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Graph spaces of first-order linear partial differential operators

Abstract

Symmetric positive systems of first-order linear partial differential equations were introduced by K.O. Friedrichs (1958) in order to treat the equations that change their type, like the equations modelling the transonic fluid flow.

Recently, some progress in their understanding has been made by rewriting them in terms of Hilbert spaces, characterising the admisible boundary conditions by intrinsic geometric conditions in the graph spaces. In this paper we streamline the available proofs of the properties of graph spaces (most completely presented by M. Jensen (2004)), providing some additional results in the process; thus paving the way for further study of Friedrichs' systems.

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1. Introduction

Let $\Omega \subseteq \mathbf{R}^d$ be a nonempty open set, $p \in [1,\infty]$, $\mathbf{A}^k \in \mathrm{W}^{1,\infty}(\Omega;\mathrm{M}_{l,r})$, $k \in 1..d$, and $\mathbf{C} \in \mathrm{L}^\infty(\Omega;\mathrm{M}_{l,r})$ matrix functions. We consider a linear differential operator of the form

$$\mathcal{L}\mathsf{u} := \sum_{k=1}^{d} \partial_k(\mathbf{A}^k \mathsf{u}) + \mathbf{C}\mathsf{u}\,. \tag{1}$$

Such an operator $\mathcal{L}$ is called *symmetric positive* if $\mathbf{A}^k$ are symmetric (or hermitian in the complex case; i.e. $(\mathbf{A}^k)^* = \mathbf{A}^k$) and $\mathbf{C} + \mathbf{C}^* + \sum \partial_k \mathbf{A}^k$ is uniformly positive.

The class of symmetric positive systems of first-order partial differential equations was introduced by Kurt Otto Friedrichs [F] in an attempt to unify the treatment of equations of various types (elliptic, parabolic or hyperbolic). His immediate goal was the study of practical problems, like transonic flow, where the mathematical model describes the subsonic flow by an equation of elliptic type, while the equation is hyperbolic for the supersonic flow.

Of course, the inclusion of equations with so diverse character into single framework makes the study of suitable boundary (or initial) conditions particularly challenging. Specifically, the familiar Sobolev spaces are not well suited for this type of problems: the type of data usually prescribed as boundary values for such equations does not avail us of classical solutions — we can only hope for weak solutions to exist, which do not belong to Sobolev spaces, but only to the graph space of the corresponding operator $\mathcal{L}$.

Most of the function space theory, as we know it today, was developed after Friedrichs wrote his paper, and from the contemporary viewpoint we know that the traces can be taken of functions in graph spaces. However, the first systematic attempt to study the properties of such graph spaces (to the best of our knowledge) was undertaken quite recently by Max Jensen in his Ph.D. thesis [J] (which was written under the supervision of Endre Süli, see also [HMSW]).

In this paper we make a step further in this direction, by extending Jensen's results, and at the same time simplifying a number of his proofs. Our goal is to avail ourselves of technical results that we need in order to extend some recent results on Friedrichs' systems from [EGC] (see also [AB1, B]). Our original motivation has been to extend the research started in [A] to the equations which change their type; see also [AL].

The paper is organised as follows. Graph spaces are defined in the second section; in the third we extend Jensen's density results to $p = 1$, providing some simpler proofs for $p \in \langle 1,\infty\rangle$ as well. The extension of functions to the whole space is treated in the next section, while in the fifth section we briefly describe the duals of graph spaces, omitting the proofs which are quite similar to those in the case of Sobolev spaces. In the final section we introduce the trace operator on the graph space, restricting ourselves to the case $p = 2$, which is sufficient for our future application on Friedrichs' systems and, on the other hand, does not require the introduction of Besov spaces. We also provide some alternative proofs which are easier and more natural in the Hilbert space setting. In general, the trace operator turns out not to be surjective on $\mathrm{H}^{-\frac{1}{2}}$, but we can define its right inverse and identify the kernel.

Notation. For $p \in [1,\infty]$ we define p' by $\frac{1}{p} + \frac{1}{p'} = 1$, while by $\mathrm{M}_{l,r} := \mathrm{M}_{l,r}(\mathbf{C})$ we denote the space of $l \times r$ complex matrices. The complex scalar product we denote by $\cdot$; we take it to be antilinear in its second factor.

For an open set $\Omega \subseteq \mathbf{R}^d$, by $\mathcal{K}(\Omega)$ we denote the family of all compact sets in Ω, and for $K \in \mathcal{K}(\Omega)$, the space $\mathrm{C}^\infty_K(\Omega)$ stands for all smooth functions on Ω with support in K, while $\mathrm{L}^p_K(\Omega)$ stands for the space of functions from $\mathrm{L}^p(\Omega)$ with (essential) support in K.

The topological closure we denote by Cl, the interior by Int and the boundary by Fr. For $N, O \subseteq \mathbf{R}^d$, we say that N is *compactly contained* in O (and denote it by $N \Subset O$) if $\mathsf{Cl}\,N \subseteq \mathsf{Int}\,O$ and N is bounded.

A sequence of functions $\rho_n : \mathbf{R}^d \longrightarrow \mathbf{R}$ is called a *mollifying sequence* if $\rho_n(\mathbf{x}) = n^d\rho(n\mathbf{x})$, for some nonnegative function $\rho \in \mathrm{C}^\infty_c(\mathbf{R}^d;\mathbf{R})$ supported in the unit ball, with $\int \rho = 1$.

2. Definition and basic properties of graph spaces

For given matrix functions $\mathbf{A}^k \in \mathrm{W}^{1,\infty}(\Omega; \mathrm{M}_{l,r})$, $k \in 1..d$, and $\mathbf{C} \in \mathrm{L}^\infty(\Omega; \mathrm{M}_{l,r})$, we define the operator $\mathcal{L} : \mathrm{L}^p(\Omega; \mathbf{C}^r) \longrightarrow \mathcal{D}'(\Omega; \mathbf{C}^l)$ by (1), and the operator $\tilde{\mathcal{L}} : \mathrm{L}^{p'}(\Omega; \mathbf{C}^l) \longrightarrow \mathcal{D}'(\Omega; \mathbf{C}^r)$ (which is the formal adjoint of operator $\mathcal{L}$) by

$$\tilde{\mathcal{L}}\mathsf{v} := -\sum_{k=1}^{d} \partial_k((\mathbf{A}^k)^*\mathsf{v}) + \Big(\mathbf{C}^* + \sum_{k=1}^{d} \partial_k(\mathbf{A}^k)^*\Big)\mathsf{v} .$$

It is clear that $\mathcal{L}$ and $\tilde{\mathcal{L}}$ are linear operators. Note that, following Jensen [J], we are writing the operators in the divergence form, which was not the case in Friedrichs' original paper [F].

Lemma 1. *The operators $\mathcal{L}$ and $\tilde{\mathcal{L}}$ are continuous, with respect to strong topologies on the spaces L^p and $\mathrm{L}^{p'}$, and weak $*$ on the space of distributions (the codomain).*

Dem. We prove the statement for the operator $\mathcal{L}$ (as for $\tilde{\mathcal{L}}$ the proof is similar): since the derivative $\partial_k : \mathcal{D}'(\Omega; \mathbf{C}^l) \longrightarrow \mathcal{D}'(\Omega; \mathbf{C}^l)$ is continuous, it is sufficient to show that the operator of multiplication by an arbitrary $\mathbf{B} \in \mathrm{L}^\infty(\Omega; \mathrm{M}_{l,r})$, i.e. $\mathsf{u} \mapsto \mathbf{B}\mathsf{u}$, is continuous from $\mathrm{L}^p(\Omega; \mathbf{C}^r)$ to $\mathcal{D}'(\Omega; \mathbf{C}^l)$, which is an immediate consequence of the continuity of mapping $\mathsf{u} \mapsto \mathbf{B}\mathsf{u}$ from $\mathrm{L}^p(\Omega; \mathbf{C}^r)$ to $\mathrm{L}^p(\Omega; \mathbf{C}^l)$, and the continuity of embedding $\mathrm{L}^p(\Omega; \mathbf{C}^l) \hookrightarrow \mathcal{D}'(\Omega; \mathbf{C}^l)$.

Q.E.D.

Remark. The statement of the above lemma holds even if we consider L^p with the weak (for $p \in [1, \infty\rangle$), or the weak $*$ (for $p = \infty$) topology. Indeed, $\mathrm{L}^p(\Omega; \mathbf{C}^l)$ with the weak topology (we consider the case $p < \infty$) is still continuously embedded in the space of distributions, and the mapping $\mathsf{u} \mapsto \mathbf{B}\mathsf{u}$ is continuous from $\mathrm{L}^p(\Omega; \mathbf{C}^r)$ to $\mathrm{L}^p(\Omega; \mathbf{C}^l)$ even if we consider both spaces equipped with weak topologies [Br, pg. 39]. By a similar argument the statement can be proved also for $p = \infty$.

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The vector space

$$\mathrm{W}^{\mathcal{L},p}(\Omega; \mathbf{C}^r) := \{\mathsf{u} \in \mathrm{L}^p(\Omega; \mathbf{C}^r) : \mathcal{L}\mathsf{u} \in \mathrm{L}^p(\Omega; \mathbf{C}^l)\}$$

is normed, with the norm (the properties of the norm can easily be verified) defined by

$$\|\mathsf{u}\|_{\mathcal{L},p} = \|\mathsf{u}\|_{\mathcal{L},p,\Omega} := \begin{cases} \Big(\|\mathsf{u}\|^p_{\mathrm{L}^p(\Omega;\mathbf{C}^r)} + \|\mathcal{L}\mathsf{u}\|^p_{\mathrm{L}^p(\Omega;\mathbf{C}^l)}\Big)^{\frac{1}{p}}, & p \in [1, \infty\rangle \\ \max\Big\{\|\mathsf{u}\|_{\mathrm{L}^\infty(\Omega;\mathbf{C}^r)}, \|\mathcal{L}\mathsf{u}\|_{\mathrm{L}^\infty(\Omega;\mathbf{C}^l)}\Big\}, & p = \infty . \end{cases}$$

It is immediate that the space $\mathrm{W}^{\mathcal{L},p}(\Omega; \mathbf{C}^r)$ is isometrically isomorphic to the graph space of restriction of the operator $\mathcal{L}$ to $\mathrm{W}^{\mathcal{L},p}(\Omega; \mathbf{C}^r)$:

$$\Gamma_{\mathcal{L}} := \{(\mathsf{u}, \mathsf{v}) \in \mathrm{L}^p(\Omega; \mathbf{C}^r) \times \mathrm{L}^p(\Omega; \mathbf{C}^l) : \mathsf{v} = \mathcal{L}\mathsf{u}\} ,$$

with the norm induced from $\mathrm{L}^p(\Omega; \mathbf{C}^r) \times \mathrm{L}^p(\Omega; \mathbf{C}^l)$, which clarifies the use of the term *graph space* for $(\mathrm{W}^{\mathcal{L},p}(\Omega; \mathbf{C}^r), \|\cdot\|_{\mathcal{L},p})$ and *graph norm* for $\|\cdot\|_{\mathcal{L},p}$.

Following the standard notation for Sobolev and other function spaces, the subspace of $\mathrm{W}^{\mathcal{L},p}(\Omega; \mathbf{C}^r)$ containing all functions with compact support we shall denote by $\mathrm{W}^{\mathcal{L},p}_c(\Omega; \mathbf{C}^r)$.

In particular, for $p = 2$ we denote $\mathrm{H}^{\mathcal{L}}(\Omega; \mathbf{C}^r) = \mathrm{W}^{\mathcal{L},2}(\Omega; \mathbf{C}^r)$, and $\|\cdot\|_{\mathcal{L}} = \|\cdot\|_{\mathcal{L},\Omega} = \|\cdot\|_{\mathcal{L},2,\Omega}$. In this case the graph norm is induced by the following inner product:

$$\langle\, \mathsf{u} \mid \mathsf{v} \,\rangle_{\mathcal{L}} = \langle\, \mathsf{u} \mid \mathsf{v} \,\rangle_{\mathcal{L},\Omega} := \langle\, \mathsf{u} \mid \mathsf{v} \,\rangle_{\mathrm{L}^2(\Omega;\mathbf{C}^r)} + \langle\, \mathcal{L}\mathsf{u} \mid \mathcal{L}\mathsf{v} \,\rangle_{\mathrm{L}^2(\Omega;\mathbf{C}^l)} .$$

Theorem 1. *The graph space $(\mathrm{W}^{\mathcal{L},p}(\Omega; \mathbf{C}^r), \|\cdot\|_{\mathcal{L},p})$ is a Banach space. For $p \in [1, \infty\rangle$ it is separable and for $p \in \langle 1, \infty\rangle$ reflexive and uniformly convex.*

Dem. By the isomorphism, it is enough to prove the above statements for space $\Gamma_{\mathcal{L}}$. First, let us show that $\Gamma_{\mathcal{L}}$ is closed in $\mathrm{L}^p(\Omega;\mathbf{C}^r)\times\mathrm{L}^p(\Omega;\mathbf{C}^l)$. Indeed, if we take a sequence $(\mathsf{u}_n,\mathcal{L}\mathsf{u}_n)$ from $\Gamma_{\mathcal{L}}$ that converges to $(\mathsf{v},\mathsf{w})\in\mathrm{L}^p(\Omega;\mathbf{C}^r)\times\mathrm{L}^p(\Omega;\mathbf{C}^l)$, then by the continuity of operator $\mathcal{L}$ (Lemma 1) we have $\mathcal{L}\mathsf{v}=\mathsf{w}$, which proves the first statement.

Since the space $\mathrm{L}^p(\Omega;\mathbf{C}^r)\times\mathrm{L}^p(\Omega;\mathbf{C}^l)$ is separable (for $p\in[1,\infty\rangle$), and the separability is a hereditary property, so the same holds for $\Gamma_{\mathcal{L}}$.

