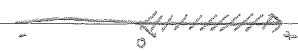
Zad. 1) a) $f(x) = \sqrt{x} + \frac{1}{x}$

$x \geq 0 \land x \neq 0 \Rightarrow x > 0 \quad \Rightarrow \mathcal{D}_f = \langle 0, +\infty \rangle$

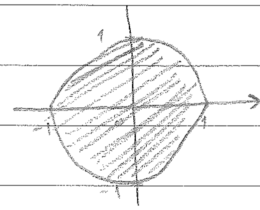


b) $f(x,y) = \sqrt{1 - x^2 - y^2}$

$1 - x^2 - y^2 \geq 0$

$x^2 + y^2 \leq 1 \quad \Rightarrow \mathcal{D}_f = \{(x,y) : x^2 + y^2 \leq 1\}$

jedinični zatvoreni krug



c) $f(x,y) = \frac{1}{x-1} + \arcsin y$

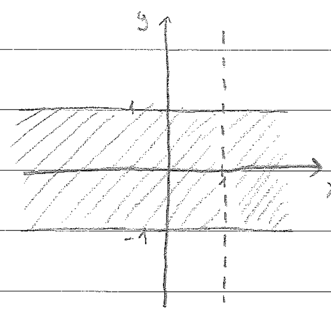
1° $x - 1 \neq 0 \Rightarrow x \neq 1$

2° Znamo da $\sin : \mathbb{R} \to [-1,1]$ nije injekcija, pa nema inverznu f-ju.

Ako napravimo restrikciju $\sin : [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1,1]$, onda možemo definirati $\arcsin : [-1,1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$

$\Rightarrow y \in [-1,1] \iff -1 \leq y \leq 1$

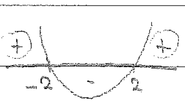
$D_f = \{(x,y) : x \neq 1 \land -1 \leq y \leq 1\}$



d) $f(x,y) = \sqrt{x^2 - 4} + \sqrt[3]{4 - y^2}$

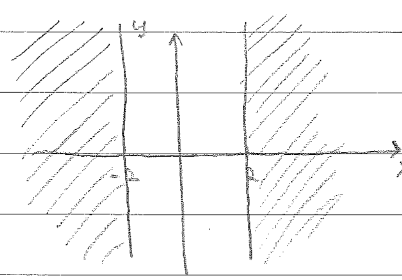
$x^2 - 4 \geq 0$

$x^2 \geq 4$



$\Rightarrow x \in \langle -\infty, -2] \cup [2, +\infty \rangle$

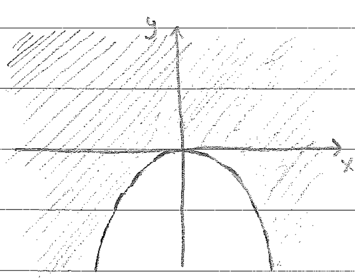
$\mathcal{D}_f = \{(x,y) \in \mathbb{R}^2 : |x| \geq 2\}$



e) $f(x,y) = \ln(x^2 + y)$

$x^2 + y > 0$

$y > -x^2$



$D_f = \{(x,y) \in \mathbb{R}^2 : y > -x^2\}$

f) $f(x,y) = \ln(x\ln(y-x))$

1° $y-x>0$

$y>x$

2° $x\ln(y-x)>0$

$\boxed{x>0}$

$\ln(y-x)>0$

$y-x>1$

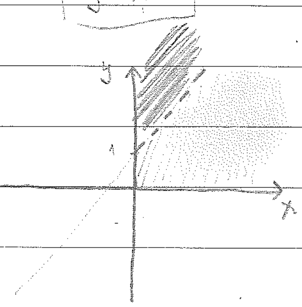
$\boxed{y>x+1}$

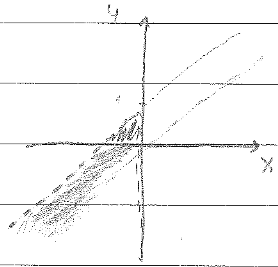
$\boxed{x<0}$

$\ln(y-x)<0$

$0<y-x<1$

$x<\boxed{y<x+1}$





$D_f = \{(x,y)\in\mathbb{R}^2 : x>0,\ y>1+x\} \cup \{(x,y)\in\mathbb{R}^2 : x<0,\ x<y<x+1\}$

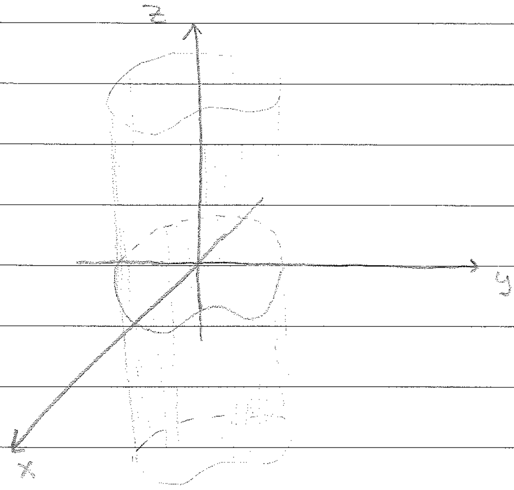
DZ $\begin{cases} \text{g)} & f(x,y) = \sqrt[4]{x^2+y^2+4x-5} \\ \text{h)} & f(x,y) = \sqrt{x^2y+2xy^2+y^3} \end{cases}$

PLOHE

1) Valjkaste (cilindrične) plohe

a) izvodnica je paralelna s osi z i prolazi kroz krivulju $F(x,y)=0$, $z=0$. Jedn. plohe glasi $F(x,y)=0$ (nema varijable z)

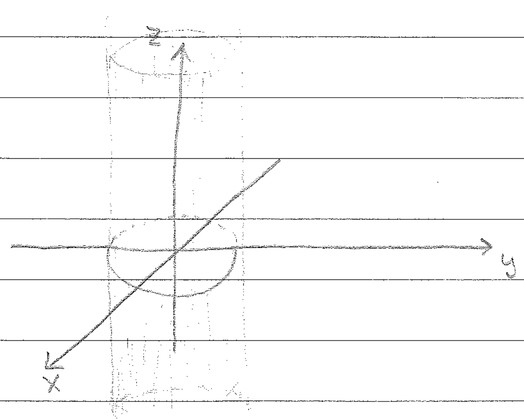
- takve plohe nastaju paralelnim pomakom pravca (izvodnice) duž neke krivulje



b) –//– s osi x i prolazi kroz krivulju $F(y,z)=0$, $x=0$.
–//– $F(y,z)=0$ (nema varijable x)

c) –//– s osi y i prolazi kroz krivulju $F(x,z)=0$, $y=0$.
–//– $F(x,z)=0$ (nema varijable y)

Primjeri I. KRUŽNI VALJAK $x^2+y^2=a^2$



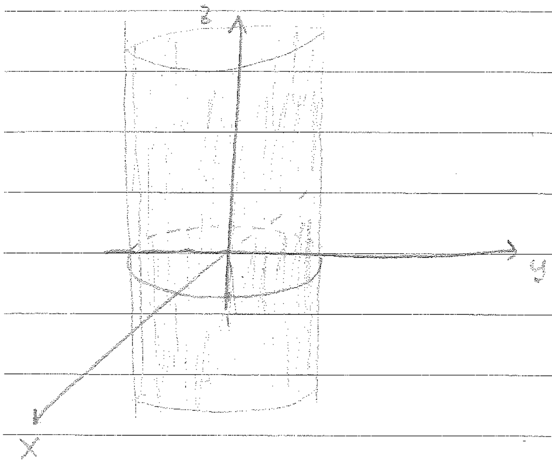
Presjek bilo koje ravnine paralelne sa $z=0$ daje kružnicu polumjera a

II) $x^2+y^2=x$

$x^2-x+y^2=0$

$(x-\frac{1}{2})^2+y^2=\frac{1}{4}$ $\quad S(\frac{1}{2},0)$, $r=\frac{1}{2}$

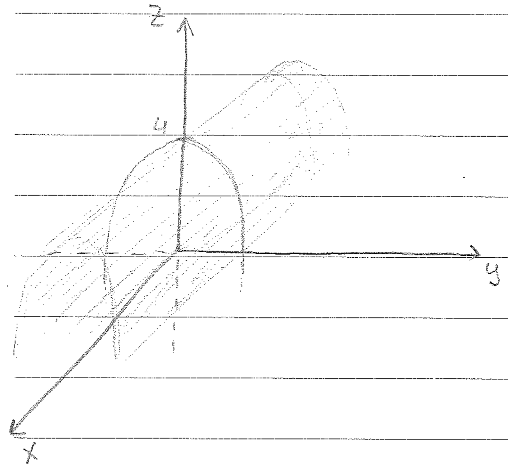
II. ELIPTIČKI VALJAK $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$



presjek s bilo kojom ravnom $z=c$ je elipsa s poluosima a i b

III. PARABOLIČKI VALJAK $z+y^2=4$ (nema varijable x)

$z=4-y^2$



u presjeku s bilo kojom ravnom paralelnom s $x=0$ imamo parabolu $z=4-y^2$

2) Stožaste (konusne) plohe

- ovdje se radi o tipu cilindričnih ploha koje proizvodi pravac (izvodnica) koji prolazi nekom fiksnom točkom i pomiče se duž krivulje

a) Izvodnice kroz ishodište i kroz točke krivulje $\begin{cases} F(x,y)=0 \\ z=1 \end{cases}$

$\Rightarrow F\left(\frac{x}{z},\frac{y}{z}\right)=0$ jednadžba stožaste plohe

b) Izvodnice kroz ishodište i kroz točke krivulje $\begin{cases} F(x,z)=0 \\ y=1 \end{cases}$

$\Rightarrow F\left(\frac{x}{y},\frac{z}{y}\right)=0$

c) Izvodnice kroz ishodište i točke krivulje $\begin{cases} F(y,z)=0 \\ x=1 \end{cases}$

$$\Rightarrow \quad F\left(\frac{y}{x}, \frac{z}{x}\right)=0$$

Primjer: I. KRUŽNI STOŽAC $\quad z^2 = a^2(x^2+y^2) \longrightarrow z = \pm a\sqrt{x^2+y^2}$

$$y^2 = a^2(x^2+z^2)$$

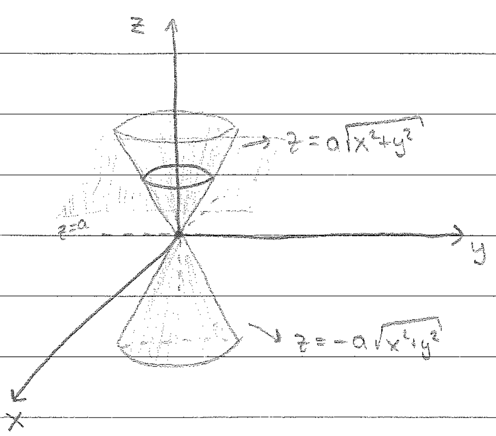
$$x^2 = a^2(y^2+z^2)$$

Što će biti presjek s ravninom $z=a$ ili $z=-a$

$$\pm a = \pm a\sqrt{x^2+y^2}$$

$$\pm 1 = \pm\sqrt{x^2+y^2} \Rightarrow x^2+y^2=1 \quad \text{kružnica } (r=1)$$

$z=0 \Rightarrow x^2+y^2=0 \iff x=y=0 \quad$ vrh $(0,0,0)$



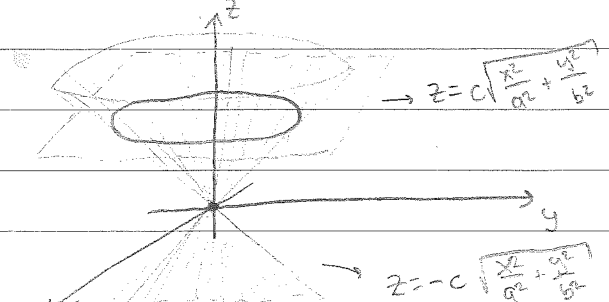
II. ELIPTIČNI STOŽAC $\quad \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-\dfrac{z^2}{c^2}=0$

$$\frac{z^2}{c^2} = \frac{x^2}{a^2}+\frac{y^2}{b^2}$$

$$z = \pm c\sqrt{\frac{x^2}{a^2}+\frac{y^2}{b^2}}$$

$\xRightarrow{z=0} \quad \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=0 \Rightarrow x=y=0 \quad (0,0,0)$

Presjek s ravninom $z=\pm c$ je elipsa s poluosima a i b



III. KRUŽNI STOŽAC (2)

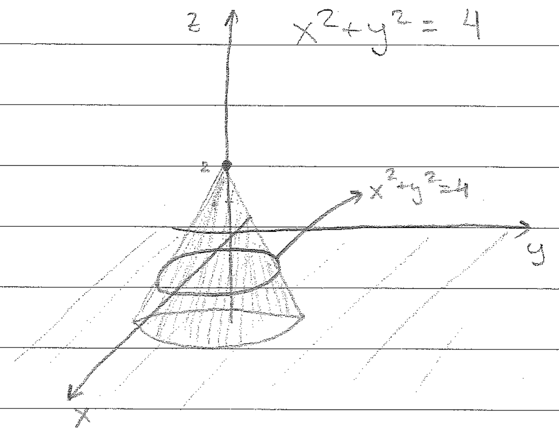
$z = 2 - \sqrt{x^2+y^2}$ $\quad z \le 2$ stožac prema dolje

$\underset{\le 0}{z-2} = \underbrace{-\sqrt{x^2+y^2}}_{\le 0}$

Za $z=2$ imamo $x, y = 0 \Rightarrow$ vrh $(0,0,2)$

Za $z=0$ $\quad -2 = -\sqrt{x^2+y^2} \; /^2$

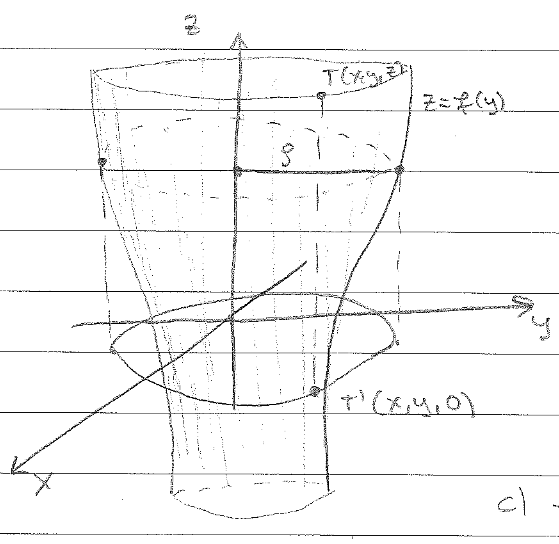
$x^2+y^2 = 4$ kružnica s radijusom $r=2$



3. ROTACIJSKE PLOHE - nastale su rotacijom neke krivulje oko osi x, y ili z

a) Naći jednadžbu rotacijske plohe koja nastaje rotacijom krivulje $z = f(y)$ oko osi z

Stavimo $\rho = \sqrt{x^2+y^2}$ udaljenost proizvoljne točke $T(x,y,z)$ rotacijske plohe od osi z. $z = f(\rho) = f(\sqrt{x^2+y^2})$ ← jedn. rotacijske plohe kojoj je os z os rotacije (ili prirodno) $F(z, x^2+y^2) = 0$



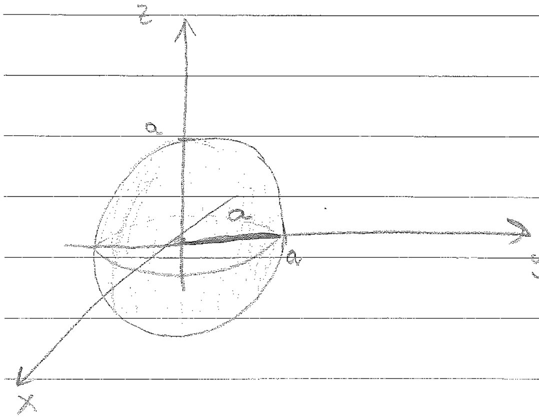
b) jedn. plohe koja nastaje rotacijom krivulje $x = f(z)$ (ili $f(y)$) oko ox osi je s $x = f(\rho) = f(\sqrt{y^2+z^2})$

$F(x, \sqrt{y^2+z^2})$

c) -//- oko y-osi

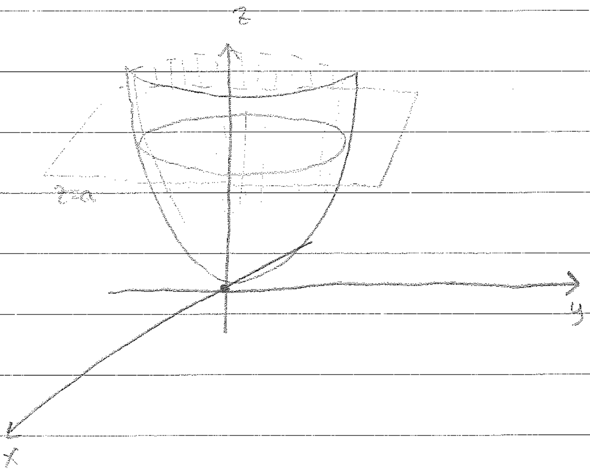
Primjeri: I. Sfera (kuglina ploha)

$x^2+y^2+z^2=a^2$ nastala rotacijom polukružnica (ili kružnica) oko sve tri osi



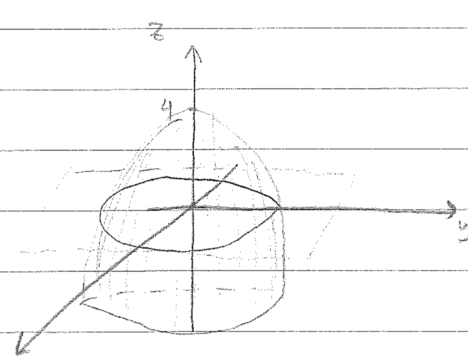
II. Rotacijski paraboloid $z=a(x^2+y^2)$, $a>0$

$x=0$ | $y=0$ | $z=a$
$z=ay^2$ (parabola) | $z=ax^2$ (parabola) | $x^2+y^2=1$ (kružnica)



Nastaje rotacijom krivulje $z=ay^2$ ili $z=ax^2$ oko osi z

3) $z=4-x^2-y^2$ | $z-4\le 0$ | $z=4 \Rightarrow x^2+y^2=0$
$z-4=-(x^2+y^2)$ | $z\le 4$ | vrh $(0,0,4)$



$z=0 \quad -4=-(x^2+y^2)$
$\Rightarrow x^2+y^2=4$

krovna oko osi bez kvadrata

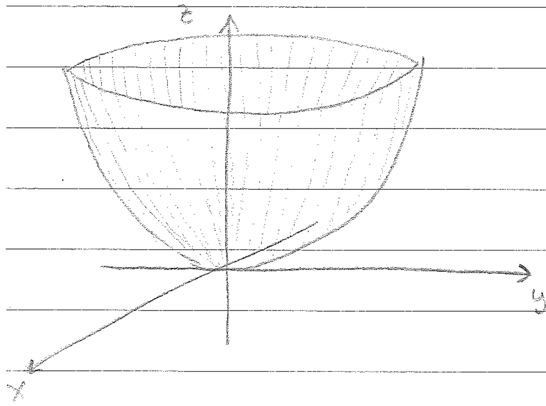
Jednadžbe nekih ploha

① Eliptički paraboloid $z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$

* c=0 dobijemo vrh

Presjek s bilo kojom ravninom $z=c,\ c>0$ daje elipsu

Presjek s bilo kojom ravninom $x=0$ ili $y=0$ daje parabolu

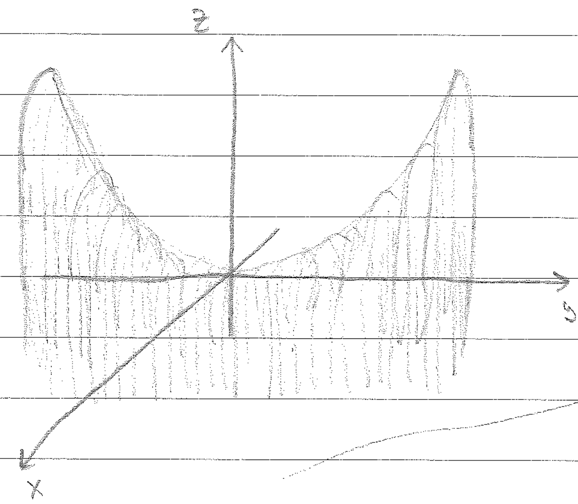


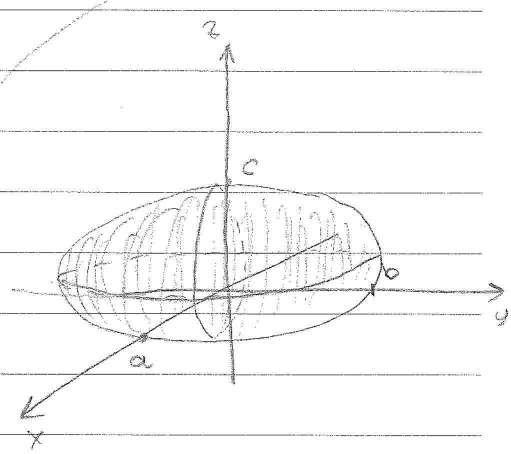
② Hiperbolički paraboloid (sedlasta ploha) $z = \frac{x^2}{a^2} - \frac{y^2}{b^2}$

U presjeku s $y=c$ ili $x=c$ su parabole

U presjeku sa $z=c,\ c>0$ dobivamo hiperbole

„nastaje klizanjem parabole po paraboli"





③ Elipsoid (troosni) $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

$a>b>c>0$

a – velika poluos

b – srednja poluos

c – mala poluos

* Ako je $a=b=c$ imamo sferu

Limes i neprekidnost

Tm (Heineova karakterizacija)

$f: D \subseteq \mathbb{R}^2 \to \mathbb{R}$ nep. u $P_0=(a,b) \in D'$ $\Leftrightarrow$ $\forall$ niz (x_n) $x_n \to P_0$

$f(x_n) \to f(P_0)$

Zad Ispitajte nep. f-e $f(x,y) = (x^2+y^2)\cdot \sin\left(\frac{1}{xy}\right)$

$D_f = \{(x,y)\in\mathbb{R}^2 : x\neq 0 \wedge y\neq 0\}$

Funkcija je produkt polinoma i kompozicije funkcije sinus i racionalne funkcije. Sve su ovo nep. f-e na svojoj domeni pa je i f neprekidna

Što je s limesom u $(0,0)$?

$\cancel{\lim_{\substack{x\to 0\\ y\to 0}} (x^2+y^2)\sin\left(\frac{1}{xy}\right)}$

Dokažimo da je limes jednak 0

$$|f(x,y)-0| = \left|(x^2+y^2)\sin\left(\frac{1}{xy}\right)\right| = \underbrace{|x^2+y^2|}_{\geq 0}\underbrace{\left|\sin\left(\frac{1}{xy}\right)\right|}_{\leq 1} \leq x^2+y^2 < \delta^2 = \varepsilon$$

$\delta := \sqrt{\varepsilon}$

$d((x,y),(0,0)) < \delta$

$\sqrt{x^2+y^2} < \delta$ uzmemo $\varepsilon = \delta^2$

$x^2+y^2 < \delta^2$

$$\Rightarrow \lim_{\substack{x\to 0\\ y\to 0}} \left[(x^2+y^2)\sin\left(\frac{1}{xy}\right)\right] = 0 \quad //$$

2. domaća zadaća (predavanja) Molnar, Borna

Zadatak 1. Dokažite da definicija neprekidnosti ne ovisi o izboru norme na $\mathbb{R}^n$ i $\mathbb{R}^m$.

Dokaz. Neka su $\|\cdot\|_a$ i $\|\cdot\|_b$ neke norme na $\mathbb{R}^n$ i $\mathbb{R}^m$ te neka je $f: \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^m$ neprekidna funkcija u P_0. Tada za $\|\cdot\|_a$ vrijedi:

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall P \in \Omega)\ \|P - P_0\|_a < \delta \Rightarrow \|f(P) - f(P_0)\|_a < \varepsilon$$

Pokažimo da gornja implikacija vrijedi i za $\|\cdot\|_b$. Ove dvije norme definirane su na konačnodimenzionalnim vektorskim prostorima, pa su ekvivalentne, odnosno vrijedi (za svaki $x \in \mathbb{R}^n$, odnosno $\mathbb{R}^m$)

$$m\|x\|_b \le \|x\|_a \le M\|x\|_b \iff \frac{1}{M}\|x\|_a \le \|x\|_b \le \frac{1}{m}\|x\|_a, \qquad m, M > 0$$

Sada imamo $m\|P - P_0\|_b \le \|P - P_0\|_a < \delta \Rightarrow \|P - P_0\|_b < \frac{\delta}{m} < \delta$ i $\|f(P) - f(P_0)\|_b \le \frac{1}{m}\|f(P) - f(P_0)\|_a < \frac{\varepsilon}{m} < \varepsilon$. Dakle

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall P \in \Omega)\ \|P - P_0\|_b < \delta \Rightarrow \|f(P) - f(P_0)\|_b < \varepsilon \qquad \text{Q.E.D}$$

Zadatak 2. Dokažite Heineovu karakterizaciju neprekidnosti.

- Teorem: Neka je $\Omega \subseteq \mathbb{R}^n$ otvoren skup i $f: \Omega \to \mathbb{R}^m$ dana funkcija. Funkcija f neprekidna je u točki $P_0 \in \Omega$ ako i samo ako vrijedi $\lim_{P \to P_0} f(P) = f(P_0)$.

Dokaz. ($\Rightarrow$) Neka je f neprekidna u $P_0 \in \Omega$, tada vrijedi

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall P \in \Omega)\ 0 \le \|P - P_0\| < \delta \Rightarrow \|f(P) - f(P_0)\| < \varepsilon$$

što prema definiciji limesa znači da f ima limes $f(P_0)$ u P_0.

($\Leftarrow$) Neka je $\lim_{P \to P_0} f(P) = f(P_0)$, tada vrijedi

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall P \in \Omega) \quad 0 < \|P - P_0\| < \delta \Rightarrow \|f(P) - f(P_0)\| < \varepsilon$$

Još treba pokazati da nije problem uzeti $P = P_0$ jer za neprekidnost nemamo uvjet $\|P - P_0\| > 0$. Ako bismo uzeli $P = P_0$, dobili bismo $0 = \|P - P_0\| < \delta$ i $0 = \|f(P) - f(P_0)\| < \varepsilon$ što je u skladu s uvjetima na δ i ε u definiciji neprekidnosti. Q.E.D

$\boxed{\text{Zadatak 3.}}$ a) Ako postoji dvostruki limes, postoje li uzastopni limesi i jesu li jednaki? (Š. Ungar, Matematička analiza 3)

→ Ne moraju postojati, npr. promotrimo funkciju $f(x,y) = (x+y)\sin\frac{1}{x}\sin\frac{1}{y}$. Pokažimo da je $\lim_{(x,y)\to(0,0)} f(x,y) = 0$, ali da uzastopni limesi ne postoje.

