a'^2 < (m - n)^2 + h^2 + k^2 + 2hk,$$

i.e.

$$(4)\qquad [a'^2 - (m - n)^2 - h^2 - k^2]^2 < 4h^2k^2.$$

As we have

$$h^2 = \frac{1}{4a^2}(2a^2b^2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4),$$

$$k^2 = \frac{1}{4a^2}(2a^2b'^2 + 2a^2c'^2 + 2b'^2c'^2 - a^4 - b'^4 - c'^4),$$

$$m = \frac{1}{2a}(a^2 + c^2 - b^2), \qquad n = \frac{1}{2a}(a^2 + b'^2 - c'^2),$$

the condition (4) is equivalent to (3).

O. Bottema [8]; R. Fritsch [15]; J. F. Rigby [48].

1.2. $a + a' + b + b' > c + c', \quad a + a' + c + c' > b + b',$

$$b + b' + c + c' > a + a'.$$

Proof. The first inequality follows by addition of the inequalities $a + b > c$, $a + b' > c'$, $a' + b > c'$, $a' + b' > c$. The other two are proved similarly.

M. S. Klamkin [33]; M. Lascu [39].

1.3. $u + v > w, \quad u + w > v, \quad v + w > u.$

Proof. Let $\vec{BC} = \underline{a}$, $\vec{CA} = \underline{b}$, $\vec{AB} = \underline{c}$, $\vec{DA} = \underline{a}'$, $\vec{DB} = \underline{b}'$, $\vec{DC} = \underline{c}'$. Then we obtain successively

(5) $$\underline{a} + \underline{a}' = \underline{c}' - \underline{c}, \quad \underline{b} + \underline{b}' = \underline{a}' - \underline{a}, \quad \underline{c} + \underline{c}' = \underline{b}' - \underline{b},$$

$$(\underline{a} + \underline{a}')^2 + (\underline{b} + \underline{b}')^2 - (\underline{c} + \underline{c}')^2 =$$

$$= (\underline{c}' - \underline{c})^2 + (\underline{a}' - \underline{a})^2 - (\underline{b}' - \underline{b})^2,$$

$$2\underline{a} \cdot \underline{a}' = c^2 + c'^2 - b^2 - b'^2,$$

$$u + v - w = a^2 + a'^2 - 2\underline{a} \cdot \underline{a}' = (\underline{a} - \underline{a}')^2 > 0.$$

Bhattacharya and H. W. Curjel [6]; E. A. Jasinovij [28]; Š. Arslanagić [3]; M. Lascu [39].

Remark. If M, N are the midpoints of edges $\overline{AB}$, $\overline{CD}$, then $\underline{a} - \underline{a}' = 2\vec{MN}$ and (5) implies $a^2 + a'^2 + b^2 + b'^2 - c^2 - c'^2 = 4MN^2$. This identity holds also in the case of planar quadrangle ABCD and was known to L. Euler (Nov. Comm. Petrop. 1747-1748, I, 49).

1.4. $$\frac{a}{b' + c'} + \frac{b}{c' + a'} > \frac{c}{a' + b'}, \quad \frac{b}{c' + a'} + \frac{c}{a' + b'} > \frac{a}{b' + c'},$$

$$\frac{c}{a' + b'} + \frac{a}{b' + c'} > \frac{b}{c' + a'}.$$

Proof. Suppose $a' \geqslant b' \geqslant c'$. Since $a + b > c$, we obtain

$$\frac{a}{b' + c'} + \frac{b}{c' + a'} \geqslant \frac{a}{b' + a'} + \frac{b}{b' + a'} = \frac{a + b}{a' + b'} > \frac{c}{a' + b'}.$$

The inequalities

$$\frac{b}{c' + a'} \geqslant \frac{bb'}{a'(b' + c')}, \quad \frac{c}{a' + b'} \geqslant \frac{cc'}{a'(b' + c')}$$

are equivalent to $a' \geqslant b'$ and $a' \geqslant c'$, respectively, and from them we obtain by 1.6 that

$$\frac{b}{c' + a'} + \frac{c}{a' + b'} \geqslant \frac{bb' + cc'}{a'(b' + c')} > \frac{aa'}{a'(b' + c')} = \frac{a}{b' + c'} .$$

If we have

$$\frac{c}{a' + b'} + \frac{a}{b' + c'} \geqslant \frac{c + a}{c' + a'} ,$$

then by $c + a > b$ the third inequality 1.4 follows. But, if we have

$$\frac{c}{a' + b'} + \frac{a}{b' + c'} < \frac{c + a}{c' + a'} ,$$

then this can be written in the equivalent form $c(b'^2 - c'^2) > a(a'^2 - b'^2)$, from which we get $ca'(b'^2 - c'^2) > ac'(a'^2 - b'^2)$, because of $a' \geqslant c'$. This is equivalent to the inequality

$$\frac{c}{a' + b'} + \frac{a}{b' + c'} > \frac{cc' + aa'}{b'(c' + a')} ,$$

which implies by 1.6

$$\frac{c}{a' + b'} + \frac{a}{b' + c'} > \frac{bb'}{b'(c' + a')} = \frac{b}{c' + a'} .$$

M. S. Klamkin [34]; D. Skordev [56].

1.5. $$\frac{a}{\max(b', c')} + \frac{b}{\max(c', a')} > \frac{c}{\max(a', b')} ,$$

$$\frac{b}{\max(c', a')} + \frac{c}{\max(a', b')} > \frac{a}{\max(b', c')} ,$$

$$\frac{c}{\max(a', b')} + \frac{a}{\max(b', c')} > \frac{b}{\max(c', a')} .$$

Proof. Suppose $a' \geqslant b' \geqslant c'$. Then

$$\frac{a}{\max(b', c')} = \frac{a}{b'} , \quad \frac{b}{\max(c', a')} = \frac{b}{a'} , \quad \frac{c}{\max(a', b')} = \frac{c}{a'} .$$

Because of $a + b > c$ we have

$$\frac{a}{b'} + \frac{b}{a'} = \frac{aa' + bb'}{a'b'} \geqslant \frac{ab' + bb'}{a'b'} = \frac{a + b}{a'} > \frac{c}{a'} ,$$

and by 1.6 we obtain

$$\frac{b}{a'} + \frac{c}{a'} = \frac{bb' + cb'}{a'b'} \geqslant \frac{bb' + cc'}{a'b'} > \frac{aa'}{a'b'} = \frac{a}{b'} ,$$

$$\frac{c}{a'} + \frac{a}{b'} = \frac{aa' + cb'}{a'b'} \geqslant \frac{aa' + cc'}{a'b'} > \frac{bb'}{a'b'} = \frac{b}{a'} .$$

D. Skordev [55].

1.6. $aa' + bb' > cc', \quad bb' + cc' > aa', \quad cc' + aa' > bb'.$

Proof. The inversion with respect to the sphere with the centre D and radius 1 maps the points A, B, C into the points A', B', C' and we have

$$B'C' = \frac{a}{b'c'}, \quad C'A' = \frac{b}{c'a'}, \quad A'B' = \frac{c}{a'b'}.$$

Therefore, from $B'C' + C'A' > A'B'$ we obtain

$$\frac{a}{b'c'} + \frac{b}{c'a'} > \frac{c}{a'b'},$$

i.e. $aa' + bb' > cc'$. Analogously, we can prove the other two inequalities 1.6.

Bhattacharya and H. W. Curjel [6]; V. Thébault [68]; W. Hansen [22]; T. M. Apostol [2]; D. Veljan [70]; N. Zhu [76]; M. Lascu [39].

1.7. If $H = aa'bb' + bb'cc' + cc'aa'$, then

$$H \leqslant (aa')^2 + (bb')^2 + (cc')^2 < 2H \quad \{T_{ID}\}.$$

Proof. This follows by applying the inequality GI 1.1 to the triangle with the sides aa', bb', cc', which exists according to 1.6.

D. Nikolov [42].

1.8. $$(-aa' + bb' + cc')(aa' - bb' + cc')(aa' + bb' - cc') \leqslant abca'b'c' \quad \{T_{ID}\}.$$

Proof. Follows by applying of GI 1.3 to the triangle with sides aa', bb', cc'.

A. Oppenheim [43]; M. Lascu [39].

1.9. $$\left(\frac{a'}{a}\right)^2 + \left(\frac{b'}{b}\right)' + \left(\frac{c'}{c}\right)^2 > 1.$$

Proof. It is sufficient to prove the inequality

$$\left(\frac{AP}{a}\right)^2 + \left(\frac{BP}{b}\right)^2 + \left(\frac{CP}{c}\right)^2 \geqslant 1, \tag{6}$$

where P is the orthogonal projection of the point D onto the plane ABC. With

$$n = 2, \quad \lambda_1 = \frac{1}{a^2}, \quad \lambda_2 = \frac{1}{b^2}, \quad \lambda_3 = \frac{1}{c^2},$$

according to XVIII.2.24, the left-hand side of (6) has the form

$$\lambda_1 \cdot PA^2 + \lambda_2 \cdot PB^2 + \lambda_3 \cdot PC^2 \geqslant$$

$$\geqslant \frac{1}{(\lambda_1 + \lambda_2 + \lambda_3)^2}\left[\frac{\lambda_2\lambda_3(\lambda_2 + \lambda_3)}{\lambda_1} + \frac{\lambda_3\lambda_1(\lambda_3 + \lambda_1)}{\lambda_2} + \right.$$

$$\left. + \frac{\lambda_1\lambda_2(\lambda_1 + \lambda_2)}{\lambda_3} + \lambda_1\lambda_2\lambda_3\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3}\right)\right] =$$

$$= \frac{1}{\lambda_1 + \lambda_2 + \lambda_3}\left(\frac{\lambda_2\lambda_3}{\lambda_1} + \frac{\lambda_3\lambda_1}{\lambda_2} + \frac{\lambda_1\lambda_2}{\lambda_3}\right) =$$

$$= \frac{1}{2(\lambda_1 + \lambda_2 + \lambda_3)}\left[\lambda_1\left(\frac{\lambda_2}{\lambda_3} + \frac{\lambda_3}{\lambda_2}\right) + \lambda_2\left(\frac{\lambda_3}{\lambda_1} + \frac{\lambda_1}{\lambda_3}\right) + \lambda_3\left(\frac{\lambda_1}{\lambda_2} + \frac{\lambda_2}{\lambda_1}\right)\right]$$

$$\geqslant \frac{1}{2(\lambda_1 + \lambda_2 + \lambda_3)}(2\lambda_1 + 2\lambda_2 + 2\lambda_3) = 1.$$

J. Szikszai [66].

1.10. $$\frac{a'^2}{bc} + \frac{b'^2}{ca} + \frac{c'^2}{ab} > 1.$$

Proof. It is sufficient to prove the inequality $a \cdot AP^2 + b \cdot BP^2 + c \cdot CP^2 \geqslant abc$, where P is the orthogonal projection of D onto the plane ABC. With $n = 2$, $\lambda_1 = a$, $\lambda_2 = b$, $\lambda_3 = c$ according to XVIII.2.24 we obtain

$$a \cdot PA^2 + b \cdot PB^2 + c \cdot CP^2 \geqslant \frac{1}{(a + b + c)^2}[bc(b + c)a^2 +$$

$$+ ca(c + a)b^2 + ab(a + b)c^2 + abc(a^2 + b^2 + c^2)] = abc.$$

1.11. $$\frac{b'c'}{bc} + \frac{c'a'}{ca} + \frac{a'b'}{ab} > 1.$$

J. Zh. Zhang and L. Yang [75].

1.12. $P \leqslant 2Q < 2P \quad \{T_R\}.$

Proof. Follows by addition of the four inequalities obtained by application of GI 1.1 to the triangles BCD, CDA, DAB, ABC.

F. Harant [23].

1.13. $$Q \geq \frac{9}{26}[P + 4(\frac{abc}{a + b + c} + \frac{ab'c'}{a + b' + c'} + \frac{a'bc'}{a' + b + c'} + \frac{a'b'c}{a' + b' + c})] \quad \{T_R\}.$$

Proof. Follows by application of GI 1.2 to triangles ABC, BCD, CDA, ABC and addition of the inequalities obtained.

F. Harant [23].

1.14. $$F_A < F_B + F_C + F_D, \quad F_B < F_C + F_D + F_A,$$
$$F_C < F_D + F_A + F_B, \quad F_D < F_A + F_B + F_C.$$

M. S. Klamkin [32].

Remark. Inequalities 1.14 are necessary and sufficient conditions for the existence of a tetrahedron with face areas F_A, F_B, F_C, F_D.

F. Harant [23]; R. Fritsch [15].

1.15. $$3P - 5Q \leq 2F\sqrt{3} \leq P - Q \quad \{T_R\}.$$

Proof. The inequalities GI 4.7 for a triangle with sides a, b, c and area F can be written in an equivalent form

$$3(bc + ca + ab) - \frac{5}{2}(a^2 + b^2 + c^2) \leq 2F\sqrt{3} \leq bc + ca + ab - \frac{1}{2}(a^2 + b^2 + c^2) \quad \{a = b = c\}.$$

By application of these inequalities to triangles BCD, CDA, DAB, ABC and addition of the inequalities obtained we get 1.15.

F. Harant [23].

Remark 1. The second inequality 1.15 can be written in an equivalent form

1.15' $$E^2 \geq 12\sqrt{3}F + 2K + 3L \quad \{T_R\},$$

where $K = (a + a' - b - b')^2 + (b + b' - c - c')^2 + (c + c' - a - a')^2$, $L = (a - a')^2 + (b - b')^2 + (c - c')^2$. Inequality 1.15' is an improvement of the third inequality 2.1.

D. O. Škljarskij, N. N. Čencov, and I. M. Jaglom [54, p. 47, 268-269].

Remark 2. According to the first inequality 1.12 the second inequality 1.15 implies $2\sqrt{3}F \leq Q$, i.e.

1.15'' $$F \leq \frac{\sqrt{3}}{6} Q \quad \{T_R\}.$$

This inequality is also a consequence of the third inequality 2.1 by the

use of the A-Q-inequality for means.
N. Teodorescu [67].

1.16. $F_A^2 + F_B^2 + F_C^2 + F_D^2 \leqslant \frac{1}{4}[(aa')^2 + (bb')^2 + (cc')^2]$ $\{T_O\}$.

Proof. From

$$(7) \qquad 16F_A^2 = 2a^2b'^2 + 2a^2c'^2 + 2b'^2c'^2 - a^4 - b'^4 - c'^4,$$

$$16F_B^2 = 2a'^2b^2 + 2a'^2c'^2 + 2b^2c'^2 - a'^4 - b^4 - c'^4,$$

$$16F_C^2 = 2a'^2b'^2 + 2a'^2c^2 + 2b'^2c^2 - a'^4 - b'^4 - c^4,$$

$$16F_D^2 = 2a^2b^2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4$$

we obtain

$$4[(aa')^2 + (bb')^2 + (cc')^2] - 16(F_A^2 + F_B^2 + F_C^2 + F_D^2) =$$

$$= (a^2 + a'^2 - b^2 - b'^2)^2 + (b^2 + b'^2 - c^2 - c'^2)^2 +$$

$$+ (c^2 + c'^2 - a^2 - a'^2)^2 \geqslant 0.$$

G. P. Bevz [5].

1.17. $F_A^2 + F_B^2 + F_C^2 + F_D^2 \leqslant \frac{1}{48}(a^2 + b^2 + c^2 + a'^2 + b'^2 + c'^2)^2$ $\{T_R\}$.

Proof. From (7) we obtain

$$(a^2 + b^2 + c^2 + a'^2 + b'^2 + c'^2)^2 - 48(F_A^2 + F_B^2 + F_C^2 + F_D^2) =$$

$$= 2(a^2 + a'^2 - b^2 - b'^2)^2 + 2(b^2 + b'^2 - c^2 - c'^2)^2 +$$

$$+ 2(c^2 + c'^2 - a^2 - a'^2)^2 + 3(a^2 - a'^2)^2 + 3(b^2 - b'^2)^2 +$$

$$+ 3(c^2 - c'^2)^2 \geqslant 0.$$

Y. Aihara [1].

Comment by W. Janous. The right-hand sides (RHS) of 1.16 and 1.17 are incomparable in general. Indeed, if $c = n$, $c' \approx 0$, $a \approx a' \approx b \approx b' \approx n/2$, then RHS (1.16) $\approx n^4/32$ and RHS (1.17) $\approx n^4/12$. On the other hand, for $a \approx a' \approx n$, $b \approx b' \approx n$, $c \approx c' \approx 0$ we get RHS (1.16) $\approx n^4/12$ and

RHS (1.17) $\approx n^4/3$.

It should be noted that because of 1.3-1.6 all inequalities for the sides of a triangle only yield inequalities linking the edges of a tetrahedron. So e.g. 1.3 and GI 1.3 lead to

$$(v + w - u)(w + u - v)(u + v - w) \leqslant uvw \quad \{T_O\}.$$

2. Inequalities for the Volume, the Circumradius and Other Elements of a Tetrahedron

With the notation from 1 let V, R and r be the volume, the circumradius and the inradius respectively, of the tetrahedron T and $\overline{F}$ the area of the triangle with sides aa', bb', cc', which exists by 1.6. The sign $\{T_I\}$ means that equality holds iff T is an isosceles tetrahedron, i.e. iff a = a', b = b', c = c'.

2.1. $$3r \leqslant \frac{1}{2}\sqrt[3]{9\sqrt{3}V} \leqslant \frac{1}{4}\sqrt{2\sqrt{3}F} \leqslant \frac{\sqrt{6}}{24} E \leqslant R \quad \{T_R\}.$$

Proof. The first inequality is XVIII.2.40 (with n = 3), the third inequality is a consequence of 1.15' and the fourth inequality is XVIII.3.2 (with n = 3, k = 1). The second inequality is equivalent to the inequality

$$F^2 \geqslant 36V\sqrt[3]{3V},$$

which is a consequence of the inequality

$$\frac{2d^3 + 3V}{d^2} \geqslant 3\sqrt[3]{3V},$$

because of 2.19. But, the last inequality is equivalent successively to

$$(2d^3 + 3V)^3 - 81d^6V \geqslant 0,$$

$$(d^3 - 3V)^2(8d^3 + 3V) \geqslant 0.$$

Equality holds iff the conditions for equality in 2.19 are satisfied and $d^3 = 3V$. According to the equalities 6V = aa'd, a = a' we obtain $2d^2 = a^2$ and this together with the conditions for equality in 2.19 implies that T is a regular tetrahedron.

J. Steiner [58], [59]; R. Sturm [63], [64, p. 115-116]; G. Sansone [51]; O. Chisini [10]; E. Steinitz [60]; Y. Aihara [1]; K. S. K. Iyengar [25]; B. R. Venkataraman [72]; K. S. K. Iyengar and K. V. Iyengar [27]; G. Krammer [38]; A. Heppes [24]; Z. A. Melzak [40]; G. Kalajdžić [29]; D. O. Škljarskij, N. N. Čencov, and I. M. Jaglom [54, p. 46, 47, 50, 257-262, 280]; F. Harant [23].

2.2. If M = max (a, b, c, a', b', c'), then

$$V \leqslant \frac{1}{8} M^3 \quad \{a = b = a' = b' = c' = M,\ c = \frac{\sqrt{6}}{2} M\}.$$

Proof. Let $M = 1$. If $h_a = AA'$ is the altitude of the face ACD, then $CA' \geqslant c'/2$ or $DA' \geqslant c'/2$. Let $CA' \geqslant c'/2$. Then

$$h_a = \sqrt{AC^2 - CA'^2} \leqslant \sqrt{1 - \frac{c'^2}{4}}, \quad F_B = \frac{1}{2} c' h_a \leqslant \frac{1}{2} c' \sqrt{1 - \frac{c'^2}{4}}.$$

If $h_b = BB'$ is the altitude of the face BCD and h_B the altitude of T (onto ACD), then analogously

$$h_b \leqslant \sqrt{1 - \frac{c'^2}{4}}$$

and $h_B \leqslant h_b$. Therefore,

$$V = \frac{1}{3} F_B h_B \leqslant \frac{1}{6} c' \left(1 - \frac{c'^2}{4}\right).$$

Now, for $0 < c' \leqslant 1$ we have

$$\frac{1}{8} - V \geqslant \frac{1}{8} - \frac{1}{24} c'(4 - c'^2) = \frac{1}{24}(c' - 1)(c'^2 + c' - 3) \geqslant 0$$

because of $c' - 1 \leqslant 0$, $c'^2 + c' - 3 < 0$. Equality holds iff ACD and BCD are equilateral triangles with side lengths equal 1 and the dihedral angle at the edge of CD is right, i.e. $c = AB = (\sqrt{3}/2) \cdot \sqrt{2} = \sqrt{6}/2$.

D. O. Škljarskij, N. N. Čencov, and I. M. Jaglom [54, p. 32, 166-167].

2.3. If $F_A \leqslant F_B \leqslant F_C \leqslant F_D$, then

$$V^2 \leqslant \frac{2}{9} F_A F_B F_C \left[1 - \frac{(F_A^2 + F_B^2 + F_C^2 - F_D^2)^2}{12(F_B^2 F_C^2 + F_C^2 F_A^2 + F_A^2 F_B^2)}\right] \quad \{T_0\}.$$

K. S. K. Iyengar [26].

Remark. K. Venkatachaliengar [71] has a weaker result

$$V^2 \leqslant \frac{2}{9} F_A F_B F_C \quad \{T_0\}.$$

2.4. If R_D is the circumradius of the face ABC, then

$$6 R_D V \leqslant \overline{F}$$

with equality iff the circumcentre of T lies in the plane ABC.

Proof. The inversion with centre D and radius 1 maps the points A, B, C onto the points A_1, B_1, C_1. Let V_1 be the volume of the tetrahedron $A_1B_1C_1D$, h_1 the altitude from D of this tetrahedron, F_1 the area of the triangle $A_1B_1C_1$ and $a_1 = B_1C_1$, $b_1 = C_1A_1$, $c_1 = A_1B_1$, $a_1' = A_1D$, $b_1' = B_1D$, $c_1' = C_1D$. Then

$$(1) \qquad a_1 = \frac{a}{b'c'}, \quad b_1 = \frac{b}{c'a'}, \quad c_1 = \frac{c}{a'b'},$$

$$h_1 \cdot 2R = 1, \quad 3V_1 = h_1F_1, \quad \frac{V_1}{V} = \frac{a_1'b_1'c_1'}{a'b'c'},$$

and therefore

$$(2) \qquad 3\,\frac{a_1'b_1'c_1'}{a'b'c'}\,V = \frac{F_1}{2R}.$$

Since from (1) we have

$$a_1 = \frac{1}{a'b'c'}\,aa', \quad b_1 = \frac{1}{a'b'c'}\,bb', \quad c_1 = \frac{1}{a'b'c'}\,cc',$$

$$(3) \qquad \frac{F_1}{\overline{F}} = \left(\frac{1}{a'b'c'}\right)^2$$

follows. From (2), (3) and the equalities $a'a_1' = b'b_1' = c'c_1' = 1$ we obtain

$$(4) \qquad 6RV = \overline{F}.$$

From (4) according to $R \geqslant R_D$ we obtain 2.4.

A. Oppenheim [43]; M. Lascu [39].

2.5. $72V^2 \leqslant$

$$\frac{(aa' + bb' + cc')(-aa' + bb' + cc')(aa' - bb' + cc')(aa' + bb' - cc')}{\sqrt{a^2 + b^2 + c^2}\sqrt{a'^2 + b'^2 + c'^2}}$$

$\{T_I\}$.

Proof. Follows by the inequality 2.13, equality (4) and the equality

$$16\overline{F}^2 = (aa' + bb' + cc')(-aa' + bb' + cc')(aa' - bb' + cc')(aa' + bb' - cc'),$$

which can be obtained by application of Heron's formula.

A. Oppenheim [43]; M. Lascu [39].

2.6. $72V^2 \leqslant (-aa' + bb' + cc')(aa' - bb' + cc')(aa' + bb' - cc') \quad \{T_I\}$.

Proof. Follows from 2.5 according to Cauchy's inequality

$$aa' + bb' + cc' \leqslant \sqrt{a^2 + b^2 + c^2}\sqrt{a'^2 + b'^2 + c'^2}.$$

A. Oppenheim [43]; M. Lascu [39].

2.7. $1944V^2 \leqslant (aa' + bb' + cc')^3 \quad \{T_R\}$.

Proof. Follows from 2.8 according to the A-G-inequality for means.
G. Kalajdžić [29].

2.8. $72V^2 \leqslant abca'b'c' \quad \{T_R\}$.

Proof. Follows from 2.6 and 1.8.
D. Voiculescu and F. Leuenberger [73]; N. Şaganai [50]; M. Lascu [39].

Remark. B. R. Venkataraman [72] has a weaker inequality

$$36V^2 < abca'b'c'.$$

2.9. $V < \frac{R^2\sqrt{2}}{6}(\sqrt{F_A} + \sqrt{F_B} + \sqrt{F_C} + \sqrt{F_D})$.

Proof. Let O be the circumcentre of T, further V_A, V_B, V_C, V_D the volumes of tetrahedra OBCD, ODAB, OABC and R_A, R_B, R_C, R_D the circumradii of the triangles BDC, CDA, DAB, ABC. According to 2.8 we have

$$V_A \leqslant \frac{\sqrt{2}}{12}\sqrt{R^3ab'c'} = \frac{\sqrt{2}}{12}\sqrt{R^3 \cdot 4F_AR_A} \leqslant \frac{\sqrt{2}}{12}\sqrt{R^3 \cdot 4F_AR} = \frac{R^2\sqrt{2}}{6}\sqrt{F_A}$$

and analogously

$$V_B \leqslant \frac{R^2\sqrt{2}}{6}\sqrt{F_B}, \quad V_C \leqslant \frac{R^2\sqrt{2}}{6}\sqrt{F_C}, \quad V_D \leqslant \frac{R^2\sqrt{2}}{6}\sqrt{F_D}.$$

By addition of these four inequalities 2.9 follows. Here equality cannot hold, because only one of four equalities $R_A = R$, $R_B = R$, $R_C = R$, $R_D = R$ can hold.

Remark. B. R. Venkataraman [72] has a weaker inequality

$$V < \frac{R^2}{3}(\sqrt{F_A} + \sqrt{F_B} + \sqrt{F_C} + \sqrt{F_D}).$$

2.10. $$F_B + F_C \geq \sqrt{\frac{9V^2 s(s - a)}{F_D^2} + F_A^2 + F_D^2 - \sqrt{4F_A^2F_D^2 - 9V^2a^2}},$$

where $2s = a + b + c$. Equality holds iff $\sphericalangle$ (ABC, ABD) = $\sphericalangle$ (ABC, BCD), i.e. iff the foot D' of the altitude DD' lies on the bisector of the angle $\sphericalangle$ BAC.

E. Kötter [37]*.

2.11. $$F_B + F_C \geq \sqrt{\frac{36V^2}{a^2} + F_A^2 + F_D^2 - \sqrt{4F_A^2F_D^2 - 9V^2a^2}}$$

with equality iff the points A and D lie in the plane of symmetry of the segment $\overline{BC}$.

J. Steiner [58]*, [59]*.

Remark. 2.11 follows from 2.10 according to the inequality

$$\frac{s(s - a)}{F_D^2} = \frac{1}{(s - b)(s - c)} = \frac{4}{a^2 - (b - c)^2} \geq \frac{4}{a^2} \quad \{b = c\}.$$

2.12. If A', B', C', A'', B'', C'' are the midpoints of the edges $\overline{BC}$, $\overline{CA}$, $\overline{AB}$, $\overline{AD}$, $\overline{BD}$, $\overline{CD}$ and if a'' = A'A'', b'' = B'B'', c'' = C'C'', then

$$V \leq \frac{1}{3} a''b''c'' \quad \{T_I\}.$$

Proof. The centroid G of T is the common midpoint of segments $\overline{A'A''}$, $\overline{B'B''}$, $\overline{C'C''}$ and so we have

$$GA' = \frac{a''}{2}, \quad GB' = \frac{b''}{2}, \quad GC' = \frac{c''}{2}.$$

Let h_D be the altitude from the vertex D of T and let V' be the volume of the tetrahedron T' = A'B'C'G. Then $\frac{1}{4} h_D$ is the altitude from the vertex G of T' and $\frac{1}{4} F_D$ is the area of the triangle A'B'C'. Therefore, we get successively

$$V = 16V' = 16 \cdot \frac{1}{3}(\frac{1}{2} GA' \cdot GB' \cdot \sin\phi) \cdot GC' \cdot \sin\psi =$$

$$= \frac{1}{3} a''b''c'' \sin\phi \sin\psi \leq \frac{1}{3} a''b''c'',$$

where $\phi = \sphericalangle$ A'GB' and ψ is the angle between the line GC' and the plane GA'B'. Equality holds iff $\sin\phi = \sin\psi = 1$, i.e. iff the lines GA', GB', GC' are mutually orthogonal. This means that B'C'' = B''C'', and therefore a = 2B''C'' = 2B'C'' = a'. Analogously b = b', c = c'.

E. G. Gotman [18].

2.13. $(a^2 + b^2 + c^2)(a'^2 + b'^2 + c'^2) \leqslant 64R^4 \quad \{T_I\}$.

Proof. According to XVIII.3.1 (with n = 3) we have

$$(5) \qquad a^2 + b^2 + c^2 + a'^2 + b'^2 + c'^2 \leqslant 16R^2$$

with equality if $m_A = m_B = m_C = m_D$, i.e. iff $a = a'$, $b = b'$, $c = c'$, because for the medians we have the equalities

$$(6) \qquad 9m_A^2 = 3(a'^2 + b^2 + c^2) - (a^2 + b'^2 + c'^2),$$

$$9m_B^2 = 3(a^2 + b'^2 + c^2) - (a'^2 + b^2 + c'^2),$$

$$9m_C^2 = 3(a^2 + b^2 + c'^2) - (a'^2 + b'^2 + c^2),$$

$$9m_D^2 = 3(a'^2 + b'^2 + c'^2) - (a^2 + b^2 + c^2).$$

Now, 2.13 follows by (5) and the inequality

$$(a^2 + b^2 + c^2)(a'^2 + b'^2 + c'^2) \leqslant \frac{1}{4}(a^2 + b^2 + c^2 + a'^2 + b'^2 + c'^2)^2.$$

A. Oppenheim [43]; M. Lascu [39].

2.14. $aa' + bb' + cc' \leqslant 8R^2 \quad \{T_I\}$.

Proof. Follows from (5) because of

$$a^2 + a'^2 \geqslant 2aa', \quad b^2 + b'^2 \geqslant 2bb', \quad c^2 + c'^2 \geqslant 2cc'.$$

2.15. $R^2 \geqslant \frac{4\sqrt{3}}{9} \max(F_A, F_B, F_C, F_D)$

with equality iff among the triangles BCD, CDA, DAB, ABC that one with the maximal area is equilateral and the circumcentre of T lies inside this triangle.

Proof. Let R_A, R_B, R_C, R_D be the circumradii of triangles BCD, CDA, DAB, ABC. According to GI 7.9 for every $X \in \{A, B, C, D\}$ we have

$$R^2 \geqslant \frac{4\sqrt{3}}{9} F_X,$$

and 2.15 is a consequence of the inequality $R \geqslant \max(R_A, R_B, R_C, R_D)$.

B. R. Venkataraman [72].

2.16. Two segments $\overline{BC}$ and $\overline{AD}$ of the lengths a and a' lie inside a given sphere Σ with centre O and radius R. If F is the total area of tetrahedron ABCD, then

$$F \leqslant a\sqrt{a''^2 + \frac{a'^2}{4}} + a'\sqrt{a''^2 + \frac{a^2}{4}},$$

where

$$a'' = \sqrt{R^2 - \frac{a^2}{4}} + \sqrt{R^2 - \frac{a'^2}{4}}.$$

Equality holds iff $\overline{BC}$ and $\overline{AD}$ are chords of Σ and each of them lies in the plane of symmetry of the other one and the midpoints of these two chords lie on opposite sides of the point O.

A. Heppes [24].

2.17. Let p and p' be two skew lines, d = PP' the distance between these lines, B, C two points of p and A, D two points of p' such that BC = a, AD = a'. If F is the total area of the tetrahedron ABCD, then

$$F \geqslant a\sqrt{d^2 + \frac{a'^2}{4}} + a'\sqrt{d^2 + \frac{a^2}{4}} \quad \{PB = PC,\ P'A = P'D\}.$$

Proof. According to XX.7.7 we have

$$F_A + F_D \geqslant a\sqrt{d^2 + \frac{a'^2}{4}} \quad \{P'A = P'D\},$$

$$F_B + F_C \geqslant a'\sqrt{d^2 + \frac{a^2}{4}} \quad \{PB = PC\},$$

and by adding these inequalities we obtain 2.17.

R. Sturm [63]*, [64, p. 113-114]*.

2.18. Let d = PP' be the distance and ϕ the angle between two skew lines p and p'. Let B, C be two points of p and A, D two points of p'. If V is the volume and F the total area of the tetrahedron ABCD, then

$$F^2 \geqslant \frac{12V}{d^2 \sin^2 \phi}(3V + 2d^3 \sin \phi) \quad \{BC = AD,\ PB = PC,\ P'A = P'D\}.$$

Proof. Let BC = a, AD = a'. Using the A-G-inequality for means and the Cauchy's inequality, we get successively

$$\left(a\sqrt{d^2 + \frac{a'^2}{4}} + a'\sqrt{d^2 + \frac{a^2}{4}}\right)^2 \geqslant 4aa'\sqrt{\left(d^2 + \frac{a'^2}{4}\right)\left(d^2 + \frac{a^2}{4}\right)} \geqslant$$

$$\geq 4aa'(d^2 + \frac{aa'}{4}) \quad \{a = a'\}.$$

The equality $6V = aa'd \sin \phi$ and the inequality 2.17 imply therefore 2.18.

2.19. With the notation from 2.18 we have

$$F^2 \geq \frac{12V(3V + 2d^3)}{d^2} \quad \{BC = AD, PB = PC, P'A = P'D, p \perp p'\}.$$

Proof. Follows from 2.18, because the right-hand side of 2.18 is a decreasing function of $\sin \phi$ on R^+.

R. Sturm [64, p. 114-115]*.

3. Other Inequalities for Tetrahedra

3.1. If h_D is the altitude from the vertex D of T and if $a \leq b \leq c$; $a \leq b', c'$; $b \leq c', a'$; $c \leq a', b'$, then

$$h_D \geq \frac{\sqrt{6}}{3} b.$$

A. Rosenblatt [49].

3.2. Let R be the circumradius and $\rho_A, \rho_B, \rho_C, \rho_D$ the radii of the escribed spheres of T and let $\rho_A \leq \rho_B \leq \rho_C \leq \rho_D$. Then

$$\rho_A \leq \frac{2}{3} R, \quad \rho_B \leq \sqrt{2\sqrt{3} - 3}\, R, \quad \rho_C < \frac{4\sqrt{3}}{9} R, \quad \rho_D < R.$$

The first equality holds iff T is a regular tetrahedron. The second equality holds iff T is a regular triangular pyramid with the vertex A, whose lateral faces form with the base an angle $\phi \approx 70°50'29''$ such that $x = \cos \phi$ is the unique positive solution of $(9n + 6)x^3 + 3nx^2 + (3n - 6)x + n = 0$, where $n = \sqrt{2\sqrt{3} - 3}$.

H. J. Baron [4]; D. O. Škljarskij, N. N. Čencov, and I. M. Jaglom [54, p. 29-30, 153-154].

3.3. With the notation from 1 and 3.2 let $F_A \leq F_B \leq F_C \leq F_D$, $F_A + F_D \leq F_B + F_C$. If r is the inradius of T and if $\rho_{AB}, \rho_{AC}, \rho_{AD}$ are the radii of the spheres inscribed in the outer regions of T alongside the edges $\overline{AB}$, $\overline{AC}$, $\overline{AD}$, then

$$r < \rho_A \leq 2r, \quad r < \rho_B < 3r, \quad \frac{3}{2} r < \rho_C, \quad 2r \leq r_D,$$

$$r < \rho_{AB}, \quad 2r < \rho_{AC}, \quad 3r < \rho_{AD}$$

with equalities iff $F_A = F_B = F_C = F_D$. (ρ_{AB}, ρ_{AC}, ρ_{AD} exist if $F_A + F_B < F_C + F_D$, $F_A + F_C < F_B + F_D$, $F_A + F_D < F_B + F_C$, respectively.)

Proof. According to $4F_A \leqslant F$ we get

$$\frac{\rho_A}{r} = \frac{F}{F - 2F_A} \leqslant \frac{F}{F - \frac{1}{2}F} = 2.$$

From $3F_B \leqslant F_B + F_C + F_D < F$ we obtain

$$\frac{\rho_B}{r} = \frac{F}{F - 2F_B} < \frac{F}{F - \frac{2}{3}F} = 3.$$

The fourth inequality 1.14 implies

$$6F_C \geqslant 2(F_A + F_B + F_C) > F_A + F_B + F_C + F_D = F$$

and

$$\frac{\rho_C}{r} = \frac{F}{F - 2F_C} > \frac{F}{F - \frac{1}{3}F} = \frac{3}{2}.$$

According to $4F_D \geqslant F$ we obtain

$$\frac{\rho_D}{r} = \frac{F}{F - 2F_D} \geqslant \frac{F}{F - \frac{1}{2}F} = 2.$$

The inequalities $4F_B < 2(F_A + F_B + F_C) = 2(F - F_D)$ and $2F_D < F$ imply

$$4(F_B + F_D) < 2(F + F_D) < 3F \quad \text{and} \quad 4(F_A + F_C) > F.$$

Therefore

$$\frac{\rho_{AC}}{r} = \frac{F}{F - 2(F_A + F_C)} > \frac{F}{F - \frac{1}{2}F} = 2.$$

Since $3(F_B + F_C) \leqslant 2(F_B + F_C + F_D) < 2F$, we get $3(F_A + F_D) > F$ and therefore

$$\frac{\rho_{AD}}{r} = \frac{F}{F - 2(F_A + F_D)} > \frac{F}{F - \frac{2}{3}F} = 3.$$

The remaining inequalities $r < \rho_A$, $r < \rho_B$, $r < \rho_{AB}$ are trivial consequences of

$$\frac{\rho_A}{r} = \frac{F}{F - 2F_A}, \quad \frac{\rho_B}{r} = \frac{F}{F - 2F_B}, \quad \frac{\rho_{AB}}{r} = \frac{F}{F - 2(F_A + F_B)}.$$

D. O. Škljarskij, N. N. Čencov, and I. M. Jaglom [54, p. 32, 152-157].

3.4. If R_A, R_B, R_C, R_D are the circumradii and r_A, r_B, r_C, r_D the inradii of triangles BCD, CDA, DAB, ABC, then with the notation from 1 we have

$$27(R_A r_A + R_B r_B + R_C r_C + R_D r_D) \leqslant P + Q \quad \{\mathcal{T}_R\}.$$

Proof. Follows by application of GI 5.12 to the triangles BCD, CDA, DAB, ABC and by addition of the four inequalities obtained.
F. Harant [23].

3.5. Let P be any point inside the tetrahedron $A_1A_2A_3A_4$, let $R_i = A_iP$ for i = 1, 2, 3, 4 and let r_1, r_2, r_3, r_4 be the distances from the point P to the planes $A_2A_3A_4$, $A_3A_4A_1$, $A_4A_1A_2$, $A_1A_2A_3$, respectively. Then

$$\sum_{i=1}^{4} R_i > 2\sqrt{2} \sum_{i=1}^{4} r_i$$

and $2\sqrt{2}$ is the best factor.
N. D. Kazarinoff [31].

3.6. Let $\mathcal{T}$ = ABCD be a tetrahedron such that for its dihedral angles α, β, γ, α', β', γ' at the edges BC, CA, AB, AD, BD, CD we have

$$\alpha' + \beta + \gamma < 2\pi, \quad \alpha + \beta' + \gamma < 2\pi,$$

$$\alpha + \beta + \gamma' < 2\pi, \quad \alpha' + \beta' + \gamma' < 2\pi.$$

Inside the tetrahedron $\mathcal{T}$ there is a unique point M such that

$$\measuredangle AMB = \measuredangle CMD, \quad \measuredangle AMC = \measuredangle BMD, \quad \measuredangle AMD = \measuredangle BMC$$

(if $\mathcal{T}$ is an isosceles tetrahedron, then M is the circumcentre of $\mathcal{T}$). For any point P ($P \neq M$) we have

$$AP + BP + CP + DP > AM + BM + CM + DM.$$

If $\alpha' + \beta' + \gamma' \geqslant 2\pi$, then for any point P ($P \neq D$) we have

$$AP + BP + CP + DP > AD + BD + CD.$$

R. Sturm [63], [64], [65]; K. Simon [53]; E. Weiszfeld [74]; E. Egerváry [13]; P. Erdniev [14] ($AB \perp CD$, $EF \perp AB$, $EF \perp CD$, where E and

F are the midpoints of $\overline{AB}$ and $\overline{CD}$); R. Spira [57].

3.7. There is a unique sphere of minimal radius which contains a given tetrahedron ABCD. Let O be the centre and ρ the radius of this sphere. For any point P we have

$$\max (PA, PB, PC, PD) \geqslant \rho.$$

E. Egerváry [13].

3.8. Let M and M' be two points inside a tetrahedron $A = A_1A_2A_3A_4$, which are isogonal with respect to A (i.e. the planes A_1A_2M and A_1A_2M' are symmetric with respect to the symmetral plane of the planes $A_1A_2A_3$ and $A_1A_2A_4$, and analogously for the other five edges of A), and let f_1, f_2, f_3, f_4 be the face areas of the pedal tetrahedron of the point M. For any point P inside of A

$$\sum_{i=1}^{4} f_i \cdot A_iP \geqslant \sum_{i=1}^{4} f_i \cdot A_iM' \quad \{P = M'\}.$$

E. Egerváry [13].

3.9. Let $A = A_1A_2A_3A_4$ be a given tetrahedron and M a given point. For any cyclic permutation (i, j, k, l) of (1, 2, 3, 4) let Φ_i be the trihedron with edges MA_j, MA_k, MA_l and let

$$\sin \Phi_i = \frac{|(\overrightarrow{MA_j}, \overrightarrow{MA_k}, \overrightarrow{MA_l})|}{MA_j \cdot MA_k \cdot MA_l} .$$

where $(\overrightarrow{MA_j}, \overrightarrow{MA_k}, \overrightarrow{MA_l})$ is the scalar triple product of vectors $\overrightarrow{MA_j}$, $\overrightarrow{MA_k}$, $\overrightarrow{MA_l}$. Let $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in R^+$ and let

$$L_i = \lambda_j^2 + \lambda_k^2 + \lambda_l^2 + 2\lambda_j\lambda_k \cos \alpha_{jik} + 2\lambda_k\lambda_l \cos \alpha_{kil} + + 2\lambda_l\lambda_j \cos \alpha_{lij} - \lambda_i^2,$$

where e.g. $\alpha_{jik} = \measuredangle A_jA_iA_k$. At least three of the four numbers L_1, L_2, L_3, L_4 are positive. If $L_1, L_2, L_3, L_4 > 0$, then inside the tetrahedron A there is a unique point M such that

$$\sin \Phi_1 : \sin \Phi_2 : \sin \Phi_3 : \sin \Phi_4 = \lambda_1 : \lambda_2 : \lambda_3 : \lambda_4$$

and for any point P ($P \neq M$) we have

$$\sum_{h=1}^{4} \lambda_h \cdot PA_h > \sum_{h=1}^{4} \lambda_h \cdot MA_h .$$

If $L_i \leqslant 0$, then for any point P ($P \neq A_i$) we have

$$\sum_{h=1}^{4} \lambda_h \cdot PA_h > \sum_{\substack{h=1 \\ h\neq i}}^{4} \lambda_h \cdot A_iA_h .$$

E. Egerváry [13].

3.10. For any point P inside a tetrahedron T = ABCD we have

$$\measuredangle APB + \measuredangle APC + \measuredangle APD + \measuredangle BPC + \measuredangle BPD + \measuredangle CPD > 3\pi .$$

Proof. Let E be the point of intersection of the line DP and the plane ABC and F the point of intersection of the line CE and the edge AB. From

$$\measuredangle BPF + \measuredangle EPF > \measuredangle BPE,$$

$$\measuredangle APC + \measuredangle APF > \measuredangle CPF = \measuredangle CPE + \measuredangle EPF$$

we obtain by addition

$$\measuredangle APB + \measuredangle APC > \measuredangle BPE + \measuredangle CPE$$

because of the equality $\measuredangle APF + \measuredangle BPF = \measuredangle APB$. Analogously, we have

$$\measuredangle APB + \measuredangle BPC > \measuredangle APE + \measuredangle CPE,$$

$$\measuredangle APC + \measuredangle BPC > \measuredangle APE + \measuredangle BPE.$$

By addition of the last three inequalities we obtain

$$\measuredangle APB + \measuredangle APC + \measuredangle BPC > \measuredangle APE + \measuredangle BPE + \measuredangle CPE =$$

$$= (\pi - \measuredangle APD) + (\pi - \measuredangle BPD) + (\pi - \measuredangle CPD).$$

W. Janous 1980 (private communication).

3.11. With the notation from 1 let m_A, m_B, m_C, m_D be the lengths of the medians of T and let

$$S = (a - b)^2 + (a - b')^2 + (a' - b)^2 + (a' - b')^2 +$$

$$+ (b - c)^2 + (b - c')^2 + (b' - c)^2 + (b' - c')^2 +$$

$$+ (c - a)^2 + (c - a')^2 + (c' - a)^2 + (c' - a')^2 .$$

Then

$$\frac{1}{9}(2S + 8\sqrt{3}F) \leqslant m_A^2 + m_B^2 + m_C^2 + m_D^2 \leqslant \frac{1}{9}(6S + 8\sqrt{3}F) \qquad \{T_R\}.$$

Proof. By addition of equalities (6) from 2 we obtain

$$m_A^2 + m_B^2 + m_C^2 + m_D^2 = \frac{4}{9} Q,$$

and since $S = 4Q - 2P$, 3.11 follows from 1.15.