For $p\in\langle 1,\infty\rangle$ the space $\mathrm{L}^p(\Omega;\mathbf{C}^r)\times\mathrm{L}^p(\Omega;\mathbf{C}^l)$ is uniformly convex, and therefore $\Gamma_{\mathcal{L}}$ is uniformly convex as well. Finally, by the Milman-Pettis theorem each uniformly convex Banach space is reflexive [Br, pg. 51], which concludes the proof.

Q.E.D.

For $p\in\langle 1,\infty\rangle$ the above theorem was proved in [J, pg. 20 and pg. 32] by the use of density result (Theorem 3 below).

Examples. If $\mathcal{L}$ is one of the operators ∇, div, or rot, then the corresponding graph spaces are $\mathrm{W}^{1,p}(\Omega)$, $\mathrm{L}^p_{\mathrm{div}}(\Omega)=\{\mathsf{u}\in\mathrm{L}^p(\Omega;\mathbf{C}^d):\mathsf{div}\,\mathsf{u}\in\mathrm{L}^p(\Omega)\}$, or $\mathrm{L}^p_{\mathrm{rot}}(\Omega)=\{\mathsf{u}\in\mathrm{L}^p(\Omega;\mathbf{C}^3):\mathsf{rot}\,\mathsf{u}\in\mathrm{L}^p(\Omega;\mathbf{C}^3)\}$ respectively, which are well known spaces in the theory of partial differential equations.

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3. Density of smooth functions

Graph spaces share many properties with the *classical* Sobolev spaces, but often such properties are more difficult to prove. One of these properties is the density of smooth functions, and the complexity of the proof for graph spaces arises from the loss of strong convergence in the L^p space. Instead, we have to use the Banach-Steinhaus theorem to get the weak convergence in L^p (for $p=1$ we also use the Dunford—Petis—de la Vallée-Poussin theorem), and then achieve the strong convergence by an application of Mazur's theorem.

Theorem 2. *Let $p\in[1,\infty\rangle$ and $\mathsf{v}\in\mathrm{W}^{\mathcal{L},p}_c(\Omega;\mathbf{C}^r)$. Then v can be approximated in the norm by a $\mathrm{C}^\infty_c(\Omega;\mathbf{C}^r)$ function; i.e.*

$$(\forall\,\varepsilon>0)(\exists\,\mathsf{v}_\varepsilon\in\mathrm{C}^\infty_c(\Omega;\mathbf{C}^r))\quad\|\mathsf{v}-\mathsf{v}_\varepsilon\|_{\mathcal{L},p}<\varepsilon\,.$$

Dem. Let us denote by (ρ_n) a mollifying sequence and choose n large enough (such that the distance $d(\mathsf{Fr}\,\Omega,\operatorname{supp}\mathsf{v})>\frac{2}{n}$). Then the convolution

$$(\rho_n*\mathsf{v})(\mathbf{x})=\int_\Omega\rho_n(\mathbf{x}-\mathbf{y})\mathsf{v}(\mathbf{y})d\mathbf{y}\,,\quad \mathbf{x}\in\mathbf{R}^d$$

is well defined and $\rho_n*\mathsf{v}\in\mathrm{C}^\infty_K(\Omega;\mathbf{C}^r)$, for some $K\in\mathcal{K}(\Omega)$. Moreover, $\rho_n*\mathsf{v}\longrightarrow\mathsf{v}$ in $\mathrm{L}^p(\Omega;\mathbf{C}^r)$ [LL, pg. 58]). By Lemma 1 we have

$$\mathcal{L}(\rho_n*\mathsf{v})\rightharpoonup\mathcal{L}\mathsf{v}\quad\text{in}\quad\mathcal{D}'(\Omega;\mathbf{C}^l)\,.\tag{2}$$

The Leibniz formula implies

$$\mathcal{L}(\rho_n*\mathsf{v})=\sum_{k=1}^d\Big[(\partial_k\mathbf{A}^k)(\rho_n*\mathsf{v})+\mathbf{A}^k\partial_k(\rho_n*\mathsf{v})\Big]+\mathbf{C}(\rho_n*\mathsf{v})\,,$$

and since $\rho_n*\mathsf{v}$, $\partial_k(\rho_n*\mathsf{v})\in\mathrm{C}^\infty_K(\Omega;\mathbf{C}^r)$ and $\partial_k\mathbf{A}^k,\mathbf{A}^k,\mathbf{C}\in\mathrm{L}^\infty(\Omega;\mathrm{M}_{l,r})$ (for $k\in 1..d$), it follows that the sequence $(\mathcal{L}(\rho_n*\mathsf{v}))$ is contained in $\mathrm{L}^\infty_K(\Omega;\mathbf{C}^l)$. Now from (2) it follows

$$(\forall\,\boldsymbol{\varphi}\in\mathrm{C}^\infty_c(\Omega;\mathbf{C}^l))\qquad\int_\Omega\mathcal{L}(\rho_n*\mathsf{v})\cdot\boldsymbol{\varphi}\longrightarrow\int_\Omega\mathcal{L}\mathsf{v}\cdot\boldsymbol{\varphi}\,,$$

and since $\mathrm{C}_c^\infty(\Omega; \mathbf{C}^l)$ is dense in $\mathrm{L}^1(\Omega; \mathbf{C}^l)$, the Banach-Steinhaus theorem implies that the sequence $(\mathcal{L}(\rho_n * \mathsf{v}))$ is bounded in $\mathrm{L}_K^\infty(\Omega; \mathbf{C}^l)$, and therefore also in $\mathrm{L}^p(\Omega; \mathbf{C}^l)$. This is enough to conclude that the sequence $(\mathcal{L}(\rho_n * \mathsf{v}))$ is sequentially precompact in $\mathrm{L}^p(\Omega; \mathbf{C}^l)$, if $p \in \langle 1, \infty\rangle$, while for $p = 1$ we use the Dunford–Petis–de la Vallée-Poussin theorem [Br, pg. 76] to conclude the same (since the sequence $(\mathcal{L}(\rho_n * \mathsf{v}))$ is bounded in $\mathrm{L}_K^\infty(\Omega; \mathbf{C}^l)$, it is also uniformly absolutely continuous). Combining this and (2) we easily conclude that

$$(3) \qquad \mathcal{L}(\rho_n * \mathsf{v}) \rightharpoonup \mathcal{L}\mathsf{v} \quad \text{in} \quad \mathrm{L}^p(\Omega; \mathbf{C}^l)\,.$$

Now fix $\varepsilon > 0$ and choose a subsequence of the mollifying sequence (ρ_n) (which, for simplicity, we denote the same), such that

$$(4) \qquad (\forall n \in \mathbf{N}) \qquad \|\rho_n * \mathsf{v} - \mathsf{v}\|_{\mathrm{L}^p(\Omega;\mathbf{C}^r)} < \frac{\varepsilon}{2} \cdot \frac{1}{2^n}\,.$$

From (3) and the Mazur theorem [Ru, pg. 67] it follows that there are numbers $s \in \mathbf{N}$ and $\lambda_1, \lambda_2, \ldots, \lambda_s \in [0, 1]$, with the property that $\sum_{i=1}^s \lambda_i = 1$, as well as numbers $n(i) \in \mathbf{N}$, $i \in 1..s$, such that

$$(5) \qquad \Big\|\sum_{i=1}^s \lambda_i \mathcal{L}(\rho_{n(i)} * \mathsf{v}) - \mathcal{L}\mathsf{v}\Big\|_{\mathrm{L}^p(\Omega;\mathbf{C}^l)} < \frac{\varepsilon}{2}\,.$$

If we denote $\mathsf{v}_\varepsilon := \sum_{i=1}^s \lambda_i \rho_{n(i)} * \mathsf{v} \in \mathrm{C}_c^\infty(\Omega; \mathbf{C}^r)$, we have

$$\mathcal{L}\mathsf{v}_\varepsilon = \sum_{i=1}^s \lambda_i \mathcal{L}(\rho_{n(i)} * \mathsf{v})\,,$$

and the formula (5) reads

$$\Big\|\mathcal{L}\mathsf{v}_\varepsilon - \mathcal{L}\mathsf{v}\Big\|_{\mathrm{L}^p(\Omega;\mathbf{C}^l)} < \frac{\varepsilon}{2}\,.$$

Since by (4) we have

$$\begin{aligned} \|\mathsf{v}_\varepsilon - \mathsf{v}\|_{\mathrm{L}^p(\Omega;\mathbf{C}^r)} &= \Big\|\sum_{i=1}^s \lambda_i(\rho_{n(i)} * \mathsf{v} - \mathsf{v})\Big\|_{\mathrm{L}^p(\Omega;\mathbf{C}^r)} \\ &\leqslant \sum_{i=1}^s \lambda_i \|(\rho_{n(i)} * \mathsf{v} - \mathsf{v})\|_{\mathrm{L}^p(\Omega;\mathbf{C}^r)} < \frac{\varepsilon}{2} \sum_{i=1}^s \frac{1}{2^{n(i)}} < \frac{\varepsilon}{2} \sum_{i=1}^\infty \frac{1}{2^i} = \frac{\varepsilon}{2}\,, \end{aligned}$$

from the definition of norm on $\mathrm{W}^{\mathcal{L},p}(\Omega; \mathbf{C}^r)$ we finally get

$$\|\mathsf{v}_\varepsilon - \mathsf{v}\|_{\mathcal{L},p} < \varepsilon\Big(\frac{1}{2}\Big)^{\frac{p-1}{p}} \leqslant \varepsilon\,.$$

Q.E.D.

For $p \in \langle 1, \infty\rangle$ the above theorem was proved in [J, pg. 15]. However, the proof of the weak convergence $\mathcal{L}(\rho_n * \mathsf{v}) \rightharpoonup \mathcal{L}\mathsf{v}$ there is much more complicated and cannot be extended to cover the case $p = 1$.

Remark. Note that the function v_ε in the previous theorem can be chosen to be of the form $\mathsf{v}_\varepsilon = \varphi * \mathsf{v}$, where $\varphi \in \mathrm{C}_c^\infty(\mathrm{K}(0, \delta))$ and $\delta > 0$ is arbitrary small. Indeed, for given δ we get such v_ε by choosing φ to be a convex combination as in the proof of the previous theorem:

$$\varphi := \sum_{i=1}^s \lambda_i \rho_{n(i)}\,,$$

with previously taken subsequence of the mollifying sequence, whose each member has the support contained in the open ball $\mathrm{K}(0, \delta)$.

∎

The proof of the following theorem is analogous to the corresponding proof for the Sobolev spaces [AF, pg. 67] and (for $p \in \langle 1, \infty \rangle$) it can be found in [J, pg. 19], which is the reason for omitting it here.

Theorem 3. *For $p \in [1, \infty\rangle$ the space $\mathrm{C}^\infty(\Omega; \mathbf{C}^r) \cap \mathrm{W}^{\mathcal{L},p}(\Omega; \mathbf{C}^r)$ is dense in $\mathrm{W}^{\mathcal{L},p}(\Omega; \mathbf{C}^r)$.* ∎

In order to investigate further the density properties, we need additional assumptions on the open set Ω. By $\mathcal{U}_\mathbf{x}$ we denote the family of all neighbourhoods of the point $\mathbf{x} \in \mathbf{R}^d$. We say that a set $\Omega \subseteq \mathbf{R}^d$ has the *segment property* if there exists a family $\mathcal{N} = \{(N_\mathbf{x}, \mathbf{y}_\mathbf{x}) \in \mathcal{U}_\mathbf{x} \times \mathbf{R}^d : \mathbf{x} \in \mathsf{Fr}\,\Omega\}$ such that

$$(\forall \mathbf{x} \in \mathsf{Fr}\,\Omega)(\forall \mathbf{z} \in \mathsf{Cl}\,\Omega \cap N_\mathbf{x})(\forall \tau \in \langle 0, 1\rangle) \quad \mathbf{z} + \tau \mathbf{y}_\mathbf{x} \in \Omega\,.$$

The geometric interpretation of the segment property is that Ω cannot simultaneously be on both sides of its boundary.

Theorem 4. *If Ω has the segment property and $p \in [1, \infty\rangle$, then the restrictions on Ω of functions from the space $\mathrm{C}_c^\infty(\mathbf{R}^d; \mathbf{C}^r)$ form a dense set in $\mathrm{W}^{\mathcal{L},p}(\Omega; \mathbf{C}^r)$.*

Dem. We divide the proof into three steps.

I. cutting by smooth functions with bounded support

Let $f \in \mathrm{C}_c^\infty(\mathbf{R}^d)$, such that

$$f(x) = \begin{cases} 1, & \text{for} \quad |\mathbf{x}| < 1 \\ 0, & \text{for} \quad |\mathbf{x}| > 2 \end{cases},$$

and $f_n(\mathbf{x}) := f(\frac{\mathbf{x}}{n})$, for $n \in \mathbf{N}$ (Figure 1).

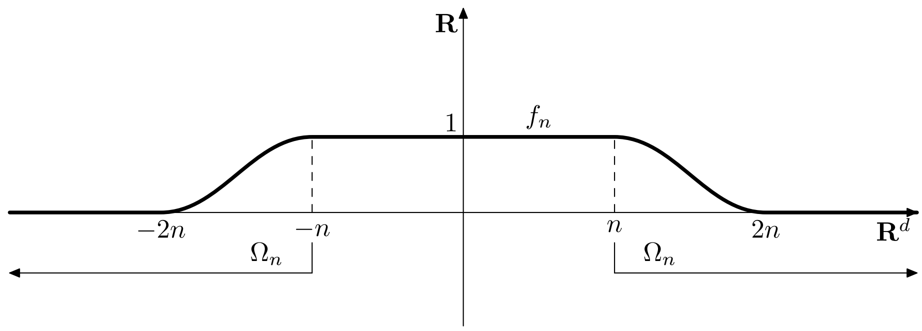

Figure 1. Cutoff function f_n and set Ω_n.