$\lim_{(x,y)\to(0,0)} f(x,y) = 0$ znači da kada $(x,y) \to (0,0)$, $|f(x,y) - 0| \to 0$, pa pokažimo da to vrijedi

$$|f(x,y)| = |x+y||\sin\tfrac{1}{x}||\sin\tfrac{1}{y}|$$

$$\Rightarrow \quad 0 \leq |x+y|\underbrace{|\sin\tfrac{1}{x}||\sin\tfrac{1}{y}|}_{\in[0,1]} \leq x+y$$

Prema teoremu o sendviču:

$$\lim_{(x,y)\to(0,0)} 0 \leq \lim_{(x,y)\to(0,0)} |x+y||\sin\tfrac{1}{x}||\sin\tfrac{1}{y}| \leq \lim_{(x,y)\to(0,0)} (x+y)$$

$$0 \leq \lim_{(x,y)\to(0,0)} |x+y||\sin\tfrac{1}{x}||\sin\tfrac{1}{y}| \leq 0 \Rightarrow \lim_{(x,y)\to(0,0)} f(x,y) = 0$$

Pogledajmo što je s uzastopnim limesima

$$\lim_{x\to 0}\lim_{y\to 0} f(x,y) = \lim_{x\to 0}\lim_{y\to 0} (x+y)\sin\tfrac{1}{x}\sin\tfrac{1}{y} = \lim_{x\to 0}\lim_{y\to 0}\left(x\sin\tfrac{1}{x}\sin\tfrac{1}{y} + y\sin\tfrac{1}{x}\sin\tfrac{1}{y}\right)$$

$$= \lim_{x\to 0}\left[\lim_{y\to 0} x\sin\tfrac{1}{x}\sin\underbrace{\tfrac{1}{y}}_{\to\infty} + \lim_{y\to 0} y\sin\tfrac{1}{x}\sin\tfrac{1}{y}\right]$$

$\nexists \sin\infty \Rightarrow \nexists \lim$.

Analogno za drugi limes.

b) Ako postoje uzastopni limesi i jednaki su, postoji li i dvostruki?

→ Ne mora postojati dvostruki, npr. promotrimo funkciju $f: \mathbb{R}^2 \setminus \{0,0\} \to \mathbb{R}$ zadanu formulom $f(x,y) = \frac{x^2y^2}{x^2y^2+(x-y)^2}$. Lako se pokaže da su uzastopni limesi kad $(x,y) \to (0,0)$ jednaki 0. Pokažimo sada da $\lim_{(x,y)\to(0,0)} f(x,y)$ ne postoji.

Teorem (o limesu restrikcije) Neka je $f: \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^m$ dana funkcija i $\Sigma \subseteq \Omega$. Ako f ima limes u točki $P_0 \in \Sigma'$, onda i restrikcija $f|_\Sigma : \Sigma \to \mathbb{R}^m$ ima limes u P_0 i ta su dva limesa jednaka.

Dakle, ako pretpostavimo da f ima limes u $(0,0)$, onda bi svaka restrikcija trebala imati taj isti limes u $(0,0)$. Uzmimo recimo $S = \{(x,0) \in \mathbb{R}^2 : x \in \mathbb{R} \setminus \{0\}\} \subseteq \mathbb{R}^2 \setminus \{0,0\}$ i pogledajmo $\lim_{(x,0)\to(0,0)} f|_S$.

$$\lim_{(x,0)\to(0,0)} \frac{x^2(0)^2}{x^2(0)^2+(x-0)^2} = \lim_{(x,0)\to(0,0)} \frac{0}{x^2} = 0$$

Uzmimo sada $T = \{(x,x) \in \mathbb{R}^2 : x \in \mathbb{R} \setminus \{0\}\} \subseteq \mathbb{R}^2 \setminus \{0,0\}$ i pogledajmo $\lim_{(x,x)\to(0,0)} f|_T$:

$$\lim_{(x,x)\to(0,0)} \frac{x^2x^2}{x^2x^2+(x-x)^2} = \lim_{(x,x)\to(0,0)} \frac{x^4}{x^4} = 1 \qquad \lightning$$

Ta dva limesa nisu ista, pa ne postoji ni početni limes.

1. domaća zadaća (vježbe) Molnar, Borna

$\boxed{\text{Zadatak 1.}}$ Odredite i skicirajte domenu funkcije $f(x,y)=\sqrt{\dfrac{x^2+y^2-25}{x+y}}+\dfrac{1}{\ln(x^2-25)}$

1. uvjet: $x+y\neq 0 \Rightarrow y\neq -x$

2. uvjet: $\dfrac{x^2+y^2-25}{x+y}\geq 0$

1° $x^2+y^2-25\geq 0$ $\qquad$ $x^2+y^2-25\leq 0$

$x^2+y^2\geq 25$ $\qquad$ $x^2+y^2\leq 25$

& $x+y>0$ $\qquad$ & $x+y<0$

$y>-x$ $\qquad$ $y<-x$

3. uvjet: $\ln(x^2-25)\neq 0$

$x^2-25\neq 1$

$x^2\neq 26$

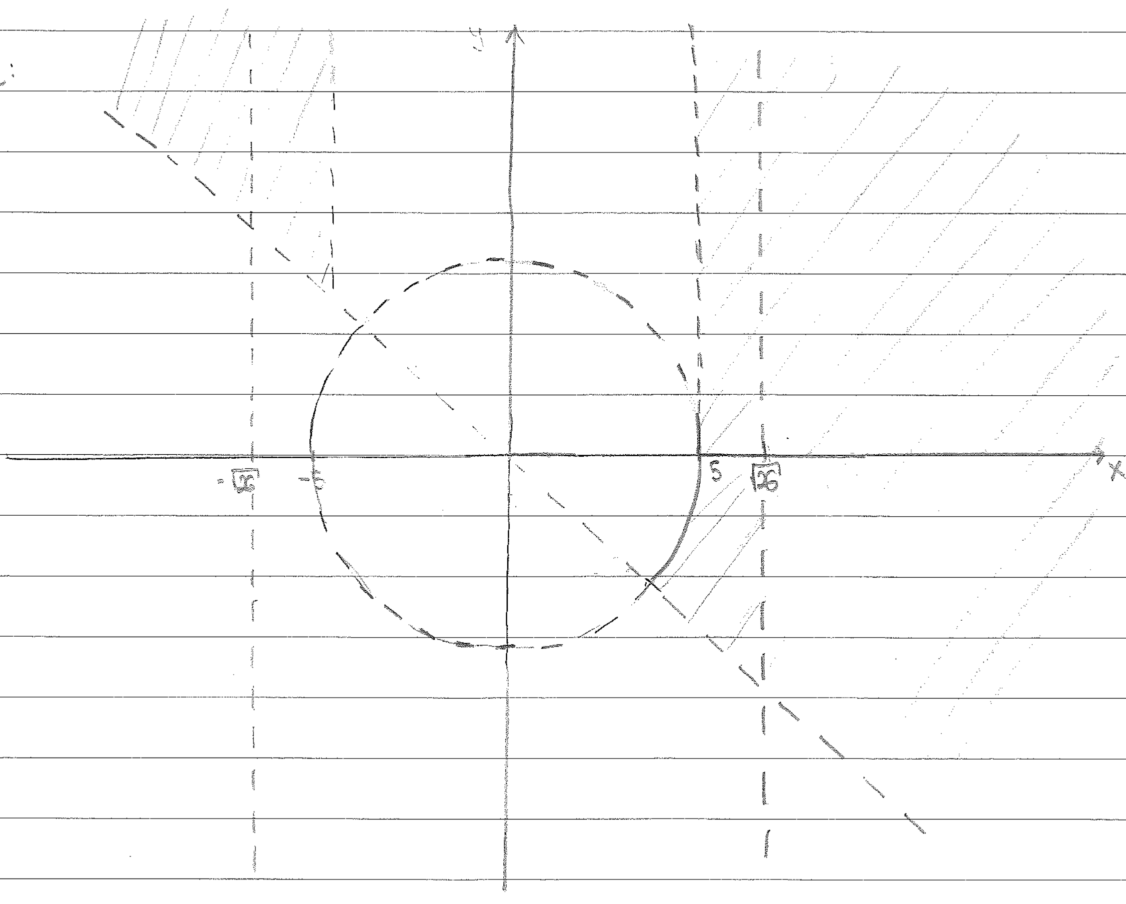
$x\neq \pm\sqrt{26}$

4. uvjet $x^2-25>0$

$x^2>25 \Rightarrow x\in\langle-\infty,-5\rangle\cup\langle 5,+\infty\rangle$

$\Rightarrow D_f=\{(x,y)\in\mathbb{R}^2:\ x^2+y^2\geq 25,\ y>-x \ \wedge\ x^2+y^2\leq 25,\ y<-x\ \wedge\ x\neq\pm\sqrt{26}\ \wedge\ x\in\langle-\infty,-5\rangle\cup\langle 5,\infty\rangle\}$

Skica:



Zadatak 2.

a) $z^2 = 25((x-2)^2 + y^2)$ kružni stožac

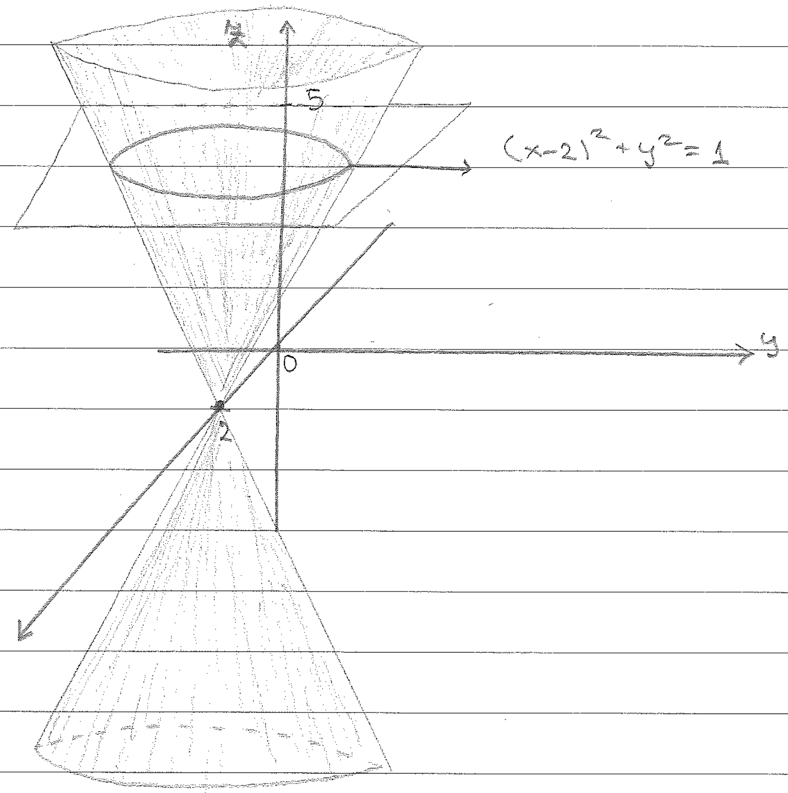
$\Rightarrow z = \pm 5\sqrt{(x-2)^2 + y^2}$ Presjek s ravninama $z = \pm 5$

$\pm 5 = \pm 5\sqrt{(x-2)^2 + y^2}$

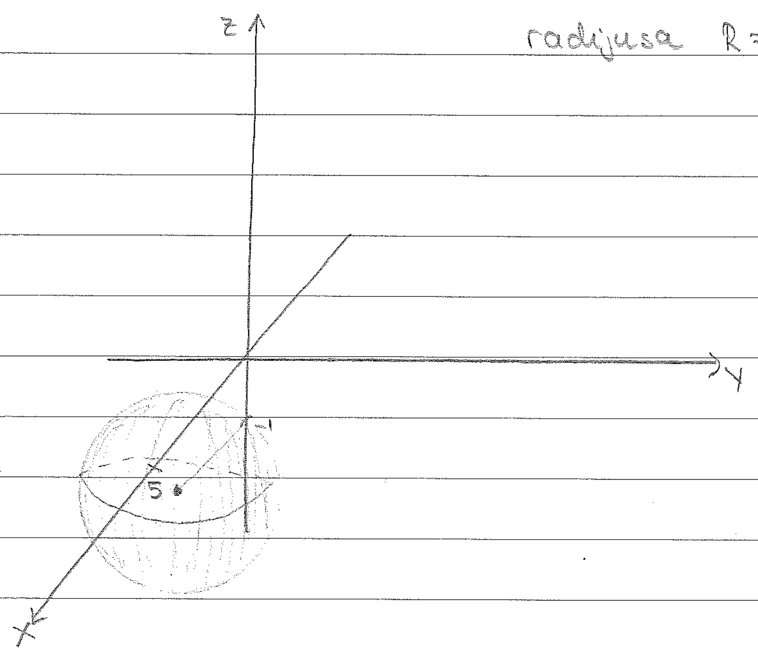
$\Rightarrow (x-2)^2 + y^2 = 1$ kružnica

Za $z = 0 \Rightarrow (x-2)^2 + y^2 = 0 \Rightarrow x = 2,\ y = 0 \Rightarrow$ vrh u $(2,0,0)$

Skica:



b) $(x-5)^2 + y^2 + (z+1)^2 = 25$ sfera sa središtem u $(5,0,-1)$ radijusa $R = 5$.



3. VJEŽBE 24.10.2024.

Zad.1. Postoji li $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$?

Promatramo restrikciju na $y = kx$, $k \in \mathbb{R}$

$$\lim_{(x,kx)\to(0,0)} \frac{x^2-k^2x^2}{x^2+k^2x^2} = \lim_{(x,kx)\to(0,0)} \frac{(1-k^2)\cancel{x^2}}{(1+k^2)\cancel{x^2}} = \frac{1-k^2}{1+k^2}$$

limes ovisi o izboru $k \in \mathbb{R}$, pa $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ ne postoji!

Zad.2. Dokažite da je $\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2} = 0$.

Treba pokazati

$$(\forall \varepsilon>0)(\exists \delta>0)(\forall (x,y)\in \Omega) \quad 0 < \|(x,y)-(0,0)\| < \delta \Rightarrow |f(x,y) - \cancel{0}| < \varepsilon$$

$$\sqrt{x^2+y^2} < \delta$$

$$|f(x,y)| = \left|\frac{x^3+y^3}{x^2+y^2}\right| = \frac{|x^3+y^3|}{x^2+y^2} \overset{\text{n.t.}}{\leq} \frac{|x^3|+|y^3|}{x^2+y^2} = \frac{x^2|x|+y^2|y|}{x^2+y^2}$$

$$= \underbrace{\frac{x^2}{x^2+y^2}}_{\leq 1}|x| + \underbrace{\frac{y^2}{x^2+y^2}}_{\leq 1}|y| \leq |x|+|y| = \sqrt{x^2}+\sqrt{y^2} \leq \sqrt{x^2+y^2}+\sqrt{x^2+y^2} <$$

$$= 2\sqrt{x^2+y^2} = 2\delta = \varepsilon \quad \Rightarrow \quad \delta := \frac{\varepsilon}{2}$$

Q.E.D.

Zad.3. Dokažite da je $\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2} = 0$.

$\|(x,y)-(0,0)\| < \delta$

$\sqrt{x^2+y^2} < \delta$

$$|f(x,y)-0| = \left|\frac{x^2y}{x^2+y^2}\right| = \frac{|x^2y|}{x^2+y^2}$$

$$= \frac{x^2|y|}{x^2+y^2} \leq |y| = \sqrt{y^2} \leq \sqrt{x^2+y^2} < \delta = \varepsilon$$

Uzmemo $\delta := \varepsilon$.

Q.E.D

Zad. 4) Koristeći Heineovu karakterizaciju neprekidnosti, dokažite da ne postoji $\lim\limits_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$, tj. da se f ne može proširiti u $(0,0)$ po neprekidnosti

$x_n \to (0,0) \quad f(x_n) \to * = f(0,0)$ ↯ s definicijom funkcije
$x_n' \to (0,0) \quad f(x_n') \to ** = f(0,0)$

Uzmemo nizove $x_n = \left(\frac{1}{n}, \frac{1}{n}\right)_{n\in\mathbb{N}}$ i $x_n' = \left(\frac{1}{n}, -\frac{1}{n}\right)_{n\in\mathbb{N}}$

$$\left(\frac{1}{n}, \frac{1}{n}\right) \xrightarrow{n\to\infty} (0,0) \Rightarrow f(x_n) \to f(0,0)$$

$$f(x_n) = \frac{\frac{1}{n^2}}{\frac{1}{n^2}+\frac{1}{n^2}} = \frac{1}{2} \xrightarrow{n\to\infty} \frac{1}{2} = f(0,0)$$

$$\left(\frac{1}{n}, -\frac{1}{n}\right) \xrightarrow{n\to\infty} (0,0) \Rightarrow f(x_n') \to f(0,0)$$

$$f(x_n') = \frac{-\frac{1}{n^2}}{\frac{1}{n^2}+\frac{1}{n^2}} = -\frac{1}{2} \xrightarrow{n\to\infty} -\frac{1}{2} = f(0,0) \Rightarrow \frac{1}{2} = -\frac{1}{2} \quad ↯$$

dakle, limes funkcije u $(0,0)$ ne postoji

Zad. Odredite limes ako postoji

DZ. $$\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2y^2+(x-y)^4}$$

Zad. Ispitajte neprekidnost u točki $(0,0)$:

a) $$f(x,y) = \begin{cases} \frac{x^4-y^4}{x^2+y^2}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0) \end{cases}$$

$$\lim_{(x,y)\to(0,0)} \frac{x^4-y^4}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{(x^2-y^2)\cancel{(x^2+y^2)}}{\cancel{x^2+y^2}} = \lim_{(x,y)\to(0,0)} (x^2-y^2) = 0 = f(0,0)$$

$\Rightarrow f$ je neprekidna u $(0,0)$ //

b) $f(x,y) = \begin{cases} \dfrac{x^2 y}{x^4+y^4}, & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

Rj. $x = r\cos\varphi \qquad y = r\sin\varphi$ (polarne kordinate)

$(x,y) \to (0,0) \Rightarrow r \to 0$

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^4} \overset{\text{supstitucija}}{=} \lim_{r\to 0} \frac{r^2\cos^2\varphi \cdot r\sin\varphi}{r^4\cos^4\varphi + r^4\sin^4\varphi} = \lim_{r\to 0} \frac{r^3\cos^2\varphi\sin\varphi}{r^4(\cos^4\varphi+\sin^4\varphi)}$$

$$= \lim_{r\to 0} \underbrace{\frac{1}{r}}_{\to\infty} \cdot \frac{\cos^2\varphi\sin\varphi}{\cos^4\varphi+\sin^4\varphi}$$

Kako god biramo φ, ovaj limes je ∞ pa $\nexists$.
$\Rightarrow f$ ima prekid u $(0,0)$

Zad. Koristeći polarne kordinate, dokažite da ne postoji $\displaystyle\lim_{(x,y)\to(2,1)} \frac{1-4x+x^2+6y-3y^2}{5-4x+x^2-2y+y^2}$

Rj. $x = r\cos\varphi + 2$
$y = r\sin\varphi + 1$

$$\lim_{(x,y)\to(2,1)} \frac{(x-2)^2-3+6y-3y^2}{(x-2)^2+1-2y+y^2}$$

$$\Rightarrow \lim_{(x,y)\to(2,1)} \frac{(x-2)^2-3(y-1)^2}{(x-2)^2+(y-1)^2} \Rightarrow \lim_{r\to 0} \frac{r^2\cos^2\varphi - 3r^2\sin^2\varphi}{r^2\cos^2\varphi + r^2\sin^2\varphi}$$

$$= \lim_{r\to 0} \frac{\cos^2\varphi - 3\sin^2\varphi}{\underbrace{\cos^2\varphi+\sin^2\varphi}_{1}} = \cos^2\varphi - 3\sin^2\varphi$$

limes ovisi o putu, tj. o $\varphi \Rightarrow \nexists \lim$ u toč. $(2,1)$

DZ dokažite $f:\mathbb{R}^2\to\mathbb{R}$ $f(x,y)=\begin{cases}\dfrac{x^2-2y^2-3x-y}{x^2+y}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$

ima prekid u $(0,0)$.

DZ ispitajte postoji li limes f-je $f(x,y)=\dfrac{x-y+x^2+y^2}{x+y}$ u $(0,0)$

Zad. Može li se f-ja $f(x,y)=xy\cdot\dfrac{y^2-(x+y)^2}{x^4+3y^4}$ dodefinirati

u točki $(0,0)$ tako da bude neprekidna na $\mathbb{R}^2$?

Probajmo pokazati da limes ne postoji

$y=kx$ $\qquad \lim\limits_{x\to0} x\cdot kx\,\dfrac{k^2x^2-(x+kx)^2}{x^4+3k^4x^4}=\lim\limits_{x\to0} kx^2\,\dfrac{k^2x^2-(1+k)^2x^2}{x^4+3k^4x^4}$

$$=\lim_{x\to0}\frac{k^3x^4-k\overbrace{(1+k)^2}^{1+2k+k^2}x^4}{x^4+3k^4x^4}=\lim_{x\to0}\frac{(\cancel{k^3}-k-2k^2-\cancel{k^3})\cancel{x^4}}{(1+3k^4)\cancel{x^4}}$$

$=-\dfrac{2k^2+k}{1+3k^4}$ $\Rightarrow$ limes ovisi o k, pa ne postoji

$\Rightarrow$ f se ne može dodefinirati

(Pr 1) Izračunajte parcijalne derivacije funkcije $f(x,y) = \ln \operatorname{tg} \frac{x}{y}$

$$\frac{\partial f(x,y)}{\partial x} = \frac{1}{\operatorname{tg}\frac{x}{y}} \cdot \frac{\partial}{\partial x}\left(\operatorname{tg}\frac{x}{y}\right) = \frac{1}{\operatorname{tg}\frac{x}{y}} \cdot \frac{1}{\cos^2\frac{x}{y}} \cdot \frac{1}{y} \quad //$$

$$\frac{\partial f(x,y)}{\partial y} = \frac{1}{\operatorname{tg}\frac{x}{y}} \cdot \frac{1}{\cos^2\frac{x}{y}} \cdot \left(-\frac{x}{y^2}\right) \quad //$$

(Zad.) 1) $f(x,y,z) = (x\cdot y)^z$

$$\frac{\partial f}{\partial x} = z(xy)^{z-1} \cdot y \quad // \qquad \frac{\partial f}{\partial y} = z(xy)^{z-1} \cdot x$$

$$\frac{\partial f}{\partial z} = (xy)^z \ln(xy) \quad //$$

DZ
2) $f(x,y,z) = x^y y^z \cdot z^{xyz}$

3) $f(x,y) = \sqrt{xy + \frac{x}{y}}$

$$\frac{\partial f}{\partial x} = \frac{1}{2\sqrt{xy+\frac{x}{y}}} \cdot \frac{\partial}{\partial x}\left(xy + \frac{x}{y}\right) = \frac{1}{2\sqrt{xy+\frac{x}{y}}}\left(y + \frac{1}{y}\right) \quad //$$

$$\frac{\partial f}{\partial y} = \frac{1}{2\sqrt{xy+\frac{x}{y}}} \cdot \frac{\partial}{\partial y}\left(xy + \frac{x}{y}\right) = \frac{1}{2}\left(xy + \frac{x}{y}\right)^{-\frac{1}{2}} \cdot \left(x - \frac{x}{y^2}\right) \quad //$$

Diferencijabilnost

$$\begin{tikzcd}
 & \text{nep.} \arrow[dl, shift left, "\not"] \arrow[dr, shift left, "\not"] & \\
\text{dif} \arrow[ur, shift left] \arrow[rr, shift left] & & \text{parc der} \arrow[ll, shift left, "\not"] \arrow[ul, shift left, "\not"]
\end{tikzcd}$$

vrijedi samo ako su neprekidne

(Zad.) Izračunajte gradijent i diferencijal f-je $f(x,y) = e^{\sin\frac{y}{x}}$ u $(1,\pi)$

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial x} \end{bmatrix} = \begin{bmatrix} -e^{\sin\frac{y}{x}} \cos\frac{y}{x} \cdot \frac{y}{x^2} \\ e^{\sin\frac{y}{x}} \cos\frac{y}{x} \cdot \frac{1}{x} \end{bmatrix}\Bigg|_{(1,\pi)} = \begin{bmatrix} \pi \\ -1 \end{bmatrix}$$

$$Df(1,\pi)(x_0,y_0) = \pi x_0 - y_0$$

Zad. $f(x,y,z) = x^3y^2z + 2x - 3y + z + 5$ u $(1,2,0)$

$$\nabla f(1,2,0) = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{bmatrix} = \begin{bmatrix} 3x^2y^2z + 2 \\ 2x^3yz - 3 \\ x^3y^2 + 1 \end{bmatrix}\Bigg|_{(1,2,0)} = \begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix}$$

$$Df(1,2,0)(x_1,x_2,x_3) = 2x_1 - 3x_2 + 5x_3$$

Zad. Dana je f-ja $f: \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = \begin{cases} \frac{2x^2y - y^3}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

a) Izračunajte sve parcijalne derivacije f-je f na $\mathbb{R}^2$

b) Odredite sve točke u kojima je f diferencijabilna. Je li f klase C^1 na $\mathbb{R}^2$

c) Izračunajte $Df(1,-1)(2,1)$

Rj: a) $\frac{\partial f}{\partial x} = \begin{cases} \frac{4xy\cdot(x^2+y^2) - (2x^2y - y^3)\cdot 2x}{(x^2+y^2)^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$ $= \frac{6xy^3}{(x^2+y^2)^2}$ kada se sredi

$$\frac{\partial f}{\partial y} = \begin{cases} \frac{(2x^2 - 3y^2)\cdot(x^2+y^2) - (2x^2y - y^3)\cdot 2y}{(x^2+y^2)^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

b) Ako parcijalne derivacije postoje i neprekidne su u P_0, onda je f diferencijabilna u P_0. Parcijalne derivacije $\partial_x f$ i $\partial_y f$ neprekidne su sigurno u svim točkama iz $\mathbb{R}^2 \setminus \{0,0\}$. Moramo dodatno provjeriti jesu li neprekidne u $(0,0)$.

$$\lim_{(x,y)\to(0,0)} \frac{6xy^3}{(x^2+y^2)^2} \overset{\text{po putu } y=kx}{=} \lim_{x\to 0} \frac{6xk^3x^3}{(x^2+k^2x^2)^2} = \lim_{x\to 0} \frac{6k^3\cancel{x^4}}{(1+k^2)^2\cancel{x^4}} = \frac{6k^3}{(1+k^2)^2}$$

Limes ovisi o putu (o izboru $k \in \mathbb{R}$), pa $\partial_x f$ ima prekid u $(0,0)$

$\Rightarrow$ ne može se zaključiti dif. u $(0,0)$ $\Rightarrow$ f nije klase C^1 na $\mathbb{R}^2$

