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Graph representation for asymptotic expansion in homogenisation of nonlinear first-order equations

Abstract

Homogenisation of a linear transport equation leads to an integro-differential equation with the differential part of the same type as the starting equation. The (nonperiodic) homogenisation of semilinear transport equations is open.

In order to pinpoint technical difficulties, as a first step in that direction, following the approach of Tartar we consider an ordinary differential equation with an oscillating coefficient a:

$$\begin{cases} u' + au^2 = f \\ \qquad u(0) = v \end{cases}$$

instead, and expand the solution in terms of a small parameter (the size of oscillations in a). The crucial observation we made is a correspondence between multiple integrals representing the terms in asymptotic expansion of the solution and certain graphs, which allows easy manipulation of otherwise highly complicated expressions, and leads to efficient computation of the terms in expansion.

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1. Introduction

The homogenisation of semilinear transport equations is still an open problem; for example, consider the initial value problem of the form:

$$\begin{cases} \dfrac{\partial u}{\partial t} + c\dfrac{\partial u}{\partial x} + au^2 = f \\ \qquad\qquad u(\cdot;0) = v. \end{cases}$$

Youcef Amirat, Kamel Hamdache and Abdelhamid Ziani [AHZ] studied the linear case ($a = 0$), with c depending only on another variable y, and oscillating. The effective equation contains an additional nonlocal term.

As the first step towards understanding what happens in the nonlinear case, Luc Tartar [T2] suggested to look at a sequence of Cauchy problems for ordinary differential equations:

$$\tag{1} \begin{cases} \dfrac{\partial u_n}{\partial t}(x,t) + a_n(x,t)u_n^2(x,t) = f(x,t) \quad \text{in } \Omega \times \langle 0, \infty\rangle \\ \qquad\qquad u_n(x,0) = v(x) \quad \text{in } \Omega\ , \end{cases}$$

where Ω in general denotes a nonatomic finite measure space (the reader may think of Ω as a segment of the real line with the Lebesgue measure). Let us stress that x has no a priori physical meaning; it is just a parameter in (1).

Of course, the equation (1) is what one gets when trying to solve the above transport equation via characteristics. The quadratic nonlinearity in (1) could be replaced by an arbitrary integer power u_n^p, and the presented method would work, in spite of much more tedious details.

It is assumed that $0 < \alpha \leqslant a_n(x,t) \leqslant \beta$ and $|a_n(x,t) - a_n(x,s)| \leqslant \varepsilon(|t-s|)$, (*a.e.* $x \in \Omega$), $t, s \in \mathbf{R}^+$, where $\varepsilon(\sigma) \to 0$ as $\sigma \to 0$. After passing to a subsequence if necessary, we thus have $a_n(\cdot,t) \longrightarrow a_\infty(\cdot,t)$ in $\mathrm{L}^\infty(\Omega)$ weak $*$, for any $t \in \mathbf{R}^+$.

For the right hand side we suppose that

$$\tag{2} \begin{aligned} 0 \leqslant v \leqslant M_0 \quad & a.e. \text{ in } \Omega\ , \\ 0 \leqslant f \leqslant F \quad & a.e. \text{ in } \Omega \times \langle 0,\infty\rangle\ , \quad \text{and} \\ \int_0^\infty f(\cdot,t)dt \leqslant M_1 \quad & a.e. \text{ in } \Omega\ , \end{aligned}$$

for some $M_0, F, M_1 > 0$. Finally, let us denote $b_n := a_n - a_\infty$.

The above assumptions guarantee that there exists a solution $u_n = \Phi_n(v,f)$, which is nonnegative and satisfies $0 \leqslant u_n \leqslant M_0 + t\,F$ and $0 \leqslant u_n \leqslant M = \max\left\{M_0, \sqrt{F/\alpha}\right\}$. Our goal is to determine the equation satisfied by u_∞, a weak accumulation point of the sequence (u_n).

In a particular case where the coefficients a_n do not depend on t, and $f = 0$, we can solve (1) explicitly:

$$u_n(x,t) = \frac{v(x)}{1 + tv(x)a_n(x)}\ ,$$

and after passing to a subsequence such that (a_{n_k}) determines the Young measure ν, we can write the limit

$$u_\infty(x,t) = \int_{[\alpha,\beta]} \frac{v(x)}{1+tv(x)a} d\nu_x(a)\ .$$

Even in this case we do not know the equation satisfied by u_∞, which is related to the sequence of problems (1). And this relation is the crucial question in Tartar's approach to homogenisation, as it should model the passage from microscale to macroscale.

For linear problems, methods based on the Laplace transform or the Nevanlinna functions were succsessful; for nonlinear problems only the old idea of using a perturbation expansion has shown some potential (see [A], [Ar], [T1], [T3]).

For a small parameter γ, the unique global solution of

$$\begin{cases} \dfrac{\partial U_n(\cdot,\cdot;\gamma)}{\partial t} + (a_\infty + \gamma b_n) U_n^2(\cdot,\cdot;\gamma) = f \\ \hfill U_n(\cdot,0;\gamma) = v \end{cases}$$

depends analytically on γ, so it admits the Taylor expansion

$$U_n(x,t;\gamma) = U_0(x,t) + \sum_{k=1}^{\infty} \gamma^k U_k^n(x,t)\,.$$

Here $U_0^n := U_0$ satisfies

$$\begin{cases} \dfrac{\partial U_0}{\partial t} + a_\infty U_0^2 = f \\ \hfill U_0(\cdot,0) = v, \end{cases} \tag{3}$$

while U_k^n are defined by induction:

$$\begin{cases} \dfrac{\partial U_k^n}{\partial t} + 2a_\infty U_0 U_k^n = W_k^n \\ \hfill U_k^n(\cdot,0) = 0, \end{cases} \tag{4}$$

with

$$W_k^n := -a_\infty \sum_{j=1}^{k-1} U_j^n U_{k-j}^n - b_n \sum_{j=0}^{k-1} U_j^n U_{k-1-j}^n\,. \tag{5}$$

Clearly, after denoting $R(x,s,t) = e^{-\int_s^t 2a_\infty(x,\sigma)U_0(x,\sigma)d\sigma}$, the solution of (4) can be written by formula

$$U_k^n(x,t) = \int_0^t R(x,s,t) W_k^n(x,s) ds\,.$$

For the missing details we refer the reader to [T2] or [AL].

In the second section we briefly recall the idea presented in [AL] regarding the representation of U_k^n by graphs, with a number of modifications which made it more feasible for applications to other expressions, and more economical regarding the implementation on a computer (some ideas we got by reading [K]). For the (simple) proofs that this representation is correct, we invite the reader to try it by himself, or slightly modify the proofs given in [AL]. Some rough estimates on graph complexity are given as well. In the next section we pass to the limit in the terms appearing in the asymptotic expansion, noticing the same structure of expressions. The last step is to replace all appearances of auxilliary functions like U_0 by the limit, at least up to an error of order $O(\gamma^{K+1})$. Prior to that, we illustrate the situation on an example.

2. Graphs for W_k^n and U_k^n

It has already been noticed in [AL] that graphs (collections of binary trees) are a natural way to represent W_k^n and U_k^n. This follows from the structure of recursion (for convenience we shall fix the index n and omit it, as well as the variable x, in writting in this section, and write a for a_∞)

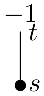
$$\begin{aligned} W_k(s) &= -\,a(s)\sum_{j=1}^{k-1} U_j(s)U_{k-j}(s) - b(s)\sum_{j=0}^{k-1} U_j(s)U_{k-1-j}(s) \\ U_k(t) &= \int_0^t R(s,t)W_k(s)ds\,, \end{aligned} \tag{6}$$

where U_0 is given as the unique solution of equation (3). Then the graph for $W_1(s) = -b(s)U_0^2(s)$ is

The formal rule is that a black circle in a vertex denotes function b in appropriate variable, and if the vertex has got no children, then there is U_0^2 in appropriate variable. We write the coefficient at top (or left) of the tree. The graph for

$$U_1(t) = \int_0^t R(s,t)W_1(s)ds = -\int_0^t R(s,t)b(s)U_0^2(s)ds$$

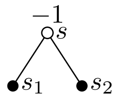
is



In general, the edge $|_s^t$ will denote $\int_0^t R(s,t)F(s)ds$, where $F(s)$ is the formula represented by subtree whose root is the vertex in variable s. For representing $W_2(s) = -a(s)U_1^2(s) - 2b(s)U_0(s)U_1(s)$ we use two binary trees, each of them representing one of the terms in the above equality. The first term, that is

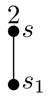
$$-a(s)\left(\int_0^s R(s_1,s)b(s_1)U_0^2(s_1)ds_1\right)\left(\int_0^s R(s_2,s)b(s_2)U_0^2(s_2)ds_2\right),$$

we write as



Here the empty circle stands for function $a(s)$ (this is another general rule), and the fact that two graphs for U_1 are connected in a new vertex represents the term U_1^2. So, the rule is that a product of type U_iU_j we represent by a tree such that subtrees of the root are graphs for U_i and U_j. From recursive relation (6) it is obvious that such type of product always comes either with function a or b as a factor, and that determines whether we have empty or black circle in the root vertex.

