2 Nejednadžbe

Napomena.

$$\begin{aligned} 0 \le a \le b &\Leftrightarrow 0 \le a^{2n} \le b^{2n}, \quad n \in \mathbb{N} \\ a \le b \le 0 &\Rightarrow a^{2n} \ge b^{2n} \ge 0, \quad n \in \mathbb{N} \\ a \le b &\Leftrightarrow a^{2n+1} \le b^{2n+1}, \quad n \in \mathbb{N} \end{aligned}$$

$$\begin{aligned} a \cdot b \ge 0 &\Leftrightarrow (a \ge 0 \wedge b \ge 0) \vee (a \le 0 \wedge b \le 0) \\ a \cdot b \le 0 &\Leftrightarrow (a \ge 0 \wedge b \le 0) \vee (a \le 0 \wedge b \ge 0) \end{aligned}$$

Za $a > 0$ vrijedi

$$\begin{aligned} x^2 \le a^2 &\Leftrightarrow |x| \le a \Leftrightarrow -a \le x \le a \\ x^2 \ge a^2 &\Leftrightarrow |x| \ge a \Leftrightarrow x \le -a \vee x \ge a \end{aligned}$$

Primjer 1. *Riješite sljedeće nejednadžbe*

a) $x^2 \le 100$

b) $x^2 \ge 30$

Primjer 2. *Dokažite da za sve realne brojeve x vrijedi*

$$x \cdot (1 - x) \le \frac{1}{4}.$$

2.1 Nejednadžbe s apsolutnim vrijednostima

Zadatak 1. Riješite sljedeće nejednadžbe:

1. $\left|3x + \frac{1}{2}\right| \le 2$
2. $|2x - 1| \ge 1$
3. $(x + 1)^2 \le 49$
4. $(x - 3)^2 \ge 16$
5. $|4x + 1| \ge -x + 2$
6. $|3x + 1| < 2x + 5$
7. $|x^2 - x - 6| > 4$
8. $|x^2 - 2x - 15| \le x + 13$
9. $|-x^2 + 2x + 3| \ge x + 4$
10. $|-x^2 + x| < 3x - 8$

Zadatak 2. Riješite sljedeće nejednadžbe:

1. $|2x-1|-|x+2| \leq -\frac{1}{3}x+1$
2. $\left|\frac{2}{3}x-2\right|+\left|\frac{1}{2}x+1\right| > -3x+4$
3. $|3x+5|-2|x-1| \leq -4x$
4. $\left|-\frac{1}{2}x+3\right|+\left|\frac{3}{2}x+4\right| > 5$

2.2 Racionalne i iracionalne nejednadžbe

Napomena. *Nejednadžbe tipa*

$$(x-a_1)(x-a_2)\cdot\dots\cdot(x-a_n) > 0,$$

gdje su $a_1, a_2, \dots, a_n \in \mathbb{R}, a_1 < a_2 < \dots < a_n$ *mogu se riješiti tzv. metodom intervala. Na brojevnom pravcu nanesemo brojeve* $a_1, a_2, \dots, a_n$ *u rastućem poretku. Na intervalu iza najvećeg broja* a_n *stavimo znak plus, a na sljedećem lijevom intervalu znak minus, pa zatim opet znak plus, pa minus itd. Rješenje nejednadžbe je unija intervala nad kojim je znak plus. Na sličan način možemo riješiti nejednadžbe tipa*

$$\frac{(x-a_1)(x-a_2)\cdot\dots\cdot(x-a_n)}{(x-b_1)(x-b_2)\cdot\dots\cdot(x-b_n)} > 0,$$

gdje su svi realni brojevi a_i, b_j *međusobno različiti.*

Zadatak 3. Riješite sljedeće nejednadžbe:

1. $\frac{7x+1}{4x-3} \geq -2$
2. $\frac{5-9x}{2x-5} < 3$
3. $\frac{(x-2)(x-5)(x-6)}{(x-3)(x-4)} > 0$
4. $\frac{(x-5)(x+1)^2(x+2)}{(x-1)(x+6)} \leq 0$
5. $\frac{(x^2+x+5)(x+1)(x-3)}{(x+7)(x-4)} \geq 0$
6. $\frac{(-x+2)(x-1)}{(-x-3)(x-5)(-x+3)} > 0$
7. $\frac{(x+1)^3(x-2)}{(x-4)(x+3)(x-5)^2} < 0$

Zadatak 4. Riješite sljedeće nejednadžbe:

1. $\dfrac{x^2-2x}{x^2+8x+15} \geq 0$
2. $\dfrac{x^2-x}{x^2+9x+14} \geq 0$
3. $\dfrac{x^2+x-2}{x^2+9x+20} \leq 0$
4. $\dfrac{x^2-2x+3}{x^2-4x+3} < -3$
5. $\dfrac{x^2-x}{x^2+x+1} < 2$
6. $\dfrac{x^2-1}{x^2+x} < \dfrac{1}{2}$

Napomena.

$$\sqrt[2n]{f(x)} > g(x) \Leftrightarrow \left[(f(x) \geq 0 \wedge g(x) < 0) \vee \left(f(x) > (g(x))^{2n} \wedge g(x) \geq 0\right)\right], \quad n \in \mathbb{N}.$$

Zadatak 5. Riješite sljedeće nejednadžbe:

1. $\sqrt{x-1} > x-3$
2. $\sqrt{x+5} > 7-x$
3. $\sqrt{1-4x^2} \geq 1-3x$
4. $\sqrt{x^2-3x} > x+1$
5. $\sqrt{3x^2-2x-1} \geq x$
6. $\sqrt{5-\sqrt{x+4}} < 2$
7. $\sqrt{2+\sqrt{x^2-2}} > 3$

Napomena.

$$\sqrt[2n]{f(x)} < g(x) \Leftrightarrow \left[f(x) < (g(x))^{2n} \wedge f(x) \geq 0 \wedge g(x) > 0\right], \quad n \in \mathbb{N}.$$

Zadatak 6. Riješite sljedeće nejednadžbe:

1. $\sqrt{4x+10} < 2x+1$
2. $\sqrt{x^2-3x+2} < x$
3. $\sqrt{x^2-5x+4} \leq x-3$
4. $\sqrt{-x^2-x+2} < x+4$
5. $\sqrt{x^2+4x+4} < -x-2$

Zadatak 7. Riješite sljedeće nejednadžbe:

1. $\sqrt{x+1}+\sqrt{x+6}\leq 5$
2. $\sqrt{x-3}+\sqrt{x+4}\leq 2$
3. $\sqrt{5x+4}+\sqrt{5x-4}>\sqrt{10x-6}$
4. $\sqrt{x+1}-\sqrt{2x-3}<\sqrt{x+2}$
5. $\dfrac{1-\sqrt{1-8x^2}}{2x}<1$
6. $\sqrt[3]{5x^2+2x+3}\leq x+1$
7. $\sqrt[3]{3x^2+15x+5}\geq x+2$

2.3 Eksponencijalne i logaritamske nejednadžbe

Napomena.

$$a>1 \Longrightarrow (a^x\leq a^y \Leftrightarrow x\leq y)$$
$$a\in(0,1) \Longrightarrow (a^x\leq a^y \Leftrightarrow x\geq y)$$

Zadatak 8. Riješite sljedeće nejednadžbe:

1. $\left(\dfrac{1}{2}\right)^x<4$
2. $10^{x-2}>\sqrt{10}$
3. $11^{\frac{1}{x}}<\left(\dfrac{1}{11}\right)^{\frac{1}{3-x}}$
4. $0.8\cdot\left(\dfrac{4}{5}\right)^{\frac{1}{x-1}}<\left(\dfrac{5}{4}\right)^{x+\frac{1}{2}}$
5. $3^{\frac{1}{x}}+3^{3+\frac{1}{x}}>84$
6. $2^{2x+5}-3\cdot 2^{x+2}+1\leq 0$
7. $3\cdot 2^{1-x}\geq 1+2^x$
8. $0.4^x-2.5^{x+1}\geq 1.5$
9. $6^{2x+3}<2^{x+7}\cdot 3^{3x-1}$

Zadatak 9. Riješite sljedeće nejednadžbe:

1. $\dfrac{9^x-3^{x+1}+2}{\sqrt{2-x}}\geq 0$

2. $\dfrac{8^x - 4^x - 2^{x+1}}{\sqrt{9 - x^2}} \leq 0$

3. $\dfrac{3^{2x+1} - 4 \cdot 3^x + 1}{3^x - 9^x} \leq 0$

4. $\dfrac{2^{2x+3} - 3 \cdot 2^{x+1} + 1}{2^{1-x} - 1} > 0$

5. $\dfrac{3^x}{3^x - 1} - \dfrac{1}{3^x + 1} \leq 0$

6. $\dfrac{2^x}{5^{x-1}} + 3 < \dfrac{5^x}{2^{x-1}}$

7. $\dfrac{1}{2^x - 4} > \dfrac{1}{2^x - 1}$

Napomena.

$$a > 1 \Longrightarrow (\log_a x \leq \log_a y \Leftrightarrow x \leq y)$$
$$a \in (0, 1) \Longrightarrow (\log_a x \leq \log_a y \Leftrightarrow x \geq y)$$