Let us denote

$$\Omega_n := \{\mathbf{x} \in \Omega : |\mathbf{x}| > n\}\,,$$
$$a := \max_{k \in 1..d} \|\mathbf{A}^k\|_{\mathrm{L}^\infty(\Omega; \mathrm{M}_{l,r})}\,,$$

and $\mathsf{v}_n := f_n \mathsf{v}$, for given $\mathsf{v} \in \mathrm{W}^{\mathcal{L},p}(\Omega; \mathbf{C}^r)$. By using the Leibniz rule for the derivative of the product (for a smooth function multiplying a distribution) we have:

$$\begin{aligned} \|\mathcal{L}\mathsf{v}_n\|_{\mathrm{L}^p(\Omega_n; \mathbf{C}^l)} &= \Big\| \sum_{k=1}^d (\partial_k f_n) \mathbf{A}^k \mathsf{v} + f_n \sum_{k=1}^d \partial_k(\mathbf{A}^k \mathsf{v}) + f_n \mathbf{C}\mathsf{v} \Big\|_{\mathrm{L}^p(\Omega_n; \mathbf{C}^l)} \\ &\leqslant C_1 a \|\nabla f_n\|_{\mathrm{L}^\infty(\Omega_n; \mathbf{R}^d)} \|\mathsf{v}\|_{\mathrm{L}^p(\Omega_n; \mathbf{C}^r)} + \|f_n\|_{\mathrm{L}^\infty(\Omega_n)} \|\mathcal{L}\mathsf{v}\|_{\mathrm{L}^p(\Omega_n; \mathbf{C}^l)} \\ &\leqslant C_1' \|f_n\|_{\mathrm{W}^{1,\infty}(\Omega_n)} \|\mathsf{v}\|_{\mathcal{L},p,\Omega_n} \\ &\leqslant C_2 \|f\|_{\mathrm{W}^{1,\infty}(\Omega)} \|\mathsf{v}\|_{\mathcal{L},p,\Omega_n}\,, \end{aligned}$$

for some positive constants C_1, C_1', C_2 (depending neither on n nor on v). This implies (for a $C_3 > 0$)

$$\begin{aligned} \|\mathsf{v} - \mathsf{v}_n\|_{\mathcal{L},p,\Omega} &= \|\mathsf{v} - \mathsf{v}_n\|_{\mathcal{L},p,\Omega_n} \\ &\leqslant \|\mathsf{v}\|_{\mathcal{L},p,\Omega_n} + \|\mathsf{v}_n\|_{\mathcal{L},p,\Omega_n} \leqslant C_3 \|\mathsf{v}\|_{\mathcal{L},p,\Omega_n}\,. \end{aligned}$$

Since (by the Lebesgue dominated convergence theorem) $\lim_{n\to\infty}\|\mathsf{v}\|_{\mathcal{L},p,\Omega_n}=0$, it follows that each function from $\mathrm{W}^{\mathcal{L},p}(\Omega;\mathbf{C}^r)$ can be approximated in the norm by a function from $\mathrm{W}^{\mathcal{L},p}(\Omega;\mathbf{C}^r)$ with bounded support. Taking into account Theorem 3 it follows that each function from $\mathrm{W}^{\mathcal{L},p}(\Omega;\mathbf{C}^r)$ can be approximated by a *smooth* function from $\mathrm{W}^{\mathcal{L},p}(\Omega;\mathbf{C}^r)$ with bounded support. Therefore, it is enough to prove the theorem for such functions.

II. reduction to the local approximation

In the sequel let v be from $\mathrm{C}^\infty(\Omega;\mathbf{C}^r)\cap\mathrm{W}^{\mathcal{L},p}(\Omega;\mathbf{C}^r)$, with bounded support. If we denote by $\mathcal{N}$ a family from the definition of segment property, then the set

$$F:=\operatorname{supp}\mathsf{v}\setminus\Big(\bigcup_{\mathbf{x}\in\mathsf{Fr}\,\Omega}N_{\mathbf{x}}\Big)\subseteq\Omega$$

is compact in $\mathbf{R}^d$. Therefore, there is an open set N_0, such that $F\subseteq N_0\Subset\Omega$. The family $\{N_0\}\cup\{N_{\mathbf{x}}:\mathbf{x}\in\mathsf{Fr}\,\Omega\}$ is then an open cover of the compact set $\operatorname{supp}\mathsf{v}$ in $\mathbf{R}^d$, so there is a finite subcover $\{N_0,N_1,\dots,N_K\}$. Choose open sets $\dot N_i$ such that $\dot N_i\Subset N_i$, $i\in 0..K$, and $\operatorname{supp}\mathsf{v}\subseteq\dot N_0\cup\dot N_1\cup\ldots\cup\dot N_K$. Let $\mathcal{F}=\{f_\alpha:\alpha\in 0..K\}$ be a partition of unity which is subordinate to the open cover $\{\dot N_0,\dot N_1,\dots,\dot N_K\}$ of the set $\operatorname{supp}\mathsf{v}$, and denote by $\dot f_i$ the sum of all $f_\alpha\in\mathcal{F}$ for which i is the smallest index satisfying $\operatorname{supp}f_\alpha\subseteq\dot N_i$. Note that then $\sum_{i=1}^K\dot f_i=1$ on $\operatorname{supp}\mathsf{v}$, and denote $\mathsf{v}_i:=\dot f_i\mathsf{v}$.

If for each $i\in 0..K$ and $\varepsilon>0$ we could find $\mathsf{v}_{\varepsilon,i}\in\mathrm{C}_c^\infty(\mathbf{R}^d;\mathbf{C}^r)$, such that

$$\|\mathsf{v}-\mathsf{v}_{\varepsilon,i}\|_{\mathcal{L},p}<\frac{\varepsilon}{K+1}\,,$$

then for $\mathsf{v}_\varepsilon:=\sum_{i=0}^K\mathsf{v}_{\varepsilon,i}$ we would have

$$\|\mathsf{v}-\mathsf{v}_\varepsilon\|_{\mathcal{L},p}=\Big\|\sum_{i=0}^K\mathsf{v}_i-\sum_{i=0}^K\mathsf{v}_{\varepsilon,i}\Big\|_{\mathcal{L},p}\leqslant\sum_{i=0}^K\|\mathsf{v}_i-\mathsf{v}_{\varepsilon,i}\|_{\mathcal{L},p}<\varepsilon\,,$$

and the theorem would be proved.

III. construction of a local approximation $\mathsf{v}_{\varepsilon,i}$

First note that each v_i belongs to $\mathrm{C}^\infty(\Omega;\mathbf{C}^r)\cap\mathrm{W}^{\mathcal{L},p}(\Omega;\mathbf{C}^r)$, with $\operatorname{supp}\mathsf{v}_i\subseteq\dot N_i\cap\operatorname{supp}\mathsf{v}$. In particular, $\operatorname{supp}\mathsf{v}_0\subseteq\dot N_0$ is compactly contained in Ω, which implies $\mathsf{v}_0\in\mathrm{C}_c^\infty(\Omega;\mathbf{C}^r)$, and allows us to take $\mathsf{v}_{\varepsilon,0}=\mathsf{v}_0$.

Let us fix $i\in 1..K$, and extend v_i by zero outside Ω. Then we have $\mathsf{v}_i\in\mathrm{C}^\infty(\mathbf{R}^d\setminus\Gamma;\mathbf{C}^r)$, where $\Gamma:=\mathsf{Fr}\,\Omega\cap\mathsf{Cl}\,\dot N_i$. Denote by $\mathbf{y}\neq\mathsf{0}$ a vector corresponding to the set N_i from the definition of segment property, and for $n\in\mathbf{N}$ define

$$\Gamma_n:=\Gamma-\frac{1}{n}\mathbf{y}=\{\mathbf{x}-\frac{1}{n}\mathbf{y}:\mathbf{x}\in\Gamma\}\,,$$
$$\mathsf{v}^n(\mathbf{x}):=\mathsf{v}_i(\mathbf{x}+\frac{1}{n}\mathbf{y})\,.$$

For n large enough (such that $\frac{1}{n}<\min\{1,\frac{d(\dot N_i,\mathsf{Fr}\,N_i)}{|\mathbf{y}|}\}$) we have $\Gamma_n\subseteq N_i$, and by the segment property it holds $\Gamma_n\cap\mathsf{Cl}\,\Omega=\emptyset$ (Figure 2).

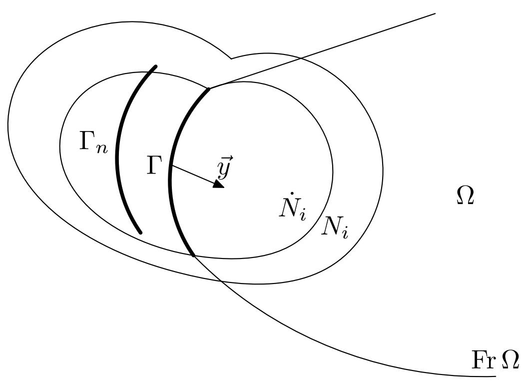

Figure 2. Construction of a local approximation.

We can easily see that $\operatorname{supp}\mathsf{v}^n$ is compactly contained in N_i, $\mathsf{v}^n \in \mathrm{C}^\infty(\mathbf{R}^d \setminus \Gamma_n; \mathbf{C}^r)$, and v^n is bounded on $\mathsf{Cl}\,\Omega$. This implies that

$$\mathcal{L}\mathsf{v}^n = \sum_{k=1}^{d}\Big[(\partial_k\mathbf{A}^k)\mathsf{v}^n + \mathbf{A}^k\partial_k\mathsf{v}^n\Big] + \mathbf{C}\mathsf{v}^n$$

is bounded on Ω, and therefore $\mathsf{v}^n \in \mathrm{W}^{\mathcal{L},p}(\Omega;\mathbf{C}^r)$. Since the translation $(\tau_{\mathsf{c}}g)(\mathbf{x}) = g(\mathbf{x}+\mathsf{c})$ is continuous (in c) on L^p, it follows

$$\mathsf{v}^n \longrightarrow \mathsf{v}_i \quad \text{in} \quad \mathrm{L}^p(\Omega;\mathbf{C}^r) \quad \text{when} \quad n \longrightarrow \infty\,.$$

Similarly as in the proof of Theorem 2 we get

$$\mathcal{L}\mathsf{v}^n \longrightarrow \mathcal{L}\mathsf{v}_i \quad \text{in} \quad \mathrm{L}^p(\Omega;\mathbf{C}^l) \quad \text{when} \quad n \longrightarrow \infty\,.$$

Let us fix $\varepsilon > 0$ and choose a subsequence of (v^n) (which, for simplicity, we denote the same), such that

$$(6) \qquad (\forall\, n \in \mathbf{N}) \qquad \|\mathsf{v}^n - \mathsf{v}_i\|_{\mathrm{L}^p(\Omega;\mathbf{C}^r)} < \frac{1}{4}\cdot\frac{\varepsilon}{K+1}\cdot\frac{1}{2^n}\,.$$

Now using the Mazur theorem, as in the proof of Theorem 2, we choose a convex combination of members of the sequence $(\mathcal{L}\mathsf{v}^n)$, such that

$$\Big\|\sum_{j=1}^{s}\lambda_j\mathcal{L}\mathsf{v}^{n(j)} - \mathcal{L}\mathsf{v}_i\Big\|_{\mathrm{L}^p(\Omega;\mathbf{C}^l)} < \frac{1}{4}\cdot\frac{\varepsilon}{K+1}\,.$$

Denote $\mathsf{u}_{\varepsilon,i} := \sum_{j=1}^{s}\lambda_j\mathsf{v}^{n(j)}$; then $\mathcal{L}\mathsf{u}_{\varepsilon,i} = \sum_{j=1}^{s}\lambda_j\mathcal{L}\mathsf{v}^{n(j)}$ and

$$(7) \qquad \Big\|\mathcal{L}\mathsf{u}_{\varepsilon,i} - \mathcal{L}\mathsf{v}_i\Big\|_{\mathrm{L}^p(\Omega;\mathbf{C}^l)} < \frac{1}{4}\cdot\frac{\varepsilon}{K+1}\,.$$

By (6) we have

$$\begin{aligned}
\|\mathsf{u}_{\varepsilon,i} - \mathsf{v}_i\|_{\mathrm{L}^p(\Omega;\mathbf{C}^r)} &= \Big\|\sum_{j=1}^{s}\lambda_j(\mathsf{v}^{n(j)} - \mathsf{v}_i)\Big\|_{\mathrm{L}^p(\Omega;\mathbf{C}^r)} \\
&\leqslant \sum_{j=1}^{s}\lambda_j\|(\mathsf{v}^{n(j)} - \mathsf{v}_i)\|_{\mathrm{L}^p(\Omega;\mathbf{C}^r)} \\
&< \frac{1}{4}\cdot\frac{\varepsilon}{K+1}\sum_{j=1}^{s}\frac{1}{2^{n(j)}} < \frac{1}{4}\cdot\frac{\varepsilon}{K+1}\sum_{j=1}^{\infty}\frac{1}{2^j} = \frac{1}{4}\cdot\frac{\varepsilon}{K+1}\,,
\end{aligned}$$

which together with (7) gives

$$\|\mathsf{u}_{\varepsilon,i} - \mathsf{v}_i\|_{\mathcal{L},p} < \frac{1}{4}\cdot\frac{\varepsilon}{K+1}\cdot 2^{\frac{1}{p}} < \frac{1}{2}\cdot\frac{\varepsilon}{K+1}\,.$$

From the definition of function $\mathsf{u}_{\varepsilon,i}$ and the properties of functions $\mathsf{v}^{n(j)}$ it follows that $\mathsf{u}_{\varepsilon,i}$ belongs to $\mathrm{C}^\infty(\mathbf{R}^d \setminus (\cup_{j=1}^{s}\Gamma_{n(j)}); \mathbf{C}^r)$ and $\operatorname{supp}\mathsf{u}_{\varepsilon,i}$ is compactly contained in N_i. Since $\mathsf{Cl}\,\Omega \cap N_i$ is compactly contained in $\mathbf{R}^d \setminus (\cup_{j=1}^{s}\Gamma_{n(j)})$, by the Urysohn lemma we can choose a function $\varphi \in \mathrm{C}_c^\infty(\mathbf{R}^d)$ such that $\varphi \equiv 1$ on $\mathsf{Cl}\,\Omega \cap \operatorname{supp}\mathsf{u}_{\varepsilon,i}$ and $\varphi \equiv 0$ outside of some compact set that does not contain $\cup_{j=1}^{s}\Gamma_{n(j)}$. Then the function $\mathsf{v}_{\varepsilon,i} := \varphi\mathsf{u}_{\varepsilon,i}$ satisfies all the required properties.