D. M. Milošević and B. Milisavljević [41].

3.12. Let O and I be the centres and R and r the radii of the circumscribed and inscribed sphere, respectively, of a tetrahedron, and let $d = OI$. Then

$$d^2 \leqslant (R + r)(R - 3r) \qquad \{T_R\}.$$

J. H. Grace [20]; G. Danielsson [11].

Remark. Inequality 3.12 is the necessary and sufficient condition for the existence of a tetrahedron inscribed in a given sphere with radius R and circumscribed about a second given sphere with radius r contained inside the first sphere, if d is the distance of the centres of these two spheres.

3.13. Let L be the Lemoine point of a tetrahedron T, i.e. let the barycentric coordinates of L be proportional to $F_A^2, F_B^2, F_C^2, F_D^2$, with the notation from 1. Let Q' and Q_L be the sums of squares of edge lengths of any tetrahedron T' inscribed in T and of the pedal tetrahedron T_L of L with respect to T. Then

$$Q' \geqslant Q_L \qquad \{T' = T_L\}.$$

E. Egerváry [13].

3.14. With the notation from 1 and 3.6 we have

$$\frac{\pi}{3} < \frac{a\alpha + b\beta + c\gamma + a'\alpha' + b'\beta' + c'\gamma'}{a + b + c + a' + b' + c'} < \frac{\pi}{2} .$$

G. Pólya [46]; G. Pólya and G. Szegő [47, p. 166, 393]; [79].

Remark. L. A. Santaló [52] proved that in elliptic and hyperbolic geometry the analogous inequalities hold, where to the first and third term of 3.14 we must add the quantity $2KV/(a + b + c + a' + b' + c')$ and where V is the volume of T and K the space curvature.

3.15. If $\varepsilon_A, \varepsilon_B, \varepsilon_C, \varepsilon_D$ are the sums of the two least angles of triangles BCD, CDA, DAB, ABC in a tetrahedron T, respectively, then

$$\varepsilon_A < \varepsilon_B + \varepsilon_C + \varepsilon_D, \quad \varepsilon_B < \varepsilon_C + \varepsilon_D + \varepsilon_A,$$

$$\varepsilon_C < \varepsilon_D + \varepsilon_A + \varepsilon_B, \quad \varepsilon_D < \varepsilon_A + \varepsilon_B + \varepsilon_C.$$

C. Pauc [44]; L. M. Blumenthal [7].

3.16. If M is the midpoint of the edge $\overline{AB}$ of a tetrahedron $\mathcal{T}$ = ABCD and F_A, F_B, F_M are the areas of triangles BCD, ACD, MCD, then

$$F_M < \frac{1}{2}(F_A + F_B).$$

Proof. Let A' and B' be the orthogonal projections of the points A and B onto the plane MCD. Then M is the midpoint of the segment $\overline{A'B'}$. Therefore, if F'_A, F'_B are the areas of triangles B'CD, A'CD, then we have

$$F_M = \frac{1}{2}(F'_A + F'_B).$$

This inequality implies 3.16 because of the obvious inequalities $F'_A < F_A$, $F'_B < F_B$.

J. Steiner [59]; R. Sturm [64, p. 136]; A. Gruia [21].

3.17. If F_a is the area of any intersection of a tetrahedron $\mathcal{T}$ = ABCD by a plane parallel to the lines AD and BC, then

$$F_a \leqslant \frac{1}{4} aa'$$

with equality iff AD ⊥ BC and the considered plane bisects the edges $\overline{AB}$, $\overline{AC}$, $\overline{BD}$, $\overline{CD}$.

Proof. If the considered plane intersects the edges $\overline{AB}$, $\overline{AC}$, $\overline{DC}$, $\overline{DB}$ in the points Q, R, S, T and if

$$\lambda = \frac{AQ}{QB} = \frac{AR}{RC} = \frac{DS}{SC} = \frac{DT}{TB},$$

then QRST is a parallelogram with

$$x = QR = ST = \frac{\lambda a}{\lambda + 1}, \quad y = QT = RS = \frac{a'}{\lambda + 1},$$

and therefore $a'x + ay = aa'$. Now we obtain successively

$$F_a = xy \sin \sphericalangle RQT \leqslant xy = \frac{1}{aa'} \cdot a'x \cdot ay \leqslant$$

$$\leqslant \frac{1}{aa'}\left(\frac{a'x + ay}{2}\right)^2 = \frac{1}{4} aa'$$

with equality iff QR ⊥ QT and $a'x = ay$, i.e. iff AD ⊥ BC and $\lambda = 1$.

F. G.-M. [77, p. 917-918].

Remark 1. With analogous notation we also have the inequalities

$$F_b \leqslant \frac{1}{4} bb', \quad F_c \leqslant \frac{1}{4} cc'$$

and therefore

3.17' $$F_a + F_b + F_c \leqslant \frac{1}{4}(aa' + bb' + cc')$$

with equality iff T is an orthocentric tetrahedron and each of the three considered planes bisects the four corresponding edges of T.

I. Dimovski [12].

Remark 2. From 3.17' and 2.14 we obtain

3.17'' $$F_a + F_b + F_c \leqslant 2R^2$$

with equality iff T is a regular tetrahedron and each of the three considered planes bisects the four corresponding edges of T.

4. Inequalities for Special Tetrahedra

4.1. Let T = ABCD be an orthocentric tetrahedron with the orthocentre H. If we have (with notation from 1)

$$F_A^2 < F_B^2 + F_C^2 + F_D^2, \quad F_B^2 < F_C^2 + F_D^2 + F_A^2,$$

$$F_C^2 < F_D^2 + F_A^2 + F_B^2, \quad F_D^2 < F_A^2 + F_B^2 + F_C^2,$$

then for any point P inside T the inequality

$$F_A \cdot PA + F_B \cdot PB + F_C \cdot PC + F_D \cdot PD \geqslant F_A \cdot HA + F_B \cdot HB + F_C \cdot HC + F_D \cdot HD \quad \{P = H\}$$

holds, but if e.g.

$$F_A^2 \geqslant F_B^2 + F_C^2 + F_D^2,$$

then

$$F_A \cdot PA + F_B \cdot PB + F_C \cdot PC + F_D \cdot PD > F_B \cdot AB + F_C \cdot AC + F_D \cdot AD.$$

E. Egerváry [13].

4.2. If we have CD ⊥ AC, BC and AC ⊥ AB in a tetrahedron ABCD and if $\phi = \measuredangle ACB$, $\psi = \measuredangle CAD$, $\chi = \measuredangle CBD$, then

$$\sin^2 \psi - \sin^2 \chi \leqslant \tan^2 \frac{\phi}{2} \quad \{\psi + \chi = \frac{\pi}{2}\}.$$

Proof. Let E be the point of intersection of the edge $\overline{AB}$ and the bisector of the angle $\measuredangle ACB$. We have (with the notation from 1)

$$\sin^2 \psi = \frac{c'^2}{a'^2} = \frac{c'^2}{b^2 + c'^2}, \quad \sin^2 \chi = \frac{c'^2}{b'^2} = \frac{c'^2}{a^2 + c'^2},$$

$$\tan \frac{\phi}{2} = \frac{AE}{b} = \frac{BE}{a} = \frac{c}{a + b} = \frac{\sqrt{a^2 - b^2}}{a + b}$$

and 4.2 is equivalent successively to the inequalities

$$\frac{c'^2}{b^2 + c'^2} - \frac{c'^2}{a^2 + c'^2} \leqslant \frac{a^2 - b^2}{(a + b)^2},$$

$$c'^2(a + b)^2 \leqslant (a^2 + c'^2)(b^2 + c'^2), \quad (ab - c'^2)^2 \geqslant 0.$$

Equality holds iff $ab = c'^2$, i.e. iff

$$\tan \psi = \frac{c'}{a} = \frac{b}{c'} = \tan(\frac{\pi}{2} - \chi), \quad \text{i.e.} \quad \psi + \chi = \frac{\pi}{2}.$$

V. S. Pokrovskij and R. P. Ušakov [45].

4.3. If AD ⊥ BD ⊥ CD ⊥ AD in a tetrahedron ABCD, then (with the notation from 1)

$$(a + b + c)^2 \leqslant 6(a'^2 + b'^2 + c'^2) \quad \{a = b = c,\ a' = b' = c'\}.$$

Proof. We have

$$a^2 + b^2 + c^2 = 2(a'^2 + b'^2 + c'^2)$$

and 4.3 follows by the A-Q-inequality for means.
[78].

4.4. With the conditions from 4.3 and the notation from 2 we have

$$R \geqslant \frac{3}{2}(\sqrt{3} + 1)r \quad \{a = b = c,\ a' = b' = c'\}.$$

Proof. From the equalities

$$2F_A = b'c', \quad 2F_B = c'a', \quad 2F_C = a'b',$$

$$2F_D = \sqrt{a'^2b'^2 + b'^2c'^2 + c'^2a'^2}, \quad 6V = a'b'c'$$

we obtain

$$r = \frac{3V}{F_A + F_B + F_C + F_D} =$$

$$= \frac{a'b'c'}{a'b' + b'c' + c'a' + \sqrt{a'^2b'^2 + b'^2c'^2 + c'^2a'^2}}.$$

R is the circumradius of the rectangular parallelepiped spanned by vectors $\overrightarrow{DA}$, $\overrightarrow{DB}$, $\overrightarrow{DC}$ and so we have

$$2R = \sqrt{a'^2 + b'^2 + c'^2}.$$

Therefore, the inequalities

$$a'b' + b'c' + c'a' \geqslant 3\sqrt[3]{a'^2b'^2c'^2},$$

$$a'^2 + b'^2 + c'^2 \geqslant 3\sqrt[3]{a'^2b'^2c'^2},$$

$$a'^2b'^2 + b'^2c'^2 + c'^2a'^2 \geqslant 3\sqrt[3]{a'^4b'^4c'^4}$$

imply

$$\frac{2R}{r} = \frac{\sqrt{a'^2 + b'^2 + c'^2}}{a'b'c'}(a'b' + b'c' + c'a' +$$

$$+ \sqrt{a'^2b'^2 + b'^2c'^2 + c'^2a'^2}) \geqslant$$

$$\geqslant \frac{3\sqrt[3]{a'b'c'}}{a'b'c'}(3\sqrt[3]{a'^2b'^2c'^2} + 3\sqrt[3]{a'^2b'^2c'^2}) = 3(\sqrt{3} + 1).$$

Ju. I. Gerasimov [16]; R. P. Ušakov [69].

4.5. With the conditions of 4.3 let l be any line through the point D. If s is the sum of the distances from the points A, B, C to the line l, then

$$s \leqslant \sqrt{2(a'^2 + b'^2 + c'^2)}$$

with equality iff

$$\sin\alpha = \frac{a'\sqrt{2}}{d}, \quad \sin\beta = \frac{b'\sqrt{2}}{d}, \quad \sin\gamma = \frac{c'\sqrt{2}}{d},$$

where α, β, γ are the angles between l and the lines AD, BD, CD, respectively and

$$d = \sqrt{a'^2 + b'^2 + c'^2}.$$

Proof. The equalities

$$\sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma = 2,$$

$$s = a' \sin \alpha + b' \sin \beta + c' \sin \gamma$$

imply by Cauchy's inequality

$$s \leqslant \sqrt{a'^2 + b'^2 + c'^2} \cdot \sqrt{\sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma} =$$

$$= \sqrt{2(a'^2 + b'^2 + c'^2)}.$$

Equality holds iff

$$\frac{\sin \alpha}{a'} = \frac{\sin \beta}{b'} = \frac{\sin \gamma}{c'}$$

i.e. iff

$$\frac{\sin^2 \alpha}{a'^2} = \frac{\sin^2 \beta}{b'^2} = \frac{\sin^2 \gamma}{c'^2} = \frac{2}{a'^2 + b'^2 + c'^2}.$$

E. G. Gotman [19]; [80].

4.6. Let DA ⊥ DB, DC and let a'' be the distance of the point A from the line BC. If ϕ is the angle between the planes ABC and BCD, then

$$\frac{aV^2}{a''F_D^3} \leqslant \frac{4\sqrt{3}}{81} \qquad \{\cos \phi = \frac{\sqrt{3}}{3}\}.$$

Proof. From equalities

(1) $$V = \frac{1}{6} aa''^2 \tan \phi, \qquad F_D = \frac{1}{2} aa'' \sec \phi$$

and the inequality

$$\sin^2 \phi \cos \phi \leqslant \frac{2\sqrt{3}}{9} \qquad \{\cos \phi = \frac{\sqrt{3}}{3}\}.$$

proved in the proof of XX.3.9 we obtain

$$\frac{V^2}{F_D^3} = \frac{2a''}{9a} \sin^2 \phi \cos \phi \leqslant \frac{4\sqrt{3}}{81} \cdot \frac{a''}{a} \qquad \{\cos \phi = \frac{\sqrt{3}}{3}\}.$$

J. Steiner [59] ($a'':a$ = const.).

4.7. With the same conditions as in 4.6, we have

$$\frac{aV^2}{a''(F_A + F_D)^3} \leq \frac{1}{36} \quad \{\cos \phi = \frac{1}{3}\}.$$

Proof. We have the equalities (1) and the equality

$$F_A = \frac{1}{2} aa''.$$

Therefore

$$\frac{V^2}{(F_A + F_D)^3} = \frac{2a''}{9a} \cdot \frac{\sin^2 \phi \cos \phi}{(1 + \cos \phi)^3} =$$

$$= \frac{a''}{9a} \cdot \frac{2 \cos \phi \cdot (1 - \cos \phi)}{(1 + \cos \phi)^2} \leq$$

$$\leq \frac{a''}{9a(1 + \cos \phi)^2}\left[\frac{1}{2}(2 \cos \phi + 1 - \cos \phi)\right]^2 = \frac{a''}{36a}$$

with equality iff $2 \cos \phi = 1 - \cos \phi$, i.e. $\cos \phi = 1/3$.
J. Steiner [59]*.

4.8. In a tetrahedron T = ABCD with BC = CA = AB = a, AD = a', BD = b', CD = c' and $a' < b' + c'$, $b' < c' + a'$, $c' < a' + b'$ let F'_D be the area of a triangle with sides a', b', c'. Then

$$2\sqrt{3}V \leq aF'_D$$

with equality iff the circumcentre of T lies in the plane ABC.
Proof. Follows from 2.4 because of

$$\overline{F} = a^2 F'_D, \quad R_D = \frac{a\sqrt{3}}{3}.$$

A. Oppenheim [43]; M. Lascu [39].

4.9. In an isodynamic tetrahedron T = ABCD with aa' = bb' = cc' = k we have

$$\frac{a^2b^2c^2(a^2 + b^2 + c^2 - 4F_D\sqrt{3})}{(a^2 - b^2)^2 + (b^2 - c^2)^2 + (c^2 - a^2)^2} < k^2 <$$

$$< \frac{a^2b^2c^2(a^2 + b^2 + c^2 + 4F_D\sqrt{3})}{(a^2 - b^2)^2 + (b^2 - c^2)^2 + (c^2 - a^2)^2}.$$

O. Bottema and G. R. Veldkamp [9].

4.10. In an isosceles tetrahedron $\mathcal{T}$ = ABCD with BC = AD = a, CA = BD = b, AB = CD = c we have

$$V \leqslant \frac{\sqrt{2}}{12}\,abc \leqslant \frac{\sqrt{2}}{324}(a + b + c)^3 \qquad \{\mathcal{T}_R\}.$$

Proof. Follows from 2.8.
O. Stolz and E. Fauquemberque [61]; Ju. I. Gerasimov [17].

4.11. With the conditions from 4.10 let r be the inradius of the tetrahedron $\mathcal{T}$ and R_D the circumradius of any of its faces. Then

$$R_D \geqslant 2\sqrt{2}r \qquad \{\mathcal{T}_R\}.$$

Proof. Follows from 4.10 according to the equalities

$$V = \frac{4F_D r}{3}, \qquad 4R_D F_D = abc.$$

Ju. I. Gerasimov [17].

4.12. With the notation from 4.10 and 4.11 we have

$$V \leqslant \frac{\sqrt{6}}{4}\,R_D^3 \qquad \{\mathcal{T}_R\}.$$

Proof. Follows from 4.10, since GI 5.27 implies abc $\leqslant 3R_D^3\sqrt{3}$ with equality iff a = b = c.
K. Kolarov [36].

4.13. With the conditions of 4.10 let α, β, γ be the dihedral angles at the edges with lengths a, b, c. Then

$$\alpha + \beta + \gamma \leqslant 3 \arccos\frac{1}{3} \qquad \{\mathcal{T}_R\}.$$

F. Kárteszi [30].

4.14. Let P be a point inside a regular tetrahedron $A_1A_2A_3A_4$ and Q a point inside the tetrahedron $\mathcal{P} = A_1A_2A_3P$. If F_P, F_Q are the total areas and V_P, V_Q the volumes of the tetrahedra $\mathcal{P}$ and $\mathcal{Q} = A_1A_2A_3Q$, then

$$\frac{V_P^2}{F_P^3} > \frac{V_Q^2}{F_Q^3}.$$

M. S. Klamkin [35].

References

1. Y. Aihara, 'On Some Inequalities Among the Elements of a Tetrahedron' (Japanese), J. Phys.-Math. Soc. Japan 4 (1930), 1-4, 147-151.
2. T. M. Apostol, 'Ptolemy's Inequality and the Chordal Metric', Math. Mag. 40 (1967), 233-235.
3. Š. Arslanagić, 'Dokazivanje nejednakosti pomoću vektora', Matematika 12 (1983), No. 1, 65-72.
4. H. J. Baron, 'Die Inkugeln des Tetraeders in Beziehung zur Umkugel', Tôhoku Math. J. 48 (1941), 185-192.
5. G. P. Bevz, 'Zadača 924', Mat. v Škole 1972, No. 1, 78-79.
6. Bhattacharya and H. W. Curjel, 'Question 11523', Educ. Times 59 (1893), 36.
7. L. M. Blumenthal, 'A Budget of Curiosa Metrica', Amer. Math. Monthly 66 (1959), 453-460.
8. O. Bottema, 'Aan welke ongelijkheden voldoen de ribben van een viervlak', Nieuw Tijdschr. Wisk. 48 (1960-1961), 166-170.
9. O. Bottema and G. R. Veldkamp, 'Problem 514', Nieuw Arch. Wisk. (3) 27 (1979), 277-278.
10. O. Chisini, 'Questione 98', Period Mat. (4) 5 (1925), 123, 294-299, 371.
11. G. Danielsson, 'Proof of the Inequality $d^2 \leqslant (R + r)(R - 3r)$ for the Distance Between the Centres of the Circumscribed and Inscribed Spheres of a Tetrahedron', 11. Skand. Kongr. (Trondheim 1949), Oslo 1952, 101-105.
12. I. Dimovski, 'Zadača 4', Matematika (Sofija) 7 (1968), No. 3, 37.
13. E. Egerváry, 'Über ein Minimumproblem der Elementargeometrie', J. reine angew. Math. 178 (1938), 174-186.
14. P. Erdniev, 'Zadača 21', Mat. v Škole 1954, No. 5, 90-91.
15. R. Fritsch, '"Dreiecks"-Ungleichungen für Tetraeder', Math. naturwiss. Unterr. 34 (1981), 274-278.
16. Ju. I. Gerasimov, 'Zadača 136', Mat. v Škole 1965, No. 6, 82.
17. Ju. I. Gerasimov, 'Zadača 950', Ibid. 1972, No. 2, 79-80.
18. E. G. Gotman, 'Tetraedr, vpisannij v drugoj tetraedr', Ibid. 1975, No. 6, 71-73.
19. E. G. Gotman, 'Zadača 2096', Ibid. 1979, No. 6, 61.
20. J. H. Grace, 'Tetrahedra in Relation to Spheres and Quadrics', Proc. London Math. Soc. (2) 17 (1918), 259-271.
21. A. Gruia, 'Problema 16520 a)', Gaz. Mat. 82 (1977), 78.
22. W. Hansen, 'Ein Beweis der Ptolemäischen Ungleichung', Arch. Math. 7 (1956), 320-322.
23. F. Harant, 'Některé nerovnosti a úloha pro čtyřstěn v E_3', Acta polytechn. (4) 7 (1978), No. 3, 19-33.
24. A. Heppes, 'Egy tetraéder felszínére vonatkozó szélsőértékfeladat', Mat. Lapok 12 (1961), 59-61.
25. K. S. K. Iyengar, 'On Mr. Narasinga Rao's Problem and Its Reciprocal', Math. Student 5 (1938), 164-167.
26. K. S. K. Iyengar, 'A Note on Narasinga Rao's Problem Relating to Tetrahedra', Proc. Indian Acad. Sci. A 7 (1938), 269-278.
27. K. S. K. Iyengar and K. V. Iyengar, 'On a Problem Relating to a Tetrahedron', Ibid. A 7 (1938), 305-311.
28. E. A. Jasinovij, 'Primenenie vektorov', Mat. v Škole 1979, No. 3, 71-73.
29. G. Kalajdžić, 'Some Inequalities Concerning a Tetrahedron', Publ.

Elektrotehn. Fak. Univ. Beograd. No. 320-328 (1970), 51-58.
30. F. Kárteszi, 'Egy geometriai szélsőértékfeladat', Mat. Lapok 18 (1967), 67-74.
31. N. D. Kazarinoff, 'D. K. Kazarinoff's Inequality for Tetrahedra', Michigan Math. J. 4 (1957), 99-104.
32. M. S. Klamkin, 'Vector Proofs in Solid Geometry', Amer. Math. Monthly 77 (1970), 1051-1065.
33. M. S. Klamkin, 'Problem S12', Ibid. 87 (1980), 576.
34. M. S. Klamkin, 'The Olympiad Corner: 18, Practice-Set 14-3', Crux Math. 6 (1980), 248-249.
35. M. S. Klamkin, 'Problem 83-5', SIAM Rev. 26 (1984), 275-276.
36. K. Kolarov, 'Zadača 4', Matematika-Fizika (Sofija) 1968, No. 4, 55-56.
37. E. Kötter, 'Über diejenigen Polyeder, die bei gegebener Gattung und gegebenem Volumen die kleinste Oberfläche besitzen', J. reine angew. Math. 110 (1892), 198-230.
38. G. Krammer, 'A szabályos tetraéder egy szélsőértéktulajdonsága', Mat. Lapok 12 (1961), 54-58.
39. M. Lascu, 'Asupra unor inegalităţi într-un tetraedru', Gaz. Mat. 89 (1984), 397-399.
40. Z. A. Melzak, 'An Isoperimetric Inequality for Tetrahedra', Canad. Math. Bull. 9 (1966), 667-669.
41. D. M. Milošević and B. Milisavljević, 'Problem 390', Mat. Vesnik 2 (15) (30) (1978), 201.
42. D. Nikolov, 'Zadača 3', Matematika (Sofija) 1981, No. 9, 34 and 1982, No. 3, 36.
43. A. Oppenheim, 'Inequalities Involving Elements of Triangles, Quadrilaterals or Tetrahedra', Publ. Elektrotehn. Fak. Univ. Beograd. No. 461-497 (1974), 257-263.
44. C. Pauc, Les méthodes directes en géométrie différentielle, Paris 1941.
45. V. S. Pokrovskij and R. P. Ušakov, 'Zadača 2856', Mat. v Škole 1986, No. 2, 72.
46. G. Pólya, 'Aufgabe 565', Arch. Math. Phys (3) 27 (1918), 162.
47. G. Pólya and G. Szegő, Aufgaben und Lehrsätze aus der Analysis, II, Berlin 1925.
48. J. F. Rigby, 'The Edge Lengths of a Tetrahedron'. To appear.
49. A. Rosenblatt, 'Sobre algunas desigualdades elementales para el tetraedro', Actas Acad. Ci. Lima 4 (1941), 213-230.
50. N. Şaganai, 'Problema E:7464', Gaz. Mat. 86 (1981), 466.
51. G. Sansone, 'Sulle espressioni del volume del tetraedro e su qualche problema di massimo', Period. Mat. (4) 3 (1923), 20-50.
52. L. A. Santaló, 'Some Inequalities Between the Elements of Tetrahedron in Non-Euclidean Geometry', Math. Notae 9 (1949), 113-117.
53. K. Simon, Über den Punkt kleinster Entfernungssume und die Flächen r_n = const., Halle 1887.
54. D. O. Škljarskij, N. N. Čencov, and I. M. Jaglom, Geometričeskie neravenstva i zadači na maksimum i minimum, Moskva 1970.
55. D. Skordev, 'Zadača 4', Obzornik Mat. (Sofija) 1983, No. 4, 62 and 1984, No. 5, 60-62.
56. D. Skordev, 'Zadača 4', Ibid. 1983, No. 5, 61 and 1984, No. 5, 63-64.
57. R. Spira, 'The Isogonic and Erdős-Mordell Points of a Simplex', Amer. Math. Monthly 78 (1971), 856-864.

58. J. Steiner, 'Einfache Beweise der Isoperimetrischen Hauptsätze', J. reine angew. Math. 18 (1838), 281-296; Ges. Werke II, 75-91.
59. J. Steiner, 'Über Maximum und Minimum bei den Figuren in der Ebene, auf der Kugelfläche und im Raume überhaupt. II', Ibid. 24 (1842), 189-250; Ges. Werke II, 243-308.
60. E. Steinitz, 'Über isoperimetrische Probleme bei konvexen Polyedern', Ibid. 158 (1927), 129-153.
61. O. Stolz and E. Fauquembergue, Interméd. Math. 1 (1894), 173, 201.
62. R. Sturm, 'Würfel und reguläres Tetraeder als Maximum und Minimum', J. reine angew. Math. 97 (1884), 1-12.
63. R. Sturm, 'Über den Punkt kleinster Entfernungssumme von gegebenen Punkten', Ibid. 97 (1884), 49-61.
64. R. Sturm, Maxima und Minima in der elementaren Geometrie, Leipzig-Berlin, 1910.
65. R. Sturm, 'Existenzbeweis des Punktes kleinster Entfernungssumme von vier gegebenen Punkten', J. reine angew. Math. 143 (1913), 241-249.
66. J. Szikszai, 'Problem 1856', Közöp. Mat. Lapok 47 (1973), 121-123.
67. N. Teodorescu, 'Teme de concursuri şi cercuri comentate metodologic', Gaz. Mat. 88 (1983), 98-103.
68. V. Thébault, 'Sur un théorème de M. D. Pompeiu', Bull. Math. Phys. École Polytechn. Bucarest 10 (1938-1939), 38-42.
69. R. P. Ušakov, 'Zadača 2836', Mat. v škole 1986, No. 1, 65-66.
70. D. Veljan, 'Zadatak 1018', Mat.-fiz. list 21 (1970-1971), 78-79.
71. K. Venkatachaliengar, 'On a Problem of the Tetrahedron', Proc. Indian Acad. Sci. A 7 (1938), 257-260.
72. B. R. Venkataraman, 'Inequalities Connected with a Tetrahedron', Math. Student 6 (1938), 78-79.
73. D. Voiculescu and F. Leuenberger, 'Aufgabe 583', Elem. Math. 24 (1969), 116-117.
74. E. Weiszfeld, 'Sur un problème de minimum dans l'espace', Tôhoku Math. J. 42 (1936), 274-280.
75. J. Zh. Zhang and L. Yang, 'A Problem of Stolarsky', Kexue Tongbao 30 (1985), 1426-1430.
76. N. Zhu, 'A Tetrahedron Inequality', Math. Gaz. 68 (1984), 200-202.
77. F. G.-M., Exercises de géométrie, 4. éd., Tours-Paris 1912.
78. 'Feladat 1742', Közöp. Mat. Lapok 41 (1970), 125-126, 157.
79. 'Problem 252', Ibid. 50 (1975), 222.
80. 'Feladat 2215', Közöp. Mat. Lapok 59 (1979), 30.

Chapter XX

OTHER INEQUALITIES IN E^n ($n \geqslant 2$)

1. Inequalities for Convex Polyhedra

Let P be a convex polyhedron with v vertices, e edges and f faces. Let V be the volume and F the total area of P, further I and r the centre and radius of the biggest sphere σ contained in P and O and R the centre and radius of the smallest sphere Σ containing P. If $a_1, a_2, \ldots, a_e$ are the edge lengths and $\alpha_1, \alpha_2, \ldots, \alpha_e$ the angles between the exterior normals of faces which contain these edges, then

$$E = \sum_{i=1}^{e} a_i, \quad M = \sum_{i=1}^{e} a_i\alpha_i$$

are the total edge length and the 'edge curvature' of P. For every $k \in \{2, 3, \ldots\}$ let v_k, respectively f_k, be the number of vertices respectively faces which are incident with exactly k edges. Obviously

(1) $$v = \sum_{k=3}^{\infty} v_k, \qquad (2) \quad f = \sum_{k=3}^{\infty} f_k,$$

(3) $$\sum_{k=3}^{\infty} kv_k = 2e, \qquad (4) \quad \sum_{k=3}^{\infty} kf_k = 2e,$$

and moreover we have Euler's formula

(5) $$v - e + f = 2.$$

Equality $v = v_3$ respectively $f = f_3$ characterizes the polyhedra with only trihedral vertices respectively triangular faces.

Let

$$p = \frac{2e}{f}, \quad q = \frac{2e}{v}$$

be the average valency of the faces respectively vertices and let

$$\omega_k = \frac{\pi k}{6(k-2)}.$$

Let

$$G = \sum_{i=1}^{e} a_i \tan \frac{\alpha_i}{2}$$

(obviously $G > M$) and let U be the volume of a polyhedron circumscribed about a unit sphere with the faces parallel to the corresponding faces of P.

Let $\{P = P_R\}$ mean that the equality sign appearsiff P is a regular polyhedron. Especially, T_R, H_R, O_R, D_R and I_R are the signs for regular tetrahedron, hexahedron, octahedron, dodecahedron and icosahedron. Further, let $\{v = v_3\}$ mean that the equality sign appears iff $v = v_3$, etc.

1.1. $3v \leq 2e \quad \{v = v_3\}$.

Proof. Follows from (1) and (3).

L. Euler [23]; E. Steinitz [106, p. 16]; L. Fejes Tóth [33, p. 15].

Remark. Inequality 1.1 can be written in the form $q \geq 3$.

1.2. $3f \leq 2e \quad \{f = f_3\}$.

Proof. Follows from (2) and (4).

L. Euler [23]; E. Steinitz [106, p. 16]; L. Fejes Tóth [33, p. 15].

Remark 1. Inequality 1.2 can be written in the form $p \geq 3$.

Remark 2. Inequalities 1.1, 1.2 and equality (5) are the necessary and sufficient conditions for the existence of a polyhedron with v vertices, e edges and f faces.

E. Steinitz [105].

1.3. $e \leq 3f - 6 \quad \{v = v_3\}$.

Proof. Follows from 1.1 according to (5).

L. Euler [23]; E. Steinitz [106, p. 17]; L. Fejes Tóth [33, p. 15].

Remark. Inequalities 1.2 and 1.3 are the necessary and sufficient conditions for the existence of a polyhedron with e edges and f faces.

E. Steinitz [105].

1.4. $e \leq 3v - 6 \quad \{f = f_3\}$.

Proof. Follows from 1.2 according to (5).

L. Euler [23]; E. Steinitz [106, p. 17]; L. Fejes Tóth [33, p. 15].

Remark. Inequalities 1.1 and 1.4 are the necessary and sufficient conditions for the existence of a polyhedron with v vertices and e edges.

E. Steinitz [105].

1.5. $v \leq 2f - 4 \quad \{v = v_3\}$.

Proof. Follows from 1.1 according to (5).

L. Euler [23]; E. Steinitz [106, p 17].

1.6. $f \leqslant 2v - 4 \quad \{f = f_3\}$.

Proof. Follows from 1.2 according to (5).

L. Euler [23]; E. Steinitz [106, p. 17].

Remark. Inequalities 1.5 and 1.6 are the necessary and sufficient conditions for the existence of a polyhedron with v vertices and f faces.

E. Steinitz [105].

1.7. $v_3 + f_3 \geqslant 8 \quad \{v = v_3 + v_4,\ f = f_3 + f_4\}$.

Proof. Adding equalities (1) and (2) multiplied by 4 and equalities (3) and (4) we obtain

$$4v + 4f - 4e = v_3 + f_3 - \sum_{k=5}^{\infty} (k - 4)(v_k + f_k)$$

and it suffices to apply equality (5).

A. M. Legendre [72]; E. Catalan [10]; E. Steinitz [106, p. 18]; L. Fejes Tóth [33, p. 15].

1.8. $3v_3 + 2v_4 + v_5 \geqslant 12 \quad \{v = v_3 + v_4 + v_5 + v_6,\ f = f_3\}$.

Proof. Adding equalities (1) and (2) multiplied by 6, equality (3) and equality (4) multiplied by 2 we obtain

$$6v + 6f - 6e = 3v_3 + 2v_4 + v_5 - \sum_{k=7}^{\infty} (k - 6)v_k - \sum_{k=4}^{\infty} (2k - 6)f_k$$

and (5) implies 1.8.

E. Catalan [10]; E. Steinitz [106, p. 18].

1.9. $3f_3 + 2f_4 + f_5 \geqslant 12 \quad \{v = v_3,\ f = f_3 + f_4 + f_5 + f_6\}$.

Proof. Adding equalities (1) and (2) multiplied by 6, equality (3) multiplied by 2 and equality (4) according to (5) we obtain 1.9.

E. Catalan [10]; E. Steinitz [106, p. 18].

1.10. $4v_3 + 2v_4 + f_3 \geqslant 20 \quad \{v = v_3 + v_4 + v_5,\ f = f_3\}$.

Proof. Adding equalities (1) and (2) multiplied by 10, equality (3) multiplied by 2 and equality (4) multiplied by 3 we get

$$10\,v + 10f - 10e = 4v_3 + 2v_4 + f_3 - \sum_{k=6}^{\infty} (2k - 10)v_k -$$

$$- \sum_{k=4}^{\infty} (3k - 10)f_k$$

and (5) implies 1.10.
E. Catalan [10]; E. Steinitz [106, p. 18].

1.11. $4f_3 + 2f_4 + v_3 \geqslant 20 \quad \{v = v_3,\ f = f_3 + f_4 + f_5\}.$

Proof. Dually to the proof of 1.10.
E. Catalan [10]; E. Steinitz [106, p. 18].

1.12. $p \leqslant \dfrac{6(f-2)}{f} < 6 \quad \{v = v_3\}.$

Proof. Follows from 1.3.
L. Fejes Tóth [33, p. 15].

1.13. $q \leqslant \dfrac{6(v-2)}{v} < 6 \quad \{f = f_3\}.$

Proof. Follows from 1.4.
L. Fejes Tóth [33, p. 15].

1.14. $\dfrac{R}{r} \geqslant \tan\dfrac{\pi}{p}\tan\dfrac{\pi}{q} \quad \{P = P_R\}.$

Proof. Follows from 1.24 and 1.25.
L. Fejes Tóth [31], [33, p. 130], [38, p. 247]; A. Florian [42], [43].

1.15. $\dfrac{R}{r} \geqslant \sqrt{3}\tan\omega_v \quad \{P = T_R,\ O_R,\ I_R\}.$

Proof. Follows by applying 1.14 to the polyhedron P realized as a polyhedron with only triangular faces (this can always be done by adding of diagonals of non-triangular faces as new edges), for which we have $p = 3$, $q = \dfrac{6(v-2)}{v}$ because of 1.2 and 1.13.
L. Fejes Tóth [24], [33, p. 117, 131], [38, p. 247]; A. Florian [43].
Remark. There is a more general inequality

1.15' $\dfrac{V_E}{V_e} \geqslant 3\sqrt{3}\tan^3\omega_v,$

where V_E and V_e are the volumes of two ellipsoids, the first of which contains P and the second contained in P. Equality appears only for the circumscribed and inscribed ellipsoid of an affine-regular tetrahedron (any tetrahedron is such a one), octahedron or icosahedron. In the case of equality both ellipsoids are concentric with the centres at the centroid of P.
L. Fejes Tóth [26], [29], [33, p. 131].

1.16. $\dfrac{R}{r} \geqslant \sqrt{3}\tan\omega_f \quad \{P = T_R,\ H_R,\ D_R\}.$

Proof. Follows by applying 1.14 to the polyhedron P realized as a polyhedron with only trihedral vertices (a procedure dual to that one in the proof of 1.15), for which we have $q = 3$, $p = \frac{6(f-2)}{f}$ because of 1.1 and 1.12.

L. Fejes Tóth [24], [33, p. 117, 131], [38, p. 247], [39]; A. Florian [43].

Remark. There is a more general inequality

1.16' $$\frac{V_E}{V_e} \geqslant 3\sqrt{3}\tan^3 \omega_f$$

where V_E and V_e are the volumes of two ellipsoids, the first of which contains P and the second contained in P. Equality appears only for the circumscribed and inscribed ellipsoid of an affine-regular tetrahedron, hexahedron or dodecahedron. In the case of equality both ellipsoids are concentric with the centres at the centroid of P.

L. Fejes Tóth [26], [29], [33, p. 131].

1.17. $$\frac{R}{r} \geqslant \tan^2 \frac{(e+2)\pi}{4e} \qquad \{P = T_R\}.$$

Proof. Equality

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{e} + \frac{1}{2},$$

which is equivalent to (5), implies

(6) $$\tan\frac{\pi}{p}\tan\frac{\pi}{q} \geqslant \tan^2 \frac{1}{2}\left(\frac{\pi}{p} + \frac{\pi}{q}\right) = \tan^2 \frac{\pi}{2}\left(\frac{1}{e} + \frac{1}{2}\right) = \tan^2 \frac{(e+2)\pi}{4e}$$

and 1.17 is a consequence of 1.14. Let us prove the inequality in (6). If we put

$$u = \tan\frac{\pi}{2p}, \qquad v = \tan\frac{\pi}{2q},$$

then $u, v \in (0, 1)$ because of $p \geqslant 3$, $q \geqslant 3$, and inequality in (6) can be written in the form

$$\frac{2u}{1-u^2} \cdot \frac{2v}{1-v^2} \geqslant \left(\frac{u+v}{1-uv}\right)^2,$$

which is equivalent to

$$(u-v)^2[(u+v)^2 - (1-uv)^2] \geqslant 0,$$

i.e.

$$\left(\tan\frac{\pi}{2p} - \tan\frac{\pi}{2q}\right)^2\left[\tan^2\left(\frac{\pi}{2p} + \frac{\pi}{2q}\right) - 1\right] \geqslant 0.$$

But

$$\tan\left(\frac{\pi}{2p} + \frac{\pi}{2q}\right) = \tan \frac{\pi}{2}\left(\frac{1}{e} + \frac{1}{2}\right) > 1$$

because of

$$\frac{\pi}{4} < \frac{\pi}{2}\left(\frac{1}{e} + \frac{1}{2}\right) \leqslant \frac{\pi}{3} .$$

Equality in 1.17 holds iff p = q and this equality characterizes the regular tetrahedra among all polyhedra.

A. Florian [42], [43].

Remark. L. Fejes Tóth [33, p. 119] has the weaker inequality

$$\frac{R}{r} \geqslant \sqrt{3} \tan \omega_e .$$

1.18. $E \geqslant 24r \quad \{P = H_R\}.$

A. S. Besicovitch and H. G. Eggleston [6]; L. Fejes Tóth [33, p. 143] (P polyhedron with equal face areas).

Remark. L. Fejes Tóth [27], [33, p. 143-144] has the weaker result

$$E > 20r$$

and J. M. Hammersley [59] has the weaker inequality

$$E > \frac{10}{3} r\sqrt{m\pi + \tan \frac{\pi}{m}} ,$$

where $m \in \{3, 4, \ldots\}$ is such that

$$\sum_{k=m+1}^{\infty} f_k = 0.$$

1.19. If $f = f_3$, then

$$E \geqslant 12\sqrt{6}r \quad \{P = T_R, O_R\}.$$

J. Linhart [78].

Remark. L. Fejes Tóth [27], [33, p. 144-145] has the weaker result

$$E > 28r.$$

1.20. $F \geqslant e \sin \frac{2\pi}{p}\left(\tan^2 \frac{\pi}{p} \tan^2 \frac{\pi}{q} - 1\right) r^2 \quad \{P = P_R\}.$

Proof. We can suppose that r = 1. Let $A_1, A_2, \ldots, A_f$ be the faces of P and $A_1', A_2', \ldots, A_f'$ their central projections onto the sphere σ from its centre I. If F_i is the area and e_i the number of sides of a polygon A_i and ϕ_i the area of spherical polygon A_i', then by 3.1 for any $i \in \{1, 2, \ldots, f\}$ we have

$$F_i \geqslant \frac{e_i}{2} \sin \frac{2\pi}{e_i}\left(\tan^2 \frac{\pi}{e_i} \cot an^2 \frac{2\pi - \phi_i}{2e_i} - 1\right)$$

with equality iff A_i is a regular polygon with the centre on σ. The function

$$F'(u, v) = v \sin \frac{2\pi}{v}(\tan^2 \frac{\pi}{v} \operatorname{cotan}^2 \frac{2\pi - u}{2v} - 1)$$

is convex for $u \geqslant 0$, $v \geqslant 3$ and according to the equalities

(7) $$\frac{1}{f} \sum_{i=1}^{f} e_i = \frac{2e}{f} = p,$$

(8) $$\frac{1}{f} \sum_{i=1}^{f} (2\pi - \phi_i) = \frac{2\pi(f - 2)}{f},$$

(9) $$\frac{2\pi(f - 2)}{4e} = \frac{\pi(e - v)}{2e} = \frac{\pi}{2} - \frac{\pi}{q}$$

it follows

$$F = \sum_{i=1}^{f} F_i \geqslant \frac{f}{2} \cdot \frac{2e}{f} \sin \frac{2\pi f}{2e}[\tan^2 \frac{\pi f}{2e} \operatorname{cotan}^2 \frac{2\pi(f - 2)}{4e} - 1] =$$

$$= e \sin \frac{2\pi}{p}(\tan^2 \frac{\pi}{p} \tan^2 \frac{\pi}{q} - 1) \qquad \{P = P_R\}.$$

L. Fejes Tóth [31], [33, p. 154], [38, p. 261].

1.21. If the feet of perpendiculars drawn from O to the face planes and edge lines of P lie on the corresponding faces and lines, then

$$F \leqslant e \sin \frac{2\pi}{p}(1 - \operatorname{cotan}^2 \frac{\pi}{p} \operatorname{cotan}^2 \frac{\pi}{q})R^2 \qquad \{P = P_R\}.$$

Proof. Let $R = 1$. With the same notations as in the proof of 1.20 (we project from O onto Σ) for any $i \in \{1, 2, \ldots, f\}$ we have by 3.4

$$F_i \leqslant \frac{e_i}{2} \sin \frac{2\pi}{e_i}\left(1 - \operatorname{cotan}^2 \frac{\pi}{e_i} \tan^2 \frac{2\pi - \phi_i}{2e_i}\right)$$

with equality iff A_i is a regular polygon inscribed in Σ. The function

$$G'(u, v) = v \sin \frac{2\pi}{v}(1 - \operatorname{cotan}^2 \frac{\pi}{v} \tan^2 \frac{2\pi - u}{2v})$$

is concave for $u \geqslant 0$, $v \geqslant 3$ and by (7)-(9) it follows

$$F = \sum_{i=1}^{f} F_i \leqslant \frac{f}{2} \cdot \frac{2e}{f} \cdot \sin\frac{2\pi f}{2e}[1 - \cot^2\frac{\pi f}{2e}\tan^2\frac{2\pi(f-2)}{4e}] =$$

$$= e \sin\frac{2\pi}{p}(1 - \cot^2\frac{\pi}{p}\cot^2\frac{\pi}{q})$$

with $\{P = P_R\}$.

L. Fejes Tóth [31].

Remark. J. Linhart [80] has shown the superfluity of the condition that the feet of perpendiculars drawn from O to the face planes lie on the corresponding faces.

L. Fejes Tóth [33, p. 154-155], [38, p. 261-263], [39].

1.22. $F \geqslant 6(f-2)\tan\omega_f(4\sin^2\omega_f - 1)\cdot r^2 \quad \{P = T_R, H_R, D_R\}$.

L. Fejes Tóth [28].

1.23. $F \leqslant \frac{3\sqrt{3}}{2}(v-2)(1 - \frac{1}{3}\cot^2\omega_v)\cdot R^2 \quad \{P = T_R, O_R, I_R\}$.

Proof. Follows by applying 1.21 to polyhedra P with $f = f_3$ for which we have by 1.4, 1.2 and 1.13

(10) $$e = 3(v-2), \quad p = 3, \quad q = \frac{6(v-2)}{v}.$$

J. Linhart [80].

1.24. $V \geqslant \frac{e}{3}\sin\frac{2\pi}{p}(\tan^2\frac{\pi}{p}\tan^2\frac{\pi}{q} - 1)\cdot r^3 \quad \{P = P_R\}$.