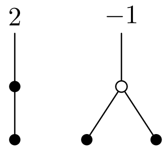
The second term in the expression for $W_2(s)$ is $2b(s)U_0(s)\int_0^s R(s_1,s)b(s_1)U_0^2(s_1)ds_1$ and the corresponding graph is



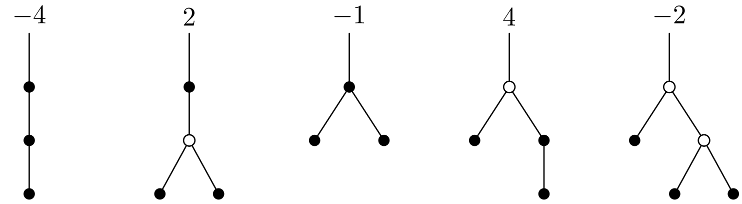
where we have applied the rule for products described above. Since the graph for U_0 is simply a single vertex, the root does not have two children, but only one. The rule is that if we have vertex with black or empty circle, and if it has only one child, then we assume that there is U_0 in appropriate variable as the second child. Now, using a binary tree for each term in recursion for W_k, and using the additivity of integral we can inductively draw graphs for each W_k and U_k. Before we do that let us make one simplification. Note that we can omit variables in vertices, because they are inner variables of integration (and only the root variable is visible). With this, the graphs for U_1, U_2 and U_3 are (these graphs are a streamlined version of graphs in [AL]):

U_1 :

U_2 :



U_3 :



Remark. We wrote a program and ran it on a personal computer which calculated and drew graphs for U_k and W_k. For $k = 13$ the run took one second and 100 MB of RAM was used. If we denote the number of different graphs appearing in U_k by $n(k)$, while the total number of graphs is $m(k)$, then for the first ten graphs we have

k	1	2	3	4	5	6	7	8	9	10
$m(k)$	1	3	13	67	381	2307	14589	95235	636925	4341763
$n(k)$	1	2	5	15	48	166	596	2221	8472	32995

The table was computed using recursive formulae for $m(k)$ and $n(k)$ (which can easily be obtained from (6); in particular, we take $m(0) := n(0) := 1$):

$$m(k) = \sum_{j=1}^{k-1} m(j)m(k-j) + \sum_{j=0}^{k-1} m(j)m(k-1-j)$$

$$n(k) = \sum_{j=1}^{[\frac{k-1}{2}]} n(j)n(k-j) + \sum_{j=0}^{[\frac{k-2}{2}]} n(j)n(k-1-j) + \frac{n([\frac{k}{2}])(n([\frac{k}{2}])+1)}{2} .$$

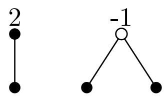
Here $[r] := \max\{n \in \mathbf{Z} : n \leqslant r\}$ denotes the largest integer function. Using Maple we checked that $\frac{n(k+1)}{n(k)}$ is between 4 and 5 for $k \in 11..5000$, and that it always grows. From this we guess that the number of different binary trees grows exponentially in k.

Another interesting fact (the proof is immediate) is that all coefficients are powers of 2, while their sum is $(-1)^k$. We note that in graphs for W_k the number of vertices varies from k to $2k-1$. ∎

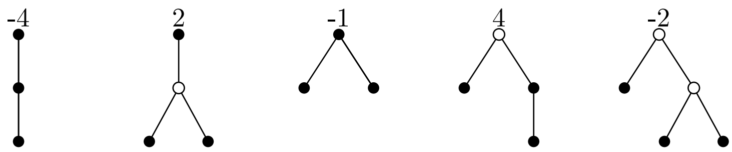
Because of the exponential growth of the number of component graphs ($n(k)$ in the Remark), it is more convenient not to distribute products. Let us introduce some modifications and try to do some further simplifications. Note that expressions for W_k differ only in one integral from the expressions for U_k. So, the graphs for W_1, W_2 and W_3 are:

W_1 :

W_2 :

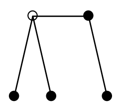


W_3 :

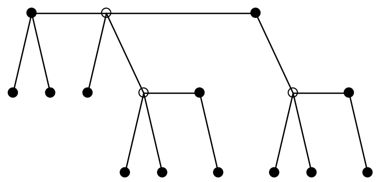


Now let us try to avoid such large numbers of trees: to this end we introduce a new rule that a horizontal line is the sign for a sum. The graphs for W_2, W_3 and W_4 thus become:

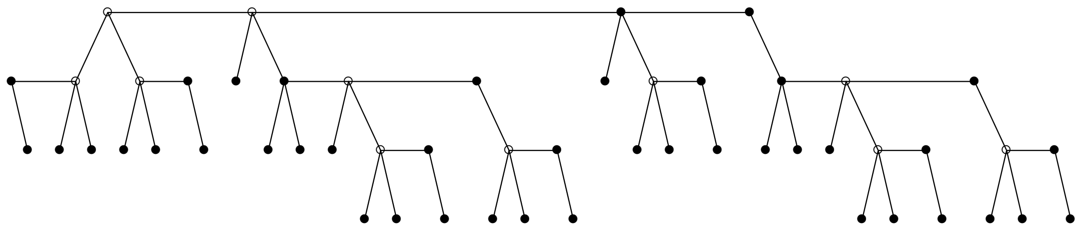
W_2 :



W_3 :

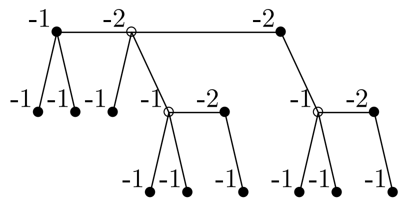


W_4 :



Note that we did not write any coefficients in these graphs. From the recursion (6) it is clear that each term but one appears twice. So, one term in the sum has coefficient -1 while all others have -2. The rule is to write the one with -1 as the first term in summation, and the corresponding vertex as the one connected with its parent, or if there is no parent, as the first (leftmost) vertex on the top horizontal line. For example, if we explicitly write down all coefficients in the graph for W_3, it will look like this:

W_3 :



The coefficients are then multiplied as the vertices enter into products.

By using this representation for graphs in a computer program, we save on memory, but automatic drawing becomes highly nontrivial.

Remark. The asymptotic expansion

$$U_n(x,t;\gamma) = U_0(x,t) + \sum_{k=1}^{\infty} \gamma^k U_k^n(x,t)$$

in general converges only for small γ, while we are interested in the case $\gamma = 1$. Before proceeding further, we would like to show that the whole expansion converges in ordinary sense for $\gamma = 1$, at least with some additional assumptions.

Let us prove that the above series converges for $|\gamma| \leqslant 1$, at least for $|t| < \rho$, for some small $\rho > 0$. In order to do that, we assume that additional estimates (uniformly in x, t and n) are valid: $\alpha \leqslant a_\infty \leqslant \beta$, $|b_n| \leqslant \beta - \alpha$, $0 \leqslant U_0 \leqslant M$ and $0 \leqslant R \leqslant 1$. (For simplicity, we drop the index n in writing below.) Using (5) we can get

$$|W_1| \leqslant (\beta - \alpha) M^2 \qquad \text{and} \qquad |U_1| \leqslant (\beta - \alpha) M^2 t\,,$$

and inductively

$$|U_k| \leqslant M \sigma^k \sum_{j=0}^{k-1} c_{k,j} \tau^j\,,$$

where $\sigma := (\beta - \alpha) M t$, $\tau := \beta M t$ and $c_{k,j}$ are some coefficients. Taking $\beta M \leqslant 1$, and $\rho = 1/2$, we obtain an absolutely convergent series.