Provjerimo preko definicije

$$\lim_{H\to 0} \frac{f(P_0+H)-f(P_0)-\mathcal{A}H}{\|H\|}=0 \qquad H=(h_1,h_2)\in\mathbb{R}^2$$

$$\mathcal{A}=Df(P_0)$$

Uzmemo $P_0=(0,0)$; $H=(x,y)\in\mathbb{R}^2$

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y)-f(0,0)-\overbrace{\mathcal{A}(x,y)}^{\langle\nabla f(0,0),(x,y)\rangle}}{\|(x,y)\|}=0$$

$$\lim_{(x,y)\to(0,0)} \frac{\frac{2x^2y-y^3}{x^2+y^2}-0-0}{\sqrt{x^2+y^2}}=0$$

$$\lim_{(x,y)\to(0,0)} \frac{2x^2y-y^3}{(x^2+y^2)\sqrt{x^2+y^2}}=0$$

$x=r\cos\varphi$	$x,y\to 0$	$\lim_{r\to 0} \frac{2r^2\cos^2\varphi\, r\sin\varphi-r^3\sin^3\varphi}{r^2\cdot r}$
$y=r\sin\varphi$	$r\to 0$	

$$=2\cos^2\varphi\sin\varphi-\sin^3\varphi\neq 0$$

$\Rightarrow$ f nije diferencijabilna !

c) $Df(1,-1)(2,1)=\langle\nabla f(1,-1),(2,1)\rangle=\frac{\partial f(1,-1)}{\partial x}\cdot 2+\frac{\partial f(1,-1)}{\partial y}\cdot 1$

$$=\frac{6\cdot 1\cdot(-1)^3}{(1^2+(-1)^2)^2}\cdot 2+\frac{(2\cdot 1^2-3\cdot(-1)^2)\cdot(1^2+(-1)^2)-(2\cdot 1^2\cdot(-1)-(-1)^3)\cdot 2(-1)}{(1^2+(-1)^2)^2}$$

$\dots=-4$

Zad 1. $f(x,y,z) = y^x x^z z^{xyz}$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(y^x x^z) \cdot z^{xyz} + (y^x x^z)\frac{\partial}{\partial x}(z^{xyz})$$
$$= \left[\frac{\partial}{\partial x}(y^x)\cdot x^z + y^x \frac{\partial}{\partial x}(x^z)\right]\cdot z^{xyz} + (y^x x^z)\frac{\partial}{\partial x}(z^{xyz})$$
$$= (y^x \ln y \cdot x^z + y^x \cdot z x^{z-1}) \cdot z^{xyz} + y^x x^z \cdot z^{xyz} \ln z \cdot yz$$

$$= y^x x^z \cdot z^{xyz} \cdot \ln y + y^x z^{xyz+1} x^{z-1} + y^{x+1} x^z \cdot z^{xyz+1} \cdot \ln z \quad //$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(y^x x^z) \cdot z^{xyz} + (y^x x^z)\frac{\partial}{\partial y}(z^{xyz})$$
$$= x^{z+1} y^{x-1} \cdot z^{xyz} + y^x x^z z^{xyz} \cdot \ln z \cdot xz$$
$$= x^{z+1} y^{x-1} \cdot z^{xyz} + y^x x^{z+1} z^{xyz+1} \ln z \quad //$$

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial z}(y^x x^z) \cdot z^{xyz} + y^x x^z \underbrace{\frac{\partial f}{\partial z}(z^{xyz})}_{\alpha} = \ldots$$

Odredimo α:

$$w(z) = z^{xyz} \quad / \ln$$
$$\ln w(z) = xyz \ln z \quad / \frac{\partial}{\partial z}$$

$$\frac{\partial}{\partial z}(\ln w(z)) = xy \frac{\partial}{\partial z}(z \ln z)$$
$$\frac{1}{w(z)} \cdot \frac{\partial}{\partial z} w(z) = xy\left(\frac{\partial}{\partial z}(z) \cdot \ln z + z \cdot \frac{\partial}{\partial z}(\ln z)\right)$$
$$\frac{\partial}{\partial z} w(z) = w(z) \cdot xy(\ln z + 1) = z^{xyz} \cdot xy(\ln z + 1)$$
$$= z^{xyz} xy \ln z + z^{xyz} xy = \alpha$$

$$\ldots = y^x \cdot x^z \cdot \ln x \cdot z^{xyz} + y^x x^z (z^{xyz} xy \ln z + z^{xyz} xy)$$

$$\frac{\partial f}{\partial z} = y^x x^z z^{xyz} \ln x + y^{x+1} x^{z+1} z^{xyz} \ln z + y^{x+1} x^{z+1} z^{xyz} \quad //$$

Zad 2. $g: \mathbb{R}^2 \to \mathbb{R}$ $\quad g(x,y) = \begin{cases} \frac{x^2y}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

Odredimo parcijalne derivacije funkcije g

$$\frac{\partial g}{\partial x} = \begin{cases} \frac{\partial}{\partial x}\left(\frac{x^2y}{x^2+y^2}\right), & (x,y) \neq (0,0) \\ \frac{\partial}{\partial x}(0), & (x,y) = (0,0) \end{cases} = \begin{cases} \frac{2xy(x^2+y^2) - x^2y \cdot 2x}{(x^2+y^2)^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

$$= \begin{cases} \frac{2xy^3}{(x^2+y^2)^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

$$\frac{\partial g}{\partial y} = \begin{cases} \frac{\partial}{\partial y}\left(\frac{x^2y}{x^2+y^2}\right), & (x,y) \neq (0,0) \\ \frac{\partial}{\partial y}(0), & (x,y) = (0,0) \end{cases} = \begin{cases} \frac{x^2(x^2+y^2) - x^2y \cdot 2y}{(x^2+y^2)^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

$$= \begin{cases} \frac{x^4 - x^2y^2}{(x^2+y^2)^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

Uočimo kako su $\partial_x g$ i $\partial_y g$ neprekidne u svim točkama osim možda u $(0,0)$, što bismo trebali provjeriti. Provjerimo je li $\lim\limits_{\substack{x\to 0 \\ y \to 0}} \partial_x g = 0$ i $\lim\limits_{\substack{x\to 0 \\ y \to 0}} \partial_y g = 0$

$$\lim_{(x,y)\to(0,0)} \frac{2xy^3}{(x^2+y^2)^2} \overset{y=kx}{=} \lim_{x\to 0} \frac{2xk^3x^3}{(x^2+k^2x^2)^2} = \lim_{x \to 0} \frac{2k^3 \cancel{x^4}}{(1+k^2)^2 \cancel{x^4}}$$

$= \frac{2k^3}{(1+k^2)^2}$. Limes ovisi o izboru $k \in \mathbb{R}$, pa $\partial_x g$ ima prekid u $(0,0) \Rightarrow$ ne možemo zaključiti diferencijabilnost u $(0,0)$

Pokušajmo pokazati diferencijabilnost preko definicije.

$$\lim_{(x,y)\to(0,0)} \frac{g(x,y) - g(0,0) - Dg(0,0)(x,y)}{\|(x,y)\|} \overset{?}{=} 0$$

$$\lim_{(x,y)\to(0,0)} \frac{\frac{x^2y}{x^2+y^2} - 0 - \langle \nabla g(0,0), (x,y)\rangle}{\sqrt{x^2+y^2}} \qquad \nabla g(0,0) = [0,0]^T \Rightarrow \langle \nabla g(0,0), (x,y)\rangle = 0$$

$$= \lim_{(x,y)\to(0,0)} \frac{x^2y}{(x^2+y^2)\sqrt{x^2+y^2}} = \left| \begin{array}{l} x = r\cos\theta \ , \ y = r\sin\theta \\ (x,y)\to(0,0) \Rightarrow r\to 0 \\ |r| = r \text{ jer } r>0 \end{array} \right|$$

$$= \lim_{r\to 0} \frac{r^2\cos^2\theta \, r\sin\theta}{r^2 \cdot r} = \lim_{r\to 0} \frac{\cancel{r^3}\cos^2\theta\sin\theta}{\cancel{r^3}}$$

$= \cos^2\theta \sin\theta$ Limes ovisi o $\theta \in [0, 2\pi\rangle$

$\Rightarrow \cos^2\theta\sin\theta \neq 0$ za npr. $\theta = \frac{\pi}{4}$

$\Rightarrow$ funkcija g diferencijabilna je u svakoj točki svoje domene OSIM u $(0,0)$!

$$Dg(3,1)(2,2) = \langle \nabla g(3,1), (2,2) \rangle = \frac{\partial g}{\partial x}(3,1)\cdot 2 + \frac{\partial g}{\partial y}(3,1)\cdot 2$$

$$= \frac{2\cdot 3\cdot 1^3}{(3^2+1^2)^2}\cdot 2 + \frac{3^4 - 3^2\cdot 1^2}{(3^2+1^2)^2}\cdot 2 = \frac{6}{100}\cdot 2 + \frac{72}{100}\cdot 2 = \frac{39}{25} \quad //$$

5.VJEŽBE 14.11.2024.

(Zad) $f(x,y)=x^y$ provjerite $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$.

$\frac{\partial f}{\partial x} = y x^{y-1}$ $\quad \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(y x^{y-1}) = x^{y-1} + y x^{y-1} \cdot \ln x$ //

$\frac{\partial f}{\partial y} = x^y \ln x$ $\quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}(x^y \ln x) = y x^{y-1} \ln x + x^y \frac{1}{x} = x^{y-1} + y x^{y-1} \ln x$

Jednaki su

(DZ) $f(x,y)=\ln(xy + x^2y^2 + x^3y^3)$ $\quad \frac{\partial^2 f}{\partial x^2}(1,1) + \frac{\partial^2 f}{\partial y^2}(1,1) = ?$

(Zad) $f(x,y)=\sin(xy)$ $\quad \frac{\partial^3 f}{\partial x \partial y^2}$

$\frac{\partial f}{\partial y} = x\cos(xy)$, $\quad \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}(x\cos(xy)) = -x^2 \sin(xy)$

$\frac{\partial^3 f}{\partial x \partial y^2} = \frac{\partial}{\partial x}(-x^2 \sin(xy)) = -2x\sin(xy) - x^2 y \cos(xy)$ //

(DZ) Pokažite da funkcija $f(x,y) = \operatorname{arctg}\frac{x}{y}$ zadovoljava Laplaceovu jednadžbu $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$.

Derivacija vektorske funkcije

(Zad.) Zadana je vektorska f-ja $\vec{f}: \mathbb{R}^2 \to \mathbb{R}^2$ s $\vec{f}(x,y) = (\overset{f_1}{e^x \cos y}, \overset{f_2}{e^x \sin y})$
Odredite joj diferencijal u točki $(0, \frac{\pi}{2})$

$$D\vec{f}\left(0,\frac{\pi}{2}\right) = \begin{bmatrix} \partial_x f_1 & \partial_y f_1 \\ \partial_x f_2 & \partial_y f_2 \end{bmatrix} = \begin{bmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{bmatrix}\Bigg|_{(0,\frac{\pi}{2})} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Diferencijali višeg reda

Primjer $f(x,y) = x^2 + y^2$ $\quad P_0(x,y)$

$Df(x,y)(H) = \nabla f(x,y) \cdot H = (2x, 2y) \cdot H = (2x)h_1 + (2y)h_2$

$D(Df(x,y)(H))(H) = (2h, 2k)\cdot(h,k) = 2h^2 + 2k^2$ //

Može se pokazati: $D^2 f(x,y) = \begin{bmatrix} f''_{xx} & f''_{xy} \\ f''_{yx} & f''_{yy} \end{bmatrix}$ operator $(f: \mathbb{R}^2 \to \mathbb{R})$

Općenito vrijedi formula $d^n f = \left(\frac{\partial}{\partial x}dx + \frac{\partial}{\partial y}dy\right)^n f$

diferencijal n-tog reda funkcije dvije varijable (zapisano djelovanje na vektor)

$$d^2 f = \left(\frac{\partial}{\partial x}dx + \frac{\partial}{\partial y}dy\right)^2 f = \frac{\partial^2 f}{\partial x^2}(dx)^2 + \frac{\partial^2 f}{\partial y^2}(dy)^2 + 2\frac{\partial^2 f}{\partial x \partial y}(dx\,dy)$$

Zad. Odredite diferencijal drugog reda f-je

1) $f(x,y) = \ln(x^2+y)$

$$\frac{\partial f}{\partial x} = \frac{1}{x^2+y}\cdot 2x = \frac{2x}{x^2+y}, \quad \frac{\partial^2 f}{\partial x^2} = \frac{2(x^2+y) - 2x\cdot 2x}{(x^2+y)^2} = \frac{2y-2x^2}{(x^2+y)^2}$$

$$\frac{\partial f}{\partial y} = \frac{1}{x^2+y}, \quad \frac{\partial^2 f}{\partial y^2} = -\frac{1}{(x^2+y)^2}, \quad \frac{\partial^2 f}{\partial x \partial y} = -\frac{2x}{(x^2+y)^2}$$

svejedno zbog Schwarzove leme

$$d^2 f(x,y) = \frac{2y-2x^2}{(x^2+y)^2}(dx)^2 - \frac{1}{(x^2+y)^2}(dy)^2 - \frac{4x}{(x^2+y)^2}(dx\,dy) \quad //$$

2) DZ $f(x,y) = x^2 + xy + y^2 - 4\ln x - 10\ln y$ u $T(1,2)$

3) $f(x,y,z) = x^2 + 2y^2 + 3z^2 - 2xy + 4xz + 2yz$ u $T(0,0,0)$

$$\frac{\partial f}{\partial x} = 2x - 2y + 4z, \quad \frac{\partial f}{\partial y} = 4y - 2x + 2z, \quad \frac{\partial f}{\partial z} = 6z + 4x + 2y$$

$$\frac{\partial^2 f}{\partial x^2} = 2, \quad \frac{\partial^2 f}{\partial y^2} = 4, \quad \frac{\partial^2 f}{\partial z^2} = 6$$

$$\frac{\partial^2 f}{\partial x \partial y} = -2, \quad \frac{\partial^2 f}{\partial x \partial z} = 4, \quad \frac{\partial^2 f}{\partial y \partial z} = 2$$

$$d^2 f(0,0) = \frac{\partial^2 f}{\partial x^2}(dx)^2 + \frac{\partial^2 f}{\partial y^2}(dy)^2 + \frac{\partial^2 f}{\partial z}(dz)^2 + 2\frac{\partial^2 f}{\partial x \partial y}(dx\,dy) + 2\frac{\partial^2 f}{\partial x \partial z}(dx\,dz) + 2\frac{\partial^2 f}{\partial y \partial z}(dy\,dz)$$

$$= 2(dx)^2 + 4(dy)^2 + 6(dz)^2 - 4(dx\,dy) + 8(dx\,dz) + 4(dy\,dz)$$

Derivacija u smjeru vektora

Za zadani vektor $\vec{a}$ označimo s $\vec{u}$ jedinični vektor $\vec{u} = \frac{\vec{a}}{\|\vec{a}\|}$.
Usmjerena derivacija skalarnog polja $f: \Omega \subseteq \mathbb{R}^3 \to \mathbb{R}$ u smjeru vektora $\vec{a}$ računa se po formuli:

$$\frac{\partial f}{\partial \vec{a}} = \nabla f \cdot \vec{u}$$

Zad. Odredite usmjerenu derivaciju skalarnog polja $f(x,y,z) = \frac{zy}{x}$ u smjeru vektora $\vec{a} = 4\vec{i} + 4\vec{j} + 2\vec{k}$

$\|\vec{a}\| = \sqrt{16+16+4} = 6$, $\vec{u} = \frac{\vec{a}}{\|\vec{a}\|} = \frac{2}{3}\vec{i} + \frac{2}{3}\vec{j} + \frac{1}{3}\vec{k}$

$$\nabla f = \begin{bmatrix} \partial_x f \\ \partial_y f \\ \partial_z f \end{bmatrix} = \begin{bmatrix} -\frac{zy}{x^2} \\ \frac{z}{x} \\ \frac{y}{x} \end{bmatrix} \Rightarrow \frac{\partial f}{\partial \vec{a}} = \nabla f \cdot \vec{u} = -\frac{2zy}{3x^2} + \frac{2z}{3x} + \frac{y}{3x} \quad //$$

DZ. Odredite usmjerenu derivaciju skalarnog polja $f(x,y,z) = x^3 + y^2 \sin z$ u smjeru vektora $\vec{a} = \vec{i} - \vec{j} + \vec{k}$ u točki $T(1,1,0)$

Zad. Odredite derivaciju s.p. $f(x,y,z) = xe^{y^2} \ln z$ u smjeru vektora $\vec{a} = \vec{k}$ u točki $T(1,1,e)$

$$\frac{\partial f}{\partial z} = \frac{xe^{y^2}}{z} \Big|_{(1,1,e)} = \frac{1 \cdot e^{1^2}}{e} = 1 \quad //$$

Deriviranje složenih funkcija

1) slučaj jedne nezavisne varijable

- Neka je $z = f(x,y)$ diferencijabilna i $x = \varphi(t)$, $y = \psi(t)$ derivabilne

$$z = f(x,y) = f(\varphi(t), \psi(t)) = g(t)$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

Analogno $u = f(x_1, \ldots, x_n)$, $x_i = \varphi_i(t)$

$$\frac{du}{dt} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \cdot \frac{dx_i}{dt}$$

Specijalno, $z = f(x,y) = f(x, \varphi(x)) = g(x)$

$$\frac{dz}{dx} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dx}$$

Zad. $f(x,y) = \frac{x}{y}$, $x(t) = e^t$, $y(t) = \ln t$

1)

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} = \frac{1}{y} \cdot e^t - \frac{x}{y^2} \cdot \frac{1}{t} = \frac{1}{\ln t} \cdot e^t - \frac{e^t}{\ln^2 t} \cdot \frac{1}{t}$$

$$= \frac{e^t (t \ln t - 1)}{t \ln^2 t} \quad //$$

2) $f(u,v) = u^v$, $u = \sin x$, $v = \cos x$

$$\frac{df}{dx} = \frac{\partial f}{\partial u} \cdot \frac{du}{dx} + \frac{\partial f}{\partial v} \cdot \frac{dv}{dx} = v(u^{v-1}) \cdot \cos x + u^v \ln u \cdot (-\sin x)$$

$$= \cos x \left((\sin x)^{\cos x - 1} \right) \cdot \cos x + (\sin x)^{\cos x} \cdot \ln \sin x \, (-\sin x)$$

$$= \sin^{\cos x - 1} x \cdot \cos^2 x - \sin^{\cos x + 1} x \cdot \ln(\sin x) \quad //$$

3) $f(x,y) = \arcsin(x-y)$, $x(t) = 3t$, $y(t) = 4t^3$ DZ $\frac{df}{dt}$

4) $f(x,y) = x^y$, $y = \varphi(x)$ DZ?

$$\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \varphi'(x) =$$

5) $f(x,y) = \ln(e^x + e^y)$, $y = x^3$

$$\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx} = \frac{1}{e^x + e^y} \cdot e^x + \frac{1}{e^x + e^y} \cdot e^y \cdot 3x^2$$

$$= \frac{e^x + e^{x^3} \cdot 3x^2}{e^x + e^{x^3}} \quad //$$

2) slučaj više nezavisnih varijabli

- Ako je $z=f(x,y)$, $x=\varphi(u,v)$, $y=\psi(u,v)$

$$z=f(x,y)=f(\varphi(u,v),\psi(u,v))=g(u,v)$$

$$\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}, \quad \frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$$

Zad. $f(x,y)=\operatorname{arctg}\frac{x}{y}$, $x=u\sin v$, $y=u\cos v$

$$\frac{\partial f}{\partial u}=\frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial u}+\frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial u}=\frac{1}{1+(\frac{x}{y})^2}\cdot\frac{1}{y}\cdot\sin v+\frac{1}{1+(\frac{x}{y})^2}\cdot\left(-\frac{x}{y^2}\right)\cdot\cos v$$

$$=\frac{1}{1+tg^2 v}\cdot\frac{tg\, v}{u}-\frac{1}{1+tg^2 v}\cdot\frac{tg\, v}{u}=0$$

$$\frac{\partial f}{\partial v}=\frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial v}+\frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial v}=\frac{1}{1+(\frac{x}{y})^2}\cdot\cancel{\frac{1}{y}}\cdot\cancel{u\cos v}+\frac{1}{1+(\frac{x}{y})^2}\left(-\frac{x}{y^2}\right)\left(-\cancel{u}\sin v\right)$$

$$=\frac{1}{1+tg^2 v}+\frac{1}{1+tg^2 v}\cdot\frac{1}{u^2\cos^2 v} \quad //$$

Zad. $u=f(x,y,z)$, $y=\varphi(x)$, $z=\psi(x,y)$

$$\frac{du}{dx}=\frac{\partial u}{\partial x}\cdot\cancelto{1}{\frac{dx}{dx}}+\frac{\partial u}{\partial y}\cdot\frac{dy}{dx}+\frac{\partial u}{\partial z}\cdot\frac{dz}{dx}$$

$$=\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\cdot\varphi'(x)+\frac{\partial u}{\partial z}\left(\frac{\partial z}{\partial x}\cdot\frac{dx}{dx}+\frac{\partial z}{\partial y}\cdot\frac{dy}{dx}\right)$$

$$=\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\cdot\varphi'(x)+\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial x}+\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial y}\cdot\varphi'(x)$$

DZ Pokazati da f-ja $z=y\cdot\varphi(x^2-y^2)$ zadovoljava jednadžbu
$\frac{1}{x}\frac{\partial z}{\partial x}+\frac{1}{y}\frac{\partial z}{\partial y}=\frac{z}{y^2}$

Derivacije višeg reda složenih f-ja

Zad. $z=f(u,v)$, $u=\varphi(x,y,t)$, $v=\psi(x,y,t)$

Odredimo $\frac{\partial^2 z}{\partial x\partial t}$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial t} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial t}.$$

$$\frac{\partial^2 z}{\partial x \partial t} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial t}\right) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial t} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial t}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial t}\right) + \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial t}\right)$$

$$= \left(\frac{\partial^2 z}{\partial u^2}\cdot\frac{\partial u}{\partial x} + \frac{\partial^2 f}{\partial v \partial u}\cdot\frac{\partial v}{\partial x}\right)\cdot\frac{\partial u}{\partial t} + \frac{\partial z}{\partial u}\cdot\frac{\partial^2 u}{\partial x \partial t}$$

$$+ \left(\frac{\partial^2 z}{\partial u \partial v}\cdot\frac{\partial u}{\partial x} + \frac{\partial^2 z}{\partial v^2}\cdot\frac{\partial v}{\partial x}\right)\cdot\frac{\partial v}{\partial t} + \frac{\partial z}{\partial v}\cdot\frac{\partial^2 v}{\partial x \partial t} \quad //$$

DZ. $z = f(u,v)$, $u = x^2 + y^2$, $v = xy$

Izračunaj $\frac{\partial^2 f}{\partial y \partial x}$.

Zad. $z = f(u,v)$, $u = \varphi(x,y)$, $v = \psi(x,y)$

Naći $\frac{\partial^2 f}{\partial y^2}$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial y}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial y}\right)$$

$$= \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial y}\right)$$

$$= \left(\frac{\partial^2 f}{\partial u^2}\cdot\frac{\partial u}{\partial y} + \frac{\partial^2 f}{\partial v \partial u}\cdot\frac{\partial v}{\partial y}\right)\cdot\frac{\partial u}{\partial y} + \frac{\partial f}{\partial u}\cdot\frac{\partial^2 u}{\partial y^2}$$

$$+ \left(\frac{\partial^2 f}{\partial u \partial v}\cdot\frac{\partial u}{\partial y} + \frac{\partial^2 f}{\partial v^2}\cdot\frac{\partial v}{\partial y}\right)\cdot\frac{\partial v}{\partial y} + \frac{\partial f}{\partial v}\cdot\frac{\partial^2 v}{\partial y^2} \quad //$$

Derivacije implicitno zadane f-je

1) slučaj jedne nezavisne varijable

$f(x,y) = 0$ Neka je f diferencijabilna i neka $y = y(x)$

Ako je $\frac{\partial f}{\partial y}(x,y) \neq 0$, tada $\frac{dy}{dx} = -\frac{\partial_x f(x,y)}{\partial_y f(x,y)}$

2

Pr. $y = 1 + y^x \Rightarrow 1 + y^x - y = 0$, $y = y(x)$

$\frac{dy}{dx} = ?$ $\quad f(x,y) = 1 + y^x - y$

$$\frac{dy}{dx} = -\frac{\partial_x f}{\partial_y f} = -\frac{y^x \ln y}{x y^{x-1} - 1}$$

2) slučaj više nezavisnih varijabli

$F(x,y,z) = 0$ $\quad$ F diferencijabilna, a $z = z(x,y)$

$\frac{\partial F}{\partial z}(x,y,z) \neq 0$

$$\frac{\partial z}{\partial x} = -\frac{\partial_x F}{\partial_z F}, \quad \frac{\partial z}{\partial y} = -\frac{\partial_y F}{\partial_z F}$$

Primjer $x\cos y + y\cos z + z\cos x = 1$ $\quad z = z(x,y)$

$F(x,y,z) = x\cos y + y\cos z + z\cos x - 1$

$$\frac{\partial z}{\partial x} = -\frac{\partial_x F}{\partial_z F} = -\frac{\cos y - z\sin x}{-y\sin z + \cos x}$$

Zad. $x^2 + y^2 + z^2 = 3a^2$, $z = z(x,y)$ u $T(a,a,a)$

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

$$d^2 z = \frac{\partial^2 z}{\partial x^2}(dx)^2 + \frac{2\partial^2 z}{\partial x \partial y}(dx\,dy) + \frac{\partial^2 z}{\partial y^2}(dy)^2$$

$F(x,y,z) = x^2 + y^2 + z^2 - 3a^2$

$$\frac{\partial z}{\partial x} = -\frac{\partial_x F}{\partial_z F} = -\frac{2x}{2z} = -\frac{x}{z}$$

$$\frac{\partial z}{\partial y} = -\frac{\partial_y F}{\partial_z F} = -\frac{2y}{2z} = -\frac{y}{z}$$

$$dz = -\frac{x}{z}\,dx - \frac{y}{z}\,dy \quad //$$

$$dz(a,a,a) = -dx - dy$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(-\frac{x}{z}\right)$$

$$= \frac{-1\cdot z + x\cdot\left(-\frac{x}{z}\right)}{z^2} = \frac{-z - \frac{x^2}{z}}{z^2}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(-\frac{y}{z}\right) = \frac{-z + y\cdot\left(-\frac{y}{z}\right)}{z^2} = \frac{-z - \frac{y^2}{z}}{z^2}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(-\frac{y}{z}\right) = \frac{y\cdot\left(-\frac{x}{z}\right)}{z^2} = -\frac{xy}{z^3}$$

$$d^2 z = \frac{-z - \frac{x^2}{z}}{z^2}(dx)^2 - \frac{2xy}{z^3}(dxdy) + \frac{-z - \frac{y^2}{z}}{z^2}(dy)^2$$

$$d^2 z|_T = -\frac{2}{a}(dx)^2 - \frac{2}{a}(dxdy) - \frac{2}{a}(dy)^2 \quad //$$

Zad. 1. Pokažite da je svaki linearan operator ograničen

Rješenje. Treba pokazati kako postoji $\lambda > 0$ takav da za proizvoljan linearan operator $\mathcal{A} \in \mathcal{L}(\mathbb{R}^n; \mathbb{R}^m)$ vrijedi $\|\mathcal{A}(H)\| \leq \lambda \|H\|$, pri čemu je $H \in \mathbb{R}^n$ vektor. Neka je $\{e_1, \dots, e_n\}$ kanonska baza za $\mathbb{R}^n$, a $\{f_1, \dots, f_m\}$ za $\mathbb{R}^m$. Prema fundamentalnom teoremu linearne algebre $(\forall H \in \mathbb{R}^n)(\exists! \alpha_1, \dots \alpha_n \in \mathbb{R})$ $H = \sum_{k=1}^{n} \alpha_k e_k$. Kako je $\mathcal{A}$ linearan operator, vrijedi

$$\mathcal{A}(H) = \sum_{k=1}^{n} \alpha_k \mathcal{A}(e_k) \qquad (1)$$

pri čemu je $\mathcal{A}(e_k)$ neki vektor iz $\mathbb{R}^m$, pa opet prema FTLA:

$$\mathcal{A}(e_k) = \sum_{i=1}^{m} \beta_i f_i$$

Uvrštavanjem u (1) imamo $\mathcal{A}(H) = \sum_{k=1}^{n} \alpha_k \left(\sum_{i=1}^{m} \beta_i f_i \right) = \sum_{i=1}^{m} \left(\sum_{k=1}^{n} \alpha_k \beta_i \right) f_i$

Budući da su sve norme na konačnodimenzionalnom vektorskom prostoru međusobno ekvivalentne, tvrdnju ćemo pokazati za Čebiševljevu normu.

$$\|\mathcal{A}(H)\|_\infty = \left\| \sum_{i=1}^{m} \left(\sum_{k=1}^{n} \alpha_k \beta_i \right) f_i \right\|_\infty = \max_{i=1,\dots,m} \left| \sum_{k=1}^{n} \alpha_k \beta_i \right| \leq \max_{i=1,\dots,m} \sum_{k=1}^{n} |\alpha_k| |\beta_i|$$

$$= \left(\sum_{k=1}^{n} |\alpha_k| \right) \cdot \max_{i=1,\dots,m} |\beta_i|$$

Uočimo kako je $\sum_{k=1}^{n} |\alpha_k| = \|H\|_1$, ali zbog ekvivalentnosti normi na $\mathbb{R}^n$ postoji $M > 0$ takav da je $\|H\|_1 \leq M \|H\|_\infty$. Uzmemo li sada $\lambda := M \cdot \max_{i=1,\dots,m} |\beta_i| > 0$ imamo

$$\|\mathcal{A}(H)\|_\infty \leq \left(\sum_{k=1}^{n} |\alpha_k| \right) \cdot \max_{i=1,\dots,m} |\beta_i| \leq M \|H\|_\infty \cdot \max_{i=1,\dots,m} |\beta_i| = \lambda \|H\|_\infty$$

Čime je tvrdnja dokazana. Q.E.D

Zad 2. Neka je dana ploha $z = x^2 + y^2$. Nađite jednadžbu normale na plohu koja prolazi točkom $P(3,3,1)$.

