1 Jednadžbe

1.1 Simetrične jednadžbe

Pripadni polinom simetrične jednadžbe parnog stupnja se može predočiti u varijabli $x+\frac{1}{x}$. Pripadni polinom simetrične jednadžbe neparnog stupnja djeljiv je s $x+1$, a kvocijent je simetrični polinom parnog stupnja.

$$\begin{aligned} a^3 \pm b^3 &= (a \pm b)(a^2 \mp ab + b^2) \\ a^5 \pm b^5 &= (a \pm b)(a^4 \mp a^3b + a^2b^2 \mp ab^3 + b^4) \\ a^{2n+1} \pm b^{2n+1} &= (a \pm b)(a^{2n} \mp a^{2n-1}b + a^{2n-2}b^2 \mp \cdots \mp ab^{2n-1} + b^{2n}) \end{aligned}$$

Zadatak 1. Riješite sljedeće simetrične jednadžbe parnog stupnja:

1. $2x^4 - 3x^3 - x^2 - 3x + 2 = 0$
2. $6x^4 + 5x^3 - 38x^2 + 5x + 6 = 0$
3. $18x^4 - 21x^3 - 94x^2 - 21x + 18 = 0$
4. $x^6 - 6x^5 + 14x^4 - 18x^3 + 14x^2 - 6x + 1 = 0$
5. $x^6 - 16x^4 - 30x^3 - 16x^2 + 1 = 0$

Zadatak 2. Riješite sljedeće simetrične jednadžbe neparnog stupnja:

1. $2x^3 - 3x^2 - 3x + 2 = 0$
2. $2x^5 + 5x^4 - 13x^3 - 13x^2 + 5x + 2 = 0$
3. $3x^5 - 7x^4 - 4x^3 - 4x^2 - 7x + 3 = 0$
4. $8x^5 - 6x^4 - 83x^3 - 83x^2 - 6x + 8 = 0$
5. $x^7 - x^6 - 3x^5 + 3x^4 + 3x^3 - 3x^2 - x + 1$

1.2 Jednadžbe s apsolutnim vrijednostima

Zadatak 3. U skupu $\mathbb{R}$ riješite sljedeće jednadžbe:

1. $|5x + 1| = 2$
2. $||x - 3| - 5| = 4$
3. $\dfrac{1 + |x|}{1 - |x|} - \dfrac{1 - |x|}{1 + |x|} = 0$
4. $|3 - x| = 2x - 1$

5. $|2x+3| = 4x$
6. $\left|4 + \frac{x}{2}\right| = \frac{x-1}{3}$
7. $\frac{|x-2|}{4} + x = 4|x-2|$
8. $|4x-3| = x^2$
9. $|x-1| + |x-2| = 1$
10. $\left|\frac{x}{x+1}\right| - \left|\frac{2x}{1+x}\right| = -1$
11. $|x+1| - |2x-3| = 2$
12. $|4x-1| - |2x-3| + |x-2| = 0$
13. $|2x-3| + |x+1| = 5x+1$
14. $|x-2| - |x+4| = 2x$
15. $\left|x + \left|\frac{1}{2}x + \left|\frac{1}{3}x - 1\right|\right|\right| = 1$
16. Za koje vrijednosti parametra $a \in \mathbb{R}$ jednadžba $|2x+3| + |2x-3| = ax+6$ ima barem dva rješenja?
17. $x^2 - |x-1| - 1 = 0$
18. $|x^2 + 5x + 4| = -x$
19. $|x^2 - x - 2| = x^2 - x - 2$
20. $\left|\frac{x-3}{x+2}\right| = \frac{x-3}{x+2}$
21. $|x^3 + 5x^2 + 2x - 5| = 3$

1.3 Racionalne i iracionalne jednadžbe

Zadatak 4. Riješite sljedeće jednadžbe:

1. $\frac{2}{x} - \frac{3x}{9-x^2} = \frac{5x-15}{x^2+6x+9}$
2. $\frac{x-2}{2x^2+4x} - \frac{x+2}{3x^2-6x} = \frac{4}{4-x^2}$
3. $\frac{x+1}{x-1} - \frac{x-1}{x+1} = \frac{16}{x^2-1}$
4. $\frac{1}{x+2} + \frac{1}{x+3} = \frac{1}{x+1} + \frac{1}{x+4}$

5. $\dfrac{1}{1-4x^2}+\dfrac{x+1}{2x^2+x}=\dfrac{x-1}{2x^2-x}$

6. $1-\dfrac{x}{1+\frac{x}{1-x}}=x^2$

Zadatak 5. Uz raspravu o ovisnosti rješenja o realnim parametrima a i b riješite sljedeće jednadžbe:

1. $\dfrac{a+\frac{x}{1-a}}{a-\frac{x}{1+a}}=\dfrac{1+a}{1-a}$

2. $\dfrac{x-a}{x^2-1}+\dfrac{a}{x^2+x}=\dfrac{x+a}{x^2-x}$

3. $\frac{x}{a^2}+\frac{x}{b^2}+\frac{x}{ab}=a^3-b^3$

4. $\frac{x+a}{a-b}+\frac{x-a}{a-b}=0$

5. $\frac{a-5}{x+1}-\frac{7+3a}{x-2}=\frac{2ax-5}{x^2-x-2}$

6. $\frac{1}{a-2}-\frac{1}{a(a-2)}=\frac{2}{(a-2)x}+\frac{1}{ax(a-2)}$

Napomena.

$$\sqrt[2n]{f(x)}=g(x)\Leftrightarrow\left[f(x)=(g(x))^{2n}\wedge g(x)\geq 0\right],\quad n\in\mathbb{N}.$$

Zadatak 6. Riješite sljedeće jednadžbe:

1. $\sqrt{x+7}=x+1$

2. $\sqrt{4-x}+\sqrt{5+x}=3$

3. $\sqrt{x-1}+\sqrt{2x+5}=8$

4. $\sqrt{2x+1}+\sqrt{x-3}=4$

5. $\sqrt{2x+6}-\sqrt{x-1}=\sqrt{3x-11}$

Zadatak 7. Riješite sljedeće jednadžbe:

1. $\sqrt{3+\sqrt{x-3}}=2$

2. $\sqrt{2-\sqrt{2x-1}}=1$

3. $\sqrt{3-\sqrt{2+\sqrt{3x+1}}}=1$

4. $\sqrt{6-\sqrt{5-\sqrt{3-\sqrt{x+1}}}}=2$

5. $\sqrt{x-1}+1=\sqrt{x+\sqrt{x+2}}$

6. $\sqrt{x+\sqrt{x+11}}+\sqrt{x-\sqrt{x+11}}=4$

Zadatak 8. Riješite sljedeće jednadžbe:

1. $\sqrt{x^2+2x}=2x-1$
2. $\sqrt{x^2+5x+3}+2=4-x$
3. $\sqrt{2x+3}\cdot\sqrt{2x-3}=4$
4. $\sqrt{x+1}\cdot\sqrt{5-x}=\sqrt{x+3}\cdot\sqrt{4-x}$
5. $\dfrac{1}{1-\sqrt{1-x}}-\dfrac{1}{1+\sqrt{1-x}}=\dfrac{\sqrt{3}}{x}$
6. $\dfrac{x-1}{\sqrt{x}+1}=4+\dfrac{\sqrt{x}-1}{2}$
7. $\sqrt{x+\sqrt{x+3}}=3$
8. $\sqrt{4x+13}+\sqrt{x+1}=\sqrt{3x+12}$
9. $2x=19-\sqrt{3x+4}$
10. $\sqrt{x+1}-\sqrt[3]{2x-6}=2$