Indeed, this convergence follows by noting that the numbers $u_0 := 1$, $u_1 := 1$, and inductively:

$$\begin{cases} w_k := \sum_{j=1}^{k-1} u_j u_{k-j} + \sum_{j=0}^{k-1} u_j u_{k-j-1} \\ u_k := \dfrac{w_k}{k} \ , \end{cases}$$

satisfy $u_k \leqslant 2^{k-1}$ for $k \in \mathbf{N}$, while $\sum_{j=0}^{k-1} c_{k,j} \leqslant u_k$. ∎

3. Passing to the limit

Next we would like to describe an accumulation point of the sequence (u_n). Note that for a fixed x, using the (strong) L^1 topology for v and f (as L^∞ is not separable), and for M_0, F given, the restriction of Φ_n to the set of v, f satisfying (2) is Lipschitz continuous with values in $\mathrm{L}^\infty(\langle 0, \infty\rangle)$, the Lipschitz bound depending only on M_0, F, α and β. Using this we can extract a subsequence such that for each v, f satisfying (2), $k \in \mathbf{N}$ and $s_1, \dots, s_k \in \mathbf{R}^+$

$$\begin{aligned} \Phi_n(v,f) &\rightharpoonup u_\infty = \Phi_\infty(v,f) \qquad \mathrm{L}^\infty \text{ weak } * \\ b_n(\cdot, s_1)\cdots b_n(\cdot, s_k) &\rightharpoonup M_k(\cdot, s_1, \dots, s_k) \qquad \mathrm{L}^\infty \text{ weak } * . \end{aligned} \tag{7}$$

This allows us to pass to the limit (as $n \to \infty$) in equation (4), thus obtaining:

$$\begin{cases} \dfrac{\partial U_k^\infty}{\partial t} + 2a_\infty U_0 U_k^\infty = W_k^\infty \\ U_k^\infty(\cdot, 0) = 0. \end{cases} \tag{8}$$

The recurrence relations (6) are still valid for $n = \infty$ if we replace the product $b_n(\cdot, s_1)\cdots b_n(\cdot, s_k)$ by the appropriate weak limit $M_k(\cdot, s_1, \dots, s_k)$. For example:

$$\begin{aligned} W_2^n(s) = 2b_n(s)U_0(s) &\int_0^s R(s_1, s) b_n(s_1) U_0^2(s_1) ds_1 \\ &- a_\infty(s) \int_0^s \int_0^s R(s_1, s) R(s_2, s) b_n(s_1) b_n(s_2) U_0^2(s_1) U_0^2(s_2) ds_1 ds_2 \ , \end{aligned}$$

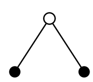
while

$$\begin{aligned} W_2^\infty(s) = 2U_0(s) &\int_0^s R(s_1, s) M_2(s_1, s) U_0^2(s_1) ds_1 \\ &- a_\infty(s) \int_0^s \int_0^s R(s_1, s) R(s_2, s) M_2(s_1, s_2) U_0^2(s_1) U_0^2(s_2) ds_1 ds_2 \ . \end{aligned}$$

Note that, in general, the expressions for weak limits W_k^∞ and U_k^∞ of sequences (W_k^n) and (U_k^n) are formally the same as expressions for W_k^n and U_k^n, the only difference being that in W_k^∞ and U_k^∞ we have $M_j(\cdot, s_1, s_2, ..., s_j)$ instead of the product $b_n(\cdot, s_1) b_n(\cdot, s_2)\cdots b_n(\cdot, s_j)$ in each summation term. So, from the graphs for W_k^n and U_k^n we can easily *read* their weak limits.

Thus we choose to represent W_k^∞ and U_k^∞ by the same graphs as W_k^n and U_k^n.

However, as we want to use graphs for manipulating such expressions, one difficulty does arise: if we want to multiply, say U_1^∞ by U_1^∞ and a_∞, we would expect the graph to be



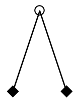
On the other hand, following the rules described in Section 2 and the convention adopted above, this graph describes:

$$a_\infty(t) \int_0^t \int_0^t R(s,t) U_0^2(s) R(s_1, t) U_0^2(s_1) M_2(s, s_1) ds ds_1 \ ,$$

while we actually want:

$$a_\infty(t)\int_0^t R(s,t)U_0^2(s)M_1(s)ds\int_0^t R(s_1,t)U_0^2(s_1)M_1(s_1)ds_1\,.$$

Therefore, we add another general rule: If we multiply two graphs where we have already passed to the limit, we replace the top circle in each child by a rhombus (solid ♦ or empty ♢):



After illustrating this point, let us mention that in fact $M_1 = 0$, as $b_n \xrightarrow{*} 0$. However, for $k \geqslant 2$, M_k will in general be nontrivial; in fact, we shall need this distinction to represent, for example, $a_\infty(U_2^\infty)^2$ below.

For the convenience of the reader, all nine rules of corespondence between multiple integrals and graphs are atated in the Appendix (the remaining three rules will be explained in Section 5).

Let us summarise what we have achieved so far as a lemma.

Lemma 1. *The graphs described above can be used to recursively compute W_k^∞ and U_k^∞ to any desired order k.* ∎

Knowing U_k^∞, we define (after noting that $U_1^\infty = 0$, because of $M_1 = 0$ as shown above)

$$U_\infty(x,t;\gamma) := U_0(x,t) + \sum_{k=2}^{\infty}\gamma^k U_k^\infty(x,t)\,. \tag{9}$$

Our goal is to determine the equation (the macroscopic law) satisfied by $U_\infty(\cdot,\cdot;1)$. We shall not achieve this goal, in the sense as it was possible in the linear case [T2], but only present a method allowing us to write the equation which is correct only to an error of order γ^{K+1}. Multiplying (8) by γ^k, and adding for $k\in\mathbf{N}_0$ (for $k=0$ we use (3)), we get

$$\partial_t U_\infty + a_\infty U_0^2 + 2a_\infty U_0\sum_{k=2}^{\infty}\gamma^k U_k^\infty = \sum_{k=2}^{\infty}\gamma^k W_k^\infty + f\,.$$

Expressing U_0 from (9), one gets the following

$$\begin{aligned}\partial_t U_\infty + a_\infty U_\infty^2 &= f + \sum_{k=2}^{\infty}\gamma^k W_k^\infty + a_\infty\Big(\sum_{k=2}^{\infty}\gamma^k U_k^\infty\Big)^2\\ &= f + \sum_{k=2}^{\infty}\gamma^k W_k^\infty + a_\infty\sum_{k=4}^{\infty}\gamma^k\sum_{j=2}^{k-2}U_j^\infty U_{k-j}^\infty\,.\end{aligned}\tag{10}$$

It is clear that the equation for U_∞ up to an error of order γ^2 is

$$\partial_t U_\infty + a_\infty U_\infty^2 = f + O(\gamma^2)\,,$$

while for the error of order γ^3 we have to add the term $\gamma^2 W_2^\infty$. The problem is that W_2^∞ is just an auxiliary construct; we look for an equation for U_∞ expressed only in terms of functions given at the beginning, while W_2^∞ contains U_0.

4. An example

The expressions we obtain are getting more and more complicated. In order to make the algorithm more transparent, let us compute the various quantities in an example, and see better how to proceed further.

As it was mentioned in the Introduction, if we take a_n independent of t and $f = 0$, we can solve each Cauchy problem (1) explicitly. To specify the example, we take $x \in \Omega := [0,1]$ (the Lebesgue measure assumed) and $a_n(x) := a(nx)$, where $a = \frac{1}{2}\chi_{[0,\frac{1}{2}\rangle} + \frac{3}{2}\chi_{[\frac{1}{2},1\rangle}$ on $[0,1\rangle$, and extended periodically to $\mathbf{R}_0^+$. Thus we have $a_n \xrightarrow{*} 1$, and the whole sequence determines a homogeneous (independent of x) Young measure $\nu = \frac{1}{2}\delta_{\frac{1}{2}} + \frac{1}{2}\delta_{\frac{3}{2}}$.