Proof. Follows from 1.20 because of $3V \geqslant Fr$.

L. Fejes Tóth [31], [33, p. 123-126], [38, p. 246-248]; A. Florian [42].

Remark. Inequality 1.24 is valid also if we replace V by the volume of the intersection of P and the sphere with radius $r\tan\frac{\pi}{p}\tan\frac{\pi}{q}$ concentric to the sphere σ.

1.25. If the conditions of 1.21 are satisfied, then

$$V \leqslant \frac{2e}{3}\cos^2\frac{\pi}{p}\cot\frac{\pi}{q}(1 - \cot^2\frac{\pi}{p}\cot^2\frac{\pi}{q})\cdot R^3 \quad \{P = P_R\}.$$

Proof. Follows from 3.6 analogously as 1.21 follows from 3.4.

L. Fejes Tóth [31], [33, p. 128-130], [36], [38, p. 246, 248-260]; A. Florian [42], [43], [45].

Remark. Inequality 1.25 is valid also without the additional conditions from 1.21.

1.26. $(f - 2) \sin 2\omega_f (3 \tan^2 \omega_f - 1) r^3 \leqslant v \leqslant$

$$\leqslant \frac{2\sqrt{3}}{9}(f - 2) \cos^2 \omega_f (3 - \cot^2 \omega_f) R^3 \qquad \{\mathcal{P} = \mathcal{T}_R, \mathcal{H}_R, \mathcal{D}_R\}.$$

Proof. Follows by applying of inequalities 1.24 and 1.25 to polyhedra $\mathcal{P}$ with $v = v_3$ for which we have by 1.3, 1.12 and 1.1

(11) $$e = 3(f - 2), \quad p = \frac{6(f - 2)}{f}, \quad q = 3.$$

L. Fejes Tóth [33, p. 123], [38, p. 246-247]; A. Florian [42].

Remark. The sequence

$$f[(f - 2) \sin 2\omega_f (3 \tan^2 \omega_f - 1) - \frac{4\pi}{3}] \quad (f = 1, 2, \ldots)$$

monotonically converges to the limit $20\sqrt{3}\pi^2/27$. Therefore, we have

1.26' $$v > \left(\frac{4\pi}{3} + \frac{20\sqrt{3}\pi^2}{27f}\right) r^3.$$

L. Fejes Tóth [33, p. 123-124].

1.27. $\frac{\sqrt{3}}{2}(v - 2)(3 \tan^2 \omega_v - 1) r^3 \leqslant v \leqslant$

$$\leqslant \frac{1}{6}(v - 2) \cot \omega_v (3 - \cot^2 \omega_v) R^3 \qquad \{\mathcal{P} = \mathcal{T}_R, \mathcal{O}_R, \mathcal{I}_R\}.$$

Proof. Follows by applying 1.24 and 1.25 to polyhedra $\mathcal{P}$ with $f = f_3$, for which we have the equalities (10).

L. Fejes Tóth [25], [29], [33, p. 123, 126-127], [38, p. 246-247]; A. Florian [42].

Remark. The sequence

$$v[\frac{\sqrt{3}}{2}(v - 2)(3 \tan^2 \omega_v - 1) - \frac{4\pi}{3}] \quad (v = 1, 2, \ldots)$$

monotonically converges to the limit $4\sqrt{3}\pi^2/9$. Therefore, we have

1.27' $$v > \left(\frac{4\pi}{3} + \frac{4\sqrt{3}\pi^2}{9v}\right) r^3.$$

L. Fejes Tóth [33, p. 123-124].

1.28. If f_1, f_2, v_1, v_2 are integers defined by

$$f_1 + f_2 = f, \quad v_1 - v_2 = 1, \quad f_1 v_1 + f_2 v_2 = k$$

and if

$$U(u, v) = \frac{v}{3} \cos^2 \frac{\pi}{v} \tan \frac{2\pi - u}{2v}\left(1 - \cot^2 \frac{\pi}{v} \tan^2 \frac{2\pi - u}{2v}\right),$$

then

$$V \leqslant \max \{f_1 U(u_1, v_1) + f_2 U(u_2, v_2)\} \cdot R^3,$$

where the maximum is taken over the real values u_1 and u_2 defined by $u_1 \geqslant 0$, $u_2 \geqslant 0$, $f_1 u_1 + f_2 u_2 \leqslant 4\pi$.

L. Fejes Tóth [36], [38, p. 259].

1.29. Let I' be the centre of a sphere with radius r' contained in P. If V' is the volume of the intersection of P and the sphere with centre I' and radius $\sqrt{3} \tan \omega_f \cdot r'$, then

$$V' \geqslant (f - 2) \sin 2\omega_f (3 \tan^2 \omega_f - 1) r^3.$$

L. Fejes Tóth [33, p. 140].

1.30. $$V \leqslant \frac{v}{3} \sin \frac{2\pi}{v} \cdot R^3 \quad (v \in \{5, 6, 7\})$$

with equality iff P is a regular bipyramid inscribed in a sphere with radius R.

J. D. Berman and K. Hanes [4].

1.31. $$V \leqslant \frac{1}{50} \sqrt{4750 + 290\sqrt{145}} \cdot R^3 \quad (v = 8)$$

with equality iff P has the vertices in a coordinate system:

$$A_1 = (R \sin 3\phi, 0, R \cos 3\phi), \quad A_2 = (R \sin \phi, 0, R \cos \phi),$$

$$A_3 = (-R \sin \phi, 0, R \cos \phi), \quad A_4 = (-R \sin 3\phi, 0, R \cos 3\phi),$$

$$A_5 = (0, -R \sin 3\phi, -R \cos 3\phi), \quad A_6 = (0, -R \sin \phi, -R \cos \phi),$$

$$A_7 = (0, R \sin \phi, -R \cos \phi), \quad A_8 = (0, R \sin 3\phi, -R \cos 3\phi),$$

where

$$\cos \phi = \frac{1}{20} \sqrt{150 + 10\sqrt{145}}.$$

In the case of equality the 18 edges of P are

$$A_1A_2 = A_2A_3 = A_3A_4 = A_5A_6 = A_6A_7 = A_7A_8 = 2R \sin \phi =$$
$$= \frac{R}{10} \sqrt{250 - 10\sqrt{145}},$$
$$A_1A_5 = A_1A_8 = A_4A_5 = A_4A_8 = R\sqrt{2 + 2 \cos^2 3\phi} = \frac{R}{5} \sqrt{65 - \sqrt{145}},$$
$$A_1A_6 = A_1A_7 = A_2A_5 = A_2A_8 = A_3A_5 = A_3A_8 = A_4A_6 = A_4A_7 =$$
$$= R\sqrt{2 + \cos \phi \cos 3\phi} = \frac{2R}{5} \sqrt{10}.$$

J. D. Berman and K. Hanes [4].

1.32. $$6\pi < \inf \frac{E^2}{F} \leqslant 2\sqrt{3} + \frac{16}{3} \pi.$$

O. Aberth [1]; M. Kömhoff [71].

1.33. $$\frac{F^3}{V^2} \geqslant 9e \sin \frac{2\pi}{p}(\tan^2 \frac{\pi}{p} \tan^2 \frac{\pi}{q} - 1) \qquad \{P = P_R\}.$$

Proof. If P is a polyhedron circumscribed about a sphere, then 1.33 follows from 1.24 because of the equality 3V = Fr, and then for any polyhedron P inequality 1.33 follows according to 1.44.

L. Fejes Tóth [31].

1.34. $$\frac{F^3}{V^2} \geqslant 54(f - 2) \tan \omega_f (4 \sin^2 \omega_f - 1) \qquad \{P = T_R, H_R, D_R\}.$$

Proof. If P is a polyhedron circumscribed about a sphere, then 1.34 follows from 1.26 because of 3V = Fr, and then for any polyhedron P inequality 1.34 follows according to 1.44.

M. Goldberg [50]; L. Fejes Tóth [25], [28], [31], [33, p. 135-137], [38, p. 264-265]; D. O. Škljarskij, N. N. Čencov, and I. M. Jaglom [102, p. 262] (P parallelepiped.

1.35. $$\frac{F^3}{V^2} \geqslant 162\sqrt{3} \qquad (f = 5)$$

with equality iff P is a regular triangular prism circumscribed about a sphere.

J. Sucksdorff [111].

1.36. $$\frac{E^2}{F} \geqslant 12\sqrt{2} \qquad (f = f_3) \qquad \{P = T_R\}.$$

M. Kömhoff [70].

1.37. $\frac{E^3}{V} \geq \frac{81}{2}(11\sqrt{11} + 21\sqrt{3}) \quad (f = f_3 = 6)$

with equality iff P is a regular triangular bipyramid whose base edges and lateral edges have the lengths

$$\frac{E}{6}(\sqrt{33} - 5), \quad \frac{E}{12}(7 - \sqrt{33}).$$

A. Procissi [94].

1.38. $\frac{E^3}{V} \geq 2592\sqrt{2} \quad (v = v_4 = 6,\ f = f_3 = 8) \quad \{P = O_R\}.$

A. Procissi [94]; G. Sansone [97].

1.39. $$M \geq 2e \sin\frac{\pi}{p}\sqrt{\tan^2\frac{\pi}{p}\tan^2\frac{\pi}{q} - 1}\ \cdot$$

$$\cdot\ \arccos\left(\cos\frac{\pi}{q}\operatorname{cosec}\frac{\pi}{p}\right) \cdot r \quad \{P = P_R\}.$$

Proof. Follows from 3.3 analogously as 1.20 follows from 3.1.

A. Florian [43]; L. Fejes Tóth [38, p. 261-263]; A. Florian and J. Linhart [46]; J. Linhart [79].

1.40. $$M \geq 6(f - 2)\sin\omega_f\sqrt{3\tan^2\omega_f - 1}\ \cdot$$

$$\cdot\ \arccos\left(\frac{1}{2}\operatorname{cosec}\omega_f\right) \cdot r \quad \{P = T_R, H_R, I_R\}.$$

Proof. Follows by applying 1.39 to polyhedra with $v = v_3$, for which we have the equalities (11).

A. Florian [44].

1.41. If the conditions of 1.21 are satisfied, then

$$M \leq 2e \sin\frac{\pi}{p}\sqrt{1 - \cot^2\frac{\pi}{p}\cot^2\frac{\pi}{q}}\ \cdot$$

$$\cdot\ \arccos\left(\cos\frac{\pi}{q}\operatorname{cosec}\frac{\pi}{p}\right) \cdot R \quad \{P = P_R\}.$$

Proof. With the same conditions and notations as in the proof of 1.21 for any $i \in \{1, 2, \ldots, f\}$ we have by 3.7

$$M_i \leq e_i \sin\frac{\pi}{e_i}\sqrt{1 - \cot^2\frac{\pi}{e_i}\cot^2\frac{2\pi - \phi_i}{2e_i}}\ \cdot$$

$$\cdot \ \arccos\left(\sin \frac{2\pi - \phi_i}{2e_i} \operatorname{cosec} \frac{\pi}{e_i}\right)$$

with equality iff A_i is a regular polygon inscribed in Σ, where M_i is the base curvature (its definition is given in 3) of the pyramid with base A_i and the vertex O. The function

$$M'(u, v) = v \sin \frac{\pi}{v} \sqrt{1 - \cotan^2 \frac{\pi}{v} \tan^2 \frac{2\pi - u}{2v}} \cdot$$

$$\cdot \ \arccos(\sin \frac{2\pi - u}{2v} \operatorname{cosec} \frac{\pi}{v})$$

is concave for $u \geqslant 0$, $v \geqslant 3$, and by (7)-(9) it follows

$$M = \sum_{i=1}^{f} M_i \leqslant f \cdot \frac{2e}{f} \cdot \sin \frac{\pi f}{2e} \sqrt{1 - \cotan^2 \frac{\pi f}{2e} \tan^2 \frac{2\pi(f-2)}{4e}} \cdot$$

$$\cdot \ \arccos(\sin \frac{2\pi(f-2)}{4e} \operatorname{cosec} \frac{\pi f}{2e}) =$$

$$= 2e \sin \frac{\pi}{p} \sqrt{1 - \cotan^2 \frac{\pi}{p} \cotan^2 \frac{\pi}{q}} \cdot$$

$$\cdot \ \arccos(\cos \frac{\pi}{q} \operatorname{cosec} \frac{\pi}{p}) \qquad \{P = P_R\}.$$

A. Florian [43]; L. Fejes Tóth [38, p. 261-263].

Remark. J. Linhart [77] has shown that the additional conditions from 1.21. are superfluous.

L. Fejes Tóth [39].

1.42. $F^2 \geqslant 3GV$

with equality iff P is a polyhedron circumscribed about a sphere.

L. Fejes Tóth [38, p. 268-269].

1.43. $G^2 \geqslant 3UF$

with equality iff P is a polyhedron circumscribed about a sphere.

L. Fejes Tóth [38, p. 268-269].

1.44. $\frac{F^3}{V^2} \geqslant 27U$

with equality iff P is a polyhedron circumscribed about a sphere.

L. Lindelöf [76]; L. Fejes Tóth [38, p. 269].

1.45. If the set of edges of P encloses an unit sphere without permitting it to slide out, then

$$E > \frac{8}{3}\pi + 2\sqrt{3}$$

and this is the best possible result.

A. S. Besicovitch [5]; O. Aberth [1].

1.46. If the set of edges of P encloses an unit sphere without permitting it to slide out and if all these edges touch the sphere, then

$$E > 15$$

and this is the best possible result.

Proof. Let $x^2 + y^2 + z^2 = 1$ be the equation of the considered sphere Ω and let $0 < \alpha < 1$. In the plane $z = -\alpha$ we consider the square Q = ABCD whose sides touch Ω and moreover the lines AB, CD are parallel to the x-axis. Through these lines we construct two planes σ_1, σ_2 which contain the point (0, 0, 1). The line $l = \sigma_1 \cap \sigma_2$ has the equation $y = 0$, $z = 1$ and touches the sphere Ω. The intersections $\sigma_1 \cap \Omega$, $\sigma_2 \cap \Omega$ are two circles c_1, c_2. The second tangents from A and D to these circles c_1 and c_2 intersect the line l in the same point F and the second tangents from B and C onto c_1 and c_2 intersect line l in the same point G. The obtained polyhedron is a truncated prism with the bases ADF and BCG and this prism is symmetric with respect to the xz-plane and with respect to the yz-plane. The edges of this prism touch the sphere Ω and enclose it, because none of the planes ABCD, ABGF, BCG, CDFG, DAF contains the centre of Ω. Now, if α approaches to 0, then the prism P = ABCDFG converges to a limit position $P_o = A_oB_oC_oD_oF_oG_o$, where we have

$$A_o = (1, 1, 0), \quad B_o = (-1, 1, 0), \quad C_o = (-1, -1, 0),$$

$$D_0 = (1, -1, 0), \quad F_o = (\tfrac{1}{2}, 0, 1), \quad G_o = (-\tfrac{1}{2}, 0, 1),$$

$$A_oB_o = B_oC_o = C_oD_o = D_oA_o = 2,$$

$$A_oF_o = B_oG_o = C_oG_o = D_oF_o = \frac{3}{2}, \quad F_oG_o = 1,$$

and therefore finally $E_o = 15$.

G. Valette [116].

Remark. G. C. Shephard [101] has the weaker inequality

$$E > 3\pi + 6.$$

1.47. If ρ is the radius of the largest circle which can be contained in any face of P, then

$$E \geqslant 12\sqrt{3}\rho \qquad \{\mathcal{P} = \mathcal{T}_R\}.$$

L. Fejes Tóth [33, p. 145].

1.48. If E_k is the sum of k-th powers of edge lengths of $\mathcal{P}$, then with ρ from 1.47 we have

$$E_2 \geqslant 48\rho^2 \qquad \{\mathcal{P} = \mathcal{H}_R\}.$$

L. Fejes Tóth [32], [33, p. 145].

1.49. With the notations from 1.48 we have

$$E_{4,5} \geqslant 30(2 \tan \frac{\pi}{5} \cdot \rho)^{4,5} \qquad \{\mathcal{P} = \mathcal{D}_R\}.$$

L. Fejes Tóth [33, p. 145].

1.50. If $\rho_1, \rho_2, \ldots, \rho_f$ are the radii of the largest circles contained in the different faces of $\mathcal{P}$, then

$$\left(\sum_{i=1}^{f} \rho_i\right)^2 \leqslant \frac{f^2 F}{2e} \operatorname{cotan} \frac{\pi}{p} \qquad \{\mathcal{P} = \mathcal{P}_R\}.$$

L. Fejes Tóth [33, p. 146].

1.51. If $v = v_3$, then $v = 2f - 4$ by 1.5. Let $e_1, e_2, \ldots, e_f$ be the numbers of sides of the different faces of $\mathcal{P}$. If $r_1, r_2, \ldots, r_f$ are the distances of an inner point P of $\mathcal{P}$ from the corresponding faces and R_1, $R_2, \ldots, R_{2f-4}$ the distances of P from the vertices, then

$$\frac{A_{2f-4}(R_i)}{H_f(r_i;e_i)} \geqslant \sqrt{3} \tan \omega_f$$

with equality iff $\mathcal{P}$ is a regular tetrahedron, hexahedron or dodecahedron having the centre P.

L. Fejes Tóth [25], [30], [33, p. 121].

1.52. If $f = f_3$, then $f = 2v - 4$ by 1.6. Let $e_1, e_2, \ldots, e_v$ be the numbers of edges running into the different vertices. If $R_1, R_2, \ldots, R_v$ are the distances of an inner point P of $\mathcal{P}$ from the corresponding vertices and $r_1, r_2, \ldots, r_{2v-4}$ the distances of P from the faces, then

$$\frac{A_v(R_i;e_i)}{H_{2v-4}(r_i)} \geqslant \sqrt{3} \tan \omega_v$$

with equality iff P is a regular tetrahedron, octahedron or icosahedron having the centre P.

L. Fejes Tóth [25], [30], [33, p. 121-122].

1.53. If η is the maximal distance of a boundary point of P from a sphere Ω with the radius ρ and if v = k or f = k, then

$$\eta \geqslant \frac{\sin(\omega_k - \frac{\pi}{6})}{\sin(\omega_k + \frac{\pi}{6})}$$

with equality iff k = 4, $P = T_R$ or k = 6, $P = H_R$ or k = 12, $P = O_R$, I_R and $2\rho = R + r$.

Proof. From 1.15 and 1.16 with $R = \rho + \eta$, $r = \rho - \eta$ it follows

$$\frac{\rho + \eta}{\rho - \eta} \geqslant \sqrt{3} \tan \omega_k,$$

and therefore

$$\frac{\eta}{\rho} \geqslant \frac{\sqrt{3} \tan \omega_k - 1}{\sqrt{3} \tan \omega_k + 1} = \frac{\sin(\omega_k - \frac{\pi}{6})}{\sin(\omega_k + \frac{\pi}{6})} .$$

L. Fejes Tóth [33, p. 119].

1.54. If $\beta_1, \beta_2, \ldots, \beta_e$ are the dihedral angles of P, then

$$\sum_{i=1}^{e} \beta_i > (f - 2)\pi.$$

Proof. Inequality 7.13 can be written in an equivalent form for the polyhedral angles. Applying this inequality to the different vertices of P, adding the obtained v inequalities and dividing by 2, we obtain

$$(12) \qquad \sum_{i=1}^{k} \beta_i > (e - v)\pi.$$

From (12) it follows 1.54.

A. Pinciu [91].

Remark. Because of $\beta_i = \pi - \alpha_i$ (i = 1, 2, ..., e) the inequality (12) can be written in the equivalent form

$$1.54' \qquad \sum_{i=1}^{e} \alpha_i < v\pi.$$

2. Inequalities for Prisms

Let $\mathcal{P}$ be a convex n-angular prism ($n \in \{3, 4, \ldots\}$) with volume V, total area F, base area B, total lateral area L, total edge length E, perimeter p of the base, and altitude h. Let $a_1, a_2, \ldots, a_n$ be the lengths of base edges and $h_1, h_2, \ldots, h_n$ the altitudes and $L_1, L_2, \ldots, L_n$ the areas of the corresponding lateral faces $\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_n$. In the case of a regular prism $\mathcal{P}_R$ let $a_1 = a_2 = \ldots = a_n = a$ and let r be the inradius of the base. Let $\mathcal{P} = \mathcal{P}_\perp$ mean that $\mathcal{P}$ is a right prism.

If $\mathcal{P}$ is a rectangular parallelepided, then let a, b, c be the lengths of three edges from a vertex of $\mathcal{P}$, and let d be the length of a diagonal of $\mathcal{P}$. Let $\mathcal{P} = \mathcal{H}_R$ mean that $\mathcal{P}$ is a cube.

2.1. $\frac{E^2}{F} \geq 2n(4 - \cot\frac{\pi}{n})$ $(n \leq 12)$ $\{\mathcal{P} = \mathcal{P}_R,\ n \in \{3, 4, 5, 6\},$ $h = \frac{p}{n}(2 - \cot\frac{\pi}{n})\}$.

Proof. For given n and given edge lengths of $\mathcal{P}$ the maximal total area obviously has a regular n-angular prism. For such a prism from equalities

$$E = 2p + nh, \quad F = p\left(\frac{p}{2n}\cot\frac{\pi}{n} + h\right) \tag{1}$$

it follows successively

$$\frac{F}{E^2} = \frac{2n}{4 - \cot\frac{\pi}{n}} \cdot \frac{\left(\frac{2}{n} - \frac{1}{2n}\cot\frac{\pi}{n}\right)p \cdot \left(\frac{p}{2n}\cot\frac{\pi}{n} + h\right)}{(2p + nh)^2} \leq$$

$$\leq \frac{2n}{(4 - \cot\frac{\pi}{n})(2p + nh)^2} \cdot \frac{1}{4}\left[\left(\frac{2}{n} - \frac{1}{2n}\cot\frac{\pi}{n}\right)p + \right.$$

$$\left. + \left(\frac{p}{2n}\cot\frac{\pi}{n} + h\right)\right]^2 = \frac{n}{2(4 - \cot\frac{\pi}{n})(2p + nh)^2}\left(\frac{2p}{n} + h\right)^2 =$$

$$= \frac{1}{2n(4 - \cot\frac{\pi}{n})}$$

with equality iff

$$\left(\frac{2}{n} - \frac{1}{2n}\cot\frac{\pi}{n}\right)p = \frac{p}{2n}\cot\frac{\pi}{n} + h,$$

i.e.

$$h = \frac{p}{n}(2 - \cot\frac{\pi}{n}).$$

The conditions

$$4 - \cot\frac{\pi}{n} > 0, \quad 2 - \cot\frac{\pi}{n} > 0$$

are satisfied for $n \in \{3, 4, \ldots, 12\}$ respectively for $n \in \{3, 4, 5, 6\}$.

2.2. $$\frac{E^2}{F} \geq 2(12 - \sqrt{3}) \quad \{n = 3,\ h = \frac{p}{9}(6 - \sqrt{3})\}.$$

M. Kömhoff [71].

2.3. $$\frac{E^3}{V} \geq 108n^2 \tan\frac{\pi}{n} \quad \{P = P_R,\ h = a\}.$$

Proof. For given edge lengths the maximal volume has the right prism and for given base perimeter p the maximal volume has the prism with regular base. Therefore, it suffices to consider the regular prisms. For such a prism from equalities

$$V = \frac{n}{4}\cot\frac{\pi}{n}\cdot a^2h, \quad E = n(2a + h)$$

it follows successively

$$V = \frac{n}{4}\cot\frac{\pi}{n}\cdot a\cdot a\cdot h \leq \frac{n}{4}\cot\frac{\pi}{n}\cdot\left(\frac{a + a + h}{3}\right)^3 =$$

$$= \frac{n}{4}\cot\frac{\pi}{n}\cdot\left(\frac{E}{3n}\right)^3 = \frac{1}{108n^2}\cot\frac{\pi}{n}\cdot E^3 \quad \{h = a\}.$$

A. Procissi [94]; F. G.-M. [125, p. 195-196] (P paralellepiped); D. O. Škljarskij, N. N. Čencov, and I. M. Jaglom [102, p. 262] (P parallelepiped).

Remark. The right-hand side of 2.3 is an increasing function of n and it follows the inequality

2.3' $$\frac{E^3}{V} \geq 972\sqrt{3} \quad \{P = P_R,\ n = 3,\ h = a\}.$$

A. Procissi [94].

2.4. $$L \geq hp \quad \{P = P_\perp\}.$$

Proof. We have

$$L = \sum_{i=1}^{n} a_i h_i \geq \sum_{i=1}^{n} a_i h = hp.$$

J. Steiner [104]; R. Sturm [109, p. 97].

Remark. Inequality 2.4 can be written in the equivalent form

2.4' $BL \geqslant Vp \quad \{P = P_{\perp}\}.$

2.5. $F^2 \geqslant 8Vp \quad \{P = P_{\perp},\ L = 2B\}.$

Proof. Follows from 2.4' according to $F^2 = (2B + L)^2$ with $\{L = 2B\}$.

2.6. $\dfrac{L^2B}{V^2} = \dfrac{L^2}{Vh} \geqslant 4n \tan\dfrac{\pi}{n} \quad \{P = P_R\}.$

Proof. The isoperimetric inequality $p^2/B \geqslant 4n \tan(\pi/n)$ and 2.4 imply

$$\frac{L^2B}{V^2} \geqslant \frac{h^2p^2B}{h^2B^2} = \frac{p^2}{B} \geqslant 4n \tan\frac{\pi}{n} \quad \{P = P_R\}.$$

S. A. J. Lhuilier [75]; J. Steiner [104]*; R. Sturm [109, p. 98-99]*.

2.7. $\dfrac{F^3}{V^2} \geqslant 54n \tan\dfrac{\pi}{n} \quad \{P = P_R,\ L = 4B\}.$

Proof. Because of 2.6 we have

$$\frac{F^3}{V^2} = \frac{27}{8V^2}\left(\frac{2F}{3}\right)^3 = \frac{27}{8V^2}\left(\frac{L + L + 4B}{3}\right)^3 \geqslant \frac{27}{8V^2} \cdot L \cdot L \cdot 4B =$$

$$= \frac{27}{2V^2} L^2B \geqslant 54n \tan\frac{\pi}{n} \ \{P = P_R,\ L = 4B\}.$$

S. A. J. Lhuilier [75]; J. Steiner [104]*; R. Sturm [109, p. 101-102]*, [110] (n = 4); E. Steinitz [107]; F. G.-M. [125, p. 197] (P rectangular parallelepiped); D. O. Škljarskij, N. N. Čencov, and I. M. Jaglom [102, p. 262] (P parallelepiped).

Remark 1. Because of

$$\lim_{n\to\infty} n \tan\frac{\pi}{n} = \pi$$

from 2.7 an analogous result follows for a cylinder of revolution:

2.7' $\dfrac{F^3}{V^2} \geqslant 54\pi \quad \{L = 4B\}.$

S. A. J. Lhuilier [75]; J. Steiner [104]*; R. Sturm [109, p. 102]*;

E. Steinitz [107]; T. J. Fletcher [41]; F. G.-M. [125 , p. 916, 1005].

Remark 2. For a regular prism (respectively for a cylinder of revolution) we have

(2) $L = ph, \quad 2B = pr,$

and the equality $L = 4B$ is equivalent to $h = 2r$, which means that the prism (respectively the cylinder) is circumscribed to a sphere of radius r.

2.8. If $H = B + L$, then

$$\frac{H^3}{V^2} \geq 27n \tan\frac{\pi}{n} \quad \{P = P_R,\ h = r\}.$$

Proof. According to 2.6 it follows

$$\frac{H^3}{V^2} = \frac{27}{V^2}\left[\frac{1}{3}\left(\frac{L}{2} + \frac{L}{2} + B\right)\right]^3 \geq \frac{27}{V^2} \cdot \frac{L}{2} \cdot \frac{L}{2} \cdot B =$$

$$= \frac{27}{4} \cdot \frac{L^2 B}{V^2} \geq \frac{27}{4} \cdot 4n \tan\frac{\pi}{n} = 27n \tan\frac{\pi}{n} \quad \{P = P_R,\ L = 2B\},$$

and because of (2) the equality $L = 2B$ is equivalent to $h = r$.

J. Steiner [104]*.

Remark. For $n \to \infty$ an analogous result follows for a cylinder of revolution

2.8' $$\frac{H^3}{V^2} \geq 27\pi \quad \{h = r\}.$$

F. G.-M. [125, p. 916].

2.9. If $F_n = 2B + L_1 + L_2 + \dots + L_{n-1}$, then

$$\frac{F_n^3}{V^2} \geq 54(n - 1) \tan\frac{\pi}{2(n - 1)}$$

with equality iff P is a prism inscribed in a cylinder of revolution with an axial section L_n and $F_n = 6B$, $L_1 = L_2 = \dots = L_{n-1}$.

Proof. Let P' be the $2(n - 1)$-angular prism, which is the join of the prism P and of its symmetric image with respect to the centre of the face L_n. If V' is the volume and F' the total area of P' and if B' and L' are the base area and the total lateral area of P', then we have by 2.7

(3) $$\frac{F'^3}{V'^2} \geqslant 108(n-1) \tan \frac{\pi}{2(n-1)} \qquad \{\mathcal{P}' = \mathcal{P}_R,\ L' = 4B'\}.$$

Because of the equalities $V' = 2V$, $F' = 2F_n$, $B' = 2B$, $L' = 2F_n - 4B$ we obtain finally 2.9.

J. Steiner [104]*; R. Sturm [109, p. 106-107]*.

2.10. If $G_n = B + L_1 + L_2 + \dots + L_{n-1}$, then

$$\frac{G_n^3}{V^2} \geqslant 27(n-1) \tan \frac{\pi}{2(n-1)}$$

with equality iff $\mathcal{P}$ is a prism inscribed in a cylinder of revolution with an axial section $\mathcal{L}_n$ and $G_n = 3B$, $L_1 = L_2 = \dots = L_{n-1}$.

Proof. Analogous as the proof of 2.9.

J. Steiner [104]*; R. Sturm [109, p. 107]*; L. E. Bush [9] (n = 3).

2.11. If $K_1 = 2B + L_1$, then

$$\frac{BK_1^3}{a_1^2V^2} \geqslant \frac{27}{2}$$

with equality iff the lateral face $\mathcal{L}_1$ is orthogonal to the base and $L_1 = 4B$.

Proof. From $V = Bh$, $L_1 = a_1h_1$, $h \leqslant h_1$ it follows that

$$V \leqslant Bh_1 = \frac{BL_1}{a_1} = \frac{B(K_1 - 2B)}{a_1}$$

with equality iff the face $\mathcal{L}_1$ is orthogonal to the base. Further

$$\frac{4a_1^2V^2}{B} \leqslant 4B(K_1 - 2B)^2 \leqslant \left[\frac{4B + 2(K_1 - 2B)}{3}\right]^3 = \frac{8}{27} K_1^3,$$

where the second inequality becomes equality iff $4B = K_1 - 2B$, i.e. $L_1 = 4B$.

J. Steiner [104]*.

2.12. If $H_1 = B + L_1$, then

$$\frac{BH_1^3}{a_1^2V^2} \geqslant \frac{27}{4}$$

with equality iff the lateral face L_1 is orthogonal to the base and $L_1 = 2B$.

Proof. Analogous to the proof of 2.11.

F. G.-M. [125, p. 197-198] (P a regular quadrangular prism).

If P is a prism circumscribed about a sphere of volume V' and surface area F', then we have the following three inequalities:

2.13. $\frac{F}{F'} = \frac{V}{V'} \geqslant \frac{3n}{2\pi} \tan \frac{\pi}{n} \quad \{P = P_R\},$

2.14. $\frac{B}{F'} \geqslant \frac{n}{4\pi} \tan \frac{\pi}{n} \quad \{P = P_R\},$

2.15. $\frac{L}{F'} \geqslant \frac{n}{\pi} \tan \frac{\pi}{n} \quad \{P = P_R\}.$

R. Sturm [109, p. 110-111].

Remark. For $n \to \infty$ we obtain for a cylinder of revolution the equalities $2V = 3V'$, $2F = 3F'$, $4B = F'$, $L = F'$.

With the same conditions as in 2.13-2.15 we have further:

2.16. $\frac{F}{F'} = \frac{V}{V'} > \frac{3}{2},$

2.17. $\frac{B}{F'} > \frac{1}{4},$

2.18. $\frac{L}{F'} > 1,$

R. Sturm [110].

2.19. If P is a prism inscribed into a cylinder of revolution with the radius r and the altitude h and if V' is the volume of the sphere circumscribed about P, then

$$\frac{V}{V'} \leqslant \frac{\sqrt{3}n}{6\pi} \sin \frac{2\pi}{n} \quad \{h = \sqrt{2}r\}.$$

Proof. If we replace the base of P by a regular n-gon inscribed in the same circle, then the volume of P increases. Therefore, it is sufficient to consider a regular prism. If R is the radius of the circumscribed sphere, then

$$V = \frac{n}{2} r^2 h \sin \frac{2\pi}{n}, \quad V' = \frac{4\pi}{3} R^3,$$

(4) $$R^2 = r^2 + \frac{h^2}{4},$$

and therefore it follows

$$V^2 = \frac{n^2}{2}\sin^2\frac{2\pi}{n}\cdot r^2\cdot r^2\cdot\frac{h^2}{2} \leqslant \frac{n^2}{2}\sin^2\frac{2\pi}{n}\left[\frac{1}{3}(r^2 + r^2 + \frac{h^2}{2})\right]^3 =$$

$$= \frac{n^2}{54}\sin^2\frac{2\pi}{n}(2R^2)^3 = \frac{4n^2}{27}\sin^2\frac{2\pi}{n}\cdot R^6 = \frac{n^2}{12\pi^2}\sin^2\frac{2\pi}{n}\cdot V'^2$$

with equality iff $r^2 = h^2/2$, i.e. $h = \sqrt{2}r$.

R. Sturm [110]; F. G.-M. [125, p. 207-209].

Remark. For $n \to \infty$ we obtain an analogous result for a cylinder of revolution with the radius r and the altitude h and the circumscribed sphere:

2.19' $\frac{V}{V'} \leqslant \frac{\sqrt{3}}{3}$ $\{h = \sqrt{2}r\}$.

R. Sturm [110]; T. J. Fletcher [41]; F. G.-M. [125, p. 208, 996-997, 1006].

2.20. If the conditions of 2.19 are satisfied and if F' is the area of surface of the circumscribed sphere, then

$$\frac{L}{F'} \leqslant \frac{n}{2\pi}\sin\frac{\pi}{n} \quad \{h = 2r\}.$$

Proof. Analogously as in 2.19 it is sufficient to consider only the regular prisms. From the equalities

$$L = 2nrh\sin\frac{\pi}{n}, \quad F' = 4\pi R^2$$

and the equality (4) it follows

$$L^2 = n^2\sin^2\frac{\pi}{n}\cdot 4r^2\cdot h^2 \leqslant n^2\sin^2\frac{\pi}{n}\left(\frac{4r^2 + h^2}{2}\right)^2 =$$

$$= 4n^2\sin^2\frac{\pi}{n}\cdot R^4 = \frac{n^2}{4\pi^2}\sin^2\frac{\pi}{n}\cdot F'^2$$

with equality iff $4r^2 = h^2$, i.e. $h = 2r$.

R. Sturm [110]; F. G.-M. [125, p. 1003] ($n = 3$).

Remark. For $n \to \infty$ we obtain an analogous result for a cylinder of revolution:

2.20' $\frac{L}{F'} \leqslant \frac{1}{2}$ $\{h = 2r\}$.

R. Sturm [110]; F. G.-M. [125, p. 1003].

2.21. If h_n is the altitude, V_n the volume, L_n the total lateral area of a regular n-prism and B_n the area of its base, then for $n, m \in \{3, 4, \dots\}$

($n < m$) we have

$$\frac{L_n^2 B_n}{V_n^2} = \frac{L_n^2}{V_n h_n} > \frac{L_m^2}{V_m h_m} = \frac{L_m^2 B_m}{V_m^2} .$$

Proof. According to 2.6 it follows

$$\frac{L_n^2}{V_n h_n} = 4n \tan \frac{\pi}{n}$$

and the right-hand side of this equality is a decreasing function of n.

J. Steiner [104]*; R. Sturm [109, p. 99]*.

Remark. If h_∞, V_∞, L_∞, B_∞ are the corresponding quantities for a cylinder of revolution, then for any $n \in \{3, 4, \ldots\}$ we have

2.21' $$\frac{L_n^2 B_n}{V_n^2} = \frac{L^2}{V_n h_n} > \frac{L_\infty^2 B_\infty}{V_\infty^2} = \frac{L_\infty^2}{V_\infty h_\infty} = 4.$$

2.22. The planes α, β, γ, δ intersect the given infinite n-angular prism at the polygons $\mathcal{A} = A_1A_2 \ldots A_n$, $\mathcal{B} = B_1B_2 \ldots B_n$, $\mathcal{C} = C_1C_2 \ldots C_n$, $\mathcal{D} = D_1D_2 \ldots D_n$ such that $A_iB_i = C_iD_i$ ($i = 1, 2, \ldots, n$) and such that the plane of symmetry of the planes α and β is orthogonal to the lateral edges of the prism. If F_α, F_β, F_γ, F_δ are the areas of the polygons $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, then

$$F_\alpha + F_\beta \leqslant F_\gamma + F_\delta$$

with equality if the plane of symmetry of the planes γ and δ is orthogonal to the lateral edges of the prism.

J. Steiner [103], [104]; R. Sturm [109, p. 135-136].

2.23. The planes α, β intersect the given infinite n-prism at the polygons $\mathcal{A} = A_1A_2 \ldots A_n$, $\mathcal{B} = B_1B_2 \ldots B_n$. For any $i \in \{1, 2, \ldots, n\}$ let C_i be the midpoint of the segment $\overline{A_iB_i}$. The points $C_1, C_2, \ldots, C_n$ form a polygon $\mathcal{C}$ in a plane γ. If F_α, F_β, F_γ are the areas of $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, then

$$F_\alpha + F_\beta > 2F_\gamma .$$

J. Steiner [103]; R. Sturm [109, p. 136-137].

Remark. Inequality 2.23 generalizes the inequality XIX.3.16.

2.24. If a, b, c are the edge lengths, d the length of the diagonal of a parallelepiped and e, f, g the lengths of the diagonals of its faces, where all these segments have one common end point, then

$$a + b + c + d > e + f + g.$$

H. Hornich [61]; G. R. Veldkamp [117]; AI 2.52.2.

2.25. If $\mathcal{P}$ is a rectangular parallelepiped, then

$$6\sqrt[3]{V^2} \leqslant F \leqslant \frac{1}{24} E^2 \leqslant 2d^2 \qquad \{\mathcal{P} = \mathcal{H}_R\}.$$

Proof. We must prove the inequalities

$$6\sqrt[3]{a^2b^2c^2} \leqslant 2(bc + ca + ab) \leqslant \frac{2}{3}(a + b + c)^2 \leqslant$$

$$\leqslant 2(a^2 + b^2 + c^2) \qquad \{a = b = c\}.$$

The first inequality is the A-G-inequality for means and the second and third inequality are equivalent to the inequality $bc + ca + ab \leqslant a^2 + b^2 + c^2$, i.e. to the inequality $(b - c)^2 + (c - a)^2 + (a - b)^2 \geqslant 0$.

N. Schaumberger [98]; F. G.-M. [125, p. 201-202 (inequality $3\sqrt{3}V \leqslant d^3$), p. 915 (inequalities $F \leqslant 2d^2$, $E \leqslant 4\sqrt{3}d$)].

2.26. If $\mathcal{P}$ is a rectangular parallelepiped, then

$$L \leqslant \sqrt{2}d^2$$

with equality iff the base of $\mathcal{P}$ is a square and the altitude of $\mathcal{P}$ is equal to the diagonal of this square.

Proof. From the equalities $d^2 = a^2 + b^2 + c^2$, $L = 2(a + b)c$ we obtain

$$d^2 = a^2 + b^2 + c^2 \geqslant \frac{(a + b)^2 + 2c^2}{2} \geqslant \sqrt{(a + b)^2 \cdot 2c^2} =$$

$$= \sqrt{2}(a + b)c = \frac{1}{\sqrt{2}} L.$$

with equality iff $a = b$, $a + b = \sqrt{2}c$, i.e. $a = b$, $c = \sqrt{2}a$.

E. G. Gotman [53].

2.27. If F_1, F_2, F_3 are the areas of three non-parallel faces of a rectangular parallelepiped $\mathcal{P}$, then

$$F_1^2 + F_2^2 + F_3^2 \geqslant \sqrt{3}\,Vd \qquad \{\mathcal{P} = \mathcal{H}_R\}.$$

Proof. Because of $F_1 = ab$, $F_2 = ac$, $F_3 = bc$, $V = abc$, $d^2 = a^2 + b^2 + c^2$ we get

$$(F_1^2 + F_2^2 + F_3^2)^2 \geqslant 3(F_1^2F_2^2 + F_1^2F_3^2 + F_2^2F_3^2) =$$

$$= 3a^2b^2c^2(a^2 + b^2 + c^2) = 3V^2d^2 \quad \{a = b = c\},$$

where the inequality is equivalent to the inequality

$$(F_1^2 - F_2^2)^2 + (F_1^2 - F_3^2)^2 + (F_2^2 - F_3^2)^2 \geqslant 0 \quad \{a = b = c\}.$$

S. G. Guba [55].

2.28. If P is a rectangular parallelepiped and, if we have the notations from 2.24, then

$$e + f + g \geqslant \sqrt{2}(a + b + c) \quad \{P = H_R\}.$$

Proof. Follows by adding of the inequalities

$$e = \sqrt{a^2 + b^2} \geqslant \frac{\sqrt{2}}{2}(a + b) \quad \{a = b\},$$

$$f = \sqrt{a^2 + c^2} \geqslant \frac{\sqrt{2}}{2}(a + c) \quad \{a = c\},$$

$$g = \sqrt{b^2 + c^2} \geqslant \frac{\sqrt{2}}{2}(b + c) \quad \{b = c\}.$$

G. Ionescu-Ţiu [62].

2.29. If α, β, γ are the angles of a diagonal of a rectangular parallelepiped P with its edges, then

$$\tan \alpha \tan \beta \tan \gamma \geqslant 2\sqrt{2} \quad \{P = H_R\}.$$

Proof. Because of the equality $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$ we obtain

$$\tan^2 \alpha \tan^2 \beta \tan^2 \gamma = \frac{(1 - \cos^2 \alpha)(1 - \cos^2 \beta)(1 - \cos^2 \gamma)}{\cos^2 \alpha \cos^2 \beta \cos^2 \gamma} =$$

$$= \frac{\cos^2 \beta \cos^2 \gamma + \cos^2 \gamma \cos^2 \alpha + \cos^2 \alpha \cos^2 \beta - \cos^2 \alpha \cos^2 \beta \cos^2 \gamma}{\cos^2 \alpha \cos^2 \beta \cos^2 \gamma}$$

$$= \frac{1}{\cos^2 \alpha} + \frac{1}{\cos^2 \beta} + \frac{1}{\cos^2 \gamma} - 1 = \left(\frac{1}{\cos^2 \alpha} + \frac{1}{\cos^2 \beta} + \frac{1}{\cos^2 \gamma}\right)\cdot$$

$$\cdot(\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma) - 1 \geqslant 9 - 1 = 8 \quad \{\alpha = \beta = \gamma\}.$$

I. Voicu [119].

2.30. If ϕ and ψ are the angles between a diagonal of a rectangular parallelepiped and two diagonals of its faces, where all three diagonals have one common end point, then

$$\phi + \psi < \frac{\pi}{2}.$$

Proof. From equalities $a = d \sin \phi$, $b = d \sin \psi$, $a^2 + b^2 = d^2 - c^2$ it follows

$$\sin^2 \phi < 1 - \sin^2 \psi = \sin^2 (\frac{\pi}{2} - \psi),$$

i.e.

$$\phi < \frac{\pi}{2} - \psi.$$

D. O. Škljarskij, N. N. Čencov, and I. M. Jaglom [102, p. 31, 164-165].

2.31. If P is a rectangular parallelepiped and if $a = b$, $a + c = s$, then

$$\frac{s^3}{V} \geq \frac{27}{4} \quad \{a = 2c\}.$$

F. G.-M. [125, p. 196].

2.32. If P is a rectangular parallelepiped, $F_1 = ab$ the area of a face of P and $F' = F - F_1$, then

$$\frac{F'^3}{V^2} \geq 108 \quad \{a = b = 2c\}.$$

Proof. Follows by applying the inequality 2.7 (with $n = 4$) to a rectangular parallelepiped with edge lengths $a_1 = a$, $b_1 = b$, $c_1 = 2c$, the total area $F_1 = 2F'$ and the volume $V_1 = 2V$.