Q.E.D.

The proof of Theorem 4 is modelled after the proof of Theorem 4 in [J, pg. 21], for $p \in \langle 1, \infty\rangle$, with some differences in part III. In fact, Jensen's proof uses the density of test functions in the dual of L^p, which is not the case for $p = 1$. The same applies to our Theorem 5 below, which compares to [J, Theorem 6, pp. 28–29].

4. Extension to the whole space $\mathbf{R}^d$

As it is the case for Sobolev spaces, many properties of graph spaces depend on regularity of the boundary of set Ω. The most frequently encountered requirement is that Ω is a Lipschitz open set (i.e. an open set with Lipschitz boundary). In the literature one can find different, although quite similar, definitions of this term (for some references see [BC, pg. 350]). Here we adopt the definition from [W, pg. 38], where the term $N^{0,1}$ property is used.

An open set $\Omega \subseteq \mathbf{R}^d$ is called a *Lipschitz set* (or a *set with a Lipschitz boundary*) if for each $\mathbf{x} \in \mathsf{Fr}\,\Omega$ there is a neighbourhood $N_{\mathbf{x}} \in \mathcal{U}_{\mathbf{x}}$, an orthogonal coordinate transformation $\mathbf{K}_{\mathbf{x}} : \mathbf{R}^d \longrightarrow \mathbf{R}^d$, real numbers $\alpha_{\mathbf{x}}, \beta_{\mathbf{x}} > 0$, and a function $f_{\mathbf{x}} : \mathbf{R}^{d-1} \longrightarrow \mathbf{R}$ such that, after denoting by $(y_1, y_2, \dots, y_d)$ the new coordinates given by the transformation $\mathbf{K}_{\mathbf{x}}$, and by

$$P(\alpha_{\mathbf{x}}) := \{(y_1, y_2, \dots, y_{d-1}) : |y_i| < \alpha_{\mathbf{x}}\,,\ i \in 1..d-1\}$$

the open cube in $\mathbf{R}^{d-1}$, centred at the origin, with sides of length $2\alpha_{\mathbf{x}}$, the following four properties hold:

a) $f_{\mathbf{x}}|_{P(\alpha_{\mathbf{x}})}$ is a Lipschitz function;

b) The part $\mathsf{Fr}\,\Omega \cap N_{\mathbf{x}}$ of the boundary is the graph of function $f_{\mathbf{x}}|_{P(\alpha_{\mathbf{x}})}$ in the new coordinates:

$$\mathsf{Fr}\,\Omega \cap N_{\mathbf{x}} = \Big\{ \Big(y_1, y_2, \dots, y_{d-1}, f_{\mathbf{x}}(y_1, y_2, \dots, y_{d-1})\Big) : (y_1, y_2, \dots, y_{d-1}) \in P(\alpha_{\mathbf{x}}) \Big\}\,;$$

c)

$$\Omega \cap N_{\mathbf{x}} = \Big\{ (y_1, y_2, \dots, y_d) \in \mathbf{R}^d : (y_1, y_2, \dots, y_{d-1}) \in P(\alpha_{\mathbf{x}}) \quad \& \\ f_{\mathbf{x}}(y_1, y_2, \dots, y_{d-1}) < y_d < f_{\mathbf{x}}(y_1, y_2, \dots, y_{d-1}) + \beta_{\mathbf{x}} \Big\}\,;$$

d)

$$N_{\mathbf{x}} \setminus \mathsf{Cl}\,\Omega = \Big\{ (y_1, y_2, \dots, y_d) \in \mathbf{R}^d : (y_1, y_2, \dots, y_{d-1}) \in P(\alpha_{\mathbf{x}}) \quad \& \\ f_{\mathbf{x}}(y_1, y_2, \dots, y_{d-1}) - \beta_{\mathbf{x}} < y_d < f_{\mathbf{x}}(y_1, y_2, \dots, y_{d-1}) \Big\}\,.$$

One can easily check that any Lipschitz set also satisfies the segment property.

In the sequel by $\mathrm{W}_0^{\mathcal{L},p}(\Omega; \mathbf{C}^r)$ we denote the closure of the space $\mathrm{C}_c^\infty(\Omega; \mathbf{C}^r)$ in $\mathrm{W}^{\mathcal{L},p}(\Omega; \mathbf{C}^r)$, while for a given measurable function $\mathsf{v} : \Omega \longrightarrow \mathbf{C}^r$, by $\check{\mathsf{v}} : \mathbf{R}^d \longrightarrow \mathbf{C}^r$ we denote its extension by zero to $\mathbf{R}^d$:

$$\check{\mathsf{v}}(\mathbf{x}) := \begin{cases} \mathsf{v}(\mathbf{x}), & \mathbf{x} \in \Omega \\ 0, & \text{otherwise} \end{cases}.$$

Before we proceed further, let us recall that any Lipschitz function on Ω can be extended to a Lipschitz function (with the same Lipschitz constant) on the whole $\mathbf{R}^d$ (the Kirszbraun theorem [Fe, 2.10.43]).

Theorem 5. *Let Ω be a Lipschitz domain (connected open set) and let $\mathbf{A}_e^k \in \mathrm{W}^{1,\infty}(\mathbf{R}^d; \mathrm{M}_{l,r})$ (for $k \in 1..d$) be some Lipschitz extensions of matrix functions $\mathbf{A}^k \in \mathrm{W}^{1,\infty}(\Omega; \mathrm{M}_{l,r})$. Denote by $\mathbf{C}_e$ some bounded extension of function $\mathbf{C}$ to $\mathbf{R}^d$, and define an operator $\mathcal{L}_e : \mathrm{L}^p(\mathbf{R}^d; \mathbf{C}^r) \longrightarrow \mathcal{D}'(\mathbf{R}^d; \mathbf{C}^l)$ by the formula*

$$\mathcal{L}_e \mathsf{u} := \sum_{k=1}^d \partial_k(\mathbf{A}_e^k \mathsf{u}) + \mathbf{C}_e \mathsf{u}\,.$$

Then the following statements for $\mathsf{v} \in \mathrm{W}^{\mathcal{L},p}(\Omega; \mathbf{C}^r)$ are equivalent:

a) $\mathsf{v} \in \mathrm{W}_0^{\mathcal{L},p}(\Omega; \mathbf{C}^r)$,

b) $\check{\mathsf{v}} \in \mathrm{W}^{\mathcal{L}_e,p}(\mathbf{R}^d; \mathbf{C}^r)$.

Dem. Let us show that the first statement implies the second one: for given $\mathsf{v} \in \mathrm{W}_0^{\mathcal{L},p}(\Omega;\mathbf{C}^r)$ it is clear that $\check{\mathsf{v}} \in \mathrm{L}^p(\mathbf{R}^d;\mathbf{C}^r)$, and it remains to be proven that $\mathcal{L}_e\check{\mathsf{v}} \in \mathrm{L}^p(\mathbf{R}^d;\mathbf{C}^l)$. Let us denote by (φ_n) a sequence in $\mathrm{C}_c^\infty(\Omega;\mathbf{C}^r)$ such that

$$\varphi_n \longrightarrow \mathsf{v} \quad \text{in} \quad \mathrm{W}^{\mathcal{L},p}(\Omega;\mathbf{C}^r)\,.$$

It is immediate that $\check{\varphi}_n \longrightarrow \check{\mathsf{v}}$ in $\mathrm{L}^p(\mathbf{R}^d;\mathbf{C}^r)$ and $\mathcal{L}\varphi_n \longrightarrow \mathcal{L}\mathsf{v}$ in $\mathrm{L}^p(\Omega;\mathbf{C}^l)$. Note that

$$\mathcal{L}_e\check{\varphi}_n = (\mathcal{L}\varphi_n)\check{} \longrightarrow (\mathcal{L}\mathsf{v})\check{} \quad \text{in} \quad \mathrm{L}^p(\mathbf{R}^d;\mathbf{C}^l)\,,$$

while Lemma 1 implies

$$\mathcal{L}_e\check{\varphi}_n \rightharpoonup \mathcal{L}_e\check{\mathsf{v}} \quad \text{in} \quad \mathcal{D}'(\mathbf{R}^d;\mathbf{C}^l)\,.$$

The uniqueness of the limit (in the space of distributions) implies $\mathcal{L}_e\check{\mathsf{v}} = (\mathcal{L}\mathsf{v})\check{} \in \mathrm{L}^p(\mathbf{R}^d;\mathbf{C}^l)$, which proves one implication. Note that we have not yet used the fact that Ω is Lipschitz.

In order to prove the other implication we proceed similarly as in the proof of Theorem 4: first we use the sequence (f_n) in order to reduce the problem to the case where $\check{\mathsf{v}}$ has bounded support, then proceeding in the same fashion we get the family $\mathcal{N}$, the sets F, $N_0, N_1, \ldots, N_K$, $\dot{N}_0, \dot{N}_1, \ldots, \dot{N}_K$, the partition of unity $\{f_\alpha : \alpha \in A\}$, and the functions $\dot{f}_i$, $i \in 1..K$. Denote $\dot{\mathsf{v}}_i := \dot{f}_i\check{\mathsf{v}}$, $i \in 0..K$, such that $\sum_{i=0}^K \dot{\mathsf{v}}_i = \check{\mathsf{v}}$ and note that Theorem 2 implies $\dot{\mathsf{v}}_0 \in \mathrm{W}_0^{\mathcal{L},p}(\Omega;\mathbf{C}^r)$. To prove the statement it is enough, for a fixed $i \in 1..K$, to approximate $\dot{\mathsf{v}}_i$ by a sequence from the space $\mathrm{C}_c^\infty(\Omega;\mathbf{C}^r)$ in the $\mathrm{W}^{\mathcal{L},p}(\Omega;\mathbf{C}^r)$ norm.

Let $(y_1, y_2, \ldots, y_d)$ be an orthogonal coordinate system in $\mathbf{R}^d$ and $f_{\mathbf{x}_i} : \mathbf{R}^{d-1} \longrightarrow \mathbf{R}$ the Lipschitz function (with constant L) associated to the pair $(\mathbf{x}_i, N_i)$ from the definition of the Lipschitz open set. Denote by

$$C := \Big\{(y_1, y_2, \ldots, y_d) \in \mathbf{R}^d : y_d > L\Big(\sum_{j=1}^{d-1} y_j^2\Big)^{\frac12}\Big\}$$

a cone in the new coordinates, and by

$$C_{\mathbf{x}} := \Big\{(y_1, y_2, \ldots, y_d) \in \mathbf{R}^d : y_d - y_d^{\mathbf{x}} > L\Big(\sum_{j=1}^{d-1} (y_j - y_j^{\mathbf{x}})^2\Big)^{\frac12}\Big\}$$

the corresponding translated cone with the vertex in $\mathbf{x} \in \mathsf{Fr}\,\Omega \cap N_i$, where $(y_1^{\mathbf{x}}, y_2^{\mathbf{x}}, \ldots, y_d^{\mathbf{x}})$ stands for the transformed coordinates of $\mathbf{x}$. Let us show that for each $\mathbf{x} \in \mathsf{Fr}\,\Omega \cap N_i$ we have the inclusion $C_{\mathbf{x}} \cap N_i \subseteq \Omega$. Indeed, for an arbitrary point $\mathbf{z} = (y_1^{\mathbf{z}}, y_2^{\mathbf{z}}, \ldots, y_d^{\mathbf{z}}) \in C_{\mathbf{x}} \cap N_i$ we have

$$\begin{aligned} y_d^{\mathbf{z}} - y_d^{\mathbf{x}} > L\Big(\sum_{j=1}^{d-1} (y_j^{\mathbf{z}} - y_j^{\mathbf{x}})^2\Big)^{\frac12} &> \Big|f_{\mathbf{x}_i}(y_1^{\mathbf{z}}, y_2^{\mathbf{z}}, \ldots, y_{d-1}^{\mathbf{z}}) - f_{\mathbf{x}_i}(y_1^{\mathbf{x}}, y_2^{\mathbf{x}}, \ldots, y_{d-1}^{\mathbf{x}})\Big| \\ &\geqslant f_{\mathbf{x}_i}(y_1^{\mathbf{z}}, y_2^{\mathbf{z}}, \ldots, y_{d-1}^{\mathbf{z}}) - f_{\mathbf{x}_i}(y_1^{\mathbf{x}}, y_2^{\mathbf{x}}, \ldots, y_{d-1}^{\mathbf{x}})\,, \end{aligned}$$

which together with $y_d^{\mathbf{x}} = f_{\mathbf{x}_i}(y_1^{\mathbf{x}}, y_2^{\mathbf{x}}, \ldots, y_{d-1}^{\mathbf{x}})$ (as $\mathbf{x} \in \mathsf{Fr}\,\Omega \cap N_i$) imply $y_d^{\mathbf{z}} > f_{\mathbf{x}_i}(y_1^{\mathbf{z}}, y_2^{\mathbf{z}}, \ldots, y_{d-1}^{\mathbf{z}})$ or, in other words, $\mathbf{z} \in \Omega \cap N_i$.

