On the rank of elliptic curves over $\mathbb{Q}(\sqrt{-3})$ with torsion groups $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ and $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$

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Abstract

We construct elliptic curves over the field $\mathbb{Q}(\sqrt{-3})$ with torsion group $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ and ranks equal to 7 and an elliptic curve over the same field with torsion group $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$ and rank equal to 6.

1 Introduction

Let as suppose that E is an elliptic curve defined over a number field K. According to the Mordell-Weil theorem, the group of K-rational points $E(K)$ of E is a finitely generated abelian group. Therefore,

$$E(K) \simeq E(K)_{\text{tors}} \times \mathbb{Z}^r,$$

where $E(K)_{\text{tors}}$ is the torsion group and integer $r \geq 0$ is the rank of E. By Mazur's theorem [9], when $K = \mathbb{Q}$, the torsion group is one of the following 15 groups: $\mathbb{Z}/n\mathbb{Z}$ with $1 \leq n \leq 10$ or $n = 12$, $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2m\mathbb{Z}$ with $1 \leq m \leq 4$. If $K = \mathbb{Q}(\sqrt{-3})$, Najman [11, 12] recently showed that possible torsion group is either one of the groups from Mazur's theorem, or $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ or $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$ (the last two groups are possible only over this quadratic field [6, 7]). It is not known which values of rank are possible. In the case of field $K = \mathbb{Q}$, elliptic curves of rank greater from 28 haven't yet been found (current records of ranks for each of 15 possible torsion groups can be found at `http://web.math.hr/~duje/tors/tors.html`) but the conjecture that there is no upper bound for the rank of elliptic curve is widely accepted.

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In this paper we focused on elliptic curves over the field $\mathbb{Q}(\sqrt{-3})$ with torsion groups $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$, $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$. Rabarison [14] constructed elliptic curves with these torsion groups and ranks ≥ 2, ≥ 3, respectively. We have improved these results and described how we find elliptic curves over the field $\mathbb{Q}(\sqrt{-3})$ with ranks equal to 7 for torsion group $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ and an elliptic curve with rank equal to 6 for torsion group $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$. It is interesting to mention that curves with torsion group $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$ and positive rank are used for factoring numbers of the form $a^{3n} \pm b^{3n}$ (see [2]).

Our main tool for calculating the rank over $\mathbb{Q}(\sqrt{-3})$ is the fact (see, for example, [15]) that if E is an elliptic curve over $\mathbb{Q}$, then the rank of E over $\mathbb{Q}(\sqrt{-3})$ is given by

$$\operatorname{rank}(E(\mathbb{Q}(\sqrt{-3}))) = \operatorname{rank}(E(\mathbb{Q})) + \operatorname{rank}(E_{-3}(\mathbb{Q})), \tag{1}$$

where E_{-3} is the (-3)-twist of E over $\mathbb{Q}$. Searching methods that we used are similar as methods used in [5] (we have implemented them in `PARI/GP` [13]).

We started with a family of elliptic curves $E(t)$ and for curves $E \in E(t)$ with property $t = t_1/t_2$, $|t_1| \leq 100, t_2 \leq 500$, we maximize the sum $S(N,E) + S(N,E_{-3})$ (it is experimentally known that sum $S(N,E)$ is relatively large for curve E with large rank, see [10]), where

$$S(N,E) = \sum_{p \leq N, p \text{ prime}} \frac{2-a_p}{p+1-a_p}, \quad a_p = a_p(E) = p + 1 - \#E(\mathbb{F}_p),$$

and we used $N = 1999$.

In the next step we used Mestre's conditional upper bound [8] for the rank: if

$$G_\lambda(E) = \frac{\pi^2}{8\lambda}\Big(\log N - 2\sum_{p^m \leq e^\lambda} b(p^m) F_\lambda(m \log p)\frac{\log p}{p^m} - M_\lambda\Big),$$

where N is the conductor, $b(p^m) = a_p^m$ if $p \mid N$ and $b(p^m) = \alpha_p^m + {\alpha'}_p^m$ if $p \nmid N$ where α_p, α'_p are the roots of $x^2 - a_p x + p$,

$$M_\lambda = 2\Big(\log 2\pi + \int_0^{+\infty} (F_\lambda(x)/(e^x - 1) - e^{-x}/x)dx\Big),$$