Further, we take $v = 1$, and the sequence of Cauchy problems (1) reads:

$$\begin{cases} \dfrac{\partial u_n}{\partial t}(x,t) + a_n(x)u_n^2(x,t) = 0 \\ \hfill u_n(x,0) = 1 \ . \end{cases}$$

We have $0 < \frac{1}{2} \leqslant a_n \leqslant \frac{3}{2}$, and

$$u_n(x,t) = \frac{1}{1 + ta_n(x)} \ .$$

Passing to the limit we get

$$u_\infty(t) = \int \frac{1}{1+ta}\, d\nu(a) = \frac{1+t}{(1+\frac{1}{2}t)(1+\frac{3}{2}t)} \ ,$$

while for $b_n = a_n - 1$ we have $b_n \xrightarrow{*} 0$, or for higher powers k

$$b_n^k \xrightarrow{*} \begin{cases} 0\,, & k \text{ odd} \\ \frac{1}{2^k}\,, & k \text{ even} \end{cases} \ . \tag{11}$$

The Cauchy problem (3) for U_0 reduces to

$$\begin{cases} \dfrac{\partial U_0}{\partial t} + U_0^2 = 0 \\ \quad U_0(\cdot,0) = 1, \end{cases}$$

and the solution is independent of x: $U_0(t) = \frac{1}{1+t}$. This gives:

$$R(s,t) = e^{-2\int_s^t \frac{d\sigma}{1+\sigma}} = \left(\frac{1+s}{1+t}\right)^2 \ ,$$

and then $W_1^n(s) = -\frac{1}{1+t}$ and $U_1^n(t) = -\frac{t}{(1+t)^2}$. For W_k^n and U_k^n we get simple expressions (the dependence on x is only through b_n) as well:

Lemma 2.

$$\begin{aligned} W_k^n(s) &= b_n^k(-1)^k\left[\frac{ks^{k-1}}{(1+s)^{k+1}} - \frac{(k-1)s^k}{(1+s)^{k+2}}\right] \\ U_k^n(t) &= b_n^k(-1)^k\frac{t^k}{(1+t)^{k+1}} \ . \end{aligned}$$

Dem. By induction, from (6) we get:

$$W_k^n(s) = -\sum_{j=1}^{k-1}\frac{(-tb_n)^k}{(1+t)^{k+2}} - \sum_{j=0}^{k-1}\frac{(-tb_n)^{k-1}}{(1+t)^{k+1}} = b_n^k(-1)^k\left[\frac{ks^{k-1}}{(1+s)^{k+1}} - \frac{(k-1)s^k}{(1+s)^{k+2}}\right] \ ,$$

and then by integration we get the required expression for U_k^n.

Q.E.D.

Thus we are able to compute the functions U_n:

$$U_n(t;\gamma) = \frac{1}{1+t}\sum_{k=0}^{\infty}\left(\frac{-\gamma t b_n}{1+t}\right)^k = \frac{1}{1+t+\gamma b_n t} \; ,$$

the equality being uniformly valid for $|\gamma| < 2-\varepsilon$, for any $\varepsilon > 0$.

After passing to the limit in n, we have for even k:

$$\begin{aligned} W_k^\infty(s) &= \frac{1}{2^k}\left[\frac{ks^{k-1}}{(1+s)^{k+1}} - \frac{(k-1)s^k}{(1+s)^{k+2}}\right] \\ U_k^\infty(t) &= \frac{1}{2^k}\frac{t^k}{(1+t)^{k+1}} \; , \end{aligned}$$

while for odd k we get noughts, by (11). Thus

$$U_\infty(t;\gamma) = \frac{1}{1+t}\sum_{l=0}^{\infty}\left(\frac{\gamma^2 t^2}{4(1+t)^2}\right)^l = \frac{1+t}{(1+t)^2-\gamma^2t^2/4} \; ,$$

which, for $\gamma = 1$, coincides with u_∞ computed above as the limit of u_n.

The final task is, of course, to determine the right equation satisfied by $u_\infty = U_\infty(\cdot;1)$, or, in the spirit of asymptotic expansion approach, at least the approximate equations satisfied up to certain order $O(\gamma^K)$ (i.e. up to an error of order $O(\gamma^{K+1})$).

Remark. Let us find the equation valid up to order γ^4; to this end we take:

$$U_{(4)} := U_0 + \gamma^2 U_2^\infty + \gamma^4 U_4^\infty = U_\infty(\cdot;\gamma) + O(\gamma^5) \; .$$

In (10) we have to replace all occurences of U_0 and U_k^∞ by $U_{(4)}$, and R by

$$R_\infty(s,t) = e^{-2\int_s^t U_\infty(\sigma)}d\sigma = e^{-2\int_s^t U_{(4)}(\sigma)}d\sigma + O(\gamma^5) \; .$$

As $W_3^\infty = 0$ for this example, the terms of interest are W_2^∞ (up to order γ^2), as well as W_4^∞ and $(U_2^\infty)^2$ (only the term with γ^0).

In the latter case, we have to substitute $U_{(4)}, R_\infty$ and $1/2^k$ (for k even) for U_0, R and M_k in the expressions for W_4^∞ and U_2^∞ obtained in Section 3. For simplicity of notation, we denote R_∞ by $\mathbf{r}$, and $U_{(4)}$ by $\mathbf{u}$ in the formulae below. This gives us:

$$\begin{aligned} U_2^\infty(t) &= \frac{1}{2}\int_0^t\int_0^{s_1} \mathbf{r}(s_1,t)\mathbf{r}(s_2,s_1)\mathbf{u}(s_1)\mathbf{u}^2(s_2)ds_2ds_1 \\ &- \frac{1}{4}\int_0^t\int_0^{s_1}\int_0^{s_1} \mathbf{r}(s_1,t)\mathbf{r}(s_2,s_1)\mathbf{r}(s_3,s_1)\mathbf{u}^2(s_2)\mathbf{u}^2(s_3)ds_3ds_2ds_1 + O(\gamma^2) \; , \end{aligned} \tag{12}$$

and

$$\begin{aligned} W_4^\infty(t) = &-\frac{1}{4}\int_0^t\int_0^t\int_0^{s_1}\int_0^{s_1}\int_0^{s_4}\int_0^{s_4} \mathbf{r}(s_1,t)\mathbf{r}(s_4,t)\mathbf{r}(s_2,s_1)\mathbf{r}(s_3,s_1)\mathbf{r}(s_5,s_4) \\ &\qquad \mathbf{r}(s_6,s_4)\mathbf{u}^2(s_2)\mathbf{u}^2(s_3)\mathbf{u}^2(s_5)\mathbf{u}^2(s_6)ds_6ds_5ds_3ds_2ds_4ds_1 \\ &+ \frac{1}{2}\int_0^t\int_0^t\int_0^{s_1}\int_0^{s_1}\int_0^{s_4} \mathbf{r}(s_1,t)\mathbf{r}(s_4,t)\mathbf{r}(s_2,s_1)\mathbf{r}(s_3,s_1)\mathbf{r}(s_5,s_4)\mathbf{u}^2(s_2)\mathbf{u}^2(s_3)\mathbf{u}(s_4)\mathbf{u}^2(s_5)ds_5ds_3ds_2ds_4ds_1 \\ &+ \frac{1}{2}\int_0^t\int_0^t\int_0^{s_1}\int_0^{s_3}\int_0^{s_3} \mathbf{r}(s_1,t)\mathbf{r}(s_3,t)\mathbf{r}(s_2,s_1)\mathbf{r}(s_4,s_3)\mathbf{r}(s_5,s_3)\mathbf{u}(s_1)\mathbf{u}^2(s_2)\mathbf{u}^2(s_4)\mathbf{u}^2(s_5)ds_5ds_4ds_2ds_3ds_1 \\ &- \int_0^t\int_0^t\int_0^{s_1}\int_0^{s_3} \mathbf{r}(s_1,t)\mathbf{r}(s_3,t)\mathbf{r}(s_2,s_1)\mathbf{r}(s_4,s_3)\mathbf{u}(s_1)\mathbf{u}^2(s_2)\mathbf{u}(s_3)\mathbf{u}^2(s_4)ds_4ds_2ds_3ds_1 \\ &- \frac{1}{2}\int_0^t\int_0^t\int_0^{s_2}\int_0^{s_2} \mathbf{r}(s_1,t)\mathbf{r}(s_2,t)\mathbf{r}(s_3,s_2)\mathbf{r}(s_4,s_2)\mathbf{u}^2(s_1)\mathbf{u}^2(s_3)\mathbf{u}^2(s_4)ds_4ds_3ds_2ds_1 \end{aligned}$$