F. G.-M. [125, p. 197] ($a = b$).

3. Inequalities for Pyramids

Let P be a convex n-angular pyramid ($n \geq 3$) with vertex O, base $A = A_1A_2 \ldots A_n$, altitude $h = OC$, total area F, lateral area L, volume V, and polyhedral angle ω at the vertex O. Let p be the perimeter and B the area of the base A and Π the plane of the base. For any $i \in \{1, 2, \ldots, n\}$, where the indices are taken modulo n from the set $\{1, 2, \ldots, n\}$, let $a_i = A_iA_{i+1}$, $R_i = CA_i$, $\delta_i = \measuredangle COA_i$, let β_i, $\alpha_{i,i+1}$, $\alpha_{i+1,i}$ be the angles between the plane OA_iA_{i+1} and the planes Π, OCA_i, OCA_{i+1}, respectively. Further if B_i is the foot of the normal from C onto the line A_iA_{i+1}, then let $a_{i,i+1} = A_iB_i$, $a_{i+1,i} = A_{i+1}B_i$, $r_i = CB_i$. Let $\gamma_{i,i+1}$, $\gamma_{i+1,i}$ be the angles between the plane OCB_i and the planes OCA_i, OCA_{i+1}, respectively, and let $B_{i,i+1}$, $B_{i+1,i}$ be the areas of the triangles CA_iB_i, $CA_{i+1}B_i$, respectively. Let

(1) $$M_{i,i+1} = \frac{1}{2} a_{i,i+1}(\frac{\pi}{2} - \beta_i),$$

$$M_{i+1,i} = \frac{1}{2} a_{i+1,i}(\frac{\pi}{2} - \beta_i),$$

and let

$$M = \sum_{i=1}^{n} a_i(\frac{\pi}{2} - \beta_i)$$

be the 'base curvature' of $\mathcal{P}$.

We shall say that $\mathcal{P}$ is a 'vertical' pyramid if C is an internal point of the base $\mathcal{A}$ and B_1, B_2, ..., B_n are internal points of the segments $\overline{A_1A_2}$, $\overline{A_2A_3}$, ..., $\overline{A_nA_1}$, respectively. For a vertical pyramid we have $a_i = a_{i,i+1} + a_{i+1,i}$,

(2) $$\sum_{i=1}^{n} (\gamma_{i,i+1} + \gamma_{i+1,i}) = 2\pi,$$

(3) $$\sum_{i=1}^{n} (\alpha_{i,i+1} + \alpha_{i+1,i}) = (n - 2)\pi + \omega,$$

(4) $$\sum_{i=1}^{n} (B_{i,i+1} + B_{i+1,i}) = B,$$

(5) $$\sum_{i=1}^{n} (M_{i,i+1} + M_{i+1,i}) = M.$$

Applying spherical trigonometry for any $i \in \{1, 2, ..., n\}$ we have

(6) $$\cos(\frac{\pi}{2} - \beta_i) = \frac{\cos \alpha_{i,i+1}}{\cos \gamma_{i,i+1}}$$

and

(7) $$\cos \delta_i = \operatorname{cotan} \alpha_{i,i+1} \operatorname{cotan} \gamma_{i,i+1},$$

i.e

(8) $$\tan \delta_i = \sqrt{\tan^2 \alpha_{i,i+1} \tan^2 \gamma_{i,i+1} - 1}.$$

From the equalities

$$B_{i,i+1} = \frac{1}{2} a_{i,i+1} r_i,$$

(9) $$a_{i,i+1} = R_i \sin \gamma_{i,i+1}, \quad r_i = R_i \cos \gamma_{i,i+1},$$

(10) $$R_i = h \tan \delta_i$$

it follows according to (8) that

(11) $$B_{i,i+1} = \frac{h^2}{2} \sin 2\gamma_{i,i+1} (\tan^2 \alpha_{i,i+1} \tan^2 \gamma_{i,i+1} - 1),$$

and analogously we have

(12) $$B_{i+1,i} = \frac{h^2}{2} \sin 2\gamma_{i+1,i} (\tan^2 \alpha_{i+1,i} \tan^2 \gamma_{i+1,i} - 1).$$

From (1), (9), (10), (8) and (6) we obtain

(13) $$M_{i,i+1} = \frac{h}{2} \sin \gamma_{i,i+1} \sqrt{\tan^2 \alpha_{i,i+1} \tan^2 \gamma_{i,i+1} - 1} \cdot \arccos\left(\frac{\cos \alpha_{i,i+1}}{\sin \gamma_{i,i+1}}\right)$$

and analogously

(14) $$M_{i+1,i} = \frac{h}{2} \sin \gamma_{i+1,i} \sqrt{\tan^2 \alpha_{i+1,i} \tan^2 \gamma_{i+1,i} - 1} \cdot \arccos\left(\frac{\cos \alpha_{i+1,i}}{\sin \gamma_{i+1,i}}\right).$$

If $r_1 = r_2 = \dots = r_n = r$, then the pyramid $\mathcal{P}$ is circumscribed about a circular cone $\mathcal{C}$ with the radius r and the altitude h. If $\mathcal{C}$ is a cone of revolution, then $\beta_1 = \beta_2 = \dots = \beta_n = \beta$. For $n \to \infty$ the pyramid $\mathcal{P}$ converges to the cone $\mathcal{C}$.

If the pyramid $\mathcal{P}$ is inscribed in a circular cone, then let $R_1 = R_2 = \dots = R_n = R$. The notation $\{\mathcal{P} = \mathcal{P}_R\}$ respectively $\{\mathcal{C} = \mathcal{C}_R\}$ means that in a considered inequality the equality sign occurs iff P is a regular pyramid respectively C is a cone of revolution.

3.1. If Σ is a sphere with the centre O and the radius s and if $\Sigma \cap \Pi = \emptyset$, then

$$B \geq \frac{n}{2} \sin \frac{2\pi}{n}(\tan^2 \frac{\pi}{n} \cot^2 \frac{2\pi - \omega}{2n} - 1)s^2 \quad \{P = P_R,\ h = s\}.$$

Proof. We can suppose that P is a vertical pyramid, because in the contrary the n-gon A can be replaced by an n-gon of smaller area but with the same polyhedral angle ω and the same altitude h. From (4), (11) and (12) because of $h \geq s$ it follows the inequality

$$B \geq \frac{s^2}{4} \sum_{i=1}^{n} [\sin 2\gamma_{i,i+1}(\tan^2 \alpha_{i,i+1} \tan^2 \gamma_{i,i+1} - 1) +$$

$$+ \sin 2\gamma_{i+1,i}(\tan^2 \alpha_{i+1,i} \tan^2 \gamma_{i+1,i} - 1)]$$

with equality iff $h = s$. The function $B(u, v) = \sin 2u(\tan^2 u \tan^2 v - 1)$ is convex for $0 < u,\ v < \frac{\pi}{2}$, $u + v > \frac{\pi}{2}$, because of

$$B_{uu}B_{vv} - B_{uv}^2 = \frac{32 \tan^4 u}{\cos^6 v}(\cos^2 u - \sin^2 v)^2 > 0,$$

$$B_{vv} = 2 \sin 2u \tan^2 u \frac{1 + 2 \sin^2 v}{\cos^4 v} > 0.$$

Therefore, from (2) and (3) we obtain

$$B \geq \frac{1}{4}\, 2n \sin \frac{4\pi}{2n}[\tan^2 \frac{(n-2)\pi + \omega}{2n} \tan^2 \frac{2\pi}{2n} - 1] \geq$$

$$\geq \frac{n}{2} \sin \frac{2\pi}{n}(\tan^2 \frac{\pi}{n} \cot^2 \frac{2\pi - \omega}{2n} - 1)$$

with equality iff $h = s$ and all triangles CA_iB_i and $CA_{i+1}B_i$ are congruent.

L. Fejes Tóth [31], [33, p. 124-125], [29] (n = 3); A. Florian [42].

3.2. If the conditions of 3.1 are satisfied, then

$$V \geq \frac{n}{6} \sin \frac{2\pi}{n}(\tan^2 \frac{\pi}{n} \cot^2 \frac{2\pi - \omega}{2n} - 1)s^3 \quad \{P = P_R,\ h = s\}.$$

Proof. Follows from 3.1 because of $3V = Bh$ and $h \geq s$.

3.3. If the conditions of 3.1 are satisfied and if P is a vertical pyramid, then

$$M \geqslant n \sin \frac{\pi}{n} \sqrt{\tan^2 \frac{\pi}{n} \operatorname{cotan}^2 \frac{2\pi - \omega}{2n} - 1} \cdot$$

$$\cdot \arccos(\sin \frac{2\pi - \omega}{2n} \operatorname{cosec} \frac{\pi}{n}) \cdot s \qquad \{P = P_R,\ h = s\}.$$

Proof. From (5), (13) and (14) and $h \geqslant s$ it follows the inequality

$$M \geqslant \frac{s}{2} \sum_{i=1}^{n} [\sin \gamma_{i,i+1} \sqrt{\tan^2 \alpha_{i,i+1} \tan^2 \gamma_{i,i+1} - 1} \cdot$$

$$\cdot \arccos(\cos \alpha_{i,i+1} \operatorname{cosec} \gamma_{i,i+1}) +$$

$$+ \sin \gamma_{i+1,i} \sqrt{\tan^2 \alpha_{i+1,i} \tan^2 \gamma_{i+1,i} - 1} \cdot$$

$$\cdot \arccos(\cos \alpha_{i+1,i} \operatorname{cosec} \gamma_{i+1,i})]$$

with equality iff $h = s$. As $M(u, v) = \sin u \sqrt{\tan^2 u \tan^2 v - 1} \cdot \arccos(\cos v \operatorname{cosec} u)$ is a convex function for $0 < u, v < \frac{\pi}{2}$, $u + v > \frac{\pi}{2}$, by (2) and (3) it follows 3.3.

A. Florian [43], [44]; L. Fejes Tóth [38, p. 261-263].

3.4. If Σ is a sphere with the centre O and the radius s which contains a vertical pyramid P, then

$$B \leqslant \frac{n}{2} \sin \frac{2\pi}{n}(1 - \operatorname{cotan}^2 \frac{\pi}{n} \tan^2 \frac{2\pi - \omega}{2n}) s^2 \qquad \{P = P_R;\ A_1, A_2, \ldots, A_n \in \Sigma\}.$$

Proof. From (4), (11) and (12) according to the inequality

(15) $$h \leqslant s \cos \delta_i \qquad \{A_i \in \Sigma\},$$

the equality (7) and the similar equality

$$\cos \delta_{i+1} = \operatorname{cotan} \alpha_{i+1,i} \operatorname{cotan} \gamma_{i+1,i}$$

for every $i \in \{1, 2, \ldots, n\}$ we obtain

$$B \leqslant \frac{s^2}{4} \sum_{i=1}^{n} [\sin 2\gamma_{i,i+1}(1 - \operatorname{cotan}^2 \alpha_{i,i+1} \operatorname{cotan}^2 \gamma_{i,i+1}) +$$

$$+ \sin 2\gamma_{i+1,i}(1 - \operatorname{cotan}^2 \alpha_{i+1,i} \operatorname{cotan}^2 \gamma_{i+1,i})]$$

with equality iff $A_1, A_2, \ldots, A_n \in \Sigma$. The function $B'(u, v) =$

$= \sin 2u(1 - \mathrm{cotan}^2 u \,\mathrm{cotan}^2 v)$ is concave for $0 < u, v < \frac{\pi}{2}$, $u + v > \frac{\pi}{2}$ and according to (2) and (3) it follows 3.4.

L. Fejes Tóth [31] $(A_1, A_2, \ldots, A_n \in \Sigma)$.

3.5. If the conditions of 3.4 are satisfied, then

$$h \leqslant \mathrm{cotan}\,\frac{\pi}{n}\tan\frac{2\pi - \omega}{2n}\cdot s \qquad \{P = P_R;\ A_1, A_2, \ldots, A_n \in \Sigma\}.$$

<u>Proof</u>. For any $i \in \{1, 2, \ldots, n\}$ let $\omega_{i,i+1}$ and $\omega_{i+1,i}$ be the trihedral angles defined by the vertex O and the triangles CA_iB_i and $CA_{i+1}B_i$. From equalities

$$\omega_{i,i+1} = \alpha_{i,i+1} + \gamma_{i,i+1} - \frac{\pi}{2}$$

according to (7) and (15) we obtain

$$\text{(16)} \qquad \gamma_{i,i+1} - \omega_{i,i+1} \geqslant \arctan\left(\frac{h}{s}\tan\gamma_{i,i+1}\right) \qquad (i = 1, 2, \ldots, n)\ \{A_i \in \Sigma\},$$

and analogously

$$\text{(17)} \qquad \gamma_{i+1,i} - \omega_{i+1,i} \geqslant \arctan\left(\frac{h}{s}\tan\gamma_{i+1,i}\right) \qquad (i = 1, 2, \ldots, n)\ \{A_{i+1} \in \Sigma\}.$$

Adding the equalities (16) and (17), because of equality (2) and the equality

$$\sum_{i=1}^{n} (\omega_{i,i+1} + \omega_{i+1,i}) = \omega,$$

it follows

$$2\pi - \omega \geqslant \sum_{i=1}^{n} \left[\arctan\left(\frac{h}{s}\tan\gamma_{i,i+1}\right) + \arctan\left(\frac{h}{s}\tan\gamma_{i+1,i}\right)\right] \qquad \{A_1, A_2, \ldots, A_n \in \Sigma\}.$$

As $f(x) = \arctan(\frac{h}{s}\tan x)$ is a convex function for $0 < x < \frac{\pi}{2}$, by (2) we get

$$2\pi - \omega \geqslant 2n\arctan\left(\frac{h}{s}\tan\frac{\pi}{n}\right),$$

i.e. the inequality 3.5. Equality occurs iff $A_1, A_2, \ldots, A_n \in \Sigma$ and all angles $\gamma_{i,i+1}$ and $\gamma_{i+1,i}$ are equal.

A. Florian [42].

3.6. If the conditions of 3.4 are satisfied, then

$$V \leqslant \frac{n}{3} \cos^2 \frac{\pi}{n} \tan \frac{2\pi - \omega}{2n} \left(1 - \cot\mathrm{an}^2 \frac{\pi}{n} \tan^2 \frac{2\pi - \omega}{2n}\right) s^3$$

$\{\underline{P = P_R;\ A_1, A_2, \ldots, A_n \in \Sigma\}}.$

Proof. Follows from 3.4 and 3.5 because of 3V = Bh.

A. Florian [42]; J. Fejes Tóth [33, p. 129], [38, p. 248-260].

3.7. If the conditions of 3.4 are satisfied, then

$$M \leqslant n \sin \frac{\pi}{n} \sqrt{1 - \cot\mathrm{an}^2 \frac{\pi}{n} \tan^2 \frac{2\pi - \omega}{2n}} \cdot$$

$$\cdot \arccos\left(\sin \frac{2\pi - \omega}{2n} \operatorname{cosec} \frac{\pi}{n}\right) \cdot s$$

with equality iff $P = P_R$ and $A_1, A_2, \ldots, A_n \in \Sigma$.

Proof. From (5), (13), (14), (15) and (7) we obtain

$$M \leqslant \frac{s}{2} \sum_{i=1}^{n} [\sin \gamma_{i,i+1} \sqrt{1 - \cot\mathrm{an}^2 \alpha_{i,i+1} \cot\mathrm{an}^2 \gamma_{i,i+1}} \cdot$$

$$\cdot \arccos(\cos \alpha_{i,i+1} \operatorname{cosec} \gamma_{i,i+1}) +$$

$$+ \sin \gamma_{i+1,i} \sqrt{1 - \cot\mathrm{an}^2 \alpha_{i+1,i} \cot\mathrm{an}^2 \gamma_{i+1,i}} \cdot$$

$$\cdot \arccos(\cos \alpha_{i+1,i} \operatorname{cosec} \gamma_{i+1,i})]$$

with equality iff $A_1, A_2, \ldots, A_n \in \Sigma$. The function $M'(u, v) = \sin u \sqrt{1 - \cot\mathrm{an}^2 u \cot\mathrm{an}^2 v} \cdot \arccos(\cos v \operatorname{cosec} u)$ is convex for $0 < u, v < \frac{\pi}{2}$, $u + v > \frac{\pi}{2}$. Therefore, by (2) and (3) it follows 3.7.

A. Florian [43]; L. Fejes Tóth [38, p. 262-263].

3.8. $$\frac{B^3 L^9}{V^8} \geqslant \frac{3^{17}}{2^{12}} n^4 \tan^4 \frac{\pi}{n} \qquad \{P = P_R,\ \beta = \arccos \frac{1}{3}\}.$$

Proof. (In the case of a pyramid P circumscribed about a cone of revolution.) If p is the perimeter of the base of P, then we have the isoperimetric inequality

(18) $$\frac{B}{p^2} \leqslant \frac{1}{4n} \cot \frac{\pi}{n} \quad \{a_1 = a_2 = \ldots = a_n\}.$$

The altitude of P is $r \tan \beta$ and the altitude of any lateral face of P is $r \sec \beta$ and therefore

(19) $$B = \frac{1}{2} pr,$$

(20) $$L = \frac{1}{2} pr \sec \beta,$$

(21) $$V = \frac{1}{6} pr^2 \tan \beta.$$

Now, we obtain successively

$$\frac{V^8}{B^3 L^9} = \frac{2^4 r^4}{3^8 p^4} \cos \beta \sin^8 \beta = \frac{2^8 B^4}{3^8 p^8} \cos \beta \sin^8 \beta \leqslant$$

$$\leqslant \frac{2^8}{3^8}\left(\frac{1}{4n} \cot \frac{\pi}{n}\right)^4 \cos \beta \sin^8 \beta =$$

$$= \frac{2^{12}}{3^8 n^4} \cot^4 \frac{\pi}{n} \cos \beta \left(\frac{1 - \cos \beta}{2}\right)^4 \left(\frac{1 + \cos \beta}{4}\right)^4 \leqslant$$

$$\leqslant \frac{2^{12}}{3^8 n^4} \cot^4 \frac{\pi}{n}\left[\frac{1}{9}\left(\cos \beta + 4 \cdot \frac{1 - \cos \beta}{2} +\right.\right.$$

$$\left.\left. + 4 \cdot \frac{1 + \cos \beta}{4}\right)\right]^9 = \frac{2^{12}}{3^{17} n^4} \cot^4 \frac{\pi}{n}$$

with equality iff $a_1 = a_2 = \ldots = a_n$ and

$$\cos \beta = \frac{1 - \cos \beta}{2} = \frac{1 + \cos \beta}{4}, \quad \text{i.e.} \quad \cos \beta = \frac{1}{3}.$$

J. Steiner [104]*; R. Sturm [109, p. 116-119]*; [126]* (A a regular n-gon).

Remark. For $n \to \infty$ we obtain an analogous result for a circular cone:

3.8' $$\frac{B^3 L^9}{V^8} \geqslant \frac{3^{17}}{2^{12}} \pi^4 \quad \{C = C_R,\ \beta = \arccos \frac{1}{3}\}.$$

R. Sturm [109, p. 122]*.

3.9. $\dfrac{L^3}{V^2} \geqslant \dfrac{27\sqrt{3}}{2}\, n \tan \dfrac{\pi}{n} \quad \{P = P_R,\ \beta = \arccos \dfrac{\sqrt{3}}{3}\}.$

Proof. (In the case of a pyramid P circumscribed about a cone of revolution.) According to (18)-(21) we obtain

$$\frac{V^2}{L^3} = \frac{2r}{9p} \cos \beta \sin^2 \beta = \frac{4B}{9p^2} \cos \beta \sin^2 \beta \leqslant$$

$$\leqslant \frac{4}{9}\left(\frac{1}{4n} \operatorname{cotan} \frac{\pi}{n}\right) \cos \beta \sin^2 \beta = \frac{1}{36n} \operatorname{cotan} \frac{\pi}{n} \cdot 2 \cos \beta \cdot$$

$$\cdot (\sqrt{3} + 1)(1 - \cos\beta) \cdot (\sqrt{3} - 1)(1 + \cos \beta) \leqslant$$

$$\leqslant \frac{1}{36n} \operatorname{cotan} \frac{\pi}{n}\Big\{\frac{1}{3}[2\cos \beta + (\sqrt{3} + 1)(1 - \cos \beta) +$$

$$+ (\sqrt{3} - 1)(1 + \cos \beta)]\Big\}^3 = \frac{1}{36n} \operatorname{cotan} \frac{\pi}{n} \cdot \left(\frac{2\sqrt{3}}{3}\right)^3 =$$

$$= \frac{2\sqrt{3}}{81n} \operatorname{cotan} \frac{\pi}{n}$$

with equality iff $a_1 = a_2 = \ldots = a_n$ and

$$2 \cos \beta = (\sqrt{3} + 1)(1 - \cos\beta) = (\sqrt{3} - 1)(1 + \cos \beta)$$

i.e.

$$\cos \beta = \frac{\sqrt{3}}{3}.$$

J. Steiner [104]*; R. Sturm [109, p. 122-123]*.

Remark. For $n \to \infty$ we obtain an analogous result for a circular cone

3.9' $\dfrac{L^3}{V^2} \geqslant \dfrac{27\sqrt{3}}{2}\, \pi \quad \{C = C_R,\ \beta = \arccos \dfrac{\sqrt{3}}{3}\}.$

3.10. $\dfrac{F^3}{V^2} \geqslant 72\, n \tan \dfrac{\pi}{n} \quad \{P = P_R,\ \beta = \arccos \dfrac{1}{3}\}.$

Proof. (In the case of a pyramid P circumscribed about a cone of revolution.) We have

$$BL^3 = 27 \cdot B \cdot \frac{L}{3} \cdot \frac{L}{3} \cdot \frac{L}{3} \leqslant 27\left[\frac{1}{4}\left(B + 3 \cdot \frac{L}{3}\right)\right]^4 =$$

$$= 27\left(\frac{1}{4} F\right)^4 = \frac{27}{256} F^4 \quad \{L = 3B\}, .$$

and according to 3.8 it follows

$$\frac{3^9}{2^{24}}\frac{F^{12}}{V^8} \geqslant \frac{1}{V^8}(BL^3)^3 \geqslant \frac{3^{17}}{2^{12}} n^4 \tan^4 \frac{\pi}{n} \quad \{P = P_R,\ \beta = \arccos \frac{1}{3}\}.$$

i.e. 3.10, because the condition $L = 3B$ is equivalent to $\beta = \arccos \frac{1}{3}$.

S. A. J. Lhuilier [75]; R. Sturm [109]*; T. J. Fletcher [41] ($P = P_R$, $n = 4$).

Remark. For $n \to \infty$ we get an analogous result for a circular cone:

3.10' $$\frac{F^3}{V^2} \geqslant 72\pi \quad \{C = C_R,\ \beta = \arccos \frac{1}{3}\}.$$

3.11. Let P' be an n-angular prism inscribed in P such that every lateral edge of P contains one vertex of a base of P' and the second base of P' lies in the plane Π. If V' is the volume and h' the altitude of P', then

$$\frac{V'}{V} \leqslant \frac{4}{9} \quad \{h = 3h'\}.$$

Proof. If B' is the area of the base of P', then from equalities

$$V = \frac{1}{3} Bh, \qquad V' = B'h', \qquad \frac{B'}{B} = \frac{(h - h')^2}{h^2}$$

it follows

$$\frac{V'}{V} = \frac{3B'h'}{Bh} = \frac{3}{2h^3} \cdot (h - h')^2 \cdot 2h' \leqslant$$

$$\leqslant \frac{3}{2h^3}\left[\frac{2(h - h') + 2h'}{3}\right]^3 = \frac{4}{9} \quad \{h = 3h'\}.$$

F. G.-M. [125, p. 200-201].

Remark. Inequality 3.11 holds also for a circular cone and an inscribed circular cylinder.

F. G.-M. [125, p. 201, 995-996, 1005].

3.12. If P is inscribed in a circular cone and if V' is the volume of the sphere circumscribed about P, then

$$\frac{V}{V'} \leqslant \frac{4n}{27\pi} \sin \frac{2\pi}{n} \quad \{P = P_R,\ h = R\sqrt{2}\}.$$

Proof. If R' is the radius of the circumscribed sphere, then for given R, R' and n the maximal volume appears for a regular pyramid P. For such a pyramid we have

$$V = \frac{n}{6} R^2 h \sin \frac{2\pi}{n},$$

and from the equality $(h - R')^2 + R^2 = R'^2$ it follows

$$2R' = \frac{R^2 + h^2}{h}.$$

Therefore, the equality

(22) $$V' = \frac{4}{3} R'^3 \pi$$

implies

$$\frac{V}{V'} = \frac{n}{\pi} \sin \frac{2\pi}{n} \cdot \frac{R^2 h}{8R'^3} = \frac{4n}{\pi} \sin \frac{2\pi}{n} \cdot \frac{1}{(R^2 + h^2)^3} \cdot R^2 \cdot \frac{h^2}{2} \cdot \frac{h^2}{2} \leqslant$$

$$\leqslant \frac{4n}{\pi} \sin \frac{2\pi}{n} \cdot \frac{1}{(R^2 + h^2)^3} \left[\frac{1}{3}(R^2 + 2 \cdot \frac{h^2}{2})\right]^3 =$$

$$= \frac{4n}{27\pi} \sin \frac{2\pi}{n} \quad \{R^2 = \frac{h^2}{2}\}.$$

R. Sturm [110].

Remark 1. For $n \to \infty$ we get an analogous result for a circular cone:

3.12' $$\frac{V}{V'} \leqslant \frac{8}{27} \quad \{C = C_R,\ h = r\sqrt{2}\}.$$

R. Sturm [110]; F. G.-M. [125, p. 996, 999]; T. J. Fletcher [41] $(C = C_R)$.

Remark 2. D. Matei [82] has given for a cone of revolution the weaker inequality $16V < 3\sqrt{3}V'$.

3.13. If P is circumscribed about a circular cone and if F'' and V'' are the total area and the volume, respectively, of the sphere inscribed in P, then

$$\frac{V}{V''} = \frac{F}{F''} \geqslant \frac{2n}{\pi} \tan \frac{\pi}{n} \quad \{P = P_R,\ \beta = \arccos \frac{1}{3}\}.$$

Proof. For given R and n the maximal base area and maximal volume appear for a regular pyramid P. If O'' is the centre and R'' the radius of the inscribed sphere and D_i the foot of the perpendicular from O'' onto the line OB_i, then we have $O''C = OD'' = R''$, $B_iC = B_iD_i = R$, and from

$$(\sqrt{h^2 + R^2} - R):R'' = h:R$$

it follows

(23) $$h = \frac{2R^2R''}{R^2 - R''^2} .$$

Therefore, from the equalities

$$V = \frac{n}{3} R^2 h \tan \frac{\pi}{n} , \quad V'' = \frac{4}{3} R''^3 \pi$$

we obtain

$$\frac{V''}{V} = \frac{4\pi}{n} \cot an \frac{\pi}{n} \cdot \frac{R''^3}{R^2 h} = \frac{2\pi}{n} \cot an \frac{\pi}{n} \cdot \frac{1}{R^4} R''^2(R^2 - R''^2) \leqslant$$

$$\leqslant \frac{2\pi}{n} \cot an \frac{\pi}{n} \cdot \frac{1}{R^4}\left[\frac{1}{2}(R''^2 + R^2 - R''^2)\right]^2 = \frac{\pi}{2n} \cot an \frac{\pi}{n}$$

with equality iff $R''^2 = R^2 - R''^2$, i.e. by (23) $h = 2\sqrt{2}R$ or equivalently

$$\beta = \arccos \frac{R}{\sqrt{h^2 + R^2}} = \arccos \frac{1}{3} .$$

From equalities $3V = FR''$, $3V'' = F''R''$ immediately follows $V:V'' = F:F''$.

J. Steiner [104]*; R. Sturm [109, p. 123-124]*, [110]; F. G.-M. [125, p. 1000-1001] ($n = 3$); E. G. Gotman [52] ($\mathcal{P} = \mathcal{P}_R$).

Remark 1. For $n \to \infty$ we get an analogous result for a circular cone:

3.13' $$\frac{V}{V''} = \frac{F}{F''} \geqslant 2 \quad \{\mathcal{C} = \mathcal{C}_R, \ \beta = \arccos \frac{1}{3}\}.$$

S. A. J. Lhuilier [75]; F. G.-M. [125, p. 999-1000]; D. O. Škljarskij, N. N. Čencov, and I. M. Jaglom [102, p. 31, 165-166].

Remark 2. D. Matei [82] has given the following weaker inequalities for a cone of revolution:

$$8V > 3\sqrt{3}\, V', \quad 4F > 3\sqrt{3}\, F'.$$

3.14. Let $\mathcal{P}$ be circumscribed about a cone of revolution and let $\mathcal{P}'$ be a prism inscribed in $\mathcal{P}$ as in 3.11. If L' is the lateral area and h' the altitude of $\mathcal{P}'$, then

$$\frac{L'}{L} \leqslant \frac{1}{2} \sin \beta \quad \{h = 2h'\}.$$

Proof. If p' is the perimeter of the base of $\mathcal{P}'$, then the equalities

$$L = \frac{1}{2} ph \operatorname{cosec} \beta, \quad L' = p'h', \quad \frac{p'}{p} = \frac{h - h'}{h}$$

imply

$$\frac{L'}{L} = \frac{2p'h' \sin \beta}{ph} = \frac{2 \sin \beta}{h^2} h'(h - h') \leqslant$$

$$\leqslant \frac{2 \sin \beta}{h^2}\left(\frac{h' + h - h'}{2}\right)^2 = \frac{\sin \beta}{2} \qquad \{h' = h - h'\}.$$

Remark. Inequality 3.14 holds also for a cone of revolution and an inscribed cylinder of revolution.

E. G. Togliatti [114]*.

3.15. If F' is the total area of the prism P' defined as in 3.14, then (with $h > r$)

$$\frac{F'}{F} \leqslant \frac{h^2 \sin \beta}{2(h - r)(h + r \sin \beta)} \qquad \{h' = \frac{h(h - 2r)}{2(h - r)}\}.$$

Proof. If r' is the inradius of the base of P', then the equalities

$$F = \pi r(h \operatorname{cosec} \beta + r), \quad F' = 2\pi r'(h' + r'), \quad \frac{r'}{r} = \frac{h - h'}{h}$$

imply

$$\frac{F'}{F} = \frac{2r'(h' + r')}{r(h \operatorname{cosec} \beta + r)} = \frac{2(h - r) \sin \beta}{h^2(h + r \sin \beta)}(h - h')(h' + \frac{hr}{h - r}) \leqslant$$

$$\leqslant \frac{2(h - r) \sin \beta}{h^2(h + r \sin \beta)}\left[\frac{1}{2}(h - h' + h' + \frac{hr}{h - r})\right]^2 =$$

$$= \frac{h^2 \sin \beta}{2(h - r)(h + r \sin \beta)}$$

with equality iff

$$h - h' = h' + \frac{hr}{h - r}, \quad \text{i.e.} \quad h' = \frac{h(h - 2r)}{2(h - r)}.$$

Remark 1. Equality can occur only if $h > 2r$.

Remark 2. Inequality 3.15 holds also for a cone of revolution and an inscribed cylinder of revolution.

E. G. Togliatti [114]*.

3.16. If P has a regular base A, then

$$\frac{E^3}{V} \geqslant \frac{3n^3}{\sin \frac{\pi}{n} \sin \frac{2\pi}{n}}\left(2 \sin \frac{\pi}{n} + \sqrt{1 + \frac{4h^2}{a^2} \sin^2 \frac{\pi}{n}}\right)^3 \cdot \frac{a}{h} \qquad \{P = P_R\}.$$

A. Procissi [94].

3.17. If R', R'' are the radii of the circumscribed and inscribed sphere of a regular n-angular pyramid P and if h' is the altitude of any lateral face of P, then

$$\frac{R'}{R''} \geqslant 1 + \sec\frac{\pi}{n} \quad \{R + r = h'\}.$$

Proof. We have the equalities

$$R'' = r\tan\frac{\beta}{2}, \quad R = r\sec\frac{\pi}{n}, \quad h = r\tan\beta,$$

and from (22) we get

$$R' = \frac{r}{2}\cdot\frac{\tan^2\beta + \sec^2\frac{\pi}{n}}{\tan\beta}.$$

According to the identity

$$\tan^2\beta - 2\tan\beta\tan\frac{\beta}{2} = \tan\beta\left(\frac{2\tan\frac{\beta}{2}}{1-\tan^2\frac{\beta}{2}} - 2\tan\frac{\beta}{2}\right) =$$

$$= \frac{2\tan\beta\tan^3\frac{\beta}{2}}{1-\tan^2\frac{\beta}{2}} = \tan^2\beta\tan^2\frac{\beta}{2}$$

we finally obtain

$$\frac{R'}{R''} - 1 = \frac{\tan^2\beta + \sec^2\frac{\pi}{n}}{2\tan\beta\tan\frac{\beta}{2}} - 1 = \frac{\tan^2\beta\tan^2\frac{\beta}{2} + \sec^2\frac{\pi}{n}}{2\tan\beta\tan\frac{\beta}{2}} \geqslant$$

$$\geqslant \frac{2\sqrt{\tan^2\beta\tan^2\frac{\beta}{2}\cdot\sec^2\frac{\pi}{n}}}{2\tan\beta\tan\frac{\beta}{2}} = \sec\frac{\pi}{n}$$

with equality iff

$$\frac{R}{r} = \sec\frac{\pi}{n} = \tan\beta\tan\frac{\beta}{2} = \tan\beta\cdot\frac{1-\cos\beta}{\sin\beta} = \sec\beta - 1 = \frac{h'}{r} - 1,$$

i.e. $R = h' - r$.

[128]; D. S. Mitrinović, P. M. Vasić, R. Ž. Djordjević, and R. R. Janić [85 , p. 89-90] ($n = 4$); XVIII.2.51 ($n = 3$).

3.18. If P is a regular pyramid and if χ is the angle between a lateral edge and the base of P, then

$$\sin^2 \beta - \sin^2 \chi \leqslant \tan^2 \frac{\pi}{2n} \quad \{\beta + \chi = \frac{\pi}{2}\}.$$

Proof. Follows from XIX.4.2 (with $\phi = \frac{\pi}{n}$, $\psi = \beta$).

V. S. Pokrovskij and R. P. Ušakov [92].

3.19. $$\frac{E^3}{V} \geqslant 24\sqrt{3842 + 1066\sqrt{13}}$$

with equality iff $P = P_R$ and the ratio of lengths of lateral and base edges of P equals $(\sqrt{13} + 1):4$.

A. Procissi [94].

3.20. Let A = ABCD be a trapezoid and let M and N be the midpoints of its bases $\overline{AB}$ and $\overline{CD}$. If F_1, F_2, F_m are the areas of the triangles OAD, OBC, OMN, respectively, then

$$F_m < \frac{1}{2}(F_1 + F_2).$$

Proof. Follows by applying XIX.3.16 to the tetrahedron ABOE, where E is the common point of the lines AD, BC, MN, because (with AB > CD) we have AD:AE = BC:BE = MN:MC. If AB = CD, then 3.20 is a consequence of the obvious inequality $2 \cdot OM' < OA' + OB'$, where A', B', M' are the points of intersection of the lines AD, BC, MN and the plane which contains the point O and is orthogonal to those three lines.

J. Steiner [104].

3.21. Let the conditions of 3.20 be satisfied and let AB = a, CD = c. If F_1, F_2, F_3, F_4 are the areas of the triangles OAB, OBC, OAD and of the trapezoid ABCD, respectively, then

$$F_2 + F_3 \geqslant \sqrt{\frac{36V^2}{a^2} + \frac{(a - c)^2 F_1^2}{a^2} + \frac{a^2 F_4^2}{(a + c)^2} - \sqrt{\frac{a^2 F_1^2 F_4^2}{(a + c)^2} - 9a^2 V^2}}$$

with equality iff the plane OMN is orthogonal to the lines AB, CD.

J. Steiner [103]*, [104]*.

3.22. If P is a frustum of a regular n-angular pyramid inscribed into a sphere of radius R and circumscribed about a sphere of radius r, then

$$\frac{R}{r} \geqslant \sqrt{1 + \sec^2 \frac{\pi}{n}}$$

with equality iff P is a regular n-angular prism circumscribed about a sphere.

Proof. Let R_1 and R_2 be the circumradii and r_1 and r_2 the inradii of the two bases of P and let $R_1 \geqslant R_2$. Then

(24) $R_i = r_i \sec \frac{\pi}{n} \quad (i = 1, 2).$

The altitude of P is $h = 2r$ and the altitude of any lateral face of P is $r_1 + r_2$ and we have the equality $(r_1 + r_2)^2 - (r_1 - r_2)^2 = 4r^2$, i.e.

(25) $r_1 r_2 = r^2.$

From the equality (the upper sign appears if the centre M of circumsphere of P is inside P and the lower sign appears if M is outside P)

$$\sqrt{R^2 - R_2^2} \pm \sqrt{R^2 - R_1^2} = 2r$$

after two squarings and a straight-forward calculation we obtain

$$16r^2R^2 = 16r^4 + 8r^2(R_1^2 + R_2^2) + (R_1^2 - R_2^2)^2 \geqslant$$

$$\geqslant 16r^4 + 16r^2R_1R_2 \quad \{R_1 = R_2\},$$

i.e. by (24) and (25) finally

$$R^2 \geqslant r^2(1 + \sec^2 \frac{\pi}{n}) \quad \{R_1 = R_2\}.$$

Remark. For $n \to \infty$ we get an analogous result for a frustum of a cone of revolution circumscribed about a sphere:

3.22' $\frac{R}{r} \geqslant \sqrt{2}$

with equality only for a cylinder of revolution circumscribed about a sphere.

3.23. Let the conditions of 3.22 be satisfied. If F' and F'' are the surface areas of the circumsphere and the insphere of P, respectively, then

$$F \leqslant \frac{n}{2\pi}(F' \sin \frac{2\pi}{n} - F'' \tan \frac{\pi}{n} \cos \frac{2\pi}{n})$$

with equality iff P is a regular n-prism circumscribed about a sphere.

Proof. With the notation from the proof of 3.22 we have the inequality

$$16r^2R^2 \geqslant 16r^4 + 8r^2(R_1^2 + R_2^2) \quad \{R_1 = R_2\},$$

i.e. by (24) the inequality

(26) $$r_1^2 + r_2^2 \leqslant 2(R^2 - r^2)\cos^2\frac{\pi}{n} \quad \{r_1 = r_2\}.$$

The equalities

$$F = 2n\tan\frac{\pi}{n}(r_1^2 + r_2^2 + r^2), \quad F' = 4\pi R^2, \quad F'' = 4\pi r^2$$

and (26) imply

$$F \leqslant 2n\tan\frac{\pi}{n}[2(R^2 - r^2)\cos^2\frac{\pi}{n} + r^2] =$$

$$= 2n(R^2\sin\frac{2\pi}{n} - r^2\tan\frac{\pi}{n}\cos\frac{2\pi}{n}) =$$

$$= \frac{n}{2\pi}(F'\sin\frac{2\pi}{n} - F''\tan\frac{\pi}{n}\cos\frac{2\pi}{n}) \quad \{r_1 = r_2\}.$$

Remark 1. For $n \to \infty$ we get an analogous result for a frustum of a cone of revolution circumscribed about a sphere:

3.23' $$F \leqslant F' - \frac{1}{2}F''$$

with equality only for a cylinder of revolution circumscribed about a sphere.

Remark 2. By 3.22' we have $2F'' \leqslant F'$ and 3.23' implies the weaker inequality

3.23'' $$F \leqslant 2F' - \frac{5}{2}F''$$

with equality only for a cylinder of revolution circumscribed about a sphere.

Hr. Lesov [73].

3.24. Let the conditions of 3.22 be satisfied. If V' and V'' are the volumes of the circumsphere and the insphere of P, respectively, then

$$V \leqslant \frac{n}{2\pi}\left(V'\frac{\cos\frac{\pi}{n}\sin\frac{2\pi}{n}}{\sqrt{1 + \cos^2\frac{\pi}{n}}} - V''\tan\frac{\pi}{n}\cos\frac{2\pi}{n}\right)$$

with equality iff P is a regular n-prism circumscribed about a sphere.

Proof. According to (25) we have the equality

(27) $$V = \frac{2n}{3}\tan\frac{\pi}{n}(r_1^2 + r_2^2 + r^2)r.$$

Therefore, the equalities

(28) $$V' = \frac{4\pi}{3} R^3, \quad V'' = \frac{4\pi}{3} r^3$$

and inequalities (26) and 3.22 imply successively

$$V \leqslant \frac{2n}{3} \tan \frac{\pi}{n}[2(R^2 - r^2) \cos^2 \frac{\pi}{n} + r^2]r =$$

$$= \frac{2n}{3} \tan \frac{\pi}{n}(2R^2 r \cos^2 \frac{\pi}{n} - r^3 \cos \frac{2\pi}{n}) \leqslant$$

$$\leqslant \frac{2n}{3} \tan \frac{\pi}{n}\left(2R^3 \frac{\cos^2 \frac{\pi}{n}}{\sqrt{1 + \sec^2 \frac{\pi}{n}}} - r^3 \cos \frac{2\pi}{n}\right) =$$

$$= \frac{n}{2\pi}\left(V' \frac{\cos \frac{\pi}{n} \sin \frac{2\pi}{n}}{\sqrt{1 + \cos^2 \frac{\pi}{n}}} - V'' \tan \frac{\pi}{n} \cos \frac{2\pi}{n}\right)$$

with equality iff $r_1 = r_2$.

Remark 1. For $n \to \infty$ we get an analogous result for a frustum of a cone of revolution circumscribed about a sphere:

3.24' $$V \leqslant \frac{\sqrt{2}}{2} V' - \frac{1}{2} V''$$

with equality only for a cylinder of revolution circumscribed about a sphere.

Remark 2. By 3.22' we have $4V'' \leqslant \sqrt{2}V'$ and from 3.24' we obtain the weaker inequality

3.24'' $$V \leqslant \sqrt{2}V' - \frac{5}{2} V''$$

with equality only for a cylinder of revolution circumscribed about a sphere.

Hr. Lesov [73].

3.25. With the conditions of 3.22-3.24 we have

$$\frac{V}{V''} = \frac{F}{F''} \geqslant \frac{3n}{2\pi} \tan \frac{\pi}{n}$$

with equality iff P is a regular n-prism circumscribed about a sphere.

Proof. According to equalities (25), (27), (28) and the inequality

$$r_1^2 + r_1 r_2 + r_2^2 \geqslant 3r_1 r_2 \quad \{r_1 = r_2\}$$

we obtain

$$\frac{V}{V''} = \frac{n}{2\pi} \tan\frac{\pi}{n} \frac{r_1^2 + r_1 r_2 + r_2^2}{r_1 r_2} \geqslant \frac{3n}{2\pi} \tan\frac{\pi}{n} \quad \{r_1 = r_2\}.$$

From equalities $3V = Fr$, $3V'' = F''r$ it follows $V:V'' = F:F''$.

Hr. Lesov [74].

Remark. For $n \to \infty$ we obtain an analogous result for a frustum of a cone of revolution circumscribed to a sphere:

3.25' $$\frac{V}{V''} = \frac{F}{F''} \geqslant \frac{3}{2}$$

with equality only for a cylinder of revolution circumscribed about a sphere.

Hr. Lesov [73].

4. Inequalities for Regular Convex Polyhedra and Polyhedra Isomorphic to Them

Let $\mathcal{T}_R$, $\mathcal{H}_R$, $\mathcal{O}_R$, $\mathcal{D}_R$, $\mathcal{I}_R$ be the signs for the following convex regular polyhedra: tetrahedron, hexahedron, octahedron, dodecahedron and icosahedron, respectively. In the inequalities 4.12-4.25 let $\mathcal{P}'_R$ be a regular polyhedron contained in a regular polyhedron $\mathcal{P}_R$ and let V and V' be the volumes of $\mathcal{P}_R$ and $\mathcal{P}'_R$, respectively. The sign $\{\mathcal{H} = \mathcal{H}_R\}$ means that an inequality becomes equality iff $\mathcal{H}$ is a cube, etc.

In 4.7, 4.10 and 4.11 for every face of a regular polyhedron $\mathcal{P}_R$ we consider a vector parallel to this face. Let ϕ be the maximal angle of a pair of these vectors.

4.1. If $\mathcal{P}_R$ is a convex regular polyhedron with f faces and the total area F and if F' is the area of the orthogonal projection of $\mathcal{P}_R$ onto any plane α, then

$$F' \leqslant \frac{\sqrt{3}}{4} F$$

with equality iff $\mathcal{P}_R = \mathcal{T}_R$ and α is parallel to two opposite edges of $\mathcal{T}_R$ or $\mathcal{P}_R = \mathcal{H}_R$ or $\mathcal{P}_R = \mathcal{O}_R$ and α is orthogonal to a diagonal of $\mathcal{H}_R$ or $\mathcal{O}_R$.

H. Vogler [118].

4.2. Let $\mathcal{H}$ be a polyhedron isomorphic to $\mathcal{H}_R$. If V is the volume and E the total edge length of $\mathcal{H}$, then

$$E^3 \geqslant 1728V \quad \{\mathcal{H} = \mathcal{H}_R\}.$$

G. Sansone [97].

4.3. If a is the length of the side of a square $\mathcal{Q}$ contained in a cube $\mathcal{H}_R$ with edge length e, then

$$a \leqslant \frac{3\sqrt{2}}{4} e$$

with equality iff $\mathcal{Q}$ and $\mathcal{H}_R$ are concentric and two adjacent vertices of $\mathcal{Q}$ lie on two adjacent edges of $\mathcal{H}_R$ at the equal distances $\frac{3}{4}$ e from the common end point of these edges.
M. Dedo [16].

4.4. If F is the area of a plane section of a cube $\mathcal{H}_R$ with the edge length e, then

$$F \leqslant \sqrt{2}\, e^2$$

with equality iff the considered plane contains two opposite edges of $\mathcal{H}_R$.
E. Ehrhart [20].