Zadatak 9. Riješite sljedeće jednadžbe:

1. $\sqrt[3]{x+2}-\sqrt[3]{x-5}=1$
2. $\sqrt[3]{3-x}+\sqrt[3]{6+x}=3$
3. $\sqrt[3]{x+1}+\sqrt[3]{3x+1}=\sqrt[3]{x-1}$
4. $\sqrt[3]{x+1}+\sqrt[3]{2x-3}=\sqrt[3]{3x-2}$

1.4 Eksponencijalne i logaritamske jednadžbe

Zadatak 10. Riješite sljedeće jednadžbe:

1. $3^{\frac{3x-5}{3-x}}=3^{2x}$
2. $\left(\dfrac{1}{3}\right)^{x}\left(\dfrac{3}{4}\right)^{x+1}\left(\dfrac{4}{9}\right)^{x+2}=12$
3. $2^{x^2+4x}-4^{-x^2+2}=0$
4. $4^{\frac{1}{2}x+2}=8^{-\frac{1}{2}x+2}$
5. $3^{2x-3}\cdot 2^{4x-10}=0.75\cdot 12^{x}$
6. $2^{x+1}\cdot 5^{2x-1}=40\cdot 0.1^{x+2}$

7. $27 \cdot \sqrt{3 \cdot 9^{2x-1}} = 9 \cdot 3^{x+1}$
8. $5^x + 3 \cdot 5^{x-2} = 140$
9. $5 \cdot 3^{2x-1} + 9^x = 8$
10. $4^x + 4^{x+1} + 4^{x+2} = 100$

Zadatak 11. Riješite sljedeće jednadžbe:

1. $9^x - 3^{x+1} = 4$
2. $7^{2x} - 7^{x-2} = 1$
3. $36^x = 3^{x+2} \cdot 2^x - 18$
4. $5 \cdot 2^{2x+2} - 2 \cdot 6^{x+1} = 3^{2x+3}$
5. $2^x + 2 \cdot 2^{-x} = 3$
6. $2^x + 2^{x+1} + 2^{x+2} = 7^{x-2} + 7^{x-1}$
7. $3^{2x-1} + 9^{x+1} = 2^{2x-1} + 4^{x+1}$
8. $\dfrac{2^x - 2^{-x}}{2^x + 2^{-x}} = \dfrac{1}{3}$
9. $\left(\dfrac{3}{5}\right)^{x-1} + \left(\dfrac{3}{5}\right)^{1-x} = 2$
10. $\left(\dfrac{3}{4}\right)^{x-1} \cdot \left(\dfrac{4}{3}\right)^{\frac{2}{x}} = \dfrac{16}{9}$
11. $|3^{x+2} - 1| = 3^{x+2} - 1$
12. $0.2 \cdot 5^{|2-x|} = 25^{x-1}$
13. $0.4^{|2x^2-3|} = 2.5^x$

Zadatak 12. Riješite sljedeće jednadžbe:

1. $\log_4(x+6) - \log_4(x-1) = \log_4 10 - \log_4 2$
2. $\log(2x+5) - \log(3x-1) = \log(4x+8) - \log(6x-7)$
3. $2 - \log_{\frac{1}{4}} x = \log_4\left(x^2 + 10x + 6\right)$
4. $\dfrac{1}{2}\log_5(x-2) = 3\log_5 2 - \dfrac{3}{2}\log_5(x-2)$
5. $3\sqrt{\log x} + 2\log\sqrt{\dfrac{1}{x}} = 2$
6. $\log(7x-9)^2 + \log(3x-4)^2 = 2$

7. $\log x^4 + \log(x+4)^4 = 4\log 3$

Zadatak 13. Riješite sljedeće jednadžbe:

1. $\dfrac{1}{5-\log x} + \dfrac{4}{1+\log x} = 3$
2. $\dfrac{2\log x}{\log x - 1} - \log x = \dfrac{2}{\log x - 1}$
3. $\log^2 x + 2\log(0.1x) = 1$
4. $\log(0.1x^2) \cdot \log \dfrac{10}{x} = -3$
5. $\log[3 + 2\log(x+1)] = 0$
6. $\log_4[4 - 2\log_5(4-x)] = \dfrac{1}{2}$
7. $\log_3\{1 + \log_2[1 + \log_4(1 + \log_{\frac{1}{2}} x)]\} = 0$
8. $\log_{16} x + \log_8 x + \log_2 x = \dfrac{19}{36}$
9. $\log_{125} x + \log_{25} x + \log_5 x = \dfrac{11}{6}$
10. $\log_{\frac{1}{2}} x \cdot \log_2 x \cdot \log_4 x = 4$
11. $\log_3 x \cdot \log_9 x \cdot \log_{27} x \cdot \log_{81} x = \dfrac{2}{3}$
12. $\dfrac{1}{\log_x 2} \cdot \dfrac{1}{\log_{2x} 2} = \dfrac{1}{\log_{4x} 2}$

Zadatak 14. Riješite sljedeće jednadžbe:

1. $\log_3(3^x - 8) = 2 - x$
2. $\log_5(5^x - 4) = 1 - x$
3. $\log(2^x + 1) + x = x\log 5 + \log 6$
4. $\log_2(4^x + 4) = x + \log_2(2^{x+1} + 3)$
5. $\log_5(5^x + 1) \cdot \log_5(5^{x+1} + 5) = 12$
6. $|2 + \log_{\frac{1}{5}} x| + 3 = |1 + \log_5 x|$
7. $x^{2\log x - 5} = 0.01$
8. $50^{\log x} \cdot 60^{\log x} = 400$
9. $|x-1|^{\log_3 x^2 - 2\log_x 9} = (x-1)^2$

Zadatak 15. Riješite sljedeće sustave jednadžbi:

1. $\log_x y + \log_y x = \dfrac{5}{2}, \quad xy = 8$
2. $\log_2 y = \log_4(xy - 2), \quad \log_9 x^2 + \log_3(x - y) = 1$
3. $5^{\log x} = 3^{\log y}, \quad (3x)^{\log 3} = (5y)^{\log 5}$

1.5 Trigonometrijske jednadžbe

Napomena.

1. $\sin^2\alpha + \cos^2\alpha = 1$
2. $\sin 2\alpha = 2\sin\alpha\cos\alpha$
   $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$
   $$\mathrm{tg}2\alpha = \frac{2\mathrm{tg}\alpha}{1 - \mathrm{tg}^2\alpha}$$
   $$\mathrm{ctg}2\alpha = \frac{\mathrm{ctg}^2\alpha - 1}{2\mathrm{ctg}\alpha}$$
3. $\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$
   $\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$
   $$\mathrm{tg}(\alpha \pm \beta) = \frac{\mathrm{tg}\alpha \pm \mathrm{tg}\beta}{1 \mp \mathrm{tg}\alpha\mathrm{tg}\beta}$$
   $$\mathrm{ctg}(\alpha \pm \beta) = \frac{\mathrm{ctg}\alpha\mathrm{ctg}\beta \mp 1}{\mathrm{ctg}\beta \pm \mathrm{ctg}\alpha}$$

Napomena. Osnovne trigonometrijske jednadžbe su oblika

$$\sin x = p, \quad \cos x = p, \quad \mathrm{tg}\, x = r, \quad \mathrm{ctg}\, x = r, \ |p| \leq 1.$$