$$
\begin{aligned}
&- \int_0^t \int_0^t \int_0^{s_2} \int_0^{s_2} \int_0^{s_4} \int_0^{s_4} \mathbf{r}(s_1,t)\mathbf{r}(s_2,t)\mathbf{r}(s_3,s_2)\mathbf{r}(s_4,s_2)\mathbf{r}(s_5,s_4) \\
&\qquad \mathbf{r}(s_6,s_4)\mathbf{u}^2(s_1)\mathbf{u}^2(s_3)\mathbf{u}^2(s_5)\mathbf{u}^2(s_6)ds_6ds_5ds_4ds_3ds_2ds_1 \\
&+ 2\int_0^t \int_0^t \int_0^{s_2} \int_0^{s_2} \int_0^{s_4} \mathbf{r}(s_1,t)\mathbf{r}(s_2,t)\mathbf{r}(s_3,s_2)\mathbf{r}(s_4,s_2)\mathbf{r}(s_5,s_4)\mathbf{u}^2(s_1)\mathbf{u}^2(s_3)\mathbf{u}(s_4)\mathbf{u}^2(s_5)ds_5ds_4ds_3ds_2ds_1 \\
&+ \int_0^t \int_0^t \int_0^{s_2} \int_0^{s_3} \int_0^{s_3} \mathbf{r}(s_1,t)\mathbf{r}(s_2,t)\mathbf{r}(s_3,s_2)\mathbf{r}(s_4,s_3)\mathbf{r}(s_5,s_3)\mathbf{u}^2(s_1)\mathbf{u}(s_2)\mathbf{u}^2(s_4)\mathbf{u}^2(s_5)ds_5ds_4ds_3ds_2ds_1 \\
&- 2\int_0^t \int_0^t \int_0^{s_2} \int_0^{s_3} \mathbf{r}(s_1,t)\mathbf{r}(s_2,t)\mathbf{r}(s_3,s_2)\mathbf{r}(s_4,s_3)\mathbf{u}^2(s_1)\mathbf{u}(s_2)\mathbf{u}(s_3)\mathbf{u}^2(s_4)ds_4ds_3ds_2ds_1 \\
&- \frac{1}{2}\int_0^t \int_0^t \int_0^{s_2} \int_0^{s_2} \mathbf{r}(s_1,t)\mathbf{r}(s_2,t)\mathbf{r}(s_3,s_2)\mathbf{r}(s_4,s_2)\mathbf{u}^2(s_1)\mathbf{u}^2(s_3)\mathbf{u}^2(s_4)ds_4ds_3ds_2ds_1 \\
&+ \int_0^t \int_0^t \int_0^{s_2} \mathbf{r}(s_1,t)\mathbf{r}(s_2,t)\mathbf{r}(s_3,s_2)\mathbf{u}^2(s_1)\mathbf{u}(s_2)\mathbf{u}^2(s_3)ds_3ds_2ds_1 \\
&+ \frac{1}{2}\int_0^t \int_0^{s_1} \int_0^{s_1} \mathbf{r}(s_1,t)\mathbf{r}(s_2,s_1)\mathbf{r}(s_3,s_1)\mathbf{u}(t)\mathbf{u}^2(s_2)\mathbf{u}^2(s_3)ds_3ds_2ds_1 \\
&+ \int_0^t \int_0^{s_1} \int_0^{s_1} \int_0^{s_3} \int_0^{s_3} \mathbf{r}(s_1,t)\mathbf{r}(s_2,s_1)\mathbf{r}(s_3,s_1)\mathbf{r}(s_4,s_3)\mathbf{r}(s_5,s_3)\mathbf{u}(t)\mathbf{u}^2(s_2)\mathbf{u}^2(s_4)\mathbf{u}^2(s_5)ds_5ds_4ds_3ds_2ds_1 \\
&- 2\int_0^t \int_0^{s_1} \int_0^{s_1} \int_0^{s_3} \mathbf{r}(s_1,t)\mathbf{r}(s_2,s_1)\mathbf{r}(s_3,s_1)\mathbf{r}(s_4,s_3)\mathbf{u}(t)\mathbf{u}^2(s_2)\mathbf{u}(s_3)\mathbf{u}^2(s_4)ds_4ds_3ds_2ds_1 \\
&- \int_0^t \int_0^{s_1} \int_0^{s_2} \int_0^{s_2} \mathbf{r}(s_1,t)\mathbf{r}(s_2,s_1)\mathbf{r}(s_3,s_2)\mathbf{r}(s_4,s_2)\mathbf{u}(t)\mathbf{u}(s_1)\mathbf{u}^2(s_3)\mathbf{u}^2(s_4)ds_4ds_3ds_2ds_1 \\
&+ 2\int_0^t \int_0^{s_1} \int_0^{s_2} \mathbf{r}(s_1,t)\mathbf{r}(s_2,s_1)\mathbf{r}(s_3,s_2)\mathbf{u}(t)\mathbf{u}(s_1)\mathbf{u}(s_2)\mathbf{u}^2(s_3)ds_3ds_2ds_1 + O(\gamma^2)
\end{aligned}
$$

For W_2^∞ we have to first replace U_0 and R by $U_{(4)} - \gamma^2 U_2^\infty$ and $R_\infty + \gamma^2 R_\infty Q_2$, where $Q_2(s,t) := 2\int_s^t U_2^\infty(\sigma)d\sigma$; and in the next step replace U_2^∞ by (12). This gives (with the same notational simplification as above)

$$
\begin{aligned}
W_2^\infty(t) = &\frac{1}{2}\int_0^t \mathbf{r}(s_1,t)\mathbf{u}(t)\mathbf{u}^2(s_1)ds_1 \\
&+ \frac{1}{4}\int_0^t \int_0^t \mathbf{r}(s_1,t)\mathbf{r}(s_2,t)\mathbf{u}^2(s_1)\mathbf{u}^2(s_2)ds_1ds_2 \\
&- \frac{1}{4}\gamma^2\int_0^t \int_0^t \int_0^{s_2} \mathbf{r}(s_1,t)\mathbf{r}(s_2,t)\mathbf{r}(s_3,s_2)\mathbf{u}^2(s_1)\mathbf{u}(s_2)\mathbf{u}^2(s_3)ds_3ds_2ds_1 \\
&+ \frac{1}{8}\gamma^2\int_0^t \int_0^t \int_0^{s_2} \int_0^{s_2} \mathbf{r}(s_1,t)\mathbf{r}(s_2,t)\mathbf{r}(s_3,s_2)\mathbf{r}(s_4,s_2)\mathbf{u}^2(s_1)\mathbf{u}^2(s_3)\mathbf{u}^2(s_4)ds_4ds_3ds_2ds_1 \\
&- \frac{1}{2}\gamma^2\int_0^t \int_0^{s_1} \int_0^{s_2} \mathbf{r}(s_1,t)\mathbf{r}(s_2,s_1)\mathbf{r}(s_3,s_2)\mathbf{u}(t)\mathbf{u}(s_1)\mathbf{u}(s_2)\mathbf{u}^2(s_3)ds_3ds_2ds_1 \\
&+ \frac{1}{4}\gamma^2\int_0^t \int_0^{s_1} \int_0^{s_2} \int_0^{s_2} \mathbf{r}(s_1,t)\mathbf{r}(s_2,s_1)\mathbf{r}(s_3,s_2)\mathbf{r}(s_4,s_2)\mathbf{u}(s_1)\mathbf{u}^2(s_3)\mathbf{u}^2(s_4)ds_4ds_3ds_2ds_1 \\
&+ \frac{1}{2}\gamma^2\int_0^t \int_{s_1}^t \int_0^{s_2} \int_0^{s_3} \mathbf{r}(s_1,t)\mathbf{r}(s_3,s_2)\mathbf{r}(s_4,s_3)\mathbf{u}(t)\mathbf{u}^2(s_1)\mathbf{u}(s_3)\mathbf{u}^2(s_4)ds_4ds_3ds_2ds_1 \\
&- \frac{1}{4}\gamma^2\int_0^t \int_{s_1}^t \int_0^{s_2} \int_0^{s_3} \int_0^{s_3} \mathbf{r}(s_1,t)\mathbf{r}(s_3,s_2)\mathbf{r}(s_4,s_3)\mathbf{r}(s_5,s_3)\mathbf{u}(t)\mathbf{u}^2(s_1)\mathbf{u}^2(s_4)\mathbf{u}^2(s_5)ds_5ds_4ds_3ds_2ds_1 \\
&- \frac{1}{2}\gamma^2\int_0^t \int_0^t \int_0^{s_2} \int_0^{s_3} \mathbf{r}(s_1,t)\mathbf{r}(s_2,t)\mathbf{r}(s_3,s_2)\mathbf{r}(s_4,s_3)\mathbf{u}^2(s_1)\mathbf{u}(s_2)\mathbf{u}(s_3)\mathbf{u}^2(s_4)ds_4ds_3ds_2ds_1 \\
&+ \frac{1}{4}\gamma^2\int_0^t \int_0^t \int_0^{s_2} \int_0^{s_3} \int_0^{s_3} \mathbf{r}(s_1,t)\mathbf{r}(s_2,t)\mathbf{r}(s_3,s_2)\mathbf{r}(s_4,s_3)\mathbf{r}(s_5,s_3)\mathbf{u}^2(s_1)\mathbf{u}(s_2)\mathbf{u}^2(s_4)\mathbf{u}^2(s_5)ds_5ds_4ds_3ds_2ds_1 \\
&+ \frac{1}{4}\gamma^2\int_0^t \int_0^t \int_{s_1}^t \int_0^{s_3} \int_0^{s_4} \mathbf{r}(s_1,t)\mathbf{r}(s_2,t)\mathbf{r}(s_4,s_3)\mathbf{r}(s_5,s_4)\mathbf{u}^2(s_1)\mathbf{u}^2(s_2)\mathbf{u}(s_4)\mathbf{u}^2(s_5)ds_5ds_4ds_3ds_2ds_1 \\
&- \frac{1}{8}\gamma^2\int_0^t \int_0^t \int_{s_1}^t \int_0^{s_3} \int_0^{s_4} \int_0^{s_4} \mathbf{r}(s_1,t)\mathbf{r}(s_2,t)\mathbf{r}(s_4,s_3)\mathbf{r}(s_5,s_4)\mathbf{r}(s_6,s_4) \\
&\qquad \mathbf{u}^2(s_1)\mathbf{u}^2(s_2)\mathbf{u}^2(s_5)\mathbf{u}^2(s_6)ds_6ds_5ds_4ds_3ds_2ds_1 + O(\gamma^4)\,,
\end{aligned}
$$