Rješenje. Tražimo jednadžbu pravca koji prolazi dvjema točkama $P=(3,3,1)$ i $P_0=(x_0,y_0,z_0)$ koja leži na plohi, tako da je $\overrightarrow{PP_0}$ vektor normale na tangencijalnu ravninu u točki P_0. Zapišimo prvo naši jednadžbu plohe u eksplicitnom obliku, odnosno kao funkciju $F(x,y,z) = x^2+y^2-z$.
Na predavanju smo pokazali kako je gradijent te plohe u nekoj točki vektor normale na tangencijalnu ravninu u toj točki. Tada vrijedi $\overrightarrow{PP_0} \parallel \nabla F(P_0) \Rightarrow \overrightarrow{PP_0} = \alpha \nabla F(P_0)$, $\alpha \in \mathbb{R}$. Odnosno u terminima komponenata vektora:

$$\begin{bmatrix} x_0-3 \\ y_0-3 \\ z_0-1 \end{bmatrix} = \alpha \begin{bmatrix} \partial_x F \\ \partial_y F \\ \partial_z F \end{bmatrix}$$

Također, mora vrijediti $z_0 = x_0^2 + y_0^2$

$z_0 = x_0^2 + y_0^2$

$$\left.\begin{aligned} x_0 - 3 &= \alpha 2 x_0 \\ y_0 - 3 &= \alpha 2 y_0 \\ z_0 - 1 &= -\alpha \end{aligned}\right\} \Rightarrow \begin{aligned} x_0(1-2\alpha) &= 3 \\ y_0(1-2\alpha) &= 3 \\ z_0 &= 1-\alpha \end{aligned} \qquad \begin{aligned} x_0 &= \frac{3}{1-2\alpha} = 1 \\ y_0 &= \frac{3}{1-2\alpha} = 1 \\ z_0 &= 2 \end{aligned}$$

$$\Rightarrow \left(\frac{3}{1-2\alpha}\right)^2 + \left(\frac{3}{1-2\alpha}\right)^2 = 1-\alpha$$

$$2 \cdot \frac{3}{1-2\alpha} = 1-\alpha \Rightarrow \alpha = -1$$

$$\Rightarrow \quad n \ldots \quad \frac{x-1}{2} = \frac{y-1}{2} = \frac{z-2}{-1} \quad //$$

Zad. 1. Odredite gradijent i drugi diferencijal funkcije $f(x,y,z) = x^3z^2\sin^2 y - 5xy + x^4 z - 20$ u točki $T(1,0,2)$.

Rješenje: Odredimo prvo parcijalne derivacije prvog reda:

$$\frac{\partial f}{\partial x} = 3x^2z^2\sin^2 y - 5y + 4x^3 z, \quad \frac{\partial f}{\partial y} = x^3z^2 \cdot 2\sin y\cos y - 5x, \quad \frac{\partial f}{\partial z} = 2x^3 z\sin^2 y + x^4$$

Sada je gradijent u točki $(1,0,2)$ dan s:

$$\nabla f(1,0,2) = \begin{bmatrix} 3x^2z^2\sin^2 y - 5y + 4x^3 z \\ 2x^3z^2\sin y\cos y - 5x \\ 2x^3 z\sin^2 y + x^4 \end{bmatrix}\Bigg|_T = \begin{bmatrix} 3\cdot 1^2\cdot 2^2\cdot \sin^2(0) - 5\cdot 0 + 4\cdot 1^3\cdot 2 \\ 2\cdot 1^3\cdot 2^2\cdot \sin(0)\cdot\cos(0) - 5\cdot 1 \\ 2\cdot 1^3\cdot 2\cdot \sin^2(0) + 1^4 \end{bmatrix} = \begin{bmatrix} 8 \\ -5 \\ 1 \end{bmatrix} //$$

Odredimo sada parcijalne derivacije drugog reda:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(3x^2z^2\sin^2 y - 5y + 4x^3 z\right) = 6xz^2\sin^2 y + 12x^2 z \Rightarrow \frac{\partial^2 f(1,0,2)}{\partial x^2} = 24$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(2x^3z^2\sin y\cos y - 5x\right) = 2x^3z^2(\cos^2 y - \sin^2 y) \Rightarrow \frac{\partial^2 f(1,0,2)}{\partial y^2} = 8$$

$$\frac{\partial^2 f}{\partial z^2} = \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}\right) = \frac{\partial}{\partial z}\left(2x^3 z\sin^2 y + x^4\right) = 2x^3\sin^2 y \Rightarrow \frac{\partial^2 f(1,0,2)}{\partial z^2} = 0$$

$$\frac{\partial^2 f}{\partial x\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(2x^3z^2\sin y\cos y - 5x\right) = 6xz^2\sin y\cos y - 5 \Rightarrow \frac{\partial^2 f(1,0,2)}{\partial x\partial y} = -5$$

$$\frac{\partial^2 f}{\partial x\partial z} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial z}\right) = \frac{\partial}{\partial x}\left(2x^3 z\sin^2 y + x^4\right) = 6x^2 z\sin^2 y + 4x^3 \Rightarrow \frac{\partial^2 f(1,0,2)}{\partial x\partial z} = 4$$

$$\frac{\partial^2 f}{\partial z\partial y} = \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial z}\left(2x^3z^2\sin y\cos y - 5x\right) = 6x^3 z\sin y\cos y \Rightarrow \frac{\partial^2 f(1,0,2)}{\partial z\partial y} = 0$$

Sada je drugi diferencijal (odnosno njegova matrica) dan s:

$$D^2 f(1,0,2) = \begin{bmatrix} \partial^2_{xx} f(1,0,2) & \partial^2_{xy} f(1,0,2) & \partial^2_{xz} f(1,0,2) \\ \partial^2_{yx} f(1,0,2) & \partial^2_{yy} f(1,0,2) & \partial^2_{yz} f(1,0,2) \\ \partial^2_{zx} f(1,0,2) & \partial^2_{zy} f(1,0,2) & \partial^2_{zz} f(1,0,2) \end{bmatrix} = \begin{bmatrix} 24 & -5 & 4 \\ -5 & 8 & 0 \\ 4 & 0 & 0 \end{bmatrix} //$$

Zad. 2. Neka je $F(x,y) = xy + x\varphi(x^2-y)$, $\varphi \in C^2(\mathbb{R})$. Pokažite da je

$$2x\frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 F}{\partial x\partial y} = \frac{1}{x}\frac{\partial F}{\partial y}.$$

Rješenje. Odredimo prvo sve parcijalne derivacije koje se pojavljuju u izrazu:

$$\frac{\partial F}{\partial y} = x + x\varphi'(x^2-y)\frac{\partial}{\partial y}(x^2-y) = x - x\varphi'(x^2-y)$$

$$\frac{\partial^2 F}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial y}\right) = \frac{\partial}{\partial y}\left(x - x\varphi'(x^2-y)\right) = -x\varphi''(x^2-y)\frac{\partial}{\partial y}(x^2-y) = x\varphi''(x^2-y)$$

$$\frac{\partial^2 F}{\partial x\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right) = \frac{\partial}{\partial x}\left(x - x\varphi'(x^2-y)\right) = 1 - \left(\varphi'(x^2-y) + x\varphi''(x^2-y)\frac{\partial}{\partial x}(x^2-y)\right)$$

$$= 1 - \varphi'(x^2-y) - 2x^2\varphi''(x^2-y)$$

Sada imamo: $2x\frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 F}{\partial x\partial y} = \frac{1}{x}\cdot\frac{\partial F}{\partial y}$

$$2x(x\varphi''(x^2-y)) + 1 - \varphi'(x^2-y) - 2x^2\varphi''(x^2-y) = \frac{1}{x}(x - x\varphi'(x^2-y))$$

$$\cancel{2x^2\varphi''(x^2-y)} + 1 - \varphi'(x^2-y) - \cancel{2x^2\varphi''(x^2-y)} = \frac{1}{\cancel{x}} * \cancel{x}(1-\varphi'(x^2-y))$$

$$1 - \varphi'(x^2-y) = 1 - \varphi'(x^2-y) \quad //$$

Zad 3. Neka je $z=f(u,v)$, pri čemu je $u=x^2+y^2$ i $v=x\cdot y$. Odredite $\frac{\partial^2 z}{\partial y \partial x}$.

Rješenje: Odredimo prvo $\frac{\partial z}{\partial x}$:

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}$$

Sada imamo:

$$\frac{\partial^2 z}{\partial y\partial x}=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}\right)=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}\right)$$

$$=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right)\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial u}\cdot\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial v}\right)\cdot\frac{\partial v}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}\right)$$

$$=\left(\frac{\partial^2 z}{\partial u^2}\cdot\frac{\partial u}{\partial y}+\frac{\partial^2 z}{\partial v\partial u}\cdot\frac{\partial v}{\partial y}\right)\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial u}\cdot\frac{\partial^2 u}{\partial y\partial x}$$

$$+\left(\frac{\partial^2 z}{\partial u\partial v}\cdot\frac{\partial u}{\partial y}+\frac{\partial^2 z}{\partial v^2}\cdot\frac{\partial v}{\partial y}\right)\cdot\frac{\partial v}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial^2 v}{\partial y\partial x}$$

$$=\left(2y\frac{\partial^2 z}{\partial u^2}+x\frac{\partial^2 z}{\partial v\partial u}\right)\cdot 2x+\left(2y\frac{\partial^2 z}{\partial u\partial v}+x\frac{\partial^2 z}{\partial v^2}\right)\cdot y+\frac{\partial z}{\partial v} \quad //$$

Zad 4. Neka je $\underset{*}{\cos(xy)+\sin z+x^4+zy^2}=0$ implicitno definirana funkcija $z(x,y)$. Odredite gradijent i drugi diferencijal funkcije z.

Rješenje: Odredimo prvo parcijalne derivacije prvog reda, $F(x,y)=*$

$$\frac{\partial z}{\partial x}=-\frac{\partial_x F}{\partial_z F}=-\frac{-y\sin(xy)+4x^3}{\cos z+y^2}=\frac{y\sin(xy)-4x^3}{\cos z+y^2}$$

$$\frac{\partial z}{\partial y}=-\frac{\partial_y F}{\partial_z F}=-\frac{-x\sin(xy)+2zy}{\cos z+y^2}=\frac{x\sin(xy)-2zy}{\cos z+y^2}$$

Sada je gradijent funkcije dan s: $\nabla z=\begin{bmatrix}\frac{y\sin(xy)-4x^3}{\cos z+y^2}\\ \frac{x\sin(xy)-2zy}{\cos z+y^2}\end{bmatrix}$

Odredimo sada parcijalne derivacije drugog reda:

$$\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial}{\partial x}\left(\frac{y\sin(xy)-4x^3}{\cos z+y^2}\right)=\frac{(y^2\cos(xy)-12x^2)(\cos z+y^2)-(y\sin(xy)-4x^3)\frac{\partial}{\partial x}(\cos z+y^2)}{(\cos z+y^2)^2}$$

$$=\frac{(y^2\cos(xy)-12x^2)(\cos z+y^2)-(y\sin(xy)-4x^3)\left(-\sin z\cdot\frac{\partial z}{\partial x}\right)}{(\cos z+y^2)^2}$$

$$=\frac{(y^2\cos(xy)-12x^2)(\cos z+y^2)+(y\sin(xy)-4x^3)\left(\sin z\cdot\frac{y\sin(xy)-4x^3}{\cos z+y^2}\right)}{(\cos z+y^2)^2}=\alpha$$

$$\frac{\partial^2 z}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial}{\partial y}\left(\frac{x\sin(xy)-2zy}{\cos z+y^2}\right)=$$

$$=\frac{\left(x^2\cos(xy)-2\left(\frac{\partial z}{\partial y}\cdot y+z\right)\right)(\cos z+y^2)-(x\sin(xy)-2zy)\frac{\partial}{\partial y}(\cos z+y^2)}{(\cos z+y^2)^2}$$

$$= \frac{\left(x^2\cos(xy) - 2\frac{x\sin(xy)-2zy}{\cos z + y^2}y - 2z\right)(\cos z + y^2) - (x\sin(xy)-2zy)\left(-\sin z \cdot \frac{\partial z}{\partial y} + 2y\right)}{(\cos z + y^2)^2}$$

$$= \frac{\left(x^2\cos(xy) - \frac{2yx\sin(xy) - 4zy^2}{\cos z + y^2} - 2z\right)(\cos z + y^2) - (x\sin(xy)-2zy)\left(-\sin z\frac{x\sin(xy)-2zy}{\cos z + y^2} + 2y\right)}{(\cos z + y^2)^2} = \beta$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{y\sin(xy) - 4x^3}{\cos z + y^2}\right)$$

$$= \frac{(\sin(xy) + x\cos(xy))(\cos z + y^2) - (y\sin(xy) - 4x^3)\left(-\sin z\frac{\partial z}{\partial x} + 2y\right)}{(\cos z + y^2)^2}$$

$$= \frac{(\sin(xy) + x\cos(xy))(\cos z + y^2) - (y\sin(xy) - 4x^3)\left(-\sin z\frac{x\sin(xy) - 2zy}{\cos z + y^2} + 2y\right)}{(\cos z + y^2)^2} = \gamma$$

Sada je $D^2 z = \begin{bmatrix} \partial^2_{xx} z & \partial^2_{xy} z \\ \partial^2_{yx} z & \partial^2_{yy} z \end{bmatrix} = \begin{bmatrix} \alpha & \gamma \\ \gamma & \beta \end{bmatrix}$. //

6. VJEŽBE 21.11.2024.

Sustavi implicitno zadanih f-ja

- Neka sustav

$$(*)\begin{cases} F_1(x_1, x_2, \dots, x_k, u_1, u_2, \dots, u_r) = 0 \\ \vdots \\ F_r(x_1, x_2, \dots, x_k, u_1, u_2, \dots, u_r) = 0 \end{cases}$$

definira $u_1, \dots, u_r$ kao diferencijabilne funkcije od $x_1, \dots, x_k$ i neka je Jacobijan

$$\begin{vmatrix} \frac{\partial F_1}{\partial u_1} & \cdots & \frac{\partial F_1}{\partial u_r} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_r}{\partial u_1} & \cdots & \frac{\partial F_r}{\partial u_r} \end{vmatrix} \neq 0.$$

Tada $du_1, du_2, \dots, du_r$ (a time i $\frac{\partial u_i}{\partial x_j}$, $i=1,\dots,r$, $j=1,\dots,k$) možemo naći iz sustava linearnih jednadžbi

$$(**)\ \left\{ \frac{\partial F_i}{\partial x_1} dx_1 + \dots + \frac{\partial F_i}{\partial x_k} dx_k + \frac{\partial F_i}{\partial u_1} du_1 + \dots + \frac{\partial F_i}{\partial u_r} du_r = 0 \right., \quad i = 1, \dots, r$$

koji dobivamo iz (*) diferenciranjem

Zad Neka je $\left.\begin{matrix} x = u^2 - v^2 \\ y = u \cdot v \end{matrix}\right\}$ u i v funkcije od x i y

Nađi $\frac{\partial v}{\partial y}$ i $\frac{\partial v}{\partial x}$

$$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{y} dy \qquad \nabla v \cdot (dx, dy)$$

$$x = u^2 - v^2 \ /d$$
$$y = u \cdot v \ /d$$

$$\begin{matrix} dx = 2u\,du - 2v\,dv \ /\cdot v \\ dy = du \cdot v + u \cdot dv \ /(-2u) \end{matrix} \Rightarrow \left.\begin{matrix} v\,dx = 2uv\,du - 2v^2 dv \\ -2u\,dy = -2uv\,du - 2u^2 dv \end{matrix}\right\} (+)$$

$$v\,dx - 2u\,dy = -2v^2 dv - 2u^2 dv$$

$$dv = \frac{v}{-2v^2-2u^2}dx - \frac{2u}{-2v^2-2u^2}dy \Rightarrow dv = \underbrace{-\frac{v}{2v^2+2u^2}}_{\frac{\partial v}{\partial x}}dx + \underbrace{\frac{2u}{2v^2+2u^2}}_{\frac{\partial v}{\partial y}}dy$$

Kad se izračuna:
...

$$du = \frac{u}{2u^2+2v^2}dx + \frac{2v}{2u^2+2v^2}dy$$

DZ Izračunati $\frac{\partial z}{\partial x}$ i $\frac{\partial z}{\partial y}$ u T za koji je $u=1$, $v=1$ ako je $x = u+\ln v$, $y = v-\ln u$, $z = 2u+v$, $u=u(x,y)$, $v=v(x,y)$, $z=(x,y)$

Zad. Izračunati $\frac{\partial z}{\partial x}$ i $\frac{\partial z}{\partial y}$ u točki T za koji je $u=1$, $v=1$ ako je $x = \frac{u^3+v^3}{3}$, $y = \frac{u^3-v^3}{3}$, $z = u\cdot v$ $(z=z(x,y))$

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

$$\begin{cases} x = \frac{u^3}{3} + \frac{v^3}{3} \\ y = \frac{u^3}{3} - \frac{v^3}{3} \\ z = u\cdot v \end{cases} \overset{d}{\Rightarrow} \begin{aligned} dx &= u^2du + v^2dv \\ dy &= u^2du - v^2dv \\ dz &= du\cdot v + u\cdot dv \end{aligned}$$

$$\begin{cases} dx = u^2du + v^2dv \\ dy = u^2du - v^2dv \end{cases} \Big| + \Rightarrow \begin{aligned} dx+dy &= 2u^2du \\ du &= \frac{1}{2u^2}dx + \frac{1}{2u^2}dy \end{aligned}$$

$$dx-dy = 2v^2dv$$
$$\Rightarrow dv = \frac{1}{2v^2}dx - \frac{1}{2v^2}dy$$

$$dz = v\,du + u\,dv = \underbrace{\left(\frac{v}{2u^2} + \frac{u}{2v^2}\right)}_{\frac{\partial z}{\partial x}}dx + \underbrace{\left(\frac{v}{2u^2} - \frac{u}{2v^2}\right)}_{\frac{\partial z}{\partial y}}dy$$

uvrstimo $u=1$, $v=1$ $\Rightarrow \frac{\partial z}{\partial x}(T) = 1$, $\frac{\partial z}{\partial y}(T) = 0$ //

Tangencijalna ravnina
i normala na plohu

$z = f(x,y)$

$$\vec{n} = \frac{\partial z}{\partial x}(T_0)\vec{i} + \frac{\partial z}{\partial y}(T_0)\vec{j} - \vec{k}$$

$\Pi\ldots$ $\partial_x z\,(x-x_0) + \partial_y z\,(y-y_0) - (z-z_0) = 0$ jedn. tangencijalne ravnine

$n\ldots$ $\frac{x-x_0}{\partial_x z} = \frac{y-y_0}{\partial_y z} = \frac{z-z_0}{-1}$

$2°$ $F(x,y,z)=0$ $\quad T_0(x_0,y_0,z_0)$ $\quad$ analogno

$$\vec{n} = \frac{\partial F}{\partial x}(T_0)\vec{i} + \frac{\partial F}{\partial y}(T_0)\cdot\vec{j} + \frac{\partial F}{\partial z}(T_0)\vec{k}$$

$$\pi \ldots \quad \partial_x F(T_0)(x-x_0) + \partial_y F(T_0)(y-y_0) + \partial_z F(T_0)(z-z_0) = 0$$

$$n \ldots \quad \frac{x-x_0}{\partial_x F(T_0)} = \frac{y-y_0}{\partial_y F(T_0)} = \frac{z-z_0}{\partial_z F(T_0)}$$

Zad. 1. Nađi jednadžbu tangencijalne ravnine i normale na plohu $z-1=x^2y^2+y$ u $(0,0,1)$

$$F(x,y,z) = x^2y^2+y-z+1$$

$$\frac{\partial F}{\partial x} = 2xy^2, \quad \frac{\partial F}{\partial y} = 2x^2y+1, \quad \frac{\partial F}{\partial z} = -1$$

$$|_T = 0 \qquad |_T = 1$$

$$\pi \ldots \quad (y-0)-(z-1)=0 \qquad n \ldots \quad \frac{x-0}{0} = \frac{y-0}{1} = \frac{z-1}{-1}$$

DZ $\quad 3yx+z^2=4$ u $(1,1,1)$

DZ $\quad x^2+y^2+z^2=2Rz$ u $M(R\cos\alpha, R\sin\alpha, R)$

Zad.