Njihova rješenja su:

a) $\sin x = p \Longrightarrow x_{1k} = \arcsin p + 2k\pi,\ k \in \mathbb{Z}$ i $x_{2k} = \pi - \arcsin p + 2k\pi,\ k \in \mathbb{Z}$

b) $\cos x = p \Longrightarrow x_{1k} = \arccos p + 2k\pi,\ k \in \mathbb{Z}$ i $x_{2k} = -\arccos p + 2k\pi,\ k \in \mathbb{Z}$

c) $\mathrm{tg}\, x = r \Longrightarrow x_k = \mathrm{arctg}\, r + k\pi,\ k \in \mathbb{Z}$

d) $\mathrm{ctg}\, x = r \Longrightarrow x_k = \mathrm{arcctg}\, r + k\pi,\ k \in \mathbb{Z}$

Zadatak 16. Riješite sljedeće trigonometrijske jednadžbe:

1. $2\sin(x - \dfrac{\pi}{3}) = \sqrt{3}$
2. $\dfrac{\sqrt{3}}{\cos(3x - \frac{\pi}{3})} = 2$

3. $3\mathrm{tg}\,(x-\frac{\pi}{6})=-\sqrt{3}$
4. $\mathrm{tg}\,(\frac{\pi}{3}+x)+\mathrm{ctg}\,(\frac{\pi}{6}-x)=\frac{2}{\sqrt{3}}$
5. $\cos(\frac{\pi}{4}+x)\cdot\sin(\frac{\pi}{4}+x)=\frac{3}{4}$
6. $\sin x+\sin 3x=0$
7. $\sin 5x=\cos 10x$
8. $\sin^3 2x-\sin 2x=0$
9. $\sin 2x+\sqrt{3}\cos 2x=0$
10. $\sin(\frac{\pi}{2}+x)+\mathrm{ctg}\,(2\pi-x)=0$
11. $2\sin^2 x+5\sin x+2=0$
12. $2\cos^2 x-3\sin x=0$
13. $\mathrm{tg}^2 x-(\sqrt{3}+1)\mathrm{tg}\,x+\sqrt{3}=0$
14. $2\sin^2 2x-\cos^2 2x=5\cos 2x$
15. $11\mathrm{ctg}\,x-5\mathrm{tg}\,x=\frac{16}{\sin x}$
16. $\mathrm{tg}^2 x+\mathrm{ctg}^2 x=2$
17. $4\cos^4 3x+8=11\sin^2 3x$

Napomena. Homogena trigonometrijska jednadžba ima oblik

$$a_n\sin^n x+a_{n-1}\sin^{n-1}x\cos x+\cdots+a_1\sin x\cos^{n-1}x+a_0\cos^n x=0,$$

gdje su $a_0,a_1,\ldots,a_n$ realni brojevi, a rješava se dijeljenjem s $\cos^n x$.

Zadatak 17. Riješite sljedeće homogene trigonometrijske jednadžbe:

1. $2\sin^2 x-5\sin x\cos x+3\cos^2 x=0$
2. $3\sin^2 x-4\sin x\cos x+5\cos^2 x=2$
3. $\sin^2 x+9\cos^2 x=5\sin 2x$
4. $8\sin^2 x+\sin x\cos x+\cos^2 x=4$
5. $2\sin^4 2x-9\sin^3 2x\cdot\cos 2x+7\sin^2 2x\cdot\cos^2 2x=0$

Zadatak 18. Riješite sljedeće trigonometrijske jednadžbe:

1. $2 + \cos 4x = 2\sin^2 x$
2. $\sin 3x \cdot \sin x + 1 = 0$
3. $\mathrm{tg}^3\, x + \mathrm{tg}^2\, x - 2\mathrm{tg}\, x - 2 = 0$
4. $\dfrac{1 - \cos x}{\sin x} + \mathrm{tg}^3 \dfrac{x}{2} = 0$
5. $\mathrm{tg}\, 3x \cdot \cos x = 0$
6. $\cos x - 2\cos 3x + \cos 5x = 0$
7. $\sin^2 x + \cos^2 2x + \sin^2 3x = \dfrac{3}{2}$
8. $\dfrac{1}{\cos x} + \dfrac{1}{\sin x} = 2\sqrt{2}$
9. $\sin 13x + \cos 13x = \sqrt{2}\sin 17x$