and putting all this together we get an approximation of the equation for u_∞, correct up to order γ^4. Besides the unknown u_∞, this equation involves only arithmetic operations, exponential function, integrals and derivatives (it is a complicated integro-differential equation for u_∞).

Even in this simplified example, it is clear that the classical mathematical notation for integrals, sums and products is useless for such expressions, even for relatively small powers of γ. In the next section we shall describe a much better notation, and rules of manipulation. However, any practical calculations should better be left to computers.

5. Substitutions for the asymptotic expansion

Our immediate goal is to obtain an equation for U_∞ that is correct up to γ^K for some given K. In order to do that we need to replace U_0 in the equation

$$\partial_t U_\infty + a_\infty U_\infty^2 = f + \sum_{k=2}^{\infty} \gamma^k W_k^\infty + a_\infty \sum_{k=4}^{\infty} \gamma^k \sum_{j=2}^{k-2} U_j^\infty U_{k-j}^\infty \tag{10}$$

by $U_\infty - \sum_{k=2}^{\infty} \gamma^k U_k^\infty$. Unfortunately, U_0 appears also in R in a way that is more difficult to handle. First we define:

$$\begin{aligned} Q_k(s,t) :=& 2\int_s^t a_\infty(\sigma) U_k^\infty(\sigma) d\sigma\,, \\ R_\infty(s,t) :=& e^{-2\int_s^t a_\infty(\sigma) U_\infty(\sigma) d\sigma}\,, \end{aligned}$$

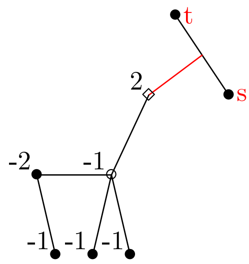
and then note that (for simplicity we omit the variables in writting)

$$\begin{aligned} R &= R_\infty e^{\sum_{k=2}^{\infty} \gamma^k Q_k} \\ &= R_\infty \left(1 + \sum_{m=1}^{\infty} \frac{1}{m!} \Big(\sum_{k=2}^{\infty} \gamma^k Q_k \Big)^m \right) \\ &= R_\infty \left(1 + \sum_{l=2}^{\infty} \gamma^l \sum_{m=1}^{[l/2]} \sum_{\substack{\alpha_2+\cdots+\alpha_{l-2m+2}=m \\ \sum_{i=2}^{l-2m+2} i\alpha_i = l}} \frac{1}{\alpha_2! \cdots \alpha_{l-2m+2}!} Q_2^{\alpha_2} \cdots Q_{l-2m+2}^{\alpha_{l-2m+2}} \right), \end{aligned} \tag{13}$$

where $\alpha_i \in \mathbf{N}_0$. Thus, in order to get a correct equation for U_∞ we need to replace R by the above expression. Since U_k^∞ (as well as Q_k) contains U_0 (and R) we will have to repeat this procedure a sufficient number of times. Note that with each replacement of U_0 and R we achieve that auxilary functions appear in the equation with a higher power of γ than before the replacements. This ensures that after finitely many replacements we obtain an equation for U_∞ such that the terms appearing with γ^k, for $k \leqslant K$, do not contain U_0. It is also clear that our equation is going to be complicated for manipulation (as the number of multiple integrals increases with each replacement), and for this reason we would like to write an equation for U_∞ with the aide of graphs. This task is not going to be trivial, as it will be clear from the algorithm described in the next section.

We note that taking into account the terms of order up to γ^3 (the same is true if we are interested in the terms of order up to γ^2), it is enough to replace U_0 by U_∞ and R by R_∞ in the expressions for W_2^∞ and W_3^∞. Other corrections can be included in $O(\gamma^4)$ term. And these terms can be computed using the graphs described above. The first case was computed in [T2], and the second (using an earlier version of graphs presented here) in [AL]. It is clear that by including the terms with higher powers of γ we change nothing to lower order terms.

How to represent the correction of order γ^4, or higher? Clearly, we have to adjust our graphs to incorporate representation of Q_k, which allways appears in an integral (as Q_k is part of the substitution for R which is in the integrand), and which is a function of two variables, the upper and lower bound of mentioned integral. In a way, Q_k should be attached to a pair of computed vertices, or better, to the edge connecting them. For example, the graph



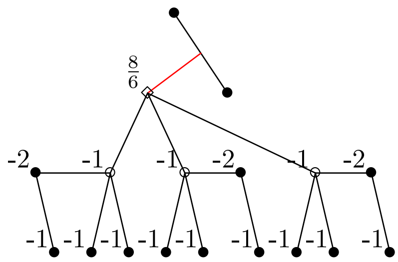
describes

$$\int_0^t R(s,t)Q_2(s,t)M_2(s,t)U_0(t)U_0^2(s)ds = \int_0^t R(s,t)2\int_s^t a_\infty(\sigma)U_2^\infty(\sigma)d\sigma M_2(s,t)U_0(t)U_0^2(s)ds\,.$$

In particular, the red edge and its subgraph describe $Q_2(s,t)$. More precisely, this means that we multiply $2a_\infty$ by U_2^∞, take the integral from s to t, and multiply the integrand over $[0,t]$ (integration in s) by this integral. The graph representation of Q_k we get by replacing the graph for U_2^∞ by the graph for U_k^∞ in the figure representing Q_2.

The rule is that an edge (red coloured above) connected to a pair of computed vertices (actually, to the edge connecting them) contains an integral without R with bounds being the variables (which we will not draw in the future as they are inner variables of integration) that belong to given vertices.

Aditional adjustments should be made in order to represent multiple products of Q_k appearing in expresion (13) for R. A new rule states that if a certain number of graphs is connected in one vertex, then the integrals generated by them are multiplied. Additionally, if a product appears immediately after a red coloured edge, then each factor contains this *red integral* and the vertex below it. For example, the graph



represents

$$\int_0^t R(s,t)\frac{1}{3!}(Q_2(s,t))^3 M_2(s,t)U_0(t)U_0^2(s)ds =$$
$$= \int_0^t R(s,t)\frac{1}{3!}\Big(2\int_s^t a_\infty(\sigma)U_2^\infty(\sigma)d\sigma\Big)^3 M_2(s,t)U_0(t)U_0^2(s)ds\,,$$

of which the red edge and its subgraph stands for $\frac{1}{3!}Q_2Q_2Q_2$. At this point we have a necessary tool (the graphs) for describing an equation correct up to γ^K.

6. Details of the substitution

Next we want to describe an algorithm for systematic replacement of U_0 and R in the expressions.

In order to obtain an equation correct up to the terms with γ^K, we need W_k^∞ and U_k^∞ correct up to order γ^{K-k}. Let us denote by $W_{k,N}^\infty$, $U_{k,N}^\infty$, $Q_{k,N}$ and R_N such N*-correct* expressions obtained form W_k^∞, U_k^∞, Q_k and R, where the terms up to γ^N contain only known functions, or the unknown U_∞.