4.5. If L is the perimeter of a plane section of a cube $\mathcal{H}_R$ with the edge length e, then

$$L \leqslant 2(\sqrt{2} + 1)e$$

with equality iff the considered plane contains two opposite edges of $\mathcal{H}_R$.
E. Ehrhart and J. M. Faure [21].

4.6. A cube of a given edge length e can be perforated in such a way that a second cube of the edge length e' may pass through the hole. Then

$$e' < \frac{3\sqrt{2}}{4} e \approx 1{,}06066\ e.$$

J. H. van Swinden [112, p. 608-610]; J. H. van Swinden and C. F. Jacobi [113, p. 542]; D. J. E. Schrek [99].
Remark. The connection between 4.3 and 4.6 is obvious.

4.7. If $\mathcal{P}_R = \mathcal{H}_R$ or $\mathcal{P}_R = \mathcal{O}_R$, then

$$\phi \geqslant \frac{\pi}{3}\ .$$

H. T. Croft [14].

4.8. Let $\mathcal{O}$ be any polyhedron isomorphic to $\mathcal{O}_R$. If V is the volume and E the total edge length of $\mathcal{O}$, then

$$E^3 \geqslant 2592\sqrt{2}V \quad \{\mathcal{O} = \mathcal{O}_R\}.$$

A. Procissi [94]; G. Sansone [97].

4.9. Let the conditions of 4.8 be satisfied and let R be the radius of a minimal sphere containing $\mathcal{O}$. Then

$$V \leqslant \frac{4}{3} R^3 \qquad \{\mathcal{O} = \mathcal{O}_R\}.$$

B. Drachmann [19].

4.10. If $\mathcal{P}_R = \mathcal{D}_R$, then

$$\phi \geqslant \arccos \frac{5 - 2\sqrt{5}}{10} .$$

H. T. Croft [14].

4.11. If $\mathcal{P}_R = \mathcal{I}_R$, then

$$\phi \geqslant \arccos \frac{9 - \sqrt{5}}{12} .$$

H. T. Croft [14].

4.12. If $\mathcal{P}_R = \mathcal{T}_R$, $\mathcal{P}'_R = \mathcal{H}_R$, then

$$\frac{V}{V'} \geqslant \frac{1}{216}(216 + 171\sqrt{2} + 153\sqrt{3} + 106\sqrt{6}) \approx 4.54852.$$

Let Δ be a triangle with the side length

$$d = \frac{1}{167}(122 - 42\sqrt{2} + 72\sqrt{3} - 33\sqrt{6})e \approx 0.63759\ e,$$

whose vertices lie on three edges of $\mathcal{T}_R$ with one common end point and have the distance d from this end point, where e is the edge length of $\mathcal{T}_R$.

Equality in 4.12 occurs iff a face of $\mathcal{H}_R$ is a square with side length

$$\frac{1}{167}(66 - 72\sqrt{2} + 28\sqrt{3} + 15\sqrt{6})e \approx 0.29591\ e$$

inscribed in Δ (such that two vertices of the square lie on a side of Δ and the other two vertices of the square lie on other two sides of Δ) and the opposite face of $\mathcal{H}_R$ lies on the face of $\mathcal{T}_R$ parallel to the plane of Δ. There are 12 extremal cubes inscribed in $\mathcal{T}_R$.

C. S. Ogilvy and D. P. Robbins [88]; J. H. Croft [15].

4.13. If $\mathcal{P}_R = \mathcal{H}_R$, $\mathcal{P}'_R = \mathcal{T}_R$, then

$$\frac{V}{V'} \geqslant 3$$

with equality iff the vertices of T_R are four vertices of H_R, any two of which are non-adjacent.

H. T. Croft [15].

4.14. If $P_R = T_R$, $P'_R = O_R$, then

$$\frac{V}{V'} \geq 2$$

with equality iff the vertices of O_R are the midpoints of the edges of T_R.

H. T. Croft [15].

4.15. If $P_R = O_R$, $P'_R = T_R$, then

$$\frac{V}{V'} \geq 4$$

with equality iff O_R and T_R have a common face and the fourth vertex of T_R is the centre of the opposite face of O_R.

H. T. Croft [15].

4.16. If $P_R = H_R$, $P'_R = O_R$, then

$$\frac{V}{V'} \geq \frac{16}{9}$$

with equality iff H_R and O_R are concentric and the vertices of a face of O_R lie on three edges of H_R with one common end point and have the distance $\frac{3}{4}$ e from this end point, where e is the edge length of H_R.

H. T. Croft [15].

Remark. The connection between 4.3 and 4.16 is obvious.

4.17. If $P_R = O_R$, $P'_R = H_R$, then

$$\frac{V}{V'} \geq \frac{1}{6}(7 + 5\sqrt{2})$$

with equality iff O_R and H_R are concentric and the vertices of a face of H_R lie on four edges of O_R with common end point and have the distance $(2 - \sqrt{2})$e from this end point, where e is the edge length of O_R.

H. T. Croft [15].

Remark. Inequality 4.17 is valid also if V' is the volume of any parallelepiped P' contained in O_R and equality occurs iff P' is the extremal cube H_R from 4.17.

4.18. If $P_R = D_R$, $P'_R = T_R$, then

$$\frac{V}{V'} \geq \frac{3}{4}(5 + \sqrt{5})$$

with equality iff the vertices of $\mathcal{T}_R$ are four of the vertices of $\mathcal{D}_R$ such that any two vertices of $\mathcal{T}_R$ are connected by a path composed of three edges of $\mathcal{D}_R$ which are not in a same plane.

H. T. Croft [15].

4.19. If $\mathcal{P}_R = \mathcal{D}_R$, $\mathcal{P}'_R = \mathcal{H}_R$, then

$$\frac{V}{V'} \geq \frac{1}{4}(5 + \sqrt{5})$$

with equality iff the vertices of $\mathcal{H}_R$ are eight of the vertices of $\mathcal{D}_R$ such that any edge of $\mathcal{H}_R$ is a diagonal of a face of $\mathcal{D}_R$.

H. T. Croft [15].

4.20. If $\mathcal{P}_R = \mathcal{H}_R$, $\mathcal{P}'_R = \mathcal{D}_R$, then

$$\frac{V}{V'} \geq \frac{1}{3645}(1755 + 1755\sqrt{2} + 621\sqrt{5} + 675\sqrt{10}) \approx 2.12897.$$

Equality occurs iff $\mathcal{H}_R$ and $\mathcal{D}_R$ have in a certain coordinate system the following vertices (here always hold all upper or all lower signs):

$$(\pm \tfrac{1}{2}, \pm \tfrac{1}{2}, \pm \tfrac{1}{2}), \quad (\pm \tfrac{1}{2}, \pm \tfrac{1}{2}, \mp \tfrac{1}{2}),$$

$$(\pm \tfrac{1}{2}, \mp \tfrac{1}{2}, \pm \tfrac{1}{2}), \quad (\pm \tfrac{1}{2}, \mp \tfrac{1}{2}, \mp \tfrac{1}{2})$$

respectively

$$(\pm a, \pm a, \pm a), \quad (\pm b, \pm b, \pm c), \quad (\pm b, \pm c, \pm b),$$

$$(\pm c, \pm b, \pm b), \quad (\pm d, \pm e, \pm \tfrac{1}{2}), \quad (\pm e, \pm d, \pm \tfrac{1}{2}), (\pm \tfrac{1}{2}, \pm d, \pm e),$$

$$(\pm \tfrac{1}{2}, \pm e, \pm d), \quad (\pm e, \pm \tfrac{1}{2}, \pm d), \quad (\pm d, \pm \tfrac{1}{2}, \pm e),$$

where

$$a = \frac{1}{4}(4 - \sqrt{2} - 2\sqrt{5} + \sqrt{10}), \quad b = \frac{1}{4}(-4 + 3\sqrt{2} + 2\sqrt{5} - \sqrt{10}),$$

$$c = -\frac{1}{4}(2 + \sqrt{2} - \sqrt{10}), \quad d = \frac{1}{2}(2 - 2\sqrt{2} - \sqrt{5} + \sqrt{10}),$$

$$e = -\frac{1}{4}(2 - 3\sqrt{2} + \sqrt{10}).$$

In this extremal case every face of H_R contains an edge of D_R and a diagonal of this face is the perpendicular bisector of that edge. Moreover, two vertices of D_R lie on a diagonal of H_R and the polyhedra H_R and D_R are concentric. The edge length of H_R is 1 and the edge length of D_R is

$$\frac{1}{2}(-7 + 3\sqrt{2} + 3\sqrt{5} - \sqrt{10}) \approx 0.39428.$$

The distance of midpoints of two opposite edges of D_R is

$$\frac{1}{2}(-3 + 2\sqrt{2} + \sqrt{5}) \approx 1.03225.$$

H. T. Croft [15].

4.21. If $P_R = D_R$, $P'_R = O_R$, then

$$\frac{V}{V'} \geq \frac{3}{2}(3\sqrt{5} - 5)$$

with equality iff the vertices of O_R are midpoints of six edges of D_R such that any two of these edges are either parallel or orthogonal.
H. T. Croft [15].

4.22. If $P_R = T_R$, $P'_R = I_R$, then

$$\frac{V}{V'} \geq \frac{1}{5}(7 + 3\sqrt{5})$$

with equality iff the vertices of I_R lie on the edges of an O_R, whose vertices are the midpoints of the edges of T_R, so that the vertices of I_R divide the edges on any face of O_R in the same ratio $(3 - \sqrt{5}):(\sqrt{5} - 1)$ and in the same cyclical order.
H. T. Croft [15].

4.23. If $P_R = H_R$, $P'_R = I_R$, then

$$\frac{V}{V'} \geq \frac{3}{5}(1 + \sqrt{5})$$

with equality iff every face of H_R contains an edge of I_R so that any two of these six edges are either parallel or orthogonal and any of these edges is either parallel or orthogonal to particular face of H_R.
H. T. Croft [15].

4.24. If $P_R = O_R$, $P'_R = I_R$, then

$$\frac{V}{V'} \geqslant \frac{1}{10}(7 + 3\sqrt{5})$$

with equality iff the vertices of I_R lie on the edges of O_R (one vertex on every edge) so that the vertices of I_R divide the edges on any face of O_R in the same ratio $(3 - \sqrt{5}):(\sqrt{5} - 1)$ and in the same cyclical order.

H. T. Croft [15].

4.25. If $P_R = I_R$, $P'_R = O_R$, then

$$\frac{V}{V'} \geqslant \frac{5}{54}(19 + 15\sqrt{2} - 3\sqrt{5} - \sqrt{10}) \approx 2.80951$$

with equality iff I_R and O_R are concentric and three adjacent vertices of O_R divide the edges $\overline{AB}$, $\overline{CD}$, $\overline{EF}$ of I_R in the same ratio $1:(\sqrt{2} - 1)$, where $\overline{AB}$, $\overline{AC}$, $\overline{AD}$, $\overline{AE}$, $\overline{AF}$ are (in this cyclical order) the edges of I_R with a common vertex A.

H. T. Croft [15].

5. Inequalities for Quadric Surfaces

Let us remind the reader of the previously stated inequalities 3.8', 3.9', 3.10', 3.11, 3.12', 3.13', 3.14, 3.15 for a circular cone, 3.20', 3.21', 3.21'', 3.22', 3.22'', 3.23' for a frustum of a cone of revolution and 2.7', 2.8', 2.19', 2.20' for a cylinder of revolution.

Let C be a cone of revolution with radius r, altitude h, length of a generator g, volume V, total area F and lateral area L. If S is the sphere circumscribed about C and R the radius, V' the volume and F' the surface area of S, then we have the following equalities:

(1) $L = \pi rg,$

(2) $F = \pi r(g + r),$

(3) $V = \frac{\pi}{3} r^2 h,$

(4) $g^2 = h^2 + r^2,$

(5) $F' = 4\pi R^2,$

(6) $V' = \frac{4\pi}{3} R^3,$

(7) $2Rh = g^2.$

5.1. $V \leqslant \frac{2\pi\sqrt{3}}{27} g^3 \quad \{r = \sqrt{2}h\}.$

Proof. From (3) and (4) we obtain

$$V^2 = \frac{\pi^2}{9} r^4 h^2 = \frac{4\pi^2}{9} \cdot \frac{r^2}{2} \cdot \frac{r^2}{2} \cdot (g^2 - r^2) \leqslant$$

$$\leqslant \frac{4\pi^2}{9}\left[\frac{1}{3}\left(\frac{r^2}{2} + \frac{r^2}{2} + g^2 - r^2\right)\right]^3 = \frac{4\pi^2}{243} g^6$$

with equality iff

$$\frac{r^2}{2} = g^2 - r^2 = h^2, \quad \text{i.e.} \quad r = \sqrt{2}h.$$

F. G.-M. [125, p. 995, 997]; T. J. Fletcher [41].

5.2. $$\frac{L^3}{V^2} \geqslant \frac{27\sqrt{3}\pi}{2} \quad \{h = \sqrt{2}r\}.$$

Proof. From (1), (3) and (4) we get

$$2L^3 - 27\sqrt{3}\pi V^2 = 2\pi^3 r^3 g^3 - 3\sqrt{3}\pi^3 r^4 (g^2 - r^2) =$$

$$= \pi^3 r^3 (2g^3 - 3\sqrt{3}g^2 r + 3\sqrt{3}r^3) = \pi^3 r^3 (g - \sqrt{3}r)^2 (2g + \sqrt{3}r) \geqslant 0$$

with equality iff $g = \sqrt{3}\, r$ or equivalently iff $h = \sqrt{2}\, r$ because of (4).
[127]; I. Dimovski [18].

5.3. $$\frac{F^3}{V^2} \geqslant 72\pi \quad \{g = 3r\}.$$

Proof. From (2)-(4) we obtain

$$18\pi V^2 = F \cdot 2\pi r^2 (F - 2\pi r^2) \leqslant F \cdot \left(\frac{2\pi r^2 + F - 2\pi r^2}{2}\right)^2 = \frac{F^3}{4}$$

with equality iff $2\pi r^2 = F - 2\pi r^2$, i.e. $g = 3r$ according to (2).

5.4. $$\frac{F}{F'} \leqslant \frac{107 + 51\sqrt{17}}{512} \quad \{g = \frac{\sqrt{17} - 1}{2} r\}.$$

Proof. From (4) and (7) we obtain

(8) $$4R^2 = \frac{g^4}{g^2 - r^2}.$$

Therefore, according to (2) and (5) we have

$$\frac{F}{F'} = \frac{r(g + r)(g^2 - r^2)}{g^4} = \frac{1}{4(1 + \sqrt{17})(5 + \sqrt{17})g^4} \cdot$$

$$\cdot (1 + \sqrt{17})r \cdot (2g + 2r)^2 \cdot (5 + \sqrt{17})(g - r) \leqslant$$

$$\leqslant \frac{1}{8(11 + 3\sqrt{17})g^4}\left[\frac{(1 + \sqrt{17})r + 2(2g + 2r) + (5 + \sqrt{17})(g - r)}{4}\right]^4 =$$

$$= \frac{1}{8(11 + 3\sqrt{17})g^4}\left[\frac{(9 + \sqrt{17})g}{4}\right]^4 = \frac{107 + 51\sqrt{17}}{512}$$

with equality iff

$$(1 + \sqrt{17})r = 2g + 2r = (5 + \sqrt{17})(g - r),$$

i.e.

$$2g = (\sqrt{17} - 1)r.$$

Remark. D. Matei [82] has a weaker inequality

$$8F < 3\sqrt{3}\,F'.$$

5.5. $\frac{L}{F'} \leqslant \frac{2\sqrt{3}}{9} \quad \{g = \sqrt{3}r\}.$

Proof. From (1), (5) and (8) we obtain

$$\frac{L}{F'} = \frac{r(g^2 - r^2)}{g^3} = \frac{1}{4g^3} \cdot 2r \cdot (\sqrt{3} + 1)(g - r) \cdot (\sqrt{3} - 1)(g + r) \leqslant$$

$$\leqslant \frac{1}{4g^3}\left[\frac{2r + (\sqrt{3} + 1)(g - r) + (\sqrt{3} - 1)(g + r)}{3}\right]^3 =$$

$$= \frac{1}{4g^3}\left(\frac{2\sqrt{3}g}{3}\right)^3 = \frac{2\sqrt{3}}{9}$$

with equality iff

$$2r = (\sqrt{3} + 1)(g - r) = (\sqrt{3} - 1)(g + r), \quad \text{i.e.} \quad g = \sqrt{3}r.$$

R. Sturm [110]; A. Kiefer [65].

5.6. Let α be any plane parallel to the plane of the base of C such that α intersects C and S. If d is the distance of α from the vertex of C and if S and S' are the areas of the circles $\alpha \cap C$ and $\alpha \cap S$, respectively, then

$$S' - S \leqslant \frac{\pi}{2} Rh \quad \{d = \frac{h}{2}\}.$$

Proof. If x and y are the radii of the circles $\alpha \cap S$ and $\alpha \cap C$, respectively, then the equalities

$$r^2 = h(2R - h), \quad x^2 = d(2R - d),$$

$$y = \frac{r}{h} d, \quad S' = x^2\pi, \quad S = y^2\pi$$

imply

$$S' - S = \pi(x^2 - y^2) = [d(2R - d) - \frac{2R - h}{d} d^2] =$$

$$= \frac{2\pi R}{h} d(h - d) \leqslant \frac{2\pi R}{h}(\frac{d + h - d}{2})^2 = \frac{\pi}{2} Rh$$

with equality iff $d = h - d$.
F. G.-M. [125, p. 1007-1008].

5.7. If β is the angle between a generator and the base of C and if the centre of the base is the centre of a hemisphere of volume V'' inscribed in C, then

$$\frac{V}{V''} \geqslant \frac{3\sqrt{3}}{4} \quad \{\beta = \arccos \frac{\sqrt{3}}{3}\}.$$

Proof. If R' is the radius of the hemisphere, then from the equality (3) and the equalities

$$r = \frac{R'}{\sin\beta}, \quad h = \frac{R'}{\cos \beta}, \quad V'' = \frac{2\pi}{3} R'^3$$

we obtain

$$\frac{V''}{V} = \frac{2R'^3}{r^2 h} = 2 \sin^2 \beta \cos \beta,$$

and 5.7 is a consequence of the inequality

$$\sin^2 \beta \cos \beta \leqslant \frac{2\sqrt{3}}{9} \quad \{\beta = \arccos \frac{\sqrt{3}}{3}\}.$$

proved in the proof of 3.9.

5.8. Let O be the vertex, C the centre of the base of C and S the area of the section of C by a plane α. If $r \leqslant h$, then

$$S \leqslant rh$$

with equality iff α contains the axis OC. If $r > h$, then

$$S \leqslant \frac{1}{2}(r^2 + h^2)$$

with equality iff α contains the vertex O and intersects the lateral surface of C at two generators making the angle

$$\arccos\left(-\frac{h^2}{r^2}\right).$$

E. G. Gotman [51].

5.9. If V is the volume, h the altitude of a frustum C of a cone of revolution and r the radius of its midcircle, then

$$\pi r^2 h \leqslant V \leqslant \frac{4\pi}{3} r^2 h.$$

The first equality occurs only in the case of a cylinder of revolution and the second equality occurs only in the case of a cone of revolution.

Proof. If r_1, r_2 are the radii of the bases ($r_1 \geqslant r_2$), then $r_1 + r_2 = 2r$ and therefore

$$\pi r^2 h = \frac{\pi}{3}(r_1 + r_2)^2 h - \frac{\pi}{12}(r_1 + r_2)^2 h \leqslant \frac{\pi}{3}(r_1 + r_2)^2 h -$$

$$- \frac{\pi}{12} \cdot 4r_1 r_2 h = \frac{\pi h}{3}(r_1^2 + r_1 r_2 + r_2^2) = V =$$

$$= \frac{\pi h}{3}[(r_1 + r_2)^2 - r_1 r_2] \leqslant \frac{\pi h}{3}(r_1 + r_2)^2 =$$

$$= \frac{4\pi}{3} r^2 h \quad \{r_1 = r_2,\ r_2 = 0\}.$$

M. Petrović [90]; B. Martić [81].

5.10. With the notation from 5.9 let V' be the volume of the intersection of two cones, where any of these two cones has one base of C as the base and the centre of the other base of C as the vertex. Then

$$\frac{r_2^2}{r_1^2} V \leqslant 12V' \leqslant \frac{r_1^2}{r_2^2} V \quad \{r_1 = r_2\}.$$

Proof. The intersection of the two cones considered is the sum of two other cones having as common base a circle of radius

$$r = \frac{r_1 r_2}{r_1 + r_2}$$

and with the altitudes h_1 and h_2 such that $h_1 + h_2 = h$. Therefore

$$V = \frac{\pi}{3} h(r_1^2 + r_1r_2 + r_2^2), \quad V' = \frac{\pi}{3} hr^2 = \frac{\pi h}{3} \cdot \frac{r_1^2r_2^2}{(r_1 + r_2)^2}$$

and the inequality $r_1 \geqslant r_2$ implies successively

$$\frac{r_2^2}{r_1^2} V = \frac{r_2^2}{r_1^2} \frac{\pi h}{3}(r_1^2 + r_1r_2 + r_2^2) \leqslant \frac{r_2^2}{r_1^2} \pi hr_1^2 \leqslant \frac{4\pi hr_1^2r_2^2}{(r_1 + r_2)^2} = 12V' =$$

$$= \frac{4\pi hr_1^2r_2^2}{(r_1 + r_2)^2} \leqslant \frac{r_1^2}{r_2^2} \cdot \pi hr_2^2 \leqslant \frac{r_1^2}{r_2^2} \frac{\pi h}{3}(r_1^2 + r_1r_2 + r_2^2) =$$

$$= \frac{r_1^2}{r_2^2} V \quad \{r_1 = r_2\}.$$

C. Ciotloş [12].

5.11. Let C be a cylinder of revolution with radius r and altitude h inscribed in a sphere S. If F and F' are the surface areas of C and S, respectively, then

$$\frac{F}{F'} \leqslant \frac{1}{4}(\sqrt{5} + 1) \quad \{h = (\sqrt{5} - 1)r\}.$$

Proof. If R is the radius of S, then the equalities

$$F = 2\pi r(r + h), \quad F' = 4\pi R^2, \quad 4R^2 = 4r^2 + h^2$$

imply

$$\frac{F}{F'} = \frac{2r(r + h)}{4r^2 + h^2} \leqslant \frac{1}{4}(\sqrt{5} + 1) \quad \{h = (\sqrt{5} - 1)r\}.$$

because the last inequality is equivalent to

$$[h - (\sqrt{5} - 1)r]^2 \geqslant 0.$$

F. G.-M. [125, p. 1004].

Remark. Inequality 5.11 was known already to P. Fermat.

5.12. Let S be a sphere with radius R and let α be any plane which intersects S. The circle $\alpha \cap S$ is a base of a cylinder of revolution C and the second base of C lies in a tangential plane of S parallel to α. If V and V' are the volumes of C and S, respectively, and if h is the altitude of C, then

$$\frac{V}{V'} \leqslant \frac{8}{9} \quad \{h = \frac{4}{3} R\}.$$

Proof. Let r be the radius of C. The equalities

$$V = \pi h r^2, \quad V' = \frac{4\pi}{3} R^3, \quad r^2 = h(2R - h)$$

imply

$$\frac{V}{V'} = \frac{3}{R^3} \cdot \frac{h}{2} \cdot \frac{h}{2} \cdot (2R - h) \leqslant \frac{3}{R^3}[\frac{1}{3}(\frac{h}{2} + \frac{h}{2} + 2R - h)]^3 = \frac{8}{9}$$

with equality iff $\frac{h}{2} = 2R - h$, i.e. $h = \frac{4}{3} R$.

F. G.-M. [125, p. 998-999].

5.13. An axial section of a cylinder of revolution is a rectangle ABCD, where AB is a diameter of a base. If l is the length of the arc AC of a geodesic on the lateral surface of the cylinder and if $\phi = \measuredangle BAC$, then

$$\phi \lesseqgtr \arctan \frac{\pi^2 - 4}{8}$$

implies

$$AB + BC \lesseqgtr l.$$

M. M. Sebaršin [100].

5.14. Into a sphere S with the radius R we inscribe a truncated elliptic cylinder C, whose bases are two congruent circles with radii r and lieing in planes symmetric with respect to a diametral plane δ of S so that 2ϕ is the angle between these two planes. If V and V' are the volumes of C and S, respectively, then

$$\frac{V}{V'} \leqslant \frac{\sqrt{3}\cos\phi}{3} \quad \{r = \frac{\sqrt{6}}{3} R\}.$$

Proof. Let d be the distance of the base of C from the centre of S. Then $2d \cos \phi$ is the length of the axis of C and the area of the ellipse $\delta \cap C$ equals $\pi r^2 \cos \phi$. Therefore

$$V = 2\pi d r^2 \cos \phi, \quad V' = \frac{4\pi}{3} R^3.$$

Moreover, we have the equality

$$(9) \qquad d^2 + r^2 = R^2.$$

So, we obtain successively

$$\frac{V^2}{V'^2} = \frac{9\cos^2\phi}{4} \cdot \frac{d^2r^4}{R^6} = \frac{9\cos^2\phi}{R^6} \cdot d^2 \cdot \frac{r^2}{2} \cdot \frac{r^2}{2} \leqslant$$

$$\leqslant \frac{9\cos^2\phi}{R^6}\left[\frac{1}{3}\left(d^2 + \frac{r^2}{2} + \frac{r^2}{2}\right)\right]^3 = \frac{\cos^2\phi}{3}$$

with equality iff $r^2 = 2d^2$ and this is equivalent to $r = R\sqrt{6}/3$, because of (9).

F. G.-M. [125, p. 209].

Remark. 5.14 with $\phi = 0$ implies 2.19'.

5.15. If L is the area of a spherical cap and V the volume of the corresponding spherical segment, then

$$\frac{L^3}{V^2} \geqslant 18\pi$$

with equality only for a hemisphere.

Proof. If R is the radius of the sphere and h the altitude of the spherical cap, then the equalities $L = 2\pi Rh$ and

$$\text{(10)} \qquad V = \frac{\pi}{3} h^2(3R - h)$$

imply

$$\frac{V^2}{L^3} = \frac{1}{144\pi R^3} \cdot 2h(3R - h)^2 \leqslant \frac{1}{144\pi R^3}\left[\frac{2h + 2(3R - h)}{3}\right]^3 =$$

$$= \frac{1}{18\pi} \quad \{2h = 3R - h\}.$$

F. G.-M. [125, p. 1013]*.

5.16. If V is the volume and F the total surface area of a spherical segment (of one base), then

$$\frac{F^3}{V^2} \geqslant 36\pi$$

with equality only in the case of the whole sphere.

Proof. If R is the radius of the sphere, h the altitude of the segment and r the radius of its base, then the equalities (10) and

$$F = 2\pi hR + \pi r^2, \quad r^2 = h(2R - h)$$

imply

$$\frac{V^2}{F^3} = \frac{2}{9\pi(4R - h)^3} \cdot \frac{h}{2} \cdot (3R - h)^2 \leqslant$$

$$\leqslant \frac{2}{9\pi(4R - h)^3}\left\{\frac{1}{3}\left[\frac{h}{2} + 2(3R - h)\right]\right\}^3 = \frac{2}{9\pi(4R - h)^3}\left(\frac{4R - h}{2}\right)^3 = \frac{1}{36\pi}$$

with equality iff $\frac{h}{2} = 3R - h$, i.e. $h = 2R$.

5.17. If 2ϕ is the central angle of a spherical sector, R the radius of the corresponding sphere, r the radius and L the lateral area of a cylinder of revolution inscribed into that sector such that the cylinder and the sector have their axes in common, then

$$L \leqslant \pi \tan \frac{\phi}{2} \cdot R^2 \quad \{r = R \sin \frac{\phi}{2}\}.$$

Proof. If h is the altitude of the cylinder, then $L = 2\pi rh$ and from $(h + r \cot \phi)^2 + r^2 = R^2$ we obtain

$$h^2 + 2rh \cot \phi + r^2 \operatorname{cosec}^2 \phi = R^2. \tag{11}$$

Therefore

$$R^2 - \frac{L}{\pi} \cot \frac{\phi}{2} = h^2 + 2rh \cot \phi + r^2 \operatorname{cosec}^2 \phi - 2rh \cot \frac{\phi}{2} =$$

$$= (h - r \operatorname{cosec} \phi)^2 \geqslant 0$$

according to the identity

$$\cot \phi - \cot \frac{\phi}{2} = - \operatorname{cosec} \phi.$$

Equality occurs iff $h = r \operatorname{cosec} \phi$, i.e. according to (11) iff

$$R^2 = 2r^2 \operatorname{cosec}^2 \phi + 2r^2 \cot \phi \operatorname{cosec} \phi =$$

$$= 2r^2 \operatorname{cosec}^2 \phi(1 + \cos \phi) = 4r^2 \operatorname{cosec}^2 \phi \cos^2 \frac{\phi}{2} =$$

$$= r^2 \operatorname{cosec}^2 \frac{\phi}{2}.$$

F. G.-M. [125, p. 1004].

5.18. Let P be a segment of a paraboloid of revolution determined by a plane orthogonal to the axis of this paraboloid and let C be a cylinder of revolution inscribed in P. If V is the volume and h the altitude of P and if V' is the volume and h' the altitude of C, then

$$\frac{V}{V'} \geqslant 2 \quad \{h = 2h'\}.$$

<u>Proof</u>. If r and r' are the radii of the bases of P and C, respectively, then

$$V = \frac{\pi}{2} r^2 h, \quad V' = \pi r'^2 h'.$$

If $x^2 + y^2 = 2pz$ is the equation of the paraboloid, then

$$r^2 = 2ph, \quad r'^2 = 2p(h - h').$$

Therefore, we obtain

$$\frac{V'}{V} = \frac{2r'^2 h'}{r^2 h} = \frac{2h'(h - h')}{h^2} \leqslant \frac{2}{h^2}\left(\frac{h' + h - h'}{2}\right)^2 = \frac{1}{2} \quad \{h' = h - h'\}.$$

F. G.-M. [125, p. 206].

5.19. Let E_1, E_2, E_3 denote three ellipsoids, E_1 and E_3 being polar reciprocals of each other with respect to E_2. Between the volumes V_1, V_2 V_3 of these ellipsoids the following inequality holds

$$\frac{V_3}{V_2} \geqslant \frac{V_2}{V_1} .$$

Equality occurs iff E_1, E_2, E_3 are concentric.

<u>Proof</u>. It may be assumed (because of an affinity) that E_2 is the unit sphere with its centre at the origin, and that the coordinate axes are parallel to the principal axes 2a, 2b, 2c of E_1, respectively. Let (ξ, η, ζ) be the centre of E_1. Since the ellipsoid E_1 is carried by polarity with respect to E_2 into an ellipsoid, it follows that E_1 contains the centre of E_2, i.e. $|\xi| < a$, $|\xi| < b$, $|\xi| < c$. The polarity

$$ux + vy - 1 = 0 \rightarrow (u, v)$$

applied to the tangent planes $x = \xi \pm a$ of E_1 at the endpoints $(\xi \pm a, 0)$ of the axis of length 2a, yields two points

$$\left(\frac{1}{\xi \pm a}, 0\right)$$

of E_3 lying on the x-axis, the distance of which is given by

$$\frac{1}{\xi + a} - \frac{1}{\xi - a} = \frac{2a}{a^2 - \xi^2} \geqslant \frac{2}{a}\,.$$

If 2α, 2β, 2γ are the diameters of E_3 parallel to coordinate axes, then $2\alpha \geqslant 2/a$ and analogously $2\beta \geqslant 2/b$, $2\gamma \geqslant 2/c$. If V is the volume of the octahedron O with diameters 2α, 2β, 2γ, then

$$V = \frac{1}{6} \cdot 2\alpha \cdot 2\beta \cdot 2\gamma \geqslant \frac{4}{3abc}\,.$$

Let us replace 2α by the diameter $2\alpha'$ of E_3 conjugate with respect to E_3 to the diametral plane $\beta\gamma$. Similarly, let us replace 2β by the diameter $2\beta'$ conjugate to the diametral plane $\alpha'\gamma$. The volume of O has been increased by both steps. The diameters $2\alpha'$, $2\beta'$, 2γ of the new octahedron O' (with volume V') are pair by pair conjugate with respect to E_3 and thus $\alpha'\beta'\gamma = \bar{\alpha}\,\bar{\beta}\,\bar{\gamma}$, where $2\bar{\alpha}$, $2\bar{\beta}$, $2\bar{\gamma}$ denote the principal axes of E_3. Since

$$\frac{4}{3}\,\bar{\alpha}\,\bar{\beta}\,\bar{\gamma} = \frac{4}{3}\,\alpha'\beta'\gamma = V' \geqslant V \geqslant \frac{4}{3abc}\,,$$

we have

$$V_1 V_3 = \frac{4\pi}{3}\,abc \cdot \frac{4\pi}{3}\,\bar{\alpha}\,\bar{\beta}\,\bar{\gamma} \geqslant \left(\frac{4\pi}{3}\right)^2 = V_2^2.$$

Equality holds iff $\xi = \eta = \zeta = 0$.

L. Fejes Tóth [26], [29], [33, p. 4-5].

6. Inequalities for Spherical Triangles

Let T = ABC be a spherical triangle (on a unit sphere S with the centre O) with the sides $a = BC$, $b = CA$, $c = AB$ and angles α, β, γ at the vertices A, B, C, respectively. Let $2s = a + b + c$, $2\sigma = \alpha + \beta + \gamma$ and let $\varepsilon = 2\sigma - \pi$ be the spherical excess of T. If $\overline{AA'}$, $\overline{BB'}$, $\overline{CC'}$ are the diameters of S, then let R, R_a, R_b, R_c be the circumradii and r, r_a, r_b, r_c the inradii of the triangles T, A'BC, AB'C, ABC', respectively. The sign T_R means $a = b = c$.

6.1. The relations $a \lesseqgtr b$ are equivalent to the corresponding relations $\alpha \lesseqgtr \beta$ (analogously for the other pairs of sides and angles).

E. Rouché and Ch. de Comberousse [95, p. 36]; F. G.-M. [125, p. 867-868].

6.2. $\pi < 2\sigma < 3\pi$.

Proof. Follows by applying the inequalities $0 < 2s < 2\pi$ to the polar triangle (with the sides $\pi - \alpha$, $\pi - \beta$, $\pi - \gamma$) of T.

E. Rouché and Ch. de Comberousse [95, p. 42]; F. G.-M. [125, p. 869].

Remark. Inequality 6.2 can be written in the equivalent form $0 < \varepsilon < 2\pi$.

6.3. $\pi + \alpha > \beta + \gamma, \quad \pi + \beta > \gamma + \alpha, \quad \pi + \gamma > \alpha + \beta.$

Proof. Follows by applying of the inequalities $b + c > a$, $c + a > b$, $a + b > c$ to the polar triangle of T.

E. Rouché and Ch. de Comberousse [95, p. 42]; F. G.-M. [125, p. 869].

Remark 1. Inequalities 6.3 can be written in the equivalent form $\varepsilon < \min(2\alpha, 2\beta, 2\gamma)$.

Remark 2. Inequalities 6.2 and 6.3 are necessary and sufficient conditions for the existence of a spherical triangle with angles α, β, γ.

E. Rouché and Ch. de Comberousse [95, p. 42-43]; F. G.-M. [125, p. 869-870].

6.4. $\frac{1}{3}\pi^2 < \alpha^2 + \beta^2 + \gamma^2 < 9\pi^2.$

Proof. Follows from 6.2 using the inequalities

$$\frac{1}{3}(\alpha + \beta + \gamma)^2 \leqslant \alpha^2 + \beta^2 + \gamma^2 \leqslant (\alpha + \beta + \gamma)^2.$$

M. Petrović [90].

6.5. $\cos a + \cos b + \cos c > -\frac{3}{2}.$

R. Sturm [109, p. 67-68].

6.6. $\sin\frac{a}{2}\sin\frac{b}{2}\sin\frac{c}{2} \geqslant \sin(s - a)\sin(s - b)\sin(s - c) \quad \{T_R\}.$

Proof. With $x = s - a$, $y = s - b$, $z = s - c$ we must prove the inequality

$$\sin\frac{y + z}{2}\sin\frac{z + x}{2}\sin\frac{x + y}{2} \geqslant \sin x \sin y \sin z \quad \{x = y = z\}.$$

But this inequality follows by multiplication of three inequalities, the first of which is

$$(1) \qquad \sin^2\frac{y + z}{2} = \frac{1}{2}[1 - \cos(y + z)] \geqslant \frac{1}{2}[\cos(y - z) - \cos(y + z)] =$$

$$= \sin y \sin z \quad \{y = z\}.$$

M. S. Klamkin [67].

Remark. Inequality (1) means that $x \to \log \sin x$ is a concave function on $(0, \pi)$.

6.7. $\tan R \geq 2 \tan r \quad \{T_R\}.$

Proof. Follows from 6.6 according to the equalities

(2) $$\tan R = \frac{2}{\sqrt{\sin s \sin(s - a)\sin(s - b)\sin(s - c)}} \sin \frac{a}{2} \sin \frac{b}{2} \sin \frac{c}{2},$$

(3) $$\tan r = \sqrt{\frac{\sin(s - a)\sin(s - b)\sin(s - c)}{\sin s}}.$$

M. S. Klamkin [67].

6.8. $\tan R \tan r \leq \frac{2}{\sin s} \sin^3 \frac{s}{3} \quad \{T_R\}.$

Proof. Follows from the concavity of the function $x \to \log \sin x$ for $0 < x < \pi$ according to the equality

$$\tan R \tan r = \frac{2}{\sin s} \sin \frac{a}{2} \sin \frac{b}{2} \sin \frac{c}{2},$$

which is a consequence of (2) and (3).
M. S. Klamkin [67].

6.9. $\tan^2 r \leq \frac{1}{\sin s} \sin^3 \frac{s}{3} \quad \{T_R\}.$

Proof. Follows from 6.7 and 6.8.
M. S. Klamkin [67].

6.10. $\tan \frac{s - a}{2} \tan \frac{s - b}{2} \tan \frac{s - c}{2} \leq \tan^3 \frac{s}{6} \quad \{T_R\}.$

Proof. For every $x, y \geq 0$, $x + y \leq \frac{\pi}{2}$ we have the inequality

(4) $$\tan x \tan y \leq \tan^2 \frac{x + y}{2} \quad \{x = y\}.$$

Indeed, this inequality is equivalent to

$$\frac{1 - \cos(x + y)}{1 + \cos(x + y)} \geq \frac{\sin x \sin y}{\cos x \cos y},$$

i.e. to

$$[1 - \cos(x - y)] \cos(x + y) \geq 0 \quad \{x = y\}.$$

According to (4) we obtain successively

$$\tan \frac{s - a}{2} \tan \frac{s - b}{2} \tan \frac{s - c}{2} \tan \frac{s}{6} \leq$$

$$\leqslant \tan^2 \frac{c}{4} \tan^2 \frac{2a + 2b - c}{12} \leqslant \tan^4 \frac{s}{6},$$

since

$$\frac{1}{2}\left(\frac{s - a}{2} + \frac{s - b}{2}\right) = \frac{c}{4}, \quad \frac{1}{2}\left(\frac{s - c}{2} + \frac{s}{6}\right) = \frac{2a + 2b - c}{12},$$

$$\frac{1}{2}\left(\frac{c}{4} + \frac{2a + 2b - c}{12}\right) = \frac{s}{6}.$$

Equality holds iff

$$s - a = s - b, \quad s - c = \frac{s}{3}, \quad c = \frac{2a + 2b - c}{3},$$

i.e.

$$a = b = c.$$

M. S. Klamkin [67].

6.11. $$\tan^2 \frac{\varepsilon}{4} \leqslant \tan \frac{s}{2} \tan^3 \frac{s}{6} \quad \{T_R\}.$$

Proof. Follows by 6.10 from Lhuilier's formula

$$\tan^2 \frac{\varepsilon}{4} = \tan \frac{s}{2} \tan \frac{s - a}{2} \tan \frac{s - b}{2} \tan \frac{s - c}{2}.$$

J. Steiner [104]*; M. S. Klamkin [67].

6.12. $$\sin^2 \frac{\varepsilon}{4} \leqslant \frac{\sin \frac{s}{2} \sin^3 \frac{s}{6}}{\cos^3 \frac{s}{6}} \quad \{T_R\}.$$

Proof. According to 6.11 and the identity

(5) $$\cos 3x \cos^3 x + \sin 3x \sin^3 x = \cos^3 2x.$$

we obtain

$$\sin^2 \frac{\varepsilon}{4} = \frac{1}{1 + \cot^2 \frac{\varepsilon}{4}} \leqslant \frac{1}{1 + \cot \frac{s}{2} \cot^3 \frac{s}{6}} =$$

$$= \frac{\sin \frac{s}{2} \sin^3 \frac{s}{6}}{\sin \frac{s}{2} \sin^3 \frac{s}{6} + \cos \frac{s}{2} \cos^3 \frac{s}{6}} = \frac{\sin \frac{s}{2} \sin^3 \frac{s}{6}}{\cos^3 \frac{s}{3}}.$$

6.13. $\cos^2 \frac{\varepsilon}{4} \geqslant \frac{\cos \frac{s}{2} \cos^3 \frac{s}{6}}{\cos^3 \frac{s}{3}}$ $\{T_R\}$.

Proof. According to 6.11 and (5) we obtain

$$\cos^2 \frac{\varepsilon}{4} = \frac{1}{1 + \tan^2 \frac{\varepsilon}{4}} \geqslant \frac{1}{1 + \tan \frac{s}{2} \tan^3 \frac{s}{6}} =$$

$$= \frac{\cos \frac{s}{2} \cos^3 \frac{s}{6}}{\cos \frac{s}{2} \cos^3 \frac{s}{6} + \sin \frac{s}{2} \sin^3 \frac{s}{6}} = \frac{\cos \frac{s}{2} \cos^3 \frac{s}{6}}{\cos^3 \frac{s}{3}} .$$

6.14. $\frac{\sin^3 \frac{s}{6}}{\cos^3 \frac{s}{3}} \geqslant \frac{\sin \frac{s-a}{2} \sin \frac{s-b}{2} \sin \frac{s-c}{2}}{\cos \frac{a}{2} \cos \frac{b}{2} \cos \frac{c}{2}}$ $\{T_R\}$.

Proof. Follows from 6.12 according to the equality

$$\frac{\sin^2 \frac{\varepsilon}{4}}{\sin \frac{s}{2}} = \frac{\sin \frac{s-a}{2} \sin \frac{s-b}{2} \sin \frac{s-c}{2}}{\cos \frac{a}{2} \cos \frac{b}{2} \cos \frac{c}{2}} .$$

M. S. Klamkin [67].

6.15. $\frac{\cos^3 \frac{s}{6}}{\cos^3 \frac{s}{3}} \leqslant \frac{\cos \frac{s-a}{2} \cos \frac{s-b}{2} \cos \frac{s-c}{2}}{\cos \frac{a}{2} \cos \frac{b}{2} \cos \frac{c}{2}}$ $\{T_R\}$.

Proof. Follows from 6.13 according to the equality

$$\frac{\cos^2 \frac{\varepsilon}{4}}{\cos \frac{s}{2}} = \frac{\cos \frac{s-a}{2} \cos \frac{s-b}{2} \cos \frac{s-c}{2}}{\cos \frac{a}{2} \cos \frac{b}{2} \cos \frac{c}{2}} .$$

M. S. Klamkin [67].

6.16. $\tan r_a + \tan r_b + \tan r_c \geqslant \sin s \tan \frac{\sigma}{3}$ $\{T_R\}$.

Proof. Follows by adding of the equalities

(6) $\tan r_a = \sin s \tan \frac{\alpha}{2}$, $\tan r_b = \sin s \tan \frac{\beta}{2}$,

$\tan r_c = \sin s \tan \frac{\gamma}{2}$

according to the convexity of the function $x \to \tan x$.
M. S. Klamkin [68].

6.17. $\tan^2 r_a + \tan^2 r_b + \tan^2 r_c \geqslant \sin^2 s \quad \{T_R\}.$

Proof. Follows from (6) according the inequality of GI 2.35

$$\tan^2 \frac{\alpha}{2} + \tan^2 \frac{\beta}{2} + \tan^2 \frac{\gamma}{2} \geqslant 1,$$

which holds in the case when $\alpha + \beta + \gamma = \pi$ and therefore in the case $\alpha + \beta + \gamma > \pi$, too.
M. S. Klamkin [68].

6.18. $\tan r_a \tan r_b \tan r_c \leqslant$

$$\leqslant \frac{4 \sin s \tan r}{\sin \frac{\varepsilon}{2}} \sin^3 \frac{2\pi - \varepsilon}{6} \sin \frac{a}{2} \sin \frac{b}{2} \sin \frac{c}{2} \quad \{T_R\}.$$

Proof. Follows from 6.29 according to the equality

$$\tan r_a \tan r_b \tan r_c = \frac{2 \sin s}{\tan R} \sin \frac{a}{2} \sin \frac{b}{2} \sin \frac{c}{2}.$$

M. S. Klamkin [68].

The 'colunar' transformation maps the elements of the triangle T onto the corresponding elements of the triangle A'BC:

$$a \to a, \quad b \to \pi - b, \quad c \to \pi - c, \quad s \to \pi - (s - a),$$

$$s - b \leftrightarrow s - c, \ \alpha \to \alpha, \ \beta \to \pi - \beta, \ \gamma \to \pi - \gamma, \ \sigma \to \pi - (\sigma - \alpha),$$

$$\varepsilon \to 2\alpha - \varepsilon, \quad r \leftrightarrow r_a, \quad r_b \leftrightarrow r_c, \quad R \leftrightarrow R_a.$$

Using this transformation, from 6.6-6.18 we obtain the inequalities 6.19-6.31 with equalities iff $\pi - a = b = c$.