$F_\lambda(x) = F(x/\lambda)$ and

$$F(x) = \begin{cases} (1-x)\cos(\pi x) + \sin(\pi x)/\pi, & x \in [-1,1] \\ 0, & \text{elsewhere} \end{cases},$$

then the rank of elliptic curve E over $\mathbb{Q}$ is $\leq G_\lambda(E)$ (assuming the Birch and Swinnerton-Dyer conjecture and GRH). For curves with large value of $S(N,E)+S(N,E_{-3})$ we used Mestre's upper bound with parameter $\lambda = 12$ to select elliptic curve E which has potentially large rank over $\mathbb{Q}$. To find independent points of infinite order on elliptic curves E and E_{-3} over $\mathbb{Q}$ we used Cremona's program `MWRANK` [4] for curves which have rational 2-torsion points. For other elliptic curves we used Conell's `APECS` [1] or Stoll's `RATPOINTS` [16].

2 Elliptic curves over $\mathbb{Q}(\sqrt{-3})$ with torsion subgroup $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$

According to Rabarison's article [14], a general form of elliptic curves over $\mathbb{Q}(\sqrt{-3})$ with torsion group $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ is

$$y^2+s(ts^2-12)xy+4t(144s^2-24ts^4+432ts+432t^2+t^2s^6-36t^2s^3)y = x^3. \quad (2)$$

The torsion group is generated by the points $T_1 = [0,0]$ and (with corrections of misprints in Rabarison's article)

$$T_2 = \left[12t^2s^3 - 144t^2 + 8ts^4 - 144ts - 48s^2 - \frac{t^2s^6}{3},\right.$$

$$\left.\frac{(3+\sqrt{-3})(ts^3+(6\sqrt{-3}-18)t-12s)(ts^3-(6\sqrt{-3}+18)t-12s)^2}{18}\right].$$

For fixed small s'es we used searching methods that we described earlier, with the following alternation: after calculating Mestre's upper bound for rank, for selected curves we calculated analytic rank by `MAGMA`. For example, for $s = 1$ we get five elliptic curves with the ranks equal to 6 ($t = -51/20, -97/425, -95/389, -74/297, 69/262$), and for $s = 2$ we get two elliptic curves with ranks equal to 6 ($t = -53/247, 33/313$) and two elliptic curves with ranks equal to 7 ($t = 43/171, 97/133$). We will now proceed with giving details only for the last two curves.

When we put $s = 2$ in equation (2) we get the following family of elliptic curves

$$y^2 + (8t-24)xy + 64t(36+30t+13t^2)y = x^3$$

and family of (-3)-twists

$$y^2 + (8t-24)xy + 64t(36+30t+13t^2)y = x^3 - 64(-3+t)^2x^2$$

$$+2048t(-108-54t-9t^2+13t^3)x-28672t^2(36+30t+13t^2)^2.$$

For $t = 43/171$ we have the elliptic curve with minimal Weierstrass equation

$$E: \quad y^2+y=x^3+x^2-42484096963x+3506965787198963,$$

and independent points of infinite order

$$[-114191, 82881102], [-71449, 78598173],$$
$$[127323, 12721285], [277613, 114491289],$$
$$\left[-\frac{742000}{3}, -\frac{1}{2}-\frac{347102063}{18}\sqrt{-3}\right], \left[-\frac{14508298}{3}, -\frac{1}{2}-\frac{110421361925}{18}\sqrt{-3}\right],$$
$$\left[-\frac{522977855649598}{2051310603}, -\frac{1}{2}-\frac{8780527001022491644615}{321838325747082}\sqrt{-3}\right].$$

For $t = 97/133$ we have the elliptic curve with minimal Weierstrass equation

$$E': \quad y^2+y=x^3+x^2-36348070599x+4166981243028849.$$

Independent points of infinite order are

$$[-138469, 80901936], [2068591, 2963211943],$$
$$\left[\frac{17313869}{4}, \frac{71974844343}{8}\right], \left[\frac{15483229569}{64}, \frac{1926602838263967}{512}\right],$$
$$\left[-\frac{81123124}{27}, -\frac{1}{2}-\frac{1458267957839}{486}\sqrt{-3}\right],$$
$$\left[-\frac{706816}{3}, -\frac{1}{2}-\frac{193750613}{18}\sqrt{-3}\right],$$
$$\left[-\frac{147741896293}{164268}, -\frac{1}{2}-\frac{55330636651296391}{115316136}\sqrt{-3}\right].$$

The curves E and E' have ranks equal to 4 over $\mathbb{Q}$ and twisted curves have ranks equal to 3 (we searched for the points on these or isogenous curves by `RATPOINTS` and the numbers of found independent points on all of these curves coincide with their 2-Selmer ranks calculated by `MAGMA`).