Let us take an arbitrary $\varepsilon > 0$ and denote $\delta := \frac13 d(\mathsf{Fr}\,N_i, \mathsf{Fr}\,\dot{N}_i)$. Furthermore, let $\varphi : \mathbf{R}^d \longrightarrow \mathbf{R}$ be a smooth function, such that $\operatorname{supp}\varphi \subseteq \mathrm{K}(0,\delta) \cap C$ and

$$\|\dot{\mathsf{v}}_i - \varphi * \dot{\mathsf{v}}_i\|_{\mathcal{L},p} < \frac{\varepsilon}{2(K+1)} \tag{8}$$

(see the remark after Theorem 2). Then

$$\operatorname{supp}\varphi * \dot{\mathsf{v}}_i \subseteq \operatorname{supp}\dot{\mathsf{v}}_i + (\mathrm{K}(0,\delta) \cap C) \subseteq (\mathsf{Cl}\,\Omega \cap \dot{N}_i) + (\mathrm{K}(0,\delta) \cap C) \subseteq \mathsf{Cl}\,\Omega \cap N_i\,,$$

so that the sequence $(\dot{\mathsf{v}}^n)$, defined by

$$\dot{\mathsf{v}}^n(\mathbf{y}) := (\varphi * \dot{\mathsf{v}}_i)\Big(\mathbf{y} - \frac{1}{n|y_d|}(\underbrace{0,0,\dots,0}_{d-1}, y_d)\Big)\,, \quad \mathbf{y} \in \mathbf{R}^d\,,$$

satisfies $\operatorname{supp}\dot{\mathsf{v}}^n \subseteq \Omega \cap N_i$, for n large enough (such that $\frac{1}{n} < \delta$), which implies $\dot{\mathsf{v}}^n \in \mathrm{C}_c^\infty(\Omega;\mathbf{C}^r)$. We now proceed similarly as in the third part of the proof of Theorem 4 by constructing a function $\dot{\mathsf{v}}_{\varepsilon,i} \in \mathrm{C}_c^\infty(\Omega;\mathbf{C}^r)$, which is a convex combination of members of the sequence $(\dot{\mathsf{v}}^n)$, such that

$$\|\varphi * \dot{\mathsf{v}}_i - \dot{\mathsf{v}}_{\varepsilon,i}\|_{\mathcal{L},p} < \frac{\varepsilon}{2(K+1)}\,,$$

and together with (8) this finally implies

$$\|\dot{\mathsf{v}}_i - \dot{\mathsf{v}}_{\varepsilon,i}\|_{\mathcal{L},p} < \frac{\varepsilon}{K+1}\,.$$

Q.E.D.

5. Dual space

Dual spaces of $\mathrm{W}^{\mathcal{L},p}(\Omega;\mathbf{C}^r)$ and $\mathrm{W}_0^{\mathcal{L},p}(\Omega;\mathbf{C}^r)$ can be characterised in the same way as in the case of Sobolev spaces [AF, pp. 62–64]. We write these results in the next two theorems, omitting the proofs as they are analogous to those for the Sobolev spaces and can be found in [J, pp. 60–62] and [B, pp. 18–20].

As in the previous section, we assume that Ω is Lipschitz and $p \in [1,\infty\rangle$.

Theorem 6. *For each $S \in \mathrm{W}^{\mathcal{L},p}(\Omega;\mathbf{C}^r)'$ there are $\mathsf{w}_1 \in \mathrm{L}^{p'}(\Omega;\mathbf{C}^r)$ and $\mathsf{w}_2 \in \mathrm{L}^{p'}(\Omega;\mathbf{C}^l)$, such that*

$$(9) \qquad (\forall\, \mathsf{v} \in \mathrm{W}^{\mathcal{L},p}(\Omega;\mathbf{C}^r)) \qquad S\mathsf{v} = \int_\Omega \mathsf{w}_1 \cdot \mathsf{v}\, d\mathbf{x} + \int_\Omega \mathsf{w}_2 \cdot \mathcal{L}\mathsf{v}\, d\mathbf{x}\,.$$

If we denote by V_S the set of all $(\mathsf{w}_1,\mathsf{w}_2) \in \mathrm{L}^{p'}(\Omega;\mathbf{C}^r) \times \mathrm{L}^{p'}(\Omega;\mathbf{C}^l)$ satisfying (9), then there exists $\tilde{\mathsf{w}} \in V_S$ such that

$$\|S\|_{\mathrm{W}^{\mathcal{L},p}(\Omega;\mathbf{C}^r)'} = \|\tilde{\mathsf{w}}\|_{\mathrm{L}^{p'}(\Omega;\mathbf{C}^r)\times\mathrm{L}^{p'}(\Omega;\mathbf{C}^l)} = \min\{\|\mathsf{w}\|_{\mathrm{L}^{p'}(\Omega;\mathbf{C}^r)\times\mathrm{L}^{p'}(\Omega;\mathbf{C}^l)} : \mathsf{w} \in V_S\}\,.$$

Moreover, for $p \in \langle 1,\infty\rangle$ such $\tilde{\mathsf{w}}$ is unique in V_S. ∎

Theorem 7. *Let us denote by U the vector space of all distributions $T \in \mathcal{D}'(\Omega;\mathbf{C}^r)$ for which there are $\mathsf{w}_1 \in \mathrm{L}^{p'}(\Omega;\mathbf{C}^r)$ and $\mathsf{w}_2 \in \mathrm{L}^{p'}(\Omega;\mathbf{C}^l)$, such that*

$$T = \mathsf{w}_1 + \tilde{\mathcal{L}}\mathsf{w}_2\,.$$

Then it holds:

a)

$$(\forall\, S \in \mathrm{W}^{\mathcal{L},p}(\Omega;\mathbf{C}^r)')\,(\exists!\, T \in U) \qquad T = S_{|\mathcal{D}(\Omega;\mathbf{C}^r)}\,.$$

If V_S is as in the previous theorem, then for each $(\mathsf{w}_1,\mathsf{w}_2) \in V_S$ we have $T = \mathsf{w}_1 + \tilde{\mathcal{L}}\mathsf{w}_2$.

b) $\mathrm{W}_0^{\mathcal{L},p}(\Omega;\mathbf{C}^r)'$ is isometrically isomorphic to the space U, with the norm on U defined by

$$\|T\|_U = \inf\Big\{\|(\mathsf{w}_1,\mathsf{w}_2)\|_{\mathrm{L}^{p'}(\Omega;\mathbf{C}^r)\times\mathrm{L}^{p'}(\Omega;\mathbf{C}^l)} : (\mathsf{w}_1,\mathsf{w}_2) \in \mathrm{L}^{p'}(\Omega;\mathbf{C}^r) \times \mathrm{L}^{p'}(\Omega;\mathbf{C}^l) \;\;\&\;\; T = \mathsf{w}_1 + \tilde{\mathcal{L}}\mathsf{w}_2\Big\}\,.$$

∎

For some further results regarding better description of the set V_S, and in particular its element $\tilde{\mathsf{w}}$, we refer to the work of Jensen [J, J1].

6. Trace operator

Before we define the trace operator for graph spaces, we shall prove two technical results.

Lemma 2. *The Sobolev space $\mathrm{W}^{1,p}(\Omega;\mathbf{C}^r)$ is continuously imbedded in $\mathrm{W}^{\mathcal{L},p}(\Omega;\mathbf{C}^r)$, for $p\in[1,\infty]$. If $p\in[1,\infty\rangle$ and Ω satisfies the segment property, then this imbedding is dense.*

Dem. For $\mathsf{u}\in\mathrm{W}^{1,p}(\Omega;\mathbf{C}^r)$ we have $\mathsf{u},\partial_k\mathsf{u}\in\mathrm{L}^p(\Omega;\mathbf{C}^r)$, $k\in 1..d$, and by an application of the Leibniz formula we get

$$\mathcal{L}\mathsf{u}=\sum_{k=1}^d\Big[\partial_k\mathbf{A}^k\mathsf{u}+\mathbf{A}^k\partial_k\mathsf{u}\Big]+\mathbf{C}\mathsf{u}\in\mathrm{L}^p(\Omega;\mathbf{C}^l)\,,$$

which implies $\mathsf{u}\in\mathrm{W}^{\mathcal{L},p}(\Omega;\mathbf{C}^r)$. It is easy to check that

$$\|\mathcal{L}\mathsf{u}\|_{\mathrm{L}^p(\Omega;\mathbf{C}^l)}\leqslant C_1\|\mathsf{u}\|_{\mathrm{L}^p(\Omega;\mathbf{C}^r)}+C_2\sum_{k=1}^d\|\partial_k\mathsf{u}\|_{\mathrm{L}^p(\Omega;\mathbf{C}^r)}\leqslant C_3\|\mathsf{u}\|_{\mathrm{W}^{1,p}(\Omega;\mathbf{C}^r)}\,,$$

and therefore

$$\|\mathsf{u}\|_{\mathcal{L},p}\leqslant C_4\|\mathsf{u}\|_{\mathrm{W}^{1,p}(\Omega;\mathbf{C}^r)}\,,$$

for some positive constants C_1,C_2,C_3,C_4, which proves the first statement.

If $p\in[1,\infty\rangle$ and Ω satisfies the segment property, then by Theorem 4 the space $\mathrm{C}_c^\infty(\mathbf{R}^d;\mathbf{C}^r)$ is dense in $\mathrm{W}^{\mathcal{L},p}(\Omega;\mathbf{C}^r)$. As $\mathrm{C}_c^\infty(\mathbf{R}^d;\mathbf{C}^r)\subseteq\mathrm{W}^{1,p}(\Omega;\mathbf{C}^r)$, the second statement holds as well.

Q.E.D.

Lemma 3. *For $\mathsf{v}\in\mathrm{L}^p(\Omega;\mathbf{C}^r)$ $(p\in[1,\infty])$ and $\boldsymbol{\varphi}\in\mathcal{D}(\Omega;\mathbf{C}^l)$ we have*

$${}_{\mathcal{D}'(\Omega;\mathbf{C}^l)}\langle\,\mathcal{L}\mathsf{v},\boldsymbol{\varphi}\,\rangle_{\mathcal{D}(\Omega;\mathbf{C}^l)}=\int_\Omega\mathsf{v}\cdot\tilde{\mathcal{L}}\boldsymbol{\varphi}\,d\mathbf{x}\,.$$

Dem. First note that the above formula makes sense, as $\mathcal{L}\mathsf{v}\in\mathcal{D}'(\Omega;\mathbf{C}^l)$, and $\tilde{\mathcal{L}}\boldsymbol{\varphi}\in\mathrm{L}^\infty(\Omega;\mathbf{C}^r)$ has a compact support. Using the Leibniz formula and the definition of distributional derivative we get

$$\begin{aligned}\int_\Omega\mathsf{v}\cdot\tilde{\mathcal{L}}\boldsymbol{\varphi}\,d\mathbf{x}&=\int_\Omega\mathsf{v}\cdot\Big[-\sum_{k=1}^d\partial_k((\mathbf{A}^k)^*\boldsymbol{\varphi})+\Big(\mathbf{C}^*+\sum_{k=1}^d\partial_k(\mathbf{A}^k)^*\Big)\boldsymbol{\varphi}\Big]\,d\mathbf{x}\\&=\int_\Omega\mathsf{v}\cdot\Big[-\sum_{k=1}^d(\mathbf{A}^k)^*\partial_k\boldsymbol{\varphi}+\mathbf{C}^*\boldsymbol{\varphi}\Big]\,d\mathbf{x}\\&=\int_\Omega\Big[-\sum_{k=1}^d\mathbf{A}^k\mathsf{v}\cdot\partial_k\boldsymbol{\varphi}+\mathbf{C}\mathsf{v}\cdot\boldsymbol{\varphi}\Big]\,d\mathbf{x}\\&=-\sum_{k=1}^d{}_{\mathcal{D}'(\Omega;\mathbf{C}^l)}\langle\,\mathbf{A}^k\mathsf{v},\partial_k\boldsymbol{\varphi}\,\rangle_{\mathcal{D}(\Omega;\mathbf{C}^l)}+{}_{\mathcal{D}'(\Omega;\mathbf{C}^l)}\langle\,\mathbf{C}\mathsf{v},\boldsymbol{\varphi}\,\rangle_{\mathcal{D}(\Omega;\mathbf{C}^l)}\\&=\sum_{k=1}^d{}_{\mathcal{D}'(\Omega;\mathbf{C}^l)}\langle\,\partial_k(\mathbf{A}^k\mathsf{v}),\boldsymbol{\varphi}\,\rangle_{\mathcal{D}(\Omega;\mathbf{C}^l)}+{}_{\mathcal{D}'(\Omega;\mathbf{C}^l)}\langle\,\mathbf{C}\mathsf{v},\boldsymbol{\varphi}\,\rangle_{\mathcal{D}(\Omega;\mathbf{C}^l)}\\&={}_{\mathcal{D}'(\Omega;\mathbf{C}^l)}\langle\,\mathcal{L}\mathsf{v},\boldsymbol{\varphi}\,\rangle_{\mathcal{D}(\Omega;\mathbf{C}^l)}\,.\end{aligned}$$

Q.E.D.

Note that the statement of the above lemma still holds if we change the rôles of the operators $\mathcal{L}$ and $\tilde{\mathcal{L}}$.

In the rest of this section we shall consider only the case $p = 2$, and therefore define the trace operator only for the space $\mathrm{H}^{\mathcal{L}}(\Omega; \mathbf{C}^r)$ (most results for general $\mathrm{W}^{\mathcal{L},p}(\Omega; \mathbf{C}^r)$ can be found in [J, pp. 30–36]). In this way we avoid the introduction of Besov spaces, at the same time simplifying a number of arguments by using the Hilber space structure (compare our proof of Theorem 11 and the construction of lifting operator $\mathcal{E}$ to [J, pp. 31–32, 36).