Zadatak 19. Riješite sljedeće trigonometrijske jednadžbe:

1. $|\sin x| = \sin x + 2$
2. $|\sin(2x)| = \cos x$
3. $|\mathrm{tg}x| = \mathrm{tg}x - \dfrac{1}{\cos x}$

Napomena. Linearne trigonometrijske jednadžbe imaju oblik

$$a\sin x + b\cos x = c, \quad a, b, c \in \mathbb{R},\ a, b \neq 0 \tag{1}$$

a mogu se rješavati na sljedeće načine:

1. Podijelimo li ovu jednadžbu s $\sqrt{a^2 + b^2}$ dobivamo:

   $$\frac{a}{\sqrt{a^2+b^2}}\sin x + \frac{b}{\sqrt{a^2+b^2}}\cos x = \frac{c}{\sqrt{a^2+b^2}}.$$

   Kako je

   $$\left|\frac{a}{\sqrt{a^2+b^2}}\right|, \left|\frac{b}{\sqrt{a^2+b^2}}\right| \leq 1$$

   postoji kut φ takav da vrijedi

   $$\cos\varphi = \frac{a}{\sqrt{a^2+b^2}}, \quad \sin\varphi = \frac{b}{\sqrt{a^2+b^2}}.$$

   Gornja jednadžba prelazi u

   $$\sin(x + \varphi) = \frac{c}{\sqrt{a^2+b^2}}.$$

2. Jednadžba (1) se može riješavati i uvođenjem univerzalne zamjene

$$t = \operatorname{tg} \frac{x}{2}.$$

(Tu se mora provjeriti je li $x = \pi + 2k\pi$, $k \in \mathbb{Z}$, (tj. x sa svojstvom $\cos \frac{x}{2} = 0$) rješenje jednadžbe.) Tada je

$$\sin x = \frac{2t}{1+t^2}, \quad \cos x = \frac{1-t^2}{1+t^2}$$

i jednadžba se svodi na kvadratnu po nepoznanici t.

Zadatak 20. Riješite sljedeće linearne jednadžbe:

1. $\sin x - 3\cos x = 1$
2. $\sqrt{3}\cos x - \sin x = 0$
3. $2\sin x - \cos x = 1$
4. $8\sin x - 15\cos x = 17$
5. $3\sin x + 4\cos x = 6$
6. $\sin x + 7\cos x = -7$
7. $2\sqrt{3}\sin^2 x - \sin 2x = 2\sin x$

Napomena. Jednadžbu oblika

$$a(\sin x + \cos x) + b\sin x \cdot \cos x + c = 0$$

zovemo **simetričnom** jednadžbom i rješavamo ju uvodeći supstituciju $t = \sin x + \cos x$, a kvadriranjem dobivamo $\frac{t^2-1}{2} = \sin x \cdot \cos x$. Tako se početna jednadžba svodi na kvadratnu po nepoznanici t.

Zadatak 21. Riješite sljedeće jednadžbe:

1. $2\sin x \cdot \cos x - \sin x - \cos x = 1$
2. $3(\sin x + \cos x) + 2\sin x \cdot \cos x - 3 = 0$
3. $\operatorname{tg} x + 1 = \frac{1}{\cos x} - 2\sin(x)$

Zadatak 22. Riješite sljedeće sustave jednadžbi:

1. $\sin^2 x + \sin^2 y = \frac{1}{2}, \quad x - y = \frac{4\pi}{3}$
2. $\sin x + \sin y = 1, \quad x + y = \frac{\pi}{2}$
3. $\sin x \cdot \sin y = \frac{1}{4}, \quad \cos x \cdot \cos y = \frac{3}{4}$
4. $\operatorname{tg}x + \operatorname{tg}y = 5, \quad \operatorname{tg}(x+y) = -1$