Nađite sve točke na plohi $\overbrace{x^2+2y^2+3z^2=21}^{\vec{n_2}}$ u kojima je tangencijalna ravnina paralelna s ravninom $\underbrace{x+4y+6z=5}_{\vec{n_1}}$

Dvije ravnine su paralelne kada su im vektori normale kolinearni

$$\vec{n_2} = \alpha \vec{n_1} \qquad \frac{\partial F}{\partial x} = 2x, \quad \frac{\partial F}{\partial y} = 4y, \quad \frac{\partial F}{\partial z} = 6z$$

$T_0(x_0,y_0,z_0)$ nepoznata točka

$$\vec{n_2} = 2x_0\vec{i} + 4y_0\vec{j} + 6z_0\vec{k} \qquad \vec{n_1} = \vec{i} + 4\vec{j} + 6\vec{k}$$

$$\Rightarrow \quad \begin{matrix} 2x_0 = \alpha \\ 4y_0 = 4\alpha \\ 6z_0 = 6\alpha \end{matrix} \quad \Rightarrow \quad \begin{cases} x_0 = \frac{1}{2}\alpha \\ y_0 = \alpha \\ z_0 = \alpha \end{cases} \qquad T_0\left(\tfrac{1}{2}\alpha, \alpha, \alpha\right),\ \alpha \in \mathbb{R}$$

$\frac{1}{4}\alpha^2 + 2\alpha^2 + 3\alpha^2 = 21$

$\frac{1+8+12}{4}\alpha^2 = 21$

$\frac{21}{4}\alpha^2 = 21 \quad / \cdot \frac{4}{21}$

$\alpha^2 = 4 \quad \Rightarrow \quad T_1(1,2,2),\ T_2(-1,-2,-2)$ //

Zad. Nađite točku na plohi $x^2+y^2+4z^2=4$ sa svojstvom da normala na plohu u toj točki bude paralelna s ravninama $\pi_1 \ldots\ x+y-z=3$ i $\pi_2 \ldots\ x-2y+z=2$ $\quad T(x_0, y_0, z_0)$

$\vec{n} \perp \vec{n_1}$, $\vec{n} \perp \vec{n_2}$, $\vec{n} = 2x_0\vec{i} + 2y_0\vec{j} + 8z_0\vec{k}$

$\vec{n_1} = \vec{i} + \vec{j} - \vec{k}$

$\vec{n}\cdot\vec{n_1} = 0$ $\quad \vec{n_2} = \vec{i} - 2\vec{j} + \vec{k}$

$\vec{n}\cdot\vec{n_2} = 0$

$$\begin{cases} 2x_0 + 2y_0 - 8z_0 = 0 & \Rightarrow \cancel{x_0 = \frac{-2y_0+8z_0}{2} = -y_0+4z_0 = -2z_0+4z_0 = 2z_0} \\ 2x_0 - 4y_0 + 8z_0 = 0 & \rightarrow \cancel{2(-y_0+4z_0)-4y_0+0z_0 = 0 \Rightarrow -8y_0 = -16z_0} \\ x_0^2 + y_0^2 + 4z_0^2 = 4 & \cancel{y_0 = 2z_0} \end{cases}$$

$\cancel{4z_0^2 + 4z_0^2 + 4z_0^2 = 4}$ $\quad \cancel{x_0 = \pm\frac{2\sqrt{3}}{3}}$

$\cancel{12z_0^2 = 4}$ $\quad \cancel{y_0 = \pm\frac{2\sqrt{3}}{3}}$

$\cancel{z_0^2 = \frac{1}{3}}$

$\cancel{z_0 = \pm\frac{\sqrt{3}}{3}}$

$4x_0 - 2y_0 = 0 \quad \Rightarrow \quad y_0 = 2x_0 \quad z_0 = \frac{3}{4}x_0$

$x_0^2 + 4x_0^2 + \cancel{4}^{1}\cdot\frac{9}{\cancel{16}_4}x_0^2 = 4$

$\frac{4+16+9}{4}x_0^2 = 4$

$\frac{29}{4}x_0^2 = 4 \quad / \cdot \frac{4}{29}$

$x_0^2 = \frac{16}{29}$

$x_0 = \pm\sqrt{\frac{16}{29}} \quad y_0 = \pm 2\sqrt{\frac{16}{29}} \quad z_0 = \pm\frac{3}{4}\sqrt{\frac{16}{29}}$ //

Taylorova formula za FVV

u okolini točke (x_0, y_0)

$$f(x,y) = f(x_0,y_0) + \sum_{j=1}^{n} \frac{1}{j!}\left[(x-x_0)\frac{\partial}{\partial x} + (y-y_0)\frac{\partial}{\partial y}\right]^j f(x_0,y_0)$$

$$+ R_n(x_0,y_0) \longrightarrow \text{ostatak}$$

Za $x_0 = y_0 = 0$ imamo MacLaurinovu formulu

(Zad) Napisati Taylorovu formulu za f-ju $f(x,y) = x^2 + 2xy + 3y^2 - 6x - 2y - 4$

u okolini točke $T(-2,1)$

$$f(x,y) = f(-2,1) + \sum_{j=1}^{n} \frac{1}{j!}\left[(x+2)\frac{\partial}{\partial x} + (y-1)\frac{\partial}{\partial y}\right]^j f(-2,1) + R_n(-2,1)$$

od treće derivacije su postaju nula

$\frac{\partial f}{\partial x} = 2x + 2y - 6 \underset{T}{=} 0$ $\quad \frac{\partial^2 f}{\partial x^2} = -2$ $\quad \frac{\partial^2 f}{\partial y^2} = 6$

$\frac{\partial f}{\partial y} = 2x + 6y - 2 \underset{T}{=} 0$ $\quad \frac{\partial^2 f}{\partial x \partial y} = 2$ $\quad \frac{\partial^3 f}{\partial x^3} = \frac{\partial^3 f}{\partial y^3} = \frac{\partial^3 f}{\partial x^2 \partial y} = \frac{\partial^3 f}{\partial y \partial x^2} = 0$

$$f(x,y) = 1 + \frac{1}{1!}\left(\cancel{\frac{\partial f(-2,1)}{\partial x}(x+2)} + \cancel{\frac{\partial f(-2,1)}{\partial y}(y-1)}\right) + \frac{1}{2!}\left(\frac{\partial^2 f(-2,1)}{\partial x^2}(x+2)^2 + 2\frac{\partial^2 f(-2,1)}{\partial x \partial y}(x+2)(y-1)\right.$$
$$\left. + \frac{\partial^2 f}{\partial y^2}(y-1)^2\right)$$
$$= 1 + \frac{1}{2!}\left(-2(x+2)^2 + 4(x+2)(y-1) + 6(y-1)^2\right) + 0$$

(Zad) Razviti po MacLaurinovoj formuli do uključivo članova trećeg reda f-ju $f(x,y) = e^x \sin y$

$f(0,0) = 0$ $\quad \frac{\partial f}{\partial x} = \underset{=0}{e^x \sin y}$ $\quad \frac{\partial f}{\partial y} = \underset{=1}{e^x \cos y}$

$\frac{\partial^2 f}{\partial x^2} = \underset{=0}{e^x \sin y}$ $\quad \frac{\partial^2 f}{\partial y^2} = \underset{=0}{-e^x \sin y}$ $\quad \frac{\partial^2 f}{\partial x \partial y} = \underset{=1}{e^x \cos y}$

$\frac{\partial^3 f}{x^3} = e^x \sin y$ $= 0$ $\quad \frac{\partial^3 f}{\partial y^3} = -e^x \cos y$ $= -1$ $\quad \frac{\partial^3 f}{\partial x^2 \partial y} = e^x \cos y$ $= 1$ $\quad \frac{\partial^3 f}{\partial y^2 \partial x} = -e^x \sin y$ $= 0$

$$f(x,y) = f(0,0) + \sum_{j=1}^{3} \frac{1}{j!}\left[x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right]^j f(0,0) + R_3(0,0)$$

$$= 0 + \frac{1}{1!}(0\cdot x + 1\cdot y) + \frac{1}{2!}(0\cdot x^2 + 2\cdot 1\cdot xy + 0\cdot y^2) +$$
$$+ \frac{1}{3!}(0\cdot x^3 - 1\cdot y^3 + 3\cdot 1\, x^2 y + 3\cdot 0\, xy^2) + R_3(0,0) \,//$$

DZ: 1) $f(x,y) = \cos x \cos y$ u Maclaurinov red do trećeg reda

2) $f(x,y) = e^{x+y}$ u okolini točke $(1,-1)$ do uključno članova trećeg reda

TOTALNI PRIRAST

I TOTALNI DIFERENCIJAL

Neka je $z = f(x,y)$ diferencijabilna u $T(x,y)$. Totalni prirast funkcije z: $\Delta z = f(x+\Delta x, y+\Delta y) - f(x,y)$

govori nam za koliko se promijenila vrijednost funkcije ako se vrijednost varijable x promijenila za Δx, a vrijednost varijable y za Δy

Totalni diferencijal funkcije f je $dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$
Može se pokazati da je za dovoljno male Δx i Δy: $\Delta z \approx dz$.

Zad. Za koliko se približno promijeni vrijednost f-je $z = f(x,y) = x\cdot e^y$ ako $x=1$ poraste na $1{,}15$, a $y=1$ padne na $0{,}9$?

$$\Delta z = f(1{,}15, 0{,}9) - f(1,1) = 1{,}15\cdot e^{0{,}9} - e \approx 0{,}110262$$

Ekstremi f-ja više varijabli

Zad. 1. Nađite, ako postoje, ekstreme funkcije $f(x,y)=x^2-y^4$

Nužan uvjet: $\partial_x f = 0$, $\partial_y f = 0$

$$\underbrace{2x=0 \qquad -4y^3=0}_{S(0,0)}$$

kandidat stacionarna točka

$$\frac{\partial^2 f}{\partial x^2}=2 \qquad \frac{\partial^2 f}{\partial y^2}=-12y^2 \qquad \frac{\partial^2 f}{\partial x \partial y}=0$$

1. način $\quad d^2 f = \frac{\partial^2 f}{\partial x^2}(dx)^2 + \frac{\partial^2 f}{\partial x \partial y}dx\,dy + \frac{\partial^2 f}{\partial y^2}(dy)^2$

$$d^2 f(S) = 2(dx)^2$$

Treba ispitati ponašanje drugog diferencijala uz $(dx)^2+(dy)^2>0$

$\Rightarrow$ $(dx)^2>0$ i $(dy)^2=0$ ili $(dx)^2=0$ i $(dy)^2>0$ ili $(dx)^2\neq 0$ i $(dy)^2\neq 0$

$d^2 f(S)>0$ | $d^2 f(S)=0$ | $d^2 f(S)>0$

slučaj (6) $\Rightarrow$ nema odluke

Silvesterov kriterij: drugom diferencijalu pridružimo matricu

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad a_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}(S)$$

$$\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \qquad 2>0 \qquad \begin{vmatrix} 2 & 0 \\ 0 & 0 \end{vmatrix}=0 \Rightarrow \text{nema odluke (slučaj (4))}$$

Preko definicije: pogledajmo ponašanje totalnog prirasta u točki (0,0) funkcije f

$$(\Delta f)_S = f(0+\Delta x, 0+\Delta y) - f(0,0)$$
$$= (\Delta x)^2 - (\Delta y)^4$$

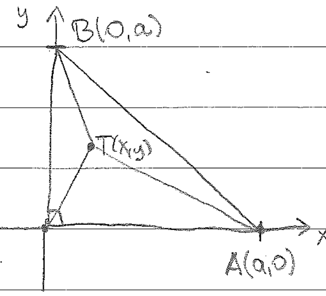
gledamo predznak od $(\Delta f)_S$ na dovoljno maloj okolini točke S

za $\Delta x \neq 0$, $\Delta y = 0$ imamo $(\Delta f)_S = (\Delta x)^2 > 0$ $\Big\}$ $(\Delta f)_S$ mijenja predznak
za $\Delta x = 0$, $\Delta y \neq 0$ imamo $(\Delta f)_S = -(\Delta y)^4 < 0$ pa nema ekstrema!

7. VJEŽBE 28.11.2024.

(DZ) Nađite ekstreme f-je $f(x,y)=x^3+y^3-3xy$

1. U pravokutnom jednakokračnom trokutu naći točku za koju je suma kvadrata udaljenosti do vrhova trokuta najmanja.



$$d^2(T,O)+d^2(T,A)+d^2(T,B)=$$
$$=x^2+y^2+(x-a)^2+y^2+x^2+(y-a)^2$$
$$=x^2+y^2+x^2-2xa+a^2+y^2+x^2+y^2-2ya+a^2$$
$$=3x^2+3y^2-2xa-2ya+2a^2=f(x,y)$$

Tražimo stacionarne točke

$$\partial_x f=6x-2a=0 \quad \Rightarrow \quad x=\frac{1}{3}a$$
$$\partial_y f=6y-2a=0 \quad \Rightarrow \quad y=\frac{1}{3}a$$

Kandidat $(\frac{1}{3}a, \frac{1}{3}a)$ za minimum

$$\partial_{xx}^2 f=6, \quad \partial_{xy}^2 f=0, \quad \partial_{yy}^2 f=6$$

$$a_{11}=6>0 \qquad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} 6 & 0 \\ 0 & 6 \end{vmatrix} = 36>0$$

Silvesterov kriterij $\Rightarrow$ $(\frac{1}{3}a, \frac{1}{3}a)$ je minimum f-je f.

(DZ) Nađi ekstreme f-je $f(x,y)=(x-y)^2+(y-1)^3$ Rj. $(1,1)$ nije točka ekstrema

(DZ) Odredite lokalne ekstreme f-je $f(x,y)=e^{x-y}(x^2-2y^2)$

$S_1(0,0)$ nema $\qquad$ $S_2(-4,-2)$ max

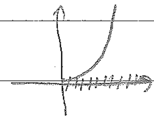
2. Odredite najkraću udaljenost mimosmjernih pravaca, tj. točke na pravcima za koje je udaljenost najmanja

$$p_1 \ldots \frac{x-5}{1}=\frac{y}{-16}=\frac{z+4}{2}=\lambda \qquad p_2 \ldots \frac{x-27}{2}=\frac{y+25}{1}=\frac{z-1}{-2}=\mu \qquad \lambda,\mu\in\mathbb{R}$$

Rj. p_1 $A(\lambda+5,-16\lambda,2\lambda-4)$ $\qquad$ $p_2 \ldots$ $B(2\mu+27,\mu-25,-2\mu+1)$

$$f(\lambda,\mu)=d(A,B)=\sqrt{(2\mu+27-\lambda-5)^2+(\mu-25+16\lambda)^2+(-2\mu+1-2\lambda+4)^2}$$

$x_1 < x_2 \Rightarrow x_1^2 < x_2^2$, $x_1, x_2 \in [0, -\infty\rangle$



Tražimo minimum od $f^2 = g$

$$g(\lambda,\mu) = (2\mu-\lambda+22)^2+(\mu+16\lambda-25)^2+(-2\mu-2\lambda+5)^2$$

$$\partial_\lambda g = -2(2\mu-\lambda+22)+32(\mu+16\lambda-25)-4(-2\mu-2\lambda+5)=0$$
$$\partial_\mu g = 4(2\mu-\lambda+22)+2(\mu+16\lambda-25)-4(-2\mu-2\lambda+5)=0$$

Stac. točka $S(2,-5)$ kandidat

$$\partial^2_{\lambda\lambda} g = 2+32\cdot16+8 = 522$$
$$\partial^2_{\mu\mu} g = 8+2+8 = 18$$
$$\partial^2_{\mu\lambda} g = -4+32+8 = 36$$

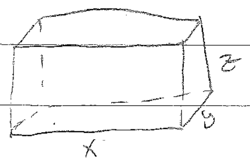
Silvester $a_{11} = 522 > 0$

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} 522 & 36 \\ 36 & 18 \end{vmatrix} = 522\cdot18-36^2>0$$

S je točka minimuma

$A(7,-32,0)$, $B(17,-30,11)$ //

3. Odredite pravokutni kvadar najvećeg volumena ako je zbroj duljina njegovih svih bridova zadan i iznosi $12a$



$V = xyz \rightarrow$ maksimizirat

$4(x+y+z)=12a \Rightarrow x+y+z=3a$ uvjet $\varphi(x,y,z) = x+y+z-3a$

Lagrangeova funkcija $F(x,y,z,\lambda) = V(x,y,z)+\lambda\varphi(x,y,z)$
$$= xyz+\lambda(x+y+z-3a)$$

$$\left.\begin{aligned} \partial_x F &= yz+\lambda = 0 & \lambda &= -yz \\ \partial_y F &= xz+\lambda = 0 & \lambda &= -xz \\ \partial_z F &= xy+\lambda = 0 & \lambda &= -xy \end{aligned}\right\} \Rightarrow x=y=z$$
$$x+y+z-3a=0$$

$3x=3a$, $x=a$

$x=y=z=a$

$S(a,a,a)$

$\lambda = -a^2$

$\partial^2_{xx} F = \partial^2_{yy} F = \partial^2_{zz} F = 0$

$\partial^2_{xy} F = z \qquad \partial^2_{xz} = y \qquad \partial^2_{yz} = x$

$$d^2F = \partial^2_{xx} F (dx)^2 + \partial^2_{yy} F (dy)^2 + \partial^2_{zz} F (dz)^2$$
$$+ 2\partial^2_{xy} \, dx\,dy + 2\partial^2_{xz} \, dx\,dz + 2\partial^2_{yz} \, dy\,dz$$

$$d^2F|_S = 2\underset{a}{z}\, dx\,dy + 2\underset{a}{y}\, dx\,dz + 2\underset{a}{x}\, dy\,dz$$

$$= 2a(dx\,dy + dx\,dz + dy\,dz)$$

Diferenciranjem uvjeta i izjednačavanjem s 0

$$d\varphi = dx + dy + dz = 0$$
$$dx = -dy - dz$$

$$d^2F|_S = 2a(-(dy)^2 - dy\,dz - \cancel{dy\,dz} - (dz)^2 + \cancel{dy\,dz})$$

$$= -2a((dy)^2 + dy\,dz + (dz)^2) = \underbrace{-2a}_{<0}\Big(\underbrace{(dy + \tfrac{1}{2}dz)^2}_{\geq 0} + \underbrace{\tfrac{3}{4}(dz)^2}_{\geq 0}\Big) < 0$$

f-ja f ima uvjetni maksimum u $S(a,a,a)$

$\rightarrow$ to je kocka brida a

(4.) Na pravcu $p \begin{cases} 2x - 5y + 8 = 0 \\ 3y + 2z - 12 = 0 \end{cases}$ odredite točku $T(x,y,z)$ koja je najmanje udaljena od ishodišta

$f(x,y,z) = \sqrt{x^2 + y^2 + z^2}$ treba minimizirati

$$F(x,y,z,\lambda,\mu) = x^2 + y^2 + z^2 + \lambda(2x - 5y + 8) + \mu(3y + 2z - 12)$$

$$\left.\begin{aligned} \partial_x F &= 2x + 2\lambda = 0 \\ \partial_y F &= 2y - 5\lambda + 3\mu = 0 \\ \partial_z F &= 2z + 2\mu = 0 \\ 2x - 5y + 8 &= 0 \\ 3y + 2z - 12 &= 0 \end{aligned}\right\}$$

$\lambda = -1, \quad \mu = -3 \qquad S(1,2,3)$

$\partial^2_{xx} F = 2 \qquad \partial^2_{yy} F = 2 \qquad \partial^2_{zz} = 2$

$\partial^2_{xy} F = 0 \qquad \partial^2_{xz} = 0 \qquad \partial^2_{yz} = 0$

$$d^2F = 2(dx)^2 + 2(dy)^2 + 2(dz)^2 = 2\left((dx)^2 + (dy)^2 + (dz)^2\right) = \dots > 0$$

$2x - 5y + 8 = 0 \quad /d$

$3y + 2z - 12 = 0 \quad /d$

$2dx - 5dy + 8 = 0 \quad \Rightarrow \quad dy = \frac{2}{5}dx + \frac{8}{5}$

$3dy + 2dz - 12 = 0 \qquad\quad dy = -\frac{2}{3}dz + 4$

$\Rightarrow$ $S(1,2,3)$ uvjetni minimum f-je f.

Materijali iz Funkcija više varijabli
Uvjetni ekstremi funkcija više varijabli

Uvjetnim ekstremom funkcije $u = f(x_1, \ldots, x_n)$ nazivamo maksimum ili minimum te funkcije dostignut pod uvjetom da su varijable $x_1, \ldots, x_n$ funkcije f povezane jednadžbama $\varphi_i(x_1, \ldots, x_n) = 0$, $i = 1, \ldots, m$, $m < n$.
Traženje uvjetnog ekstrema svodi se na traženje običnog ekstrema funkcije F (tzv. **Lagrangeove funkcije**),

$$F(x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_n) = f(x_1, \ldots, x_n) + \lambda_1\varphi_1(x_1, \ldots, x_n) + \cdots + \lambda_m\varphi_m(x_1, \ldots, x_n),$$

gdje su λ_i, $i = 1, \ldots, m$ konstantni faktori.

NUŽNI UVJETI EKSTREMA

$$\begin{aligned}
\frac{\partial F}{\partial x_1} = \frac{\partial f}{\partial x_1} + \lambda_1\frac{\partial \varphi_1}{\partial x_1} + \cdots + \lambda_m\frac{\partial \varphi_m}{\partial x_1} &= 0\\
&\vdots\\
\frac{\partial F}{\partial x_n} = \frac{\partial f}{\partial x_n} + \lambda_1\frac{\partial \varphi_1}{\partial x_n} + \cdots + \lambda_m\frac{\partial \varphi_m}{\partial x_n} &= 0\\
\varphi_1(x_1, \ldots, x_n) &= 0\\
&\vdots\\
\varphi_m(x_1, \ldots, x_n) &= 0.
\end{aligned}$$

Ako je $(x_1^*, \ldots, x_n^*, \lambda_1^*, \ldots, \lambda_m^*)$ jedno rješenje gornjeg sustava, tada funkcija f može imati ekstrem $f(x_1^*, \ldots, x_n^*)$ u točki $T(x_1^*, \ldots, x_n^*)$.

DOVOLJNI UVJETI EKSTREMA

Postojanje i karakter uvjetnog ekstrema istražujemo na osnovu predznaka drugog diferencijala Lagrangeove funkcije.

$$d^2F(x_1, \ldots, x_n) = (\frac{\partial}{\partial x_1}dx_1 + \cdots + \frac{\partial}{\partial x_n}dx_n)^2 F(x_1, \ldots, x_n)$$

za vrijednosti $(x_1^*, \ldots, x_n^*, \lambda_1^*, \ldots, \lambda_m^*)$ pod uvjetom da su $dx_1, \ldots, dx_n$ vezani jednadžbama

$$\begin{aligned}
d\varphi_1 &= \frac{\partial \varphi_1}{\partial x_1}dx_1 + \cdots + \frac{\partial \varphi_1}{\partial x_n}dx_n = 0\\
&\vdots\\
d\varphi_m &= \frac{\partial \varphi_m}{\partial x_1}dx_1 + \cdots + \frac{\partial \varphi_m}{\partial x_n}dx_n = 0
\end{aligned}$$

Iz posljednjih m jednadžbi možemo izračunati m diferencijala dx_i, $i = 1, \ldots, m$ kao funkcije od preostalih $n-m$ diferencijala $dx_{m+1}, \ldots, dx_n$ te ih uvrstiti u drugi diferencijal funkcije F. Time se broj diferencijala u d^2F smanji i tada ispitujemo predznak od d^2F za $(x_1^*, \ldots, x_n^*, \lambda_1^*, \ldots, \lambda_m^*)$.
Ako je

(a) $d^2F(x_1^*, \ldots, x_n^*) < 0$, funkcija f ima uvjetni maksimum u $T(x_1^*, \ldots, x_n^*)$.

(b) $d^2F(x_1^*, \ldots, x_n^*) > 0$, funkcija f ima uvjetni minimum u $T(x_1^*, \ldots, x_n^*)$.

(c) $d^2F(x_1^*, \ldots, x_n^*)$ mijenja predznak , funkcija f nema ekstrem u $T(x_1^*, \ldots, x_n^*)$.

(d) $d^2F(x_1^*, \ldots, x_n^*) \leq 0$ ili $d^2F(x_1^*, \ldots, x_n^*) \geq 0$, tada nema odluke.

Materijali iz Funkcija više varijabli
Ekstremi funkcija više varijabli

Definicija 1 *Funkcija $u = f(x_1, \dots, x_n)$ u točki $T(a_1, \dots, a_n)$ ima*
a) lokalni minimum $f(a_1, \dots, a_n)$ ako za svaku točku $T'(a_1 + dx_1, \dots, a_n + dx_n)$, $T' \neq T$ iz dovoljno male okoline točke T vrijedi nejednakost

$$f(T') > f(T),$$

tj. kada je totalni prirast funkcije f u točki T, $\Delta f = f(T') - f(T) > 0$.
b) lokalni maksimum $f(a_1, \dots, a_n)$ ako za svaku točku $T'(a_1 + dx_1, \dots, a_n + dx_n)$, $T' \neq T$ iz dovoljno male okoline točke T vrijedi nejednakost

$$f(T') < f(T),$$

tj. kada je totalni prirast funkcije f u točki T, $\Delta f = f(T') - f(T) < 0$.
c) nema ekstrema ako Δf mijenja predznak u točki T.

NUŽAN UVJET POSTOJANJA EKSTREMA

Ako diferencijabilna funkcija $u = f(x_1, \dots, x_n)$ ima ekstrem u točki T, tada nužno vrijedi:

$$\frac{\partial f}{\partial x_1}(T) = 0, \quad , \frac{\partial f}{\partial x_2}(T) = 0, \quad \dots, \quad \frac{\partial f}{\partial x_n}(T) = 0, \tag{1}$$

što je pak ekvivalentno sa $df(x_1, \dots, x_n)(T) = 0$.
Rješavanjem sustava jednadžbi (1) dobivamo točke $T_1, T_2, \dots, T_n$ koje zovemo *stacionarnim točkama.* One su kandidati za točke ekstrema.

DOVOLJNI UVJETI EKSTREMA

Prvi način. Neka je $T(a_1, \dots, a_n)$ stacionarna točka funkcije $u = f(x_1, \dots, x_n)$, tj. $df(x_1, \dots, x_n)(T) = 0$. Ako u nekoj dovoljno maloj okoliko točke T vrijedi

(1) $d^2 f(a_1, \dots, a_n) > 0$ za $(dx_1)^2 + \dots + (dx_n)^2 > 0$, tada funkcija f ima minimum u točki T.

(2) $d^2 f(a_1, \dots, a_n) < 0$ za $(dx_1)^2 + \dots + (dx_n)^2 > 0$, tada funkcija f ima maksimum u točki T.

(3) $d^2 f(a_1, \dots, a_n)$ mijenja predznak za $(dx_1)^2 + \dots + (dx_n)^2 > 0$, tada funkcija f nema ekstrem u točki T.

(4) $d^2 f(a_1, \dots, a_n) \geq 0$ ili $d^2 f(a_1, \dots, a_n) \leq 0$ za $(dx_1)^2 + \dots + (dx_n)^2 > 0$, tj. kada za neku kombinaciju diferencijala $d^2 f(a_1, \dots, a_n)$ može biti jednak nuli, tada nema odluke (moramo dodatno ispitati ekstreme).

(5) ako u točki T vrijedi $df(a_1, \dots, a_n) = 0$, $d^2 f(a_1, \dots, a_n) = 0, \dots, d^{n-1} f(a_1, \dots, a_n) = 0$, $d^n f(a_1, \dots, a_n) \neq 0$, tada:

  - za n neparan, T nije točka ekstrema
  - za n paran, T je točka ekstrema i to :
    (i) T je točka minimuma ako je $d^n f(a_1, \dots, a_n) > 0$
    (ii) T je točka maksimuma ako je $d^n f(a_1, \dots, a_n) < 0$

(6) ako je $d^2 f(a_1, \dots, a_n) = 0$ za $(dx_1)^2 + \dots + (dx_n)^2 > 0$, tada nema odluke.

Drugi način. Neka je $T(a_1, \dots, a_n)$ stacionarna točka funkcije $u = f(x_1, \dots, x_n)$. Pridružimo drugom diferencijalu funkcije računatom u točki T matricu

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \qquad a_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}(T), \quad i, j = 1, 2, \dots, n.$$

Tada vrijede sljedeće tvrdnje (**Silvesterov kriterij**):

Materijali iz Funkcija više varijabli
Uvjetni ekstremi funkcija više varijabli

Uvjetnim ekstremom funkcije $u = f(x_1, \ldots, x_n)$ nazivamo maksimum ili minimum te funkcije dostignut pod uvjetom da su varijable $x_1, \ldots, x_n$ funkcije f povezane jednadžbama $\varphi_i(x_1, \ldots, x_n) = 0$, $i = 1, \ldots, m$, $m < n$.
Traženje uvjetnog ekstrema svodi se na traženje običnog ekstrema funkcije F (tzv. **Lagrangeove funkcije**),

$$F(x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_n) = f(x_1, \ldots, x_n) + \lambda_1\varphi_1(x_1, \ldots, x_n) + \cdots + \lambda_m\varphi_m(x_1, \ldots, x_n),$$

gdje su λ_i, $i = 1, \ldots, m$ konstantni faktori.

NUŽNI UVJETI EKSTREMA

$$\begin{aligned}
\frac{\partial F}{\partial x_1} = \frac{\partial f}{\partial x_1} + \lambda_1\frac{\partial \varphi_1}{\partial x_1} + \cdots + \lambda_m\frac{\partial \varphi_m}{\partial x_1} &= 0\\
&\vdots\\
\frac{\partial F}{\partial x_n} = \frac{\partial f}{\partial x_n} + \lambda_1\frac{\partial \varphi_1}{\partial x_n} + \cdots + \lambda_m\frac{\partial \varphi_m}{\partial x_n} &= 0\\
\varphi_1(x_1, \ldots, x_n) &= 0\\
&\vdots\\
\varphi_m(x_1, \ldots, x_n) &= 0.
\end{aligned}$$

Ako je $(x_1^*, \ldots, x_n^*, \lambda_1^*, \ldots, \lambda_m^*)$ jedno rješenje gornjeg sustava, tada funkcija f može imati ekstrem $f(x_1^*, \ldots, x_n^*)$ u točki $T(x_1^*, \ldots, x_n^*)$.