We additionally suppose that Ω is a bounded Lipschitz set, so that the Rademacher theorem (stating that Lipschitz functions have first derivatives almost everywhere [EG, pg. 81]) ensures the existence of the unit outward normal $\boldsymbol{\nu} = (\nu_1, \nu_2, \ldots, \nu_d) \in \mathrm{L}^\infty(\mathsf{Fr}\,\Omega; \mathbf{R}^d)$ on the boundary. Define the mapping $\mathbf{A}_{\boldsymbol{\nu}} \in \mathrm{L}^\infty(\mathsf{Fr}\,\Omega; \mathrm{M}_{l,r})$ by

$$\mathbf{A}_{\boldsymbol{\nu}}(\mathbf{x}) := \sum_{k=1}^{d} \nu_k(\mathbf{x}) \mathbf{A}^k(\mathbf{x}) \,.$$

For $m \in \mathbf{N}$, by $\mathcal{T}_{\mathrm{H}^1} : \mathrm{H}^1(\Omega; \mathbf{C}^m) \longrightarrow \mathrm{H}^{\frac{1}{2}}(\mathsf{Fr}\,\Omega; \mathbf{C}^m)$ we denote the trace operator, and by $\mathcal{E}_{\mathrm{H}^1} : \mathrm{H}^{\frac{1}{2}}(\mathsf{Fr}\,\Omega; \mathbf{C}^m) \longrightarrow \mathrm{H}^1(\Omega; \mathbf{C}^m)$ its right inverse (for example, see [W, pp. 120–133]).

Theorem 8. *For* $\mathsf{u} \in \mathrm{H}^1(\Omega; \mathbf{C}^r)$ *and* $\mathsf{v} \in \mathrm{H}^1(\Omega; \mathbf{C}^l)$ *we have*

$$\langle\, \mathcal{L}\mathsf{u} \mid \mathsf{v} \,\rangle_{\mathrm{L}^2(\Omega;\mathbf{C}^l)} - \langle\, \mathsf{u} \mid \tilde{\mathcal{L}}\mathsf{v} \,\rangle_{\mathrm{L}^2(\Omega;\mathbf{C}^r)} = \langle\, \mathbf{A}_{\boldsymbol{\nu}} \mathcal{T}_{\mathrm{H}^1}\mathsf{u} \mid \mathcal{T}_{\mathrm{H}^1}\mathsf{v} \,\rangle_{\mathrm{L}^2(\mathsf{Fr}\,\Omega;\mathbf{C}^l)} \,.$$

Dem. First we prove the statement for $\mathsf{u} \in \mathrm{C}_c^\infty(\mathbf{R}^d; \mathbf{C}^r)$ and $\mathsf{v} \in \mathrm{C}_c^\infty(\mathbf{R}^d; \mathbf{C}^l)$. After applying the Leibniz formula several times we achieve

$$\begin{aligned}
\langle\, \mathcal{L}\mathsf{u} \mid \mathsf{v} \,\rangle_{\mathrm{L}^2(\Omega;\mathbf{C}^l)} - \langle\, \mathsf{u} \mid \tilde{\mathcal{L}}\mathsf{v} \,\rangle_{\mathrm{L}^2(\Omega;\mathbf{C}^r)} &= \int_\Omega \sum_{k=1}^{d} \partial_k(\mathbf{A}^k\mathsf{u}) \cdot \mathsf{v} \, d\mathbf{x} + \int_\Omega \mathsf{u} \cdot \sum_{k=1}^{d} (\mathbf{A}^k)^* \partial_k \mathsf{v} \, d\mathbf{x} \\
&= \int_\Omega \sum_{k=1}^{d} \Bigl[\partial_k(\mathbf{A}^k\mathsf{u}) \cdot \mathsf{v} + \mathbf{A}^k\mathsf{u} \cdot \partial_k\mathsf{v} \Bigr] \, d\mathbf{x} \\
&= \int_\Omega \sum_{k=1}^{d} \partial_k(\mathbf{A}^k\mathsf{u} \cdot \mathsf{v}) \, d\mathbf{x} \,,
\end{aligned}$$

which together with the divergence theorem implies

$$\langle\, \mathcal{L}\mathsf{u} \mid \mathsf{v} \,\rangle_{\mathrm{L}^2(\Omega;\mathbf{C}^l)} - \langle\, \mathsf{u} \mid \tilde{\mathcal{L}}\mathsf{v} \,\rangle_{\mathrm{L}^2(\Omega;\mathbf{C}^r)} = \int_{\mathsf{Fr}\,\Omega} \sum_{k=1}^{d} \nu_k (\mathbf{A}^k\mathsf{u} \cdot \mathsf{v})_{|\mathsf{Fr}\,\Omega} \, dS = \int_{\mathsf{Fr}\,\Omega} \mathbf{A}_{\boldsymbol{\nu}} \mathsf{u}_{|\mathsf{Fr}\,\Omega} \cdot \mathsf{v}_{|\mathsf{Fr}\,\Omega} \, dS \,,$$

and proves the statement in this case.

Take now $\mathsf{u} \in \mathrm{C}_c^\infty(\mathbf{R}^d; \mathbf{C}^r)$, $\mathsf{v} \in \mathrm{H}^1(\Omega; \mathbf{C}^l)$, and let (v_n) be a sequence in $\mathrm{C}_c^\infty(\mathbf{R}^d; \mathbf{C}^l)$ that converges to v in $\mathrm{H}^1(\Omega; \mathbf{C}^l)$. Then

$$\langle\, \mathcal{L}\mathsf{u} \mid \mathsf{v}_n \,\rangle_{\mathrm{L}^2(\Omega;\mathbf{C}^l)} \longrightarrow \langle\, \mathcal{L}\mathsf{u} \mid \mathsf{v} \,\rangle_{\mathrm{L}^2(\Omega;\mathbf{C}^l)} \,,$$

and by using $\tilde{\mathcal{L}}\mathsf{v}_n \longrightarrow \tilde{\mathcal{L}}\mathsf{v}$ in $\mathrm{L}^2(\Omega; \mathbf{C}^r)$ (which follows from Lemma 2 and the definition of $\tilde{\mathcal{L}}$–norm), we get

$$\langle\, \mathsf{u} \mid \tilde{\mathcal{L}}\mathsf{v}_n \,\rangle_{\mathrm{L}^2(\Omega;\mathbf{C}^r)} \longrightarrow \langle\, \mathsf{u} \mid \tilde{\mathcal{L}}\mathsf{v} \,\rangle_{\mathrm{L}^2(\Omega;\mathbf{C}^r)} \,.$$

On the other hand, from $\mathcal{T}_{\mathrm{H}^1}\mathsf{v}_n \longrightarrow \mathcal{T}_{\mathrm{H}^1}\mathsf{v}$ in $\mathrm{H}^{\frac{1}{2}}(\mathsf{Fr}\,\Omega; \mathbf{C}^l)$, we derive that $\mathcal{T}_{\mathrm{H}^1}\mathsf{v}_n \longrightarrow \mathcal{T}_{\mathrm{H}^1}\mathsf{v}$ in $\mathrm{L}^2(\mathsf{Fr}\,\Omega; \mathbf{C}^l)$, which implies

$$\langle\, \mathbf{A}_{\boldsymbol{\nu}} \mathcal{T}_{\mathrm{H}^1}\mathsf{u} \mid \mathcal{T}_{\mathrm{H}^1}\mathsf{v}_n \,\rangle_{\mathrm{L}^2(\mathsf{Fr}\,\Omega;\mathbf{C}^l)} \longrightarrow \langle\, \mathbf{A}_{\boldsymbol{\nu}} \mathcal{T}_{\mathrm{H}^1}\mathsf{u} \mid \mathcal{T}_{\mathrm{H}^1}\mathsf{v} \,\rangle_{\mathrm{L}^2(\mathsf{Fr}\,\Omega;\mathbf{C}^l)} \,,$$

and proves the statement in this case.

Approximating $\mathsf{u} \in \mathrm{H}^1(\Omega; \mathbf{C}^r)$ by functions from $\mathrm{C}_c^\infty(\mathbf{R}^d; \mathbf{C}^r)$, and applying a similar argument as before, we prove the statement for $\mathsf{u} \in \mathrm{H}^1(\Omega; \mathbf{C}^r)$ and $\mathsf{v} \in \mathrm{H}^1(\Omega; \mathbf{C}^l)$.

Q.E.D.

Let $\mathsf{u} \in \mathrm{H}^1(\Omega; \mathbf{C}^r)$ and $\mathsf{v} \in \mathrm{H}^1(\Omega; \mathbf{C}^l)$. Using the Cauchy–Schwartz inequality (for L^2 inner product) we get

$$\begin{aligned}\Big|\langle \mathbf{A}_{\boldsymbol{\nu}} \mathcal{T}_{\mathrm{H}^1}\mathsf{u} \mid \mathcal{T}_{\mathrm{H}^1}\mathsf{v} \rangle_{\mathrm{L}^2(\mathsf{Fr}\,\Omega;\mathbf{C}^l)}\Big| &\leqslant \Big|\langle \mathcal{L}\mathsf{u} \mid \mathsf{v} \rangle_{\mathrm{L}^2(\Omega;\mathbf{C}^l)}\Big| + \Big|\langle \mathsf{u} \mid \tilde{\mathcal{L}}\mathsf{v} \rangle_{\mathrm{L}^2(\Omega;\mathbf{C}^r)}\Big| \\ &\leqslant \|\mathcal{L}\mathsf{u}\|_{\mathrm{L}^2(\Omega;\mathbf{C}^l)}\|\mathsf{v}\|_{\mathrm{L}^2(\Omega;\mathbf{C}^l)} + \|\mathsf{u}\|_{\mathrm{L}^2(\Omega;\mathbf{C}^r)}\|\tilde{\mathcal{L}}\mathsf{v}\|_{\mathrm{L}^2(\Omega;\mathbf{C}^r)} \\ &\leqslant \|\mathsf{u}\|_{\mathcal{L}}\|\mathsf{v}\|_{\tilde{\mathcal{L}}}\,.\end{aligned}$$

Therefore, for $\mathsf{z} \in \mathrm{H}^{\frac{1}{2}}(\mathsf{Fr}\,\Omega; \mathbf{C}^l)$ we have

$$\begin{aligned}\langle \mathbf{A}_{\boldsymbol{\nu}} \mathcal{T}_{\mathrm{H}^1}\mathsf{u} \mid \mathsf{z} \rangle_{\mathrm{L}^2(\mathsf{Fr}\,\Omega;\mathbf{C}^l)} &\leqslant \|\mathsf{u}\|_{\mathcal{L}} \cdot \|\mathcal{E}_{\mathrm{H}^1}\mathsf{z}\|_{\tilde{\mathcal{L}}} \\ &\leqslant C_1\|\mathsf{u}\|_{\mathcal{L}} \cdot \|\mathcal{E}_{\mathrm{H}^1}\mathsf{z}\|_{\mathrm{H}^1(\Omega;\mathbf{C}^l)} \leqslant C_1\|\mathsf{u}\|_{\mathcal{L}}\|\mathcal{E}_{\mathrm{H}^1}\| \cdot \|\mathsf{z}\|_{\mathrm{H}^{\frac{1}{2}}(\mathsf{Fr}\,\Omega;\mathbf{C}^l)}\,,\end{aligned}$$

for some constant $C_1 > 0$. This implies that the mapping

$$\mathsf{z} \mapsto \langle \mathbf{A}_{\boldsymbol{\nu}} \mathcal{T}_{\mathrm{H}^1}\mathsf{u} \mid \mathsf{z} \rangle_{\mathrm{L}^2(\mathsf{Fr}\,\Omega;\mathbf{C}^l)}$$

is a continuous antilinear functional on $\mathrm{H}^{\frac{1}{2}}(\mathsf{Fr}\,\Omega; \mathbf{C}^l)$, with norm being less or equal $C_1\|\mathsf{u}\|_{\mathcal{L}}\|\mathcal{E}_{\mathrm{H}^1}\|$. In other words, for fixed $\mathsf{u} \in \mathrm{H}^1(\Omega; \mathbf{C}^r)$, the mapping $\langle \mathbf{A}_{\boldsymbol{\nu}} \mathcal{T}_{\mathrm{H}^1}\mathsf{u} \mid \cdot\rangle_{\mathrm{L}^2(\mathsf{Fr}\,\Omega;\mathbf{C}^l)}$ belongs to the space $\mathrm{H}^{-\frac{1}{2}}(\mathsf{Fr}\,\Omega; \mathbf{C}^l)$. This defines the mapping $\dot{\mathcal{T}}_{\mathcal{L}} : \mathrm{H}^1(\Omega; \mathbf{C}^r) \longrightarrow \mathrm{H}^{-\frac{1}{2}}(\mathsf{Fr}\,\Omega; \mathbf{C}^l)$ by setting

$$\dot{\mathcal{T}}_{\mathcal{L}}\mathsf{u} := \langle \mathbf{A}_{\boldsymbol{\nu}} \mathcal{T}_{\mathrm{H}^1}\mathsf{u} \mid \cdot\rangle_{\mathrm{L}^2(\mathsf{Fr}\,\Omega;\mathbf{C}^l)}\,.$$

It is immediate that this mapping is linear and continuous if we equip $\mathrm{H}^1(\Omega; \mathbf{C}^r)$ with the norm $\|\cdot\|_{\mathcal{L}}$. Since $\mathrm{H}^1(\Omega; \mathbf{C}^r)$ is dense in $\mathrm{H}^{\mathcal{L}}(\Omega; \mathbf{C}^r)$, we can extend $\dot{\mathcal{T}}_{\mathcal{L}}$ to a unique continuous linear operator on the whole $\mathrm{H}^{\mathcal{L}}(\Omega; \mathbf{C}^r)$. The operator $\dot{\mathcal{T}}_{\mathcal{L}}$ will not be surjective in general (see Example 1 below), which is why we denote its image by

$$\mathrm{H}^{\mathcal{L}}_{\mathcal{T}}(\mathsf{Fr}\,\Omega; \mathbf{C}^l) := \operatorname{Im} \dot{\mathcal{T}}_{\mathcal{L}} \subseteq \mathrm{H}^{-\frac{1}{2}}(\mathsf{Fr}\,\Omega; \mathbf{C}^l)\,,$$

and look at the surjective trace operator $\mathcal{T}_{\mathcal{L}} : \mathrm{H}^{\mathcal{L}}(\Omega; \mathbf{C}^r) \longrightarrow \mathrm{H}^{\mathcal{L}}_{\mathcal{T}}(\mathsf{Fr}\,\Omega; \mathbf{C}^l)$, defined as the restriction of $\dot{\mathcal{T}}_{\mathcal{L}}$ on $\mathrm{H}^{\mathcal{L}}(\Omega; \mathbf{C}^r)$.