6.19. $\sin \frac{a}{2} \cos \frac{b}{2} \cos \frac{c}{2} \geqslant \sin s \sin(s - b) \sin(s - c).$

6.20. $\tan R_a \geqslant 2 \tan r_a.$

6.21. $\tan R_a \tan r_a \leqslant \frac{2}{\sin(s - a)} \sin^3 \frac{\pi + a - s}{3}.$

M. S. Klamkin [68].

6.22. $\tan^2 r_a \leqslant \frac{1}{\sin(s - a)} \sin^3 \frac{\pi + a - s}{3}.$

M. S. Klamkin [68].

6.23. $$\operatorname{cotan}\frac{s}{2}\tan\frac{s-b}{2}\tan\frac{s-c}{2} \leqslant \tan^3\frac{\pi+a-s}{6}.$$

M. S. Klamkin [68].

6.24. $$\tan^2\frac{2\alpha-\varepsilon}{4} \leqslant \operatorname{cotan}\frac{s-a}{2}\tan^3\frac{\pi+a-s}{6}.$$

6.25. $$\tan^2\frac{2\alpha-\varepsilon}{4} \leqslant \frac{\cos\frac{s-a}{2}\sin^3\frac{\pi+a-s}{6}}{\cos^3\frac{\pi+a-s}{3}}.$$

6.26. $$\cos^2\frac{2\alpha-\varepsilon}{4} \geqslant \frac{\sin\frac{s-a}{2}\cos^3\frac{\pi+a-s}{6}}{\cos^3\frac{\pi+a-s}{3}}.$$

6.27. $$\frac{\sin^3\frac{\pi+a-s}{6}}{\cos^3\frac{\pi+a-s}{3}} \geqslant \frac{\cos\frac{s}{2}\sin\frac{s-b}{2}\sin\frac{s-c}{2}}{\cos\frac{a}{2}\sin\frac{b}{2}\sin\frac{c}{2}}.$$

M. S. Klamkin [68].

6.28. $$\frac{\cos^3\frac{\pi+a-s}{6}}{\cos^3\frac{\pi+a-s}{3}} \leqslant \frac{\sin\frac{s}{2}\cos\frac{s-b}{2}\cos\frac{s-c}{2}}{\cos\frac{a}{2}\sin\frac{b}{2}\sin\frac{c}{2}}.$$

M. S. Klamkin [68].

6.29. $$\tan r + \tan r_b + \tan r_c \geqslant \sin(s-a)\tan\frac{\pi+\alpha-\sigma}{3}.$$

M. S. Klamkin [68].

6.30. $$\tan^2 r + \tan^2 r_b + \tan^2 r_c \geqslant \sin^2(s-a).$$

M. S. Klamkin [68].

6.31. $$\tan r \tan r_b \tan r_c \leqslant$$

$$\leqslant \frac{4\sin(s-a)\tan r_a}{\sin\frac{2\alpha-\varepsilon}{2}}\sin^3\frac{2\pi-2\alpha+\varepsilon}{6}\sin\frac{a}{2}\cos\frac{b}{2}\cos\frac{c}{2}.$$

M. S. Klamkin [68].

Using the polar transformation

$$a \to \pi - \alpha, \quad b \to \pi - \beta, \quad c \to \pi - \gamma, \quad \varepsilon \to 2\pi - 2s,$$

$$s \to \frac{3\pi}{2} - \sigma = \pi - \frac{\varepsilon}{2}, \quad s - a \to \frac{\pi}{2} + \alpha - \sigma = \alpha - \frac{\varepsilon}{2},$$

$$s - b \to \frac{\pi}{2} + \beta - \sigma = \beta - \frac{\varepsilon}{2}, \quad s - c \to \frac{\pi}{2} + \gamma - \sigma = \gamma - \frac{\varepsilon}{2},$$

$$\alpha \to \pi - a, \quad \beta \to \pi - b, \quad \gamma \to \pi - c, \quad \sigma \to \frac{3\pi}{2} - s,$$

$$r \to \frac{\pi}{2} - R, \quad r_a \to \frac{\pi}{2} - R_a, \quad r_b \to \frac{\pi}{2} - R_b,$$

$$r_c \to \frac{\pi}{2} - R_c, \quad R \to \frac{\pi}{2} - r, \quad R_a \to \frac{\pi}{2} - r_a,$$

from the inequalities 6.6, 6.8-6.19, 6.21-6.31 we obtain the inequalities 6.32-6.55. In 6.44-6.55 equalities hold iff $\pi - \alpha = \beta = \gamma$.

6.32. $$\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2} \geqslant \cos(\alpha - \sigma)\cos(\beta - \sigma)\cos(\gamma - \sigma) \quad \{T_R\}.$$

6.33. $$\tan R \tan r \geqslant \frac{\sin\frac{\varepsilon}{2}}{2\sin^3\frac{2\pi - \varepsilon}{6}} \quad \{T_R\}.$$

M. S. Klamkin [68].

6.34. $$\tan^2 R \geqslant \frac{\sin\frac{\varepsilon}{2}}{\sin^3\frac{2\pi - \varepsilon}{6}} \quad \{T_R\}.$$

M. S. Klamkin [68].

6.35. $$\tan\frac{2\alpha - \varepsilon}{4}\tan\frac{2\beta - \varepsilon}{4}\tan\frac{2\gamma - \varepsilon}{4} \leqslant \tan^3\frac{2\pi - \varepsilon}{12} \quad \{T_R\}.$$

M. S. Klamkin [68].

6.36. $$\tan^2\frac{s}{2} \geqslant \tan\frac{\varepsilon}{4}\cot an^3\frac{2\pi - \varepsilon}{12} \quad \{T_R\}.$$

6.37 $$\cos^2\frac{s}{2} \leqslant \frac{\cos\frac{\varepsilon}{4}\sin^3\frac{2\pi - \varepsilon}{12}}{\cos^3\frac{2\pi - \varepsilon}{6}} \quad \{T_R\}.$$

6.38. $$\sin^2 \frac{s}{2} \geq \frac{\sin \frac{\varepsilon}{4} \cos^3 \frac{2\pi - \varepsilon}{12}}{\cos^3 \frac{2\pi - \varepsilon}{6}} \quad \{T_R\}.$$

6.39. $$\frac{\sin^3 \frac{2\pi - \varepsilon}{12}}{\cos^3 \frac{2\pi - \varepsilon}{6}} \geq \frac{\sin \frac{2\alpha - \varepsilon}{4} \sin \frac{2\beta - \varepsilon}{4} \sin \frac{2\gamma - \varepsilon}{4}}{\sin \frac{\alpha}{2} \sin \frac{\beta}{2} \sin \frac{\gamma}{2}} \quad \{T_R\}.$$

M. S. Klamkin [68].

6.40. $$\frac{\cos^3 \frac{2\pi - \varepsilon}{12}}{\cos^3 \frac{2\pi - \varepsilon}{6}} \leq \frac{\cos \frac{2\alpha - \varepsilon}{4} \cos \frac{2\beta - \varepsilon}{4} \cos \frac{2\gamma - \varepsilon}{4}}{\sin \frac{\alpha}{2} \sin \frac{\beta}{2} \sin \frac{\gamma}{2}} \quad \{T_R\}.$$

M. S. Klamkin [68].

6.41. $$\cot R_a + \cot R_b + \cot R_c \geq \sin \frac{\varepsilon}{2} \cot \frac{s}{3} \quad \{T_R\}.$$

M. S. Klamkin [68].

6.42. $$\cot^2 R_a + \cot^2 R_b + \cot^2 R_c \geq \sin^2 \frac{\varepsilon}{2} \quad \{T_R\}.$$

M. S. Klamkin [68].

6.43. $$\cot R_a \cot R_b \cot R_c \leq$$

$$\leq \frac{4 \sin \frac{\varepsilon}{2} \cot R}{\sin s} \sin^3 \frac{s}{3} \cos \frac{\alpha}{2} \cos \frac{\beta}{2} \cos \frac{\gamma}{2} \quad \{T_R\}.$$

M. S. Klamkin [68].

6.44. $$\cos \frac{\alpha}{2} \sin \frac{\beta}{2} \sin \frac{\gamma}{2} \geq \sin \frac{\varepsilon}{2} \sin(\beta - \frac{\varepsilon}{2}) \sin(\gamma - \frac{\varepsilon}{2}).$$

6.45. $$\tan R_a \tan r_a \geq \frac{\sin(\alpha - \frac{\varepsilon}{2})}{2 \sin^3 \frac{2\pi - 2\alpha + \varepsilon}{6}}.$$

M. S. Klamkin [68].

6.46. $$\tan^2 R_a \geq \frac{\sin(\alpha - \frac{\varepsilon}{2})}{\sin^3 \frac{2\pi - 2\alpha + \varepsilon}{6}}.$$

M. S. Klamkin [68].

6.47. $\tan \frac{\varepsilon}{4} \tan \frac{2\beta - \varepsilon}{4} \tan \frac{2\gamma - \varepsilon}{4} \leqslant \tan^3 \frac{2\pi - 2\alpha + \varepsilon}{12}$.

M. S. Klamkin [68].

6.48. $\tan^2 \frac{s - a}{2} \leqslant \operatorname{cotan} \frac{2\alpha - \varepsilon}{4} \tan^3 \frac{2\pi - 2\alpha + \varepsilon}{12}$.

6.49. $$\sin^2 \frac{s - a}{2} \leqslant \frac{\cos \frac{2\alpha - \varepsilon}{4} \sin^3 \frac{2\pi - 2\alpha + \varepsilon}{12}}{\cos^3 \frac{2\pi - 2\alpha + \varepsilon}{6}}.$$

6.50. $$\cos^2 \frac{s - a}{2} \geqslant \frac{\sin \frac{2\alpha - \varepsilon}{4} \cos^3 \frac{2\pi - 2\alpha + \varepsilon}{12}}{\cos^3 \frac{2\pi - 2\alpha + \varepsilon}{6}}.$$

6.51. $$\frac{\sin^3 \frac{2\pi - 2\alpha + \varepsilon}{12}}{\cos^3 \frac{2\pi - 2\alpha + \varepsilon}{6}} \geqslant \frac{\sin \frac{\varepsilon}{4} \sin \frac{2\beta - \varepsilon}{4} \sin \frac{2\gamma - \varepsilon}{4}}{\sin \frac{\alpha}{2} \cos \frac{\beta}{2} \cos \frac{\gamma}{2}}.$$

M. S. Klamkin [68].

6.52. $$\frac{\cos^3 \frac{2\pi - 2\alpha + \varepsilon}{12}}{\cos^3 \frac{2\pi - 2\alpha + \varepsilon}{6}} \leqslant \frac{\cos \frac{\varepsilon}{2} \cos \frac{2\beta - \varepsilon}{4} \cos \frac{2\gamma - \varepsilon}{4}}{\sin \frac{\alpha}{2} \cos \frac{\beta}{2} \cos \frac{\gamma}{2}}.$$

M. S. Klamkin [68].

6.53. $\operatorname{cotan} R + \operatorname{cotan} R_b + \operatorname{cotan} R_c \geqslant \sin(\alpha - \frac{\varepsilon}{2}) \tan \frac{\pi - 2a + 2s}{6}$.

6.54. $\operatorname{cotan}^2 R + \operatorname{cotan}^2 R_b + \operatorname{cotan}^2 R_c \geqslant \sin^2 (\alpha - \frac{\varepsilon}{2})$.

M. S. Klamkin [68].

6.55. $\operatorname{cotan} R \operatorname{cotan} R_b \operatorname{cotan} R_c \leqslant$

$$\leqslant \frac{4 \sin(\alpha - \frac{\varepsilon}{2}) \operatorname{cotan} R_a}{\sin(s - a)} \sin^3 \frac{\pi - s + a}{3} \cos \frac{\alpha}{2} \sin \frac{\beta}{2} \sin \frac{\gamma}{2}.$$

Let P be an internal point of a spherical triangle ABC and D, E, F the intersections of the cevians AP, BP, CP with the sides BC, CA, AB, respectively. Let $R_1 = PA$, $R_2 = PB$, $R_3 = PC$, $r'_1 = PD$, $r'_2 = PE$, $r'_3 = PF$. Let r_1, r_2, r_3 be the distances of P from BC, CA, AB and w_1, w_2, w_3 the distances of P from BC, CA, AB measured along the internal bisectors of the angles ∢ BPC, ∢ CPA, ∢ APB. Obviously $r'_i \geqslant r_i$, $w_i \geqslant r_i$ (i = 1, 2, 3).

All indices are modulo 3 from the set {1, 2, 3}.

If we project the triangle ABC from O onto the tangent plane of $\underline{S}$ at the point P, then we obtain a plane triangle $\bar{A}\,\bar{B}\,\bar{C}$ and if $\bar{R}_i$, $\bar{r}_i$, $\bar{r}'_i$, $\bar{w}_i$ (i = 1, 2, 3) are the corresponding distances in the triangle $\bar{A}\,\bar{B}\,\bar{C}$, then obviously

$$\bar{R}_i = \tan R_i, \quad \bar{r}_i = \tan r_i, \quad \bar{r}'_i = \tan r'_i,$$

$$\bar{w}_i = \tan w_i \quad (i = 1, 2, 3).$$

Therefore, a great number of inequalities for R_i, r_i, r'_i, w_i (i = 1, 2, 3) from XVIII.2 (for n = 2) and from GI 12 have immediate analogues for the spherical triangle ABC. For example, from GI 12.37 we obtain:

6.56. If $\lambda_1, \lambda_2, \lambda_3 \in R^+$ and $R_i, \lambda_i R_i < \frac{\pi}{2}$ (i = 1, 2, 3), then

$$\sum_{i=1}^{3} \tan \lambda_i R_i \geq 2 \sum_{i=1}^{3} \frac{\operatorname{cotan} R_{i-1} + \operatorname{cotan} R_{i+1}}{\operatorname{cotan} \lambda_{i-1} R_{i-1} + \operatorname{cotan} \lambda_{i+1} R_{i+1}} \tan w_i$$

with equality iff $\measuredangle$ BPC = $\measuredangle$ CPA = $\measuredangle$ APB = $\frac{2\pi}{3}$, $\lambda_1 R_1 = \lambda_2 R_2 = \lambda_3 R_3$.

A. Oppenheim [89].

Remark. In 6.56 w_i can be replaced by r_i.

6.57. With the same hypothesis as in 6.56 we have

$$\sum_{i=1}^{3} \lambda_i R_i \geq 2 \sum_{i=1}^{3} \frac{\operatorname{cotan} R_{i-1} + \operatorname{cotan} R_{i+1}}{\frac{1}{\lambda_{i-1} R_{i-1}} + \frac{1}{\lambda_{i+1} R_{i+1}}} \tan w_i .$$

Proof. Follows by substituting $\lambda_i \delta$ for λ_i in 6.56, dividing by δ and letting $\delta \to 0$.

A. Oppenheim [89].

6.58. $$\sin R_1 + \sin R_2 + \sin R_3 \geq 2(\sin w_1 \cos \frac{a}{2} + \sin w_2 \cos \frac{b}{2} + \sin w_3 \cos \frac{c}{2}).$$

Equality holds only in one of the following four cases: (i) a = b = c, $R_1 = R_2 = R_3$; (ii) $0 < \pi - R_1 = R_2 = R_3 < \pi$ (and then $w_2 = w_3$; the other ends of w_1, w_2, w_3 are the midpoints of $\overline{BC}$, $\overline{CA}$, $\overline{AB}$); (iii) $0 < R_1 = \pi - R_2 = R_3 < \pi$; (iv) $0 < R_1 = R_2 = \pi - R_3 < \pi$.

A. Oppenheim [89].

Remark. In 6.58 w_i can be replaced by r_i.

Let a_i, b_i, c_i denote the sides of n spherical triangles $A_iB_iC_i$ ($i = 1, 2, \ldots, n$) on unit sphere S and let $\lambda_1, \lambda_2, \ldots, \lambda_n \in R^+$ such that $\lambda_1 + \lambda_2 + \ldots + \lambda_n = 1$. Then

$$a = \sum_{i=1}^{n} \lambda_i a_i, \quad b = \sum_{i=1}^{n} \lambda_i b_i, \quad c = \sum_{i=1}^{n} \lambda_i c_i$$

are the sides of a new spherical triangle ABC on the same sphere S. Let the indices 1, 2, ..., n denote the elements of the triangles $A_iB_iC_i$ ($i = 1, 2, \ldots, n$). $\{I_n\}$ denotes that equality holds iff the n triangles are isometric.

6.59. $$\tan^2 \frac{\varepsilon}{4} \geqslant \prod_{i=1}^{n} \tan^{2\lambda_i} \frac{\varepsilon_i}{4} \quad \{I_n\}.$$

M. S. Klamkin [68].

6.60. $$\tan^2 r \sin s \geqslant \prod_{i=1}^{n} (\tan^2 r_i \sin s_i)^{\lambda_i} \quad \{I_n\}.$$

M. S. Klamkin [68].

6.61. $$\tan R \tan r \sin s \geqslant \prod_{i=1}^{n} (\tan R_i \tan r_i \sin s_i)^{\lambda_i} \quad \{I_n\}.$$

M. S. Klamkin [68].

7. Other Inequalities in E^3

Let Ω be a convex polyhedral angle with the vertex O. If γ is any plane such that $\Omega \cap \gamma$ is a convex polygon $\mathcal{B}_\gamma$, then let B_γ be the area of $\mathcal{B}_\gamma$. If $\mathcal{P}_\gamma$ is the pyramid with the base $\mathcal{B}_\gamma$ and the vertex O, then let V_γ, L_γ, F_γ and h_γ be the volume, the lateral area, the total area and the altitude of $\mathcal{P}_\gamma$, respectively. Let H_γ be the foot of the perpendicular from O onto γ and G_γ the centroid of $\mathcal{B}_\gamma$.

7.1. For a given point P inside Ω there is one and only one plane α such that $G_\alpha = P$. If β is any plane through P, then

$$V_\beta \geqslant V_\alpha \quad \{\beta = \alpha\}.$$

J. Steiner [104]; R. Sturm [109, p. 130-131].

7.2. Let $V \in R^+$. There is one and only one plane α such that $V_\alpha = V$ and $H_\alpha = G_\alpha$. If β is any plane such that $V_\beta = V$, then

$$B_\beta \geqslant B_\alpha \quad \{\beta = \alpha\}.$$

J. Steiner [104]; R. Sturm [109, p. 131].

7.3. Let $B \in R^+$. There is one and only one plane α such that $B_\alpha = B$ and $H_\alpha = G_\alpha$. If β is any plane such that $B_\beta = B$, then

$$V_\beta \leqslant V_\alpha \quad \{\beta = \alpha\}.$$

J. Steiner [104]; R. Sturm [109, p. 131].

7.4. Let $h \in R^+$. There is one and only one plane α such that $h_\alpha = h$ and $H_\alpha = G_\alpha$. If β is any plane such that $h_\beta = h$, then

$$V_\beta \geqslant V_\alpha \quad \{\beta = \alpha\}.$$

R. Sturm [109, p. 131-132].

7.5. Let Ω be circumscribed to a cone of revolution and let $V \in R^+$. There is one and only one plane α such that $V_\alpha = V$ and such that H_α lies on the axis of the cone. If β is any plane such that $V_\beta = V$, then

$$L_\beta \geqslant L_\alpha \quad \{\beta = \alpha\}.$$

J. Steiner [104]; R. Sturm [109, p. 132-133].

7.6. Let Ω be circumscribed to a cone of revolution and let $V \in R^+$. There is one and only one plane α such that $V_\alpha = V$ and such that G_α is the point of contact of α with the inscribed sphere of P_α. If β is any plane such that $V_\beta = V$, then

$$F_\beta \geqslant F_\alpha \quad \{\beta = \alpha\}.$$

J. Steiner [104]; R. Sturm [109, p. 133].

7.7. Let p and p' be two skew lines, $d = PP'$ the distance between these two lines, B, C points of p and A, D points of p' such that $BC = a$, $AD = a'$. If F_A, F_D are the areas of the triangles BCD, ABC, respectively, then

$$F_A + F_D \geqslant a\sqrt{d^2 + \frac{a'^2}{4}} \quad \{AP' = P'D\}.$$

Proof. If AP' = x, DP' = y, then

$$F_A + F_D = \frac{a}{2}(\sqrt{d^2 + x^2} + \sqrt{d^2 + y^2}).$$

Using Minkowski's inequality we obtain

$$F_A + F_D \geq a\sqrt{d^2 + (\frac{x + y}{2})^2} \quad \{x = y\}.$$

If A and D are on the same side of P' (or A = P' or D = P'), then $x + y \geq a'$ and $x \neq y$ and if A and D are on the different sides of P', then $x + y = a'$. Therefore, we obtain 7.7 in both cases.

7.8. If in a skew quadrangle ABCD we have $AB \perp BC$, $CD \perp DA$, $\phi_1 = \sphericalangle (AB, CD)$, $\phi_2 = \sphericalangle (BC, DA)$, then

$$\cos \phi_i \leq \frac{BD}{AC} \quad (i = 1, 2)$$

with equality only for a plane quadrangle.

Proof. Let E be the fourth vertex of the rectangle ABCE. Then $\phi_1 = \sphericalangle DCE$, AC = BE. The points B, D, E lie on the sphere with a diameter $\overline{AC}$. Since BE is another diameter of this sphere S, so $BD \perp DE$. Let R be the radius of S and r the radius of the circle $\alpha \cap S$, where α is the plane CDE. Then $r \leq R$ with equality iff the plane α contains the centre of S, i.e. iff the point D lies in the plane ABC. Now, we have

$$\sin \sphericalangle DBE = \frac{DE}{2R} \leq \frac{DE}{2r} = \sin \sphericalangle DCE$$

and therefore $\cos \sphericalangle DBE \geq \cos \sphericalangle DCE$. Finally, we obtain

$$\cos \phi_1 = \cos \sphericalangle DCE \leq \cos \sphericalangle DBE = \frac{BD}{BE} = \frac{BD}{AC}.$$

Ju. Nesterenko [87].

7.9. If V is the volume and F the total area of a polyhedron isomorphic to a n-angular bipyramid, then

$$\frac{F^3}{V^2} \geq 27\sqrt{3}\, n \tan \frac{\pi}{n}$$

with equality only for a regular n-angular pyramid with the angle $\arccos \frac{\sqrt{3}}{3}$ between the base and any of its 2n lateral faces.

J. Steiner [104]*; R. Sturm [109, p. 125-128, 137-138]*.

7.10. Let $A_1B_1'A_2B_2' \ldots A_nB_n'$ be a regular 2n-gon ($n \geq 2$) inscribed in a circle with the radius r in a plane α. A translation by a vector of length h orthogonal to the plane α maps the regular n-gon $B_1'B_2' \dots B_n'$ onto a regular n-gon $B_1B_2 \dots B_n$. The regular n-gons $A_1A_2 \dots A_n$ and $B_1B_2 \dots B_n$ are the bases and the 2n congruent isosceles triangles $A_1B_1A_2$, $B_1A_2B_2$, $A_2B_2A_3$, ..., $A_nB_nA_1$, $B_nA_1B_1$ are the lateral faces of a regular n-angular antiprism P with the altitude h and the circumradius r of its bases. If V is the volume and F the total area of P, then

$$\frac{F^3}{V^2} \geqslant 72n \sin\frac{\pi}{n} \cdot \frac{v^3}{(2u+1)^2(v^2-2uv+2u-1)} \qquad \{\frac{h}{r} = \sqrt{v^2 - 2uv + 2u - 1}\},$$

where

$$u = \cos\frac{\pi}{n}, \quad v = 2u + \sqrt{4u^2 - 6u + 3}.$$

Proof. Let a be the side length and ρ the inradius of a base and h' the altitude of a lateral face of P. Then

$$\rho = r\cos\frac{\pi}{n}, \quad a = 2r\sin\frac{\pi}{n},$$

$$h' = \sqrt{(r-\rho)^2 + h^2} = \sqrt{r^2(1-\cos\frac{\pi}{n})^2 + h^2},$$

$$F = na(\rho + h') = 2n\sin\frac{\pi}{n} \cdot r(r\cos\frac{\pi}{n} + \sqrt{r^2(1-\cos\frac{\pi}{n})^2 + h^2}).$$

The symmetral plane of P, parallel to α, intersects P alongside a regular 2n-gon with the side length $\frac{a}{2}$ and the area

$$2n \cdot \frac{a}{4} \cdot \frac{a}{4}\cot\frac{\pi}{2n} = \frac{na^2}{8}\cot\frac{\pi}{2n}.$$

According to the well-known prismoidal formula we have

$$V = \frac{h}{6}(2 \cdot \frac{1}{2}na\rho + 4 \cdot \frac{na^2}{8}\cot\frac{\pi}{2n}) =$$

$$= \frac{h}{3}(nr^2\sin\frac{\pi}{n}\cos\frac{\pi}{n} + nr^2\sin^2\frac{\pi}{n}\cot\frac{\pi}{2n}) =$$

$$= \frac{nr^2h}{3}\sin\frac{\pi}{n}(\cos\frac{\pi}{n} + 2\cos^2\frac{\pi}{2n}) = \frac{nr^2h}{3}\sin\frac{\pi}{n}(1 + 2\cos\frac{\pi}{n}).$$

Therefore

$$\frac{F^3}{V^2} = \frac{72n \sin\frac{\pi}{n}}{(1 + 2\cos\frac{\pi}{n})^2} \cdot \frac{(r\cos\frac{\pi}{n} + \sqrt{r^2(1 - \cos\frac{\pi}{n})^2 + h^2})^3}{rh^2} =$$

$$= 72n \sin\frac{\pi}{n} \cdot \frac{(u + \sqrt{(1-u)^2 + (\frac{h}{r})^2})^3}{(1 + 2u)^2(\frac{h}{r})^2} .$$

Let

(1) $$u + \sqrt{(1-u)^2 + (\frac{h}{r})^2} = x.$$

Then

$$(\frac{h}{r})^2 = (x - u)^2 - (1 - u)^2 = x^2 - 2ux + 2u - 1$$

and therefore

$$\frac{F^3}{V^2} = 72n \sin\frac{\pi}{n} \cdot \frac{x^3}{(1 + 2u)^2(x^2 - 2ux + 2u - 1)} .$$

From (1) follows that x monotonically increases from 1 to $+\infty$, when $\frac{h}{r}$ increases from 0 to $+\infty$. We must determine the minimum of the function

$$f'(x) = \frac{x^2(x^2 - 4ux + 6u - 3)}{(x^2 - 2ux + 2u - 1)^2} ,$$

for $1 < x < +\infty$. Since

$$f(x) = \frac{x^2(x^2 - 4ux + 6u - 3)}{(x^2 - 2ux + 2u - 1)^2} ,$$

from $x^2 - 4ux + 6u - 3 = 0$ it follows

$$x_{1,2} = 2u \pm \sqrt{4u^2 - 6u + 3}.$$

But $u < 1$ implies

$$x_1 = 2u + \sqrt{4u^2 - 6u + 3} > 1, \quad x_2 = 2u - \sqrt{4u^2 - 6u + 3} < 1$$

and therefore only the first solution x_1 remains. Now, we have

$$f''(x) =$$

$$= \frac{8u^2x^3 - 4ux^3 + 2x^3 - 24u^2x^2 + 12ux^2 + 24u^2x - 24ux + 6x}{(x^2 - 2ux + 2u - 1)^3} =$$

$$= \frac{(8u^2 - 4u + 2)(x - 1)^2 + 6(x - 1)^2 + 12(1 - u)(x - 1) + 8(1 - u)^2}{(x - 1)^3(x - 2u + 1)^3}$$

and $f''(x_1) > 0$, according to the inequalities $1 - u > 0$, x_1 $x_1 - 1 > 0$, $8u^2 - 4u + 2 > 0$, $x - 2u + 1 > 0$. Therefore, F^3/V^2 is minimal for $x = x_1 = v$.

E. Steinitz [107].

7.11. If $a_1, a_2, \ldots, a_n$ are the side lengths and $p = a_1 + a_2 + \ldots + a_n$ the perimeter of a convex spherical n-gon A on an unit sphere with the centre O (i.e. the polyhedral angle with the vertex O, determined by A, is convex), then

$$2a_i < p < 2\pi \quad (i = 1, 2, \ldots, n).$$

E. Rouché and Ch. de Comberousse [95, p. 35-37].

Remark. Inequalities 7.11 (with n = 3) are necessary and sufficient conditions for the existence of a spherical triangle with the side lengths a_1, a_2, a_3.

E. Rouché and Ch. de Comberousse [95, p. 37-38]; F. G.-M. [125, p. 868-869].

7.12. If F_n is the area of a convex spherical n-gon, all of whose side lengths are equal to a and all of whose diagonals have lengths at least a, then

$$F_n \geqslant (n - 2)F_3 \quad \{n = 3\}.$$

W. Habicht and B. L. van der Waerden [57]; J. Molnar [86].

7.13. If σ is the sum of all angles of a convex spherical n-gon, then

$$\sigma > (n - 2)\pi.$$

A. Pinciu [91].

7.14. Let $L_1, L_2, \ldots, L_q$ be q linear complexes of lines in a three-dimensional projective space and let

$$x_{12}^{(i)}p_{34} + x_{13}^{(i)}p_{42} + x_{14}^{(i)}p_{23} + x_{34}^{(i)}p_{12} + x_{42}^{(i)}p_{13} +$$

$$+ x_{23}^{(i)}p_{14} = 0 \quad (i = 1, 2, \ldots, q)$$

be the equations of these q linear complexes, where p_{jk} are the Plücker coordinates of any line. For any i, j ∈ {1, 2, ..., q} let

$$l_{ij} = x_{12}^{(i)}x_{34}^{(j)} + x_{13}^{(i)}x_{42}^{(j)} + x_{14}^{(i)}x_{23}^{(j)} + x_{34}^{(i)}x_{12}^{(j)} + x_{42}^{(i)}x_{13}^{(j)} + x_{23}^{(i)}x_{14}^{(j)}$$

and let $C(L_1, L_2, \ldots, L_q) = \det(l_{ij})$ $(i, j = 1, 2, \ldots, q)$. For six linear complexes $L_1, L_2, \ldots, L_6$ we have

$$C(L_1, L_2, \ldots, L_6) \leqslant 0$$

with equality iff these six linear complexes are linearly dependent.
O. Bottema [8].

7.15. With the notation of 7.14 let L_1, L_2, L_3, L_4 be four linearly independent linear complexes. A pair of lines, which is the intersection of these linear complexes, consists of two real different lines, of two identical lines or of two imaginary lines iff we have the first, the second or the third sign in

$$C(L_1, L_2, L_3, L_4) \gtreqless 0.$$

O. Bottema [8].

8. Inequalities for Convex Polytopes in E^n $(n \geqslant 2)$

Let P be a convex polytope in E^n $(n \geqslant 2)$ with v vertices $A_1, A_2, \ldots, A_v$ and f 'faces' $P_1, P_2, \ldots, P_f$. Let G be the centroid and V the volume of P, further let I, O be the centres and r, R the radii of the maximal hypersphere σ contained in P and of the minimal hypersphere Σ containing P, respectively. Let $F_1, F_2, \ldots, F_f$ be the (n - 1)-contents of $P_1, P_2, \ldots, P_f$ and $F = F_1 + F_2 + \ldots + F_f$ the 'total surface area' of P. By E we denote the total edge length of P.

Let P be any point inside P. For any i ∈ {1, 2, ..., f} let r_i be the distance from the point P to the hyperplane of P_i. Obviously

$$(1) \qquad \sum_{i=1}^{f} F_i r_i = nV.$$

8.1. $\frac{V}{F} < r \leqslant \frac{nV}{F}$

with equality iff P is circumscribed about a hypersphere.

Proof. The second inequality is obvious. On every 'face' P_i (on the inner side) we construct a 'prism' with the altitude $h = V/F$. The 'prisms' on two adjacent 'faces' have common interior points and the sum of all 'prisms' has its volume smaller than $Fh = V$. Therefore, there is an inner point P of P which is not contained in any of the 'prisms', i.e. we have $r_i > h$ ($i = 1, 2, \ldots, f$). If $r' = \min(r_1, r_2, \ldots, r_f)$, then $r' > h = V/F$ and a hypersphere with the centre P and the radius r' is contained in P.

A. Kelarev [64].

8.2. If P' is a convex polytope contained in P and if V' and F' are the volume and the 'total surface area' of P', respectively, then

$$n\frac{V}{F} > \frac{V'}{F'} .$$

Proof. Let r' be the radius of the maximal hypersphere contained in P'. According to 8.1 we have successively

$$\frac{nV}{F} \geqslant r > r' > \frac{V'}{F'} .$$

A. Kelarev [64].

Remark. The constant n in 8.2 cannot be decreased. Indeed, it suffices to consider (for $n = 2$, but we can make an analogous consideration for $n > 2$) a narrow long rectangle inside an isosceles triangle with a short base and a long altitude.

8.3. Let P' be a polytope with the 'faces' parallel to the corresponding 'faces' of P and let P' be circumscribed about a unit hypersphere. If V' is the volume of P', then

$$\frac{F^n}{V^{n-1}} \geqslant n^n V'$$

with equality iff P and P' are homothetic.

H. Minkowski [84]; Lindelöf [76] ($n = 3$); S. A. J. Lhuilier [75] ($n = 2$).

8.4. If

$$\omega_k = \frac{\pi^{\frac{k}{2}}}{\Gamma(1 + \frac{k}{2})} ,$$

if χ is an auxiliary function defined by $\chi(\sigma) = \omega_{n-1} \tan^{n-1} \tau$ and if the function $\tau = \tau(\sigma)$ is defined by

$$\int_0^\tau \sin^{n-2}\alpha \, d\alpha = \frac{\sigma}{(n - 1)\omega_{n-1}},$$

then

$$\frac{F^n}{V^{n-1}} \geqslant fn^{n-1}\chi\left(\frac{n\omega_n}{f}\right).$$

H. Hadwiger [58]; M. Goldberg [50] (n = 3).

8.5. If $\lambda_1, \lambda_2, \ldots, \lambda_f \in R^+$,

$$\lambda = \sum_{i=1}^{f} \lambda_i, \quad d = \max_i \frac{F_i r_i}{\lambda_i},$$

and if $g:(0, d] \to R$ is a convex, respectively concave function, then

$$\lambda g\left(\frac{nV}{\lambda}\right) \lesseqgtr \sum_{i=1}^{f} \lambda_i g\left(\frac{F_i r_i}{\lambda_i}\right).$$

If g is strictly convex, respectively concave function, then equality holds iff

$$\frac{F_1 r_1}{\lambda_1} = \frac{F_2 r_2}{\lambda_2} = \ldots = \frac{F_f r_f}{\lambda_f}.$$

Proof. Follows from (1) using Jensen's inequality.

I. Tomescu [115] ($\lambda_i = 1$ or $\lambda_i = F_i$ for $i = 1, 2, \ldots, f$; $g(x) = x^p$, $p \in R$, $n \in \{2, 3\}$); XVIII.2.30 ($f = n + 1$, $\lambda_1 = \lambda_2 = \ldots = \lambda_f = 1$, $g(x) = \log x$); XVIII.2.32 ($f = n + 1$; $\lambda_i = (x_i F_i^p)^{\frac{1}{p+1}}$ for $i = 1, 2, \ldots, f$; $g(x) = x^{-p}$; $p > 0$); J. Neuberg [129] ($n = 3$; $\lambda_i = F_i^2$ for $i = 1, 2, \ldots, f$; $g(x) = x^2$).

8.6. $$\sum_{\substack{i,j=1\\ i<j}}^{f} F_i F_j r_i r_j \leqslant \frac{f-1}{2f} n^2 V^2 \quad \{P = G\}.$$

Proof. Using (1) we obtain

$$\sum_{\substack{i,j=1\\ i<j}}^{f} F_i F_j r_i r_j = \frac{1}{2}\left(n^2 V^2 - \sum_{i=1}^{f} F_i^2 r_i^2\right). \tag{2}$$

On the other hand, 8.5 (with $\lambda_1 = \lambda_2 = \ldots = \lambda_f$, $g(x) = x^2$) implies

$$\sum_{i=1}^{f} F_i^2 r_i^2 \geqslant \frac{n^2 V^2}{f} \tag{3}$$

with equality iff $F_1 r_1 = F_2 r_2 = \dots = F_f r_f$, i.e. iff P = G. But, from (2) and (3) inequality 8.6 follows.

8.7. If P is circumscribed about a hypersphere and if $p > 0$, then

$$\sum_{i=1}^{f} \frac{1}{r_i^p} > \frac{2}{r^p} .$$

Proof. Using 8.5 (with $\lambda_i = F_i^{\frac{p}{p+1}}$ for $i = 1, 2, \dots, f$ and $g(x) = x^{-p}$), we get

$$(4) \qquad \sum_{i=1}^{f} \frac{1}{r_i^p} \geqslant \frac{1}{(nV)^p} \left(\sum_{i=1}^{f} F_i^{\frac{p}{p+1}} \right)^{p+1} .$$

On the other hand, there holds Petrović's inequality

$$\sum_{i=1}^{f} g(x_i) > 2g \frac{1}{2} \left(\sum_{i=1}^{f} x_i \right)$$

for concave functions $g:[0, a) \to R$ with $g(0) = 0$ and

$$0 \leqslant x_j \leqslant \frac{1}{2} \sum_{i=1}^{f} x_i \quad (j = 1, 2, \dots, f).$$

For $x_i = F_i$ $(i = 1, 2, \dots, f)$ and $g(x) = x^{\frac{p}{p+1}}$ we obtain

$$(5) \qquad \left(\sum_{i=1}^{f} F_i^{\frac{p}{p+1}} \right)^{p+1} > 2 \left(\sum_{i=1}^{f} F_i \right)^p .$$

But, (4), (5) and equality $rF = nV$ imply 8.7.

8.8. Let P be a polytope isomorphic to a regular polytope P_o. If R_o is the radius of the minimal hypersphere containing P_o and r_o the radius of the maximal hypersphere contained in P_o, then

$$\frac{R}{r} \geqslant \frac{R_o}{r_o}$$

with equality iff the polytopes P and P_o are similar.

L. Fejes Tóth [34], [35], [38, p. 292].

Remark. If P_o is a hypercube or a cross-polytope, then $R_o/r_o = \sqrt{n}$, and if P_o is a regular simplex, then $R_o/r_o = n$ (XVIII. 2.51).

8.9. Let $n = 4$ and P_o be the regular polytope circumscribed about a hypersphere with the radius R and bounded by the regular tetrahedra or octahedra, and let P be a polytope isomorphic to P_o and circumscribed about the same hypersphere. If V_o is the volume of P_o, then

$$V \geqslant V_o$$

with equality iff the polytopes P and P_o are congruent.

L. Fejes Tóth [34].

8.10. With the same conditions as in 8.9 let F_o be the total area of P_o. Then

$$F \geqslant F_o$$

with equality iff the polytopes P and P_o are congruent.

L. Fejes Tóth [34].

8.11. If $n = 4$, then

$$F > 67.5r^2.$$

L. Fejes Tóth [37].

8.12. If $n = 4$ and P has tetrahedral cells only, then

$$F > 110.4r^2.$$

L. Fejes Tóth [37].

8.13. If $n = 4$ and P has octahedral cells only, then

$$F > 81.6r^2.$$

L. Fejes Tóth [37].

8.14. If P is a parallelotope, then

$$\frac{E^n}{V} \geqslant 2^{n(n-1)}n^n$$

with equality iff P is a hypercube.

Proof. Follows from XVIII.3.20' (with $k = 1$).

D. O. Škljarskij, N. N. Čencov, and I. M. Jaglom [102, p. 262] ($n = 3$).

8.15. Let P be a set of 2^m points ($m \geq 2$) labelled as if they were the vertices of an m-parallelotope. Let Q be the sum of the squares of the $2^{m-1}m$ 'edges' and D the sum of the squares of the 2^{m-1} 'diagonals'. Then

$$D \leq Q$$

with equality iff P is the vertex set of an m-parallelotope.
L. Gerber [48].

8.16. Let P be a parallelotope that contains the origin O in its interior. Let P' be the cross-polytope polar reciprocal to P with respect to the unit sphere Σ centered at O. If V' is the volume of P', then

$$VV' \geq \frac{4^n}{n!}$$

with equality iff O is the centre of P.

Proof. Let the 'faces' of P be denoted by $P_1, P_2, \ldots, P_n$ and $P_1', P_2', \ldots, P_n'$, where $P_i \parallel P_i'$ ($i = 1, 2, \ldots, n$). By a suitable choice of orthogonal coordinate axes we can suppose that the foot of the perpendicular from O to P_i' has coordinates $(a_{i1}, \ldots, a_{ii}, 0, \ldots, 0)$. Then the foot of the perpendicular from O to P_i' has coordinates $(-k_1 a_{i1}, \ldots, -k_i a_{ii}, 0, \ldots, 0)$, where $k_i > 0$. The equations of P_i and P_i' have the form

$$a_{i1}x_1 + \ldots + a_{ii}x_i = a_{i1}^2 + \ldots + a_{ii}^2,$$

$$a_{i1}x_1 + \ldots + a_{ii}x_i = -k_1(a_{i1}^2 + \ldots + a_{ii}^2),$$

and we have

$$V' = \prod_{i=1}^{n} \frac{(1 + k_i)(a_{i1}^2 + \ldots + a_{ii}^2)}{|a_{ii}|}.$$

The poles P_i, P_i' of P_i, P_i' with respect to Σ have coordinates

$$\left(\frac{a_{i1}}{a_{i1}^2 + \ldots + a_{ii}^2}, \ldots, \frac{a_{ii}}{a_{i1}^2 + \ldots + a_{ii}^2}, 0, \ldots, 0\right)$$

and

$$\left(\frac{-a_{i1}}{k_i(a_{i1}^2 + \ldots + a_{ii}^2)}, \ldots, \frac{-a_{ii}}{k_i(a_{i1}^2 + \ldots + a_{ii}^2)}, 0, \ldots, 0\right)$$

respectively. So we have

$$V' = \frac{1}{n!} \prod_{i=1}^{n} \frac{(1 + \frac{1}{k_i})|a_{ii}|}{(a_{i1}^2 + \ldots + a_{ii}^2)}$$

and therefore

$$VV' = \frac{1}{n!} \prod_{i=1}^{n} \frac{(1 + k_i)^2}{k_i} \geqslant \frac{1}{n!} 4^n$$

with equality iff $k_1 = k_2 = \ldots = k_n = 1$.

R. P. Bambah [3].

9. Other Inequalities in E^n $(n \geqslant 2)$

Let $\mathcal{P} = \{A_1, A_2, \ldots, A_m\}$ be a set of m points in E^n. For any i, j $\in$ $\{1, 2, \ldots, m\}$ $(i \neq j)$ let $a_{ij} = A_iA_j$. Let O and R be the centre and the radius of the minimal hypersphere Σ containing $\mathcal{P}$. For any point X let $\underline{X} = \overrightarrow{OX}$. Let G be the centroid of $\mathcal{P}$ and for any $i \in \{1, 2, \ldots, m\}$ let G_i be the centroid of the set $\mathcal{P} \setminus \{A_i\}$.

In some inequalities the cyclic order of the elements of $\mathcal{P}$ is given, i.e. $\mathcal{P}$ is an m-gon in E^n. In this case all indices are taken modulo m from the set $\{1, 2, \ldots, m\}$ and the sign $\{\mathcal{P} = \mathcal{P}_R\}$ respectively $\{\mathcal{P} = \mathcal{P}_{AR}\}$ means that equality holds iff $\mathcal{P}$ is a regular respectively an affine-regular plane m-gon.