Zadatak 10. Riješite sljedeće nejednadžbe:

1. $\log_3 \dfrac{x - 2}{x} < 2$

2. $\log_2 \dfrac{3x - 1}{3x + 1} > 1$

3. $\log_{\frac{1}{3}} (x + 1) - \log_3 x > \log_{\frac{1}{3}} 2$

4. $\log_2 x - \dfrac{2}{\log_2 x - 1} \leq 0$

5. $\dfrac{1}{\log x} - \dfrac{1}{\log x - 1} < 1$

6. $\dfrac{\log^2 x + 2 \log x - 6}{\log x} < 21$

Zadatak 11. Riješite sljedeće nejednadžbe:

1. $\log_{1+x} (2 - x) \leq 1$

2. $\log_x \dfrac{3}{8 - 2x} \geq -2$

3. $\log_x 2 \cdot \log_2 4x > 1$

4. $\log \dfrac{10}{x} \cdot \log(10x) \geq \log(\dfrac{1}{10x})$

5. $\log x^2 + \log^2 x \leq 3$

6. $\dfrac{\log_{\frac{1}{2}}(x^2+x-1)}{\log_2(x^2+1)} \geq 0$

7. $\dfrac{\log_{0.1}(2x+\frac{1}{4})}{\log_8(x^2+2)} \geq 0$

8. $\dfrac{x^2+x+3}{\log_{\frac{1}{2}}(x^2-3)} < 0$

9. $\log_{2x}|x^2-5x+6| < 1$

10. $\log|2x+3| + 4\log_{(2x+3)^2} 10 < 3$

Zadatak 12. Riješite nejednadžbu $\sqrt{2^{x-2}+1} \geq |2^{x-1}-1| + 2^x \log_x \sqrt{x}$.

2.4 Trigonometrijske nejednadžbe

Zadatak 13. Riješite sljedeće nejednadžbe:

1. $\cos x \geq -\dfrac{1}{2}$

2. $|\cos x| < \dfrac{\sqrt{3}}{2}$

3. $\mathrm{tg}x \leq 2$

4. $\dfrac{\mathrm{tg}x - \mathrm{tg}\frac{\pi}{5}}{1+\mathrm{tg}x \cdot \mathrm{tg}\frac{\pi}{5}} > -\sqrt{3}$

5. $\sin^2 2x \leq \sin 2x$

6. $\sin^2 x + 2\sin x > 0$

7. $2\cos^2 x + \cos x < 1$

8. $2\sin^4 x - 3\sin^2 x + 1 \geq 0$

9. $2\cos^2 x + 5\cos x + 2 \geq 0$

10. $\mathrm{ctg}^3 x + \mathrm{ctg}^2 x - \mathrm{ctg}x - 1 < 0$

11. $4^{\sin^2 \pi x} + 3 \cdot 4^{\cos^2 \pi x} \leq 8.$

Zadatak 14. Riješite sljedeće nejednadžbe:

1. $3\sin^2 x + 5\cos^2 x - 8\sin x \cos x > 0$

2. $6\sin^2 x - \cos^2 x - \sin x \cos x > 2$

3. $1 + \sin x + \cos x < 0$

4. $\sin x + \cos x > 1$

5. $2\sin^2 x \geq \sin 2x$
6. $\sin x + \cos x + \sin 2x > 1$
7. $\cos x - \sin x + \sin x \cos x > \frac{1}{2}$
8. $\cos 8x - \cos 4x \leq 0$
9. $\cos x + \cos 2x + \cos 3x > 0$
10. $\sin x \cdot \cos 5x < \sin 9x \cdot \cos 3x$
11. $\sin x \cdot \sin 3x > \sin 5x \cdot \sin 7x$

Zadatak 15. Riješite sljedeće nejednadžbe:

1. $\dfrac{2\cos x - 1}{5x - 3 - 2x^2} \geq 0$
2. $\sin x + \cos x < \dfrac{1}{\sin x}$
3. $\dfrac{\sin x + \cos x}{\sin x - \cos x} \geq \sqrt{3}$
4. $\dfrac{\sin 2x - \cos 2x + 1}{\sin 2x + \cos 2x - 1} > 0$
5. $|\sin x| \cdot \cos x > \dfrac{1}{4}$

Zadatak 16. Riješite sljedeće nejednadžbe:

1. $|\sin x| < |\cos x|$
2. $|\sin x| + |\cos x| > 1$
3. $|\mathrm{tg}x + \mathrm{ctg}x| < \frac{4}{\sqrt{3}}$