DOVOLJNI UVJETI EKSTREMA
Postojanje i karakter uvjetnog ekstrema istražujemo na osnovu predznaka drugog diferencijala Lagrangeove funkcije.

$$d^2F(x_1, \ldots, x_n) = (\frac{\partial}{\partial x_1}dx_1 + \cdots + \frac{\partial}{\partial x_n}dx_n)^2F(x_1, \ldots, x_n)$$

za vrijednosti $(x_1^*, \ldots, x_n^*, \lambda_1^*, \ldots, \lambda_m^*)$ pod uvjetom da su $dx_1, \ldots, dx_n$ vezani jednadžbama

$$\begin{aligned}
d\varphi_1 &= \frac{\partial \varphi_1}{\partial x_1}dx_1 + \cdots + \frac{\partial \varphi_1}{\partial x_n}dx_n = 0\\
&\vdots\\
d\varphi_m &= \frac{\partial \varphi_m}{\partial x_1}dx_1 + \cdots + \frac{\partial \varphi_m}{\partial x_n}dx_n = 0
\end{aligned}$$

Iz posljednjih m jednadžbi možemo izračunati m diferencijala dx_i, $i = 1, \ldots, m$ kao funkcije od preostalih $n-m$ diferencijala $dx_{m+1}, \ldots, dx_n$ te ih uvrstiti u drugi diferencijal funkcije F. Time se broj diferencijala u d^2F smanji i tada ispitujemo predznak od d^2F za $(x_1^*, \ldots, x_n^*, \lambda_1^*, \ldots, \lambda_m^*)$.
Ako je

(a) $d^2F(x_1^*, \ldots, x_n^*) < 0$, funkcija f ima uvjetni maksimum u $T(x_1^*, \ldots, x_n^*)$.

(b) $d^2F(x_1^*, \ldots, x_n^*) > 0$, funkcija f ima uvjetni minimum u $T(x_1^*, \ldots, x_n^*)$.

(c) $d^2F(x_1^*, \ldots, x_n^*)$ mijenja predznak , funkcija f nema ekstrem u $T(x_1^*, \ldots, x_n^*)$.

(d) $d^2F(x_1^*, \ldots, x_n^*) \leq 0$ ili $d^2F(x_1^*, \ldots, x_n^*) \geq 0$, tada nema odluke.

(1) $d^2f(a_1,\ldots,a_n) > 0$ za $(dx_1)^2 + \cdots + (dx_n)^2 > 0$ ako i samo ako je

$$a_{11} > 0, \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0, \quad \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} > 0, \quad \ldots, \quad \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} > 0$$

i tada je minimum funkcije f u točki $T(a_1,\ldots,a_n)$.

(2) $d^2f(a_1,\ldots,a_n) < 0$ za $(dx_1)^2 + \cdots + (dx_n)^2 > 0$ ako i samo ako je

$$a_{11} < 0, \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0, \quad \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} < 0, \quad \ldots, \quad (-1)^n \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} > 0$$

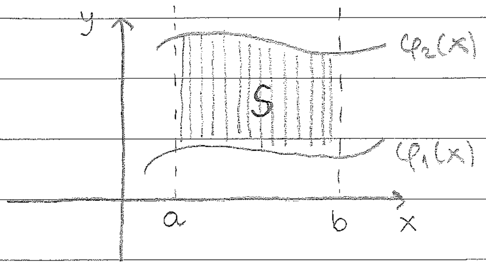
i tada je maksimum funkcije f u točki $T(a_1,\ldots,a_n)$.

(3) $d^2f(a_1,\ldots,a_n)$ mijenja predznak za $(dx_1)^2 + \cdots + (dx_n)^2 > 0$ ako su sve determinante različite od nule i ne pojavljuje se slučaj (1) ili (2) tada funkcija nema ekstrema u točki T.

(4) ako je neka od neka od determinanti jednaka nuli, tada nema odluke.

8. VJEŽBE DVOSTRUKI INTEGRALI 5.12.2024.

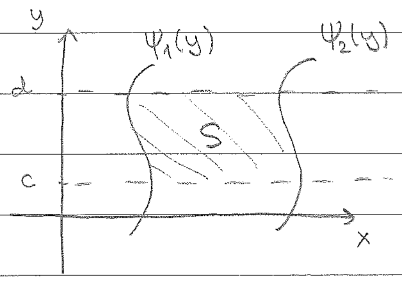
$S=\{(x,y)\in\mathbb{R}^2 : x\in[a,b],\ y\in[\varphi_1(x),\varphi_2(x)]\}$, φ_1, φ_2 neprekidne f-je s $[a,b]$ u $\mathbb{R}$ i $\varphi_1 \le \varphi_2$



Svaka nep. f-ja $f: S\to\mathbb{R}$ je R-integrabilna

$$I=\iint_S f=\int_a^b dx\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,dy$$

Može i ovako:

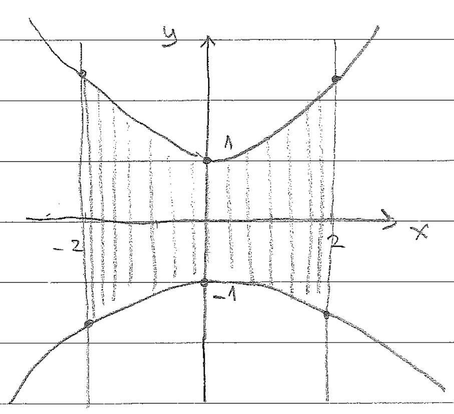


$$I=\int_c^d dy\int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\,dx$$

Napomena: Integriramo uvijek u pozitivnom smjeru koordinatnih osi, tj. u smjeru rasta koordinata

(1.) Odredite granice integracije u dvostrukom integralu $I=\iint_S f(x,y)\,dx\,dy$, ako je područje integracije S omeđeno hiperbolom $y^2-x^2=1$ i pravcima $x=2$ i $x=-2$

Rj.



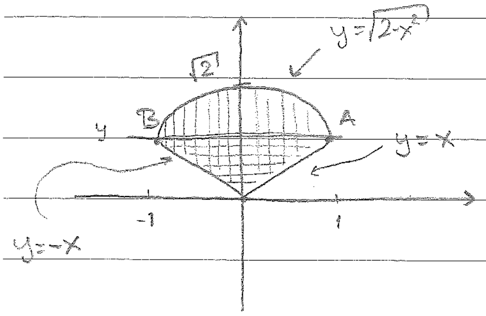
$y^2=1+x^2$

$y=\pm\sqrt{1+x^2}$ $\quad x=\pm 2$

$y^2=5 \Rightarrow y=\pm\sqrt{5}\approx\pm 2{,}23$

$$I=\int_{-2}^{2} dx\int_{-\sqrt{1+x^2}}^{\sqrt{1+x^2}} f(x,y)\,dy$$

(2.) Postavite granice integracije za oba poretka u dvostrukom integralu $I=\iint_S f(x,y)\,dx\,dy$ ako je S kružni isječak AOB s centrom u O(0,0) i s krajevima u A(1,1), B(-1,1)



$(x-p)^2+(y-q)^2=r^2$

$x^2+y^2=2$

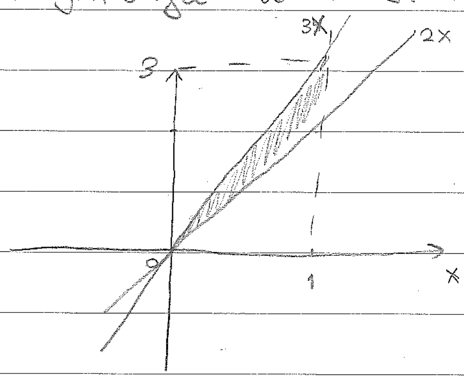
$y=\sqrt{2-x^2}$

a) $I=\int_{-1}^{0}dx\int_{-x}^{\sqrt{2-x^2}}f(x,y)\,dy+\int_{0}^{1}dx\int_{x}^{\sqrt{2-x^2}}f(x,y)\,dy$

b) $I=\int_{0}^{1}dy\int_{-y}^{y}f(x,y)\,dx+\int_{1}^{\sqrt{2}}dy\int_{-\sqrt{2-y^2}}^{\sqrt{2-y^2}}f(x,y)\,dx$

(3.) Promijenite redoslijed integriranja u dvostrukom integralu

$I=\int_{0}^{1}dx\int_{2x}^{3x}f(x,y)\,dy$



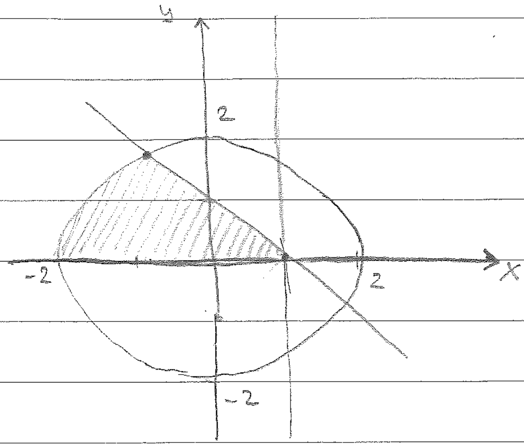
$I=\int_{0}^{2}dy\int_{\frac{y}{3}}^{\frac{y}{2}}f(x,y)\,dx$

$+\int_{2}^{3}dy\int_{\frac{y}{3}}^{1}f(x,y)\,dx$

(DZ.) Izračunajte $I=\iint_S x\,dx\,dy$ gdje je S područje omeđeno pravcem koji prolazi kroz A(2,0) i B(0,2) i lukom kružnice sa središtem u C(0,1) radijusa R=1. S je u prvom kvadrantu! Rj. $\left(\frac{1}{6}\right)$

(4.) Postavite granice integracije za $\iint_S dx\,dy$, gdje je S omeđeno s $x^2+y^2=4$, $y=0$, $y=1-x$ i $x\le 1$

Rj:



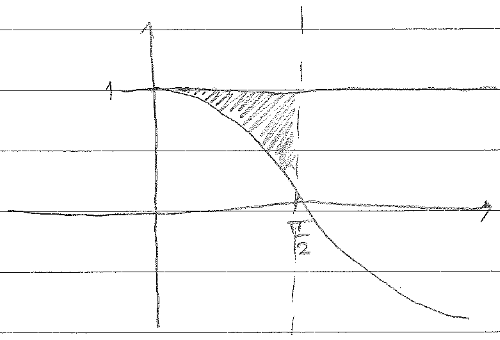
Sjecišta pravca $y=1-x$ i kružnice

$$x^2+(1-x)^2=4$$
$$x^2+1-2x+x^2=4$$
$$2x^2-2x-3=0$$
$$x_1=\frac{1+\sqrt{7}}{2} \qquad x_2=\frac{1-\sqrt{7}}{2}$$

$$I=\int_{-2}^{\frac{1-\sqrt{7}}{2}} dx \int_0^{\sqrt{4-x^2}} dy + \int_{\frac{1-\sqrt{7}}{2}}^{1} dx \int_0^{1-x} dy$$

(DZ.) Izračunajte dvostruki integral $\iint_S \frac{x}{x^2+y^2}\,dx\,dy$ gdje je S područje omeđeno s $y=\frac{x^2}{2}$ i $y=x$

(5.) Izračunajte dvostruki integral $\int_0^{\frac{\pi}{2}} dx \int_{\cos x}^{1} y^4\,dy$. Nacrtajte područje integracije



$$I=\int_0^{\pi/2} dx \int_{\cos x}^{1} y^4\,dy$$
$$=\int_0^{\pi/2} \left(\frac{y^5}{5}\right)\Big|_{\cos x}^{1} dx$$
$$=\int_0^{\pi/2}\left(\frac{1}{5}-\frac{\cos^5 x}{5}\right)dx$$

$$=\int_0^{\pi/2}\frac{1}{5}\,dx-\frac{1}{5}\int_0^{\pi/2}\cos^5 x\,dx=\left(\frac{1}{5}x\right)\Big|_0^{\pi/2}-\frac{1}{5}I_2$$

$$I_2 = \int_0^{\pi/2} \cos^5 x\,dx = \left| \begin{array}{l} t = \sin x \\ dt = \cos x\,dx \end{array} \right|$$

* Ako je jedan od njih [illegible] na pozitivnu neparnu potenciju za supstituciju uzmi onog drugog

$$= \int_0^{\pi/2} \cos^4 x \cos x\,dx = \int_0^{\pi/2} (1-\sin^2 x)^2 \cos x\,dx$$

$$= \int_0^1 (1-t^2)^2\,dt = \int_0^1 (1-2t^2+t^4)\,dt = \left(t - \frac{2t^3}{3} + \frac{t^5}{5}\right)\Bigg|_0^1 = 1 - \frac{2}{3} + \frac{1}{5}$$

$$= \frac{15-10+3}{15} = \frac{8}{15}$$

$$I = \frac{\pi}{10} - \frac{1}{5}\cdot\frac{8}{15} \quad //$$

DZ. Izračunajte $\iint_S e^{\frac{x}{y}}\,dx\,dy$ gdje je S područje omeđeno parabolom $y^2 = 3x$, pravcima $x=0$, $y=\frac{1}{2}$, $y=1$

Zamjena varijabli u dvostrukom integralu

1) Prijelaz na krivulne koordinate

$$I = \iint_S f(x,y)\,dx\,dy = \left| \begin{array}{l} x = \varphi(u,v) \\ y = \varphi(u,v) \\ dx\,dy = |J|\,du\,dv \end{array} \right| = \iint_{S'} f(\varphi(u,v), \varphi(u,v))\,|J|\,du\,dv$$

gdje je J Jacobijan $\quad J = \begin{vmatrix} \partial_u x & \partial_v x \\ \partial_u y & \partial_v y \end{vmatrix}$

2) Prijelaz na polarne koordinate

$$I = \iint_S f(x,y)\,dx\,dy = \left| \begin{array}{l} x = r\cos\varphi \\ y = r\sin\varphi \end{array},\ dx\,dy = |J|\,dr\,d\varphi \right|$$

$$= \iint_{S'} f(r\cos\varphi, r\sin\varphi)\,|J|\,dr\,d\varphi, \quad \text{gdje je } |J| = r$$

$$J = \begin{vmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{vmatrix} = r\cos^2\varphi + r\sin^2\varphi = r(\cos^2\varphi + \sin^2\varphi) = r \quad //$$

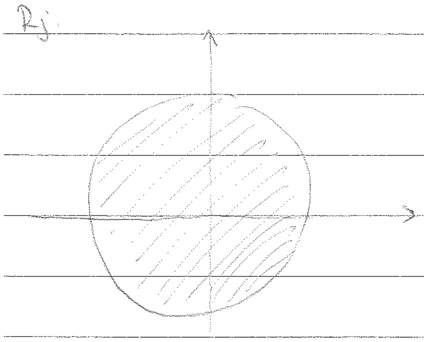
3) Prijelaz na eliptičke koordinate

$$I = \iint_S f(x,y)\,dx\,dy = \begin{vmatrix} x = ar\cos\varphi \\ y = br\sin\varphi \\ dx\,dy = abr\,dr\,d\varphi \end{vmatrix}$$

$$= \iint_S f(ar\cos\varphi, br\sin\varphi)\,abr\,dr\,d\varphi$$

Zad. $I = \iint_S \sqrt{1-x^2-y^2}\,dx\,dy$, gdje je S krug polumjera $R=1$ sa središtem u ishodištu

Rj:



$x^2+y^2 \leq 1$

$x = r\cos\varphi$ $\qquad \sqrt{1-r^2\cos^2\varphi - r^2\sin^2\varphi}$

$y = r\sin\varphi$ $\qquad = \sqrt{1-r^2}$

$dx\,dy = r\,dr\,d\varphi$

$\varphi \in [0, 2\pi\rangle$

$r \in [0, 1]$

$$I = \int_0^{2\pi}\int_0^1 \sqrt{1-r^2}\,r\,dr\,d\varphi = \begin{vmatrix} 1-r^2 = t \\ -2r\,dr = dt \\ r\,dr = -\frac{1}{2}dt \end{vmatrix}$$

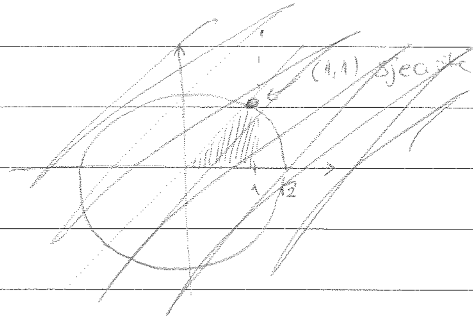
$$= \int_0^{2\pi}\int_1^0 \sqrt{t}\left(-\frac{1}{2}\right)dt\,d\varphi = -\frac{1}{2}\int_0^{2\pi}\left(\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right)\Bigg|_1^0 d\varphi$$

$$= -\frac{1}{2}\int_0^{2\pi}\left(-\frac{2}{3}\right)d\varphi = \frac{1}{3}\int_0^{2\pi} d\varphi = \frac{2\pi}{3}$$

9. VJEŽBE 12.12.2024.

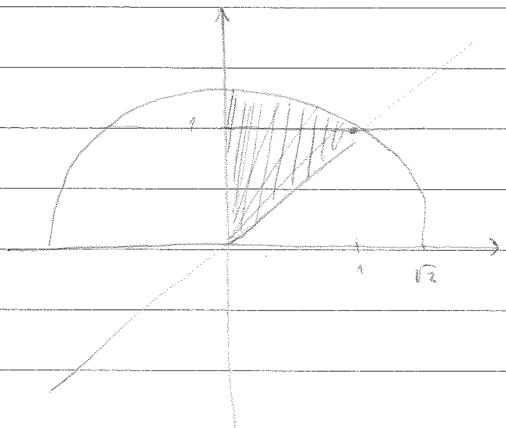
ZAD. 1) Prijelazom na polarne koordinate izračunajte

$$\int_0^1 x^2 dx \int_x^{\sqrt{2-x^2}} e^{x^2+y^2} dy$$



Rj. $x=0$ do $x=1$

$y=x$ do $y=\sqrt{2-x^2}$ $/^2$ $\quad x^2+y^2=2$



$x = r\cos\theta \qquad dxdy = r\,dr\,d\theta$

$y = r\sin\theta$

$\theta \in [\frac{\pi}{4}, \frac{\pi}{2}] \qquad r \in [0, \sqrt{2}]$

* prvo θ pa r

$$I = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} r^2\cos^2\theta\, d\theta \int_0^{\sqrt{2}} e^{r^2} r\, dr$$

$$I = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos^2\theta\, d\theta \underbrace{\int_0^{\sqrt{2}} r^3 e^{r^2} dr}_{I_1}$$

$$I_1 = \int r^3 e^{r^2} dr = \begin{vmatrix} t = r^2 \\ dt = 2r\,dr \\ dr = \frac{dt}{2r} \end{vmatrix} = \int e^t \cdot \underset{r^2=t}{r^3} \cdot \frac{dt}{2r} = \frac{1}{2}\int e^t \cdot t\, dt = \begin{vmatrix} u = t & dv = e^t dt \\ du = dt & v = e^t \end{vmatrix}$$

$$= \frac{1}{2}te^t - \frac{1}{2}\int e^t dt = \frac{1}{2}te^t - \frac{1}{2}e^t = \frac{1}{2}r^2e^{r^2} - \frac{1}{2}e^{r^2}$$

$$I = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos^2\theta\, d\theta \int_0^{\sqrt{2}} e^{r^2} r^3 dr = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos^2\theta \cancel{d\theta} \left(\frac{1}{2}r^2e^{r^2} - \frac{1}{2}e^{r^2}\right)\Big|_0^{\sqrt{2}} d\theta$$

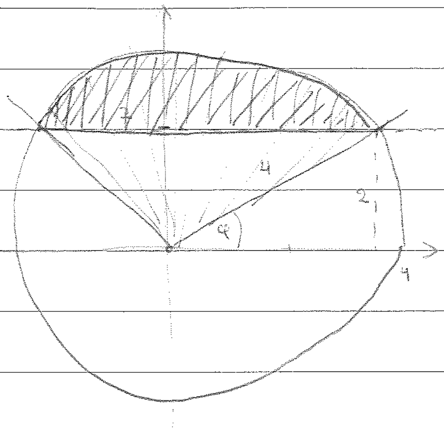
$$= \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos^2\theta\,(e^2+1)\, d\theta = \frac{1}{2}(e^2+1)\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1+\cos(2\theta)}{2} d\theta = \frac{1}{4}(e^2+1)\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (1+\cos(2\theta))d\theta$$

$$= \frac{e^2+1}{4}\left(\theta + \frac{1}{2}\sin(2\theta)\right)\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} = (e^2+1)\,\frac{\pi-2}{16}$$

ZAD 2. Izračunati $\iint_P \frac{dxdy}{(x^2+y^2)^2}$, gdje je P određeno nejednadžbama

$x^2+y^2 \le 16$, $y \ge 2$

$x = r\cos\theta \quad y = r\sin\theta \quad dxdy = r\,dr\,d\theta$



$\sin\varphi = \frac{2}{4} = \frac{1}{2} \qquad \arcsin\frac{1}{2} = \frac{\pi}{6}$

$\theta \in \left[\frac{\pi}{6}, \frac{5\pi}{6}\right]$

$y = 2$

$r\sin\theta = 2$

$r = \frac{2}{\sin\theta}$ do $r = 4$

$$I = \int_{\frac{\pi}{6}}^{5\pi/6} d\theta \int_{\frac{2}{\sin\theta}}^{4} \frac{1}{r^4}\cdot r\,dr = \int_{\pi/6}^{5\pi/6} d\theta \int_{2/\sin\theta}^{4} r^{-3}\,dr = \int_{\pi/6}^{5\pi/6} \left(\frac{r^{-2}}{-2}\right)\Bigg|_{2/\sin\theta}^{4} d\theta$$

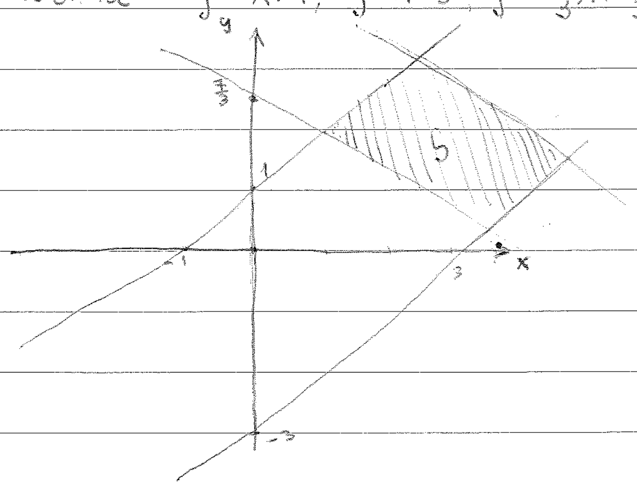
$$= -\frac{1}{2}\int_{\pi/6}^{5\pi/6} \left(\frac{1}{4^2} - \frac{\sin^2\theta}{4}\right) d\theta = -\frac{1}{2}\int_{\pi/6}^{5\pi/6} \frac{1}{16}\,d\theta + \frac{1}{8}\int_{\pi/6}^{5\pi/6} \sin^2\theta\,d\theta$$

$$= -\frac{1}{2}\,\frac{\theta}{16}\Bigg|_{\pi/6}^{5\pi/6} + \frac{1}{8}\int_{\pi/6}^{5\pi/6} \frac{1-\cos(2\theta)}{2}\,d\theta = -\frac{1}{2}\,\frac{\theta}{16}\Bigg|_{\pi/6}^{5\pi/6} + \frac{1}{8}\int_{\pi/6}^{5\pi/6} \frac{1}{2}\,d\theta$$

$$-\frac{1}{16}\int_{\pi/6}^{5\pi/6} \cos(2\theta) = \left[-\frac{1}{32}\theta + \frac{1}{16}\theta - \frac{1}{16}\cdot\frac{1}{2}\sin(2\theta)\right]\Bigg|_{\frac{\pi}{6}}^{5\pi/6} = \frac{\pi}{48} + \frac{\sqrt{3}}{32}$$

Krivuljne koordinate

Zad. Izračunajte vrijednost $I = \iint_S (y-x)\,dxdy$ ako je S područje omeđeno pravcima $y = x+1$, $y = x-3$, $y = -\frac{1}{3}x + \frac{7}{3}$, $y = -\frac{1}{3}x + 5$



Inače bismo podijelili područje na tri dijela, no ovdje imamo po dva paralelna pravca pa uvedemo nove varijable supstitucija

$$\left.\begin{array}{l} y-x=-3 \\ y-x=1 \end{array}\right\} u=y-x \qquad \left.\begin{array}{l} y+\frac{1}{3}x=7/3 \\ y+1/3x=5 \end{array}\right\} v=y+\frac{1}{3}x$$

$$x=-\frac{3}{4}u+\frac{3}{4}v \qquad y=\frac{1}{4}u+\frac{3}{4}v$$

$$-3\le u\le 1 \qquad \frac{7}{3}\le v\le 5$$

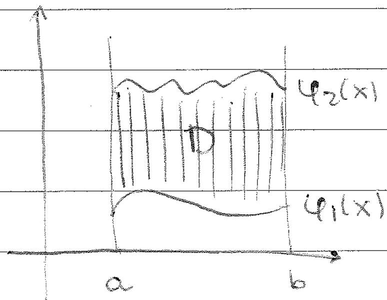
$$J=\begin{vmatrix} \partial_u x & \partial_v x \\ \partial_u y & \partial_v y \end{vmatrix}=\begin{vmatrix} -\frac{3}{4} & \frac{3}{4} \\ \frac{1}{4} & \frac{3}{4} \end{vmatrix}=-\frac{3}{4}$$

$$\Rightarrow |J|=\frac{3}{4}$$

$$I=\int_{7/3}^{5}dv\int_{-3}^{1}\frac{3}{4}u\,du=\frac{3}{4}\int_{7/3}^{5}\left(\frac{u^2}{2}\right)\Bigg|_{-3}^{1}dv=\frac{3}{4}\int_{7/3}^{5}(-4)\,dv=-3\,v\Big|_{7/3}^{5}=-8$$

Primjena dvostrukih integrala

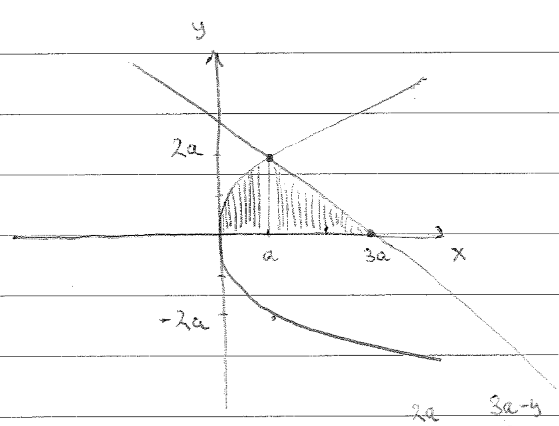
* Izračunavanje površina ravninskih likova



$$P=\iint_D dx\,dy=\int_a^b dx\int_{\varphi_1(x)}^{\varphi_2(x)} dy$$

dakle, podintegralna funkcija je $f(x,y)=1$

Zad. Izračunajte površinu lika omeđenog krivuljama $y^2=4ax$, $x+y=3a$, $y=0$, $a>0$. Lik se nalazi u prvom kvadrantu