Theorem 9. *Under the above assumptions we have*

$$\operatorname{Ker} \mathcal{T}_{\mathcal{L}} = \mathrm{H}^{\mathcal{L}}_0(\Omega; \mathbf{C}^r)\,.$$

Dem. For $\boldsymbol{\varphi} \in \mathrm{C}^\infty_c(\Omega; \mathbf{C}^r)$ it is clear that $\mathcal{T}_{\mathcal{L}}\boldsymbol{\varphi} = 0$, and since $\mathrm{C}^\infty_c(\Omega; \mathbf{C}^r)$ is dense in $\mathrm{H}^{\mathcal{L}}_0(\Omega; \mathbf{C}^r)$ and $\mathcal{T}_{\mathcal{L}}$ is continuous, we get

$$\operatorname{Ker} \mathcal{T}_{\mathcal{L}} \supseteq \mathrm{H}^{\mathcal{L}}_0(\Omega; \mathbf{C}^r)\,.$$

In order to prove the other inclusion we take $\mathsf{v} \in \operatorname{Ker} \mathcal{T}_{\mathcal{L}}$ and, as before, denote by $\check{\mathsf{v}}$ its extension by zero to $\mathbf{R}^d$. Let $\mathcal{L}_e$ be an extension of the operator $\mathcal{L}$ as in Theorem 5. Define its formal adjoint $\widetilde{\mathcal{L}_e}$ by the formula

$$\widetilde{\mathcal{L}_e}\mathsf{u} := -\sum_{k=1}^d \partial_k((\mathbf{A}^k_e)^*\mathsf{u}) + \Big((\mathbf{C}_e)^* + \sum_{k=1}^d \partial_k(\mathbf{A}^k_e)^*\Big)\mathsf{u}\,,$$

which is then also an extension (in the same sense as in Theorem 5) of the operator $\tilde{\mathcal{L}}$. By Theorem 5 it is enough to prove that $\check{\mathsf{v}} \in \mathrm{H}^{\mathcal{L}_e}(\mathbf{R}^d; \mathbf{C}^r)$. As it is clear that $\check{\mathsf{v}} \in \mathrm{L}^2(\mathbf{R}^d; \mathbf{C}^r)$, it remains to show $\mathcal{L}_e\check{\mathsf{v}} \in \mathrm{L}^2(\mathbf{R}^d; \mathbf{C}^l)$.

For $\varphi \in \mathcal{D}(\mathbf{R}^d; \mathbf{C}^l)$, from Lemma 3 and the definition of the trace operator it follows that

$$
\begin{aligned}
{}_{\mathcal{D}'(\mathbf{R}^d;\mathbf{C}^l)}\langle\, \mathcal{L}_e\check{\mathsf{v}}, \boldsymbol{\varphi}\,\rangle_{\mathcal{D}(\mathbf{R}^d;\mathbf{C}^l)} - {}_{\mathcal{D}'(\mathbf{R}^d;\mathbf{C}^l)}\langle\, (\check{\mathcal{L}\mathsf{v}}), \boldsymbol{\varphi}\,\rangle_{\mathcal{D}(\mathbf{R}^d;\mathbf{C}^l)} =& \\
&= \int_{\mathbf{R}^d} \check{\mathsf{v}} \cdot \widetilde{\mathcal{L}_e}\boldsymbol{\varphi}\, d\mathbf{x} - \int_{\mathbf{R}^d} (\check{\mathcal{L}\mathsf{v}}) \cdot \boldsymbol{\varphi}\, d\mathbf{x} \\
&= \int_{\Omega} \mathsf{v} \cdot \tilde{\mathcal{L}}\boldsymbol{\varphi}\, d\mathbf{x} - \int_{\Omega} \mathcal{L}\mathsf{v} \cdot \boldsymbol{\varphi}\, d\mathbf{x} \\
&= \langle\, \mathsf{v} \mid \tilde{\mathcal{L}}\boldsymbol{\varphi}\,\rangle_{\mathrm{L}^2(\Omega;\mathbf{C}^r)} - \langle\, \mathcal{L}\mathsf{v} \mid \boldsymbol{\varphi}\,\rangle_{\mathrm{L}^2(\Omega;\mathbf{C}^l)} \\
&= -{}_{\mathrm{H}^{-\frac{1}{2}}(\mathsf{Fr}\,\Omega;\mathbf{C}^l)}\langle\, \mathcal{T}_{\mathcal{L}}\mathsf{v}, \mathcal{T}_{\mathrm{H}^1}\boldsymbol{\varphi}\,\rangle_{\mathrm{H}^{\frac{1}{2}}(\mathsf{Fr}\,\Omega;\mathbf{C}^l)} = 0\,,
\end{aligned}
$$

since $\mathcal{T}_{\mathcal{L}}\mathsf{v} = 0$. This implies $\mathcal{L}_e\check{\mathsf{v}} = (\mathcal{L}\mathsf{v})\check{} \in \mathrm{L}^2(\mathbf{R}^d; \mathbf{C}^l)$.

Q.E.D.

We will now try to justify the name *trace operator* for the operator $\mathcal{T}_{\mathcal{L}}$. First we adopt some notation: let V be some metrisable vector space and $\mathcal{J} : \mathrm{H}^{\mathcal{L}}(\Omega; \mathbf{C}^r) \longrightarrow V$ a continuous linear operator. The operator $\mathcal{J}$ is called the *boundary operator* if

$$(\forall\, \mathsf{u}, \mathsf{v} \in \mathrm{C}_c^\infty(\mathbf{R}^d; \mathbf{C}^r)) \quad \mathsf{u}_{|\mathsf{Fr}\,\Omega} = \mathsf{v}_{|\mathsf{Fr}\,\Omega} \quad \Longrightarrow \quad \mathcal{J}\mathsf{u} = \mathcal{J}\mathsf{v}\,.$$

Theorem 10. *A continuous operator $\mathcal{J} : \mathrm{H}^{\mathcal{L}}(\Omega; \mathbf{C}^r) \longrightarrow V$ is a boundary operator if and only if*

$$(\forall\, \mathsf{u}, \mathsf{v} \in \mathrm{H}^{\mathcal{L}}(\Omega; \mathbf{C}^r)) \quad \mathcal{T}_{\mathcal{L}}\mathsf{u} = \mathcal{T}_{\mathcal{L}}\mathsf{v} \quad \Longrightarrow \quad \mathcal{J}\mathsf{u} = \mathcal{J}\mathsf{v}\,.$$

Dem. Suppose that $\mathcal{J}$ is a boundary operator and $\mathcal{T}_{\mathcal{L}}\mathsf{u} = \mathcal{T}_{\mathcal{L}}\mathsf{v}$. Then $\mathsf{u}-\mathsf{v} \in \operatorname{Ker} \mathcal{T}_{\mathcal{L}} = \mathrm{H}_0^{\mathcal{L}}(\Omega; \mathbf{C}^r)$. Take a sequence (u_n) in $\mathrm{C}_c^\infty(\Omega; \mathbf{C}^r)$ such that

$$\mathsf{u}_n \longrightarrow \mathsf{u} - \mathsf{v} \quad \text{in} \quad \mathrm{H}^{\mathcal{L}}(\Omega; \mathbf{C}^r)\,.$$

Then from $\mathcal{J}\mathsf{u}_n = 0$ and the continuity of $\mathcal{J}$ we have $\mathcal{J}\mathsf{u} = \mathcal{J}\mathsf{v}$.

To prove the converse statement note that for $\mathsf{u}, \mathsf{v} \in \mathrm{C}_c^\infty(\mathbf{R}^d; \mathbf{C}^r)$ such that $\mathsf{u}_{|\mathsf{Fr}\,\Omega} = \mathsf{v}_{|\mathsf{Fr}\,\Omega}$, by the definition of operator $\mathcal{T}_{\mathcal{L}}$, we have $\mathcal{T}_{\mathcal{L}}\mathsf{u} = \mathcal{T}_{\mathcal{L}}\mathsf{v}$ and hence $\mathcal{J}\mathsf{u} = \mathcal{J}\mathsf{v}$.

Q.E.D.

Let us show how does the trace operator look like when $\mathcal{L}$ is one of the operators of gradient, divergence or rotation.

Example 1. We have already seen that $\mathrm{H}^\nabla(\Omega) = \mathrm{H}^1(\Omega)$. However, a surjective trace operator $\mathcal{T}_{\mathrm{H}^1} : \mathrm{H}^1(\Omega) \longrightarrow \mathrm{H}^{\frac{1}{2}}(\mathsf{Fr}\,\Omega)$ is already defined on $\mathrm{H}^1(\Omega)$. Let us investigate the relationship between the operators $\mathcal{T}_{\mathrm{H}^1}$ and $\mathcal{T}_\nabla : \mathrm{H}^\nabla(\Omega) \longrightarrow \mathrm{H}_{\mathcal{T}}^\nabla(\mathsf{Fr}\,\Omega; \mathbf{C}^d) \leqslant \mathrm{H}^{-\frac{1}{2}}(\mathsf{Fr}\,\Omega; \mathbf{C}^d)$. For $\mathcal{L} = \nabla$ one can easily see that $\mathbf{A}_{\boldsymbol{\nu}} = \boldsymbol{\nu}$, so that the definition of $\mathcal{T}_\nabla$ and Theorem 8 imply (for $u \in \mathrm{H}^1(\Omega)$ and $\mathsf{z} \in \mathrm{H}^{\frac{1}{2}}(\mathsf{Fr}\,\Omega; \mathbf{C}^d)$)

$${}_{\mathrm{H}^{-\frac{1}{2}}(\mathsf{Fr}\,\Omega;\mathbf{C}^d)}\langle\, \mathcal{T}_\nabla u, \mathsf{z}\,\rangle_{\mathrm{H}^{\frac{1}{2}}(\mathsf{Fr}\,\Omega;\mathbf{C}^d)} = {}_{\mathrm{H}^{-\frac{1}{2}}(\mathsf{Fr}\,\Omega;\mathbf{C}^d)}\langle\, \boldsymbol{\nu}\mathcal{T}_{\mathrm{H}^1} u, \mathsf{z}\,\rangle_{\mathrm{H}^{\frac{1}{2}}(\mathsf{Fr}\,\Omega;\mathbf{C}^d)}\,,$$

meaning that

$$\mathcal{T}_\nabla = \boldsymbol{\nu}\mathcal{T}_{\mathrm{H}^1} \in \mathrm{H}^{-\frac{1}{2}}(\mathsf{Fr}\,\Omega; \mathbf{C}^d)\,.$$

Since the operator $\mathcal{T}_{\mathrm{H}^1}$ is surjective, we finally get

$$\operatorname{Im} \mathcal{T}_\nabla = \mathrm{H}_{\mathcal{T}}^\nabla(\mathsf{Fr}\,\Omega; \mathbf{C}^d) = \{\boldsymbol{\nu} v : v \in \mathrm{H}^{\frac{1}{2}}(\mathsf{Fr}\,\Omega)\} = \boldsymbol{\nu}\mathrm{H}^{\frac{1}{2}}(\mathsf{Fr}\,\Omega) < \mathrm{L}^2(\mathsf{Fr}\,\Omega; \mathbf{C}^d)\,.$$

∎

Example 2. If we take $\mathcal{L}\mathsf{u} = \mathsf{div}\,\mathsf{u}$, one can easily check that $\mathbf{A}_{\boldsymbol{\nu}} = \boldsymbol{\nu}^\top$, which implies that for functions from $\mathrm{L}^2_{\mathrm{div}}(\Omega)$ the *normal trace* $\mathcal{T}_{\mathsf{div}}$ is well defined, which is given (for $\mathsf{u} \in \mathrm{H}^1(\Omega; \mathbf{C}^d)$ and $z \in \mathrm{H}^{\frac{1}{2}}(\mathsf{Fr}\,\Omega)$) by the formula

$$ {}_{\mathrm{H}^{-\frac{1}{2}}(\mathsf{Fr}\,\Omega)}\langle\, \mathcal{T}_{\mathsf{div}}\,\mathsf{u}, z \,\rangle_{\mathrm{H}^{\frac{1}{2}}(\mathsf{Fr}\,\Omega)} = {}_{\mathrm{H}^{-\frac{1}{2}}(\mathsf{Fr}\,\Omega)}\langle\, \boldsymbol{\nu}^\top \mathcal{T}_{\mathrm{H}^1}\mathsf{u}, z \,\rangle_{\mathrm{H}^{\frac{1}{2}}(\mathsf{Fr}\,\Omega)} \,, $$

and then extended by density to a continuous linear operator on $\mathrm{L}^2_{\mathrm{div}}(\Omega)$. It is known that its image $\mathrm{H}^{\mathsf{div}}_{\mathcal{T}}(\mathsf{Fr}\,\Omega)$ equals the whole space $\mathrm{H}^{-\frac{1}{2}}(\mathsf{Fr}\,\Omega)$ (see [DL, IX.1.2]). Therefore, $\mathcal{T}_{\mathsf{div}} : \mathrm{L}^2_{\mathrm{div}}(\Omega) \longrightarrow \mathrm{H}^{-\frac{1}{2}}(\mathsf{Fr}\,\Omega)$ is surjective.