3 Elliptic curves over $\mathbb{Q}(\sqrt{-3})$ with torsion subgroup $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$

A general form of an elliptic curve over $\mathbb{Q}(\sqrt{-3})$ with torsion group $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$ is (see [14])

$$y^2 + 2(9t^3 - 30t^2 + 60t - 40)xy - 144(3t-2)(3t^2+4)(3t^2-6t+4)(t-2)^3 y \\ = x^3 - 16(3t-2)(3t^2+4)(3t^2-6t+4)x^2.$$

The torsion group is generated by the points $T_1 = [0,0]$ and

$$T_2 = \Big[-12(t-2)^2(3t^2-6t+4)(3t^2+4), 324(3+\sqrt{-3})(t-2)^2 \\ \times (t - \frac{2}{3}\sqrt{-3})^2 (t - 1 - \frac{1}{3}\sqrt{-3})(t - 1 + \frac{1}{3}\sqrt{-3})^2 (t + \frac{2}{3}\sqrt{-3})^2\Big].$$

The (-3)-twist of this elliptic curve is given with

$$\begin{aligned} & y^2 + 2(-40 + 60t - 30t^2 + 9t^3)xy \\ & - 144(-2+t)^3(-32 + 96t - 120t^2 + 108t^3 - 72t^4 + 27t^5)y \\ & = x^3 - 4(1984 - 5952t + 7440t^2 - 5616t^3 + 2844t^4 - 864t^5 + 81t^6)x^2 \\ & - 1152(-2+t)^3(1280 - 5760t + 11520t^2 - 14688t^3 \\ & + 13824t^4 - 9720t^5 + 4752t^6 - 1458t^7 + 243t^8)x \\ & - 145152(-2+t)^6(32 - 96t + 120t^2 - 108t^3 + 72t^4 - 27t^5)^2. \end{aligned}$$

We noticed that parameters t and $4/(3t)$ give isomorphic elliptic curves over $\mathbb{Q}(\sqrt{-3})$. For parameter $t = -74/469$ we have an elliptic curve with rank equal to 6 ($r(E(\mathbb{Q})) = r(E_{-3}(\mathbb{Q})) = 3$). The minimal Weierstrass equations is

$$\begin{aligned} E: \quad y^2 + xy + y &= x^3 - x^2 - 18764688268349002234286699902 7 \\ & -43285746898654983057699486743376155701770349. \end{aligned}$$

Independent points of infinite order are

$$\begin{aligned} P_1 &= [707406059162101, 13340697112791107526174], \\ P_2 &= \Big[\frac{770832044105421579309 79}{9162729}, \\ & \frac{213721052822394312029045911698065 42}{27735580683}\Big], \end{aligned}$$

$$
\begin{aligned}
P_3 &= \Big[\frac{19017662707559347396918 39}{63632529}, \\
&\quad \frac{262234443659859095955876141313387 2862}{507596683833}\Big], \\
P_4 &= [-216659438724672, \\
&\quad \frac{216659438724671}{2} - \frac{4131272122580067263337}{2}\sqrt{-3}\Big], \\
P_5 &= \Big[\frac{-4488915509374394670}{3481}, \\
&\quad \frac{4488915509374391189}{6962} - \frac{104608643106023193866633 28165}{410758}\sqrt{-3}\Big], \\
P_6 &= \Big[-\frac{977028563514937115787057 3}{37725681361}, \\
&\quad \frac{488514281757466671609460 6}{37725681361} \\
&\qquad - \frac{146909419394869474215515057523524 83884}{7327496816428391}\sqrt{-3}\Big],
\end{aligned}
$$

Furthemore, for parameters $t = 22/89, 35/43, 30/149, 56/117, 104/201, 138/89, -20/59, -38/153$ we have elliptic curves with ranks equal to 5 over $\mathbb{Q}(\sqrt{-3})$ (curve with parameter 22/89 and (-3)-twist of the curve with parameter 56/117 are isogenous elliptic curves and same happens with the pairs 30/149, 104/201 and 35/43, −38/153).

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