$y=3a-x$

Sjecišta pravca i parabole

$(3a-x)^2=4ax$

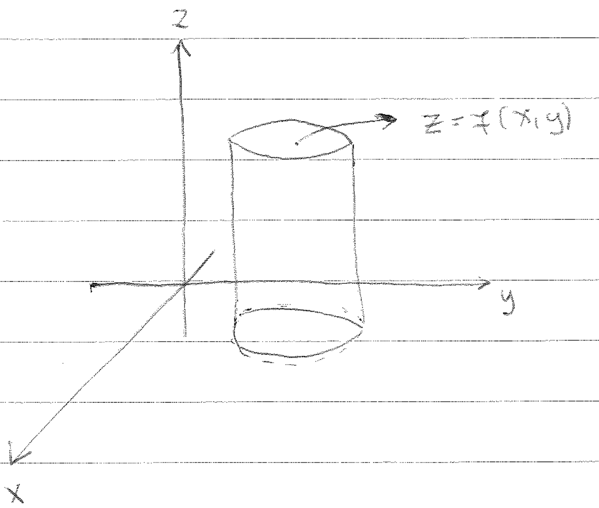
$x_1=9a \qquad x_2=a$

$$I=\int_0^{2a}dy\int_{\frac{y^2}{4a}}^{3a-y}dx=\ldots=\frac{10}{3}a^2>0$$ jer računamo površinu

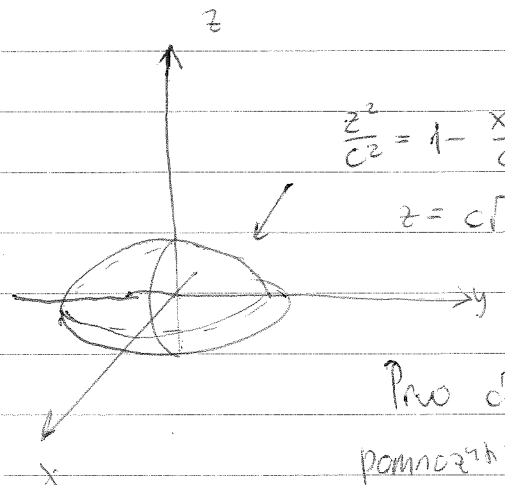
Računanje volumena tijela

Volumen cilindra V omeđenog odozgo neprekidnom plohom $z = f(x,y)$, odozdo ravninom $z=0$, a sa strane valjkastom plohom koja na ravnini xOy izrezuje omeđeno zatvoreno područje S, iznosi

$$V = \iint_S f(x,y)\,dx\,dy$$



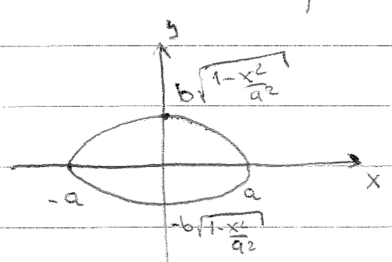
Zad. Odredite volumen elipsoida $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$



$$\frac{z^2}{c^2} = 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}$$

$$z = c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}$$

Prvo ćemo odrediti volumen iznad xOy ravnine pa pomnožiti s 2



$$I = \int_{-a}^{a} dx \int_{-b\sqrt{1-\frac{x^2}{a^2}}}^{b\sqrt{1-\frac{x^2}{a^2}}} c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}\,dy$$

$y = b\sqrt{1-\frac{x^2}{a^2}}$

$b r\sin\theta = b\sqrt{1 - r^2\cos^2\theta}$

$r^2\sin^2\theta = 1 - r^2\cos^2\theta$

$r^2 = 1$

$r = 1$

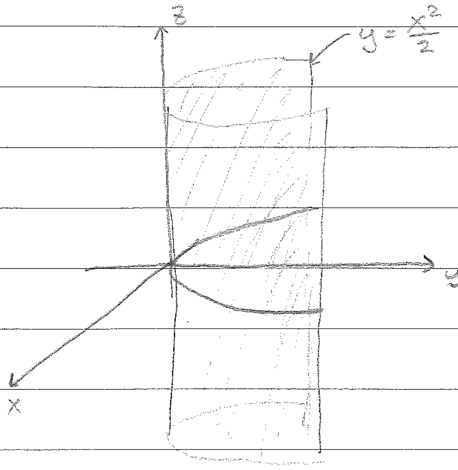
$$\left.\begin{aligned} x &= ar\cos\theta \\ y &= br\sin\theta \end{aligned}\right\} \qquad dxdy = abr\,dr\,d\theta \qquad \theta \in [0, 2\pi] \qquad r \in [0,1]$$

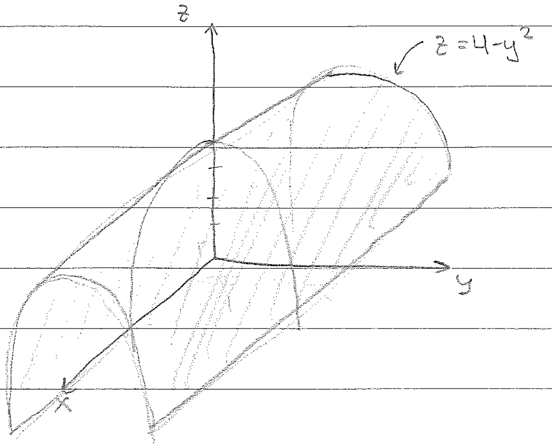
Uvrstili u jednadžbu elipse

$$I = \int_0^{2\pi} d\theta \int_0^1 c\sqrt{1-r^2}\, abr\,dr = abr \int_0^{2\pi} d\theta \int_1^0 t^{\frac{1}{2}}\left(-\frac{dt}{2}\right) = -\frac{1}{2}\int_0^{2\pi} \left.\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right|_1^0 d\theta$$

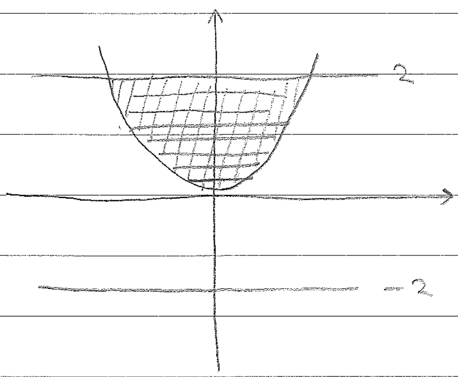
$$= \dots = \frac{2}{3}abc\pi \quad \Rightarrow \quad V = \frac{4}{3}abc\pi \;//$$

ZAD. Odredite volumen tijela omeđenog s $z=4-y^2$, $y=\frac{x^2}{2}$, $z=0$





xOy:



Za $z=0 \Rightarrow z=4-y^2$

$0=4-y^2$

$y^2=4$

$y=\pm 2$

$$V=\int_0^2 dy \int_{-\sqrt{2y}}^{\sqrt{2y}} (4-y^2)\,dx$$

$y\in[0,2]$ $\quad x^2=2y$

$x\in[-\sqrt{2y},\sqrt{2y}]$

$$=\int_0^2 (4-y^2)\, x\Big|_{-\sqrt{2y}}^{\sqrt{2y}} dy$$

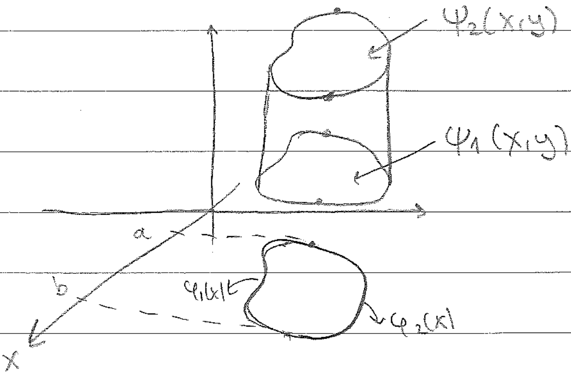
$$=\int_0^2 (4-y^2)2\sqrt{2}y^{\frac{1}{2}}dy = 2\sqrt{2}\int_0^2\left(4y^{\frac{1}{2}}-y^{\frac{5}{2}}\right)dy = 2\sqrt{2}\left(4\frac{y^{\frac{3}{2}}}{\frac{3}{2}}-\frac{y^{\frac{7}{2}}}{\frac{7}{2}}\right)\Big|_0^2 = \frac{256}{21}$$

DZ. Odredite volumen tijela omeđenog s $z=4-x^2$, $2x+y=4$, $x\geq 0, y\geq 0, z\geq 0$

Rj. $\frac{40}{3}$

TROSTRUKI INTEGRALI

$$I=\iiint_V f(x,y,z)\,dx\,dy\,dz$$



$$I=\iiint_V f(x,y,z)\,dx\,dy\,dz$$

$$=\int_a^b dx\int_{\varphi_1(x)}^{\varphi_2(x)} dy \int_{\psi_1(x,y)}^{\psi_2(x,y)} f(x,y,z)\,dz$$

Zamjena varijabli u trostrukom integralu

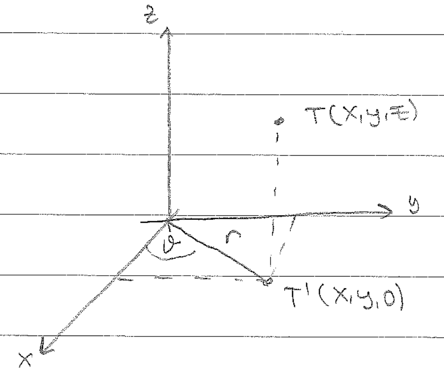
a) Krivuljne koordinate (analogno kao i za dvostruki integral)

b) Cilindrične koordinate

$x = r\cos\theta \qquad r \in [0, +\infty\rangle$

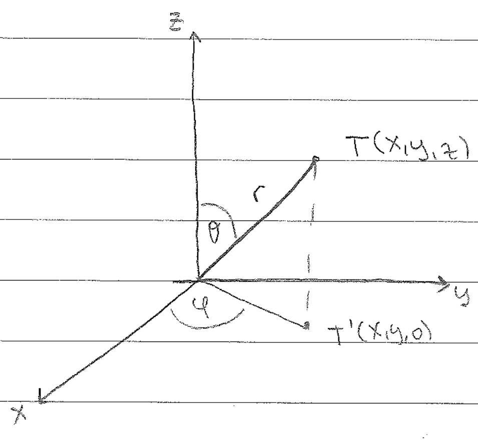
$y = r\sin\theta \qquad \theta \in [0, 2\pi]$

$z = z \qquad z \in \langle -\infty, +\infty\rangle$



$$I = \iiint_V f(x,y,z)\,dx\,dy\,dz = \iiint_{V'} f(r\cos\theta, r\sin\theta, z)\, r\,dr\,d\theta\,dz$$

c) Sferni sustav



$\varphi \in [0, 2\pi] \qquad x = r\sin\theta\cos\varphi$

$\theta \in [0, \pi] \qquad y = r\sin\theta\sin\varphi$

$r \in [0, +\infty\rangle \qquad z = r\cos\theta$

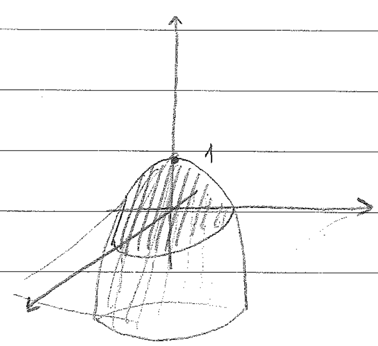
$dx\,dy\,dz = r^2 \sin\theta\, dr\,d\theta\,d\varphi$

Primjena: volumen i masa tijela

$V = \iiint_W dx\,dy\,dz \qquad m = \iiint_W \rho\,dx\,dy\,dz, \quad \rho = \rho(x,y,z)$ gustoća u točki $T(x,y,z)$

Zad. Odredite granice integracije u integralu $I = \iiint_V f(x,y,z)\,dx\,dy\,dz$ ako je V područje omeđeno plohama:

a) $z = 1 - x^2 - y^2, \; z = 0$



$z - 1 = -(x^2 + y^2)$

$z = 1 \Rightarrow x^2 + y^2 = 0$

$x = 0, \; y = 0$

vrh

10.

Presjek paraboloida i ravnine $z=0$

$1-x^2-y^2=0$

$x^2+y^2=1$ $\qquad x=r\cos\theta \qquad \theta\in[0,2\pi]$

$y=r\sin\theta \qquad r\in[0,1]$

$z=z$

$z=1-r^2\cos^2\theta-r^2\sin^2\theta=1-r^2$

$$I=\int_0^{2\pi}d\theta\int_0^1 r\,dr\int_0^{1-r^2} f(r\cos\theta, r\sin\theta, z)\,dz$$

b) stošcem $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z^2}{c^2}$ i ravninom $z=c$, $c>0$

$z=0 \Rightarrow x=0, y=0,$ vrh $(0,0,0)$

$z^2=c^2\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)$

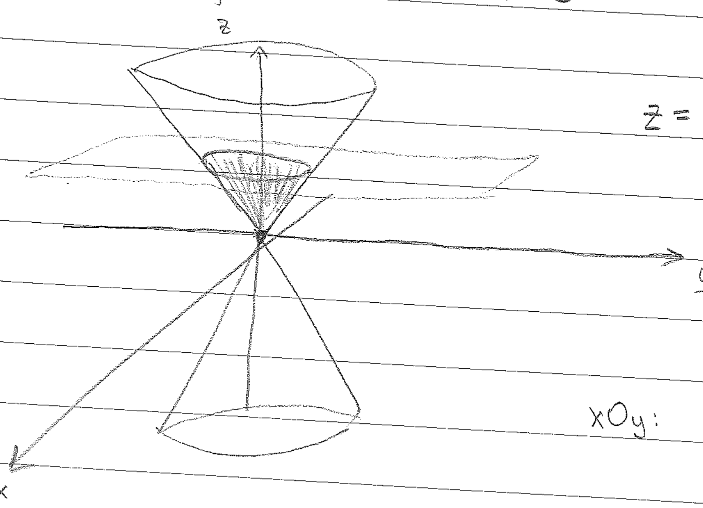
$z=\pm c\sqrt{\frac{x^2}{a^2}+\frac{y^2}{b^2}}$

Presjek stošca i ravnine

$z=c$

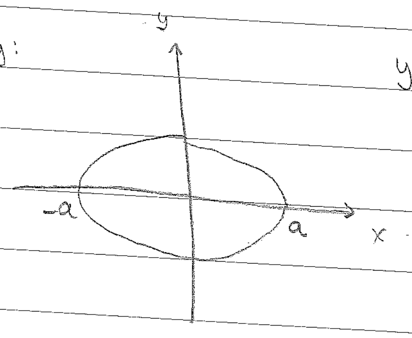
$c=c\sqrt{\frac{x^2}{a^2}+\frac{y^2}{b^2}} \Rightarrow \frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

elipsa s poluosima a i b



$z=c\sqrt{\frac{x^2}{a^2}+\frac{y^2}{b^2}}$ do c

xOy:



$y=\pm b\sqrt{1-\frac{x^2}{a^2}}$

x = od -a do a

$$I=\int_{-a}^{a}dx\int_{-b\sqrt{1-\frac{x^2}{a^2}}}^{b\sqrt{1-\frac{x^2}{a^2}}}dy\int_{c\sqrt{\frac{x^2}{a^2}+\frac{y^2}{b^2}}}^{c}dz$$

DZ c) ~~nap...~~ tetraedar omeđen ravninama $x+y+z=1$, $x=0, y=0, z=0$

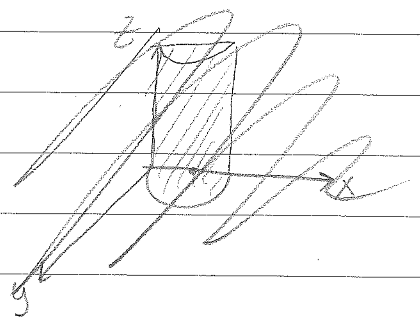
Zad.) Izračunati $I=\iiint_V z\sqrt{x^2+y^2}\,dxdydz$ gdje je V omeđeno plohama $z=0$, $z=a>0$, $y=0$, $y=\sqrt{2x-x^2}$

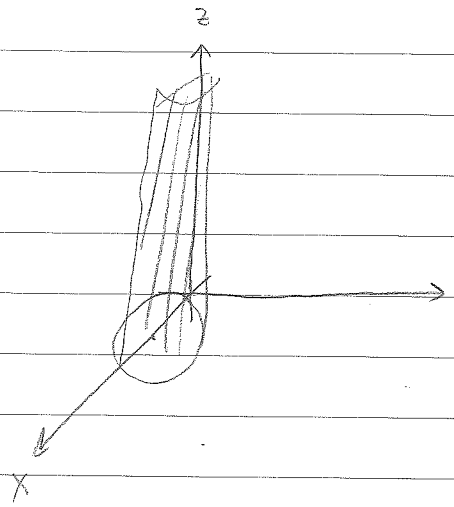
$\underset{\geq 0}{}$

$y=\sqrt{2x-x^2}\ /^2$

$x^2-2x+y^2=0$

$\boxed{(x-1)^2+y^2=1}$





z od 0 do a

$x=r\cos\theta$ $\theta\in[0,\frac{\pi}{2}]$

$y=r\sin\theta$

$z=z$

Trebamo odrediti granice za r:

$x^2+y^2=2x$

$r^2=2r\cos\theta$

$r(r-2\cos\theta)=0$

$r_1=0$ $r_2=2\cos\theta$

$$I=\int_0^{\frac{\pi}{2}}d\theta\int_0^{2\cos\theta}r\,dr\int_0^a zr\,dz$$

$$=\int_0^{\frac{\pi}{2}}d\theta\int_0^{2\cos\theta}r^2\frac{a^2}{2}\,dr$$

$$=\frac{a^2}{2}\int_0^{\frac{\pi}{2}}\frac{r^3}{3}\Big|_0^{2\cos\theta}d\theta$$

$$=\frac{a^2}{6}\int_0^{\frac{\pi}{2}}8\cos^3\theta\,d\theta$$

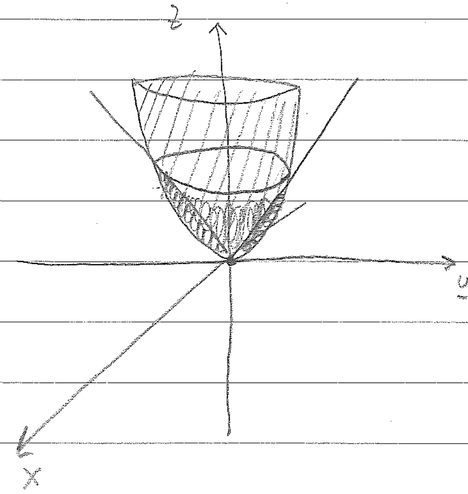
$$=\frac{4}{3}a^2\int_0^{\pi/2}\cos^3\theta\,d\theta=\left|\begin{array}{l}\sin\theta=t\\ \cos\theta d\theta=dt\\ \theta=0\Rightarrow t=0\\ \theta=\frac{\pi}{2},\ t=1\end{array}\right|=\frac{4}{3}a^2\int_0^1(1-t^2)dt=\frac{4}{3}a^2\left(t-\frac{t^3}{3}\right)\Big|_0^1=\frac{8}{9}a^2$$

(ZAD) Izračunati volumen tijela omeđenog plohama $z=x^2+y^2$
$z=2\sqrt{x^2+y^2}$

$z=x^2+y^2$

$z=0 \Rightarrow x=0, y=0$

$z^2=4(x^2+y^2)$



$x^2+y^2=2\sqrt{x^2+y^2} \quad /^2$

$(x^2+y^2)^2=4(x^2+y^2) \quad /: x^2+y^2\neq 0$

$x^2+y^2=4$ ← presjek

$z=x^2+y^2$ do $z=2\sqrt{x^2+y^2}$

$x=r\cos\theta \qquad \theta\in[0,2\pi]$

$y=r\sin\theta \qquad r\in[0,2]$

$z=z$

$$I=\int_0^{2\pi} d\theta \int_0^2 r\,dr \int_{r^2}^{2r} dz = \int_0^{2\pi} d\theta \int_0^2 r(2r-r^2)\,dr = \int_0^{2\pi} d\theta \int_0^2 (2r^2-r^3)\,dr$$

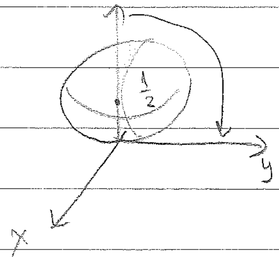
$$=\int_0^{2\pi} d\theta \left(\frac{2r^3}{3}-\frac{r^4}{4}\right)\Big|_0^2 = \dots = \frac{8\pi}{3}$$

(ZAD) Izračunati $I=\iiint_V \sqrt{x^2+y^2+z^2}\,dx\,dy\,dz$ ako je V kugla

$x^2+y^2+z^2\leq z \qquad x^2+y^2+z^2-z\leq 0$

$x^2+y^2+(z-\frac{1}{2})^2\leq\frac{1}{4} \qquad S(0,0,\frac{1}{2})$

$R=\frac{1}{2}$



$z=r\cos\theta \qquad \theta\in[0,\frac{\pi}{2}]$

$x=r\sin\theta\cos\varphi \qquad r\in[0,\cos\theta]$

$y=r\sin\theta\sin\varphi \qquad \varphi\in[0,2\pi]$

$J=r^2\sin\theta$

$\underbrace{x^2+y^2+z^2} = z$

$r^2 = z$

$r^2 = r\cos\vartheta$

$r(r-\cos\vartheta)=0$

$r_1=0 \qquad r_2=\cos\vartheta$

$$I = \int_0^{2\pi} d\varphi \int_0^{\frac{\pi}{2}} \sin\theta\, d\theta \int_0^{\cos\theta} r^3\, dr$$

$$= \frac{1}{4}\int_0^{2\pi} d\varphi \int_0^{\frac{\pi}{2}} \cos^4\vartheta \sin\vartheta\, d\theta = \left| \begin{array}{l} t=\cos\vartheta \\ dt=-\sin\vartheta d\vartheta \end{array} \right.$$

$t=0 \Rightarrow 1 \quad , \quad t=\frac{\pi}{2} \Rightarrow 0$

$$= -\frac{1}{4}\int_0^{2\pi} d\varphi \int_1^0 t^4\, dt$$

$$= -\frac{1}{4}\int_0^{2\pi} \left(-\frac{1}{5}\right) d\varphi = \frac{1}{20}\cdot 2\pi = \frac{\pi}{10}$$

11. VJEŽBE KRIVULJNI INTEGRALI 9.1.2025.

1. VRSTE

(A) Ako je krivulja zadana s $y=\varphi(x)$, $a \le x \le b$, tada $\int_{\widehat{AB}} f(x,y)\,ds$
$= \int_a^b f(x,\varphi(x))\sqrt{1+\varphi'(x)^2}\,dx$

(B) C zadana parametarski $x=\varphi(t)$, $y=\psi(t)$, $t_1 \le t \le t_2$

$$\int_{\widehat{AB}} f(x,y)\,ds = \int_{t_1}^{t_2} f(\varphi(t),\psi(t))\sqrt{\varphi'(t)^2+\psi'(t)^2}\,dt$$

ovo se lako generalizira na $C \subseteq \mathbb{R}^3$

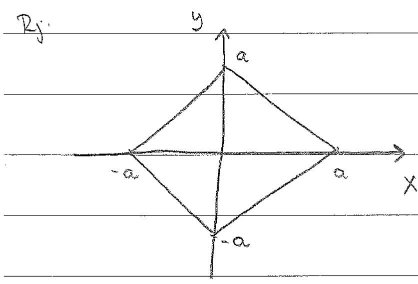
$x=\varphi(t)$, $y=\psi(t)$, $z=\xi(t)$ $\quad \int_{\widehat{AB}} f(x,y,z)\,ds = \int_{t_1}^{t_2} f(\varphi(t),\psi(t),\xi(t))\sqrt{\varphi'(t)^2+\psi'(t)^2+\xi'(t)}\,dt$

(C) C ... $r=r(\varphi)$ polarno $\alpha \le \varphi \le \beta$

$x=r(\varphi)\cos\varphi \quad y=r(\varphi)\sin\varphi \quad \int_{\widehat{AB}} f(x,y)\,ds = \int_\alpha^\beta f(r(\varphi)\cos\varphi, r(\varphi)\sin\varphi)$
$\sqrt{r(\varphi)^2+r'(\varphi)^2}\,d\varphi$

ZAD 1. Izračunajte $I=\int_C xy\,ds$ gdje je C opseg kvadrata $|x|+|y|=a$, $a>0$

Rj.