∎

Example 3. If we take $d = 3$ and $\mathcal{L}\mathsf{u} = \mathsf{rot}\,\mathsf{u}$, then from

$$ \mathbf{A}^1(\mathbf{x}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \qquad \mathbf{A}^2(\mathbf{x}) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \qquad \mathbf{A}^3(\mathbf{x}) = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, $$

it follows

$$ \mathbf{A}_{\boldsymbol{\nu}} = \begin{bmatrix} 0 & -\nu_3 & \nu_2 \\ \nu_3 & 0 & -\nu_1 \\ -\nu_2 & \nu_1 & 0 \end{bmatrix}, $$

and one can easily prove that

$$ (\forall \boldsymbol{\xi} \in \mathbf{C}^3) \qquad \mathbf{A}_{\nu}\boldsymbol{\xi} = \boldsymbol{\nu} \times \boldsymbol{\xi} \,. $$

Therefore, for functions from $\mathrm{L}^2_{\mathrm{rot}}(\Omega)$ the *tangential trace* $\mathcal{T}_{\mathsf{rot}}$ is well defined, which is (for $\mathsf{u} \in \mathrm{H}^1(\Omega; \mathbf{C}^3)$ and $\mathsf{z} \in \mathrm{H}^{\frac{1}{2}}(\mathsf{Fr}\,\Omega; \mathbf{C}^3)$) given by the formula

$$ {}_{\mathrm{H}^{-\frac{1}{2}}(\mathsf{Fr}\,\Omega;\mathbf{C}^3)}\langle\, \mathcal{T}_{\mathsf{rot}}\,\mathsf{u}, \mathsf{z} \,\rangle_{\mathrm{H}^{\frac{1}{2}}(\mathsf{Fr}\,\Omega;\mathbf{C}^3)} = {}_{\mathrm{H}^{-\frac{1}{2}}(\mathsf{Fr}\,\Omega;\mathbf{C}^3)}\langle\, \boldsymbol{\nu} \times \mathcal{T}_{\mathrm{H}^1}\mathsf{u}, \mathsf{z} \,\rangle_{\mathrm{H}^{\frac{1}{2}}(\mathsf{Fr}\,\Omega;\mathbf{C}^3)} \,, $$

and then extended by density to a continuous linear operator on $\mathrm{L}^2_{\mathrm{rot}}(\Omega)$. In this example the characterisation of the trace space is a bit more complicated (for details see [T]):

$$ \mathrm{H}^{\mathsf{rot}}_{\mathcal{T}}(\mathsf{Fr}\,\Omega; \mathbf{C}^3) = \Big\{ \mathsf{v} \in \mathrm{H}^{-\frac{1}{2}}(\mathsf{Fr}\,\Omega; \mathbf{C}^3) \,:\, (\exists\, \eta \in \mathrm{H}^{-\frac{1}{2}}(\mathsf{Fr}\,\Omega))\, (\forall\, \varphi \in \mathrm{H}^2(\Omega)) \\ {}_{\mathrm{H}^{-\frac{1}{2}}(\mathsf{Fr}\,\Omega;\mathbf{C}^3)}\langle\, \mathsf{v}, \nabla\varphi \,\rangle_{\mathrm{H}^{\frac{1}{2}}(\mathsf{Fr}\,\Omega;\mathbf{C}^3)} = {}_{\mathrm{H}^{-\frac{1}{2}}(\mathsf{Fr}\,\Omega)}\langle\, \eta, \varphi \,\rangle_{\mathrm{H}^{\frac{1}{2}}(\mathsf{Fr}\,\Omega)} \Big\} \,. $$

∎

It is immediate that the restriction $\acute{\mathcal{T}}_{\mathcal{L}}$ of the operator $\mathcal{T}_{\mathcal{L}}$ on $(\mathrm{Ker}\,\mathcal{T}_{\mathcal{L}})^\perp$ is a continuous linear bijection from $(\mathrm{Ker}\,\mathcal{T}_{\mathcal{L}})^\perp$ on $\mathrm{H}^{\mathcal{L}}_{\mathcal{T}}(\mathsf{Fr}\,\Omega; \mathbf{C}^l)$. This allows us to define the operator $\mathcal{E}_{\mathcal{L}} : \mathrm{H}^{\mathcal{L}}_{\mathcal{T}}(\mathsf{Fr}\,\Omega; \mathbf{C}^l) \longrightarrow (\mathrm{Ker}\,\mathcal{T}_{\mathcal{L}})^\perp \subseteq \mathrm{H}^{\mathcal{L}}(\Omega; \mathbf{C}^r)$ as the inverse (both left and right)

$$ \mathcal{E}_{\mathcal{L}} := \acute{\mathcal{T}}_{\mathcal{L}}^{-1} \,, $$

which is obviously linear, satisfies $\mathrm{Im}\,\mathcal{E}_{\mathcal{L}} = (\mathrm{Ker}\,\mathcal{T}_{\mathcal{L}})^\perp = \mathrm{H}^{\mathcal{L}}_0(\Omega; \mathbf{C}^r)^\perp$ and

$$ \mathcal{T}_{\mathcal{L}}\mathcal{E}_{\mathcal{L}}\mathsf{z} = \mathsf{z} \,, \qquad \mathsf{z} \in \mathrm{H}^{\mathcal{L}}_{\mathcal{T}}(\mathsf{Fr}\,\Omega; \mathbf{C}^l) \,. $$

If $\mathrm{H}^{\mathcal{L}}_{\mathcal{T}}(\mathsf{Fr}\,\Omega; \mathbf{C}^l)$ is closed in $\mathrm{H}^{-\frac{1}{2}}(\mathsf{Fr}\,\Omega; \mathbf{C}^l)$ (which is not always the case; see the remark below), then the operator $\mathcal{E}_{\mathcal{L}}$ is continuous [Br, pg. 19].

Theorem 11. *For* $\mathsf{z} \in \mathrm{H}^{\mathcal{L}}_{\mathcal{T}}(\mathsf{Fr}\,\Omega; \mathbf{C}^l)$, *the infimum of* $\{\|\mathsf{u}\|_{\mathcal{L}} : \mathsf{u} \in \mathrm{H}^{\mathcal{L}}(\Omega; \mathbf{C}^r) \quad \& \quad \mathcal{T}_{\mathcal{L}}\mathsf{u} = \mathsf{z}\}$ *is achieved in the unique point* $\mathsf{u}_{\mathsf{z}} = \mathcal{E}_{\mathcal{L}}\mathsf{z}$.

Dem. It is easy to see that $\mathcal{T}^{\leftarrow}_{\mathcal{L}}(\mathsf{z}) = \mathcal{E}_{\mathcal{L}}\mathsf{z} + \mathrm{Ker}\,\mathcal{T}_{\mathcal{L}}$, which (because $\mathcal{E}_{\mathcal{L}}\mathsf{z} \in (\mathrm{Ker}\,\mathcal{T}_{\mathcal{L}})^{\perp}$) for $\mathsf{v} \in \mathrm{Ker}\,\mathcal{T}_{\mathcal{L}}$ implies

$$\|\mathcal{E}_{\mathcal{L}}\mathsf{z} + \mathsf{v}\|_{\mathcal{L}}^2 = \langle\, \mathcal{E}_{\mathcal{L}}\mathsf{z} + \mathsf{v} \mid \mathcal{E}_{\mathcal{L}}\mathsf{z} + \mathsf{v}\,\rangle_{\mathcal{L}} = \|\mathcal{E}_{\mathcal{L}}\mathsf{z}\|_{\mathcal{L}}^2 + \|\mathsf{v}\|_{\mathcal{L}}^2 .$$

Since $0 \in \mathrm{Ker}\,\mathcal{T}_{\mathcal{L}}$, it follows

$$\inf\{\|\mathcal{E}_{\mathcal{L}}\mathsf{z} + \mathsf{v}\|_{\mathcal{L}} : \mathsf{v} \in \mathrm{Ker}\,\mathcal{T}_{\mathcal{L}}\} = \min\{\|\mathcal{E}_{\mathcal{L}}\mathsf{z} + \mathsf{v}\|_{\mathcal{L}} : \mathsf{v} \in \mathrm{Ker}\,\mathcal{T}_{\mathcal{L}}\} = \|\mathcal{E}_{\mathcal{L}}\mathsf{z}\|_{\mathcal{L}} ,$$

and $\mathcal{E}_{\mathcal{L}}\mathsf{z}$ is the only point where the above minimum is achieved.

Q.E.D.

If for $\mathsf{z}, \mathsf{w} \in \mathrm{H}^{\mathcal{L}}_{\mathcal{T}}(\mathsf{Fr}\,\Omega; \mathbf{C}^l)$ we define

$$\langle\, \mathsf{z} \mid \mathsf{w}\,\rangle_{\mathcal{T}_{\mathcal{L}}} := \langle\, \mathcal{E}\mathsf{z} \mid \mathcal{E}\mathsf{w}\,\rangle_{\mathcal{L}} ,$$

one can easily check that $\langle\,\cdot \mid \cdot\,\rangle_{\mathcal{T}_{\mathcal{L}}}$ is an inner product on $\mathrm{H}^{\mathcal{L}}_{\mathcal{T}}(\mathsf{Fr}\,\Omega; \mathbf{C}^l)$. If we equip $\mathrm{H}^{\mathcal{L}}_{\mathcal{T}}(\mathsf{Fr}\,\Omega; \mathbf{C}^l)$ with the corresponding norm

$$\|\mathsf{z}\|_{\mathcal{T}_{\mathcal{L}}} = \|\mathcal{E}_{\mathcal{L}}\mathsf{z}\|_{\mathcal{L}} ,$$

from the previous theorem it follows that operators $\mathcal{T}_{\mathcal{L}}$ and $\mathcal{E}_{\mathcal{L}}$ are continuous linear operators with operator norm equal to 1.

Remark. It is natural to investigate the relationship between two norms on $\mathrm{H}^{\mathcal{L}}_{\mathcal{T}}(\mathsf{Fr}\,\Omega; \mathbf{C}^l)$, the norm $\|\cdot\|_{\mathcal{T}_{\mathcal{L}}}$ and the one induced from $\mathrm{H}^{-\frac{1}{2}}(\mathsf{Fr}\,\Omega; \mathbf{C}^l)$. It can easily be seen that for an arbitrary $\mathsf{z} \in \mathrm{H}^{\mathcal{L}}_{\mathcal{T}}(\mathsf{Fr}\,\Omega; \mathbf{C}^l)$ it holds

$$\|\mathsf{z}\|_{\mathrm{H}^{-\frac{1}{2}}(\mathsf{Fr}\,\Omega; \mathbf{C}^l)} \leqslant \|\mathcal{T}_{\mathcal{L}}\|_{\mathcal{L}(\mathrm{H}^{\mathcal{L}}(\Omega; \mathbf{C}^r); \mathrm{H}^{-\frac{1}{2}}(\mathsf{Fr}\,\Omega; \mathbf{C}^l))} \|\mathsf{z}\|_{\mathcal{T}_{\mathcal{L}}} .$$

Therefore, the question of equivalence of these two norms on $\mathrm{H}^{\mathcal{L}}_{\mathcal{T}}(\mathsf{Fr}\,\Omega; \mathbf{C}^l)$ is actually the question of closedness of $\mathrm{H}^{\mathcal{L}}_{\mathcal{T}}(\mathsf{Fr}\,\Omega; \mathbf{C}^l)$ in $\mathrm{H}^{-\frac{1}{2}}(\mathsf{Fr}\,\Omega; \mathbf{C}^l)$ [Br, pg. 19]. We will show by a counter example that $\mathrm{H}^{\mathcal{L}}_{\mathcal{T}}(\mathsf{Fr}\,\Omega; \mathbf{C}^l)$ is not always closed in $\mathrm{H}^{-\frac{1}{2}}(\mathsf{Fr}\,\Omega; \mathbf{C}^l)$. This occurs in the case $\mathcal{L} = \nabla$, where the space $\mathrm{H}^{\mathcal{L}}_{\mathcal{T}}(\mathsf{Fr}\,\Omega; \mathbf{C}^l)$ equals $\boldsymbol{\nu}\mathrm{H}^{\frac{1}{2}}(\mathsf{Fr}\,\Omega)$. Take $f \in \mathrm{L}^2(\mathsf{Fr}\,\Omega) \setminus \mathrm{H}^{\frac{1}{2}}(\mathsf{Fr}\,\Omega)$ so that

$$\boldsymbol{\nu} f \in \mathrm{H}^{-\frac{1}{2}}(\mathsf{Fr}\,\Omega; \mathbf{C}^d) \setminus \boldsymbol{\nu}\mathrm{H}^{\frac{1}{2}}(\mathsf{Fr}\,\Omega) .$$

Let (f_n) be a sequence in $\mathrm{H}^{\frac{1}{2}}(\mathsf{Fr}\,\Omega)$ converging to f in $\mathrm{L}^2(\mathsf{Fr}\,\Omega)$ (such a sequence exists as $\mathrm{H}^{\frac{1}{2}}(\mathsf{Fr}\,\Omega)$ is dense in $\mathrm{L}^2(\mathsf{Fr}\,\Omega)$). Then we have

$$\boldsymbol{\nu} f_n \longrightarrow \boldsymbol{\nu} f$$

in $\mathrm{L}^2(\mathsf{Fr}\,\Omega; \mathbf{C}^d)$ and therefore in $\mathrm{H}^{-\frac{1}{2}}(\mathsf{Fr}\,\Omega; \mathbf{C}^d)$. As $\boldsymbol{\nu} f \notin \boldsymbol{\nu}\mathrm{H}^{\frac{1}{2}}(\mathsf{Fr}\,\Omega)$ and $\boldsymbol{\nu} f_n \in \boldsymbol{\nu}\mathrm{H}^{\frac{1}{2}}(\mathsf{Fr}\,\Omega)$, this implies that $\boldsymbol{\nu}\mathrm{H}^{\frac{1}{2}}(\mathsf{Fr}\,\Omega)$ is not closed in $\mathrm{H}^{-\frac{1}{2}}(\mathsf{Fr}\,\Omega; \mathbf{C}^d)$

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