For any points $A_1, A_2, \ldots, A_m$ in E^n let

$$R(A_1, A_2, \ldots, A_m) = (-1)^m \begin{vmatrix} 0 & 1 & 1 & \ldots & 1 & 1 \\ 1 & 0 & a_{12}^2 & \ldots & a_{1,m-1}^2 & a_{1m}^2 \\ 1 & a_{12}^2 & 0 & \ldots & a_{2,m-1}^2 & a_{2m}^2 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & a_{1,m-1}^2 & a_{2,m-1}^2 & \ldots & 0 & a_{m-1,m}^2 \\ 1 & a_{1m}^2 & a_{2m}^2 & \ldots & a_{m-1,m}^2 & 0 \end{vmatrix} .$$

Let $A_1, A_2, \ldots, A_m$ be the centres of m hyperspheres $\Sigma_1, \Sigma_2, \ldots, \Sigma_m$ with the radii $r_1, r_2, \dots, r_m$. For any $i, j \in \{1, 2, \dots, m\}$ let $p_{ij} = a_{ij}^2 - r_i^2 - r_j^2$ be the mutual potency of two hyperspheres Σ_i and Σ_j and let $t_{ij}^2 = a_{ij}^2 - (r_i \mp r_j)^2$ be the square of the length of the common tangent of two oriented hyperspheres Σ_i and Σ_j, where we take the upper or the lower sign if the orientations of Σ_i and Σ_j are equal or opposite, respectively. Let

$$P(\Sigma_1, \Sigma_2, \dots, \Sigma_m) = (-1)^{m+1} \det(p_{ij}) \quad (i, j = 1, 2, \dots, m),$$

$$T(\Sigma_1, \Sigma_2, \dots, \Sigma_m) = (-1)^m \det(t_{ij}^2) \quad (i, j = 1, 2, \dots, m).$$

9.1. Let the points $A_1, A_2, \dots, A_m$ be collinear (in this order) and let $a_1, a_2, \dots, a_m \in R^+$. If there is $k \in \{1, 2, \dots, m\}$ such that

$$\sum_{i=1}^{k-1} a_i < \frac{1}{2} \sum_{i=1}^{m} a_i, \qquad \sum_{i=1}^{k} a_i > \frac{1}{2} \sum_{i=1}^{m} a_i,$$

then let $P = A_k$, and if there is $k \in \{1, 2, \dots, m\}$ such that

$$\sum_{i=1}^{k-1} a_i < \frac{1}{2} \sum_{i=1}^{m} a_i, \qquad \sum_{i=1}^{k} a_i = \frac{1}{2} \sum_{i=1}^{m} a_i,$$

then let P be any point of the segment $\overline{A_k A_{k+1}}$. For any point Q of the space we have

$$\sum_{i=1}^{m} a_i \cdot A_i Q \geqslant \sum_{i=1}^{m} a_i \cdot A_i P \quad \{Q = P\}.$$

A. Engel [22]; H. J. Claus [13]; A. Fricke [47].

9.2. $$\sum_{i=1}^{m} a_{i,i+2}^2 \leqslant 4 \cos^2 \frac{\pi}{m} \cdot \sum_{i=1}^{m} a_{i,i+1}^2 \quad (m \geqslant 4) \quad \{P = P_{AR}\}.$$

L. Gerber [49]; J. C. Fisher, D. Ruoff, and J. Shiletto [40].

9.3 If $a_{12} = a_{23} = \dots = a_{m1} = a$, $m \geqslant 4$, $k \in \{2, \dots, m-2\}$ and $p \in [-\infty, 2]$, then

$$M_m^{[p]}(a_{i,i+k}) \leqslant \frac{a}{\sin \frac{\pi}{m}} \sin \frac{\pi k}{m} \quad \{P = P_R\}.$$

R. Alexander [2] ($p = 1$; m even, $k = m/2$); L. Yang and J. Zh. Zhang [122] ($p = 2$).

Remark. The inequality 9.3 for p = 2 implies the same inequality for any $p \in [-\infty, 2]$.

9.4. With the same hypothesis as in 9.3 we have

$$\sum_{\substack{i,j=1\\ i<j}}^{m} a_{ij} \leq \frac{ma}{2\sin\frac{\pi}{m}} \sum_{k=1}^{m-1} \sin\frac{\pi k}{m} \quad \{P = P_R\}.$$

Proof. Follows from 9.3 (with p = 1).
R. Alexander [2].

9.5. Let the closed polygonal curve

$$C \ldots \underline{r} = \underline{r}(s)$$

be parametrized by arc length $s \in [0, L]$. There is a plane closed polygonal curve

$$C' \ldots \underline{q} = \underline{q}(s)$$

parametrized by the same arc length so that for every $s_1, s_2 \in [0, L]$

$$|\underline{r}(s_1) - \underline{r}(s_2)| \leq |\underline{q}(s_1) - \underline{q}(s_2)|.$$

Moreover, if C is not plane and convex and $\underline{r}(s_1)$, $\underline{r}(s_2)$ are not on the same edge of C, then

$$|\underline{r}(s_1) - \underline{r}(s_2)| < |\underline{q}(s_1) - \underline{q}(s_2)|.$$

G. T. Sallee [96].

9.6. $$\sum_{i=1}^{m} A_iG \leq m\sqrt{R^2 - OG^2} \quad \{A_1G_1 = A_2G_2 = \ldots = A_mG_m\}.$$

G. Kalajdžić [63] (n = 3); XVIII.2.26 (m = n + 1).

9.7. $$\sum_{\substack{i,j=1\\ i<j}}^{m} a_{ij}^2 \leq m^2R^2 \quad \{P \subset \Sigma,\ G = O\}.$$

Proof. We have

$$\sum_{\substack{i,j=1\\ i<j}}^{m} a_{ij}^2 = \frac{1}{2}\sum_{i,j=1}^{m} (\underline{A}_i - \underline{A}_j)^2 \leq \sum_{i,j=1}^{m} (R^2 - \underline{A}_i\underline{A}_j) =$$

$$= m^2R^2 - \left(\sum_{i=1}^{m} \underline{A}_i\right)^2 \leqslant m^2R^2$$

with equality iff

$$|\underline{A}_i| = R \quad (i = 1, 2, \ldots, m), \qquad \sum_{i=1}^{m} \underline{A}_i = \vec{0}.$$

V. Devidé [17]; E. Hille [60]; J. H. McKay [83] (Problem A-4); G. D. Chakerian and M. S. Klamkin [11]; V. Prasolov [93].

9.8. $$\sum_{\substack{i,j=1\\ i<j}}^{m} a_{ij} \leqslant \sqrt{\frac{m(m-1)}{2}}\, mR.$$

Equality holds iff $m \leqslant n + 1$ and $A_1A_2 \ldots A_m$ is a regular simplex in E^{m+1} inscribed in a hypersphere with the radius R.

G. D. Chakerian and M. S. Klamkin [11].

9.9. If $A_1, A_2, \ldots, A_m$ are on the hypersphere Σ with the radius R and $a_1, a_2, \ldots, a_m \in R^+$ such that

(1) $$\sum_{i=1}^{m} a_i\underline{A}_i = \vec{0},$$

then for any point P

$$\sum_{i=1}^{m} a_i \cdot A_iP \geqslant R \cdot \sum_{i=1}^{m} a_i \qquad \{P = O\}.$$

<u>Proof</u>. Let R = 1, $P \neq O$. By symmetry

$$A_iP = |\underline{A}_i - \underline{P}| = |p\underline{A}_i - \frac{1}{p}\underline{P}|,$$

where $p = OP \neq 0$. Therefore

$$\sum_{i=1}^{m} a_i \cdot A_iP = \sum_{i=1}^{m} a_i|p\underline{A}_i - \frac{1}{p}\underline{P}| \geqslant$$

$$\geqslant \left|\sum_{i=1}^{m} (pa_i\underline{A}_i - \frac{a_i}{p}\underline{P})\right| = \left|\sum_{i=1}^{m} \frac{a_i}{p}\underline{P})\right| = \sum_{i=1}^{m} a_i.$$

M. S. Klamkin [66].

9.10. If $\mathcal{P}$ is on Σ with the radius R and if the convex closure of $\mathcal{P}$

contains O, then

$$\left|\sum_{i=1}^{m} \underline{A}_i\right| \leqslant (m - 2)R$$

with equality iff there are $k, l \in \{1, 2, \ldots, m\}$ $(k < l)$ such that $\underline{A}_k + \underline{A}_l = \vec{0}$ and $A_1 = \ldots = A_{k-1} = A_{k+1} = \ldots = A_{l-1} = A_{l+1} = \ldots = A_m$.

Proof. Let $R = 1$. There are numbers $a_1, a_2, \ldots, a_m$ such that $a_i \geqslant 0$ $(i = 1, 2, \ldots, m)$, $a_1 + a_2 + \ldots + a_m = 1$ and such that the equality (1) holds. Assume that $a_1 > 1/2$. Then

$$\frac{1}{2} > 1 - a_1 = \sum_{i=2}^{m} a_i \geqslant \left|\sum_{i=2}^{m} a_i\underline{A}_i\right| = |-a_1\underline{A}_1| = a_1 > \frac{1}{2},$$

a contradiction. Therefore $a_1 \leqslant 1/2$ and $a_1 = 1/2$ iff

$$\sum_{i=2}^{m} a_i = \left|\sum_{i=2}^{m} a_i\underline{A}_i\right|,$$

i.e., because of (1), iff $\underline{A}_k = -\underline{A}_1$ for every $k \in \{2, \ldots, m\}$ for which $a_k \neq 0$. We have analogous considerations for $a_2, \ldots, a_m$. Therefore, there are only three possibilities:

1° there are $k, l \in \{1, 2, \ldots, m\}$ $(k < l)$ such that $a_k = a_l = 1/2$, $a_1 = \ldots = a_{k-1} = a_{k+1} = \ldots = a_{l-1} = a_{l+1} = \ldots = a_m = 0$, $\underline{A}_k + \underline{A}_l = \vec{0}$;

2° there is $k \in \{1, 2, \ldots, m\}$ such that $a_k = 1/2$; $a_2, \ldots, a_m < 1/2$ and $\underline{A}_l = -\underline{A}_k$ for every $l \in \{1, 2, \ldots, m\} \setminus \{k\}$ for which $a_l \neq 0$;

3° $a_1, a_2, \ldots, a_m < 1/2$.

Finally

$$\left|\sum_{i=1}^{m} \underline{A}_i\right| = \left|\sum_{i=1}^{m} (1 - 2a_i)\underline{A}_i\right| \leqslant \sum_{i=1}^{m} (1 - 2a_i) = m - 2.$$

Equality holds in the case 1° iff $\underline{A}_1 = \ldots = \underline{A}_{k-1} = \underline{A}_{k+1} = \ldots = \underline{A}_{l-1} = \underline{A}_{l+1} = \ldots = \underline{A}_m$, and in the case 2° iff $\underline{A}_1 = \ldots = \underline{A}_{k-1} = \underline{A}_{k+1} = \ldots = \underline{A}_m$. In the case 3° equality cannot hold, because of the equalities $\underline{A}_1 = \underline{A}_2 = \ldots = \underline{A}_m$ are impossible according to (1).

M. S. Klamkin and D. J. Newman [69].

Remark. Inequality 9.10 can be extended for the case of radius vectors of different lengths, i.e. we have

9.10' $$\left|\sum_{i=1}^{m} \underline{A}_i\right| \leqslant \sum_{i=1}^{m} |\underline{A}_i| - 2 \min_i |\underline{A}_i|.$$

M. S. Klamkin and D. J. Newman [69].

9.11. With the same hypothesis as in 9.10 we have

$$\sum_{\substack{i,j=1\\ i<j}}^{m} a_{ij}^2 \geqslant 4(m-1)R^2$$

with the same conditions for equality as in 9.10.

Proof. From the proof of 9.7 we have the equality

$$\sum_{\substack{i,j=1\\ i<j}}^{m} a_{ij}^2 = m^2R^2 - \left(\sum_{i=1}^{m} \underline{A}_i\right)^2$$

and 9.11 is a consequence of 9.10.

G. D. Chakerian and M. S. Klamkin [11].

9.12. With the same hypothesis as in 9.10 we have

$$\sum_{\substack{i,j=1\\ i<j}}^{m} a_{ij} \geqslant 2(m-1)R$$

with equality iff there is $k \in \{1, 2, \ldots, m\}$ such that $\underline{A}_1 = \ldots = \underline{A}_{k-1} = -\underline{A}_k = \underline{A}_{k+1} = \ldots = \underline{A}_m$.

Proof. Follows from 9.11 because of $a_{ij} \leqslant 2R$ $(i, j = 1, 2, \ldots, m;$ $i < j)$ and equality holds iff $a_{ij} \neq 0$ implies $a_{ij} = 2R$.

G. D. Chakerian and M. S. Klamkin [11]; D. Wolfe [120].

9.13. If $\mathcal{P}$ is on Σ and if P is any point on Σ, then

$$2mR(R - OG) \leqslant \sum_{i=1}^{m} A_iP^2 \leqslant 2mR(R + OG)$$

with equality iff $P = P_1$ respectively $P = P_2$, where P_1 and P_2 are the intersection points of the line OG and the hypersphere Σ (the points G and P_1 on the same side of O).

Proof. Follows by $-1 \leqslant \cos \measuredangle POG \leqslant 1$ from the equality

$$\sum_{i=1}^{m} A_iP^2 = \sum_{i=1}^{m} (\underline{A}_i - \underline{P})^2 = 2mR^2 - 2\underline{P} \cdot \sum_{i=1}^{m} \underline{A}_i =$$

$$= 2mR^2 - 2m\underline{P} \cdot \underline{G} = 2mR^2 - 2mR \cdot OG \cdot \cos \measuredangle POG.$$

T. P. Grigorjeva [54] ($n = m = 3$).

9.14. If the extensions of A_iG ($i = 1, 2, \ldots, m$) intersect Σ at A_i', then

$$\sum_{i=1}^{m} GA_i' \geq \sum_{i=1}^{m} GA_i \quad \{A_1G_1 = A_2G_2 = \ldots = A_mG_m\}.$$

Proof. Same as in XVIII.2.57 (with $\alpha = 1$ and with m instead of $n + 1$).

P. Boente [7].

9.15. If $m = n + 1$, then

$$R(A_1, A_2, \ldots, A_{n+1}) \geq 0$$

with equality iff the points $A_1, A_2, \ldots, A_{n+1}$ lie on a hyperplane.

J. Haantjes [56]; O. Bottema [8].

Remark 1. A set of N points $A_1, A_2, \ldots, A_N$ of a metric space is congruent to a set of N points of E^n if for any $m \in \{2, \ldots, N\}$ and any subset $\{i_1, i_2, \ldots, i_m\}$ of the set $\mathbb{N}_N = \{1, 2, \ldots, N\}$ we have $R(A_{i_1}, A_{i_2}, \ldots, A_{i_m}) \geq 0$ and if $n + 1$ is the maximal value of m for which there is a subset $\{i_1, i_2, \ldots, i_m\} \subseteq \mathbb{N}_N$ such that $R(A_{i_1}, A_{i_2}, \ldots, A_{i_m}) > 0$.

J. Haantjes [56].

Remark 2. If $A_1A_2 \ldots A_{n+1}$ is a simplex in E^n and if V is its volume, then

$$|R(A_1, A_2, \ldots, A_{n+1})| = 2^n(n!)^2V^2.$$

Remark 3. For an elliptic respectively for a hyperbolic space we have the analogous facts as in 9.15 and in Remark 1, but we must replace $R(A_1, A_2, \ldots, A_m)$ by $\det(\cosh a_{ij})$ respectively by $\det(\cosh a_{ij})$ ($i, j = 1, 2, \ldots, m$).

J. Haantjes [56].

9.16. If $m = n + 2$, then

$$(-1)^{n+2}\det(a_{ij}^2) \leq 0 \quad (i, j = 1, 2, \ldots, n + 2)$$

with equality iff the points $A_1, A_2, \ldots, A_{n+2}$ lie on a hypersphere.

O. Bottema [8].

Remark. For $n = 2$, we obtain the Möbius-Neuberg inequality.

9.17. Let $m \geqslant n + 1$ and $a_1, a_2, \ldots, a_m \in R^+$. Denote by $V_{i_1 \ldots i_{k+1}}$ the k-dimensional content of the k-simplex $A_{i_1} \ldots A_{i_{k+1}}$ for every $k \in \{1, 2, \ldots, n\}$ and every $i_1, \ldots, i_{k+1} \in \{1, 2, \ldots, m\}$ such that $1 \leqslant i_1 < \ldots < i_{k+1} \leqslant m$. Put

$$M_k = \sum_{\substack{i_1, \ldots, i_{k+1}=1 \\ i_1 < \ldots < i_{k+1}}}^{m} a_{i_1} \ldots a_{i_{k+1}} V^2_{i_1 \ldots i_{k+1}} \quad (k \in \{1, 2, \ldots, m\}).$$

If $M_o = a_1 + a_2 + \ldots + a_m$ and $k, l \in \{1, 2, \ldots, n\}$ $(k < l)$, then

$$\frac{M_k^l}{M_l^k} \geqslant \frac{[(n - l)!(l!)^3]^k}{[(n - k)!(k!)^3]^l} (n! M_o)^{l-k}$$

with equality iff the inertial ellipsoid of the points $A_1, A_2, \ldots, A_m$ with masses $a_1, a_2, \ldots, a_m$ is a hypersphere.

Proof. Let G be the centroid of the points $A_1, A_2, \ldots, A_m$ with masses $a_1, a_2, \ldots, a_m$, respectively, and let G be the origin. Let H be any hyperplane with the unit normal vector $\underline{n}$. The moment of inertia of the points $A_1, A_2, \ldots, A_m$ with respect to H is given by

$$I = \sum_{i=1}^{m} a_i (\underline{n} \cdot \underline{A}_i)^2. \tag{2}$$

Define a vector $\underline{x} = (x_1, x_2, \ldots, x_n)$ by

$$\underline{x} = \frac{\underline{n}}{\sqrt{I}}, \quad \text{i.e. } \underline{n} = \frac{\underline{x}}{|\underline{x}|}.$$

Let $\underline{A}_i = (a_{i1}, a_{i2}, \ldots, a_{in})$ $(i = 1, 2, \ldots, m)$ and define an $n \times m$ matrix $Q = (q_{ij})$ with $q_{ij} = \sqrt{a_j} a_{ji}$ $(i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, m)$, i.e. the j-th column of Q is the vector $\sqrt{a_j}\underline{A}_j$. Since

$$\underline{n}Q = (\sqrt{a_1}\underline{n} \cdot \underline{A}_1, \sqrt{a_2}\underline{n} \cdot \underline{A}_2, \ldots, \sqrt{a_m}\underline{n} \cdot \underline{A}_m),$$

(2) can be rewritten as

$$\underline{n}QQ^T\underline{n}^T = I = \frac{1}{|\underline{x}|^2}, \tag{3}$$

where Q^T and $\underline{n}^T$ are the transposes of matrices Q and $\underline{n}$. Therefore, $\underline{x} = |\underline{x}|\underline{n}$ and (3) imply

$$(4) \qquad \underline{x}QQ^T\underline{x}^T = 1.$$

Note that QQ^T is an $n \times n$ symmetric, positive semi-definite matrix (Gram matrix of the row-vectors of Q). Hence it diagonalizes in an orthonormal basis (principal axes of inertia) and has non-negative eigenvalues. Therefore, in the non-degenerate case, (4) represents the inertial ellipsoid of $A_1, A_2, \ldots, A_m$ (in general it is an 'ellipsoid cylinder'). Now, $P = Q^TQ$ has the same non-zero eigenvalues as QQ^T, since

$$(5) \qquad \det(Q^TQ - \lambda I_m) = (-\lambda)^{m-n}\det(QQ^T - \lambda I_n).$$

We have

$$P(\lambda) = \det(Q^TQ - \lambda I_m) = $$

$$= \det(a_i a_j \underline{A}_i \underline{A}_j - \delta_{ij}\lambda) = \frac{1}{M_0} \cdot \begin{vmatrix} M_0 & 0 \ldots\ldots\ldots\ldots 0 \\ \sqrt{a_1} & \\ \vdots & \sqrt{a_i a_j}\underline{A}_i\underline{A}_j - \delta_{ij}\lambda \\ \sqrt{a_m} & \end{vmatrix}$$

where $i, j = 1, 2, \ldots, m$. By adding the $-\sqrt{a_i}$ times of the i-th row ($i = 1, 2, \ldots, m$) to the first row and using the fact that $a_1\underline{A}_1 + a_2\underline{A}_2 + \ldots + a_m\underline{A}_m = \vec{0}$, we get

$$P(\lambda) = \frac{\lambda}{M_0} \cdot \begin{vmatrix} 0 & \sqrt{a_1} \ldots\ldots\ldots \sqrt{a_m} \\ \sqrt{a_1} & \\ \vdots & \sqrt{a_i a_j}\underline{A}_i\underline{A}_j - \delta_{ij}\lambda \\ \sqrt{a_m} & \end{vmatrix} \qquad (i, j = 1, 2, \ldots, m).$$

By multiplying the first row (respectively column) by $-\sqrt{a_i}\underline{A}_i^2/2$ (respectively by $-\sqrt{a_j}\underline{A}_j^2/2$) and adding it to the i-th row (respectively to the j-th column), for all $i, j = 1, 2, \ldots, m$, we obtain

$$P(\lambda) = \frac{\lambda}{M_0} \cdot \begin{vmatrix} 0 & \sqrt{a_1} \ldots\ldots\ldots\ldots \sqrt{a_m} \\ \sqrt{a_1} & \\ \vdots & -\frac{\sqrt{a_i a_j}}{2} a_{ij}^2 - \delta_{ij}\lambda \\ \sqrt{a_m} & \end{vmatrix} \qquad (i, j = 1, 2, \ldots, m),$$

since $(\underline{A}_i - \underline{A}_j)^2 = a_{ij}^2$ (with $a_{ii} = 0$). By the well-known formula

$$V_{i_1 \ldots i_{k+1}}^2 = \frac{(-1)^{k+1}}{2^k (k!)^2} R(A_{i_1}, A_{i_2}, \ldots, A_{i_{k+1}})$$

we obtain

$$\begin{vmatrix} 0 & \sqrt{a_{i_1}} \; \cdots \; \sqrt{a_{i_{k+1}}} \\ \sqrt{a_{i_1}} & \\ \vdots & -\frac{1}{2}\sqrt{a_{i_p} a_{i_q}}\, a_{i_p i_q}^2 \\ \sqrt{a_{i_{k+1}}} & \end{vmatrix} = -a_{i_1} \cdots a_{i_{k+1}} V_{i_1 \ldots i_{k+1}}^2 (k!)^2,$$

where $p, q = 1, 2, \ldots, k + 1$. Therefore, from (5), the equation $P(\lambda) = 0$ is equivalent to

$$\left(\sum_{k=0}^{n} (-1)^k (k!)^2 M_k \lambda^{n-k}\right)\lambda^{m-n} = 0.$$

Suppose that QQ^T is positive. Then $P(\lambda) = 0$ has (in view of (5)) n positive roots $\lambda_1, \lambda_2, \ldots, \lambda_n$. Hence they satisfy

$$\sum_{k=0}^{n} (-1)^k (k!)^2 M_k \lambda^{n-k} = 0. \tag{6}$$

Let e_k ($k \in \{1, 2, \ldots, n\}$) be the k-th elementary symmetric function of $\lambda_1, \lambda_2, \ldots, \lambda_n$. Then (6) implies

$$e_k = (k!)^2 \frac{M_k}{M_o} \quad (k = 1, 2, \ldots, n). \tag{7}$$

From AI 2.15.1 (Theorem 4) we get for any $k, l \in \{1, 2, \ldots, n\}$ $(k < l)$

$$[\binom{n}{k}^{-1} e_k]^l \geq [\binom{n}{l}^{-1} e_l]^k,$$

i.e. the inequality 9.17, because of (7). Equality holds iff $\lambda_1 = \lambda_2 = \ldots = \lambda_n$, i.e. iff the inertial ellipsoid is a hypersphere.

J. Zh. Zhang and L. Yang [124]; L. Yang and J. Zh. Zhang [121] ($a_1 = a_2 = \ldots = a_m = 1$); XVIII.3.11 ($m = n + 1$; $a_1 = a_2 = \ldots = a_{n+1} = 1$).

9.18. With the same hypothesis as in 9.17 and with $k \in \{1, 2, \ldots, m\}$ we have

$$M_k^2 \geqslant \left(\frac{k+1}{k}\right)^3 \frac{n-k+1}{n-k} M_{k-1}M_{k+1}$$

with the same condition for equality as in 9.17.

Proof. From AI 2.15.1 (Theorem 2) we obtain for $k \in \{1, 2, \ldots, m\}$

$$\left[\binom{n}{k}^{-1} e_k\right]^2 \geqslant \left[\binom{n}{k-1} e_{k-1}\right]\left[\binom{n}{k+1} e_{k+1}\right].$$

Therefore, (7) implies 9.18.

J. Zh. Zhang and L. Yang [124]; L. Yang and J. Zh. Zhang [121] ($a_1 = a_2 = \ldots = a_m = 1$); XVIII.3.12 ($m = n + 1$; $a_1 = a_2 = \ldots = a_{n+1} = 1$).

9.19. For any $k \in \mathbb{N}$ let

(8) $$N_k = \frac{2}{m(m-1)} \sum_{\substack{i,j=1\\ i<j}}^{m} a_{ij}^k,$$

(9) $$L = \frac{2}{m(m-1)} \sum_{\substack{i,j=1\\ i<j}}^{m} (g_i^2 - g_j^2),$$

where $g_i = GA_i$ $(i = 1, 2, \ldots, m)$. Then

$$N_4 \geqslant \frac{m-1}{m}\,\frac{n+1}{n} N_2^2 + L$$

with equality iff the inertial ellipsoid of P is a hypersphere.

Proof. By a straight-forward calculation we can prove the identity

(10) $$4\left(\sum_{\substack{i,j=1\\ i<j}}^{m} a_{ij}^2\right)^2 = m^2 \sum_{\substack{i,j=1\\ i<j}}^{m} a_{ij}^4 - \sum_{\substack{j,k=1\\ j<k}}^{m} \left[\sum_{i=1}^{m} (a_{ij}^2 - a_{ik}^2)\right]^2 +$$

$$+ m \sum_{\substack{i,j,k=1\\ i<j<k}}^{m} (2a_{ij}^2 a_{ik}^2 + 2a_{ij}^2 a_{jk}^2 + 2a_{ik}^2 a_{jk}^2 -$$

$$- a_{ij}^4 - a_{ik}^4 - a_{jk}^4).$$

Let $\underline{g}_i = \overrightarrow{GA_i}$ $(i = 1, 2, \ldots, m)$. The equality $\underline{g}_1 + \underline{g}_2 + \ldots + \underline{g}_m = \vec{0}$ implies

$$\sum_{i=1}^{m} (a_{ij}^2 - a_{ik}^2) = \sum_{i=1}^{m} [(\underline{g}_i - \underline{g}_j)^2 - (\underline{g}_i - \underline{g}_k)^2] =$$

$$= m(g_j^2 - g_k^2) - 2\left(\sum_{i=1}^{m} \underline{g}_i\right)(\underline{g}_j - \underline{g}_k) = m(g_j^2 - g_k^2)$$

and by (9) we obtain

(11) $$\sum_{\substack{j,k=1\\ j<k}}^{m} \left[\sum_{i=1}^{m} (a_{ij}^2 - a_{ik}^2)\right]^2 = \frac{1}{2} m^3(m - 1)L.$$

With the notation from 9.17 let $a_1 = a_2 = \ldots = a_m = 1$. Then $M_o = m$,

(12) $$M_1 = \sum_{\substack{i,j=1\\ i<j}}^{m} a_{ij}^2 = \frac{1}{2} m(m - 1)N_2,$$

(13) $$16M_2 = \sum_{\substack{i,j,k=1\\ i<j<k}}^{m} (2a_{ij}^2 a_{ik}^2 + 2a_{ij}^2 a_{jk}^2 + 2a_{ik}^2 a_{jk}^2 - a_{ij}^4 - a_{ik}^4 - a_{jk}^4)$$

and the inequality 9.18 (with k = 1) has the form

$$M_1^2 \geqslant \frac{8n}{n - 1} M_o M_2,$$

i.e. according to (12):

(14) $$m^2(m - 1)^2 \frac{n - 1}{2n} N_2^2 \geqslant 16mM_2.$$

Applying (8) (with k = 2 and k = 4), (11) and (13), identity (10) can be written in the form

(15) $$m^2(m - 1)^2 N_2^2 = \frac{1}{2} m^3(m - 1)N_4 - \frac{1}{2} m^3(m - 1)L + 16mM_2.$$

If we eliminate $16mM_2$ from (14) and (15) we obtain 9.19.

L. Yang and J. Zh. Zhang [123].

Remark. From 9.19 (with L = O) we obtain

9.19' $$N_4 \geqslant \frac{m - 1}{m} \frac{n + 1}{n} N_2^2$$

with equality iff all points of P lie on a hypersphere with the centre G and the inertial ellipsoid of P is a hypersphere.

L. Yang and J. Zh. Zhang [123].

9.20. If m = n + 2, then

$$P(\Sigma_1, \Sigma_2, \ldots, \Sigma_{n+2}) \geqslant 0$$

with equality iff there is a hypersphere orthogonal to the hyperspheres $\Sigma_1, \Sigma_2, \ldots, \Sigma_{n+2}$.

O. Bottema [8].

Remark. If $r_1 = r_2 = \ldots = r_{n+2} = 0$, then 9.20 implies 9.16.

9.21. If m = n + 1, then

$$(-1)^{n+1} \cdot \begin{vmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & p_{11} & p_{12} & \cdots & p_{1,n+1} \\ 1 & p_{21} & p_{22} & \cdots & p_{2,n+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & p_{n+1,1} & p_{n+1,2} & \cdots & p_{n+1,n+1} \end{vmatrix} \geqslant 0$$

with equality iff the points $A_1, A_2, \ldots, A_{n+1}$ lie on a hyperplane.

O. Bottema [8].

Remark. If $r_1 = r_2 = \ldots = r_{n+1} = 0$, then 9.21 implies 9.15.

9.22. Let $m \in \{2, \ldots, n\}$. The intersection of given linearly independent hyperspheres $\Sigma_1, \Sigma_2, \ldots, \Sigma_m$ is an (n - m + 1)-sphere or a point or empty iff we have the first, the second or the third sign in

$$P(\Sigma_1, \Sigma_2, \ldots, \Sigma_m) \lesseqgtr 0.$$

O. Bottema [8].

9.23. Let m = n + 1. The common orthogonal hypersphere of given linearly independent hyperspheres $\Sigma_1, \Sigma_2, \ldots, \Sigma_{n+1}$ has positive radius, zero radius or imaginary radius iff we have the first, the second or the third sign in

$$P(\Sigma_1, \Sigma_2, \ldots, \Sigma_{n+1}) \gtreqless 0.$$

O. Bottema [8].

9.24. Let $m \in \{3, \ldots, n + 2\}$. The given linearly independent oriented hyperspheres $\Sigma_1, \Sigma_2, \ldots, \Sigma_m$ define an elliptic, a special or a hyperbolic linear (m - 2)-complex of hyperspheres iff we have the first, the second or the third sign in

$$T(\Sigma_1, \Sigma_2, \ldots, \Sigma_m) \gtreqless 0$$

O. Bottema [8].

9.25. Let m = n + 3. For the given oriented hyperspheres $\Sigma_1, \Sigma_2, \ldots, \Sigma_{n+3}$ we have

$$T(\Sigma_1, \Sigma_2, \ldots, \Sigma_{n+3}) \geqslant 0$$

with equality iff these hyperspheres are linearly dependent.
O. Bottema [8].
Remark. For $m > n + 3$ we have always $T(\Sigma_1, \Sigma_2, \ldots, \Sigma_m) = 0$.

References.

1. O. Aberth, 'An Isoperimetric Inequality for Polyhedra and Its Application to an Extremal Problem', Proc. London Math. Soc. (3) 13 (1963), 322-336.
2. R. Alexander, 'Metric Embedding Techniques Applied to Geometric Inequalities', Lecture Notes 490 (1975), 57-65.
3. R. P. Bambah, 'Polar Reciprocal Convex Bodies', Proc. Cambridge Phil. Soc. 51 (1955), 377-378.
4. J. D. Berman and K. Hanes, 'Volumes of Polyhedra Inscribed in the Unit Sphere in E^3', Math. Ann. 188 (1970), 78-84.
5. A. S. Besicovitch, 'A Crate to Hold a Unit-Sphere', Proc. Sympos. Pure Math. VII, Amer. Math. Soc. Providence, R. I., 1963, 19-20.
6. A. S. Besicovitch and H. G. Eggleston, 'The Total Length of the Edges of a Polyhedron', Quart. J. Math. (2) 8 (1957), 172-190.
7. P. Boente, 'Solution of Problem E 2959', Amer. Math. Monthly 92 (1985), 360-361.
8. O. Bottema, 'Inequalities in the Geometries of Spheres and of Lines', Proc. Koninkl. Nederl. Akad. Wetensch. A 59 (1956), 523-531.
9. L. E. Bush, 'The William Lowell Putnam Mathematical Competition', Amer. Math. Monthly 57 (1950), 467-470 (Problem I-4(a)).
10. E. Catalan, J. Éc. Polytechn. 24 (1865), 1-71.
11. G. D. Chakerian and M. S. Klamkin, 'Inequalities for Sums of Distances', Amer. Math. Monthly 80 (1973), 1009-1017.
12. C. Ciotloş, 'Problema 17881', Gaz. Mat. 84 (1979), 323.
13. H. J. Claus, 'Extremwertaufgaben in metrischen Räumen', Mathematikunterricht 28 (1982), H. 5, 28-33.
14. H. T. Croft, 'Arrows on Polyhedra', J. London Math. Soc. 37 (1962), 287-300.
15. H. T. Croft, 'On Maximal Regular Polyhedra Inscribed in a Regular Polyhedron', Proc. London Math. Soc. (3) 41 (1980), 279-296.
16. M. Dedo, 'Risposta alla Questione 859', Period. Mat. (4) 45 (1967), 327-329.
17. V. Devidé, 'Zadatak 150', Glasnik Mat.-Fiz. Astr. 7 (1952), 190-191.
18. I. Dimovski, 'Zadača 4', Obzornik Mat. (Sofija) 1983, No. 2, 63-64.
19. B. Drachmann, 'En Saetning om det regulaere Oktaeder', Nyt Tidsskr. Mat. B 25 (1915), 82-83.

20. E. Ehrhart, 'Sections maximales du cube', Rev. Math. Spéc. 76 (1965-1966), 249-251.
21. E. Ehrhart and J. M. Faure, 'Enoncé no. 4', Bull.Assoc. Prof. Math. Ens. Public 51 (1972), 336-338.
22. A. Engel, 'Geometrical Activities for the Upper Elementary School', Educ. Studies Math. 3 (1971), 372-376.
23. L. Euler, 'Elementa doctrinae solidorum', Novi Comm. Acad. Sci. Imp. Petropolit. 4 (1752), 109-140.
24. L. Fejes Tóth, 'Egy gömbfelület befedése egybevágó gömbsüvegekkel', Mat. Fiz. Lapok 50 (1943), 40-46.
25. L. Fejes Tóth, 'Über einige Extremaleigenschaften der regulären Polyeder und des gleichseitigen Dreiecksgitters', Ann. Scu. Norm. Sup. Pisa (2) 13 (1948), 51-58.
26. L. Fejes Tóth, 'On Ellipsoids Circumscribed and Inscribed to Polyhedra', Acta Univ. Szeged 11 (1948), 225-228.
27. L. Fejes Tóth, 'On the Total Length of Edges of a Polyhedron', Norske Vid. Selsk. Forh. 21 (1948), No. 8, 32-34.
28. L. Fejes Tóth, 'The Isepiphan Problem for n-hedra', Amer. J. Math. 70 (1948), 174-180.
29. L. Fejes Tóth, 'An Inequality Concerning Polyhedra', Bull. Amer. Math. Soc. 54 (1948), 139-146.
30. L. Fejes Tóth, 'Inequalities Concerning Polygons and Polyhedra', Duke Math. J. 15 (1948), 817-822.
31. L. Fejes Tóth, 'Extremum Properties of the Regular Polyhedra', Canad. J. Math. 2 (1950), 22-31.
32. L. Fejes Tóth, 'Problem E 895', Amer. Math. Monthly 57 (1950), 419-420.
33. L. Fejes Tóth, Lagerungen in der Ebene, auf der Kugel und im Raum, Berlin-Göttingen-Heidelberg 1953.
34. L. Fejes Tóth, 'Extremum Properties of the Regular Polytopes', Acta Math. Acad. Sci. Hungar. 6 (1955), 143-146.
35. L. Fejes Tóth, 'Characterization of the Nine Regular Polyhedra by Extremum Properties', Ibid. 7 (1956), 31-48.
36. L. Fejes Tóth, 'Über eine Volumenabschätzung für Polyeder', Monatsh. Math. 64 (1960), 374-377.
37. L. Fejes Tóth, 'On the Total Area of the Faces of a Four-Dimensional Polytope', Canad. J. Math. 17 (1965), 93-99.
38. L. Fejes Tóth, Reguläre Figuren, Leipzig 1965.
39. L. Fejes Tóth, 'Some Researches Inspired by H. S. M. Coxeter', The Geom. Vein, New York-Heidelberg-Berlin 1981, 271-277.
40. J. C. Fisher, D. Ruoff, and J. Shilleto, 'Perpendicular Polygons', Amer. Math. Monthly 92 (1985), 23-37.
41. T. J. Fletcher, 'Doing without Calculus', Math. Gaz. 55 (1971), 4-17.
42. A. Florian, 'Eine Ungleichung über konvexe Polyeder', Monatsh. Math. 60 (1956), 130-156.
43. A. Florian, 'Ungleichungen über konvexe Polyeder', Ibid. 60 (1956), 288-297.
44. A. Florian, 'Eine Extremaleigenschaft der regulären Dreikantpolyeder', Ibid. 70 (1966), 309-314.
45. A. Florian, 'Neuere Entwicklungen über reguläre Polyeder', Contrib. to Geom. (Proc. Geom. Sympos., Siegen 1978), Basel 1979, 238-247.
46. A. Florian and J. Linhart, 'Kantenkrümmung und Inkugelradius konvexer Polyeder', to appear.

47. A. Fricke, 'Der Punkt kleinster gewichteter Entfernungssumme von gegebenen Punkten', Mathematikunterricht 30 (1984), H. 6, 22-37.
48. L. Gerber, 'The Parallelogram Inequality', Publ. Elektrotehn. Fak. Univ. Beograd. No. 461-497 (1974), 107-109.
49. L. Gerber, 'Napoleon's Theorem and the Parallelogram Inequality for Affine-Regular Polygons', Amer. Math. Monthly 87 (1980), 644-648.
50. M. Goldberg, 'The Isoperimetric Problem for Polyhedra', Tôhoku Math. J. 40 (1935), 226-236.
51. E. G. Gotman, 'O polnote rešenija geometričeskoj zadači', Mat. v škole 1970, No. 4, 44-48.
52. E. G. Gotman, 'Zadača 1725', Ibid. 1977, No. 2, 74-75.
53. E. G. Gotman, 'Neravenstva v geometričeskih zadačah', Ibid. 1985, No. 3, 46-48.
54. T. P. Grigorjeva, 'Zadača 2220', Ibid. 1980, No. 6, 61.
55. S. G. Guba, 'Zadača 1797', Ibid. 1977, No. 5, 80.
56. J. Haantjes, 'De stelling van Ptolemeus', Simon Stevin 29 (1952), 25-31.
57. W. Habicht and B. L. van der Waerden, 'Lagerung von Punkten auf der Kugel', Math. Ann. 123 (1951), 223-234.
58. H. Hadwiger, 'Zur isoperimetrischen Ungleichung für k-dimensionale konvexe Polyeder', Nagoya Math. J. 5 (1953), 39-44.
59. J. M. Hammersley, 'The Total Length of the Edges of a Polyhedron', Compos. Math. 9 (1951), 239-240.
60. E. Hille, 'Some Geometric Extremal Problems', J. Austral. Math. Soc. 6 (1966), 122-128.
61. H. Hornich, 'Eine Ungleichung für Vektorlängen', Math. Z. 48 (1942), 268-274.
62. C. Ionescu-Ţiu, 'Problema E:6115', Gaz. Mat. 83 (1978), 12, 291.
63. G. Kalajdžić, 'Some Inequalities Concerning a Tetrahedron', Publ. Elektrotehn. Fak. Univ. Beograd. No. 320-328 (1970), 51-58.
64. A. Kelarev, 'Rešenie zadači M 690', Kvant 1982, No. 2, 29-30.
65. A. Kiefer, 'Einige Aufgaben über extreme Werte', Vierteljahrsschrift naturforsch. Ges. Zürich 71 (1926), 262-268.
66. M. S. Klamkin, 'Vector Proofs in Solid Geometry', Amer. Math. Monthly 77 (1970), 1051-1065.
67. M. S. Klamkin, 'Duality in Spherical Triangles', Ford Motor Company Preprint.
68. M. S. Klamkin, 'Partial Convexity and Triangle Inequalities', Ibid.
69. M. S. Klamkin and D. J. Newman, 'An Inequality for the Sum of Unit Vectors', Publ. Elektrotehn. Fak. Univ. Beograd No. 338-352 (1971), 47-48.
70. M. Kömhoff, 'An Isoperimetric Inequality for Convex Polyhedra with Triangular Faces', Canad. Math. Bull. 11 (1968), 723-727.
71. M. Kömhoff, 'On a 3-Dimensional Isoperimetric Problem', Ibid. 13 (1970), 447-449.
72. A. M. Legendre, Éléments de Géométrie, 8. éd., Paris 1809.
73. Hr. Lesov, 'Zadača 3', Matematika (Sofija) 1982, No. 6, 37-38.
74. Hr. Lesov, 'Zadača 5', Obzornik Mat. (Sofija) 1983, No. 3, 61-62.
75. S. A. J. Lhuilier, De relatione mutua capatitatis et terminorum figurarum, geometrice considerata: seu de maximis et minimis, Varsaviae 1782.
76. L. Lindelöf, 'Propriétés générales des polyèdres qui, sous une étendue superficielle donnée, renferment le plus grand volume', Math. Ann. 2 (1870), 150-159.

77. J. Linhart, 'Kantenkrümmung und Umkugelradius konvexer Polyeder', Studia Sci. Math. Hungar 11 (1976), 457-458.
78. J. Linhart, 'Über die Kantenlängensumme von Dreieckspolyedern', Monatsh. Math. 83 (1977), 25-36.
79. J. Linhart, 'Kantenkrümmung und Umkugelradius konvexer Polyeder', Acta Math. Acad. Sci. Hungar. 34 (1979), 1-2.
80. J. Linhart, 'Über eine Ungleichung für die Oberfläche und den Umkugelradius eines konvexen Polyeders', Arch. Math. 39 (1982), 278-284.
81. B. Martić, 'Primedba na jednu stereometrisku nejednačinu M. Petrovića', Zbornik radova SAN (Mat. Inst.) 69 (8) (1960), 131-132.
82. D. Matei, 'Problema 19623', Gaz. Mat. 88 (1983), 125.
83. J. H. McKay, 'The William Lowell Putnam Mathematical Competition', Amer. Math. Monthly 76 (1969), 909-915.
84. H. Minkowski, 'Allgemeine Lehrsätze über die konvexen Polyeder', Nachr. Ges. Wiss. Göttingen 1897, 198-219; Ges. Abh. II, 103-121.
85. D. S. Mitrinović, P. M. Vasić, R. Ž. Djordjević, and R. R. Janić, Geometrijske nejednakosti, Beograd 1966.
86. J. Molnár, 'Inhaltsabschätzung eines sphärischen Polygons', Acta Math. Acad. Sci. Hungar. 3 (1952), 67-69.
87. Ju. Nesterenko, 'Zadača M 639', Kvant 1981, No. 5, 24.
88. C. S. Ogilvy and D. P. Robbins, 'Problem E 2349', Amer. Math. Monthly 83 (1976), 54-58.
89. A. Oppenheim, 'Some Inequalities for a Spherical Triangle and an Internal Point', Publ. Elektrotehn. Fak. Univ. Beograd. No. 200-209 (1967), 13-16.
90. M. Petrović, 'Stereometrijske nejednakosti', Zbornik radova SAN (Mat. Inst.) 35 (3) (1953), 1-4.
91. A. Pinciu, 'O inegalitate geometrică', Gaz. Mat. 91 (1986), 66-67.
92. V. S. Pokrovskij and R. P. Ušakov, 'Zadača 2856', Mat. v škole 1986, No. 2, 72.
93. V. Prasolov, 'Rešenie zadači M 742', Kvant 1982, No. 10, 32.
94. A. Procissi, 'Sul massimo volume di alcune specie di poliedri dei quali è data la somma degli spigoli', Period. Mat. (4) 12 (1932), 159-172.
95. E. Rouché and Ch. de Comberousse, Traité de géométrie II, 8. éd., Paris 1912.
96. G. T. Sallee, 'Stretching Chords of Space Curves', Geom. Ded. 2 (1973), 311-315.
97. G. Sansone, 'Su una proprietà di massimo dell'ottaedro regolare e del cubo', Boll. Un. Mat. Ital. (2) 3 (1940), 140-146.
98. N. Schaumberger, 'Problem 134', Two-Year Coll. Math. J. 10 (1979), 53 and 11 (1980), 212.
99. D. E. J. Schrek, 'Prince Rupert's Problem and Its Extension by Pieter Nieuwland', Scripta Math. 16 (1950), 73-80 and 261-267.
100. M. N. Šebaršin, 'Zadača 57', Mat. v škole 1954, No. 1, 89-90.
101. G. C. Shephard, 'A Sphere in a Crate', J. London Math. Soc. 40 (1965), 433-434.
102. D. O. Škljarskij, N. N. Čencov, and I. M. Jaglom, Geometričeskie neravenstva i zadači na maksimum i minimum, Moskva 1970.
103. J. Steiner, 'Einfache Beweise der isoperimetrischen Hauptsätze', J. reine angew. Math. 18 (1838), 281-296; Ges. Werke II, 75-91.