Let us describe how to choose functions with this property. We use the definitions (we do not write the variable x, and for R_N we suppress all variables):

$$
\begin{aligned}
R_N &:= R_\infty\Bigg(1+\sum_{l=2}^{N}\gamma^l\sum_{m=1}^{[l/2]}\sum_{\substack{\alpha_2+\cdots+\alpha_{l-2m+2}=m\\ \sum_{i=2}^{l-2m+2} i\alpha_i=l}}\frac{1}{\alpha_2!\cdots\alpha_{l-2m+2}!}Q_{2,N-l}^{\alpha_2}\cdots Q_{l-2m+2,N-l}^{\alpha_{l-2m+2}}\Bigg)\\
U_{k,N}^\infty(t) &:= \int_0^t R_N(s,t)W_{k,N}^\infty(s)\,ds\\
Q_{k,N}(s,t) &:= 2\int_s^t a_\infty(\sigma)U_{k,N}^\infty(\sigma)\,d\sigma\,.
\end{aligned}
$$

Let us note that if we were to know $W_{k,N}^\infty$, these definitions would not be recursive, as R_N does not depend on $U_{k,N}^\infty$, but only on $Q_{k,i}$, for $i \leqslant N-2$. How to define $W_{k,N}^\infty$?

We take W_k^∞, and replace all occurences of R by R_N, and U_0 by

$$
U_\infty - \sum_{i=2}^{N}\gamma^i U_{i,N-i}^\infty\,.
$$

Let us look into the details for the first few steps. Starting with W_k^∞, we get

0. $W_{k,0}^\infty$ by replacing U_0 by U_∞ and R by $R_0=R_\infty$; and calculate $U_{k,0}^\infty$, $Q_{k,0}$;

1. $W_{k,1}^\infty$ by replacing U_0 by U_∞ and R by $R_1=R_\infty$; and calculate $U_{k,1}^\infty$, $Q_{k,1}$;

2. $W_{k,2}^\infty$ by replacing U_0 by $U_\infty-\gamma^2U_{2,0}^\infty$ and R by $R_2=R_\infty(1+\gamma^2Q_{2,0})$; and calculate $U_{k,2}^\infty$, $Q_{k,2}$;

3. $W_{k,3}^\infty$ by replacing U_0 by $U_\infty-\gamma^2U_{2,1}^\infty-\gamma^3U_{3,0}^\infty$ and R by $R_3=R_\infty(1+\gamma^2Q_{2,1}+\gamma^3Q_{3,0})$; and calculate $U_{k,3}^\infty$, $Q_{k,3}$;

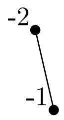
4. $W_{k,4}^\infty$ by replacing U_0 by $U_\infty-\gamma^2U_{2,2}^\infty-\gamma^3U_{3,1}^\infty-\gamma^4U_{4,0}^\infty$ and R by $R_4=R_\infty(1+\gamma^2Q_{2,2}+\gamma^3Q_{3,1}+\gamma^4Q_{4,0}+\frac{1}{2!}\gamma^4Q_{2,0}^2)$; and calculate $U_{k,4}^\infty$, $Q_{k,4}$.

As $W_{k,N}^\infty$ clearly depends only on other terms of correctness at most $N-2$, this definition is good. Using induction we can easily prove

Lemma 3. *In the above defined $W_{k,N}^\infty$, $U_{k,N}^\infty$, $Q_{k,N}$ and R_N, no term of order γ^i, for $i \leqslant N$, contains any auxiliary functions.* ∎

Let us try this algorithm on an example and illustrate it by graphs. First note that $U_{k,0}^\infty$, $W_{k,0}^\infty$ and $Q_{k,0}$ can be represented by the same graphs as U_k^∞, W_k^∞ and Q_k respectively (as we have only replaced U_0 by U_∞ and R by R_∞). The same holds for $U_{k,1}^\infty$, $W_{k,1}^\infty$ and $Q_{k,1}$ (actually $U_{k,1}^\infty=U_{k,0}^\infty$, $W_{k,1}^\infty=W_{k,0}^\infty$ and $Q_{k,1}=Q_{k,0}$). So, let us try to demonstrate the second step of the algorithm on one part of the graph for W_2^∞:

(14)



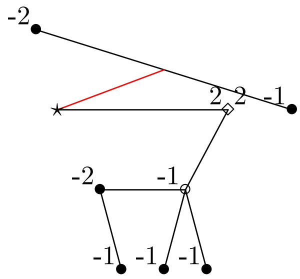
the one representing

$$
2\int_0^t R(s,t)M_2(s,t)U_0(t)U_0^2(s)ds\,.
$$

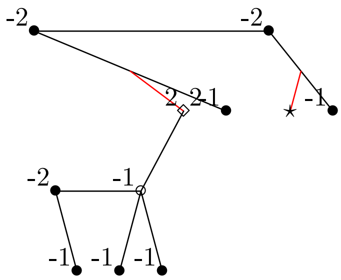
Here U_0 appears three times and needs to be replaced by $U_\infty-\gamma^2U_{2,0}^\infty$, while R appears once and should be replaced by $R_2=R_\infty(1+\gamma^2Q_{2,0})$. For the time being, let us just replace R, which gives

$$
\begin{aligned}
&2\int_0^t \Big(R_\infty(s,t)(1+\gamma^2 Q_{2,0}(s,t))\Big) M_2(s,t)U_0(t)U_0^2(s)ds = \\
&\quad = 2\gamma^2\int_0^t R_\infty(s,t)Q_{2,0}(s,t)M_2(s,t)U_0(t)U_0^2(s)ds + 2\int_0^t R_\infty(s,t)M_2(s,t)U_0(t)U_0^2(s)ds\,,
\end{aligned}
\tag{15}
$$

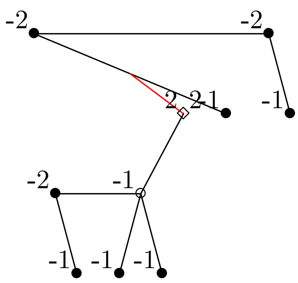
and we represent it by



The star ($\star$) in a vertex stands for R_∞, and the integral (edge) that connects a vertex denoted by star to its parent edge does not apply to this vertex, but only to the terms that are on the same summation line with it (if there are any). Also, such a vertex with a star will never have any children, but we do not assume that there are two U_0 in the appropriate variable. After partial distribution of multiplication over addition, the above graph becomes



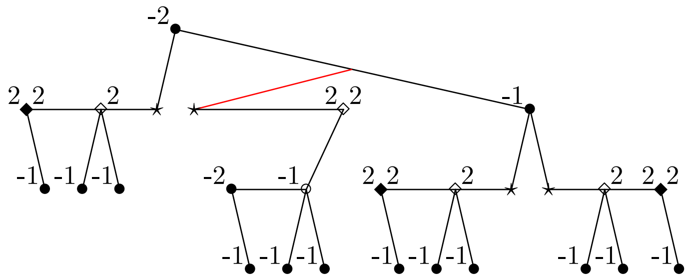
The first part in summation represents the first term in the right hand side of (15), while the second part stands for the second term. Note that since we have replaced all R's in the graph (14) (actually, there was only one R there) and therefore only R_∞ can appear in the above graph, we can omit writing stars for R_∞ and draw the above graph as



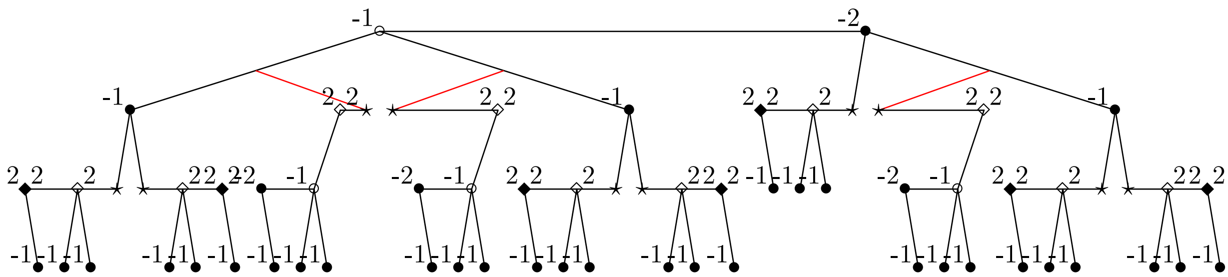
Therefore, such a vertex denoted by a star will be used only in this intermediary step while doing replacements, and after performing the complete distribution in order to eliminate higher order terms and write an approximate equation for U_∞, we will not need it any more. However, we will use it in replacements of U_0 in a similar way. The general rule is (as it is used only in an intermediary step, it is not written in the Appendix):

The star ($\star$) in a vertex stands for U_∞ or R_∞, depending whether it is connected to a vertex or an edge, and the integral (edge) that connects such a vertex to its parent vertex (or parent edge) does not apply to this vertex, but only to terms that are on the same summation line with it (if there are any). We assume that there is U_∞ instead of U_0 in the appropriate vertex, and R_∞ instead of R in the appropriate edge. Also, such a vertex will never have any children, but we do not assume that there are two U_0 in the appropriate variable.