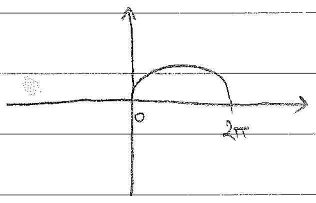
$$ds = \sqrt{1+(y')^2}\,dx = \sqrt{2}\,dx$$

$$\int_C xy\,ds = \int_0^a x(a-x)\sqrt{2}\,dx + \int_{-a}^0 x(a+x)\sqrt{2}\,dx + \int_{-a}^0 x(-a-x)\sqrt{2}\,dx + \int_0^a x(x-a)\sqrt{2}\,dx$$

$$= \int_0^a (a\sqrt{2}x-\sqrt{2}x^2)dx + \int_{-a}^0 (a\sqrt{2}x+\sqrt{2}x^2)dx + \int_{-a}^0 (-a\sqrt{2}x-\sqrt{2}x^2)dx + \int_0^a (-a\sqrt{2}x+\sqrt{2}x^2)dx$$
$$= \cancel{a\sqrt{2}\cdot\frac{x^2}{2}\Big|_0^a} - \cancel{\sqrt{2}\cdot\frac{x^3}{3}\Big|_0^a} + \cancel{a\sqrt{2}\cdot\frac{x^2}{2}\Big|_{-a}^0} + \cancel{\sqrt{2}\cdot\frac{x^3}{3}\Big|_{-a}^0} - \cancel{a\sqrt{2}\frac{x^2}{2}\Big|_{-a}^0} - \cancel{\sqrt{2}\frac{x^3}{3}\Big|_{-a}^0} - \cancel{a\sqrt{2}\frac{x^2}{2}\Big|_0^a} + \cancel{\sqrt{2}\frac{x^3}{3}\Big|_0^a}$$
$= 0$

ZAD. 2 Izračunajte $\int_C y^2\,ds$ gdje je C prvi svod cikloide $x=a(t-\sin t)$
$y=a(1-\cos t)$, $t\in[0,2\pi]$, $a>0$

zadana parametarski



$$\vec{r}(t) = (a(t-\sin t),\ a(1-\cos t))$$
$$\vec{r}\,'(t) = (a-a\cos t,\ a\sin t)$$

$$\int_c y^2 ds = \int_0^{2\pi} a^2(1-\cos t)^2\sqrt{a^2(1-\cos t)^2+a^2\sin^2 t}\; dt = \int_0^{2\pi} a^3(1-\cos t)^2\sqrt{1-2\cos t+\cos^2 t+\sin^2 t}\; dt$$

trig. identitet

$$= \int_0^{2\pi} a^3(1-\cos t)^2\sqrt{2-2\cos t}\; dt = \int_0^{2\pi} 2a^3(1-\cos t)^2\sqrt{2\sin^2\frac{t}{2}}\; dt$$

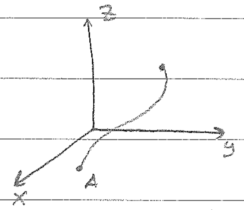
* za $t \in [0, 2\pi)$ $\sin\frac{t}{2} > 0$

$$= \int_0^{2\pi} 2a^3(1-\cos t)^2 \sin\frac{t}{2}\, dt = 2a^3\int_0^{2\pi}(1-\cos t)^2\sin\frac{t}{2}\, dt$$

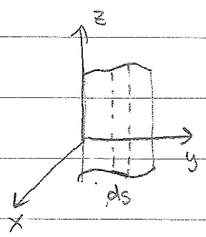
$$= 2a^3\int_0^{2\pi} 4\sin^5\frac{t}{2}\, dt = 8a^3\int_0^{2\pi}\sin^5\frac{t}{2} = \left|\begin{matrix} u=\cos\frac{t}{2} \\ du=-\frac{1}{2}\sin\frac{t}{2}\,dt\end{matrix}\right| \dots = \frac{256}{15}a^3$$

Primjena:

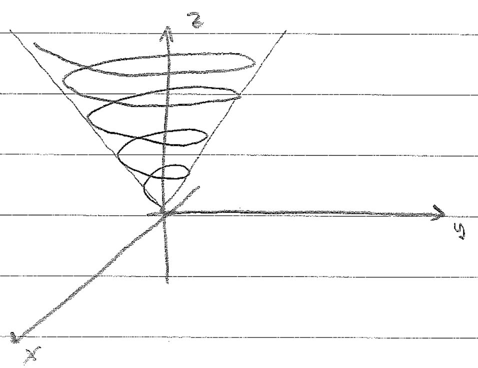
1) Duljina luka krivulje $l = \int_{\widehat{AB}} ds$



2) Računanje površine (oplošja) cilindričnih tijela $P = \int_{\widehat{AB}} z\, ds$



ZAD3. Izračunajte duljinu luka čunjaste zavojnice $x = a\cdot e^t\cos t = \varphi(t)$
od $A(0,0,0)$ do $B(a,0,a)$ $y = a\cdot e^t\cdot\sin t = \psi(t)$
$z = a\cdot e^t = \xi(t)$



$$ds = \sqrt{\varphi'(t)^2+\psi'(t)^2+\xi'(t)^2}\; dt$$
$$= \dots = a\cdot e^t\sqrt{3}\, dt$$

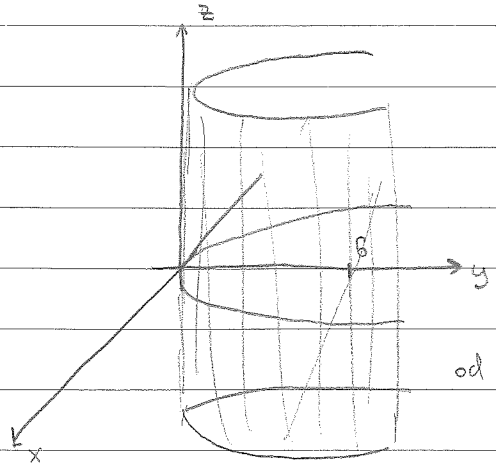
$z=0$ do $z=a$

$a\cdot e^t = 0$ do $a\cdot e^t = a$

$t=-\infty$ do $t=0$

$$l = \int_{-\infty}^{0} a\cdot e^t\sqrt{3}\, dt = \lim_{n\to-\infty}\int_n^0 a\sqrt{3}e^t dt = a\sqrt{3}\lim_{n\to-\infty}(e^0 - e^n) = a\sqrt{3} \;//$$

ZAD 4. Izračunajte površinu dijela paraboličkog cilindra $y = \frac{3}{8}x^2$
omeđenog ravninama $z=0$, $x=0$, $z=x$, $y=6$



$$ds = \sqrt{1 + \frac{9}{16}x^2}\, dx$$

trebaju nam granice za x

od $\underline{x=0}$ do $6 = \frac{3}{8}x^2 \Rightarrow x = 4$

$$P = \int_{\widehat{AB}} z\, ds = \int_0^4 x\sqrt{1+\frac{9}{16}x^2}\, dx = \left| \begin{array}{l} u = 1 + \frac{9}{16}x^2 \\ du = \frac{9}{8}x\,dx \end{array} \right| \Rightarrow \int_1^{10} \frac{8}{9}\sqrt{u}\, du$$

$$= \int_1^{10} \frac{8}{9} u^{\frac{1}{2}} du = \frac{8}{9} \left. \frac{u^{\frac{3}{2}}}{\frac{3}{2}} \right|_1^{10} = \frac{16}{27}\left(10\sqrt{10} - 1\right)$$

12. VJEŽBE 16.1.2025.

$\boxed{\text{ZAD. 1}}$ Izračunati krivuljne integrale 2. vrste

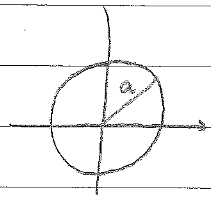
$I=\int_{\widehat{OA}} 2xy\,dx - x^2 dy$ od točke $O(0,0)$ do $A(2,1)$ po krivuljama:

a) $y=\frac{1}{2}x$
$dy=\frac{1}{2}dx$

$$I=\int_0^2\left(2x\frac{1}{2}x - x^2\frac{1}{2}\right)dx = \int_0^2\left(\frac{x^2}{2}\right)dx = \frac{1}{2}\frac{x^3}{3}\Big|_0^2 = \frac{4}{3}$$

b) DZ $\widehat{OA}\ldots\ y=\frac{1}{4}x^2$

$\boxed{\text{Zad 2.}}$ Izračunati $\oint \frac{(x+y)dx-(x-y)dy}{x^2+y^2}$ duž kružnice $x^2+y^2=a^2$ u pozitivnom smjeru.



parametrizacija kružnice: $x=a\cos t$, $t\in[0,2\pi]$
$y=a\sin t$

$dx=-a\sin t\,dt$
$dy=a\cos t\,dt$

$$I=\int_0^{2\pi}\frac{(a\cos t+a\sin t)(-a\sin t)-(a\cos t-a\sin t)(a\cos t)}{a^2}\,dt$$

$$=\frac{1}{a^2}\int_0^{2\pi}\left[-a^2\cancel{\sin t\cos t}-a^2\sin^2 t-a^2\cos^2 t+a^2\cancel{\sin t\cos t}\right]dt$$

$$=\frac{1}{a^2}\int_0^{2\pi}(-a^2)dt = -\int_0^{2\pi}dt = -2\pi$$

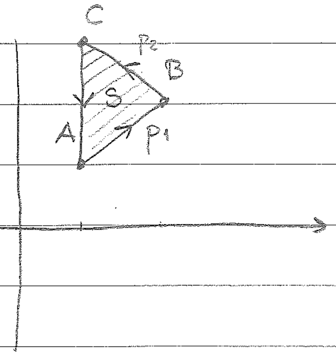
DZ Izračunajte $\int_C y^2dx+x^2dy$ gdje je C gornja polovina elipse
$\begin{cases} x=a\cos t \\ y=b\sin t \end{cases}$ u smjeru gibanja kazaljke na satu

Rj.: $\frac{4}{3}ab^2$

Greenova formula Ako je C granica područja S, $P(x,y)$, $Q(x,y)\in C^1$ u području $S\cup C$. Tada vrijedi:

$$\oint_C P\,dx+Q\,dy=\iint_S\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy$$

Zad. Izračunajte $\oint_C 2(x^2+y^2)dx + (x+y)^2 dy$, gdje je C kontura trokuta $A(1,1), B(2,2), C(1,3)$ prijeđena suprotno od smjera gibanja kazaljke na satu



$$\oint_C Pdx+Qdy = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy$$

$$= \int_1^2 dx \int_x^{-x+4} (2(x+y)-4y)dy$$

$$= \int_1^2 dx \int_x^{-x+4} (2x-2y)dy$$

$p_1 \ldots \quad y = x$

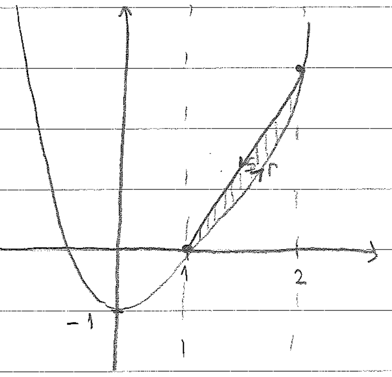
$p_2 \ldots \quad y = -x+4$

$$= \int_1^2 \left(2xy - 2\frac{y^2}{2}\right)\Big|_x^{-x+4} dx$$

$$= \int_1^2 (2x(-x+4) - (-x+4)^2 - 2x^2 + x^2)dx$$

$= \ldots = -\frac{4}{3}$

ZAD. $\oint_\Gamma (x+y)dx - (x-y)dy$, primjenom Greenove formule gdje je Γ krivulja koja se sastoji od parabole $y = x^2 - 1$, pravca $y = 3x-3$ prijeđena suprotno od kazaljke na satu



Sjecišta: $3x-3 = x^2-1$

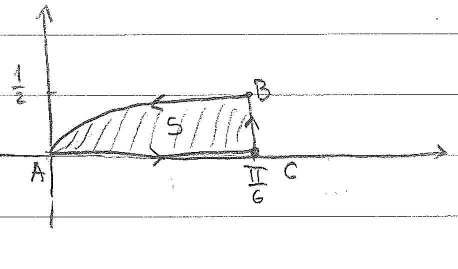
$\Rightarrow x_1 = 1 \quad x_2 = 2$

$$\oint_\Gamma (x+y)dx - (x-y)dy$$

$$= \iint_S (-1-1)\, dxdy =$$

$$= \int_1^2 dx \int_{x^2-1}^{3x-3} (-2)dy = -2\int_1^2 (3x-3-x^2+1)dx = -\frac{1}{3}$$

Zad. Primjenom Greenove formule izračunati $I = \int_{\widehat{AB}} (y+\sin x \sin y)dx - \cos x \cos y\, dy$, ako je $\widehat{AB}$ luk sinusoide $y = \sin x \quad 0 \le x \le \frac{\pi}{6}$



Da bismo mogli primijeniti Greenovu formulu moramo imati zatvorenu krivulju koja omeđuje neko jednostavno povezano područje S. Zato luk sinusoide $\overset{\frown}{AB}$ zatvorimo s $\overset{\frown}{AC}$ i $\overset{\frown}{CB}$. Rub područja Γ moramo obići u pozitivnom smjeru

$$\oint_{\Gamma} P dx + Q dy = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy$$

$$\int_{\overset{\frown}{AC}} P dx + Q dy + \int_{\overset{\frown}{CB}} P dx + Q dy + \underbrace{\int_{\overset{\frown}{BA}} P dx + Q dy}_{= -\int_{\overset{\frown}{AB}} P dx + Q dy} = \iint_S (\cancel{\sin x \cos y} - 1 - \cancel{\sin x \cos y}) dx dy = -\iint_S dx dy$$

$$\Rightarrow \int_{\overset{\frown}{AB}} P dx + Q dy = \iint_S dx dy + \int_{\overset{\frown}{AC}} P dx + Q dy + \int_{\overset{\frown}{CB}} P dx + Q dy$$

$y = 0$, x od 0 do $\frac{\pi}{6}$; $x = \frac{\pi}{6}$, y od 0 do $\frac{1}{2}$

$$= \cancel{\int_0^{\pi/6} (0 + \sin x \sin 0) dx} - \cos x \cos 0 \cdot 0 \cdot dx + \int_0^{1/2} \left((y + \sin \tfrac{\pi}{6} \sin y) \cdot 0 - \cos \tfrac{\pi}{6} \cos y \right) dy + \int_0^{\pi/6} dx \int_0^{\sin x} dy$$

$$= -\frac{\sqrt{3}}{2} \int_0^{1/2} \cos y \, dy + \int_0^{\pi/6} dx \int_0^{\sin x} dy$$

$$= -\frac{\sqrt{3}}{2} \sin y \Big|_0^{1/2} + \int_0^{\pi/6} \sin x \, dx = -\frac{\sqrt{3}}{2} \left(\sin \tfrac{1}{2} - \sin 0 \right) + \left(-\cos \tfrac{\pi}{6} + \cos 0 \right)$$

$$= 1 - \frac{\sqrt{3}}{2} \left(1 + \sin \tfrac{1}{2} \right)$$

Plošni integrali prve vrste $\iint_S f(x,y,z) \, dS = \iint_{S_{xy}} f(x,y,z(x,y)) \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} dx dy$

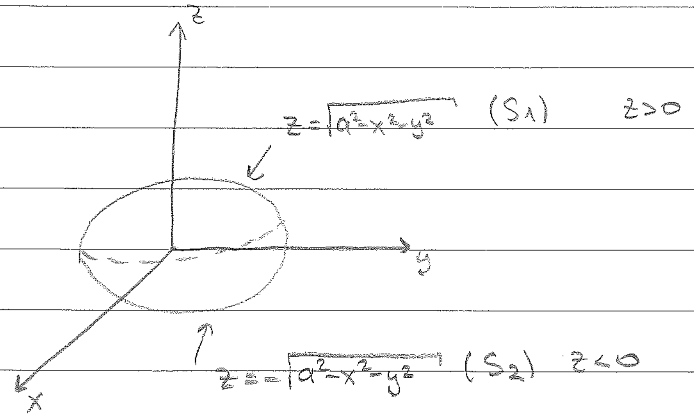
vrijednost integrala ne ovisi o izboru strane plohe S po kojoj se vrši integracija. Također možemo plohu S projicirati i na neku drugu koordinatnu ravninu, pa za dS vrijedi npr.:

$$dS = \sqrt{1 + \left(\frac{\partial x}{\partial y}\right)^2 + \left(\frac{\partial x}{\partial z}\right)^2} dy dz \qquad dS = \sqrt{1 + \left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2} dx dz$$

Primjene: površina plohe $\pi = \iint_S dS$

$\boxed{\text{ZAD.}}$ Izračunati $\iint_S (x^2 + y^2) dS$ gdje je S sfera $x^2 + y^2 + z^2 = a^2$

Rj.



$$\iint_S (x^2+y^2)\,dS = \iint_{S_1} (x^2+y^2)\,dS_1 + \iint_{S_2} (x^2+y^2)\,dS_2$$

$$dS_1 = \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2} = \frac{a}{\sqrt{a^2-x^2-y^2}}\,dx\,dy$$

Projekcija Ω od S_1 na xOy je $\Omega \ldots\ x^2+y^2=a^2$

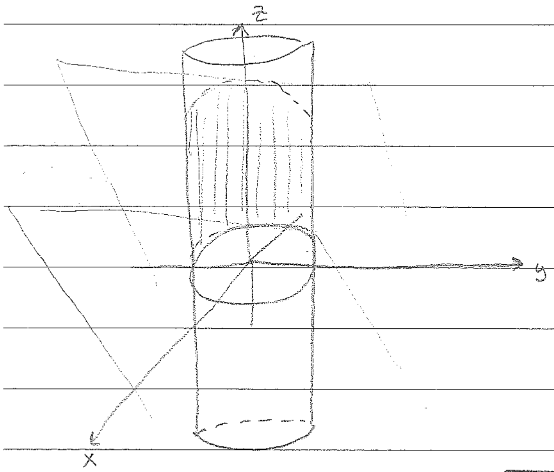
$$\iint_{S_1} (x^2+y^2)\,dS_1 = \iint_{\Omega_{xy}} (x^2+y^2)\frac{a}{\sqrt{a^2-x^2-y^2}}\,dx\,dy = \left|\begin{matrix} x = r\cos\varphi \\ y = r\sin\varphi \\ dx\,dy = r\,dr\,d\varphi \end{matrix}\right|$$

$$= \int_0^{2\pi} d\varphi \int_0^a r^3 \frac{a}{\sqrt{a^2-r^2}}\,dr = a\int_0^{2\pi} d\varphi \int_0^a \frac{r^3}{\sqrt{a^2-r^2}}\,dr = \left|\begin{matrix} a^2-r^2=u \\ -2r\,dr = du \end{matrix}\right|$$

$$= \ldots = \frac{4}{3}a^4\pi \quad \Rightarrow \iint_S (x^2+y^2)\,dS = \frac{8}{3}a^4\pi$$

23.1.2025.

$\boxed{\text{ZAD}}$ Izračunajte površinu dijela plohe valjka $x^2+y^2=R^2$, $z\geq 0$ koji se nalazi među ravninama $z=\alpha x$, $z=\beta x$, $\alpha>\beta>0$



$P=\iint_S dS$ projiciramo na xOz ravninu

dS računamo pomoću $x^2+y^2=R^2$

$y^2=R^2-x^2$

$$P=\iint_{S_1(y>0)} dS+\iint_{S_2(y<0)} dS$$

$S_1:\ y=\sqrt{R^2-x^2}$

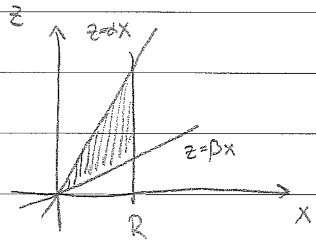
$S_2:\ y=-\sqrt{R^2-x^2}$

$$dS=\sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2+\left(\frac{\partial y}{\partial z}\right)^2}\,dx\,dz=\sqrt{1+\frac{x^2}{R^2-x^2}}\,dx\,dz=\frac{R}{\sqrt{R^2-x^2}}\,dx\,dz$$

$$\frac{\partial y}{\partial x}=\frac{-2x}{2\sqrt{R^2-x^2}}=-\frac{x}{\sqrt{R^2-x^2}}$$

$P=2\iint_{S_1} dS_1=$

za $y=0 \Rightarrow x^2+y^2=R^2 \Rightarrow x=\pm R$



$$P=2\iint_{S_1} dS_1=2\iint_{\Omega_{xy}}\frac{R}{\sqrt{R^2-x^2}}\,dx\,dz$$

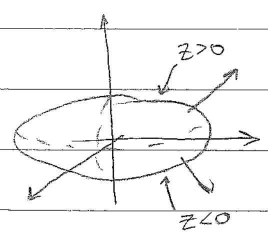
$$=2\int_0^r dx\int_{\beta x}^{\alpha x}\frac{R}{\sqrt{R^2-x^2}}\,dz=2\int_0^r\frac{R}{\sqrt{R^2-x^2}}(\alpha-\beta)x\,dx$$

$$=2R(\alpha-\beta)\int_0^r\frac{x}{\sqrt{R^2-x^2}}\,dx=\dots=2R^2(\alpha-\beta)$$

Plošni integrali druge vrste

$\boxed{\text{Zad.}}$ Izračunati $\iint_{S^+} z\,dx\,dy$ gdje je S^+ vanjska strana elipsoida $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

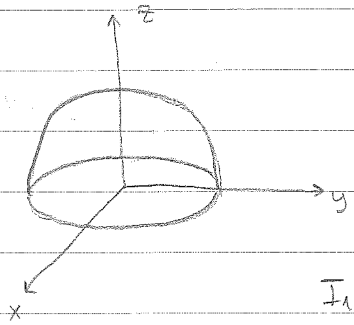
$$I=\iint_{S^+} z\,dx\,dy=\iint_{S^+_{z>0}} z\,dx\,dy+\iint_{S^+_{z<0}} z\,dx\,dy$$



$$=\iint_{\Omega_{xy}} c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\,dx\,dy-\iint_{\Omega_{xy}}\left(-c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\right)dx\,dy$$

$$I = \begin{vmatrix} x = ar\cos\theta \\ y = br\sin\theta \end{vmatrix} = 2c\int_0^{2\pi} d\theta \int abr\sqrt{1-r^2}\,dr = \dots = \frac{4}{3}abc\pi$$

(ZAD.) Izračunati $I = \iint_{S^+} x^2\,dydz + y^2\,dxdz + z^2\,dxdy$ gdje je S^+ vanjska strana polusfere $x^2+y^2+z^2=a^2,\ z>0$



$$I = \underbrace{\iint_{S^+} x^2\,dydz}_{I_1} + \underbrace{\iint_{S^+} y^2\,dxdz}_{I_2} + \underbrace{\iint_{S^+} z^2\,dxdy}_{I_3}$$

$$I_1 = \iint_{S^+_{x>0}} x^2\,dydz + \iint_{S^+_{x<0}} x^2\,dydz$$

$$= \iint_{\Omega_{yz}} (a^2-y^2-z^2)\,dydz - \iint_{\Omega_{yz}} (a^2-y^2-z^2)\,dydz = 0$$

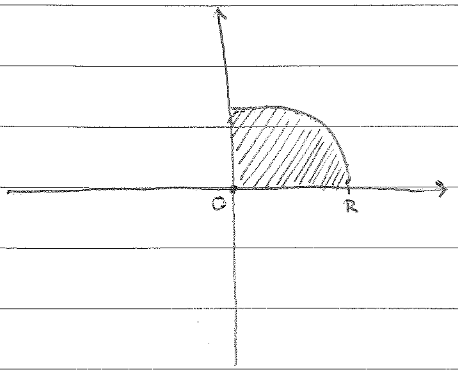
$$I_2 = \iint_{S^+} y^2\,dxdz = \iint_{S^+_{y>0}} y^2\,dxdz + \iint_{S^+_{y<0}} y^2\,dxdz = \iint_{\Omega_{xz}} (a^2-x^2-z^2)\,dxdz - \iint_{\Omega_{xz}} (a^2-x^2-z^2)\,dxdz = 0$$

$$I_3 = \iint_{S^+} z^2\,dxdy = \iint_{\Omega_{xy}} (a^2-x^2-y^2)\,dxdy = \begin{vmatrix} x = r\cos\theta \\ y = r\sin\theta \end{vmatrix}$$

$$= \int_0^{2\pi} d\theta \int_0^a r(a^2-r^2)\,dr = \begin{vmatrix} t = a^2-r^2 \\ dt = -2r\,dr \end{vmatrix} = -\frac{1}{2}\int_0^{2\pi} \frac{a^2}{2}\,d\theta = -\frac{2\pi a^2}{4} \quad \dots \quad \frac{a^4\pi}{2}$$

treba dobiti ?

(1) $\int_0^R dx \int_0^{\sqrt{R^2-x^2}} \ln(1+x^2+y^2)\,dy = \left| \begin{matrix} x = r\cos\theta & dx\,dy = r\,dr\,d\theta \\ y = r\sin\theta & \end{matrix} \right|$



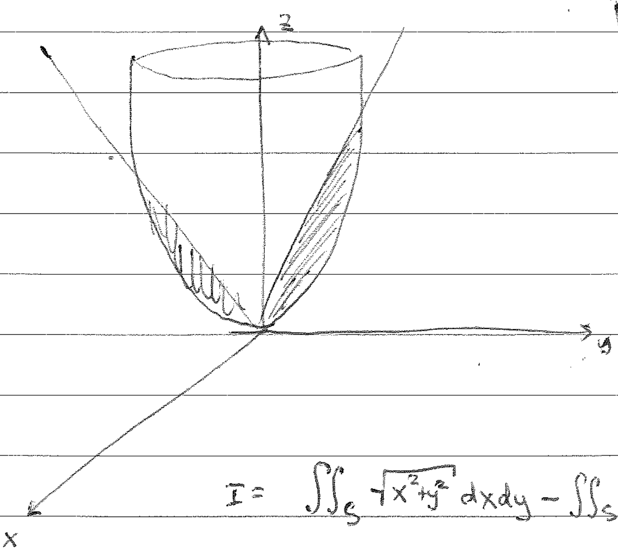
$= \int_0^{\pi/2} d\theta \int_0^R r\ln(1+r^2)\,dr = \left| \begin{matrix} t = 1+r^2 \\ dt = 2r\,dr \end{matrix} \right|$

$= \frac{1}{2}\int_0^{\pi/2} d\theta \int_1^{1+R^2} \ln t\,dt = \frac{1}{2}\int_0^{\pi/2} (t\ln t - t)\Big|_1^{1+R^2} d\theta$

$= \frac{1}{2}\int_0^{\pi/2} (1+R^2)\ln(1+R^2) - (1+R^2) + 1\,d\theta$

$= \frac{1}{2}\cdot\frac{\pi}{2}(1+R^2)\ln(1+R^2) - \frac{1}{2}\frac{\pi}{2}(1+R^2) + \frac{1}{2}\cdot\frac{\pi}{2}$

(2.) $z = \frac{x^2+y^2}{2}$, $z = \sqrt{x^2+y^2}$



Presjek tih dvaju tijela:

$\frac{x^2+y^2}{2} = \sqrt{x^2+y^2} \quad /^2$

$(x^2+y^2)^2 = 4(x^2+y^2) \quad /: x^2+y^2 \neq 0$

$x^2+y^2 = 4$ ← područje integracije

$x = r\cos\varphi$
$y = r\sin\varphi$
$dx\,dy = r\,dr\,d\varphi$

$I = \iint_S \sqrt{x^2+y^2}\,dx\,dy - \iint_S \frac{x^2+y^2}{2}\,dx\,dy$

$I = \int_0^{2\pi} d\varphi \int_0^2 r^2\,dr - \int_0^{2\pi} d\varphi \int_0^2 \frac{r^3}{2}\,dr = \int_0^{2\pi} \frac{8}{3}\,d\varphi - \int_0^{2\pi} 2\,d\varphi = 2\pi\cdot\frac{8}{3} - 2\pi\cdot 2 = \frac{4\pi}{3}$ //

(1) $x^2+y^2 = 4x$

$x^2 - 4x + y^2 = 0$

$(x-2)^2 + y^2 = 4$

$x^2+y^2 = 6x$

$x^2 - 6x + y^2 = 0$

$(x-3)^2 + y^2 = 9$

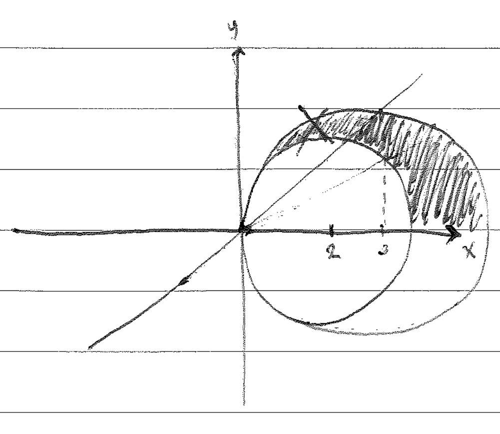
$(x-3)^2 + x^2 = 9$

$x^2 - 6x + 9 + x^2 = 9$

$2x^2 = 6x$

$x^2 = 3x$

$x = 3$



$\cancel{\int_2^3 dx \int}$ $\cancel{x = r\cos\varphi}$ $\cancel{y = r\sin\varphi}$

$\theta \in [\frac{\pi}{6}, \frac{\pi}{4}]$

$r \in [2, \frac{3}{\cos\varphi}]$

$x = 3$

$r\cos\varphi = 3$

$$\int_{\pi/6}^{\pi/4} d\varphi \int_2^{\frac{3}{\cos\varphi}} r\,dr = \int_{\pi/6}^{\pi/4} \left(\frac{9}{2\cos^2\varphi} - 2 \right) d\varphi$$

$$= \int_{\pi/6}^{\pi/4} \frac{9}{2\cos^2\varphi}\, d\varphi - \int_{\pi/6}^{\pi/4} 2\,d\varphi = \frac{9 - 3\sqrt{3}}{2} - \frac{\pi}{6}$$