104. J. Steiner, 'Über Maximum und Minimum bei den Figuren in der Ebene, auf der Kugelfläche und im Raume überhaupt. II', Ibid. 24 (1842), 189-250; Ges. Werke II, 243-308.
105. E. Steinitz, 'Über die Eulerschen Polyederrelationen', Arch. Math. Phys. (3) 11 (1907), 86-88.
106. E. Steinitz, 'Polyeder und Raumeinteilungen', Encykl. math. Wiss. III.1.2, 1-139.
107. E. Steinitz, 'Über isoperimetrische Probleme bei konvexen Polyedern', J. reine angew. Math. 158 (1927), 129-153.
108. R. Sturm, 'Würfel und reguläres Tetraeder als Maximum und Minimum', Ibid. 97 (1884), 1-12.
109. R. Sturm, Maxima und Minima in der elementaren Geometrie, Leipzig-Berlin 1910.
110. R. Sturm, 'Über Maxima und Minima bei Pyramiden und Prismen', Zeitschr. math. naturwiss. Unterr. 44 (1913), 204-207.
111. C.-G. Sucksdorff, 'Détermination du pentaèdre de volume donné dont la surface est un minimum', J. math. pures appl. (2) 2 (1857), 91-109.
112. J. H. van Swinden, Grondbeginsels der Meetkunde, Amsterdam 1816.
113. J. H. van Swinden and C. F. A. Jacobi, Elemente der Geometrie.
114. E. G. Togliatti, 'Massimi e minimi', Enciclopedia delle matematiche elementari. II.II, Milano 1950, 1-71.
115. I. Tomescu, 'Generalizări ale problemei 13027', Gaz. Mat. B 24 (1973), 583-586.
116. G. Valette, 'A propos des cages circonscrites à une sphère', Bull. Soc. Math. Belg. 21 (1969), 124-125.
117. G. R. Veldkamp, 'Een ongelijkheid uit de stereometrie', Nieuw Tijdschr. Wisk. 43 (1955-1956), 213-215.
118. H. Vogler, 'Bestimmung einer oberen Schranke für den Inhalt des Parallelrisses eines regelmässigen Körpers', Elem. Math. 22 (1967), 133-135.
119. I. Voicu, 'Problema 18666', Gaz. Mat. 86 (1981), 112.
120. D. Wolfe, 'Metric Dependence and a Sum of Distances', Lecture Notes 490 (1975), 209-211.
121. L. Yang and J. Zh. Zhang, 'A Class of Geometric Inequalities on Finite Points' (Chinese), Acta Mat. Sinica 23 (1980), 740-749.
122. L. Yang and J. Zh. Zhang, 'A Johnson Conjecture Concerning Space Curves' (Chinese), Kexue Tongbao 30 (1984), 329-333.
123. L. Yang and J. Zh. Zhang, 'Pseudo-Symmetric Sets and Related Geometric Inequalities' (Chinese), Acta Math. Sinica 29 (1986), 802-806.
124. J. Zh. Zhang and L. Yang, 'A Class of Geometric Inequalities Concerning the Mass-Points System' (Chinese), J. China Univ. Sci. Technol. 11 (1981), 1-8.
125. F. G.-M. Exercises de géométrie, 4. éd., Tours-Paris 1912.
126. 'Zadača 46', Mat. v škole 1956, No. 2, 93.
127. 'Problema 11779', Gaz. Mat. B 23 (1972), 54.
128. 'Zadača 1875', Mat. v škole 1978, No. 2, 86-87.
129. J. Neuberg, 'Mémoire sur le tétraèdre', Mémoires couronnés et autres mémoires Acad. Roy. Bruxelles 37 (1986), 1-72.

APPENDIX

1. Comments, Additions and Corrections for GI

Here follows a list of comments, additions and corrections received from the friends-mathematicians of the authors of the book GI, quoted in numerous places of this book, AGI. We want to express our gratitude to all who have shown their interest in the subject for the nice way they have formulated their criticism. We thank, in particular, Professor O. Bottema, one of the co-authors of GI, who kindly formulated, several years ago, the following text, not yet published.

Comments, additions and corrections are arranged according to the numbering in the book GI and the name of the correspondent concerned is mentioned.

Suggestions, comments, additions and corrections for improvements of AGI in a new edition will also be welcomed.

The authors of AGI consider that a new edition of GI is desirable since this book had been sold out a long time ago. Finally, GI and AGI represent a whole, i.e. AGI is in fact a continuation of GI.

1.3 (A. Bager). Here follows another proof. In view of Heron's formula and of $F = rs$, $4RF = abc$, we obtain by means of 5.1

$$8(s - a)(s - b)(s - c) = \frac{8F^2}{s} = \frac{2rabc}{R} \leqslant abc.$$

1.12 (A. Bager). The inequality is equivalent to 2.1 and to 5.3.

1.15 (A. Bager). As $r_a + r_b + r_c = r + 4R$ we obtain in view of 5.1

$$r_a + r_b + r_c \geqslant 9r$$

and if we divide both sides by F we have the given inequality. It follows also immediately from $(x + y + z)(x^{-1} + y^{-1} + z^{-1}) \geqslant 9$, keeping in mind that $(s - a) + (s - b) + (s - c) = s$.

2.0. As remarked in the preface of GI the references do not claim to be complete. This yields especially for some well-known trigonometric inequalities which have been proved again and again in the course of the years. We add here as reference a paper of R. Marcolongo, Boll. Matematica **14** (1915-1916), 182-184, in which 2.12, 2.16, 2.21 and 2.23 are proved in a simple manner with the remark that they appear already in P. Franchini, Trattato algebrico de massimi e minimi (Lucca, 1823).

A systematic determination of the maxima and minima (or the upper and lower bounds) of the 24 functions $\Sigma f(\alpha)$, $\Sigma f(\frac{\alpha}{2})$, $\Pi f(\alpha)$ and $\Pi f(\frac{\alpha}{2})$, in which f stands for one of the six elementary goniometric functions, may be found in F. Pignataro, Giorn. Matematiche, 68 (1929), 18-33; this paper contains moreover a list of other inequalities for the triangle.

Remark. α, β, γ are the angles A, B, C of the triangle ABC.

2.25 (A. Bager). The statement may be generalized: if no angle is $\pi/2$ the three numbers $|\cos\alpha|$, $|\cos\beta|$, and $\sin\gamma$ are sides of a triangle. Proof: if D is diametrically opposite C on the circumcircle the sides of ABD are

$$AD = 2R\,|\cos\alpha|,\quad BD = 2R\,|\cos\beta|,\quad AB = 2R\sin\gamma.$$

2.28 (Several readers). In the third inequality $\frac{3}{8}\sqrt{3}$ must be replaced by $\frac{1}{4}(\sqrt{2}+1)$. The equivalent in 2.2 was given correctly.

2.36 (A. Bager). The inequality may be generalized to

$$\tan^n\frac{\alpha}{2} + \tan^n\frac{\beta}{2} + \tan^n\frac{\gamma}{2} \geqslant 3^{-\frac{1}{2}(n-2)},$$

n being a natural number. It follows from 2.65.

2.41. The inequality may be generalized to

$$\cot\!an^n\frac{\alpha}{2} + \cot\!an^n\frac{\beta}{2} + \cot\!an^n\frac{\gamma}{2} \geqslant 3^{\frac{1}{2}(n+2)}$$

for integer n.

V. G. Zvezdin, Matematika v škole 1968, 65.

2.41 is the case n = 1; for n = 2 we have 2.43.

2.45 (A. Bager). This inequality is only valid for acute triangles. Same remark holds for 2.47, the first inequality in 2.55 and 2.63.

I. Paasche, Praxis Math. 28 (1986), 376 and 379.

3.11. Another proof has been given by M. G. Greening, Amer. Math. Monthly 75 (1968), 197.

4.2. This inequality is equivalent to 5.11.

4.10. This is a special case ($\lambda = 4$) of 4.21.

4.15. Add $x + y + z \geqslant 0$.

5.6. The inequality given by A. Emmerich, 'Problem 10656', Math. Questions 54 (1891), 100:

$$\cot \omega \geqslant \Sigma \tan \frac{\alpha}{2},$$

ω being Brocard's angle, and the inequality

$$\Sigma \cot \alpha \geqslant \frac{s}{3r}$$

of V. Petrov, Matematika (Sofija), 7 (1968), No. 4, 30, are equivalent to the left-hand side of 5.6 in view of

$$\cot \omega = \Sigma \cot \alpha, \quad \Sigma \tan \frac{\alpha}{2} = \frac{4R + r}{s},$$

$$\Sigma \cot \alpha = \frac{s^2 - r(4R + r)}{2rs}.$$

5.6 (A. Bager). On p. 50, second line, read $\frac{1}{6} \Omega$ for Ω.

5.8 (Several readers). In (4), p. 51, the last term should be $-\frac{2}{9} OH^2$.

5.10. These inequalities are equivalent to 7.11.

5.13 (A. Bager). If we multiply 3.14 by 2R, we obtain

$$18Rr \leqslant a^2 + b^2 + c^2 \leqslant 9R^2,$$

which is stronger than the inequality given.

5.18 (A. Bager). The chain may be enlarged to

$$36r^2 \leqslant 4F\sqrt{3} \leqslant 18Rr \leqslant 4r(5R - r) \leqslant \Sigma bc \leqslant 4(R + r)^2 \leqslant 9R^2.$$

5.37. A proof of this inequality has been given by R. W. Frucht, Amer. Math. Monthly 75 (1968), 299; it is even valid for an arbitrary triangle.

5.38 (A. Bager). This follows immediately from 5.43.

5.47 (A. Bager). The constant $\frac{3}{2}$ may be improved to $\sqrt{2}$. Proof:

$$\Sigma\sqrt{a(s - a)} = \sqrt{2}\Sigma\sqrt{\frac{1}{2} a(s - a)} \leqslant \sqrt{2}\Sigma \frac{1}{2}(\frac{1}{2} a + s - a) = \sqrt{2}s.$$

6.7. This implies the more elegant inequality

$$\frac{a^2}{h_b h_c} + \frac{b^2}{h_c h_a} + \frac{c^2}{h_a h_b} \geqslant 4.$$

Ž. Mitrović, 'Problem 176', Mat. Vesnik 6 (21) (1969), 469.

6.10 (I. Paasche). In the fourth term r must be replaced by R.

6.14 (A. Bager). The first inequality is trivial and the last one is a consequence of $r_a + r_b + r_c = 4R + r$ and $R \geqslant 2r$. The remaining one may be proved as follows:

$$w_a = \frac{2bc}{b + c} \cos \frac{\alpha}{2} \leqslant \sqrt{bc} \cos \frac{\alpha}{2} .$$

Using 5.17, 2.29 and the Cauchy-Schwarz inequality we get

$$(\Sigma w_a)^2 \leqslant (\Sigma\sqrt{bc} \cos \frac{\alpha}{2})^2 \leqslant (\Sigma bc)(\Sigma \cos^2 \frac{\alpha}{2}) \leqslant 4(R + r)^2 \cdot \frac{9}{4}$$

and therefore

$$\Sigma w_a \leqslant 3(R + r).$$

6.17 (A. Bager). The following inequality holds

$$\sqrt{h_a} + \sqrt{h_b} + \sqrt{h_c} \leqslant \sqrt{3\sqrt{3}s},$$

which is stronger than the given one in view of 5.3. The proof follows from 6.1 and $\sqrt{x} + \sqrt{y} + \sqrt{z} \leqslant \sqrt{3(x + y + z)}$.

7.3 (Several readers). The sign at the right-hand side should be +. Inequality in 7.4 is correct.

7.6 (A. Bager). The inequality is strict; equality does not hold in any (non-degenerated) triangle.

7.7 (A. Bager). From the identity

$$\Sigma a^3 = 2s^3 - 12Rrs - 6r^2s$$

and the first part of 5.8 it follows

$$\Sigma a^3 \geqslant 4F(5R - 4r),$$

which is stronger than the given inequality.

7.8 (A. Bager). There are some errors in the formula, which should read

$$4R(R - \frac{2F}{s})^3 \geqslant \left(s^2 + \frac{F^2}{s^2} - 2R^2 - \frac{10FR}{s}\right)^2 .$$

If we put F = rs it is the equivalent of 5.10.

7.11 (P. J. van Albada and A. Bager). On the left-hand (right-hand) side equality holds if ABC is isosceles and the vertex angle $\geq \pi/3$ ($\leq \pi/3$). There is equality on both sides if and only if the triangle is equilateral. Analogous remarks hold for 5.15 and 5.19.

8.2. The proof does not hold for an obtuse triangle. For this case see F. Leuenberger, Elem. Math. **16** (1961), 127-129.

8.3. As an older reference should be mentioned F. Leuenberger, Elem. Math. **13** (1958), 121-126.

8.5 (A. Bager). In the three cases the second $\leq$ sign should be $<$.

8.8. The second inequality is not correct. A counter-example is $a = c$, $\cos\alpha > 9/10$.

8.17. In the proof the reference to 5.6 should be to 5.11.

10.4 (G. R. Veldkamp). The theorem comes to this: if P is inside the equilateral triangle ABC and F_1 is the area of its pedal triangle, then $F_1 \leq \frac{1}{4}F$. This may be generalized as follows. If P is in the plane of the arbitrary triangle ABC, then according to a formula of Gergonne

$$F_1 = \frac{|R^2 - OP^2|}{4R^2}F.$$

Hence $F_1 \leq \frac{1}{4}F$ for all points P inside ABC and even for all P inside or on the circle with centre O and radius $R\sqrt{2}$.

O. Bottema, Hoofdstukken uit de elementaire wiskunde (1944), 22.

10.8 (A. Oppenheim). The inequality is essentially equivalent to 10.12, 3° and to 14.1. The latter may also be written

$$(\lambda a^2 + \mu b^2 + \nu c^2)^2 \geq 16F^2(\mu\nu + \nu\lambda + \lambda\mu).$$

For $\lambda = \mu = \nu$ we have 4.2.

11.2 (A. Bager). All inequalities but the first hold in any triangle.

12.6 (G. R. Veldkamp). The following proof is much simpler. If $a > b > c$, then $b > AD > c$, $a > BE > c$, $a > CF > b$ and therefore $AD < a$, $BE < a$, $CF < a$. Hence

$$1 = \frac{PD}{AD} + \frac{PE}{BE} + \frac{PF}{CF} > \frac{PD + PE + PF}{a}.$$

12.18 (A. Oppenheim). The inequality is a special case of

$$(x + y + z)^2 \geq 2\sqrt{3}(yz \sin\Theta_1 + zx \sin\Theta_2 + xy \sin\Theta_3),$$

$$\sin \Theta_i > 0, \quad \Theta_1 + \Theta_2 + \Theta_3 = 2\pi.$$

13.5 (A. Oppenheim). The triangle A'B'C' with sides $\cos^2 \frac{\alpha}{2}$, $\cos^2 \frac{\beta}{2}$, $\cos^2 \frac{\gamma}{2}$ is similar to that with the sides $r_b + r_c$, $r_c + r_a$, $r_a + r_b$. The repetition of the process leads to a triangle similar to ABC.

13.8. The author's name should be P. Sondat. The inequality is equivalent with 5.10.

16.12. These inequalities are the best possible.

2. Supplements to Chapters I-XX of AGI

The scope of the following text is the quotation of some geometric inequalities found in the course of corrections of proofs.

2.1. The problem stated in I.3.39 has been solved meanwhile by W. Janous as follows (see Crux Math. 13 (1987), 333-338):

(ii) If $\mu = -1$, then $\lambda = 0$.

If $-1 < \mu < 1$, then $\dfrac{\log 2}{(\log((2\mu + 2)/(\mu + 3))} \leqslant \lambda \leqslant \dfrac{\log 2}{\log(2/(\mu + 1))}$.

If $\mu = 1$, then λ is arbitrary.

If $\mu > 1$, then $\dfrac{\log 2}{\log(2/(\mu + 1))} \leqslant \lambda \leqslant \dfrac{\log 2}{\log((2\mu + 2)/(\mu + 3))}$.

For (i) we get from (ii) via $\mu = 0$:

$$-1.7 \approx \frac{\log 2}{\log(2/3)} \leqslant \lambda \leqslant 1.$$

2.2. $$\sum \frac{b^3 + c^3 - a^3}{b + c - a} \leqslant \sum bc.$$

Gh. Morghescu, 'Problem E 8769*', Gaz. Mat. (Bucharest) 91 (1986), 21.

2.3. Let u, v, w be three positive numbers. Then

$$\sum \frac{u}{v + w} \frac{bc}{s - a} \geqslant \sum a.$$

S. Bilčev and E. Velikova, 'Problem 1212', Crux Math. 13 (1987), 52.

2.4. Conjecture:

$$\sum \cos \frac{B - C}{2} \leqslant \frac{1}{\sqrt{3}}\left(\sum \sin A + \sum \cos \frac{A}{2}\right).$$

J. Garfunkel, 'Problem 1292*', Crux Math. 13 (1987), 320.

2.5. $(\Sigma \tan A)^2 \geqslant \Sigma(\sec A + 1)^2.$

C. Cooper, 'Problem 1210', Crux Math. 13 (1987), 16.

2.6. $$|\Sigma \sin n(B - C)| \begin{cases} < 1 & \text{if } n = 1, \\ < 3\sqrt{3}/2 & \text{if } n = 2, \\ \leqslant 3\sqrt{3}/2 & \text{if } n \geqslant 3. \end{cases}$$

W. Janous, 'Problem 1254', Crux Math. 13 (1987), 179.

2.7. If $k \in [0, 1]$, then

$$\cos^2 \frac{k\pi}{4} + 2 \cos \frac{k\pi}{4} \leqslant \Sigma \cos kA \cos kB \leqslant 3 \cos^2 \frac{k\pi}{6} \quad (\Delta_a),$$

$$\cos^2 \frac{k\pi}{4} + 2 \cos \frac{k\pi}{4} \leqslant \Sigma \cos kA \cos kB \leqslant 1 + 2 \cos \frac{k\pi}{2} \quad (\Delta).$$

This is Klamkin's generalization of a problem given in X.5.52. (see Crux Math. 13 (1987), 181-183).

2.8. Let $0 < \lambda \leqslant 1/2$. Then

$$\cos \lambda\pi \leqslant \Pi \cos \lambda A + \Pi \sin \lambda A \leqslant \sin^3 \frac{\lambda\pi}{3} + \cos^3 \frac{\lambda\pi}{3}.$$

M. S. Klamkin, G. Tsintsifas, and W. Janous, 'Problem 1096', Crux Math. 11 (1985), 325 and 13 (1987), 154-156.

2.9. $\Sigma \cot \frac{A}{2} \leqslant \frac{3}{2} \Sigma \operatorname{cosec} 2A \quad \{E\} \quad (\Delta_a).$

J. Garfunkel and B. Prielipp, 'Problem 1125*', Crux Math. 12 (1986), 51 and 13 (1987), 200-201.

2.10. For triangles of Bager's type I,

$$\left\{\frac{\Sigma \sin A}{\Sigma \cos(A/2)}\right\}^3 \geqslant 8\Pi \sin \frac{A}{2}.$$

J. Garfunkel and V. N. Murthy, 'Problem 1093*', Crux Math. 11 (1985), 325 and 13 (1987), 132-133.

2.11. $\Sigma a \cos A \leqslant \frac{2}{3} \Sigma m_a \sin A \leqslant s$

N. Bejlegard, 'Problem 1237*', Crux Math. 13 (1987), 119.

2.12. $\Sigma a(s - a) \leqslant 9Rr \quad \{E\}.$

S. G. Guba, 'Problem 317', Mat. v škole 1966, No. 6, 67.

2.13. 1° $\sum \cos^3 \frac{A}{2} \cos \frac{B-C}{2} \leqslant \frac{9\sqrt{3}}{8}$ {E},

2° $\sum\left(\frac{m_a}{r_a}\right)^n \geqslant 3^{1-3n/2}\left(\frac{s}{r}\right)^n \geqslant 3, \quad n \geqslant 2/3,$ {E},

3° $\frac{4\sqrt{3}}{3}\frac{7R-2r}{9R-2r} \leqslant \sum \frac{a}{h_b + h_c} \leqslant \sqrt{\frac{3R}{2r}}$ {E},

4° $3\sqrt{3r} \leqslant \sum\sqrt{h_a} \leqslant \left(1 + \frac{r}{R}\right)\sqrt{6R}$ {E}.

D. M. Milošević, 'Some Inequalities for the Triangle', Elem. Math. 42 (1987), 122-132.

2.14. 1° $\sum a^\lambda w_a \leqslant \left(abc\left(1 + \frac{r}{R}\right)\sum a^{2(\lambda-1)}\right)^{1/2}$ {E} $(\lambda \in (-\infty, +\infty))$,

2° $\sum a^2 \tan \frac{B}{2} \tan \frac{C}{2} \geqslant \frac{4F^2}{r(4R+r)}$ {E},

3° $\sum w_a^2 \leqslant s^2 + \frac{1}{2} r(2r - R)$ {E}.

Š. Arslanagić and D. Milošević, Neki problemi i napomene o nejednakostima trougla. To appear.

2.15. The following inequality for the Nagel cevians is given by R. H. Eddy:

(1) $\sum n_a \leqslant 14R - 19r.$

D. M. Milošević proved the following improvement of (1)

(2) $\sum n_a \leqslant 10R - 11r.$

If we write

(3) $\sum n_a \leqslant 9r + L(R - 2r),$

then we have (1) for $L = 14$ and (2) for $L = 10$.

W. Janous proved that the minimal L for (3) satisfies

$$6.258998\ldots \leqslant L_{min} \leqslant 6.478487\ldots$$

The lower bound is $\sup\{F(x), x > 1\}$ where

$$F(x) := 2\sqrt{x-1}\{(2x+2)\sqrt{(x-2)^2(x+1)} + 4(x-1) +$$

$$+ (x^2 - 8x)\sqrt{x - 1}\}/x(x - 2)^2;$$

the upper bound is root of $2x^3 - 3x^2 - 144x + 515 = 0$.

R. H. Eddy, 'An Upper Bound for a Sequence of Cevian Inequalities', Elem. Math. 41 (1986), 128-130.

D. M. Milošević, 'A Supplement to Eddy's Paper', Ibid. 42 (1987), 104-105.

W. Janous, A Further Remark Concerning Nagel Cevians. To appear.

2.16. Let ABC be a triangle, I the incentre, and A', B', C' the intersection of AI, BI, CI with the circumcircle. Then

$$\Sigma IA' - \Sigma IA \leqslant 2(R - 2r).$$

G. Tsintsifas, 'Problem 1282', Crux Math. 13 (1987), 289.

2.17. Let $A_1A_2A_3$ be an acute triangle with circumcentre O. Let P_1, Q_1 ($Q_1 \neq A_1$) denote the intersection of A_1O with A_2A_3 and with the circumcircle, respectively, and define P_2, Q_2, P_3, Q_3 analogously. Then

$$\Pi(OP_1/P_1Q_1) \geqslant 1; \quad \Sigma(OP_1/P_1Q_1) \geqslant 3; \quad \Pi(A_1P_1/P_1Q_1) \geqslant 27.$$

J. T. Groenman, 'Problem 1305', Crux Math. 14 (1988), 12.

2.18. $$(\Sigma a)(\Sigma \frac{1}{R_1}) > 2(3 + 2\sqrt{2});$$

$$(\Sigma \frac{1}{a})(\Sigma R_1) > \begin{cases} 5, & \text{if } \max(A, B, C) \geqslant 120°, \\ 4 + 2/\sqrt{3}, & \text{otherwise.} \end{cases}$$

All bounds cannot be improved.

W. Gmiiner and W. Janous, A Pair of General Triangle Inequalities. To appear.

2.19. Let $A_1A_2A_3$ be a triangle with sides a_1, a_2, a_3 and let P be a point in or out of the plane of the triangle. If R_1, R_2, R_3 are the distances from P to the respective vertices A_1, A_2, A_3, then

$$\Sigma a_1^2R_1^2 \geqslant \sqrt{2}a_i^2R_i^2, \quad i = 1, 2, 3,$$

which is stronger than the Möbius-Pompeiu theorem: $\Sigma a_1R_1 \geqslant 2a_iR_i$, $i = 1, 2, 3$.

M. S. Klamkin, 'Problem 1131', Crux Math. 12 (1986), 77 and 13 (1987), 223-224.

2.20. $$\Sigma ar_1^n \leqslant (2R)^{n-2}\Pi a \quad (n \geqslant 1).$$

M. S. Klamkin, 'Problem 1296', Crux Math. 13 (1987), 321.

2.21. Let ABC be a triangle and M an interior point with barycentric coordinates $(\lambda_1, \lambda_2, \lambda_3)$. The distances of M from the vertices A, B, C are x_1, x_2, x_3 and the circumradii of the triangles MBC, MCA, MAB, ABC are R_1, R_2, R_3, R. Then

$$\Sigma\lambda_1 R_1 \geqslant R \geqslant \Sigma\lambda_1 x_1.$$

G. Tsintsifas, 'Problem 1243', Crux Math. 13 (1987), 149.

2.22. Let ABC be a triangle, M an interior point, and A'B'C' its pedal triangle. Denote the sides of the two triangles by a, b, c and a', b', c', respectively. Then

$$\Sigma \frac{a'}{a} < 2.$$

G. Tsintsifas, 'Problem 1273', Crux Math. 13 (1987), 256.

2.23. Let ABC be a triangle and M an interior point with barycentric coordinates λ_1, λ_2, λ_3. We denote the pedal triangle and the cevian triangle of M by DEF and A'B'C', respectively. Then

$$\frac{[DEF]}{[A'B'C']} \geqslant 4\lambda_1\lambda_2\lambda_3(s/R)^2.$$

where s is the semiperimeter and R the circumradius of ABC, and [X] denotes the area of figure X.

G. Tsintsifas, 'Problem 1252', Crux Math. 13 (1987), 179.

2.24. Let ABC be a triangle with circumcentre O and incentre I, and let DEF be the pedal triangle of interior point M of triangle ABC (with D on BC, etc.). Then

$$OM \geqslant OI \Rightarrow r' \leqslant \frac{r}{2}$$

where r and r' are the inradii of triangles ABC and DEF, respectively.

G. T. Tsintsifas and M. S. Klamkin, 'Problem 1075', Crux Math. 11 (1985), 249 and 13 (1987), 60-61.

2.25. Let $A_1A_2A_3$ and $B_1B_2B_3$ be two triangles. Then

$$\sqrt{3}\Sigma \sin A_1/2 \geqslant 4\Sigma \sin B_1 \sin A_2/2 \sin A_3/2.$$

M. S. Klamkin, 'Problem 1271', Crux Math. 13 (1987), 256.

2.26. For any two triangles with angles A, B, C and A_1, B_1, C_1,

$$\Sigma \frac{\cos A_1}{\sin A} \leqslant \Sigma \cot an\, A.$$

R. P. Ushakov, 'Problem M 1024*', Kvant 1987, No. 1, 16, and 1987, No. 5, 26.

2.27. Let ABC and $A_1B_1C_1$ be two triangles with sides a, b, c and a_1, b_1, c_1 and inradii r and r_1, and let P be an interior point of ABC. Set AP = x, BP = y, CP = z. Then

$$\frac{\Sigma a_1 x^2}{\Sigma a} \geqslant 4rr_1.$$

G. Tsintsifas, 'Problem 1303', Crux Math. 14 (1988), 12.

2.28. Denote by K_p the intersection of three convex closed cones $x \leqslant (y^{1/p} + z^{1/p})^p$, etc. So K_p is also a convex closed cone.

If A(a, b, c) is the area of a triangle of sides a, b, c (degeneracy allowed), then, for $1 < p \leqslant 4$, the function

$$F_p(x, y, z) = 2^p A(x^{1/p}, y^{1/p}, z^{1/p})^{p/2}$$

is strictly concave (as positively defined function) on K_p.

As a consequence of this result we have Theorem P from XII.3.6.

C. Tănăsescu, A Simple Proof to Oppenheim's Inequality on Mixed Areas. To appear.

2.29. $A_1A_2A_3$ is a triangle with circumcircle Ω. Let $x_1 < X_1$ be the radii of the two circles tangent to A_1A_2, A_1A_3 and arc A_2A_3 of Ω. Let x_2, X_2, x_3, X_3 be defined analogously. Then

$$\Sigma \frac{x_1}{X_1} = 1 \quad \text{and} \quad \Sigma X_1 \geqslant 3\Sigma x_1 \geqslant 12r$$

where r is the inradius of $A_1A_2A_3$.

G. Tsintsifas, 'Problem 1224', Crux Math. 13 (1987), 86.

2.30. Let $A_1A_2A_3$ be a triangle with inscribed circle I of radius r. Let I_1 and J_1 of radii λ_1 and ν_1 be the two circles tangent to I and the lines A_1A_2 and A_2A_3. Analogously define circles I_2, J_2, I_3, J_3 of radii λ_2, ν_2, λ_3, ν_3, respectively. Then

$$\lambda_1\nu_1 = \lambda_2\nu_2 = \lambda_3\nu_3 = r^2 \quad \text{and} \quad \Sigma\lambda_1 + \Sigma\nu_1 \geqslant 10r.$$

J. T. Groenman, 'Problem 1267', Crux Math. 13 (1987), 216.

2.31. Let ABC be a triangle with medians AD, BE, CD and median point G. We denote $\Delta AGF = \Delta_1$, $\Delta BGF = \Delta_2$, $\Delta BGD = \Delta_3$, $\Delta CGD = \Delta_4$, $\Delta CGE = \Delta_5$, $\Delta AGE = \Delta_6$, and let R_i and r_i denote the circumradius and inradius of Δ_i ($i = 1, 2, \ldots, 6$). Then

(i) $R_1R_3R_5 = R_2R_4R_6$;

(ii) $\frac{15}{2r} < \frac{1}{r_1} + \frac{1}{r_3} + \frac{1}{r_5} = \frac{1}{r_2} + \frac{1}{r_4} + \frac{1}{r_6} < \frac{9}{r}$,

where r is the inradius of ΔABC.

J. T. Groenman, 'Problem 1315', Crux Math. 14 (1988), 45.

2.32. For i = 1, 2, 3 Let C_i be the centre and r_i the radius of the Malfatti circles nearest A_i in triangle $A_1A_2A_3$, and let r and r' be the inradii of triangles $A_1A_2A_3$ and $C_1C_2C_3$ respectively. Then

$$\Pi A_1C_1 \geqslant \frac{1}{3}((\Sigma r_1)^3 - 3\Pi r_1) \quad \text{and} \quad r \leqslant (1 + \sqrt{3})r'.$$

J. Garfunkel, 'Problem 1077*', Crux Math. 11 (1985), 249.
J. Garfunkel, 'Problem 1243*', Crux Math. 13 (1987), 119.

2.33. Let O be the intersection of diagonals of a quadrilateral ABCD. Then

$$\frac{AB}{CD} + \frac{CD}{AB} + \frac{BC}{AD} + \frac{AD}{BC} \leqslant \frac{OA}{OC} + \frac{OC}{OA} + \frac{OB}{OD} + \frac{OD}{OB}.$$

F. Nazarov, 'Problem 12', Kvant 1987, No. 12, 56.

2.34. Let ABCD be a convex quadrilateral with diameter d. Then

(a) $F_{ABCD} \leqslant d/2$

with equality for a square;

(b) $L_{ABCD} \leqslant d(2 + \sin \frac{\pi}{12})$

with equality for quadrilateral with AD = AB = AC = BD = d, CD = CB.

S. Golikov, 'Problem 2', Fiz. Mat. Spisanie 29 (62) (1987), 187.

2.35. Let $A_1A_2A_3A_4$ be a cyclic quadrilateral with $A_1A_2 = a_1$, $A_2A_3 = a_2$, $A_3A_4 = a_3$, $A_4A_1 = a_4$. Let r_1 be the radius of the circle outside the quadrilateral, tangent to the segment A_1A_2 and the extended lines A_2A_3 and A_4A_1. Define r_2, r_3, r_4 analogously. Then

$$\Sigma 1/r_1 \geqslant 8/\sqrt[4]{\Pi a_1}.$$

J. T. Groenman, 'Problem 1284', Crux Math. 13 (1987), 290.

2.36. A quadrilateral inscribed in a circle of radius R and circumscribed around a circle of radius r has consecutive sides a, b, c, d, semiperimeter s and area F. Then

(a) $$2\sqrt{F} \leqslant s \leqslant r + \sqrt{r^2 + 4R^2},$$

(b) $$6F \leqslant ab + ac + ad + bc + bd + cd \leqslant 4r^2 + 4R^2 + 4r\sqrt{r^2 + 4R^2},$$

(c) $$2sr^2 \leqslant abc + abd + acd + bcd \leqslant 2r(r + \sqrt{r^2 + 4R^2})^2,$$

(d) $$4Fr^2 \leqslant abcd \leqslant \frac{16}{9} r^2(r^2 + 4R^2).$$

M. N. Naydenov, 'Problem 1203', Crux Math. 13 (1987), 14-15.

2.37. The pairwise distances between six points of the plane are no greater than 1. Then there are three points among them whose pairwise distances are strictly less than 1.

S. G. Salnikov, 'Problem M 1066', Kvant 1987, No. 10, 25.

2.38. Let $A_1 \ldots A_n$ be a polygon circumscribed around a circle of area F_o. Let F_i be the area of a figure formed by this circle and the sides $A_{i-1}A_i$ and A_iA_{i+1} ($i = 1, \ldots, n$, $A_o = A_n$, $A_{n+1} = A$). Then

$$F_o \geqslant \frac{n^2\pi}{n \tan\frac{\pi}{n} - \pi}\left(\Sigma \frac{1}{F_i}\right)^{-1}$$

with equality for equilateral n-gon.

(For n = 3 see Problems 70.6 and 70.H Math. Gaz. 71 (1987), 148-149).

2.39. At any point P of an ellipse with semi-axes a and b (a > b), draw a normal line and let Q be the other meeting point. Let N be the least value of length PQ. Then

$$\min N^2 = \begin{cases} \dfrac{27a^4b^4}{(a^2 + b^2)^3} & \text{if } a^2 \geqslant 2b^2, \\ 4b^2 & \text{if } a^2 \leqslant 2b^2. \end{cases}$$

M. S. Klamkin gave a generalization of this Fukagawa's problem for n-dimensional ellipsoid.

H. Fukagawa and M. S. Klamkin, 'Problem 1986', Crux Math. 12 (1986), 140 and 13 (1987), 269-272.

2.40. Let $A = A_1A_2 \dots A_{n+1}$ be a simplex in E^n and let P be any interior point of A whose barycentric coordinates are $\lambda_1, \lambda_2, \dots, \lambda_{n+1}$. For any $i \in \{1, 2, \dots, n + 1\}$ let r_i be the distance of P to the hyperplane $A_1 \dots A_{i-1}A_{i+1} \dots A_{n+1}$. Then

$$\sum_{i=1}^{n+1} \lambda_i r_i^m \geq r^m,$$

where $m > 0$ and r is the inradius of A.

G. Tsintsifas and M. S. Klamkin, 'Problem 1102', Crux Math. 12 (1986), 11 and 13 (1987), 162-163.

2.41. Let P be a point which is inside the simplex $A = A_1A_2 \dots A_{n+1}$ and is also in the closed ball with diameter $\overline{OG}$, where O is the circumcentre and G the centroid of A. For any $i \in \{1, 2, \dots, n + 1\}$ let A_i' denote the intersection of A_iP with the circumsphere. Then

$$\sum_{i=1}^{n+1} A_iP \leq \sum_{i=1}^{n+1} A_i'P.$$

M. S. Klamkin and G. Tsintsifas, 'Problem 1086', Crux Math. 11 (1985), 289 and 13 (1987), 100-102.

2.42. Let $A = A_1A_2 \dots A_{n+1}$ be a regular simplex in E^n with the edge length a. For any $i \in \{1, 2, \dots, n + 1\}$ let B_i be a point on the hyperplane $A_1 \dots A_{i-1}A_{i+1} \dots A_{n+1}$. Then

$$\sum_{\substack{i,j=1 \\ i<j}}^{n+1} B_iB_j \geq \frac{n+1}{2} a.$$

G. Tsintsifas, 'Problem 1085', Crux Math. 11 (1985), 289 and 13 (1987), 98-99.

2.43. Let $A = A_1A_2 \dots A_{n+1}$, $A' = A_1'A_2' \dots A_{n+1}'$ be two simplexes in E^n and P an interior point of A. For any $i \in \{1, 2, \dots, n + 1\}$ let $R_i = A_iP$ and let F_i' be the $(n - 1)$-dimensional content of the $(n - 1)$-simplex $A_1' \dots A_{i-1}'A_{i+1}' \dots A_{n+1}'$. If V and V' are the volumes of A and A', respectively, then

$$\left(\sum_{i=1}^{n+1} R_iF_i'\right)^2 \geq n^4 V^{\frac{2}{n}} V'^{\frac{2n-2}{n}}.$$

G. Tsintsifas, 'A Generalization of a Two Triangle Inequality', Elem. Math. 42 (1987), 150-153.

2.44. Let L be the perimeter of a pentagon in E^3 whose convex hull has the volume V. Then

$$V \leqslant \frac{1}{384} L^3.$$

M. Kočandrlová, 'The Isoperimetric Inequality for a Pentagon in E_3 and Its Generalization in E_n Space', Čas. pěst. mat. 107 (1982), 167-174.

2.45. Let L be the perimeter of a hexagon in E^3, whose convex hull has six vertices and the volume V. Then

$$V \leqslant \frac{1}{324} L^3$$

M. Kočandrlová, 'Isoperimetrische Ungleichung für geschlossene Raumsechsecke', Čas. pěst. mat. 108 (1983), 248-257.

2.46. Let $P = A_1, A_2, \ldots, A_{n+1}$ be a point set of diameter d in E^n, i.e. $d = \max A_iA_j$ $(i, j \in \{1, 2, \ldots, n + 1\},\ i < j)$. Then P can be contained in a closed connected region bounded by two parallel hyperplanes with the distance w between these hyperplanes, where

$$w \leqslant \frac{2d}{\sqrt{2n + 2}} \quad (n \text{ odd}),$$

$$w \leqslant d\sqrt{\frac{2(n + 1)}{n(n + 2)}} \quad (n \text{ even}).$$

G. Tsintsifas, 'Problem 1162', Crux Math. 12 (1986), 178 and 13 (1987), 301-302.

2.47. Let L be the perimeter of an (n + 1)-gon in E^n, whose convex hull is a simplex A with the volume V. Then

$$V \leqslant \frac{L^n}{n!\sqrt{n^n(n + 1)^{n+1}}}$$

with equality iff A is a regular simplex.

B. Míšek, 'O (n + 1)-úhelníku v E_n s maximálním objemem konvexního obalú', Čas. pěst. mat. 84 (1959), 99-103.

2.48. Let L be an N-gon in E^{2m} $(N \geqslant 2m + 1)$ with the perimeter L. If V is the volume of the convex hull of L, then

$$V \leqslant \frac{L^{2m}}{(2m)!(2mN)^m} \cdot \prod_{j=1}^{m} \cotan \frac{\pi j}{N}$$

with equality if and only if L is congruent to the N-gon with the vertices $(u_1(k), u_2(k), \ldots, u_{2m}(k))$ $(k = 0, 1, \ldots, N - 1)$, where

$$u_{2j-1}(k) = \frac{L}{2N\sqrt{m}} \cdot \frac{\cos(2\pi jk/N)}{\sin(\pi j/N)},$$

$$u_{2j}(k) = \frac{L}{2N\sqrt{m}} \cdot \frac{\sin(2\pi jk/N)}{\sin(\pi j/N)} \quad (j = 1, 2, \ldots, m)$$

A. A. Nudeljman, 'Izoperimetričeskie zadači dlja vipuklih oboloček lomanih i krivih v mnogomernih prostranstvah', Mat. Sbornik 96 (136) (1975), 294-313.

2.49. Let L be a broken line of N sides in E^n $(N \geqslant n)$, let L be the length of L and V the volume of the convex hull of L. If $n = 2m + 1$, then

$$V \leqslant \frac{L^{2m+1}}{(2m+1)!\sqrt{(2m+1)^{2m+1}}N^m} \cdot \prod_{j=1}^{m} \cotan \frac{\pi j}{N}$$

with equality iff L is congruent to the broken line with the vertices $(u_1(k), u_2(k), \ldots, u_{2m+1}(k))$ $(k = 0, 1, \ldots, N)$, where

$$u_{2j-1}(k) = \frac{L}{N\sqrt{2(2m+1)}} \cdot \frac{\cos(2\pi jk/N)}{\sin(\pi j/N)},$$

$$u_{2j}(k) = \frac{L}{N\sqrt{2(2m+1)}} \cdot \frac{\sin(2\pi jk/N)}{\sin(\pi j/N)} \quad (j = 1, 2, \ldots, m),$$

$$u_{2m+1}(k) = \frac{kL}{N\sqrt{2m+1}}.$$

If $n = 2m$, then

$$V \leqslant \frac{L^{2m}}{(2m)!(2mN)^m} \cdot \prod_{j=1}^{m} \cotan \frac{(2j-1)\pi}{N}$$

with equality iff L is congruent to the broken line with the vertices $(u_1(k), u_2(k), \ldots, u_{2m}(k))$ $(k = 0, 1, \ldots, N)$, where

$$u_{2j-1}(k) = \frac{L}{2N\sqrt{m}} \cdot \frac{\cos((2j-1)\pi k/N)}{\sin((2j-1)\pi/2N)},$$

$$u_{2j}(k) = \frac{L}{2N\sqrt{m}} \cdot \frac{\sin((2j-1)\pi k/N)}{\sin((2j-1)\pi/2N)} \quad (j = 1, 2, \ldots, m).$$

A. A. Nudeljman, 'Isoperimetričeskie zadači dlja vipuklih oboloček lomanih i krivih v mnogomernih prostranstvah', Mat. Sbornik 96 (136) (1975), 294-313.

2.50. Let L be a broken line of N sides in E^{2m+1} $(N \geqslant 2m + 1)$, let L be the length of L and V the volume of the convex hull of L. If the end points of L have the given distance d, then

$$V \leqslant \frac{d(L^2 - d^2)^m}{(2m + 1)!(2mN)^m} \cdot \prod_{j=1}^{m} \cot \frac{\pi j}{N}$$

with equality iff L is congruent to the broken line with the vertices $(u_1(k), u_2(k), \ldots, u_{2m+1}(k))$ $(k = 0, 1, \ldots, N)$, where

$$u_{2j-1}(k) = \frac{\delta \cos(2\pi jk/N)}{2N \sin(\pi j/N)},$$

$$u_{2j}(k) = \frac{\delta \sin(2\pi jk/N)}{2N \sin(\pi j/N)} \quad (j = 1, 2, \ldots, m),$$

$$u_{2m+1}(k) = \frac{kd}{N}, \quad \delta = \sqrt{\frac{L^2 - d^2}{m}}.$$

A. A. Nudeljman, 'Izoperimetričeskie zadači dlja vipuklih oboloček lomanih i krivih v mnogomernih prostranstvah', Mat. Sbornik 96 (136) (1975), 294-313.

2.51. For any m-gon $P = A_1A_2 \ldots A_m$ in E^n there is a vector

$$\underline{F} = \frac{1}{4} \sum_{i=1}^{m} (\underline{A}_i + \underline{A}_{i+1}) \times (\underline{A}_{i+1} - \underline{A}_i),$$

where the indexes are taken modulo m and × is the cross product. This vector is independent of the choice of the origin and

$$\sum_{i=1}^{m} (A_iA_{i+1})^2 \geqslant 4F \tan \frac{\pi}{n}$$

with equality if and only if P is a regular plane m-gon.

L. Boček, 'Isoperimetrische Ungleichungen für räumliche Kurven und Polygone', Čas. pěst. mat. 104 (1979), 86-92.

2.52.O. Muškarov and H. Lesov considered the inequalities

(1) $$g_n(R, r) \leqslant p^n \leqslant G_n(R, r), (p = 2s).$$

where $g_n(x, y)$ and $G_n(x, y)$ are homogeneous polynomials of degree n. They obtained a general result wherefrom one can obtain Gerretsen's inequalities as well as:

$$256R^2r^2 - 128Rr^3 - 39r^4 \leqslant s^4 \leqslant$$

$$\leqslant 16R^4 + 32R^3r + 32R^2r^2 + 24Rr^3 + 41r^4,$$

$$4096R^3r^3 - 3072R^2r^4 - 797r^6 \leqslant s^6 \leqslant 64R^6 + 192R^5r + 288R^4r^2 +$$

$$+ 304R^3r^3 + 276R^2r^4 + 252Rr^5 + 795r^6.$$

The following result is a consequence of the last inequality:

$$s^3 \leqslant 8R^3 + 12R^2r + 9Rr^2 + (81\sqrt{3} - 130)r^3,$$

and the following of the first:

$$288R^2r^2 - 368Rr^3 + 16r^4 \leqslant \Sigma a^4 \leqslant 32R^4 - 16R^2r^2 - 16Rr^3 + 16r^4.$$

O. Muškarov and H. Lesov, 'Točni polinomialni ocenki za stepenite na poluperimet'ra na tri'g'lnik', Mat. i Matem obrazovanie (Sofija) 1988, pp. 574-578.

2.53.a) $$\sin A \sin B \sin \frac{1}{2} C \leqslant \frac{2\sqrt{3}}{9}$$

with equality for A = B = 54°44', C = 70°32';

b) $$\sin \frac{1}{2} A \sin \frac{1}{2} B \sin C \leqslant \frac{(-5 + 2\sqrt{13})\sqrt{22 + 2\sqrt{13}}}{54}$$

with equality for A = B = 64°15'.5, C = 51°29'.0;

c) $$\sin \frac{1}{4} A \sin \frac{1}{4} B \sin \frac{1}{2} C \leqslant \frac{-14 + 5\sqrt{10}}{54}$$

with equality for A = B = 61°16', C = 57°28'.

For a more general result see IX.2.2.32.

K. P. Williams, 'Note Concerning Some Trigonometric Inequalities', Amer. Math. Monthly 44 (1937), 579-583.

NAME INDEX

(Abbreviations)

SUBJECT INDEX