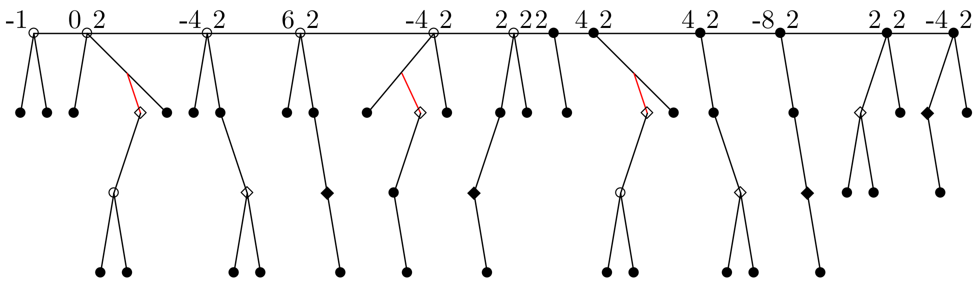
Now, if we make all needed replacements of U_0 and R in (14) we get



For all terms of $W^\infty_{2,2}$ we get a representation:



Note that after distributing, there will be some terms with powers 4 and higher present. These terms are not of interest for us, as the purpose of calculating $W^\infty_{2,2}$ is to get all second (and lower) order terms that can be derived from W^∞_2. Thus, distributing the products over sums, eliminating the terms with powers greater than 2, and using symmetries, we get



The above graph represents all second order terms which can be derived from W^∞_2.

7. The equation correct to any order K

Theorem 1. *The graphs described above can be used to compute equation for U_∞ that is correct to any desired order K.*

Dem.

From (10) we easily get

$$\partial_t U_\infty + a_\infty U_\infty^2 = f + L_K + O(\gamma^{K+1})\,,$$

where

$$L_K = \sum_{k=2}^{K} \gamma^k W^\infty_{k,K-k} + a_\infty \sum_{k=4}^{K} \gamma^k \sum_{j=2}^{k-2} U^\infty_{j,K-k} U^\infty_{k-j,K-k}\,.$$

Note that L_K contains terms with powers of γ that are higher than K. There is only one step to the required equation

$$\partial_t U_\infty + a_\infty U_\infty^2 = f + \bar{L}_K \,,$$

where $\bar{L}_K = L_K + O(\gamma^{K+1})$, and $\bar{L}_K$ does not contain the terms with powers of γ higher than K.

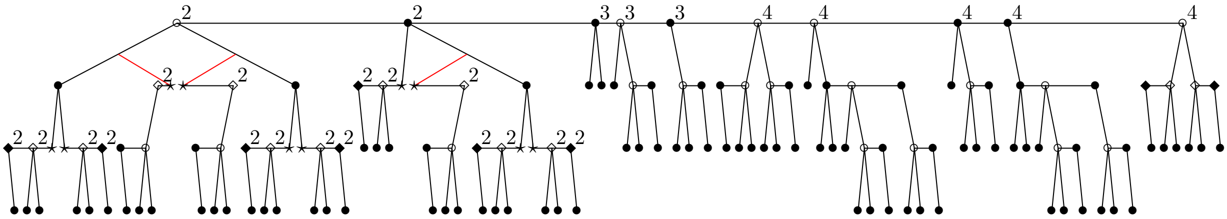
For given L_K, in order to truncate the terms with higher powers, it is enough to distribute the products over sums, and then just drop the terms with powers of γ higher than K.

Q.E.D.

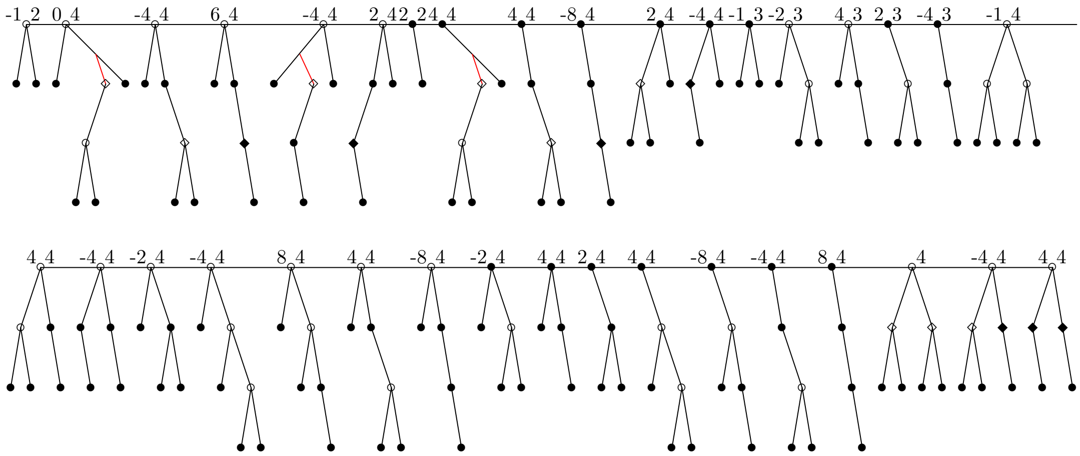
For example,

$$L_4 = \gamma^2 W_{2,2}^\infty + \gamma^3 W_{3,1}^\infty + \gamma^4 W_{4,0}^\infty + a_\infty U_{2,0}^\infty U_{2,0}^\infty$$

is represented by



(we only write the powers in the above graph), while after distribution, removal of higher powers, and taking symmetries into account we get



The above graph represents $\bar{L}_4$—the right hand side of equation for U_∞ which is correct up to γ^4.

As a conclusion, let us remark that homogenisation problems which introduce memory effects are difficult, and despite three decades of research, the available results are still restricted to particular types of equations. In this paper we hope that we made a step forward towards homogenisation of nonlinear transport equations in the non-periodic setting.

We have shown that asymptotic expansions for U_∞ can be manipulated up to order 5 by humans, and to about order 10 by the aid of a personal computer. The details of the computer algorithm will be presented elsewhere [AB].

For the starting problem in homogenisation, the graphs are getting overly complicated. Although we did not prove that, we are quite certain that this is not caused by the low efficiency of our representation (graphs), or ineffective algorithms, but by the form of the expansion chosen. However, we hope that our method might be of use in some practical situations, where for a particular problem the approximate equation could be determined.

It appears that another form of expansion should be sought, which we do not know at the time of this writting, and then a modification of our method could be applied.

Appendix

For the correspondence between multiple integrals and the graphs we use the following rules:

1. A black circle in a vertex denotes function b in an appropriate variable; if the vertex has no children, there is U_0^2 in the appropriate variable; the coefficient is given at the top (left) of the vertex (if there is no coefficient in the vertex, we take it to be 1).

2. An empty circle stands for function a, the rest being the same as in 1.

3. The edge $|_s^t$ denotes $\int_0^t R(s,t)F(s)ds$, where $F(s)$ is the formula represented by the subtree.

4. A product of type U_iU_j we represent by a tree such that the subtrees of the root are graphs for U_i and U_j.

5. If we have a vertex with a solid or empty circle, and if it has only one child, we assume that there is U_0 in the appropriate variable as the second child.

6. If we have already (separately) passed to the limit in subtrees of some graph presenting a product, then instead of circles (solid or empty) we would write rhombi in the leading child vertices.

7. A red edge conected to the edge conecting a pair of vertices contains an integral without R, with the bounds being the variables that belong to given vertices.

8. If a certain number of graphs is connected in one vertex, then the integrals generated by them are multiplied.

9. If a product appears immediately after a red coloured edge then each factor contains this *red* integral and the vertex below it.

Note: The explicit writting of the variables in vertices can be ommited, as they are dummy variables of integration, and only the root variable is important in the result.

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